# Serre modularity over real quadratic fields

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**Abstract**

We prove Serre's modularity conjecture, in every odd residual characteristic and with arbitrary conductor, over twenty-one real quadratic fields $K = \mathbf{Q}(\sqrt{d})$, under the stated lifting inputs. The proofs for $d = 2, 5, 7, 13, 17, 41$ use classical discriminant bounds. For $d = 3, 6, 33$ the proofs use specified zero-free regions without assuming GRH. In an appendix we derive the region $\Re s > 2759/3000$ from analytic results formalized in Lean; the weaker region $\Re s > 23/25$ suffices for $d = 3$. We retain the separate fixed-scale argument for $d = 6$ and the region $\Re s > 31/32$ for $d = 33$. For the remaining twelve fields we assume GRH for Dedekind zeta functions. The common methods are conductor and weight reduction through compatible systems, modularity lifting for the actual characteristic-zero source, scalar dyadic criteria, and integral trace orders.

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# 1 Introduction

Let $K$ be a real quadratic field. We study Serre's conjecture for continuous absolutely irreducible totally odd representations $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, with $p$ odd and with arbitrary conductor. Here modularity means occurrence in the reduction of the Galois representation of a regular algebraic cuspidal Hilbert modular form. We do not prescribe its weight or level.

Over $\mathbf{Q}$, the conjecture originated in Serre's questions about modular eigenvalues modulo a prime and was subsequently refined by his predictions for weight, level and character [72, Section 3, Problem 1], [73]. The lifting theorems of Wiles, Taylor–Wiles and Skinner–Wiles, together with Taylor's potential modularity result, laid the foundations for its proof [88, 81, 75, 80].

An essential step in this development was the construction of compatible systems containing a given representation. Dieulefait proved such an existence theorem over $\mathbf{Q}$, in its original crystalline range, using potential modularity and descent [22, Theorem 1.1]. This permits a change of residual characteristic while retaining a single characteristic-zero compatible system. The method was used to establish cases of the Fontaine–Mazur conjecture and small-level cases of Serre's conjecture [23, 48]. Khare and Wintenberger subsequently proved Serre's conjecture in full generality [49, 50], with Kisin's two-adic modularity lifting theorem providing the final dyadic input [52]. Later lifting theorems permit shorter versions of this proof, as in Dieulefait–Pacetti [29]. Over totally real fields, the corresponding strategy of connecting residual representations through compatible systems is formulated in the Rayuela conjecture [28].

Over a real quadratic field, discriminant bounds need not exclude all representations of small conductor and weights. For example, the elliptic curve

$$y^2 + \sqrt{6}xy + (1+\sqrt{6})y = x^3 + (1-\sqrt{6})x^2 \\ - (1+3\sqrt{6})x - 1 - 2\sqrt{6}$$

has everywhere good reduction over $\mathbf{Q}(\sqrt{6})$; its label is `2.2.24.1-1.1-a1` in [55]. Thus the base cases must establish modularity of surviving representations, rather than only nonexistence. Base cases of this kind have a natural precedent in the classification of abelian varieties with good reduction everywhere. Fontaine proved that there is no non-zero abelian variety over $\mathbf{Q}$ with good reduction at every prime [39], a result that can serve as the base case of a proof of Serre's conjecture over $\mathbf{Q}$. Schoof developed and extended these methods, using discriminant bounds together with the theory of extensions of finite flat group schemes [67, 68], and applied them to real quadratic fields of small discriminant [69, 70]. Over several of the fields considered here, these results classify the abelian varieties of conductor one: they prove that there are none, or show that they belong to a short explicit list of modular abelian varieties, which in particular gives their modularity. Many of the base cases used below rest on these results, or on adaptations of Schoof's techniques.

Our starting point is the characteristic-three method of Dieulefait–Pacetti over $\mathbf{Q}(\sqrt{13})$ [30]. Successive discriminant comparisons restrict the residual images, and a conductor reduction relates arbitrary representations to the remaining base case. We retain this distinction between the residual base and the conductor argument, remove the restriction at two, and supply new source theorems where residual modularity alone is insufficient. Fontaine's local bounds, refined local ramification calculations, and global discriminant estimates provide the arithmetic input [39, 56, 57]; finite-flat groups and integral trace orders control the scalar and dihedral cases.

For several fields we also use a weight-reduction method. Congruences pivot back and forth between a split prime and auxiliary inert primes: a congruence at an auxiliary prime changes the inertial types at the split places, and the return to the split prime lowers the Serre weights

there, until a lift with small weights is available, which in turn gives rise to a weight 2 lift of small conductor. Modularity is then transferred back along the chain of congruences.

We use the lifting and compatible-system inputs specified in Section 2. When two splits, we also use the dyadic structural and propagation results listed in Section 5, including the cited preprints. The mixed theorem is Theorem A of [31], on the residually reducible mixed ordinary–nonordinary case; it is stated as Theorem 6.4. These inputs have separate roles from the hypotheses on zeta zeros.

In both theorems, the assertion of Serre modularity over $K$ is that, for every odd prime $p$, every continuous absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is modular, with arbitrary conductor. Write $K = \mathbf{Q}(\sqrt{d})$ with $d$ squarefree.

**Theorem 1.1** (Fields without GRH)**.** *Serre modularity holds for*

$$d \in \{2, 3, 5, 6, 7, 13, 17, 33, 41\}$$

*under the compatible-system and lifting inputs of Section 2. The analytic ingredients and further source inputs are as follows.*

*(i) For $d = 2, 5, 7, 13, 17, 41$ the discriminant comparisons use the classical critical strip. For $d = 3$, the proven Theorem B.1, at scale $M_0 = 201/200$, gives $\mathrm{ZF}_M(2759/3000)$, and hence $\mathrm{ZF}_M(23/25)$, for each residual point field $M$. For $d = 6$, Theorem B.11 at scale $M_0 = 419/400$ gives $\mathrm{ZF}(0.9586)$ for its comparison fields. For $d = 33$ use $\mathrm{ZF}(31/32)$ from [17, Theorem 1.1]. The notation $\mathrm{ZF}_M(\sigma)$ and $\mathrm{ZF}(\sigma)$ is defined in Section 13.*

*(ii) For $d = 2, 3, 5, 6, 7$ use the mixed theorem (Theorem 6.4). The terminal characteristic-three proof for $d = 13$ uses [91, Theorems 5.1.1 and 6.1.1]. For $d = 17, 33, 41$ use the split-dyadic lifting and source theorems of Section 5. The scalar endpoint for $d = 17, 41$ uses [25, Theorem 9.4(b)]; for $d = 33$ it uses the general scalar propagation argument.*

**Theorem 1.2** (Fields under GRH)**.** *Assume GRH for Dedekind zeta functions and the compatible-system and lifting inputs of Section 2. Serre modularity holds for*

$$d \in \{10, 11, 14, 15, 19, 21, 29, 37, 57, 65, 73, 89\}.$$

*For $d = 15, 29$ use the mixed theorem. For $d = 57, 65, 73, 89$ use the split-dyadic inputs of Section 5.*

The proofs are grouped by analytic input. The conductor reductions and the implications supplied by modularity lifting are independent of the hypotheses on zeta zeros. In particular, the new proof for $d = 7$ uses no zero-free hypothesis. The proof for $d = 3$ uses the zero-free region proved in Appendix B in its final discriminant comparison. For $d = 29$, Proposition 29.11 describes the exact remaining ordinary source under the region $\Re s > 23/25$; Corollary 29.12 proves completion under $\Re s > 4/5$ for those remaining point fields. GRH implies both requirements and proves Theorem 29.1 without invoking the zero-free estimates of the appendix.

## 1.1 Structural theorems

Among the techniques used in the proof of the two theorems, we single out three mechanisms: conductor reduction to a specified terminal characteristic, scalar modularity from a quadratic Selmer condition, and coefficient-field rigidity from simultaneous integral trace orders. We state them separately, since their hypotheses play different roles.

**Conductor reduction.** A strongly compatible system, and the conventions for weight two, are specified in Section 2. The following theorem concerns a system already constructed in characteristic zero; it does not supply an initial lift of an arbitrary residual representation.

**Theorem 1.3** (Conductor reduction at a split terminal prime)**.** *Let $K$ be real quadratic, suppose two does not split in $K$, and let $\ell$ be an odd rational prime that splits in $K$. Assume:*

*(i) every absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ of determinant $\bar{\chi}_\ell$ and unramified outside $\ell$ is modular;*
*(ii) for every reducible representation with that determinant and ramification support, its constituent quotient is totally odd and extends to $G_{\mathbf{Q}}$.*

*Then every nondihedral potentially modular weight-two strongly compatible system over $K$, whose determinant is cyclotomic times a finite totally even character, is modular.*

The proof is given in Section 6. The quotient in hypothesis (ii) is automatically totally odd, since its two constituents have product $\bar{\chi}_\ell$. The theorem does not assume that a modular residual witness makes an arbitrary regular source modular: the weight resets and retained local types are part of its proof.

**A scalar Selmer condition.** Suppose two splits in $K$, with dyadic places $w_0, w_1$. Put

$$V^+_{w_0}(K) = \left\{ [a] \in K^\times/K^{\times 2} : \begin{array}{l} a \text{ is totally positive,} \\ v_{\mathfrak{q}}(a) \equiv 0 \pmod 2 \quad (\mathfrak{q} \nmid 2), \\ a \in K^{\times 2}_{w_1} \end{array} \right\}. \tag{1}$$

This finite squareclass space classifies totally real quadratic extensions unramified outside $w_0$ and split at $w_1$. Quadratic conjugation exchanges the two choices of dyadic place. The condition we use is

$$V^+_{w_0}(K) = 0. \tag{SV}$$

It is not an odd-narrow-class-number condition.

**Theorem 1.4** (Scalar dyadic modularity)**.** *Let $K$ be a real quadratic field in which two splits and* (SV) *holds. Under the dyadic structural, propagation and classicality inputs of Section 5, let $E/\mathbf{Q}_2$ be finite and let $r : G_K \to \mathrm{GL}_2(E)$ be continuous and absolutely irreducible, unramified outside finitely many places, with $\det r = \varepsilon_2$ and $\bar{r}^{\mathrm{ss}} = \mathbf{1} \oplus \mathbf{1}$. Suppose that at both dyadic places $r$ is absolutely irreducible, de Rham and has distinct Hodge–Tate weights. Then $r$ is automorphic.*

The theorem imposes no restriction on finite ramification or on the regular weights, and its proof uses no GRH. Sections 7–8 construct the auxiliary fields and prove the theorem. Together with a residual base at two and a two-power ordinary class number, it gives the split-dyadic Serre criterion of Theorem 10.1. The seven split-dyadic fields in the main theorem satisfy these arithmetic conditions. The Selmer vanishing is also valid when the narrow class number is even.

**Integral trace orders and symmetry.** Let $\mathcal{R}$ be a strongly compatible system of cyclotomic determinant over a real quadratic field $K$, with trace field $D$. For finitely many coefficient orbits, form their simultaneous integral matrix order $E_0$, including the trace scalars and every dyadic coefficient place, as in Section 11. Its rank is four times the sum of the trace-field degrees. The integral order, rather than a single residual two-dimensional realization, is essential here.

**Theorem 1.5** (A trace-order bound and quadratic symmetry)**.** *Let $K/\mathbf{Q}$ be real quadratic with conjugation $\sigma$, and let $\mathcal{R}$ be an absolutely irreducible strongly compatible rank-two system with cyclotomic determinant and trace field $D$. Let $h, \eta$ be restrictions of quadratic characters of $G_{\mathbf{Q}}$. Form the coefficient orbits generated by*

$$H(\mathcal{R}) = \mathcal{R} \otimes h, \qquad G(\mathcal{R}) = \mathcal{R}^\sigma \otimes \eta.$$

*Suppose* $\mathcal{R} \not\simeq \mathcal{R} \otimes h$ *and their simultaneous integral order satisfies* $\dim_{\mathbf{F}_2}(E_0/2E_0) < 32$. *Then either*

$$\mathcal{R}^{\sigma} \simeq \mathcal{R} \otimes \eta \quad \text{or} \quad \mathcal{R}^{\sigma} \simeq \mathcal{R} \otimes \eta h,$$

*or* $[D : \mathbf{Q}] \leq 2$ *or* $D/\mathbf{Q}$ *is biquadratic. In the latter alternatives every coefficient prime has residue degree at most two. In the former alternatives, a totally odd regular system is modular by quadratic descent.*

The arithmetic work in applying Theorem 1.5 lies in bounding the actual order. The finite-flat ramification theorem 11.13 and the regular-order construction in Section 11 supply different parts of this argument. The bound of twenty-six over $\mathbf{Q}(\sqrt{11})$ gives the coefficient-field alternative directly. For $d = 3$, we use the argument at eleven in Section 20.

Two further source theorems have a separate role. Proposition A.2 constructs the initial lift in the exceptional characteristic-five case over $\mathbf{Q}(\sqrt{5})$, which the general prescribed-lift theorem excludes. Theorem 28.33 proves modularity of the nonordinary reducible characteristic-three source over $\mathbf{Q}(\sqrt{21})$. Its proof kills the anti-invariant part of the full pseudodeformation cotangent space and applies Lemma 12.2; no finiteness theorem for the whole pseudodeformation ring is needed. These are source theorems, not consequences of residual-image bounds alone.

## 1.2 Field applications

| Analytic input | Radicands $d$ | Terminal characteristics in the same order |
|---|---|---|
| Classical critical strip | $2, 5, 7, 13, 17, 41$ | $7, 11, 3, 3, 2, 2$ |
| Narrower zero-free strip | $3, 6, 33$ | $11, 5, 2$ |
| GRH | $10, 11, 14, 15, 19, 21$ | $3, 5, 5, 7, 3, 5$ |
| GRH | $29, 37$ | $7 \to 5, 3$ |
| GRH | $57, 65, 73, 89$ | 2 in each field |

Part I gives the common constructions, including the conductor reductions, source-lifting statements and one explicit formula for all three analytic regimes. Part II proves Theorem 1.1, beginning with the classical cases and then the fields requiring narrower strips. Part III proves Theorem 1.2. The appendices give the longer analytic arguments and finite calculations, with their certificates cited where used.

Buzzard–Diamond–Jarvis formulated the weight conjecture at unramified residual primes [12]; see also [42]. Our modularity results yield the weight part of Serre's conjecture under the image hypotheses of Gee–Liu–Savitt [44, Theorem 6.2.1]. Namely, suppose the residual representation remains absolutely irreducible after adjoining the roots of unity of order the residual characteristic and, in characteristic five, that the projective image of this restriction is not the alternating group on five letters. Then the occurring Serre weights are exactly those prescribed by the local crystalline-lift recipes, including at ramified residual primes and for all predicted companion weights. These consequences inherit the assumptions of the corresponding modularity theorem; the weight theorem introduces no further analytic hypothesis.

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# Part I
# Common methods

## 2 Compatible systems and congruences

We use $d$ for the squarefree radicand, $D$ for a fundamental discriminant, and $q$ for the residual coefficient-field cardinality. The finite arithmetic and numerical checks are supplied in [20, 59]. The longer certificate descriptions appear in the appendices.

Throughout, $G_F = \mathrm{Gal}(\overline{F}/F)$ and the cyclotomic character $\chi_p$ has labelled Hodge–Tate weight one. Irreducibility means absolute irreducibility. A representation is totally odd if every real complex conjugation has determinant $-1$.

A system over a totally real field is a pure, regular, totally odd, absolutely irreducible, strongly compatible rank-two system. We require compatibility of the Frobenius-semisimplified Weil–Deligne representations at every finite place, including a place above the coefficient characteristic, and compatibility of labelled weights. Its conductor support consists of the places where the common Weil–Deligne representation is ramified or has nonzero monodromy. Outside this support its $p$-adic member is crystalline above $p$ and unramified at the other places. Weight two means labelled weights $\{0, 1\}$. Regular weights may include negative integers.

On the closure of a special local component the Frobenius eigenvalues have ratio $\mathrm{N}(v)$. For a pure system, monodromy cannot vanish at such a place: otherwise both eigenvalues would have the same complex absolute value, contradicting that ratio.

We write $\mathcal{R} \rightsquigarrow \mathcal{R}'$ when a congruence, possibly followed by a finite character twist, has the property that modularity of $\mathcal{R}'$ implies modularity of $\mathcal{R}$. Every such implication below is an application of a lifting theorem to the actual old characteristic-zero member. Finite twists preserve modularity. Equality of good Frobenius polynomials and Chebotarev then propagate modularity from one member to the whole system.

### 2.1 Residual witnesses and lifting

**Lemma 2.1.** *Let $F$ be totally real of even degree. An irreducible residual representation with a regular algebraic cuspidal Hilbert modular witness has a parallel-weight-two cuspidal witness, after allowing arbitrary level.*

*Proof.* Transfer the witness to the quaternion algebra over $F$ ramified at every real place and split at every finite place. At sufficiently small level $U = U^p U_p$, reduction of a primitive integral eigenvector is nonzero, with coefficient module $V$ over a finite field $k$. Its $U_p$-action factors through a finite quotient $G$; write $U_p'$ for the kernel. The map $V \hookrightarrow \mathrm{Map}(G, V)$ given by $v \mapsto (g \mapsto gv)$ is equivariant for $(hf)(g) = f(gh)$. The target is a sum of regular representations. Pointwise application and induction in stages give the same away-from-$p$ Hecke eigencharacter in scalar forms of level $U^p U_p'$. Their integral space is $\mathcal{O}^X$ for a finite double-coset set $X$, so reduction is surjective. Deligne–Serre's lifting lemma [19, Lemme 6.11] lifts the eigencharacter to characteristic zero. A norm character would have reducible residue; inverse Jacquet–Langlands therefore gives a parallel-weight-two cuspidal form.

Retaining both operators $T_v, S_v$, with polynomial $X^2 - T_v X + \mathrm{N}(v) S_v$, preserves the residual trace and determinant. Chebotarev and Brauer–Nesbitt identify the residue. No exact characteristic-zero central character or ordinary ordering is prescribed by this argument. □

Totally odd solvable residual representations have solvable Artin witnesses, followed by regular weight raising, in the scopes of automorphic induction and Langlands–Tunnell; see

[3, Theorem 1.2(1), Lemma 2.1 and proof of Theorem 2.5]. Lemma 2.1 gives weight two whenever it is needed. We use Dickson's classification and the rank-two adequacy criterion [6, Proposition A.2.1] throughout.

**Lemma 2.2.** *Let $K$ be real quadratic, $p > 2$, and let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be finitely ramified and totally odd, with modular residue whose restriction to $G_{K(\zeta_p)}$ is irreducible. Then $r$ is modular in either case:*

*(i) $p$ splits in $K$, and $r$ is potentially semistable with distinct labelled weights at the places above $p$;*
*(ii) $r$ is de Rham of labelled weights $\{0, 1\}$ above $p$, and its residue has a parallel-weight-two modular witness.*

*Proof.* Apply respectively [84, Theorem 4.7] and [3, Theorem 2.3], dualizing to their inverse-cyclotomic normalization when required. In (ii), the parallel-weight-two witness gives the required weight-zero automorphic convention. □

**Lemma 2.3.** *Let $F$ be a totally real extension of a real quadratic field, and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be finitely ramified and totally odd, with determinant $\chi_2\psi$, where $\psi$ has finite order. Suppose its residue is nonsolvable. Then $r$ is modular in either case:*

*(i) $r$ is nearly ordinary of regular weights at every dyadic place and its residue has a nearly ordinary regular modular witness;*
*(ii) $r$ is potentially Barsotti–Tate and potentially nonordinary at every dyadic place and its residue has a regular modular witness.*

*Proof.* The first case is [2, Theorem 5.2.1]. If the line and quotient have weights $b_\tau > a_\tau$ with sum one, its parameters are $w_\tau = a_\tau$ and $k_\tau = b_\tau - a_\tau + 1$, so $k_\tau + 2w_\tau = 2$. Nonsolvability excludes the CM-dihedral exception. The theorem does not require a marked residual ordinary ordering or distinct residual characters.

In the second case choose finite Galois extensions of the dyadic completions over which the source is Barsotti–Tate. The solvable local realization theorem [5, Chapter X, Section 2], in the form [16, Theorem 3.1 and Example 3.3], realizes them in a solvable totally real Galois extension. No degree is prescribed. The nonsolvable residual projective image is $\mathrm{PSL}_2(\mathbf{F}_{2^a})$, $a \geq 2$; its image after solvable Galois restriction is normal with solvable quotient and is unchanged. Solvable base change retains a cuspidal modular witness, put in weight two by Lemma 2.1. Apply [52, Theorem 3.3.5]. Its condition that a dyadic completion equal $\mathbf{Q}_2$ applies only at potentially ordinary places, and there are none.

For descent take a cyclic series. Strong multiplicity one makes the automorphic representation invariant at each step because the Galois representation extends downstairs. The descended Galois representation differs from the given one by a character of the cyclic quotient. Twisting and iterating proves modularity over $F$; see [7, Lemma 2.2.2]. □

**Lemma 2.4.** *Let $K$ be real quadratic and let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be continuous, unramified outside finitely many primes, crystalline of labelled weights $\{0, 1\}$ at two, and of determinant $\chi_2$. Suppose its residual image is $\mathrm{SL}_2(\mathbf{F}_q)$, with $q \geq 4$, and its residual point field is not totally real. If the residue is modular or $q = 4$, then $r$ is modular. If $r$ is a member of a system, that system is modular.*

*Proof.* The residual image makes $r$ absolutely irreducible, and its cyclotomic determinant makes it totally odd. For $q = 4$, [3, Theorem 2.5] supplies a regular cuspidal residual witness. The point-field hypothesis says that some real conjugation is nontrivial in the residue. Apply [83, Theorem 6.1] to $r^\vee$, whose weights match that theorem's convention, and dualize back. Strong compatibility gives the final assertion. □

**Lemma 2.5.** *Let $K$ be real quadratic and $\bar{r} : G_K \to \mathrm{SL}_2(\mathbf{F}_8)$ have full image. Suppose its point field $M$ is Galois over $\mathbf{Q}$, and some real conjugation acts nontrivially in $\bar{r}$. Then the actual residue extends to $G_{\mathbf{Q}}$ and is modular over $K$.*

*Proof.* The outer automorphism group of $\mathrm{SL}_2(\mathbf{F}_8)$ has order three. The conjugation action of $\mathrm{Gal}(M/\mathbf{Q})$ on $\mathrm{Gal}(M/K)$ is therefore inner, because it is inner on a subgroup of index two. The center is trivial, so the unique inner conjugating elements give a homomorphism $\mathrm{Gal}(M/\mathbf{Q}) \to \mathrm{Gal}(M/K)$ restricting to the identity on $\mathrm{Gal}(M/K)$. Compose with the given matrix realization to extend the actual residue to $G_{\mathbf{Q}}$. Rational Serre modularity [49, 50] and quadratic base change give a regular cuspidal witness over $K$. Its cuspidality follows from irreducibility of the restriction. □

## 2.2 Globalization and compatibility

**Lemma 2.6.** *Let $K$ be real quadratic, fix a finite set $S$ of places containing the coefficient places and all residual ramification, and let $\bar{r}$ be totally odd with irreducible cyclotomic restriction. At coefficient characteristic two assume nonsolvable image. Fix a totally odd global de Rham determinant lifting its determinant and unramified outside $S$, and nonempty compatible local components with regular weights at the coefficient places. Outside the exceptional characteristic-five case of [35, Definition 4.1.1], these components admit a global lift unramified outside $S$. Once such a lift has been constructed, it is potentially modular and belongs to a strongly compatible system in the following cases:*

- *(i) odd coefficient characteristic and parallel weight two, with irreducible cyclotomic residual restriction;*
- *(ii) odd coefficient characteristic, adequate cyclotomic residue, and regular potentially crystalline local components that are potentially diagonalizable;*
- *(iii) coefficient characteristic two, determinant $\chi_2$, and either ordinary crystalline components of regular weights at all dyadic places, ordinary semistable components of weight two at all dyadic places, or potentially Barsotti–Tate and potentially nonordinary components of weight two at all dyadic places.*

*Proof.* Existence is [35, Theorem 4.2.1 and Remark 1.1.2]. Away from the coefficient prime, inertial conditions and conductor bounds specify unions of components. At that prime, take components through the supplied points; potential diagonalizability persists after a fixed sufficiently large local extension.

In (i), first use the avoiding weight-two residual witness of [77, Theorem 5.1.1], followed by the actual-source theorem [3, Theorem 2.3]. The potential-modularity field is chosen Galois and disjoint from the residual kernel and cyclotomic fields. Thus the actual source becomes modular after this extension. This argument starts with the weight-two point already constructed; it does not assume the existence of an exceptional initial characteristic-five point.

For (ii), choose type A at all coefficient places in Snowden's avoiding theorem [77, Theorem 5.1.1]. This gives an ordinary potentially crystalline weight-two residual witness over a totally real Galois field disjoint from the residual and cyclotomic fields. Choose its finite determinant factor to lift $\det \bar{r}/\bar{\chi}_p$. Its local representations are potentially diagonalizable by [7, Lemma 1.4.3(1)]. Pass to a CM quadratic extension disjoint from the same fields and apply [27, Theorem 1] to the actual source with polarization multiplier $\det r$. The residual polarized representation is automorphic, adequacy persists, and the CM field excludes $\zeta_p$. Cyclic descent gives modularity over the totally real Galois field.

At two use the Hilbert–Blumenthal construction in the proof of [83, Proposition 6.7]. In its Moret–Bailly datum take each finite Galois dyadic field large enough to trivialize the local

residue and, in the potentially nonordinary case, to make the actual source Barsotti–Tate. The ordinary good-reduction neighbourhoods in that proof remain available after this enlargement. Put the global residual field and the prescribed avoidance field in the avoidance datum. This gives one totally real Galois extension of the original base, with the required local completions and with the nonsolvable residual image unchanged. The automorphic Hilbert–Blumenthal abelian variety in the proof already supplies a parallel-weight-two, ordinary residual witness there, before any further solvable base change to lower its level away from two. It suffices to take this witness, since the lifting theorem permits finite ramification away from two. Apply Lemma 2.3 over this Galois extension to the actual source. The ordinary case also applies to the ordinary weight-two semistable components. This gives the actual avoiding potential modularity used below.

The construction of compatible systems by potential modularity and descent originates in Dieulefait's work over $\mathbf{Q}$ [22, Theorem 1.1]. For the totally real fields and local conditions here, use the Brauer-induction construction of [77, Proposition 9.2.1 and Remark 9.2.2], retaining the finite characters in the Brauer identity. Mackey's formula and Frobenius reciprocity express the self-pairing of each virtual member in terms of pairings of restricted modular representations with finite character twists. Each restriction is absolutely irreducible: the subgroups in the Mackey formula contain the potential-modularity subgroup, on which these representations restrict to the cuspidal system, its conjugates or character twists. After extending coefficient scalars, each pairing is zero or one. Compatibility and Chebotarev decide equality by Frobenius polynomials, so these pairings are independent of the coefficient prime. At the original prime the self-pairing is one, so each virtual member is an irreducible or its negative. Its dimension two excludes the negative. Local–global compatibility for the modular systems [74, Theorem 1 and equation (1)] includes the coefficient prime and arbitrary cohomological weights. Restriction–induction identities give the common Weil representations; purity determines monodromy [7, Lemma 1.3.6(2)–(3)]. Labelled weights are compatible by the same restriction argument. □

## 2.3 Initial systems for odd residual characteristic

**Proposition 2.7.** *Let $K$ be real quadratic and $p$ odd. A continuous absolutely irreducible totally odd residual representation with solvable projective image is modular. If its projective image is nonsolvable, it is the residue of a nondihedral, potentially modular, strongly compatible weight-two system with determinant cyclotomic times a finite totally even character. For an exceptional characteristic-five representation over $\mathbf{Q}(\sqrt{5})$ use the lift of Appendix A.*

*Proof.* For solvable image, the odd solvable Artin witness and regular weight raising are as in Section 2. Otherwise the restriction to $G_{K(\zeta_p)}$ is absolutely irreducible: its projective image contains the nonsolvable derived subgroup. If $K \neq \mathbf{Q}(\sqrt{5})$, the characteristic-five exception in the prescribed-lift theorem would require the projective $S_5$ kernel field to contain $K(\zeta_5)$, a cyclic quartic extension, whereas $S_5$ has abelianization of order two. The exceptional condition for the initial lift in [3, Theorem 2.2] is also impossible, since the projective determinant-squareclass character has order at most two and $\bar{\chi}_5$ has order four over $K$.

In the nonexceptional cases use [3, Theorem 2.2] and the weight-two compatible-system construction of Lemma 2.6(i). Over $\mathbf{Q}(\sqrt{5})$ in the exceptional case, $\bar{\chi}_5$ is quadratic and cuts out $K(\zeta_5)$, so the two exceptional conditions agree. Use Proposition A.2, which also supplies avoiding potential modularity and compatibility. Nonsolvability of the given residue makes the system nondihedral. □

## 3 An auxiliary prime for congruences

In this section $K$ is real quadratic. The finite set $S$ may include any prescribed terminal reset primes.

**Proposition 3.1.** *Let $\mathcal{R}$ be a nondihedral weight-two system of determinant $\chi\psi$, where $\psi$ has finite two-power order. Fix a finite set $S$ of rational primes containing $2, 3$, all primes dividing the discriminant of $K$, and every conductor prime. There are a prime $N > \max(S \cup \{5\})$, a degree-one prime $\lambda$ of $K$ of norm $l > N$, and a congruent weight-two system $\mathcal{R}'$ such that $\mathcal{R} \rightsquigarrow \mathcal{R}'$ in characteristic $N$, its determinant remains $\chi\psi$, and its only new conductor prime is $\lambda$. At $\lambda$ its type is induced from the unramified quadratic extension with inverse inertia characters of order $N$.*

*While this type is retained and conductor is supported at places over $S \cup \{l\}$, the residual representations at every rational prime in $S$ have nonsolvable image. At odd such characteristics their cyclotomic restrictions are adequate.*

*Proof.* Let $K_S/K$ be the maximal elementary quadratic extension unramified outside the places over $S$ and infinity. It is finite. Write $2^a$ for the order of $\psi$. By [28, Proposition 4.6], choose $N > \max S$, $N > 5$, split in $K$ and the coefficient field, with $N \equiv 1 \pmod{2^{\max\{2,a+1\}}}$, such that a residual member contains $\mathrm{SL}_2(\mathbf{F}_N)$. Its determinant is surjective on inertia above $N$, so its image is $\mathrm{GL}_2(\mathbf{F}_N)$.

Let $R/K$ be the projective kernel field. Prescribe Frobenius to act trivially on $K_S$, by inversion on $K(\zeta_N)$, and as projective complex conjugation on $R$. The first two prescriptions agree on the intersection, since $K(\zeta_N)/K$ is totally ramified above $N$, whereas $K_S/K$ is unramified there. The only nontrivial solvable quotient of $\mathrm{PGL}_2(\mathbf{F}_N)$ is its determinant-squareclass quotient. Our congruence makes $\bar{\psi}$ square-valued, so that quotient is cut out by $\overline{\chi}_N^{(N-1)/2}$ and lies in $K(\zeta_N)$. Inversion fixes it, and projective complex conjugation belongs to $\mathrm{PSL}_2(\mathbf{F}_N)$, since $-1$ is square modulo $N$. The prescriptions therefore agree on $R \cap K_S K(\zeta_N)$.

Chebotarev gives a positive-density set of such prime ideals. Those of residue degree greater than one have zero prime-ideal density. We may consequently choose $\lambda$ of degree one and arbitrarily large norm $l$, outside the rational primes already used. Thus $l$ splits in $K$, $l \equiv -1 \pmod{N}$, and the residual Frobenius eigenvalues are $a, -a$.

Put $\delta = \chi_N \psi$. Choose $\alpha$ lifting $a$ with $\alpha^2 = -\delta(\mathrm{Frob}_\lambda)$, and define local inertia and Frobenius by

$$T = \begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_N^{-1} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}.$$

They satisfy $\Phi T \Phi^{-1} = T^l$ and have determinant $\delta$; reduction gives the original unramified local representation. Retain the old crystalline weight-two lifts above $N$ and compatible conditions elsewhere. Lemma 2.6 gives a global weight-two lift and its system. Lemma 2.2 at the split prime $N$ gives backward modularity.

For a later coefficient characteristic $t \in S$, the two order-$N$ inertia characters at $\lambda$ remain distinct and are exchanged by Frobenius. The residual representation is therefore irreducible and its projective image contains an element of order $N > 5$, excluding $A_4, S_4, A_5$. If it were dihedral, the inducing quadratic extension would be unramified outside $S$: at $\lambda$ its odd-order inertia lies in the rotations. But Frobenius at $\lambda$ exchanges the rotation eigenspaces, making $\lambda$ inert in that quadratic extension, contrary to its complete splitting in $K_S$. Dickson's classification forces nonsolvable projective image. Cyclotomic restriction retains its perfect derived subgroup, hence irreducibility. The order-$N$ element excludes the inadequate characteristic-three and characteristic-five cases; for $t > 5$, the usual rank-two criterion applies. In characteristic two, nonsolvability is the required conclusion. □

# 4 Local lifts and persistent inertial types

**Lemma 4.1.** *Let $F/\mathbf{Q}_p$ be finite, with $p > 2$, and let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ have determinant $\overline{\chi}_p$. Then $\overline{r}$ has a regular crystalline, potentially diagonalizable lift with determinant exactly $\chi_p$.*

*Proof.* Apply [11, Proposition 2.7] with prescribed determinant $\chi_p$. In rank two, its construction also gives potential diagonalizability. An irreducible residual representation is induced from a character of the unramified quadratic extension; the constructed lift is induced from a crystalline character, so becomes a sum of crystalline characters after restriction. A reducible representation is lifted as a crystalline extension of crystalline characters, to which [7, Lemma 1.4.3(1)] applies. The unramified scalar correction in the fixed-determinant construction preserves both properties. □

**Lemma 4.2.** *Let $F/\mathbf{Q}_p$ be finite, and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be potentially semistable with labelled Hodge–Tate weights $\{0, 1\}$ and determinant $\chi_p$ times a finite-order character. If its monodromy is nonzero, then $r$ is potentially ordinary. Thus a potentially nonordinary representation with these weights is potentially Barsotti–Tate.*

*If $F = \mathbf{Q}_p$ and $r$ is attached to a weight-two newform which is Steinberg at $p$, with determinant $\chi_p$ and $a_p = s \in \{1, -1\}$, its ordinary quotient is unramified with Frobenius value $s$.*

*Proof.* Pass to a finite extension over which $r$ is semistable and the finite determinant factor is trivial. Apply the filtered-module functor to $r^\vee$, with filtration jumps zero and one. If the residue degree is $f$ and $N \neq 0$, the relation $N\varphi = p\varphi N$ gives eigenvalues $\alpha, p^f\alpha$ for $\varphi^f$, with the first eigenline equal to $\ker N$. Equality of the total Hodge and Newton numbers gives $v_p(\alpha) = 0$.

Every labelled Hodge number on this line is zero or one. Weak admissibility bounds their average by the Newton slope zero, so all are zero. The line is weakly admissible and corresponds to a crystalline weight-zero, hence unramified, subrepresentation of $r^\vee$. Its dual is an ordinary quotient of $r$. Consequently potential nonordinarity forces $N = 0$, and the resulting potentially crystalline representation with these weights is potentially Barsotti–Tate.

In the last assertion, local–global compatibility identifies the Frobenius eigenvalues on the dual module as $s, ps$ [74, equation (1) and Theorem 1]. The same argument gives its unramified line with unit eigenvalue $s$; dualizing replaces the Frobenius value by $s^{-1} = s$. □

**Lemma 4.3.** *Let $F/\mathbf{Q}_p$ be finite with absolute ramification index $e$, and let $r : G_F \to \mathrm{GL}_2(E)$ be crystalline of labelled weights $\{0, 1\}$ and determinant $\chi_p$. Suppose the residual semisimplification is a sum of characters. Then $r$ is ordinary in either of the following cases.*

*(i) $e < p - 1$, and the two characters are unramified twists of $1$ and $\overline{\chi}_p$.*
*(ii) $p \in \{2, 3\}$, $e = p - 1$, and both characters are unramified.*

*Proof.* A stable lattice is realized by a Barsotti–Tate group with coefficient action. A nonzero étale quotient has coefficient rank one by the weights and gives ordinarity. Otherwise the group is connected. Its Cartier dual is also connected because $r^\vee(1) \simeq r$ and connectedness is invariant under isogeny. The coefficient-uniformizer torsion is then biconnected.

After an unramified base change trivializing the unramified characters, the stable residual line contains a stable $\mathbf{F}_p$-line. Its schematic closure has order $p$. In (i) its generic character is trivial or cyclotomic. The rank-one model is étale or multiplicative by finite-flat uniqueness for $e < p - 1$ [65]. In (ii) its generic character is unramified. The Oort–Tate valuations lie between zero and $p - 1$ [79, Theorem 2]. The endpoints give étale and multiplicative models. The only possible interior valuation is one for $p = 3$, and it gives a ramified quadratic generic character. In either case the closure contradicts biconnectedness, since a multiplicative subgroup gives

an étale quotient of the dual. The canonical étale quotient descends over the unramified base change. □

**Lemma 4.4.** *Let $p$ be odd and let $r : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(E)$ be crystalline with weights $\{0, k-1\}$, where $2 \le k \le p+1$. If its residual semisimplification is reducible, then $r$ is ordinary. For weights $\{a, a+k-1\}$ the same conclusion holds after the local twist by $\chi_p^{-a}$.*

*Proof.* An absolutely irreducible crystalline source of these weights is, up to an unramified twist, one of the representations $V_{k,a_p}$ with $v_p(a_p) > 0$. For $2 \le k \le p+1$, [8, Corollary 4.1.3 and Proposition 4.1.4] identify its reduction with the irreducible niveau-two representation $\mathrm{Ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \omega_2^{k-1}$, up to the same twist; see also [48, Proposition 3.2] at $k = p+1$. Thus the source is reducible after extending coefficients. It is ordinary: an extension with the smaller weight on its invariant line splits, since the crystalline extension group of a negative-weight character is zero [9, Corollary 3.8.4]. Twisting proves the last assertion. □

**Lemma 4.5.** *Let $F$ be totally real, let $p$ be odd, and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be continuous, absolutely irreducible, totally odd and unramified outside finitely many places, with determinant $\chi_p$. Suppose that $r$ is de Rham and ordinary of regular labelled weight at every place above $p$, and that*

$$\bar{r}^{\mathrm{ss}} = \bar{\psi}_1 \oplus \bar{\psi}_2.$$

*Here the ordinary filtration has the larger labelled weight on its invariant line at each embedding. Then $r$ is modular if $(\bar{\psi}_1/\bar{\psi}_2)|_{G_{F(\zeta_p)}} \neq 1$. The same conclusion holds without this quotient condition when $F/\mathbf{Q}$ is abelian, $p \ge 5$, and $p$ is unramified in $F$.*

*Proof.* Apply [32, Theorem 1.2.1 and Corollary 1.2.2] to $r^\vee$, whose determinant is $\chi_p^{-1}$. The ordinary filtration of the dual has exactly the ordering required in [32, §1.6]. The weight there is an arbitrary collection of dominant integral pairs, so the labelled weights need not be parallel. Inverting the residual quotient preserves its nontriviality on $G_{F(\zeta_p)}$. □

**Lemma 4.6.** *Let $K$ be real quadratic and let $\ell > 3$ be unramified in $K$. Suppose an absolutely irreducible characteristic-$\ell$ residue has a crystalline lift of weights $\{0, 1\}$ and determinant $\chi_\ell$ at at least one place above $\ell$. Then its restriction to $G_{K(\zeta_\ell)}$ is absolutely irreducible.*

*Proof.* A reducible cyclotomic restriction would, by Clifford theory, give induction from a character of the quadratic subextension of $K(\zeta_\ell)/K$. Since $\ell$ is unramified in $K$, this quadratic extension is ramified at every $\ell$-place. The inducing character has order prime to $\ell$, so the image has order prime to $\ell$. At the chosen place projective inertia is therefore cyclic and contains a reflection of the dihedral image; its order is two.

Let $f \in \{1, 2\}$ be the residue degree of that place. The finite-flat characters have labelled digits zero or one, and the determinant makes the two digits at each embedding complementary [65]. If the local residue is reducible, the pairs of inertial exponents are $(0, 1)$ modulo $\ell - 1$ for $f = 1$, and $(0, \ell+1)$ or $(1, \ell)$ modulo $\ell^2 - 1$ for $f = 2$. Their projective orders are respectively $\ell - 1$, and $\ell - 1$ or $\ell + 1$.

For a locally irreducible residue the niveau-$2f$ characters are conjugate by $\ell^f$. For $f = 1$ their quotient has order $\ell + 1$. For $f = 2$ the possible exponents are

$$\ell + 1, \quad \ell^2 + \ell, \quad \ell^3 + 1, \quad \ell^3 + \ell^2 \pmod{\ell^4 - 1},$$

paired by multiplication by $\ell^2$. Dividing their differences by $\ell^2 - 1$ gives $\pm 1 \pm \ell$ modulo $\ell^2 + 1$. Since $\gcd(\ell^2 + 1, \ell \pm 1) = 2$, the projective order is $(\ell^2 + 1)/2$. Every displayed order exceeds two when $\ell > 3$, a contradiction. Only the crystalline hypothesis at the chosen place was used. □

**Lemma 4.7.** *Let $w$ have norm two.*

*(i) In odd coefficient characteristic $\ell$, a local residue at $w$ of determinant $\overline{\chi}_\ell$, with unipotent inertia and Frobenius trace congruent to $\pm 3$, admits a special lift of determinant $\chi_\ell$ with unramified quadratic sign.*

*(ii) Let a weight-two system of cyclotomic determinant be good at $w$, with integral common Frobenius trace $a$. Suppose $a$ is a unit at some dyadic coefficient place and that its reduction differs from one at every dyadic coefficient place. Then an odd coefficient prime dividing $9 - a^2$ gives a residue to which (i) applies.*

*Proof.* For (i), wild inertia is trivial since the residue is unipotent on inertia and $\ell$ is odd. Frobenius has distinct eigenvalues $\delta, 2\delta$ for a sign $\delta = \pm 1$. If tame inertia is nontrivial, its invariant line is preserved by Frobenius and the tame relation makes the line eigenvalue twice the quotient eigenvalue. Thus we can choose the sign $\delta$ as the quotient value. If inertia is trivial we choose the eigenline with value $2\delta$. In this basis lift arithmetic Frobenius and a tame generator to

$$\begin{pmatrix} 2\delta & 0 \\ 0 & \delta \end{pmatrix}, \qquad \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$

with $c \neq 0$ lifting the residual parameter. These matrices satisfy the tame relation and have the required determinant. Their inertia factors through the pro-$\ell$ tame quotient. This is a special lift of the actual residual representation.

In (ii), the two Frobenius eigenvalues at every complex embedding have modulus $\sqrt{2}$ and product two. Thus $a$ is totally real, $|a| \leq 2\sqrt{2}$ at every embedding, and $9 - a^2$ has positive integral norm. Every conjugate of $9 - a^2$ is at least one. Norm one would force $a^2 = 8$, contrary to the unit hypothesis. Also $9 - a^2$ is a unit at every dyadic coefficient place by the assumption on $\bar{a}$. It therefore has an odd prime divisor. The residue at that coefficient prime is unramified at $w$ and has trace congruent to $\pm 3$ there. □

**Lemma 4.8.** *Let $F/\mathbf{Q}_2$ be quadratic. Let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ have determinant one.*

*(i) If $\overline{r}$ is reducible, it has a crystalline ordinary lift of determinant $\chi_2$ with regular labelled weights. It may retain any specified invariant residual line. Its invariant line has the larger weight at every embedding, and the lift lies on an ordinary component of the fixed-determinant crystalline space. If $F$ is ramified, the weights may be chosen to be $\{1 - n, n\}$ at both embeddings for some $n \geq 2$.*

*(ii) If $\overline{r}$ is irreducible, it has a crystalline lift of determinant $\chi_2$ and weights $\{0, 1\}$. Every potentially Barsotti–Tate lift with these weights is potentially nonordinary.*

*Proof.* In the reducible case use the proof of [11, Proposition 2.7], starting with the given residual line and quotient. Choose crystalline character lifts with product $\chi_2$. Multiplying the first by a sufficiently large positive power of $\chi_2$ and the second by the inverse power preserves their reductions and product, since $\overline{\chi}_2 = 1$. Their weights are respectively positive and negative at every embedding. The crystalline extension constructed there lifts the given extension and has those labelled weights. Its unramified scalar determinant correction reduces to one and preserves the filtration. We obtain weights $a_\tau < b_\tau$, with $a_\tau + b_\tau = 1$, and the invariant line has weight $b_\tau$.

For fixed regular weights, the ordinary crystalline deformation ring is a reduced quotient of the crystalline ring, with the same pure dimension, and both generic fibres are smooth [7, Section 1.4]. The ordinary locus is therefore a union of irreducible components of the crystalline generic fibre. Smoothness makes this union open and closed; its intersection with the fixed-determinant fibre is again a union of components. We choose a component through the constructed point.

For a ramified quadratic $F/\mathbf{Q}_2$, every odd-order character of $\mathcal{O}_F^\times$ is trivial, so the two residual diagonal characters are unramified. Starting with their Teichmuller lifts times $\chi_2^n$ and $\chi_2^{1-n}$ in the same construction gives the asserted constant labelled weights for large $n$.

Suppose that $\overline{r}$ is irreducible. Wild inertia acts trivially: its fixed space is nonzero and is stable under the whole local group. Tame inertia has two distinct characters exchanged by Frobenius.

If $F$ is ramified, Frobenius acts by squaring. If one is $\theta$, the other is $\theta^2$, and determinant one gives $\theta^3 = 1$. Thus $\theta$ has order three. A Frobenius matrix exchanges the two eigenlines; determinant one makes its square the identity, and diagonal conjugation makes both off-diagonal entries one. Hence there is exactly one such representation up to isomorphism, with image $\mathrm{GL}_2(\mathbf{F}_2) \simeq S_3$.

The elliptic curve

$$A/\mathbf{Q}: \quad y^2 + y = x^3,$$

with Cremona label `27a3` (LMFDB label `27.a4`) [55], realizes it over $\mathbf{Q}_2$. Its discriminant is $-27$, so it has good reduction at two. Its nonzero two-torsion points have $y = -1/2$ and $4x^3 + 1 = 0$, giving $\mathbf{Q}_2(A[2]) = \mathbf{Q}_2(\zeta_3, \sqrt[3]{2})$. This is an $S_3$ extension, since $X^3 - 2$ is Eisenstein and $\mathbf{Q}_2(\zeta_3)/\mathbf{Q}_2$ is unramified quadratic. Its unique quadratic subfield is therefore unramified, so it is disjoint from the ramified quadratic extension $F$. Thus $A[2]|_{G_F}$ is the required residual representation, and $V_2(A)|_{G_F}$ is a crystalline Barsotti–Tate lift by good reduction, with determinant $\chi_2$ by the Weil pairing [38, Theorem 6.2].

If $F$ is unramified, then for the unramified quadratic extension $E/F$ we may write $\overline{r} = \mathrm{Ind}_{G_E}^{G_F} \bar{\theta}$, with $\bar{\theta}|_I = \omega_4^e$, where $\omega_4$ takes values in $\mathbf{F}_{16}^\times$. Frobenius over $F$ acts by the fourth power. Determinant one gives $5e \equiv 0 \pmod{15}$, and irreducibility excludes $e = 0$, so $e \in \{3, 6, 9, 12\}$.

Index the four embeddings of $E$ by $0, 1, 2, 3$, with $i$ and $i+2$ restricting to the same embedding of $F$, and normalize $\omega_4$ so a crystalline character of weights $k_i$ reduces on inertia to exponent $\sum_i 2^i k_i$. The profiles $(k_0, k_1, k_2, k_3) = (a, b, 1-a, 1-b), \quad a, b \in \{0, 1\}$, have paired sums one and exponents $12 - 3a - 6b \in \{3, 6, 9, 12\}$. Lift $\bar{\theta}$ crystallinely with the appropriate profile and correct its unramified value. Induction along the unramified extension gives a crystalline lift with weights $\{0, 1\}$ at both embeddings of $F$. Its determinant differs from $\chi_2$ by an unramified character $\delta$ reducing to one. An unramified character $\nu$ with $\nu^2 = \delta^{-1}$ exists after enlarging coefficients and reduces to one, because characteristic two has no other square root of one. Twisting by $\nu$ fixes the determinant.

Finally, a potentially ordinary lift of these weights has, over a finite Galois extension, a line of constant labelled weight one. This line is unique, since two such lines would give weight one twice at every embedding. Its Galois conjugates are the same line, so it descends to $G_F$. Its intersection with a stable lattice is saturated; reduction gives an invariant line in $\overline{r}$, a contradiction. □

**Lemma 4.9.** *For a finite extension $F/\mathbf{Q}_l$ with $l \neq 2$, every continuous $\overline{r}: G_F \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ has a lift of determinant $\chi_2$, with finite inertia or Steinberg type. An unramified $\overline{r}$ has an unramified such lift.*

*Proof.* If $\overline{r}$ is reducible with diagonal characters $\alpha, \alpha^{-1}$ and $\alpha \neq 1$, then $H^1(G_F, \alpha^2) = 0$: local Euler characteristic and duality away from two give this from the vanishing of $H^0(G_F, \alpha^2)$ and $H^0(G_F, \alpha^{-2}\overline{\chi}_2)$, since $\overline{\chi}_2 = 1$. Thus $\overline{r}$ is split, and $\widetilde{\alpha} \oplus \chi_2 \widetilde{\alpha}^{-1}$ is the required lift.

If $\alpha = 1$, write $\overline{r} = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$. The additive character $c$ kills wild inertia and odd-order tame inertia. If $\tau$ generates pro-two tame inertia and $\Phi$ is arithmetic Frobenius, with residue

cardinality $Q$, their relation is $\Phi\tau\Phi^{-1} = \tau^Q$. Choose lifts $n, x$ of $c(\tau), c(\Phi)$ and set

$$\tau \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad \Phi \longmapsto \begin{pmatrix} Q & x \\ 0 & 1 \end{pmatrix}.$$

This satisfies the relation and has determinant $\chi_2$. Take $n = 0$ when $\overline{r}$ is unramified; otherwise the lift has nonzero rank-one monodromy.

If $\overline{r}$ is irreducible, its finite local image is solvable. Dickson's classification in characteristic two makes it odd dihedral; its scalar kernel is trivial because its determinant is one. Write $\overline{r} = \operatorname{Ind}_{G_E}^{G_F} \bar{\theta}$ for a quadratic extension $E/F$ and an odd-order character $\bar{\theta}$. The induced-determinant formula gives $\bar{\theta}|_{F^\times} = 1$. Its Teichmüller lift $\theta_0$ has the same restriction. Let $\epsilon_{E/F}$ be the quadratic character of $F^\times$. Choose a character $\phi$ of $E^\times$ reducing to one and restricting to $\chi_2\epsilon_{E/F}$ on $F^\times$ as follows.

For unramified $E/F$, take $\phi$ unramified with the required value on a common uniformizer. For ramified $E/F$, which is tamely quadratic, take the quadratic residue character on $\mathcal{O}_E^\times$. It restricts to $\epsilon_{E/F}$ on $\mathcal{O}_F^\times$. Write $\pi_F = u\pi_E^2$ and choose $\phi(\pi_E)$ to be a square root of $(\chi_2\epsilon_{E/F})(\pi_F)/\phi(u)$. All these values reduce to one. Then $\operatorname{Ind}_{G_E}^{G_F}(\theta_0\phi)$ has determinant $\epsilon_{E/F}(\theta_0\phi)|_{F^\times} = \chi_2$, the required reduction, and finite inertia. An irreducible representation over an algebraically closed field cannot be unramified, since an unramified group is procyclic. □

**Lemma 4.10.** *Let $r : G \to \mathrm{GL}_2(E)$ be a continuous absolutely irreducible representation of a compact group, where $E$ is a complete discretely valued field with finite residue field of characteristic two. Suppose its residual semisimplification is $1 \oplus 1$. Let $D \subset G$ be a closed subgroup preserving an $E$-line. There is a stable lattice whose reduction is a nonsplit extension of $1$ by $1$ and is trivial on $D$. Moreover, a binary coordinate of this extension gives a nonzero quadratic character of $G$ which is trivial on $D$.*

*Proof.* Choose a nonzero vector $v$ on the $D$-stable line and put $T = \mathcal{O}_E[G]v$. Compactness bounds the orbit, so $T$ is finitely generated, and absolute irreducibility gives $ET = E^2$. If $\lambda$ is a uniformizer, $v$ is primitive in $T$: otherwise $T \subset \lambda T$. Thus $T/\lambda T$ is generated by the orbit of $\bar{v}$. Its semisimplification is trivial, and a split trivial representation of dimension two cannot be cyclic. The reduction is therefore nonsplit. Its unique global invariant line differs from $k\bar{v}$, where $k$ is the residue field. Since $D$ preserves both lines and acts unipotently, it acts trivially.

Write the reduction as

$$g \longmapsto \begin{pmatrix} 1 & c(g) \\ 0 & 1 \end{pmatrix}, \qquad c : G \longrightarrow (k, +), \quad c \neq 0, \quad c|_D = 0.$$

Choose an $\mathbf{F}_2$-linear map $\lambda_0 : k \to \mathbf{F}_2$ nonzero on the image of $c$. Then $\lambda_0 \circ c$ is the asserted quadratic character. Its two-dimensional $\mathbf{F}_2$ representation is the subquotient

$$(ke_1 + \mathbf{F}_2 e_2)/(\ker \lambda_0)e_1.$$

In particular this subquotient has order four even when $k \neq \mathbf{F}_2$. □

**Lemma 4.11.** *Let $r$ be an absolutely irreducible dyadic representation over a totally real field with one dyadic place, whose completion is unramified over $\mathbf{Q}_2$. Assume determinant $\chi_2$, crystalline weights $\{0, 1\}$ at two, and global residual semisimplification $1 \oplus 1$. At two, $r$ is ordinary with an unramified quotient of rank one. There is a stable lattice whose reduction is a nonzero additive extension of $1$ by $1$ and is trivial at two.*

*Proof.* The dyadic restriction is ordinary by Lemma 4.3(ii). Its étale quotient has coefficient rank one, and the kernel is the ordinary line. Apply Lemma 4.10 to the global image and the dyadic decomposition subgroup. □

**Lemma 4.12** (An unramified or special replacement)**.** *Let $q$ and $p$ be distinct rational primes with $p$ odd and $q \equiv 1 \pmod{p}$. Let $\bar{s}$ be a representation of $G_{\mathbf{Q}_q}$ in characteristic $p$ of cyclotomic determinant. Suppose wild inertia acts trivially and tame inertia is trivial or cyclic unipotent of order $p$. After extending coefficients, $\bar{s}$ has a cyclotomic-determinant lift which is unramified in the first case and special with unramified quadratic sign and nonzero monodromy in the second.*

*Proof.* In the unramified case lift the whole Frobenius matrix and correct its determinant to $q$ by a scalar square root reducing to one. In the other case a tame generator acts by $T = 1 + tE_{12}$ with $t \neq 0$. The relation $\bar{\Phi} T \bar{\Phi}^{-1} = T^q = T$ and determinant one give $\bar{\Phi} = aI + bE_{12}$ with $a = \pm 1$. Use lifts $\widetilde{t}, \widetilde{b}$ and set

$$U(x) = 1 + \widetilde{t} x E_{12}, \qquad \Phi = \begin{pmatrix} qa & \widetilde{b} \\ 0 & a \end{pmatrix}.$$

These matrices satisfy $\Phi U(x) \Phi^{-1} = U(qx)$ on the tame $\mathbf{Z}_p$ quotient. They have the required determinant and preserve both the residual inertia and Frobenius parameters. The tame parameter is nonzero, so the lift is genuinely special. □

## 4.1 A type containing every relevant Serre weight

Let $F/\mathbf{Q}_p$ be finite, where $p > 2$, with residue field $k$ of size $q$. We give the cyclotomic character $\chi_p$ Hodge–Tate weight one. Put $K_0 = \mathrm{GL}_2(\mathcal{O}_F)$, choose a uniformizer $\pi$, put $K_1 = 1 + \pi M_2(\mathcal{O}_F)$, and write

$$I^+ = \left\{ \begin{pmatrix} 1 + \pi a & b \\ \pi c & 1 + \pi d \end{pmatrix} : a, b, c, d \in \mathcal{O}_F \right\}, \qquad J = \mathcal{O}_F^\times I^+.$$

Fix a nontrivial additive character $\psi : k \to \overline{\mathbf{Q}}_p^\times$. Define an affine generic character of $J$ by

$$\theta \left( z \begin{pmatrix} 1 + \pi a & b \\ \pi c & 1 + \pi d \end{pmatrix} \right) = \psi(\bar{b} + \bar{c}), \qquad z \in \mathcal{O}_F^\times.$$

The ambiguity in $z$ does not affect the two displayed residue coordinates. Let $Z$ and $U$ denote the scalar subgroup and the upper unipotent subgroup of $\mathrm{GL}_2(k)$.

**Lemma 4.13.** *Every irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(k)$ with trivial central character is a quotient of $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1$.*

*Proof.* Let $V$ be such a representation. A finite $p$-group acting on a nonzero vector space in characteristic $p$ has a nonzero fixed vector. Thus $V^U \neq 0$. Since $Z$ acts trivially, $V^{ZU} \neq 0$. Frobenius reciprocity gives a nonzero map $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1 \to V$, which is surjective by irreducibility. □

**Lemma 4.14.** *The representation $\sigma = \mathrm{Ind}_J^{K_0} \theta$ is irreducible of dimension $q^2 - 1$. It is the compact type of a simple-supercuspidal inertial type $\tau$ with determinant one. Its natural integral lattice has reduction equal to the inflation of $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1$.*

*Proof.* On $K_1 = 1 + \pi M_2(\mathcal{O}_F)$, the character is $1 + \pi A \mapsto \psi(\bar{A}_{21})$. Under conjugation by $\mathrm{GL}_2(k)$, its stabilizer is $ZU$. Indeed the trace pairing on $M_2(k)$ is nondegenerate, including after composing with the nontrivial additive character of $k$, and the centralizer of a nonzero nilpotent matrix is $ZU$. The restriction of $\sigma$ to $K_1$ is therefore a sum of $q^2 - 1$ distinct characters permuted transitively by $K_0$. This proves irreducibility and the dimension assertion.

The simple-supercuspidal construction identifies $\sigma$ as the compact type of the representations obtained by extending $\theta$ to $\langle g \rangle F^\times I^+$, where $g = \left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$; see [1, Section 3.1]. To check the

occurrence condition, let an irreducible smooth representation contain $\theta$ on $J$, and use its central character to extend $\theta$ to $F^\times I^+$. By Frobenius reciprocity the compact induction from this latter group surjects onto the representation. That compact induction is the direct sum of the two irreducible simple-supercuspidal representations in [1, Section 3.1, p. 6]. Changing the extension on $g$, or its central character on a uniformizer, gives an unramified twist. Conversely each such twist contains $\sigma$. The occurrence characterization of inertial types therefore identifies $\sigma$ with $\sigma(\tau)$, up to duality; compare [43, Theorem 1.1.3]. The Langlands parameter is the induced parameter of [1, Section 3.4 and Theorem 3.18]. The central character on $\mathcal{O}_F^\times$ is trivial, so $\det \tau = 1$.

All values of $\theta$ are $p$-power roots of unity and reduce to one. The coset basis of the induced lattice therefore reduces to the permutation representation on $K_0/J$. Since $K_1 \subset J$ and $J/K_1 = ZU$, this is the asserted inflation. Taking the dual does not change its Jordan–Hölder support. □

**Theorem 4.15** (A universal wild inertial type)**.** *Let $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be continuous with determinant $\bar{\chi}_p$. Then $\bar{r}$ admits a potentially crystalline lift of Hodge–Tate weights $\{0, 1\}$ at every embedding, determinant exactly $\chi_p$, and inertial type $\tau$ from Lemma 4.14. For $F = \mathbf{Q}_p$, write this type as $\tau_{p,w}$. Its inertia remains absolutely irreducible after reduction in every coefficient characteristic greater than $p$, for every stable lattice.*

*Proof.* We use the potentially Barsotti–Tate Breuil–Mézard theorem of Gee–Kisin, [43, Corollaries 3.5.3 and 3.5.6]. To match their conventions, apply their theorem to the dual residual representation, and dualize the resulting lift at the end. Their cyclotomic character has Hodge–Tate weight $-1$. The central-character restriction in Corollary 3.5.3 is therefore trivial in the present determinant situation.

At least one of the nonnegative multiplicities $\mu_V$ is positive. To see this, use the existence of some potentially Barsotti–Tate lift of any rank-two residual representation. This existence is explicitly justified in the proof of [43, Theorem A.2]: the irreducible case is contained in [77, Proposition 7.8.1], and the reducible case follows from the crystalline-extension construction cited there. The nonzero potentially Barsotti–Tate ring for that lift has positive Hilbert–Samuel multiplicity; the Breuil–Mézard formula makes at least one $\mu_V$ positive.

Every $V$ with positive multiplicity has trivial central character. By Lemmas 4.13 and 4.14, it occurs in the reduction of the simple-supercuspidal type. The Breuil–Mézard formula for Hodge type zero and type $\tau$ consequently has positive right side. Its potentially crystalline lifting ring is nonzero and $p$-torsion free, hence has nonempty generic fibre. Dualizing supplies a lift $r$ with the desired Hodge–Tate weights and type: a two-dimensional determinant-one type is self-dual.

Finally $\det r/\chi_p$ is unramified and reduces to one. It is unramified because it is potentially crystalline of weight zero with trivial inertial type. As $p$ is odd, its inverse has an unramified square root reducing to one. Twisting $r$ by this square root fixes its determinant without changing its reduction, weights or inertial type.

For $F = \mathbf{Q}_p$, the parameter is induced from a ramified quadratic extension. Its restriction to wild inertia has two distinct characters of order $p$, exchanged by a quadratic inertia element [1, Section 3.4 and Theorem 3.18]. In coefficient characteristic greater than $p$ the two characters remain distinct, and the exchanging element prevents either eigenline from being stable. The finite inertial image has order divisible only by 2 and $p$, so semisimplicity makes this conclusion independent of the stable lattice. □

*Remark* 4.16. For $p = 3$ this theorem applies to both unramified and ramified quadratic completions. It gives a local lift of weight two. It makes no assertion that an arbitrary-weight global source is modular whenever such a replacement is modular. A backward modularity implication must be verified separately for the actual global source.

**Lemma 4.17** (Bounded five-adic lifts)**.** *Let $\omega_2$ have niveau two and order 24 over $\mathbf{Q}_5$, and put $\tau_3 = \widetilde{\omega}_2^8 \oplus \widetilde{\omega}_2^{16}$ and $\tau_6 = \widetilde{\omega}_2^4 \oplus \widetilde{\omega}_2^{20}$. Every continuous $\bar{r} : G_{\mathbf{Q}_5} \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ of determinant $\bar{\chi}_5$ has a weight-two lift of determinant $\chi_5$ which has type $\tau_3$ or $\tau_6$, or is quadratic-special with nonzero monodromy.*

*Proof.* The weights with trivial scalar central character are

$$1, \quad \overset{2}{\det}, \quad \mathrm{Sym}^2 \det, \quad \mathrm{Sym}^2 \overset{3}{\det}, \quad \mathrm{Sym}^4, \quad \mathrm{Sym}^4 \overset{2}{\det}.$$

The cuspidal compact types satisfy

$$\overline{\sigma}(\tau_6)^{\mathrm{ss}} = 1 \oplus \mathrm{Sym}^2 \det, \qquad \overline{\sigma}(\tau_3)^{\mathrm{ss}} = \overset{2}{\det} \oplus \mathrm{Sym}^2 \overset{3}{\det}.$$

Indeed their Brauer character is zero on split regular elements, $-\theta(z) - \theta(z^5)$ on nonsplit regular elements, and $4\theta(a)$ on scalars. Substitution of the exponents four and eight proves both identities. In the full fixed-determinant Breuil–Mézard formula [84, Theorem 1.2], the multiplicities are nonnegative and at least one is positive, since a potentially Barsotti–Tate lift exists. The determinant confines them to the six displayed weights. The first four yield the cuspidal alternatives.

The last two occur in the semistable compact type for scalar quadratic inertia. A point with nonzero monodromy gives the special alternative. If every such point is crystalline, the corresponding crystalline compact type, 1 or $\det^2$, has positive multiplicity, so a cuspidal lift again exists. This applies to the actual residual extension over any finite coefficient field. The deformation rings have fixed determinant; equivalently, an unramified determinant error reducing to one is corrected by its square root. □

**Lemma 4.18.** *Let $\ell \neq 3$ be an odd prime with $\ell \equiv -1 \pmod 3$. Let $\bar{s}$ be the reduction, with any stable lattice, of a determinant-$\chi_3$ representation of $G_{\mathbf{Q}_\ell}$ which is either a quadratic twist of special type or tame dihedral with inertia of order three or six. After a finite coefficient extension, $\bar{s}$ has lifts of both kinds with determinant exactly $\chi_3$. The scalar quadratic inertial character determines whether the cuspidal lift has inertia of order three or six, and the special lift may be chosen with nonzero monodromy.*

*Proof.* Remove the scalar quadratic character. Wild inertia is trivial; tame inertia is trivial or unipotent of order three. Frobenius has characteristic polynomial $X^2 - 1$ modulo three: in the special case its eigenvalues are $\ell\alpha, \alpha$, $\alpha = \pm 1$, and in the cuspidal case its trace is zero and determinant is $\ell$. The two eigenvalues are distinct. A nonzero inertial extension has one Frobenius eigenline as its invariant line and is normalized by diagonal conjugation. There are two nonsplit orientations and the split case.

For a cuspidal lift take

$$T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Phi = bS, \quad b^2 = -\ell, \quad b \equiv 1 \pmod 3.$$

Hensel's lemma supplies $b$. We have $T^3 = 1$, $STS = T^{-1}$ and $\det \Phi = \ell$. The standard lattice gives nontrivial unipotent inertia with eigenvalue $-1$ on its invariant line; replacing $\Phi$ by $-\Phi$ gives the other orientation. For trivial residual inertia use $T = \mathrm{diag}(\zeta_3, \zeta_3^{-1})$ over $\mathbf{Q}_3(\zeta_3)$ and the same exchanging Frobenius.

For a special lift use

$$\Phi = \mathrm{diag}(\ell, 1), \qquad U(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \Phi U(t) \Phi^{-1} = U(\ell t).$$

Choose its three-primary tame parameter to be a unit multiple when residual inertia is nontrivial, and three times a unit multiple otherwise. Both choices have nonzero monodromy. An unramified quadratic sign selects either orientation. Restoring the removed quadratic character preserves the determinant and gives the required inertial order. □

## 4.2 Local constructions at a split terminal prime

We record the local characteristic-three calculations used in the three supported residual bases below. The universal existence statement is Theorem 4.15; here we also need the exact characteristic-two reduction of its type over $\mathbf{Q}_3$.

**Lemma 4.20.** *Let $K$ be a real quadratic field of ordinary class number one in which three splits. Every finite character $G_K \to \overline{\mathbf{F}}_3^\times$ unramified outside three has square one.*

*Proof.* Such a character has order prime to three and kills the principal units at the two places above three. The ray-class sequence, with both real places included, makes its group a quotient of $(\mathbf{F}_3^\times)^2 \times \{\pm 1\}^2$, which has exponent two. □

For a finite Galois extension $L/\mathbf{Q}_3$, write $d(L) = v_3(\Delta(L/\mathbf{Q}_3))/[L : \mathbf{Q}_3]$, with $v_3(3) = 1$. Unramified base extension does not change this contribution.

For the linear kernel field $L$ of a representation $G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with arbitrary determinant, Moon–Taguchi's inequality is

$$d(L) \le \frac{13}{6} - \frac{1}{Q} \tag{4}$$

when the wild ramification index is $Q > 1$; tame extensions have $d(L) < 1$ [56, Theorem 3 and the following Remark (1)].

### 4.2.1 A wild type over $\mathbf{Q}_3$

Specialize the construction of Lemma 4.14 to $F = \mathbf{Q}_3$, with uniformizer $+3$ and additive character $\psi(x) = \zeta_3^x$. Retain its notation $K_0, K_1, J$ and write $\vartheta$ for the affine generic character. The sign of this uniformizer fixes the quadratic subfield in the following lemma.

**Lemma 4.22.** *The representation $\sigma_+ := \mathrm{Ind}_J^{K_0} \vartheta$ is the irreducible eight-dimensional type of a supercuspidal inertial type $\tau_+$ with determinant one. Its reduction satisfies*

$$\overline{\sigma}_+^{\mathrm{ss}} = 1 \oplus \det \oplus \mathrm{Sym}^2 \mathbf{F}_3^2 \oplus (\det \otimes \mathrm{Sym}^2 \mathbf{F}_3^2). \tag{5}$$

*The inertia representation $\tau_+$ is irreducible after reduction in every characteristic greater than three. For a determinant-cyclotomic compatible system of this type, its characteristic-two residual local kernel field is the unique totally ramified $S_3$ extension $E_+/\mathbf{Q}_3$ with quadratic subfield $\mathbf{Q}_3(\sqrt{3})$ and normalized different $7/6$. Every continuous $\bar{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with determinant $\bar{\chi}_3$ has a potentially crystalline lift of determinant $\chi_3$, weights $\{0, 1\}$ and type $\tau_+$.*

*Proof.* The construction of Theorem 4.15 at $\ell = 3$ identifies $\sigma_+$ as the irreducible compact type of dimension eight, with determinant-one inertial parameter $\tau_+$.

In the induced integral lattice, $\vartheta$ reduces to one, so $\overline{\sigma}_+ = \mathrm{Ind}_{ZU}^{\mathrm{GL}_2(\mathbf{F}_3)} 1$. If $B$ is the upper triangular Borel, then $B/ZU$ has order two, with characters $1, \det$. Induction in stages gives

$$\overline{\sigma}_+ = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbf{F}_3)} 1 \oplus \det \otimes \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbf{F}_3)} 1.$$

The first summand is the permutation module on $\mathbf{P}^1(\mathbf{F}_3)$. Its constant line splits, since four is invertible in characteristic three, and its augmentation module is the Steinberg representation

$\operatorname{Sym}^2 \mathbf{F}_3^2$. For example, evaluating a quadratic at the four projective points identifies it with that augmentation module. This proves (5); its four constituents are exactly the Serre weights with trivial scalar central character.

By [1, Section 3.4 and Theorem 3.18], the parameter is induced from a character of a ramified quadratic extension of $\mathbf{Q}_3$, of order three on the first principal-unit quotient, with conjugate the inverse there. Its inertia contains, in a suitable basis,

$$u = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The off-diagonal matrix has this form because inertia has determinant one. These matrices generate a group of order twelve. In characteristic $l > 3$, the two $u$-eigenlines remain distinct and $v$ exchanges them. Thus reduction is irreducible; semisimplicity for this group of order prime to $l$ makes the conclusion independent of the stable lattice. We check the quadratic field in the local parameter. In the parametrization of [1, Sections 3.1, 3.4 and Theorem 3.18], the affine-generic character $b + c$ uses uniformizer $+3$. The inducing quadratic extension is consequently $\mathbf{Q}_3(\sqrt{3})$. Reduction modulo two gives the irreducible group $S_3$. Its normalizer in $\mathrm{SL}_2(\overline{\mathbf{F}}_2)$ is itself: all automorphisms of $S_3$ are inner, and its centralizer consists of determinant-one scalars, hence is trivial. Thus the entire local image equals its inertia image and the residual field has $e = 6$, $f = 1$.

The wild character has conductor two over the ramified quadratic field. The two-dimensional Artin conductor is three and remains three after reduction modulo two: the wild inertia has odd order and the inertia invariants remain zero. The lower groups of the residual field have orders $6, 3, 1$, giving normalized different $(5 + 2)/6 = 7/6$. Finally, over $\mathbf{Q}_3(\sqrt{3})$ the conductor-two anti-invariant character space has dimension one over $\mathbf{F}_3$. Its unique nonzero line gives a unique $S_3$ extension. This proves the stated uniqueness; the splitting field of $x^3 - 3x - 3$ is one realization.

The lifting assertion is Theorem 4.15 with $F = \mathbf{Q}_3$ and the affine-generic character fixed above. □

**Lemma 4.23.** *Let $\overline{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ have determinant $\overline{\chi}_3$. It has a lift of weights $\{0, 1\}$ and determinant $\chi_3$ which becomes semistable after a quadratic twist, preserving the actual residual extension. One may alternatively choose a lift of the same weights and determinant which is either a quadratic twist of a semistable representation with nonzero monodromy, or potentially crystalline of tame supercuspidal type*

$$\tau_4 = \widetilde{\omega}_2^2 \oplus \widetilde{\omega}_2^6,$$

*where $\omega_2$ is a fundamental character of niveau two. For either choice in this latter menu, in a determinant-cyclotomic compatible system the characteristic-two residual local kernel field is contained in*

$$E' = \mathbf{Q}_3(\sqrt{3}, \sqrt{-1}), \qquad e(E'/\mathbf{Q}_3) = 2, \quad f(E'/\mathbf{Q}_3) = 2, \quad d(E'/\mathbf{Q}_3) = \tfrac{1}{2}. \tag{6}$$

*Proof.* If $\overline{r}$ is reducible, its diagonal inertia characters are $1, \overline{\chi}_3$. A ramified quadratic twist, when necessary, puts its invariant character and quotient in the order $\overline{\chi}_3\lambda, \lambda^{-1}$, with $\lambda$ unramified. Let $\eta$ be the unramified Teichmüller lift of $\lambda$. If $\lambda^2 \neq 1$, local duality gives $H^2(G_{\mathbf{Q}_3}, \mathcal{O}(\chi_3\eta^2)) = 0$, so the actual residual extension lifts integrally. The local Euler characteristic and the crystalline extension formula give $\dim H^1_f(\mathbf{Q}_3, E(\chi_3\eta^2)) = \dim H^1(\mathbf{Q}_3, E(\chi_3\eta^2)) = 1$; thus the lift is crystalline. If $\lambda^2 = 1$, take $\eta^2 = 1$. Kummer theory lifts both coordinates of the residual extension in $\mathbf{Q}_3^\times/(\mathbf{Q}_3^\times)^3$. Their lifts are semistable, with crystallinity equivalent to vanishing of the valuation coordinate; the unit direction is crystalline and a nonzero valuation direction has nonzero monodromy. Undo the quadratic twist.

If $\overline{r}$ is irreducible, wild inertia acts trivially. Frobenius multiplies a tame exponent by three, and the determinant makes the sum of the paired exponents four modulo eight. The two pairs are therefore $\{1,3\}$ and $\{5,7\}$. A ramified quadratic twist interchanges them. For the first pair induce a crystalline character of labelled weights $(1,0)$ from the unramified quadratic extension, choosing its unramified value to match the full residue. The quotient of its determinant by $\chi_3$ is crystalline of weight zero, hence unramified, and reduces to one. An unramified scalar square root reducing to one corrects the determinant. Twisting back proves the first assertion; compare [77, proof of Proposition 7.8.1].

Retain this lift if its monodromy is nonzero. Otherwise its inertial type is scalar quadratic, say $\kappa \oplus \kappa$. Its potentially crystalline compact type is $\kappa \circ \det$, with reduction 1 or det. The fixed-determinant formula of [84, Theorem 1.2], throughout with determinant $\chi_3$, gives $\mu_1(\overline{r}) > 0$ or $\mu_{\det}(\overline{r}) > 0$.

The cuspidal type for $\tau_4$ has dimension two and trivial central character. It is the two-dimensional representation of $\mathrm{PGL}_2(\mathbf{F}_3) \simeq S_4$ obtained from the standard representation of $S_3$. The integral augmentation lattice in $\mathbf{Z}_3^3$ reduces to a module with a constant line and sign quotient. Sign inflates to det, so its semisimplification is $1 \oplus \det$. Breuil–Mézard for Hodge type $(1,0)$ and $\tau_4$ has multiplicity $\mu_1(\overline{r}) + \mu_{\det}(\overline{r}) > 0$. Its three-torsion-free deformation ring gives the required potentially crystalline lift, with determinant exactly $\chi_3$.

For the semistable choice, the characteristic-two local member is a quadratic twist of a special representation with diagonal characters $\chi_2\nu$ and $\nu$, where $\nu$ is unramified quadratic. Both reduce to one, as does the quadratic twist. The full residual image is therefore an elementary abelian two-group. For the tame choice, inertia and arithmetic Frobenius matrices $T, \Phi$ satisfy

$$T^2 = -I, \qquad \Phi T \Phi^{-1} = T^{-1}, \qquad \Phi^2 = -3I.$$

The last equality follows from trace zero and determinant three. Modulo two these matrices become commuting involutions, again giving an elementary abelian two-group for every stable lattice. The field $E'$ is the maximal elementary quadratic extension of $\mathbf{Q}_3$, so it contains both residual kernel fields. It is the compositum of the ramified and unramified quadratic extensions; its tame ramification index two gives the stated different. □

# 5 Dyadic automorphy and pro-modularity inputs

The local weight-two lifts and their support-preserving global construction are proved in Section 9, using the fixed-determinant Breuil–Mézard theorem [85, Corollary 8.0.2] and the prescribed-lift theorem [35, Theorem 4.2.1]. Potential modularity is established separately there. For backward modularity we use Kisin's and Tung's theorems with their local-source and residual-image hypotheses [52, Theorem 3.3.5], [85, Theorem 8.0.3].

The scalar argument uses the representation spaces, relative presentations and local geometry of [58, Section 3], and the completed-forms and local block comparisons of [58, Propositions 4.1, 4.2 and 4.4, Lemma 4.6]. The localized propagation theorem [58, Theorem 5.1] applies over totally real solvable fields of even degree completely split at two, to a potentially pro-modular characteristic-two curve whose generic representation is not virtually solvable and whose allowed odd local images are finite. In these conventions the completed-form central character is $(\det r)\varepsilon_2$ and the local block parameter is $r \otimes \varepsilon_2$. The regular de Rham locally irreducible parameters are treated by the local correspondence [14, Theorems 1.1 and 1.3].

For irreducible solvable residue we use [25, Proposition 10.3 and Theorem 10.6]. The application is over an abelian totally real field completely split at two, with every dyadic restriction of the actual source absolutely irreducible, so there are no locally reducible places. The auxiliary fields and scalar-source deduction needed here are proved in Sections 7 and 8.

# 6 Conductor reduction at a split odd prime

The local constructions above preserve the actual residual extension. The lifting implications in Section 2 apply to the old characteristic-zero source at each congruence.

**Lemma 6.1.** *Let $K$ be real quadratic with two nonsplit, and let the odd prime $\ell$ split in $K$. A nondihedral potentially modular weight-two system with determinant cyclotomic times a finite totally even character admits a chain of congruences to a system of cyclotomic determinant, good outside the two $\ell$-places and one auxiliary place $\mathfrak{q}$ of split norm $q > \max\{5, \ell\}$. The system has the wild type of Theorem 4.15 at both $\ell$-places. The type at $\mathfrak{q}$ satisfies Proposition 3.1 for every subsequent coefficient characteristic in the preparation, including $\ell$. Modularity of the last system implies modularity of the first.*

*Proof.* Twist by the inverse square root of the odd-order determinant factor, leaving determinant $\chi\psi$ with $\psi$ of two-power order. Let $S$ contain $2, 3, 5, \ell$, all primes dividing the discriminant of $K$, and all rational conductor primes. Apply Proposition 3.1; write $\mathfrak{q}$ for the auxiliary place, of split rational norm $q > N > \max S$. The order-$N$ type gives nonsolvable residual images and adequate cyclotomic restrictions at all later coefficient characteristics in $S$.

In characteristic two, use Lemma 4.8 at the unique quadratic dyadic completion to choose a crystalline target of exact determinant $\chi_2$. For a reducible local residue take an ordinary regular target, retaining the old ordinary filtration when present; for an irreducible local residue take a weight-two target, which is potentially nonordinary. Lemma 4.9 supplies determinant-$\chi_2$ lifts at odd conductor places and keeps unramified places unramified. Retain the protective type at $\mathfrak{q}$, correcting its unramified scalar. Lemma 2.6 gives the new system, good at two. For the old weight-two source, Lemma 4.2 and descent of the unique potentially ordinary line give the two cases of Lemma 2.3; in the ordinary case the target has matching residual ordering. Thus modularity returns to the old system. The new weights may be higher than two.

Reset in characteristic $\ell$, imposing the type of Theorem 4.15 at both $\ell$-places and retaining $\mathfrak{q}$ and all cleared conditions. Lemma 2.6 gives a weight-two system, with backward modularity by Lemma 2.2(i).

For each remaining odd conductor characteristic $t \neq \ell$, use the crystalline potentially diagonalizable components of Lemma 4.1, retaining the two wild $\ell$-types, the protective place and all cleared conditions. If $t$ is nonsplit, first reset at $\ell$ whenever the source is not of weight two; then apply Lemma 2.2(ii) backwards, using Lemma 2.1. If $t$ splits, use Lemma 2.2(i) without a reset. In either case Lemma 2.6 supplies the replacement system. The protective order-$N$ element ensures adequacy and excludes the exceptional characteristic-five image. Each step clears $t$, may change the weights, and introduces no conductor prime outside $S \cup \{q\}$. This gives the asserted system. □

*Proof of Theorem 1.3.* Apply Lemma 6.1. Work in coefficient characteristic $q > \max\{5, \ell\}$. The retained wild inertia at an $\ell$-place remains irreducible modulo $q$, by Theorem 4.15. It lies in the cyclotomic subgroup, since $K(\zeta_q)/K$ is unramified there. Thus the residual cyclotomic restriction is absolutely irreducible and adequate. If the residual image is solvable, an odd solvable Artin witness and split-$q$ lifting already prove modularity of the old system. Otherwise the crystalline replacements above $q$, Lemma 2.6 and split-$q$ lifting remove $\mathfrak{q}$. The resulting system has exact cyclotomic determinant, arbitrary regular weights and conductor supported at $\ell$.

Its actual $\ell$-adic member is absolutely irreducible at both $\ell$-places: a Galois line would give a Weil–Deligne line, contradicting the retained irreducible inertia type. If its residue is irreducible, (i) and [91, Theorem 6.1.1] give modularity. If it is reducible, apply (ii) and [91, Theorem 5.1.1]. Here $K$ is totally real abelian, $\ell$ splits completely, and the constituent quotient is also nontrivial

at both $\ell$-places: a residual constituent has tame inertia $\omega_\ell^a$, so the quotient has inertia $\omega_\ell^{1-2a}$, an odd power modulo the even integer $\ell - 1$. Propagate modularity backwards and undo the finite twist. □

## 6.1 A terminal source with one crystalline place

**Proposition 6.2.** *Let $K$ be real quadratic with two nonsplit, and let $\ell \geq 5$ be a rational prime that splits in $K$. Fix one of the two places above $\ell$ as $v_+$. To modularize every nondihedral potentially modular weight-two system with determinant cyclotomic times a finite totally even character, it suffices to modularize the following actual $\ell$-adic members. Their systems have determinant $\chi_\ell$ and conductor supported at one place $v_+ \mid \ell$. The member has regular labelled weights and is absolutely irreducible at $v_+$. At the other place $v_- \mid \ell$ it is crystalline of weight two. The congruence chain either proves modularity directly or has such a terminal member whose modularity implies modularity of the original system.*

*Proof.* Use Lemma 6.1 through the stage with possible conductor at $\ell$ and $\mathfrak{q}$. The conjugate $\mathfrak{q}'$ of the auxiliary place is still good. While the auxiliary type remains, switch in characteristic $\ell$. At $v_+$ prescribe the wild potentially Barsotti–Tate type of Theorem 4.15, and at $v_-$ prescribe a regular crystalline potentially diagonalizable lift by Lemma 4.1. The local representation at $v_+$ is potentially diagonalizable by [43, Lemma 3.4.1]. The auxiliary type guarantees adequacy and excludes the exceptional characteristic-five case. Lemma 2.6(ii) supplies the system and Lemma 2.2(i) gives the backward implication.

Enlarge its coefficient field $E$ to contain $K$. The weight-two label is attached to one embedding $K \hookrightarrow E$. At a split coefficient prime, a choice of embedding of $E$ into the local coefficient closure puts this label at either prime of $K$. The Weil–Deligne representation at a fixed finite place is conjugated on coefficients by this choice, so its good condition is preserved.

Choose a $q$-adic realization with the weight-two label at $\mathfrak{q}'$. There the actual member is crystalline of weight two, and hence potentially diagonalizable by [43, Lemma 3.4.1]. Retain that component and replace the component at $\mathfrak{q}$ by a regular crystalline potentially diagonalizable one. The retained wild type at $v_+$ protects irreducibility on the cyclotomic subgroup. The inertia group at $v_+$ is contained in the cyclotomic subgroup, since $K(\zeta_q)/K$ is unramified at $v_+$. Thus the restriction is irreducible. A solvable residue gives modularity of the old source by its Artin witness and split-$q$ lifting. Otherwise the rank-two adequacy criterion applies since $q > 7$. Apply Lemma 2.6(ii) and use Lemma 2.2(i) backwards. The replacement is good away from $v_+$.

Finally choose an $\ell$-adic realization placing the preserved weight-two label at $v_-$. Irreducible Weil inertia at $v_+$ precludes an actual local Galois line. This gives the asserted source, and all congruences have the required backward implications. □

**Theorem 6.4** ([31, Theorem A])**.** *Let $F$ be a totally real field, abelian over $\mathbf{Q}$, and let $p$ be an odd prime that splits completely in $F$. Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be continuous, absolutely irreducible, totally odd and unramified outside finitely many places. Let $\Sigma_p = \Sigma_{\mathrm{n}} \sqcup \Sigma_{\mathrm{o}}$ be a partition of the places above $p$, and suppose:*

(1) *for $v \in \Sigma_{\mathrm{n}}$, $\rho|_{G_{F_v}}$ is absolutely irreducible and de Rham with distinct Hodge–Tate weights;*
(2) *for $v \in \Sigma_{\mathrm{o}}$, $\rho|_{G_{F_v}}$ is reducible and de Rham with distinct Hodge–Tate weights;*
(3) *$\bar{\rho}^{\mathrm{ss}} = \bar{\chi}_1 \oplus \bar{\chi}_2$, the ratio $\bar{\chi}_1 \bar{\chi}_2^{-1}$ extends to a character of $G_{\mathbf{Q}}$, and it is nontrivial on $G_{F_v}$ for every $v \mid p$.*

*Then $\rho$ is modular.*

We call this result *the mixed theorem*. We use Theorem A of Dieulefait and X. Zhang [31], in preparation. Its reducible local alternative includes split restrictions, and its residual ratio

condition allows $\bar{\chi}_p^{\pm 1}$, also when $p = 3$. No ordinarity hypothesis is imposed at the locally absolutely irreducible places. The ratio must still extend to $G_{\mathbf{Q}}$ and be nontrivial at every place above $p$.

**Proposition 6.5** (The terminal member with one crystalline place)**.** *Let $r$ be a terminal member of Proposition 6.2 with $\ell \geq 5$. If its global residue is irreducible, it remains irreducible on $G_{K(\zeta_\ell)}$, and a modular residual witness proves modularity of $r$. In particular this applies to solvable irreducible residues. If the global residue is reducible and its constituent ratio extends to $G_{\mathbf{Q}}$ and is nontrivial on both coefficient decomposition groups, the mixed theorem (Theorem 6.4) gives modularity of $r$.*

*Proof.* The irreducible case uses Lemma 4.6 at the crystalline place $v_-$ and Lemma 2.2(i). For reducible residue, the local finite-flat characters at $v_-$ are unramified twists of 1 and $\bar{\chi}_\ell$. Thus Lemma 4.3(i) makes the actual source ordinary there. The actual representation at $v_+$ is absolutely irreducible by its inertial type. Both restrictions are regular de Rham, the field is real abelian, and $\ell$ splits completely. The ratio is totally odd by the determinant. These are hypotheses (1)–(3) of Theorem 6.4, with $\Sigma_{\mathrm{n}} = \{v_+\}$ and $\Sigma_{\mathrm{o}} = \{v_-\}$. □

**Corollary 6.6.** *Under the hypotheses of Theorem 1.3, Serre modularity holds over $K$ in odd residual characteristic with arbitrary conductor.*

*Proof.* Apply Proposition 2.7. In the nonsolvable case, apply Theorem 1.3 to the resulting system and reduce at the original coefficient place. □

## 6.2 Characters supported at the terminal prime

**Lemma 6.7.** *Let $K$ be real quadratic and $\ell$ an odd split prime. Suppose that the narrow class group of $K$ has exponent at most two and that a totally positive unit has inverse primitive residues at the two places above $\ell$. If $\theta$ is a character over $\overline{\mathbf{F}}_\ell$ unramified outside $\ell$ at finite places, then*

$$\theta = \bar{\chi}_\ell^a \eta, \qquad \frac{\theta}{\bar{\chi}_\ell \theta^{-1}} = \bar{\chi}_\ell^{2a-1},$$

*where $\eta$ is a finite-unramified character of order at most two. In particular the quotient extends to $G_{\mathbf{Q}}$, is totally odd, and is nontrivial on inertia at both places above $\ell$. If $\theta \oplus \bar{\chi}_\ell \theta^{-1}$ is finite flat at one such place, it is a quadratic twist of $1 \oplus \bar{\chi}_\ell$.*

*Proof.* The character is tame at $\ell$. Write its two inertial exponents relative to the cyclotomic character as $a_1, a_2$ modulo $\ell - 1$. Reciprocity on the unit gives $a_1 = a_2 = a$, so $\eta = \theta \bar{\chi}_\ell^{-a}$ is finite-unramified. It factors through the narrow class group. This gives the quotient formula. An odd power of the cyclotomic character has the asserted parity and has nontrivial inertia at a split prime. In the finite-flat case, the constituents over $\mathbf{Q}_\ell$ have inertia characters $1, \bar{\chi}_\ell$. Thus $a$ is zero or one and we may interchange the two characters. □

Here are the applications used below. The two residues in the last column are taken in either order.

| $d$ | $\ell$ | $\mathrm{Cl}^+(K)$ | positive unit | residues |
|---|---|---|---|---|
| 6 | 5 | $C_2$ | $5 + 2\sqrt{6}$ | $(2, 3)$ |
| 11 | 5 | $C_2$ | $10 + 3\sqrt{11}$ | $(2, 3)$ |
| 14 | 5 | $C_2$ | $15 + 4\sqrt{14}$ | $(2, 3)$ |
| 15 | 7 | $C_2^2$ | $4 + \sqrt{15}$ | $(3, 5)$ |
| 19 | 3 | $C_2$ | $170 + 39\sqrt{19}$ | $(2, 2)$ |
| 21 | 5 | $C_2$ | $(5 + \sqrt{21})/2$ | $(2, 3)$ |

All displayed units have norm one. The residue assertions follow by evaluating them at the two roots of the defining quadratic polynomial. The ordinary class groups follow from the Minkowski bound $\sqrt{D_K}/2$. For $d = 6, 11, 14, 19$, dyadic ideals are principal by the elements $2+\sqrt{6}$, $3+\sqrt{11}$, $4+\sqrt{14}$, $13+3\sqrt{19}$, respectively. At $d = 6$ the prime above three is principal by $3+\sqrt{6}$, and at $d = 19$ by $4+\sqrt{19}$. At $d = 11, 14$ three is inert. For $d = 21$ two is inert and the bound gives class number one. For $d = 15$ the primes above two and three represent the same class by $3+\sqrt{15}$, of norm $-6$. The dyadic prime is nonprincipal since neither 2 nor $-2$ is a square modulo five. Its square is generated by the totally positive element two. In all six fields reduction at a ramified prime congruent to 3 mod 4 excludes units of norm $-1$. The sign kernel therefore has order two, giving the narrow class groups in the table. For $d = 15$ this kernel and the dyadic ideal class give independent classes of order two.

Write $F_d = \mathbf{Q}(\sqrt{d})$ for the field of radicand $d$.

**Lemma 6.8.** *Let $(K, \ell) = (F_2, 7)$ or $(F_5, 11)$. Let $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ have determinant $\omega_\ell$ and reducible semisimplification, and be unramified outside $\ell$. The quotient of its constituents is $\omega_\ell^e$ for an odd $e$, up to inversion. In particular it is totally odd, extends to $G_{\mathbf{Q}}$, and is nontrivial at both $\ell$-adic places.*

*Proof.* Both fields have ordinary class number one by the quadratic Minkowski bounds $\sqrt{8}/2, \sqrt{5}/2 < 2$. Write the constituents as $\psi, \omega_\ell\psi^{-1}$. The finite character $\psi$ has prime-to-$\ell$ order, hence kills wild inertia and factors through the narrow ray modulo $\ell$ and the two real places. Write $a, b$ for its two tame exponents modulo $\ell - 1$, and $u, v \in \mathbf{Z}/2$ for its signs.

For $K = \mathbf{Q}(\sqrt{2})$ use the primitive root 3 modulo seven. At the two seven-primes the unit $1+\sqrt{2}$, of signs $(+,-)$, has residues $4 = 3^4, 5 = 3^5$. The unit relations are

$$a + b + u + v \equiv 0 \pmod 2, \qquad 4a + 5b + 3v \equiv 0 \pmod 6.$$

The second relation modulo three gives $a \equiv b \pmod 3$, so $2a \equiv 2b \pmod 6$. For $K = F_5$, put $\phi = (1+\sqrt{5})/2$. Its residues are $8 = 2^3, 4 = 2^2$ relative to the primitive root 2 modulo eleven, and its signs are $(+,-)$. The two unit relations give $a + b + u + v \equiv 0 \pmod 2$ and $3a + 2b + 5v \equiv 0 \pmod{10}$. Thus $a \equiv b \pmod 5$, $v \equiv a \pmod 2$ and $u \equiv b \pmod 2$. There are exactly twenty characters, and $2a \equiv 2b \pmod{10}$. Thus in every case $\psi^2\omega_\ell^{-2a}$ is unramified at all finite places and totally even. Ordinary class number one makes it trivial. The quotient is consequently $\omega_\ell^{1-2a}$, up to inversion. Its exponent is odd, so it is nontrivial at both split coefficient places and odd at every real place. □

# 7 Auxiliary fields with scalar Selmer vanishing

Let $K = \mathbf{Q}(\sqrt{d})$ be real quadratic, with $d > 1$ squarefree and $d \equiv 1 \pmod 8$. The space $V_{w_0}^+(K)$ and condition (SV) are defined in the introduction. No class-number hypothesis is imposed in this section.

**Lemma 7.1** (Cyclic towers with bounded odd splitting)**.** *Let $K = \mathbf{Q}(\sqrt{d})$ be real quadratic with two split, let $\Sigma$ be a finite set of odd rational primes, and let $n \geq 2$. There is an odd prime $r \notin \Sigma$ and a totally real cyclic field $L/\mathbf{Q}$ of degree $2^n$ such that:*

1. *$L$ is ramified only at $r$, where it is totally ramified;*
2. *$r$ is inert in $K$;*
3. *2 splits completely in $L$;*
4. *every $\ell \in \Sigma$ has at most two primes above it in $KL$.*

*The prime $r$ can avoid any additional prescribed finite set.*

*Proof.* Let $P$ be the union of $\Sigma$ and the prime divisors of $d$, and put

$$B_n = \mathbf{Q}(\zeta_{2^{n+1}}, 2^{1/2^n}), \qquad C_n = B_n\big(t^{1/2^n} : t \in P\big).$$

The field $B_n$ is Galois over $\mathbf{Q}$ and ramified only at two. At a prime of $B_n$ above an odd rational prime $t$, the normalized valuation of $t$ is one. These valuations show that the classes of the $t \in P$ generate a direct sum of copies of $\mathbf{Z}/2^n\mathbf{Z}$ in $B_n^\times/(B_n^\times)^{2^n}$: a product $\prod_{t\in P} t^{b_t}$ that is a $2^n$-th power has $2^n \mid b_t$ for each $t$. Kummer theory consequently gives $\mathrm{Gal}(C_n/B_n) \simeq (\mathbf{Z}/2^n\mathbf{Z})^P$, where a vector $(a_t)$ acts by $t^{1/2^n} \mapsto \zeta_{2^n}^{a_t} t^{1/2^n}$.

Let $j$ be the number of prime factors of $d$. If $j$ is odd, choose $a_t = 1$ for every $t \in P$. If $j$ is even, choose $a_t = 2$ for exactly one prime dividing $d$, and $a_t = 1$ for every other $t \in P$. Then $\sum_{t\mid d} a_t$ is odd and each $a_t$ has two-adic valuation at most one. The field $C_n$ is Galois over $\mathbf{Q}$. Apply Chebotarev to the conjugacy class of the prescribed automorphism $h \in \mathrm{Gal}(C_n/B_n)$. We obtain a prime $r$ split in $B_n$ for which the Kummer exponents have the stated 2-adic valuations. Conjugating $h$ can multiply all exponents by an odd integer, which does not change those valuations or their parity. Since $h(\sqrt{d}) = -\sqrt{d}$, the prime $r$ is inert in $K$. Since $r$ splits in $B_n$, it satisfies $r \equiv 1 \pmod{2^{n+1}}$, $\quad 2 \in (\mathbf{F}_r^\times)^{2^n}$. Take $L$ to be the unique subfield of $\mathbf{Q}(\zeta_r)$ of degree $2^n$. The first congruence makes $L$ totally real; the second makes two split completely. For $t \in P$, its Frobenius in $\mathrm{Gal}(L/\mathbf{Q})$ has order $2^n$ when $a_t$ is odd, and order $2^{n-1}$ when $a_t = 2$. This follows from the $2^n$-th power residue symbol at $r$. If $\ell \nmid d$, we chose $a_\ell = 1$, so $L$ has one prime above $\ell$ and the quadratic extension $KL/L$ yields at most two. If $\ell \mid d$, it ramifies in $K$ and is unramified in $L$. The unramified local extension supplied by $L$ and the ramified quadratic extension supplied by $K$ are disjoint. Thus the number of primes over $\ell$ in $KL$ equals their number in $L$, which is at most two. This proves all assertions. □

**Lemma 7.2** (Propagation of one-dyadic scalar Selmer vanishing)**.** *Let $K$ be a totally real field with two places $w_1, w_2$ above two, and suppose that the only totally real quadratic extension of $K$ unramified outside $w_1$ and split at $w_2$ is the trivial extension. Let $F/K$ be cyclic of degree $2^n$, split at $w_1, w_2$, unramified outside a single finite prime $\mathfrak{r} \nmid 2$, and totally ramified there. Then the only totally real quadratic extension of $F$ unramified outside the full fibre above $w_1$ and split at every place above $w_2$ is trivial.*

*Proof.* Let $V$ be the corresponding subspace of $H^1(G_F, \mathbf{F}_2)$. It is stable under the 2-group $\Gamma = \mathrm{Gal}(F/K)$. If $V \neq 0$, the elementary invariant-vector theorem for representations of a finite 2-group in characteristic two gives a nonzero $c \in V^\Gamma$. Let $H/F$ be the quadratic extension cut out by $c$. Then $H/K$ is Galois, and its group $G$ is a central extension of the cyclic group $\Gamma$ by $C_2$. Every such central extension is abelian: a lift of a generator of $\Gamma$, together with the central kernel, generates $G$.

Because $H/F$ is unramified at $\mathfrak{r}$ and $F/K$ is totally ramified there, the inertia subgroup $I \subset G$ maps isomorphically onto $\Gamma$ and intersects $\mathrm{Gal}(H/F)$ trivially. Consequently $G = I \times \mathrm{Gal}(H/F)$. The fixed field $E = H^I$ is quadratic over $K$, totally real, and unramified at $\mathfrak{r}$. At every finite place away from $w_1$ and $\mathfrak{r}$, both steps $H/F$ and $F/K$ are unramified, so $E/K$ is unramified there. At $w_2$, both steps split completely, so $E/K$ splits. This contradicts the assumed vanishing over $K$. □

**Theorem 7.3** (Uniform scalar field preparation)**.** *Let $K$ be real quadratic with two split and satisfying* (SV)*. Let $\Sigma$ be a fixed finite set of odd rational primes, and fix integers $n > a \geq 1$. There are totally real abelian two-extensions $K \subset A \subset F$, split completely at two, and a dyadic place $v_0$ of $A$, with the following properties. Put*

$$D = [F : \mathbf{Q}], \qquad M = \{v \mid v_0 \text{ in } F\}, \qquad m = |M|,$$

*and let $S_A, S_F$ consist of the real places, the dyadic places, and the places above $\Sigma$.*

1. $D = 2^{n+1}$, $[F : A] = 2^{n-a}$, *and* $m/D = 2^{-a-1}$. *The numbers* $u_A, u_F$ *of allowed odd places are at most* $2|\Sigma|$.
2. *For every finite field* $k$ *of characteristic two,*

$$\{c \in H^1(G_{F,S_F}, k) : \mathrm{loc}_v\, c = 0 \text{ for all } v \in S_F \setminus M\} = 0.$$

   *Here and below* $k$ *has trivial Galois action.*
3. *Restriction* $H^1(G_{A,S_A}, k) \longrightarrow H^1(G_{F,S_F}, k)$ *is injective.*
4. *Let* $H^1_{\mathrm{str}}(F, k)$ *have zero local condition at* $M$*, full local condition at* $S_F \setminus M$*, including the real places, and unramified condition outside* $S_F$*. Then*

$$\dim_k H^1_{\mathrm{str}}(F, k) = 2D - 3m + u_F.$$

*The degrees* $[A : \mathbf{Q}]$ *and* $[F : A]$ *can both be made arbitrarily large, while the allowed odd-place bound remains fixed.*

*Proof.* Apply Lemma 7.1, take the fields $A = KL_a$ and $F = KL$ in Lemma 7.2, using (SV) as its base vanishing hypothesis. The first three assertions follow from the two preceding lemmas and complete splitting at two. For injectivity, a class killed under restriction factors through the cyclic group $\mathrm{Gal}(F/A)$. Its only nonzero quadratic character is ramified at the fresh conductor prime $r$, which is outside $S_A$. Such a class is excluded from $H^1(G_{A,S_A}, k)$, proving injectivity.

For the last assertion use the Selmer dimension formula from local and global Tate duality, with the real local conditions included [62, Theorem 4.2]. The scalar module $k$ is self-dual with Tate twist, since the mod-two cyclotomic character is trivial. The dual local conditions are full at $M$, zero at $S_F \setminus M$, and unramified elsewhere. Their global Selmer group vanishes by the second assertion. The global invariant terms therefore cancel. Each unmarked dyadic place contributes $\dim H^1(G_{\mathbf{Q}_2}, k) - 1 = 2$, and each marked place contributes $-1$. A real place with full local condition contributes $1 - 1 = 0$. At every allowed odd finite place, local Kummer theory gives $\dim_k H^1(G_{F_v}, k) = 2$, so its contribution is one. Unramified local conditions contribute zero outside $S_F$. Consequently the strict Selmer dimension is $2(D - m) - m + u_F = 2D - 3m + u_F$, as claimed. Taking $a$ and $n - a$ large proves the final assertion. □

When $\Sigma = \varnothing$, taking $n = a + 2$ in Theorem 7.3 gives $[F : A] = 4$ with $[A : \mathbf{Q}]$ arbitrarily large. There are no odd local Euler terms, so the same Selmer dimension formula gives $\dim_k H^1_{\mathrm{str}}(F, k) = 2[F : \mathbf{Q}] - 12$.

*Remark* 7.4. Condition (SV) is a genuine restriction. For $K = \mathbf{Q}(\sqrt{113})$, the totally positive element $\alpha = (11 + \sqrt{113})/2$ has norm two and valuation one at one dyadic prime. At the other its unit value $u$ satisfies $u + 2/u = 11$, so $u \equiv 1 \pmod 8$ and is a square in $\mathbf{Q}_2$. Thus $[\alpha]$ is a nonzero element of the corresponding $V^+_{w_0}(K)$. This is a limitation of the criterion, not an obstruction to modularity over that field.

## 8 The scalar dyadic endpoint

We prove Theorem 1.4 using the field preparation of Section 7 and the dyadic inputs listed in Section 5. The argument treats arbitrary finite ramification and regular weights.

*Proof of Theorem 1.4.* First suppose $r$ is virtually solvable, in the sense that the identity component of its algebraic monodromy group is solvable. An irreducible regular two-dimensional geometric representation with this property is induced from a character of a quadratic extension of $K$. Indeed, a nontrivial unipotent radical would have a unique invariant line preserved by the whole monodromy group, while scalar identity component would give finite projective image

and, after finite extension, equal Hodge–Tate weights. The remaining torus case has two lines permuted by a quadratic quotient. The inducing quadratic field must be CM. The inducing character is algebraic. If its field had a real place, the absolute-value relation on units and Dirichlet's unit theorem would force all real algebraic exponents to agree. At the two real embeddings above any split real place of $K$, the induced weights would therefore coincide. This contradicts regularity and also excludes mixed signature. Its inducing character is de Rham and finitely ramified, hence is the 2-adic avatar of an algebraic Hecke character. Each dyadic place is nonsplit in the inducing extension: splitting would make the local induction a sum of characters, contrary to local absolute irreducibility. Quadratic automorphic induction gives a cuspidal automorphic representation with Galois representation $r$; its regular algebraic infinity type follows from the given regular Hodge–Tate weights. We henceforth assume that $r$ is not virtually solvable. Its restriction to every finite extension is then absolutely irreducible.

*The seed and the auxiliary fields.* A cuspidal modular seed with scalar semisimple reduction is available over $\mathbf{Q}$: take the compatible system of Ramanujan's form $\Delta$, twisted by $\varepsilon_2^{-5}$. Its determinant is $\varepsilon_2$, it is non-CM, and its residual semisimplification at 2 is $\mathbf{1} \oplus \mathbf{1}$. Indeed Jacobi's identity gives $\Delta \equiv \sum_{j\geq 0} q^{(2j+1)^2} \pmod 2$, so the trace at every odd prime is zero and the residual determinant is one. Base change supplies a cuspidal seed over $K$ and every abelian auxiliary field used below. Its determinant agrees exactly with the target; hence its quaternionic transfer has the required central character. It has tame level one, and the small uniform subgroups for completed forms are chosen at two.

Choose a finite set $\Sigma$ of odd rational primes supporting the ramification of $r$ and the seed, and enlarge it when necessary to include the prime factors of $d$. Theorem 7.3 supplies

$$K \subset A \subset F, \qquad D_A = [A : \mathbf{Q}], \quad D = [F : \mathbf{Q}], \qquad m = [F : A],$$

with $A/\mathbf{Q}$ and $F/\mathbf{Q}$ abelian, 2 completely split, and uniformly bounded numbers $u_A, u_F$ of allowed odd places. The marked set $M$ consists of the $m$ places of $F$ over one selected dyadic place of $A$. Both $D_A$ and $m$ may be made arbitrarily large, with $m/D = 1/D_A$. The same theorem gives injectivity of scalar residual extension classes under $A \subset F$, and gives

$$\{c \in H^1(G_{F,S_F}, k) : \mathrm{loc}_v\, c = 0 \text{ for all } v \in S_F \setminus M\} = 0. \tag{S1}$$

Here $S_F$ consists of the real and dyadic places and the allowed odd places. The coefficient field $k$ can be enlarged without changing the vanishing. The real local conditions in (S1) are zero conditions, so the assertion uses the totally positive squareclass condition (SV), not only an ordinary class-number condition.

For the strict Selmer group, Theorem 7.3(iv) gives

$$\dim_k H^1_{\mathrm{str}}(F, k) = 2D - 3m + u_F. \tag{S2}$$

Let $c$ be a fixed generator bound for the augmentation ideal of the characteristic-two universal local pseudodeformation ring of $G_{\mathbf{Q}_2}$ with residual pseudorepresentation $\mathbf{1} \oplus \mathbf{1}$ and fixed determinant one. This bound is independent of $A$: every selected dyadic completion is $\mathbf{Q}_2$, and extending the residue field preserves a fixed bound. Choose

$$D_A > \max\{4, u_A + c + 3\}, \qquad m > u_F + 4. \tag{S3}$$

The uniform bound on $u_A, u_F$ and independent growth of $D_A, m$ make these choices possible. Since $D = mD_A$, they imply $D > u_F + 4$ and $(2D - 2m - 4 - u_F) - D = m(D_A - 2) - (u_F + 4) > 0$. They also give the dimension bound needed for entry from $A$ below.

*Connected transverse charts.* Over the characteristic-two fixed-determinant pseudodeformation base for $G_F$, take the finite-type framed representation scheme and record an invariant projective

line at each mark. In its residual fibre retain the open locus where no marked line is globally invariant. At such a point the residual representation at a mark is scalar: a nonzero unipotent extension has only one invariant line, whereas the global invariant line and the chosen transverse line are different. An upper extension with diagonal characters frozen must therefore be zero locally at every mark.

At a transverse point, the product of the real and marked dyadic local line problems has dimension $2D+5m$, by [58, Lemma 3.12]. The relative global presentation of [58, Proposition 3.6] has number of generators minus relations at least $-6m$. Every component of the completed chart consequently has dimension at least $2D-m$, and its components can be joined with intersections of dimension at least $2D-m-1$, by complete-local connectedness [78, Lemma 52.14.1, Tag 0ECR].

The globally reducible locus has dimension at most $2D-2m+u_F+3$. Indeed, fixed-product diagonal characters contribute at most one parameter by Leopoldt for the abelian field $F$; moving the global invariant line and an upper coboundary contribute at most two; the strict extension class contributes at most $2D-3m+u_F$ by (S2); and moving the $m$ marked lines contributes at most $m$. This is a relative generator bound after freezing the diagonal characters. Adding the global invariant line by proper incidence and then forgetting it bounds the entire reducible image. By (S3), this bound is strictly less than $2D-m-1$.

Therefore a maximal-dimensional intersection in a component chain is generically globally irreducible. Forgetting the global frame and marked lines loses at most $3+m$ dimensions. The remaining trace image has dimension at least $2D-2m-4$. Prescribing one Frobenius trace at each allowed odd place costs at most $u_F$ equations. The consequence of (S3) displayed above leaves dimension greater than $D$. The dihedral-locus bound in [58, Lemma 3.8] and its curve-avoidance argument supply a non-virtually-solvable characteristic-two curve. The finite local-image conclusion is exactly [58, Lemma 3.9], applied after its fixed preliminary local extensions; the determinant on the characteristic-two curve is one. The prescribed residual Frobenius traces thus make all allowed odd local images finite. Theorem 5.1 of [58] then propagates potential pro-modularity across this intersection. Thus all components through a transverse residual point have the same potential pro-modularity status.

The transverse residual fibre is geometrically irreducible. Fixing the global line, its extensions form the nonzero vectors of the strict scalar cohomology space. This space contains the strict space with no odd ramification, of dimension $2D-3m>0$ by the same duality calculation. The complementary marked lines form $\mathbb{A}^m$. Conjugation gives a surjection from the product of these two spaces with $\mathrm{GL}_2$, an irreducible parameter space. Since the representation scheme is noetherian, one irreducible component $Y_0$ contains this whole transverse residual fibre. Local chart propagation at the seed and target entry points, through $Y_0$, therefore connects their components. Completion is faithfully flat, so the completed-chart conclusion applies to components of the representation scheme.

*Entering the transverse charts from the seed and target.* We include this step because connecting deformation components for one fixed residual extension would not suffice. Work first over $A$. The seed gives a pro-modular component in the fixed-determinant pseudo-base of dimension at least $1+2D_A$, by [58, Proposition 4.4]. An unrestricted fixed-determinant framed component through $r|_{G_A}$ has trace image of dimension at least $2D_A-2$, by the relative presentation over the real local rings and the loss of three frame parameters. Its generic representation is absolutely irreducible because it contains the given target.

In each component pass to the special fibre, prescribe the residual Frobenius traces at the allowed odd places, and prescribe the constant scalar local pseudorepresentation at the selected dyadic place of $A$. The last condition costs at most the $c$ local generators fixed above. The total cost is at most $1+u_A+c$ equations. Even for the smaller target component, the remaining

dimension is at least $2D_A - 2 - (1 + u_A + c) = 2D_A - u_A - c - 3 > D_A$ by (S3). The bad-locus bound and curve avoidance therefore supply a non-virtually-solvable characteristic-two curve in each component. Lemma 3.9 of [58] makes its odd local images finite, as above.

At the selected dyadic place the curve representation has an invariant line after a finite coefficient extension. Complete the normalization of the curve at its residual point. Its valuation ring has finite residue field of characteristic two, so Lemma 4.10 applies in equal characteristic. The resulting orbit lattice has nonzero additive residue and a marked vector transverse to its global invariant line. Injectivity of scalar extension classes under $A \subset F$ ensures that it remains nonzero over $F$. Transporting the local line to all marks gives transverse lines, because $F/A$ is Galois. Thus both curves enter the transverse residual fibre over $F$.

The seed curve is potentially pro-modular by solvable base change. Propagation through the transverse charts, through the common component $Y_0$, makes the target curve potentially pro-modular. Applying [58, Theorem 5.1] back over $A$ to the original target curve makes its entire containing component potentially pro-modular. In particular, $r$ becomes pro-modular over one finite solvable totally real extension of $K$, completely split at 2. Only finitely many component chains and propagation steps occur; their field extensions and tame levels can be replaced by one common Galois compositum and one common deeper tame level. This proves the scalar version of the pro-modularity input directly.

*Classicality and descent.* At every dyadic place of the resulting field the local representation is still the original absolutely irreducible regular de Rham representation over $\mathbf{Q}_2$, since the auxiliary extensions split completely above 2. Proposition 4.2 of [58], together with the all-prime local correspondence [14, Theorems 1.1 and 1.3], gives a nonzero locally algebraic vector in its Hecke eigenmodule, including when the residual local block is scalar. Proposition 4.1 identifies this vector with a regular algebraic definite-quaternionic eigensystem. Global irreducibility excludes reduced-norm characters, so Jacquet–Langlands produces a Hilbert cusp form. Local-global compatibility and Chebotarev identify its Galois representation with the restriction of $r$.

Finally descend through a tower of prime-degree cyclic extensions to $K$. At each step, Galois invariance of $r$ and strong multiplicity one make the upstairs cuspidal representation invariant under the cyclic Galois group. Cyclic automorphic descent [4, Chapter III, Theorems 4.2(d) and 5.1] yields a cuspidal representation downstairs. Its Galois representation differs from the given one by a character of the cyclic quotient, because their restrictions are absolutely irreducible and isomorphic upstairs; twisting corrects that character. The archimedean types remain regular algebraic because every real place splits in a totally real extension. Iterating gives modularity of $r$ over $K$. This is the locally irreducible part of the classicality and descent argument in [58, Section 8.8]; none of its ordinary dense-point arguments over $\mathbf{Q}$ is used here. □

# 9 Dyadic lifts and compatible systems

We construct two local types which suffice at every dyadic place and then globalize them without enlarging the prescribed ramification support. Steinberg lifts enter only in the local multiplicity calculation; the global lifts will all be potentially Barsotti–Tate and potentially nonordinary.

Let $k$ be finite of characteristic two. For a continuous $\bar{s} : G_{\mathbf{Q}_2} \to \mathrm{GL}_2(k)$ put $\psi_{\bar{s}} = \varepsilon_2[\det \bar{s}]$, with brackets denoting Teichmüller lift. Choose $E/\mathbf{Q}_2$ with residue field $k$, enlarging both when necessary, and write $\mathcal{O}$ for its ring of integers and $\varpi$ for a uniformizer. Let $\tau_0 = \widetilde{\omega}_2 \oplus \widetilde{\omega}_2^2$, where $\omega_2$ has niveau two, and let $\tau_*$ be the inertial type of the elliptic curve

$$y^2 = x^3 - 2x,$$

with Cremona label `256b1` (LMFDB label `256.b1`) [55].

**Proposition 9.1.** *Every $\bar{s}$ has a potentially Barsotti–Tate lift of determinant $\psi_{\bar{s}}$ and labelled weights $\{0,1\}$, with type $\tau_0$ if it admits a crystalline lift of those weights, and $\tau_*$ otherwise. Both types force actual local irreducibility and potential nonordinarity. The inertia image of $\tau_*$ is $Q_8$.*

*Proof.* Every character $G_{\mathbf{Q}_2} \to k^\times$ is unramified: by local class field theory its inertial image is a quotient of the pro-two group $\mathbf{Z}_2^\times$, whereas $k^\times$ has odd order. If a crystalline weight-two lift exists, its determinant differs from $\psi_{\bar{s}}$ by an unramified character $\delta$ reducing to one. After enlarging coefficients, an unramified character $\nu$ with $\nu^2 = \delta^{-1}$ also reduces to one. Twisting by $\nu$ therefore fixes the determinant without changing the residue, weights or inertial type.

We first identify the residual representations for which no such crystalline lift exists. An irreducible $\bar{s}$ has trivial wild inertia, since the fixed space of this normal two-subgroup is nonzero and $G_{\mathbf{Q}_2}$-stable. After an unramified scalar twist its determinant is one; its two tame inertia characters are exchanged by squaring and have order three. It is the unique such representation, realized by $A[2]$ for the good-reduction curve $A : y^2 + y = x^3$ used in the proof of Lemma 4.8. Thus it has a crystalline weight-two lift. A split residue $\bar{\alpha} \oplus \bar{\beta}$ lifts to $\varepsilon_2[\bar{\alpha}] \oplus [\bar{\beta}]$. For a nonsplit reducible residue, untwist its quotient and write its character ratio as $\bar{\eta}$. If $\bar{\eta} \neq 1$, local Euler characteristic and the Bloch–Kato formula [9, Corollary 3.8.4] give

$$\dim_E H^1(G_{\mathbf{Q}_2}, E(\varepsilon_2[\bar{\eta}])) = \dim_E H^1_f(G_{\mathbf{Q}_2}, E(\varepsilon_2[\bar{\eta}])) = 1.$$

Moreover $H^2(G_{\mathbf{Q}_2}, \mathcal{O}(\varepsilon_2[\bar{\eta}])) = 0$ by local Tate duality, since $[\bar{\eta}]^{-1} - 1$ takes a unit value. Reduction of $H^1$ is consequently surjective, and the actual residual extension lifts crystallinely. Hence failure of crystalline lifting is possible only for

$$\bar{s} = \bar{\alpha} \otimes \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \qquad 0 \neq c \in H^1(G_{\mathbf{Q}_2}, k),$$

with $\bar{\alpha}$ unramified.

For this last representation, Kummer theory identifies $H^1(G_{\mathbf{Q}_2}, \mathcal{O}(\varepsilon_2))$ with $\widehat{\mathbf{Q}_2^\times} \otimes_{\mathbf{Z}_2} \mathcal{O}$, where the hat denotes pro-two completion. Its reduction onto $H^1(G_{\mathbf{Q}_2}, k)$ is surjective. Choose a lift $\kappa$ of $c$ with nonzero valuation; adding a suitable $\varpi\mathcal{O}$-multiple of the Kummer class of two achieves this without changing its reduction. The extension of 1 by $\varepsilon_2$ defined by $\kappa$ is semistable and noncrystalline: Kummer classes are semistable, and the crystalline subspace is the unit subspace. Twisting by $[\bar{\alpha}]$ gives determinant $\varepsilon_2[\bar{\alpha}]^2 = \psi_{\bar{s}}$ and preserves the actual residual extension.

We now apply the full fixed-determinant Breuil–Mézard formula [85, Conjectures 5.1.2–5.1.3 and Corollary 8.0.2]. The two Serre weights are $W_0 = 1$ and $W_1 = \mathrm{Sym}^1$. Write $\mu_0(\bar{s}), \mu_1(\bar{s})$ for their nonnegative multiplicities. For a compact type $\sigma$ with semisimplified reduction containing $W_i$ with multiplicity $a_i$, the Hilbert–Samuel multiplicity of the corresponding fixed-determinant weight-two ring modulo $\varpi$ is $a_0\mu_0(\bar{s}) + a_1\mu_1(\bar{s})$. These deformation rings are taken with their reduced, $\mathcal{O}$-torsion-free structure; positivity is equivalent to a nonempty characteristic-zero fibre. The crystalline trivial type has reduction $W_0$, so $\mu_0 > 0$ exactly when the crystalline lift exists, with determinant fixed as above. The compact type of $\tau_0$ is the inflation of the sign character of $\mathrm{GL}_2(\mathbf{F}_2) \simeq S_3$, whose reduction is also $W_0$. This proves the $\tau_0$ assertion. Otherwise $\mu_0 = 0$, whereas the semistable lift just constructed gives $\mu_1 > 0$: the compact type for unrestricted monodromy and trivial inertia is the finite Steinberg representation, whose reduction is $W_1$. The determinant compatibility in both comparisons follows from $\det \tau_0 = 1$ and the unramifiedness of $[\det \bar{s}]$.

For $\tau_*$ we use one test representation. The two-torsion of the displayed curve has Kummer class two, which has odd valuation and is not finite flat [13, Definition A.3 and Lemma A.4]; in particular it has no crystalline weight-two lift. Its three-division polynomial is $3X^4 - 12X^2 - 4$.

Choose $u^2 = -3$, $i^2 = -1$, $s = iu$. Its roots are $\pm\alpha, \pm\beta$, with $\alpha^2 = 2 + 4/s$, $\beta^2 = 2 - 4/s$, $\alpha\beta = 2u/3$. They recover $s, u$, so their splitting field is $L = \mathbf{Q}_2(u, i, \alpha)$. The field $U = \mathbf{Q}_2(u)$ is unramified quadratic, and $T = U(i)$ is ramified quadratic. For $\pi = 1 - i$, one has $\alpha^2/\pi^2 = i + 2/u$ and $v_\pi(i + 2/u - 1) = 1$. This unit is nonsquare even after unramified extension of $T$: a putative square root has residue one, while its square is one modulo $\pi^2$. Hence $L/T$ is ramified quadratic, with $e(L/\mathbf{Q}_2) = 4$, $f(L/\mathbf{Q}_2) = 2$. Projective inertia on the four roots is $V_4 \subset A_4$. Its inverse image in $\mathrm{SL}_2(\mathbf{F}_3)$ is $Q_8$, with no proper subgroup surjecting onto $V_4$. The curve has potentially good reduction since $j = 1728$; its finite inertia on $T_3$ reduces injectively modulo three. Compatibility identifies its faithful irreducible $Q_8$ inertia with $\tau_*$.

The Tate module of this curve gives a nonempty type-$\tau_*$ ring, while its residue has $\mu_0 = 0$. The multiplicity formula therefore forces $a_1(\tau_*) > 0$. For every $\bar{s}$ without a crystalline weight-two lift we already have $\mu_1(\bar{s}) > 0$, so the same formula gives a nonempty type-$\tau_*$ fibre. An unramified scalar twist removes and restores $\bar{\alpha}$ without changing the compact type, giving the required determinant in general.

The two characters of $\tau_0$ are exchanged by Frobenius, and $\tau_*$ is irreducible on inertia. Their Weil parameters are therefore irreducible, which forces Galois irreducibility. An ordinary line after finite Galois base change would be the unique line of labelled weight one and would descend, a contradiction. Thus neither type is potentially ordinary. □

**Lemma 9.2.** *Let $F$ be totally real and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be finitely ramified, with nonsolvable residue, $\det r = \varepsilon_2\eta$ for a finite totally even character $\eta$, and potentially Barsotti–Tate and potentially nonordinary at every dyadic place. Then $r$ is potentially modular.*

*Proof.* A finite character twist makes the residual determinant one. Apply [83, Proposition 6.7], avoiding its kernel field, to make the residue modular without shrinking its image. A further solvable totally real extension makes every dyadic source Barsotti–Tate. Nonsolvability and residual modularity persist. Kisin's theorem [52, Theorem 3.3.5] applies: its condition that a completion equal $\mathbf{Q}_2$ concerns only potentially ordinary places, and there are none. Restore the finite twist. □

**Corollary 9.3** (Prescribed dyadic lifts in compatible systems)**.** *Let $K$ be real quadratic with two split, and let $\bar{r} : G_K \to \mathrm{SL}_2(k)$ be continuous with nonsolvable image. Fix a finite set $S$ containing the dyadic places and all residual ramification. At each odd place in $S$, choose a nonempty union of components of the generic fibre of the local lifting ring with determinant $\varepsilon_2$. Then $\bar{r}$ lifts to the two-adic member of a potentially modular weight-two strongly compatible system of cyclotomic determinant, unramified outside $S$, on the chosen odd local components, and with dyadic types prescribed by Proposition 9.1.*

*Proof.* Lemma 4.9 makes the odd local lifting rings nonempty and permits an unramified choice wherever the residue is unramified. At the dyadic places use Proposition 9.1. Choose a finite extension of each completion killing its type. In the corresponding potentially semistable generic fibre of weights $\{0, 1\}$, the type is locally constant, so the prescribed type selects a nonempty union of components. The associated Weil parameters are irreducible, so monodromy vanishes, and Proposition 9.1 gives potential nonordinarity.

The global determinant $\varepsilon_2$ lifts the residual determinant and is totally odd. Since $K(\zeta_2) = K$, the nonsolvable image gives the required absolute irreducibility. The prescribed-lift theorem [35, Theorem 4.2.1] therefore supplies a characteristic-zero point unramified outside $S$ on these components. Lemma 9.2 makes this actual lift potentially modular. The compatible-system construction in the proof of Lemma 2.6 then gives the assertion, including the local conditions and ramification support. □

# 10 A Serre criterion when two splits

**Theorem 10.1** (The split-dyadic criterion)**.** *Let $K$ be real quadratic with two split. Use the lifting inputs of Sections 2 and 5 and the solvable dyadic source theorem cited there. Suppose:*

*(i) every absolutely irreducible two-adic representation of $G_K$ unramified outside two, with cyclotomic determinant, scalar residual semisimplification $1 \oplus 1$, and both dyadic restrictions absolutely irreducible and regular de Rham, is modular;*
*(ii) the ordinary ideal class group $\mathrm{Cl}(K)$ has two-power order;*
*(iii) every continuous absolutely irreducible representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ of determinant one, unramified outside two, is modular.*

*Then every continuous absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, for every odd prime $p$ and arbitrary conductor, is modular.*

For (i), Theorem 1.4 applies whenever $V^+_{w_0}(K) = 0$ under its stated propagation inputs. Alternatively, when $d$ is prime and condition (S) holds, [25, Theorem 9.4(b)] gives (i) with both dyadic places in $\Sigma_n$. Condition (iii) is a residual base case, separate from the scalar source theorem. The proof below has no analytic assumption beyond whatever is needed to establish that condition. Strong compatibility includes the coefficient prime, so a crystalline replacement clears that conductor from every companion.

*Proof.* **10.1 Initial lifts and the protective place**

Use the global construction and compatible systems of Lemma 2.6, the modular witnesses of Lemma 2.1, and the backward lifting statements in Lemma 2.2. For arbitrary regular dyadic sources at the split completions $\mathbf{Q}_2$, we use [85, Theorem 8.0.3]. Apply Proposition 2.7. In the nonsolvable case it supplies the initial weight-two compatible system. Two splits, so $K \neq \mathbf{Q}(\sqrt{5})$ and the characteristic-five exception is absent. The subsequent protective images exclude the exceptional small images.

The protective place in Proposition 3.1 can be chosen with the additional property that the order of two modulo its rational norm is odd. We give the simultaneous field conditions, since this property is needed when its own inertia ceases to protect the last congruence.

Write $K = \mathbf{Q}(\sqrt{d})$ with $d > 1$ squarefree. A reversible finite twist removes the odd-order part of its finite determinant, leaving $\det r_t = \varepsilon_t \psi_2$, with $\psi_2$ of two-power order and with no new rational conductor prime. Choose a fresh odd reset prime $s > 5$, above the needed prime bounds and split in $K$, by Chebotarev. Include it, two, three, five, every conductor prime and every prime dividing $d$ in a finite set $S$ before choosing the protective place.

Choose $N$ and the projective residual field as in the proof of Proposition 3.1, with $N$ large enough for the fixed weight bounds. Thus $N$ splits in $K$ and the coefficient field, the projective image is $\mathrm{PGL}_2(\mathbf{F}_N)$, and $\psi_2$ is square-valued modulo $N$.

Let $K_S/K$ be the maximal elementary quadratic extension unramified outside $S$ and infinity. Choose a degree-one place $\lambda$ of rational norm $\ell$ with the standard good-dihedral Frobenius prescription and

$$\ell \equiv -1 \pmod{N}, \qquad \lambda \text{ split in } K_S, \qquad \ell \equiv 9 \pmod{16}, \qquad 2 \in (\mathbf{F}_\ell^\times)^8. \tag{7}$$

The last two conditions make $\mathrm{ord}_\ell(2)$ odd. We verify compatibility of the simultaneous conditions. The Galois group of $B = \mathbf{Q}(\zeta_8, 2^{1/8})$ has order sixteen. Writing its elements as $(a, b)$ for the actions on $2^{1/8}$ and $\zeta_8$, multiplication is $(a, b)(c, e) = (a + bc, be)$. The commutator of $(1, 3), (0, 7)$ is $(2, 1)$, generating $\mathrm{Gal}(B/\mathbf{Q}(\zeta_8))$. Thus its maximal abelian subfield is $\mathbf{Q}(\zeta_8)$. Since $B$ is ramified only at two and $K$ is real quadratic with two split, $K \cap B = \mathbf{Q}$. The same statement

holds with $\mathbf{Q}(\zeta_{16})$ in place of $B$. The compositum with the elementary extension $K_S/K$ has abelianization of exponent two: the same commutators survive with trivial $K_S$ coordinate. Hence $K(\zeta_{16}) \not\subset BK_S$ and its intersection with $BK_S$ is $K(\zeta_8)$. Splitting in $BK_S$ is compatible with Frobenius nine on $\zeta_{16}$. This two-extension is unramified at $N$; the only nontrivial solvable quotient of the projective residual field is its quadratic cyclotomic quotient, ramified at $N$. Its intersection with the compositum of the preceding two-extension and $K(\zeta_N)$ is exactly that quotient. Projective complex conjugation and inversion on $\zeta_N$ agree on it, since $N \equiv 1 \pmod 4$. Chebotarev therefore gives (7), with $\ell$ split in $K$ and arbitrarily large.

Install the order-$N$ induced type at $\lambda$ and apply the forward construction and split-$N$ backward lifting of Proposition 3.1. Its image argument makes every residue in $S$ nonsolvable, with adequate cyclotomic restriction in odd characteristic. The additional conditions (7) do not alter that argument.

### 10.2 Dyadic preparation and odd conductor

Reduce the protected system modulo two. Its determinant is one, since $\psi_2$ has two-power order. At each odd ramified place use Lemma 4.9 to choose a component of determinant $\varepsilon_2$; its type need not equal the old type. At $\lambda$ retain the order-$N$ induction: take inertia $\mathrm{diag}(\zeta_N, \zeta_N^{-1})$ and antidiagonal Frobenius with off-diagonal product $-\ell$, giving determinant $\ell$ and the same residual local representation. Corollary 9.3 gives a potentially modular weight-two system of exact cyclotomic determinant, with dyadic types $\tau_0$ or $\tau_*$ and no new conductor prime. For the backward arrow apply Tung over $K$ itself, where the dyadic completions are $\mathbf{Q}_2$: the old regular source may have arbitrary dyadic conductor. Once the replacement is modular, the common nonsolvable residue is modular and the theorem applies.

Retain the dyadic types $\tau_0, \tau_*$ in every subsequent odd-characteristic congruence. Remove all odd rational conductor primes except $s, \ell$. Before removing a nonsplit prime $t$, change coefficient characteristic to $s$ and choose a weight-two replacement, retaining the dyadic and protective types and every previously cleared place away from $s$. Weight-two local existence is [77, Proposition 7.8.1], followed by prescribed lifting. At $s$, the finite determinant correction reduces to one and has $s$-power order; its inverse square root fixes the determinant without changing the residue or weights [77, Section 7.9]. Unramified local components preserve cleared places, and the old components preserve retained types. Since $s$ splits, Tung gives the backward implication for the actual old regular source.

In characteristic $t$, replace the source by regular crystalline potentially diagonalizable lifts at all places above $t$. This clears its conductor in the new compatible system. For nonsplit $t$, the old source has weight two by the reset, and [3, Theorem 2.3], with the witness from Lemma 2.1, gives the backward implication. For split $t$, use Tung without a weight restriction on the old source. The forward construction applies in characteristic three regardless of its splitting in $K$. All conditions away from $t$ retain the dyadic and protective types. Finally clear $s$ by a regular crystalline replacement in characteristic $s$ and split-prime lifting. The weights are allowed to grow; no crystalline weight-two replacement has been asserted at a removal prime.

### 10.3 The last prime and the terminal source

Remove the protective place last, in characteristic $\ell$. Its own order-$N$ type can no longer protect that residue, so use the dyadic types. If one is $\tau_*$, its $Q_8$ inertia stays absolutely irreducible modulo odd $\ell$: the center acts as $-1$, while every one-dimensional character kills it. The local cyclotomic extension at two is unramified and preserves this inertia. If both are $\tau_0$, the two order-three characters remain distinct modulo $\ell > 3$ and Frobenius interchanges them. By (7), the cyclotomic residue degree is odd, so that interchange survives restriction to $G_{K(\zeta_\ell)}$. The

global residual cyclotomic restriction is therefore absolutely irreducible in both cases.

If the residue is solvable, its Artin lift and regular weight raising give a modular witness; Tung at the split prime $\ell$ already proves modularity of the old source, completing the backward chain. Otherwise $\ell$ is large enough for adequacy. Choose regular crystalline potentially diagonalizable lifts above $\ell$, preserving all dyadic types and cleared places. Prescribed lifting and potential modularity give the system with $\lambda$ removed. Tung gives the backward implication.

The terminal system has cyclotomic determinant, conductor supported above two, and both dyadic supercuspidal types. Its actual 2-adic member $r_2$ is locally absolutely irreducible, de Rham and regular, with no weight-two restriction. If its residue is irreducible and nonsolvable, hypothesis (iii) and Tung give modularity. If the residue is irreducible and solvable, apply [25, Theorem 10.6]: $K/\mathbf{Q}$ is abelian, two splits, all places belong to $\Sigma_n$, and $\Sigma_o = \varnothing$, so its nonsplitness requirement is vacuous. If the residual semisimplification is reducible, both characters have odd order. At the dyadic completions $\mathbf{Q}_2$ they kill the pro-two unit group, and they also kill the real sign groups. They factor through the ordinary class group and are trivial by (ii). The actual member is therefore modular by (i).

In every case the actual terminal source is modular. Compatibility identifies its entire system with the automorphic system, and all backward implications return to the initial lift. Reducing proves Theorem 10.1. □

# 11 Integral orders and finite-flat groups

For the quadratic applications write $F_d = \mathbf{Q}(\sqrt{d})$. We first retain the full coefficient information in an integral order, then establish the finite-flat bounds used on its lattices.

## 11.1 Trace orders

We use simultaneous realizations at every dyadic coefficient place. This is essential when a local calculation bounds a global trace-field degree. Let $\mathcal{R}_1, \dots, \mathcal{R}_m$ be distinct coefficient orbits of absolutely irreducible strongly compatible rank-two systems, with trace fields $D_i$ and cyclotomic determinants. Write $d_i = [D_i : \mathbf{Q}]$. Let $R_0$ be the closed $\mathbf{Z}_2$-algebra generated by their simultaneous traces and let $E_0$ be the actual order generated by their simultaneous group images and the trace scalars, in integral matrix realizations at all the coefficient places above two.

**Lemma 11.1.** *The rings $R_0, E_0$ are finite free $\mathbf{Z}_2$-modules, and*

$$R_0[1/2] = \prod_i (D_i \otimes_{\mathbf{Q}} \mathbf{Q}_2), \qquad \operatorname{rank}_{\mathbf{Z}_2} R_0 = \sum_i d_i, \qquad \operatorname{rank}_{\mathbf{Z}_2} E_0 = 4 \sum_i d_i.$$

*The order $E_0$ is the closed $\mathbf{Z}_2$-span of the group image. Its trace and determinant laws take values in $R_0$. For a subcollection of dyadic coefficient orbits, the same assertions hold with the selected local trace-field factors and the sum of their degrees over $\mathbf{Q}_2$. If every residual semisimplification is scalar, $E_0/2E_0$ is a local algebra with residue field $\mathbf{F}_2$.*

*Proof.* All the chosen matrices preserve integral lattices, so their span is contained in a finite free $\mathbf{Z}_2$-module. Cayley–Hamilton gives $T(g)1 = g + \det(g)g^{-1}$. Since $\det(g) \in \mathbf{Z}_2^\times$, the closed group span already contains every trace scalar and is an order. The determinant of a sum lies in $R_0$ by the polarization identity $\det(x+y) = \det(x) + \det(y) + T(x)T(y) - T(xy)$, first for linear combinations of group elements and then by continuity.

Good Frobenius traces generate each $D_i$. A rational subalgebra of $\prod_i D_i$ surjecting onto every factor identifies two simple factors only if an embedding matches their good traces.

Brauer–Nesbitt would then identify their coefficient orbits. The rational simultaneous trace algebra is therefore the displayed product. The same argument with the selected local factors uses good traces and coefficient conjugacy over $\mathbf{Q}_2$. On each simple trace factor, absolute irreducibility makes the image algebra central simple of dimension four; it becomes $M_2$ after a coefficient extension. This proves both ranks.

In characteristic two a scalar residual semisimplification with cyclotomic determinant is trivial. Each actual residual image is unipotent, and the principal congruence kernels are pro-two. The simultaneous image is therefore pro-two. Its finite image in $E_0/2E_0$ spans that algebra, which is consequently a quotient of a finite two-group algebra over $\mathbf{F}_2$. Such a group algebra is local with residue field $\mathbf{F}_2$. □

A finite flat group is called biconnected if its special fibre and the special fibre of its Cartier dual are connected.

**Lemma 11.2.** *Let $L/\mathbf{Q}_2$ be finite, let $E/\mathbf{Q}_2$ be a coefficient field with ring of integers $\mathcal{O}$, uniformizer $\lambda$ and residue field $k$, and let $V$ be a Barsotti–Tate representation of $G_L$ over $E$. Every $G_L$-stable $\mathcal{O}$-lattice $T \subset V$ is the Tate module of a 2-divisible group with $\mathcal{O}$-action. In particular, $T/\lambda T$, and every subquotient of it as an $\mathbf{F}_2[G_L]$-module, has a finite flat model killed by two.*

*Proof.* Start with the Tate lattice $T_0$ of a 2-divisible group $H$. After scaling $T$, assume $T_0 \subset T \subset 2^{-n}T_0$. The finite generic subgroup corresponding to $T/T_0$ has finite flat schematic closure in $H[2^n]$. Quotienting $H$ by this closure realizes the lattice $T$. Full faithfulness for 2-divisible groups over a mixed-characteristic discrete valuation ring extends the $\mathcal{O}$-action from the generic fibre. Thus $T/\lambda T$ is the generic fibre of the kernel of $[\lambda]$. Schematic closures of generic subgroups, followed by finite flat quotients, realize its subquotients. Since multiplication by two vanishes on their generic fibres, it vanishes on the finite flat models. This argument also follows from the integral classification in [51]; it uses closures and quotients in the chosen model, rather than exactness of an arbitrary preassigned sequence of integral models. □

**Lemma 11.3.** *Let $V$ be two-dimensional over $E$, Barsotti–Tate over $L$, with determinant $\chi_2$. Then either $V$ is ordinary, or every 2-divisible group realizing a stable lattice in $V$ has connected special fibre, as does its Cartier dual. In the latter case every finite flat subquotient obtained as in Lemma 11.2 has connected special fibre, as does its Cartier dual.*

*Proof.* The rational Tate module of the maximal étale quotient is an $E$-vector space. If it is nonzero, its $E$-dimension is one: dimension two would make all Hodge–Tate weights zero. The alternating pairing $\bigwedge_E^2 V \simeq E(1)$ gives self-duality up to isogeny after composing with $\mathrm{Tr}_{E/\mathbf{Q}_2}$. The étale quotient therefore has a complementary multiplicative subgroup, giving the ordinary filtration. If the étale quotient is zero, self-duality also excludes a multiplicative subgroup. The group and its dual then have connected special fibres. Connectedness is preserved under finite flat subgroups and quotients of these finite connected special fibres, and under duality with the roles of subgroups and quotients interchanged. All these assertions are invariant under isogeny. □

Let now $L/\mathbf{Q}_2$ have absolute ramification index two, let $k_L$ be its residue field, choose a uniformizer $\varpi$, and put $U_j = 1 + \varpi^j \mathcal{O}_L$. The order-two models of the constant generic group are $\mathbf{Z}/2$, $\mu_2$ and the middle model $J$. Only $J$ and its Cartier dual have connected special fibre; this follows from the order-two Oort–Tate classification [79, Theorem 2]. The following assertion permits arbitrary finite residue degree, and hence is unchanged by finite unramified base extension.

**Lemma 11.4.** *Every quadratic extension contained in the field of points of a finite flat group scheme killed by two over $\mathcal{O}_L$ has conductor exponent at most four. A quadratic class of odd valuation has conductor exponent five. If a two-dimensional unipotent representation over $\mathbf{F}_2$ has a finite flat model which is an extension of $J$ by $J$, killed by two, its quadratic class belongs to the image of $U_3$ in $L^\times/L^{\times 2}$. Such a class has conductor exponent at most two.*

*Proof.* Fontaine's bound [39, Theorem A], with its upper numbering shifted by one from classical upper numbering, bounds the shifted upper index for a group killed by $p^m$ by $v_L(p)(m+1/(p-1))$. For a ramified quadratic extension this index is its conductor exponent. Here the bound is four. If a quadratic class has odd valuation, multiply by a square to represent it by $d$ with $v_L(d) = 1$. The polynomial $X^2 - d$ is Eisenstein and its root is a uniformizer in a totally ramified quadratic extension. Its discriminant has valuation $v_L(4d) = 5$.

For the assertion about $J/J$, put $R = \mathcal{O}_L$ and define

$$\mathcal{G}^{(a)} = \operatorname{Spec} R[X, (1+aX)^{-1}], \qquad X * Y = X + Y + aXY.$$

The finite flat homomorphism of degree two

$$\psi : \mathcal{G}^{(\varpi)} \longrightarrow \mathcal{G}^{(\varpi^2)}, \qquad X \longmapsto X^2 + \frac{2}{\varpi} X$$

has kernel $J$, since $1 + \varpi^2\psi(X) = (1+\varpi X)^2$. Write the coordinate algebra of the quotient copy of $J$ as

$$A = R[t]/(t^2 + (2/\varpi)t).$$

An extension of $J$ by $J$ is a $J$-torsor over $A$ trivial on the identity section $t = 0$. The ring $A$ is complete local with residue field $k_L$. A torsor under the smooth group $\mathcal{G}_A^{(\varpi)}$ has a rational point on its special fibre, a torsor under $\mathbb{G}_a$ over $k_L$; Hensel's lemma lifts that point. The fppf sequence of $\psi$ therefore writes the $J$-torsor as

$$z^2 + \frac{2}{\varpi} z = f(t), \qquad f(t) \in A.$$

Its generic Kummer parameter is $q(t) = 1 + \varpi^2 f(t)$. Triviality at $t = 0$ makes $q(0)$ the square of a unit of the form $1 + \varpi z_0$. Dividing by this square gives $q(t) = 1 + \varpi^2 bt$ for some $b \in R$. At the nonzero generic point $t = -2/\varpi$ the parameter is

$$u = 1 - 2\varpi b \in U_3.$$

For an extension killed by two, this is precisely the quadratic class of the unipotent generic representation. Finally, $(\sqrt{u}-1)/\varpi$ is integral, with a monic quadratic polynomial of discriminant $4u/\varpi^2$, whose valuation is two. The conductor–discriminant formula proves the last claim. □

In particular, suppose $V$ is two-dimensional, Barsotti–Tate over $L$ and nonordinary, with determinant $\chi_2$ and residual semisimplification $1 \oplus 1$. Every actual two-dimensional unipotent $\mathbf{F}_2$-subquotient of its residue has a biconnected finite flat model killed by two, by Lemmas 11.2 and 11.3. Its order-two factors are both $J$, so its quadratic class lies in $\operatorname{im} U_3$ by Lemma 11.4.

Let $F/\mathbf{Q}_2$ have absolute ramification index two. The next assertion is restricted to biconnected finite flat groups with trivial generic Jordan–Hölder factors over $\mathbf{F}_2$.

**Lemma 11.5.** *Among biconnected finite flat groups killed by two whose generic Jordan–Hölder factors over $\mathbf{F}_2$ are trivial, every morphism of generic fibres extends uniquely. Generic subobjects and quotients have biconnected finite-flat models, and their exact sequences extend to those models. The simple model has Breuil–Kisin Frobenius $u$. Every self-extension in this category has generic quadratic characters of conductor at most two.*

*Proof.* Schematic closure and finite-flat quotient preserve biconnectedness. Indeed a closed subgroup of a connected finite group over the perfect residue field has local coordinate algebra and is connected; connectedness of quotients follows from surjectivity. Cartier duality gives the other two assertions.

For a group of order two, the height-one exponent is $j$ with $0 \le j \le 2$. Biconnectedness excludes $j = 0, 2$, leaving $j = 1$. The unit factor is removed by change of generator when the generic character is trivial. This is the unique middle model, including its generic identification; see the connected classification in [51, Theorem 4.1, Proposition 4.2 and Corollaries 4.3–4.4].

A generic isomorphism between two objects extends as an isomorphism by induction on their common order. Close its graph in the product. For each projection, choose a generic subgroup of order two and close it in source and target. The restriction of the projection is an isomorphism by uniqueness of the middle model. Its induced map on the finite-flat quotients is an isomorphism by induction, hence the projection is an isomorphism. For an arbitrary generic morphism, apply this to the first projection of its graph closure. The second projection then extends the morphism. Flatness gives uniqueness. Taking schematic closures and finite-flat quotients gives the asserted exact sequences.

A self-extension of the simple model is an extension of $J$ by $J$ killed by two. Lemma 11.4 gives its quadratic conductor at most two. The assertion holds over every finite unramified extension because that lemma permits arbitrary finite residue degree. □

**Lemma 11.6.** *Let $F/\mathbf{Q}_2$ have ramification index two and residue field $\mathbf{F}_2$. A unit $a \in \mathcal{O}_F^\times$ defines a quadratic character of conductor exponent at most two if and only if $a$ is a square modulo $2\mathcal{O}_F$.*

*Proof.* Let $\pi$ be a uniformizer. If $a \equiv x^2 \pmod{\pi^2}$, then $(x + \sqrt{a})/\pi$ is integral and the order it generates has discriminant valuation two. The quadratic field therefore has discriminant exponent at most two. Conversely, $\mathcal{O}_F[\sqrt{a}]$ has discriminant valuation four. If the conductor is at most two, the index of this order in the integral closure is divisible by $\pi$. A nonzero class in the $\pi$-torsion of the index quotient gives an integral element $(u + v\sqrt{a})/\pi$ with $v$ a unit. Its norm shows that $a \equiv (u/v)^2 \pmod{\pi^2}$. The assertion also holds for the trivial quadratic character. □

**Corollary 11.7.** *Let $K = \mathbf{Q}(s)$ with $s^2 = 11$, and put $\epsilon = 10 + 3s$, $a = 4 + s$ and $b = 4 - s$. Let $V$ be the group of squareclasses supported at the dyadic prime and the two primes above five, whose quadratic characters have dyadic conductor at most two. Then $[-1], [\epsilon a], [\epsilon b]$ form a basis of $V$.*

*Proof.* The field $K$ has class number one. A class with odd dyadic valuation has quadratic conductor five. Every remaining supported class has a unique expression $[-1]^i[\epsilon]^j[a]^k[b]^l$. The images of $\epsilon, a, b$ in $(\mathcal{O}_K/2)^\times$ are the same nonidentity element, so Lemma 11.6 imposes the single condition $j + k + l = 0$. Its solutions have the displayed basis; independence follows from the two odd-prime valuations and the independent unit classes. □

**Lemma 11.8.** *Let $F/\mathbf{Q}_2$ be finite, and let $V$ be a direct sum of crystalline rank-two representations, with finite coefficient fields, labelled weights $\{0, 1\}$ and determinant $\chi_2$. Every stable $\mathbf{Z}_2$-lattice in $V$ is Barsotti–Tate. If all the constituents are nonordinary, its Barsotti–Tate group and its finite-flat subquotients are biconnected. This applies, in particular, to the left regular lattice of $E_0$.*

*Proof.* The crystalline lattice classification realizes each constituent by a Barsotti–Tate group. Apply Lemma 11.2 with coefficient ring $\mathbf{Z}_2$ to their direct sum: closure of the finite isogeny kernel realizes any stable lattice in the same rational representation. For each nonordinary

constituent, Lemma 11.3 excludes an étale quotient and, by self-duality, a multiplicative subgroup. The coefficient actions preserve these decompositions. Products, isogenies, and the finite flat subquotients in that lemma therefore retain biconnectedness.

After extending coefficients, the rational regular representation of $E_0$ is a direct sum of columns of the original rank-two constituents. Applying the preceding argument to its stable left regular lattice proves the final assertion. □

Here the trace and determinant are those of the original integral order. A subsequent nonfaithful matrix quotient need not preserve the information they carry. In particular a vanishing matrix in such a quotient does not imply vanishing of its original trace.

**Lemma 11.9.** *Suppose every dyadic coefficient constituent is crystalline of labelled weights* $\{0,1\}$ *and determinant* $\chi_2$*, is nonordinary, and has scalar residual semisimplification. Suppose the dyadic completion* $F/\mathbf{Q}_2$ *is ramified quadratic. Let* $E_\pi(u)$ *be the Eisenstein polynomial of a uniformizer* $\pi$*. The regular lattice* $E_0$ *is Barsotti–Tate. Its Breuil–Kisin module is free of rank one as a right* $E_0[[u]]$*-module. If its Frobenius generator is* $\Phi(u)$*, then*

$$\mathcal{D}(\Phi(u)) \in (E_\pi(u))R_0[[u]], \qquad T(\Phi(0)) = a_2 \in R_0.$$

*The point field of* $E_0/2E_0$ *has upper breaks at most three over* $F$*.*

*Proof.* The regular lattice is Barsotti–Tate and biconnected by Lemma 11.8. Lemma 11.5 gives exactness and full faithfulness for its torsion subquotients with trivial simple generic factors. We use the covariant Breuil–Kisin functor $\mathfrak{M}_{\mathrm{cov}}(G) = \mathfrak{M}_{\mathrm{contra}}(G^D)$. Cartier duality followed by Kim's anti-equivalence is covariant and exact [51, Corollary 4.3 and Proposition 5.7.3], so it preserves the right action and the quotient used below.

The commuting right action of $E_0$ extends by Tate full faithfulness. Write $J_0 = \mathrm{rad}(E_0)$ and let $\mathfrak{M}$ be the Breuil–Kisin module of the regular group. In the scheme killed by two take the sum of the right multiplication maps for a set of generators of $J_0/2E_0$. Its finite-flat quotient has generic fibre $E_0/J_0 = \mathbf{F}_2$ and is the unique middle model. Exactness of the covariant functor gives $\mathfrak{M}/\mathfrak{M}J_0 = \mathbf{F}_2[[u]]$ with Frobenius $u$. A generator lifts by Nakayama to a surjection $E_0[[u]] \to \mathfrak{M}$ of right modules. These have the same rank over $\mathbf{Z}_2[[u]]$, so this is an isomorphism.

Height one supplies $\Psi$ with $\Phi\Psi = \Psi\Phi = E_\pi(u)$. In each integral matrix realization these two identities give a rank-two height-one matrix. Its rational étale Frobenius module is the corresponding column constituent of the regular representation, by its commuting right matrix action after extending scalars. The connected finite-flat comparison thus identifies its rank-two generic realization with the given crystalline member. Kisin's determinant comparison [52, Lemmas 2.3.2 and 2.3.4] gives divisibility of its determinant by $E_\pi(u)$. Division by this distinguished monic polynomial commutes with each actual coefficient embedding, hence its remainder already vanishes in $R_0[[u]]$. At $u = 0$ the covariant comparison gives crystalline Frobenius on $V(-1) = V^\vee$ for the rank-two member $V$. Its determinant is 2 and its trace is the arithmetic coefficient $a_2$, since $V^\vee \simeq V(-1)$ has the arithmetic Frobenius normalization. This fixes the convention in both identities. Finally Fontaine's finite-flat ramification bound for the group killed by two gives classical upper break at most three over $F$. □

**Lemma 11.9a** (Trace and determinant calculations)**.** *Let* $R$ *be a finite local* $\mathbf{F}_2$*-algebra with residue field* $\mathbf{F}_2$*, and let* $E$ *be a finite* $R$*-algebra, local with residue field* $\mathbf{F}_2$*. Write* $J = \mathrm{rad}\, E$*. Suppose* $E$ *is generated by* $A, B, C \in J$ *and carries the trace* $T$ *and determinant law* $D$ *of an actual order as above. Assume* $R$ *is generated by its group traces and*

$$T(A) = T(B) = T(C) = D(A) = D(B) = D(C) = 0,$$
$$A^2 = B^2 = C^2 = BC = CB = 0, \qquad AB + BA = X1, \quad AC + CA = Y1,$$
$$X = T(AB), \qquad Y = T(AC).$$

*Then $E$ is spanned over $R$ by $1, A, B, C, AB, AC$ and $\mathfrak{m}_R = (X, Y)$. In $R' = R/\mathfrak{m}_R^2$, with the base-changed trace and determinant laws, one has $T(J^3) = D(J^2) = 0$. For*

$$N = aA + bB + cC + Z, \qquad M = m_A A + m_B B + m_C C + Z', \qquad Z, Z' \in J^2,$$

*where $a, b, c, m_A, m_B, m_C \in \mathbf{F}_2$, one has*

$$\begin{aligned} D(N) &= abX + acY, \\ T(NM) &= (am_B + bm_A)X + (am_C + cm_A)Y. \end{aligned}$$

*If $\Phi = N + u(1 + M) + O(u^2)$ and $D(\Phi)$ is divisible by $u^2$, then $D(N) = 0$ and $T(N) = T(NM)$ in $R'$. All these are identities of the original scalar-valued laws; no faithfulness of $R \to E$ or of a later matrix quotient is required.*

*Proof.* Move $A$ to the left using $BA = X + AB$ and $CA = Y + AC$; $A^2 = B^2 = C^2 = BC = CB = 0$ gives the six stated normal words. Their traces are $0, 0, 0, 0, X, Y$. Since the group traces generate $R$, its quotient by $(X, Y)$ is $\mathbf{F}_2$, proving $\mathfrak{m}_R = (X, Y)$. Hence $T(E) \subset \mathfrak{m}_R$ and $J^2 \subset \mathfrak{m}_R E + RAB + RAC$.

Work now over $R'$. A term in $\mathfrak{m}_R E$ has zero determinant: quadratic homogeneity and polarization put every term in $\mathfrak{m}_R^2$. Its polar pairing with any element is also zero, since the traces of all products lie in $\mathfrak{m}_R$. Multiplicativity gives $D(AB) = D(AC) = 0$, and their mutual polar pairing is $XY - T(ABAC) = 0$, since $ABA = XA$. Their trace pairings with $A, B, C$ vanish: for example, $T(BAB) = XT(B) = 0$ and $T(CAB) = T(ABC) = 0$. Thus terms $Z, Z' \in J^2$ contribute nothing to the displayed norm and trace formulas. Polarizing the linear combinations of $A, B, C$ gives exactly those formulas.

Every word of length three reduces to a multiple of a trace-zero generator, for example $ABA = XA$, $ACA = YA$, and $BAC = XC = YB$. The traces of longer words contain either such a trace or a product of two elements of $(X, Y)$, and therefore vanish in $R'$. Terms from $\mathfrak{m}_R E$ have the same vanishing trace pairing. This proves $T(J^3) = 0$ and the preceding determinant calculation gives $D(J^2) = 0$.

Finally, the constant coefficient of $D(\Phi)$ is $D(N)$. Its linear coefficient is $T(N)T(1 + M) - T(N(1+M))$. The product of traces vanishes in $R'$, and $T(N(1+M)) = T(N)+T(NM)$, giving the last assertion. All computations use polarization and multiplicativity in the base-changed determinant law, rather than reading scalars from a nonfaithful matrix image. □

For the applications below, a real difference $c - 1$ has determinant zero before reduction, since $c$ has eigenvalues $1, -1$. A scalar-normalized special difference has a zero bottom row in a basis of its special line, so it also has determinant zero in the original order. Their traces vanish modulo two. These integral identities verify the trace and determinant hypotheses of the lemma without assuming that the scalar map remains faithful after reduction.

**Lemma 11.10.** *Let $F/\mathbf{Q}_2$ be ramified quadratic and let $M/F$ be the point field of a finite flat group scheme killed by two. Every positive upper break is at most three, and the normalized relative different contribution is strictly less than two, with $v_2(2) = 1$. In particular this applies to the reduction of any stable coefficient lattice of a crystalline rank-two source of weights $\{0, 1\}$.*

*Proof.* Fontaine's bound [39, Theorem A], converted to classical upper numbering, is $2e(F) - 1 = 3$. Integration of the upper filtration bounds the relative different exponent divided by the relative ramification index strictly below four. Dividing by $e(F/\mathbf{Q}_2) = 2$ gives the normalized bound. For the last assertion, realize the stable crystalline lattice by a Barsotti–Tate group. Its coefficient-uniformizer torsion is a finite-flat subgroup of rational two-torsion. □

## 11.2 Rank-one selfextensions at the endpoint exponents

**Lemma 11.11.** *Let $F/\mathbf{Q}_p$ have absolute ramification index $e$ and residue degree $f$, and let $k$ be a finite coefficient field containing its residue field. Consider a rank-one Breuil–Kisin module with coefficient factors indexed by $i \in \mathbf{Z}/f\mathbf{Z}$, whose Frobenius maps are $b_i u^{r_i}$, where $b_i \in k^\times$ and $r_i \in \{0, e\}$. On these factors Frobenius is $k$-linear and sends $u$ to $u^p$. Every height-one selfextension, exact on the underlying free $k[[u]]$-modules, has Frobenius matrices*

$$A_i = b_i u^{r_i} \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}, \qquad a_i \in k,$$

*after changing the lifts of the quotient generators. The extension classes form a one-dimensional $k$-space. Whenever the exact finite-flat comparison identifies the given models with these modules, their generic selfextensions become unramified after dividing by the common character.*

*Proof.* Before changing lifts, write $A_i = \left(\begin{smallmatrix} b_i u^{r_i} & h_i \\ 0 & b_i u^{r_i} \end{smallmatrix}\right)$. The height-one condition makes $u^e A_i^{-1}$ integral. Its upper-right entry gives $u^e \mid h_i$ when $r_i = e$; when $r_i = 0$ no divisibility is needed. Thus in either case $h_i = b_i u^{r_i} f_i(u)$ with $f_i \in k[[u]]$. Changing the quotient lift by $\mu_i$ changes $f_i$ by $\varphi(\mu_{i-1}) - \mu_i$.

On $uk[[u]]$ this cyclic coboundary map is bijective: after one cycle its solution is obtained by inverting $1 - \varphi^f$, using the convergent series $\sum_{n \geq 0} \varphi^{fn}$. Hence all positive powers of $u$ can be removed. On constants the coboundaries are $(\mu_{i-1} - \mu_i)_i$, whose cokernel is $k$, measured by $\sum_i a_i$. This proves the dimension assertion. Unramified selfextensions of the common character form a one-dimensional space and have finite-flat models: tensor the rank-one model with an unramified two-dimensional unipotent representation and descend from a finite unramified extension. The exact comparison identifies this nonzero line with the whole extension space, proving the generic assertion. □

The restriction $r_i \in \{0, e\}$ is used in the divisibility of $h_i$. The middle-exponent calculations below are separate.

## 11.3 Ramification of dyadic torsion

**Lemma 11.12.** *Let $F/\mathbf{Q}_2$ be quadratic, let $q = 2^f \geq 4$, and let $\bar{r} : G_F \to \mathrm{SL}_2(\mathbf{F}_q)$ be finite flat. If $E/F$ is its kernel field, its normalized relative dyadic discriminant contribution, with $v_2(2) = 1$, satisfies*

$$d_2(E/F) := \frac{v_2(\mathfrak{d}_{E/F})}{[E : F]} \leq 2 - \frac{2}{q}.$$

*If $F/\mathbf{Q}_2$ is unramified and both wild inertia and the tame diagonal characters are nontrivial, then $d_2(E/F) < 4/3$.*

*Proof.* We use Fontaine's ramification theorem [39, Theorem A and its corollary, pp. 515–516]. His numbering is $G^{(u)} = G^{u-1}$ in Serre's upper numbering. Thus the upper breaks are at most $2e(F/\mathbf{Q}_2) - 1$.

Suppose first that $F/\mathbf{Q}_2$ is ramified. If wild inertia is nontrivial, its unique fixed line is defined over $\mathbf{F}_q$ and is preserved by the decomposition group. The diagonal characters are unramified, since their odd-order restrictions to units factor through $\mathbf{F}_2^\times$. Inertia is therefore a translation group of order $P \leq q$. Integrating the upper-break bound three and dividing by the absolute ramification index two gives $d_2 \leq 2(1 - 1/P) \leq 2 - 2/q$. If wild inertia is trivial, a nontrivial locally irreducible tame inertia has order three, since Frobenius squares and exchanges the determinant-one eigencharacters. Its contribution is $1/3$; in the reducible tame case inertia is trivial.

Now suppose that $F/\mathbf{Q}_2$ is unramified. If wild inertia is trivial, the reducible tame characters have order dividing three. In the irreducible case they are $\theta, \theta^4$, and determinant one gives $\theta^5 = 1$. The respective contributions are at most $2/3$ and $4/5$. If wild inertia is nontrivial, write the diagonal characters as $\eta, \eta^{-1}$. Killing them gives a field $T$ over which the image is an elementary quadratic translation group of order $P \le q$. If $\eta$ is unramified, Fontaine's upper-break bound one makes every quadratic character of $E/T$ have conductor at most two. The conductor–discriminant formula gives $d_2 \le 2(1 - 1/P) \le 2 - 2/q$.

Otherwise the tame diagonal order is three and $e(T/F) = 3$. Tame conjugation acts nontrivially on the translation module and its Kummer dual. The nontrivial tame grades of the quadratic unit filtration over $T$ are one and five, giving conductors six and two; the grade-three and unramified directions are tame-invariant and cannot occur [18, §1]. A conductor-six character would have upper break $5/3$ over $F$, contrary to Fontaine's bound. Hence $d_2(E/F) \le \frac{2}{3} + \frac{2}{3}(1 - 1/P) < \frac{4}{3}$. Unramified base changes do not affect any of these normalized bounds. □

We next prove a stronger local statement for a nonordinary source.

**Theorem 11.13.** *Let $F/\mathbf{Q}_2$ be ramified quadratic, and let $r_F : G_F \to \mathrm{GL}_2(E)$ be nonordinary, crystalline of labelled weights $\{0, 1\}$ and determinant $\chi_2$. For any stable lattice, the point field of its reduction has positive upper breaks at most one and normalized relative different contribution less than one, with $v_2(2) = 1$. If the reduction is locally absolutely irreducible, its full image is $S_3$ and its wild inertia is trivial.*

*Proof.* As in Lemma 11.8, the lattice is realized by a biconnected Barsotti–Tate group $G$. The same holds after isogeny, and $G[\varpi_E]$ and all its finite-flat subquotients are biconnected.

After enlarging the residue coefficient field, suppose first that the reduction is reducible. Its diagonal characters are unramified, since $\mathcal{O}_F^\times$ is pro-two. After an unramified extension it is $1 + c(g)E_{12}$, where $c$ has values in a finite field $k$ of characteristic two. For any $\mathbf{F}_2$-linear functional $\lambda : k \to \mathbf{F}_2$, pull the quotient line back to $\mathbf{F}_2$ and quotient the invariant line by $\ker \lambda$. This gives a biconnected self-extension of the simple model with generic character $\lambda \circ c$. Lemma 11.5 bounds its conductor by two. These characters separate the elementary abelian point-field group, so every positive upper break is at most one. If its ramification index is $e$, its different exponent is at most $2(e - 1)$; division by $2e$ gives the asserted normalized bound. Unramified base change preserves these ramification bounds.

If the reduction is locally absolutely irreducible, the local classification in Lemma 4.8(ii) gives full image $S_3$ and trivial wild inertia. Tame index three contributes $1/3$ to the normalized relative different. □

**Lemma 11.14.** *Let $F/\mathbf{Q}_2$ be ramified quadratic. Consider an upper triangular representation of $G_F$ over a finite local commutative $\mathbf{F}_2$-algebra, with unramified diagonal characters and point field of upper break at most three. Let $t$ be the order of the Frobenius multiplier on the upper-right coordinate and assume that this order is odd. Write $w$ for the order of inertia. If $t = 1$, the normalized relative different is at most $5/4$. For $t > 1$ put $s = 2^{\mathrm{ord}_t(2)}$. Then $w$ is $1, s$, or $s^2$, with respective normalized different bounds*

$$0, \qquad 2 - 2/s, \qquad 2 - 1/s - 1/s^2.$$

*In particular, for the residue of an ordinary crystalline weight-two representation of cyclotomic determinant, these bounds apply with multiplier $\bar{\nu}^{\pm 2}$, where $\bar{\nu}$ is the residual unramified quotient value at Frobenius.*

*Proof.* After unramified extension of degree $t$, divide by the common diagonal character. The resulting translations define an elementary abelian extension with the same inertia and

normalized different. The upper-break hypothesis excludes odd-valuation Kummer classes, which have conductor five. At absolute ramification index two, the ramified unit classes have conductors two and four. Indeed the square map from the first principal-unit grade fills the second, and the quadratic defect formula gives conductors four and two on the surviving first and third grades, respectively. Each is a subquotient of the additive residue field. Unramified classes do not contribute to the different.

For $t > 1$, conjugation by the original Frobenius acts on translations by multiplication by the multiplier. An odd-order unit in a finite local algebra has the same order as its residue and belongs to the canonical coefficient field. Thus the Kummer dual is isotypic for a nontrivial binary simple module of dimension $\mathrm{ord}_t(2)$. Frobenius acts on the two ramified grades as on the residue field of the unramified degree-$t$ extension. This is the regular binary representation of the cyclic group of order $t$, so the given simple module occurs at most once in each grade. This proves the assertions on $w$. For $w = s^2$ there are at most $s^2 - s$ characters of conductor four, and the remaining $s - 1$ nontrivial characters have conductor two. The conductor–discriminant formula, with normalization by $2w$, gives $2 - 1/s - 1/s^2$. For $w = s$ use the conductor-four bound on all $s - 1$ nontrivial characters.

When $t = 1$, each ramified grade has dimension at most one over $\mathbf{F}_2$. In the largest case the nontrivial conductors are $4, 4, 2$, giving $5/4$ by the same formula. Smaller groups satisfy this bound as well. The final assertion follows from the ordinary filtration and Lemma 11.10. □

**Lemma 11.15.** *Let $F/\mathbf{Q}_2$ be unramified with residue field $k = \mathbf{F}_q$. For $a \in \mathcal{O}_F^\times$ put*

$$\delta_F(a) = \frac{a^{q-1} - 1}{2} \pmod 2 \in k.$$

*This induces a surjective $\mathbf{F}_2$-linear map $\mathcal{O}_F^\times/\mathcal{O}_F^{\times 2} \to k$. Its kernel is the one-dimensional unramified quadratic squareclass space. If $\delta_F(a) = 0$, set $b = a^{q/2}$ and $c = (a/b^2 - 1)/4$. Then $a$ is a square in $F$ if and only if $\mathrm{Tr}_{k/\mathbf{F}_2}(\bar{c}) = 0$.*

*Proof.* Every unit satisfies $a^{q-1} \equiv 1 \pmod 2$. Multiplication modulo four gives $\delta_F(aa') = \delta_F(a) + \delta_F(a')$, and a square has zero image. For $a = 1 + 2x$ the image is $\bar{x}$, proving surjectivity. If $\delta_F(a) = 0$, then $b^2 = a^q \equiv a \pmod 4$, so $a/b^2 = 1 + 4c$. The polynomial $X^2 - X - c$ has unit derivative and discriminant $1 + 4c$. Its reduction is split precisely when $\mathrm{Tr}_{k/\mathbf{F}_2}(\bar{c}) = 0$, and otherwise gives the unramified quadratic extension. Conversely every unramified quadratic extension is obtained in this way. The unramified quadratic extension is unique, so the kernel consists of its squareclass and the trivial class. □

**Lemma 11.16** (The cubic norm kernel)**.** *Let $A/K$ be an $S_3$ extension, let $L$ be the fixed field of a transposition, and let $V$ be the standard two-dimensional $\mathbf{F}_2$-representation of $\mathrm{Gal}(A/K)$. Let $S$ contain the dyadic places, the infinite places, and all places ramified in $A/K$. Write $L(S_L, 2)$ for the squareclasses with even valuation outside $S_L$. There is a natural isomorphism, compatible with localization,*

$$H^1(G_{K,S}, V) = \ker\bigl(L(S_L, 2) \xrightarrow{N_{L/K}} K(S, 2)\bigr).$$

*The norm is surjective. Its kernel is the image of the projector $[u] \mapsto [u/N_{L/K}(u)]$, where the norm is viewed in $L$. If $\mathrm{Cl}(\mathcal{O}_{L,S_L})[2] = 0$, the entire calculation uses $S_L$-units modulo squares. The same statements hold after scalar extension to any field of characteristic two.*

*Proof.* The permutation module on the three cosets is $\mathbf{F}_2^3 = \mathbf{F}_2 \oplus V$. The first summand is the constant line and $V$ is the kernel of the sum of coordinates. The sum restricted to the constant line is the identity because $3 = 1$ in $\mathbf{F}_2$. Shapiro's lemma identifies its first cohomology with

that of $G_{L,S_L}$, and the sum map with corestriction. Kummer theory identifies corestriction with the norm of squareclasses. Restriction followed by norm is multiplication by three, hence is the identity. This proves both the splitting and the projector formula. The Kummer exact sequence

$$0 \longrightarrow \mathcal{O}^{\times}_{L,S_L}/\mathcal{O}^{\times 2}_{L,S_L} \longrightarrow L(S_L,2) \longrightarrow \mathrm{Cl}(\mathcal{O}_{L,S_L})[2] \longrightarrow 0$$

proves the last arithmetic assertion. Shapiro's decomposition commutes with restriction to decomposition groups, giving localization. □

**Lemma 11.16a** (Solvable dyadic residues and ray characters)**.** *Let $K$ be a number field, let $S$ be a finite set of odd places, and let $\bar{r} : G_K \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ be continuous, unramified outside $S$ and the dyadic places. Suppose its inertia at each place in $S$ is a 2-group. If $\bar{r}$ is reducible, its diagonal characters factor through the odd quotient of the ray group of $K$ modulo the product of its dyadic primes. If it is irreducible and solvable, then $\bar{r} = \mathrm{Ind}_{G_L}^{G_K}\theta$ for a quadratic extension $L/K$ and a nontrivial odd-order character $\theta$ factoring through the corresponding odd dyadic ray quotient of $L$. The field $L$ is contained in the residual point field, so every proved upper-ramification bound for that field applies to $L/K$. Thickening the dyadic modulus or adding real places does not change these odd ray quotients.*

*Proof.* Diagonal characters in characteristic two have odd order. They kill the prescribed inertia at $S$ and wild dyadic inertia, and are trivial on real conjugations. Global class field theory gives the first assertion.

In the irreducible solvable case Dickson's classification gives an odd dihedral image. There is no scalar kernel, since a scalar matrix of determinant one in characteristic two is the identity. Its rotation subgroup defines $L$, and its two eigencharacters are $\theta, \theta^{-1}$. At a place in $S$, inertia has trivial intersection with the odd rotation subgroup, whether the quadratic quotient is ramified or unramified. Thus $\theta$ is unramified there; it also kills wild dyadic inertia. Its conductor divides the radical dyadic modulus of $L$. Ramification bounds pass to $L/K$ by compatibility of upper numbering with quotients. Finally, additional dyadic unit layers and real sign groups are 2-groups, so they do not alter the odd ray quotient. □

# 12 Symmetry and quadratic descent

The integral-order bounds can force either a small coefficient field or an isomorphism with a quadratic conjugate. We record the two rigidity mechanisms used below. The prime-to-characteristic cotangent argument is distinct from the dyadic order argument.

## 12.1 A marked ordinary lattice

**Lemma 12.1.** *Let $K$ be real quadratic with one dyadic place $w$, and let $S$ be a finite set of odd places. Let $V$ be the squareclasses supported at $S \cup \{w\}$ with even valuation at $w$. Suppose $\ker(V \to K_w^{\times}/K_w^{\times 2}) = \langle[\delta]\rangle$, where $\delta$ is totally negative. Let $E/\mathbf{Q}_2$ be finite and $r : G_K \to \mathrm{GL}_2(E)$ absolutely irreducible, of cyclotomic determinant, unramified outside $S \cup \{w\}$, crystalline ordinary at $w$, and with scalar residual semisimplification. For any real conjugation $c$, there is a stable lattice with a basis in which*

$$\bar{r} = 1 + \bar{\chi}_{\delta} E_{12}, \qquad r(c) = C := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{ordinary line } e_2,$$

*where $\bar{\chi}_{\delta} : G_K \to \mathbf{F}_2$ is the additive character of $K(\sqrt{\delta})/K$.*

*Proof.* The scalar semisimplification is trivial because the residual determinant is one. Apply Lemma 4.10 with $D = G_{K_w}$ and $v$ on the ordinary line. It gives an orbit lattice $\Lambda$ with $v$ primitive and a nonzero additive residue trivial on $G_{K_w}$. Every binary coordinate is a quadratic character split at $w$, hence of even dyadic valuation, and belongs to the displayed localization kernel. The extension is a nonzero multiple of $\bar{\chi}_\delta$, and $\bar{\chi}_\delta(c) = 1$. Consequently $((r(c) - 1)v, v)$ is an integral basis. The relation $r(c)^2 = 1$ gives the displayed real matrix and normalizes the multiple to one. □

**Lemma 12.1a** (First differences with fixed determinant)**.** *Let $k$ have characteristic two, let $h : G \to \mathbf{F}_2$ be additive, and put $\bar{r} = 1 + hE_{12}$. A fixed-determinant tangent, or the first difference of two integral representations with this residue and equal determinants, has the form*

$$r_\varepsilon = (1 + \varepsilon X)\bar{r}, \quad \text{or} \quad r_2 = (1 + \varpi^n X)r_1 \pmod{\varpi^{n+1}}, \qquad X = \begin{pmatrix} x & y \\ z & x \end{pmatrix}.$$

*Here $\varepsilon^2 = 0$ and $n \geq 1$. In either case $z$ and $x + y$ are additive, and $dx = dy = h \smile z$. Fixing the matrix at an element $c$ gives $X(c) = 0$. The following local consequences use the indicated markings.*

*(i) On a subgroup where $h = 0$, a fixed invariant line $e_2$ gives $y = 0$. An unramified quotient gives unramified $x$; an unchanged exact quotient gives $x = 0$. For two actual ordinary crystalline sources of weights $\{0, 1\}$ with identical local characters, the divided extension coordinate $z$ is a unit Kummer class. The same conclusion holds for a tangent whose marked finite flat extension has multiplicative subobject and étale quotient.*

*(ii) On a subgroup $H$ with $h|_H \neq 0$, suppose the marked special lines reduce to $e_1$. If their exact eigencharacters agree, variation of the line by $t_H e_2$ gives*

$$z|_H = 0, \qquad x|_H = t_H h|_H.$$

*For an integral first difference whose eigencharacters instead differ by an unramified quadratic sign $(-1)^{U_H}$, put $e = v_\varpi(2)$. These equations hold for $n < e$; at $n = e$ the second becomes $x|_H = t_H h|_H + \overline{2/\varpi^e}\, U_H$.*

*Proof.* The determinant condition gives $\operatorname{tr} X = 0$. Since $h(g)^2 = h(g)$, direct multiplication gives

$$\mathrm{Ad}(\bar{r}(g))X(g') = X(g') + h(g)z(g')(I + E_{12}).$$

The cocycle identity yields the asserted coboundaries. On the trivial residual subgroup, preservation of $e_2$ and comparison of the quotient characters give (i)'s equations.

For its crystalline assertion, subtract the two integral ordinary extension cocycles and divide by $\varpi^n$. The result is integral, and its rational class remains in the linear crystalline extension space for the same two characters. Its stable lattice is therefore ordinary Barsotti–Tate by Lemma 11.2. Its reduction is an extension of the étale model by the multiplicative model. The finite flat Kummer classification identifies its binary coordinates with unit classes. For a tangent with the stated finite flat realization, apply the same classification to that realization. This last assertion imposes no finite flat condition on a merely semistable ordinary source.

For (ii), choose $g_0 \in H$ with $h(g_0) = 1$. The unit upper-right entry of $r_i(g_0) - \lambda_i(g_0)$ recovers the special line in the chart with first coordinate one. Congruence of the matrices and eigencharacters modulo $\varpi^n$ therefore gives congruence of the lines. Subtract their eigenvector equations. If $d_H$ is the divided change of eigencharacter, the first-order equation is

$$X(g)e_1 + t_H \bar{r}(g)e_2 = t_H e_2 + d_H(g)e_1.$$

Thus $z(g) = 0$ and $x(g) = t_H h(g) + d_H(g)$. For equal characters $d_H = 0$. For the unramified sign change, its first possible nonzero divided value is $\overline{2/\varpi^e} U_H$ at $n = e$. □

## 12.2 Finite-order actions on complete local rings

**Lemma 12.2** (Cotangent rigidity)**.** *Let $\mathcal{O}$ be a complete noetherian local ring, and let $A$ be a complete noetherian local $\mathcal{O}$-algebra with the same residue field. Let a finite group $\Gamma$, with $|\Gamma| \in \mathcal{O}^\times$, act continuously by local $\mathcal{O}$-algebra automorphisms on $A$. If its action on $\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}}A)$ is trivial, its action on $A$ is trivial.*

*Proof.* Average lifts $x_1, \dots, x_n$ of a relative cotangent basis over $\Gamma$. The resulting invariant elements have the same cotangent classes. Nakayama and completeness give a surjection from $\mathcal{O}[[X_1, \dots, X_n]]$ to $A$ sending its variables to these elements. The entire image is fixed. □

The invertibility hypothesis is essential: $T \mapsto -T$ is a nontrivial involution of $\mathbf{Z}_2[[T]]$ acting trivially on its relative cotangent space. In applications to pseudodeformations, the cotangent calculation must include products of opposite extensions, not only tangent vectors of representations.

**Lemma 12.3.** *Let $k$ be a field of characteristic different from two, and let $V = ks \oplus kd$ carry the involution $\sigma(s) = s$, $\sigma(d) = -d$. On $V^\vee \otimes_k V^\vee$ put*

$$\iota(a \otimes b) = \sigma^\vee(b) \otimes \sigma^\vee(a), \qquad \ell_\pm = s^\vee \pm d^\vee.$$

*If an $\iota$-equivariant quotient kills both $\ell_+ \otimes \ell_+$ and $\ell_- \otimes \ell_-$, then $\iota$ acts trivially on that quotient.*

*Proof.* The $(-1)$-eigenspace of $\iota$ is the line generated by $s^\vee \otimes d^\vee + d^\vee \otimes s^\vee$. It is killed because

$$\ell_+ \otimes \ell_+ - \ell_- \otimes \ell_- = 2(s^\vee \otimes d^\vee + d^\vee \otimes s^\vee).$$

Since two is invertible, the two eigenspaces split and the quotient has only the eigenvalue one. □

## 12.3 Descent from a twisted conjugacy

**Lemma 12.4** (Quadratic descent)**.** *Let $K/\mathbf{Q}$ be real quadratic, with nontrivial automorphism $\sigma$. Let $\mathcal{R} = (r_\lambda)$ be a totally odd, absolutely irreducible strongly compatible rank-two system over $K$, with cyclotomic determinant and regular labelled Hodge–Tate weights. Suppose that, for one coefficient place,*

$$r_\lambda^\sigma \simeq r_\lambda \otimes \eta,$$

*where $\eta$ is the restriction of a quadratic character of $G_{\mathbf{Q}}$ (the trivial character is allowed). Then $\mathcal{R}$ is modular.*

*Proof.* The identity of good Frobenius traces transfers the displayed isomorphism to a companion $r$ in characteristic $p \geq 5$, by compatibility and Chebotarev. Write $H = G_K$ and choose $t \in G_{\mathbf{Q}} \setminus H$. There is an intertwiner $A$ with

$$A r(h) A^{-1} = \eta(h) r(tht^{-1}) \qquad (h \in H).$$

Since $\eta^\sigma \eta = 1$, Schur's lemma gives $A^2 r(t^2)^{-1}$ scalar. Thus $[r]$ and $[A]$ define a continuous projective representation of $G_{\mathbf{Q}}$.

After a finite coefficient extension, scale $A$ so that $\det A = \chi_p(t)$. Define a continuous choice of matrices by $L(h) = r(h)$ and $L(ht) = r(h)A$ for $h \in H$. Then $\det L(g) = \chi_p(g)$ for every $g$, so its scalar cocycle

$$c(g_1, g_2) = L(g_1)L(g_2)L(g_1 g_2)^{-1}$$

takes values in $\{\pm 1\}$. It is locally constant and is trivial on $H \times H$. Tate's theorem in [66, Theorem 6.3], for the *discrete trivial* $G_{\mathbf{Q}}$-module $\overline{\mathbf{Q}}^{\times}$, gives a continuous cochain $b : G_{\mathbf{Q}} \to \overline{\mathbf{Q}}^{\times}$ such that $c = \delta b$. Compactness makes the image of $b$ finite. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ and enlarge the coefficient field by these finitely many algebraic values. Then $\widetilde{r} = b^{-1}L$ is a continuous representation of $G_{\mathbf{Q}}$. Because $c|_{H\times H} = 1$, the restriction $b|_H$ is a finite-order character. In fact $c^2 = 1$ implies that $b^2$ is a finite-order character of $G_{\mathbf{Q}}$.

The representation $\widetilde{r}$ is absolutely irreducible, since its restriction to $H$ is a finite twist of $r$. The locally constant function $b$ factors through a finite Galois quotient. Outside the ramification of that quotient, $K/\mathbf{Q}$ and $r$, inertia lies in $H$ and acts trivially, so $\widetilde{r}$ is finitely ramified. Finite twists preserve de Rham representations and their weights; de Rhamness descends through finite local extensions. Thus $\widetilde{r}$ is regular de Rham at $p$. Every complex conjugation belongs to $H$, and the square of its value under $b|_H$ is one, so $\widetilde{r}$ is odd. Apply [61, Theorem 1.0.4]. Cyclic base change to $K$ is cuspidal because its Galois restriction is irreducible; untwisting by $b|_H$ proves modularity of $r$, and hence of the compatible system. □

## 12.4 Projective descent from coefficient inner twists

**Lemma 12.5** (Finite-field projective descent)**.** *Let $k/\mathbf{F}_Q$ be a finite extension of odd characteristic, of degree $n$, and write $\phi : x \mapsto x^Q$. Let $\bar{r} : G \to \mathrm{GL}_2(k)$ be absolutely irreducible. Suppose a character $\eta : G \to \{\pm 1\}$ satisfies $\eta^n = 1$ and*

$$\bar{r}^{\phi} \simeq \bar{r} \otimes \eta.$$

*Then its projectivization is conjugate to a representation with values in $\mathrm{PGL}_2(\mathbf{F}_Q)$. If $G = G_F$ for a totally real field and $\bar{r}$ is totally odd, the quadratic determinant character of the descended projective representation takes the value*

$$(-1)^{(Q-1)/2}\eta(c_v)$$

*at a real conjugation $c_v$.*

*Proof.* Choose $A \in \mathrm{GL}_2(k)$ such that $\bar{r}^{\phi} = A\bar{r}A^{-1}\eta$. Since $\eta^n = 1$, absolute irreducibility gives $A^{\phi^{n-1}} \cdots A = cI$. Cyclically permuting the factors gives $c^{\phi} = c$. The norm $k^{\times} \to \mathbf{F}_Q^{\times}$ is surjective, so rescale $A$ to make that product the identity. Matrix Hilbert 90 gives $A = B^{\phi}B^{-1}$. Then $\bar{r}' = B^{-1}\bar{r}B$ satisfies $(\bar{r}')^{\phi} = \eta\bar{r}'$. For each $g$, choose a scalar $u \in k^{\times}$ with $u^{\phi}/u = \eta(g)$. This makes $u\bar{r}'(g)$ a matrix over $\mathbf{F}_Q$, giving projective descent.

For a real conjugation, take $u = 1$ if $\eta(c_v) = 1$. If $\eta(c_v) = -1$, then $u^2 \in \mathbf{F}_Q^{\times}$ is a nonsquare: a square root in $\mathbf{F}_Q$ would force $u \in \mathbf{F}_Q$. The determinant of the chosen matrix is $-u^2$, proving the formula. □

## 12.5 Coefficient orbits and the integral-order bound

*Proof of Theorem 1.5.* The two operations commute and are involutions on isomorphism classes. Let $V = C_2^2$ be the resulting abstract group, let $m \in \{1, 2, 4\}$ be the number of coefficient orbits, and put $e = [D : \mathbf{Q}]$. Each operation preserves the trace-field degree, since it permutes the good primes and changes their traces only by rational signs. Section 11 gives $4me = \dim_{\mathbf{F}_2}(E_0/2E_0) < 32$.

The stabilizer $V_0$ of the coefficient orbit of $\mathcal{R}$ has order $4/m$. Its elements induce unique automorphisms of $D$, because the good traces generate $D$. These automorphisms form a homomorphism $V_0 \to \mathrm{Aut}(D)$ whose kernel consists of actual isomorphisms of systems. The element $H$ is not in the kernel. Any nontrivial kernel therefore contains $G$ or $GH$ and gives one of the asserted twisted-conjugacy identities.

Otherwise $V_0$ embeds in $\mathrm{Aut}(D)$. The fixed-field theorem gives $4/m \mid e$, hence $4 \mid me$. Since $0 < me < 8$, we have $me = 4$. Thus $(m, e) = (4, 1), (2, 2), (1, 4)$. In the last case $D/\mathbf{Q}$ has a Klein four group of automorphisms and degree four, so is biquadratic. A residue-field Galois group is then a cyclic quotient of a subgroup of $C_2^2$, and has order at most two. The other cases have degree at most two. The descent assertion is Lemma 12.4. □

The bounds concern the actual simultaneous order, not one residual realization. The small-residue-degree alternative must still be combined with residual modularity and a lifting theorem for the actual source.

## 13 A common discriminant comparison

For a number field $L$ of degree $n$, let $N_Q(L)$ count its prime ideals of norm $Q$ and put $\rho_Q(L) = N_Q(L)/n$. We omit $L$ when it is fixed. A lower bound for $\rho_Q$ may replace its exact value in any nonnegative Euler term below. The Euler coefficient is always $2\rho_Q \log Q$.

For $b > 0$ put $c_b = \pi/(2b)$ and define the even function

$$F_b(x) = \begin{cases} (1 - |x|/(2b)) \cos(c_b|x|) + \pi^{-1} \sin(c_b|x|), & |x| \le 2b, \\ 0, & |x| > 2b. \end{cases}$$

This is the normalized autocorrelation of $\cos(c_b t)$ on $[-b, b]$, extended by zero. Thus $F_b(0) = 1$, $0 \le F_b \le 1$, and its Fourier transform is nonnegative. Its Laplace transform is

$$J_b(s) = \int_0^{2b} F_b(x) e^{-sx}\, dx = \frac{s}{s^2 + c_b^2} + \frac{c_b^2(1 + e^{-2bs})}{b(s^2 + c_b^2)^2}. \tag{8}$$

Put

$$B_b = \int_0^\infty \frac{1 - F_b(x)}{2\sinh(x/2)}\, dx, \qquad C_b = 2\{J_b(-1/2) + J_b(1/2)\}.$$

Expansion of $(2\sinh(x/2))^{-1}$ gives $B_b = \sum_{j \ge 0}((j + 1/2)^{-1} - J_b(j + 1/2))$. Every summand lies between zero and $c_b^2(j + 1/2)^{-3}$. For $N \ge 1$ this gives

$$B_b \le B_{b,N} := \sum_{j=0}^{N-1} \Big((j + 1/2)^{-1} - J_b(j + 1/2)\Big) + \frac{c_b^2}{2(N - 1/2)^2}. \tag{9}$$

The midpoint inequality for $x^{-3}$ also gives the convenient one-term bound $B_b \le 2 - J_b(1/2) + c_b^2/2$. The alternating four-term sum

$$A_{b,4} = J_b(1/2) - J_b(3/2) + J_b(5/2) - J_b(7/2)$$

is a lower bound for $\int_0^\infty F_b(x)/(2\cosh(x/2))\, dx$.

*Definition* 13.1. For a number field $M$ and $1/2 \le \sigma < 1$, write $\mathrm{ZF}_M(\sigma)$ for the assertion that $\zeta_M(s)$ is nonzero in $\Re s > \sigma$, away from its pole. Write $\mathrm{ZF}(\sigma)$ when this is assumed for every number field. A statement restricted to comparison fields requires only their individual hypotheses. The functional equation places all nontrivial zeros in $|\Re s - 1/2| \le \sigma - 1/2$. GRH implies every such assertion. Appendix B derives two zero-free regions using (CT1)–(CT5), which are recorded there as verified Lean results: $\sigma = 2759/3000$ at scale $M_0 = 201/200$ (Theorem B.1) and $\sigma = 0.9586$ at scale $419/400$ (Theorem B.11). No additional analytic hypothesis is imposed in either theorem.

For $0 \le a \le 1/2$ set $G_{a,b}(x) = F_b(x)/\cosh(ax)$.

**Proposition 13.2.** *Suppose all nontrivial zeros of $\zeta_L$ lie in $|\Re s - 1/2| \le a$. Let $n = [L : \mathbf{Q}]$, let $r_1$ be the number of real embeddings, and let $N_Q$ count the prime ideals of norm $Q$. For $F = G_{a,b}$ put*

$$B(F) = \int_0^\infty \frac{1 - F(x)}{2\sinh(x/2)}\,dx, \quad C(F) = 4\int_0^\infty F(x)\cosh(x/2)\,dx,$$

$$A(F) = \int_0^\infty \frac{F(x)}{2\cosh(x/2)}\,dx.$$

*Then*

$$\begin{aligned}\log \mathrm{rd}(L) \ge \gamma + \log(8\pi) - B(F) - \frac{C(F)}{n} + \frac{r_1}{n}A(F) \\ + \sum_Q 2\rho_Q \log Q \sum_{m\ge 1} Q^{-m/2} F(m\log Q).\end{aligned} \tag{10}$$

*Any prime terms, and the real-place term, may be discarded. At fixed $a, b$, signature ratio and retained splitting densities, the lower bound increases with $n$.*

*Proof.* For $a > 0$ and $|\theta| < a$, with $\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx}\,dx$, we have

$$\widehat{\frac{\cosh(\theta x)}{\cosh(ax)}}(t) = \frac{\pi}{a}\,\frac{\cos(\pi\theta/(2a))\cosh(\pi t/(2a))}{\sinh^2(\pi t/(2a)) + \cos^2(\pi\theta/(2a))} \ge 0.$$

At $|\theta| = a$ the quotient is one; when $a = 0$ this is the only case. Multiplication by the positive definite $F_b$ preserves positive definiteness. If $\rho = 1/2 + \theta + i\gamma_0$, the real part of the explicit-formula transform is the Fourier transform of $F_b(x)\cosh(\theta x)/\cosh(ax)$ at $\gamma_0$, hence is nonnegative. The regularity of $F$ justifies the explicit formula and convergence of its zero sum. Apply Poitou's formula [64, equation (7) and Proposition 4], [57, §2] and discard the nonnegative zero terms. The prime and real-place terms are nonnegative, and $C(F) > 0$. □

**Proposition 13.3.** *Assume GRH for the Dedekind zeta function of $L$. Let $n = [L : \mathbf{Q}]$, let $r_1$ be its number of real embeddings, and let $N_Q$ count its prime ideals of norm $Q$. If $B_b^*$ is either upper bound for $B_b$ just given, then*

$$\log \mathrm{rd}(L) \ge \gamma + \log(8\pi) - B_b^* - \frac{C_b}{n} + \frac{r_1}{n}A_{b,4} + \sum_Q 2\rho_Q \log Q \sum_{m\ge 1} Q^{-m/2} F_b(m\log Q). \tag{11}$$

*Any subset of the prime terms may be retained. At fixed $b$, signature ratio and retained splitting densities, this lower bound increases with $n$.*

*Proof.* Apply Proposition 13.2 with $a = 0$. Replace $B_b$ by its upper bound and the real-place integral by its displayed lower bound. All omitted prime terms are nonnegative, and $C_b > 0$. The inner sums are finite because $F_b$ has compact support. □

For comparisons without GRH, use $G_b(x) = F_b(x)/\cosh(x/2)$, and write $B(G_b), C(G_b)$ for the preceding integrals with $F_b$ replaced by $G_b$. Proposition 13.2 applies with $a = 1/2$ without any hypothesis on zeros. The bound $1 - F_b(x) \le \pi^2 x^2/(8b^2)$ gives

$$B(G_b) \le \log 2 + \frac{7\pi^2\zeta(3)}{16b^2}, \qquad C(G_b) = \frac{32b}{\pi^2}, \qquad \int_0^\infty \frac{F_b(x)}{2\cosh^2(x/2)}\,dx \ge 1 - \frac{\pi^4}{24b^2}.$$

Consequently, without any hypothesis on the zeros,

$$\log \mathrm{rd}(L) \geq \log(4\pi) + \gamma - \frac{7\pi^2\zeta(3)}{16b^2} - \frac{32b}{\pi^2 n} + \frac{r_1}{n}\left(1 - \frac{\pi^4}{24b^2}\right) + \sum_{Q} 2\rho_Q \log Q \sum_{m\geq 1} \frac{2F_b(m\log Q)}{Q^m + 1}. \tag{12}$$

As before, one may discard prime terms or the real-place term. The numerical uses below take $\zeta(3) < 1.203$ and, when convenient, retain just the first prime-power term, using the same quadratic lower bound for $F_b$; their certificates are in [20, `common/verify_unconditional_refinements.py`].

The comparisons below specify the degree, signature, retained Euler densities and local discriminant upper bound. Their strict numerical inequalities include the tail in (9); no truncated infinite sum is used without a remainder bound.

### 13.1 Finite classical bounds and real places

The whole-strip kernel admits sharper finite estimates than the quadratic majorant. Put $T_b(s) = s^{-1} - J_b(s)$ for $s > 0$ and $D_b(j) = 1 - jJ_b(j)$ for positive integers $j$. Then

$$B(G_b) \leq \log 2 + 2\sum_{j=0}^{N-1} T_b(2j+1) + \frac{c_b^2}{2(2N-1)^2},$$
$$C(G_b) = \frac{32b}{\pi^2}, \qquad A(G_b) \geq 1 - 2\sum_{j=1}^{2H-1} (-1)^{j-1} D_b(j). \tag{13}$$

Indeed $0 \leq T_b(s) \leq c_b^2/s^3$, which bounds the tail in

$$B(G_b) = \log 2 + 2\sum_{j\geq 0} T_b(2j+1), \qquad C(G_b) = 32b/\pi^2. \tag{14}$$

Since $F_b$ decreases on $[0, 2b]$, $D_b(j) = \int_0^{2b} (-F_b'(x)) e^{-jx}\,dx$ is positive and decreases with $j$. Integration by parts gives $A(G_b) = 1 - 2\sum_{j\geq 1}(-1)^{j-1} D_b(j)$, so odd truncation is a lower bound. All estimates are finite elementary expressions.

**Lemma 13.4.** *The following root-discriminant bounds require no hypothesis on zeros. In the rows marked $\rho_3 = 1/12$, assume $\rho_3(L) \geq 1/12$.*

| *Signature* | $[L:\mathbf{Q}] \geq$ | $b$ | $\rho_3$ | $\mathrm{rd}(L) >$ |
|---|---|---|---|---|
| *arbitrary* | *120* | *7* | *0* | *16.96717* |
| *arbitrary* | *1008* | *10* | *0* | *20.70004* |
| *arbitrary* | *8160* | *20* | *0* | *21.93341* |
| *totally real* | *120* | *7* | *0* | *43.14954* |
| *totally real* | *120* | *7* | 1/12 | *47.16252* |
| *totally real* | *1008* | *24* | *0* | *55.46400* |
| *totally real* | *8160* | *24* | *0* | *59.34662* |
| *totally real* | *8160* | *24* | 1/12 | *65.02119* |

*Proof.* Use Proposition 13.2 with $a = 1/2$ and (13) with $N = 100$, $2H - 1 = 99$. For the marked rows retain only the first norm-three term, which is $(\log 3)F_b(\log 3)/12$. All other prime terms are discarded. The finite interval comparisons are supplied in [37, `analytic/all_q_discriminants.py`]. □

**Corollary 13.5.** *Let $q \geq 4$ be a power of two and let $L$ be a number field satisfying*

$$[L : \mathbf{Q}] \geq 2q(q^2 - 1), \qquad \mathrm{rd}(L) \leq A\, 2^{-2/q}.$$

*Then $L$ is not totally real if either*

*(i)* $A \leq 4\sqrt{220}$*; or*
*(ii)* $A \leq 4\sqrt{40}\, 3^{5/6}$ *and* $\rho_3(L) \geq 1/12$.

*Proof.* In (i), the upper bounds for $q = 4$, $q = 8$ and $q \geq 16$ are respectively less than 41.95236, 49.89004 and 59.32959. The corresponding lower bounds in Lemma 13.4 are 43.14954, 55.46400 and 59.34662. In (ii), the upper bounds are less than 44.68647, 53.14147 and 63.19621, whereas the lower bounds are 47.16252, 55.46400 and 65.02119. For $q = 8$ no prime term is needed. □

For the original finite-flat kernel fields appearing in the source arguments, Lemma 11.12 gives the factor $2^{-2/q}$. The remaining local bounds give the following values of $A$:

| $d$ | $A$ | $d$ | $A$ |
|---|---|---|---|
| 2 | $8\sqrt{14}$ | 10 | $4\sqrt{40}\, 3^{5/6}$ |
| 3 | $8\sqrt{39}$ | 11 | $4\sqrt{220}$ |
| 5 | $4\sqrt{5}\, 11^{2/3}$ | 21 | $4\sqrt{21}\, 5^{2/3}$ |
| 6 | $4\sqrt{120}$ | 37 | $4\sqrt{37}\, 3^{5/6}$ |

For $d = 10, 37$ the prescribed three-adic local field supplies $\rho_3(L) \geq 1/12$; see Theorem 23.5 and Lemma 4.22. Thus the real-place check in all eight source arguments is classical. In the $q = 4$ case for $d = 37$, even the prime term is unnecessary: the local upper bound is less than $42.97807 < 43.14954$.

## 13.2 Narrower zero-free strips

Put

$$B_\infty(a) = \int_0^\infty \frac{1 - \operatorname{sech}(ax)}{2\sinh(x/2)}\, dx, \qquad Q(a) = \int_0^\infty \frac{x^2}{2\sinh(x/2)\cosh(ax)}\, dx.$$

The derivative of $F_b$ on $[0, 2b]$ gives $1 - F_b(x) \leq \pi^2 x^2/(8b^2)$ for every $x \geq 0$. Also $\cosh(x/2)/\cosh(ax) \leq e^{(1/2-a)x}$ for $a \leq 1/2$. Consequently

$$B(G_{a,b}) \leq B_\infty(a) + \frac{\pi^2 Q(a)}{8b^2}, \qquad C(G_{a,b}) \leq 4J_b(a - 1/2). \tag{15}$$

At $a = 1/2$, these specialize to $B_\infty = \log 2$, $Q = 7\zeta(3)/2$, and $C = 32b/\pi^2$.

**Lemma 13.6.** *For $a = 15/32$, we have $B_\infty(a) < 13/20$ and $Q(a) < 6$.*

*Proof.* For $a > 0$, put $t_a(s) = 2a^2/[s(s + a)(s + 2a)]$. The identity

$$1 - \operatorname{sech}(ax) = \frac{(1 - e^{-ax})^2}{1 + e^{-2ax}}$$

and geometric expansion give $B_\infty(a) = \sum_{m \geq 0} \sum_{j \geq 0} (-1)^j t_a(m + 1/2 + 2ja)$, with each inner sum interpreted as an alternating limit. Since $t_a$ is positive and decreasing, the even partial sum $P_{80}(s) = \sum_{j=0}^{80} (-1)^j t_a(s + 2ja)$ is an upper bound for that limit and satisfies $0 < P_{80}(s) \leq t_a(s)$. Thus

$$B_\infty(a) \leq \sum_{m=0}^{499} \sum_{j=0}^{80} (-1)^j t_a(m + 1/2 + 2ja) + \frac{a^2}{(499.5)^2}. \tag{16}$$

The tail uses $t_a(s) \le 2a^2/s^3$ and the integral test. The three-term upper bound for the geometric expansion of sech similarly gives

$$Q(a) \le 4 \sum_{m=0}^{3} \sum_{j=0}^{2} \frac{(-1)^j}{(m+1/2+(2j+1)a)^3} + \frac{2}{(3+1/2+a)^2}. \tag{17}$$

At $a = 15/32$, these rational expressions give $B_\infty(a) < 0.64919 < 13/20$ and $Q(a) < 6$. The exact rational checks are supplied in [37, `analytic/all_q_discriminants.py`]. □

**Lemma 13.7.** *Let $0 < a \le 1/2$, $M \ge 1$, and let $r, s \ge 0$ be even integers. If $B_0 \ge B_\infty(a)$, then*

$$B(G_{a,b}) \le B_0 + 2 \sum_{m=0}^{M-1} \sum_{j=0}^{r} (-1)^j T_b\big(m + \tfrac{1}{2} + (2j+1)a\big) + \frac{c_b^2}{(M - \frac{1}{2} + a)^2},$$

$$C(G_{a,b}) \le 4 \sum_{j=0}^{s} (-1)^j \left\{ J_b\big((2j+1)a - \tfrac{1}{2}\big) + J_b\big((2j+1)a + \tfrac{1}{2}\big) \right\}. \tag{18}$$

*Proof.* Expand $(2\sinh(x/2))^{-1}$ and $\operatorname{sech}(ax)$. The difference $B(G_{a,b}) - B_\infty(a)$ is the corresponding alternating sum of $2T_b(m + 1/2 + (2j+1)a)$. The function $T_b$ is positive and decreasing; its even partial sums are upper bounds and are at most the first term. The omitted $m$-tail is at most $2c_b^2 \sum_{m \ge M} (m + 1/2 + a)^{-3}$, bounded by the displayed integral remainder. For the pole term, $J_b(u)$ is positive and decreasing for real $u$, so the even alternating partial sum is again an upper bound. The point $x = 0$ causes no difficulty by integration or Abel limits. □

For several one-place comparisons it is enough to use (12). At degree at least 1008, take $b = 10$. Dropping the real and prime terms gives $\mathrm{rd}(L) > 20.57608$; retaining the first norm-nine term with $\rho_9(L) \ge 1/4$ gives $\mathrm{rd}(L) > 25.32625$. At degree at least 120, take $b = 7$. The bound is $\mathrm{rd}(L) > 16.66$ in arbitrary signature and $\mathrm{rd}(L) > 41.69019$ in the totally real case. In particular $4\sqrt{5}\,11^{1/4} < 16.289$ and $5^{3/4}11^{2/5} < 8.726$. The exact rational checks are described in Appendix C.3.

## 13.3 Euler terms from a Sylow subfield

**Lemma 13.9.** *Let $M/K$ be Galois with group $\mathrm{SL}_2(\mathbf{F}_q)$ for $K$ real quadratic and $q \ge 4$ a power of two. For a Sylow two-subgroup $U$ put $L = M^U$ and $r_0 = 1/(q+1)$. Then $[L : \mathbf{Q}] = 2(q^2-1)$ and the real fraction of $L$ is at least $r_0$. At a place of $K$ of norm $P$ with elementary abelian decomposition group of ramification index $e$ and residue degree $f$, its relative different contribution in $\log \mathrm{rd}(L)$ is $1 - r_0$ times that in $\log \mathrm{rd}(M)$. The Euler coefficients supplied by this place in $L$ are $r_0$ at norm $P$ and $(1-r_0)/(ef)$ at norm $P^f$, added when $f = 1$.*

*Proof.* A nontrivial two-subgroup is contained in a unique Sylow two-subgroup, the translations fixing its line on $\mathbf{P}^1$. Distinct Sylow two-subgroups intersect trivially. It therefore fixes $q-1$ cosets on $\mathrm{SL}_2(\mathbf{F}_q)/U$ and acts freely on the others. For a trivial subgroup we may partition the cosets into blocks of the same sizes. Applying the fixed-coset count to real conjugation gives the real fraction. For every nontrivial local ramification group $J$ the normalized codimension of invariants is $(1-r_0)(1-1/|J|)$, so the conductor–discriminant formula gives the stated factor. The singleton and free orbits give the Euler coefficients. □

## 13.4 Comparisons for the fields with radicands three, seven and twenty-nine

Put $n(q) = 2q(q^2-1)$ for a power $q$ of two, and set

$$C_3 = 8\sqrt{3}, \qquad C_7 = 8\sqrt{7}, \qquad C_{29} = 4\sqrt{29}, \qquad T_3 = C_3\,11^{1/4}, \quad T_{29} = C_{29}\,5^{1/4}.$$

In the following tables $L$ is a certified lower bound supplied by Proposition 13.2, and $U$ is a certified upper bound for the proposed field. Lower endpoints are rounded down and upper endpoints up. The parameter $n_0$ is a lower bound for the absolute degree. For a range $q \geq q_0$ we keep $b$ fixed and use monotonicity in the degree; an upper bound independent of $q$ is used in that row. Thus a finite table covers every power of two.

**Proposition 13.10.** *For $d \in \{3, 7, 29\}$ there is no number field of degree $n(q)$, with $q \geq 4$ a power of two, whose root discriminant is at most $C_d 2^{-2/q}$. No zero-free hypothesis is needed.*

*Proof.* Use $a = 1/2$ and no prime terms in Proposition 13.2. The following inequalities give $L > U$ in every case:

| $d$ | $q$ | $n_0$ | $b$ | $L$ | $U$ |
|---|---|---|---|---|---|
| 3 | $\geq 4$ | 120 | 13/2 | 16.98861 | 13.85641 |
| 7 | 4 | 120 | 13/2 | 16.98861 | 14.96663 |
| 7 | 8 | 1008 | 14 | 20.88061 | 17.79843 |
| 7 | $\geq 16$ | 8160 | 28 | 21.99335 | 21.16602 |
| 29 | 4 | 120 | 13/2 | 16.98861 | 15.23155 |
| 29 | 8 | 1008 | 14 | 20.88061 | 18.11347 |
| 29 | $\geq 16$ | 8160 | 28 | 21.99335 | 21.54066 |

□

**Proposition 13.11.** *Let $q \geq 4$ be a power of two. Suppose a number field $M$ of degree $n(q)$ satisfies $\mathrm{rd}(M) \leq T_3 2^{-2/q}$. Suppose it has at least $n(q)/4$ primes of norm eleven or at least $n(q)/8$ primes of norm 121. If $\zeta_M(s)$ has no nontrivial zeros with $\mathrm{Re}\, s > 23/25$, then $M$ does not exist. Together with Proposition 13.10, this excludes every point field in Proposition 20.7 under the stated zero-free hypothesis in its ramified cases.*

*Proof.* Take $a = 21/50$ in Proposition 13.2. In these tables, $(Q, \rho_Q)$ means that the number of primes of norm $Q$ is at least $\rho_Q n(q)$, with the normalization $\rho_Q = N_Q/n$ fixed above.

| $(Q, \rho_Q)$ | $q$ | $n_0$ | $b$ | $L$ | $U$ |
|---|---|---|---|---|---|
| $(11, 1/4)$ | 4 | 120 | 13/2 | 22.01479 | 17.84366 |
| $(11, 1/4)$ | $\geq 8$ | 1008 | 12 | 28.96855 | 25.23474 |
| $(121, 1/8)$ | 4 | 120 | 6 | 18.07618 | 17.84366 |
| $(121, 1/8)$ | 8 | 1008 | 11 | 23.15901 | 21.21981 |
| $(121, 1/8)$ | 16 | 8160 | 37/2 | 24.89435 | 23.14036 |
| $(121, 1/8)$ | $\geq 32$ | 65472 | 28 | 25.46555 | 25.23474 |

Every row has $L > U$. □

**Proposition 13.12.** *The unramified point fields of Proposition 29.10 are excluded unconditionally. Its ramified fields with decomposition group $C_2$ at $w_+$ are excluded if their Dedekind zeta functions have no nontrivial zeros in $\mathrm{Re}\, s > 23/25$. Its nonsolvable nonordinary dyadic sources with $q \geq 8$ are excluded unconditionally.*

*Proof.* The first assertion is Proposition 13.10. For the second use $a = 21/50$, $Q = 5$ and $\rho_5 = 1/4$:

| $q$ | $n_0$ | $b$ | $L$ | $U$ |
|---|---|---|---|---|
| 4 | 120 | 6 | 24.90620 | 22.77648 |
| $\geq 8$ | 1008 | 12 | 32.90284 | 32.21080 |

Here $U$ bounds $T_{29}2^{-2/q}$, with $T_{29}$ itself used in the second row. In the nonordinary case the local different calculation in Proposition 29.10 gives

$$\mathrm{rd}(M) < 2^{4/3}\sqrt{29}\,5^{1/4} < 20.29154.$$

The unconditional row $a = 1/2$, $b = 14$, $n_0 = 1008$, with no prime terms, gives $\mathrm{rd}(M) > 20.88061$. □

**Lemma 13.13.** *A totally real number field of degree at least* 120 *has root discriminant greater than* 42.81537*. Suppose $M/K$ is Galois, $K$ is real quadratic, $[M : \mathbf{Q}] \geq 1008$, and $M$ has at least $[M : \mathbf{Q}]/8$ primes of norm* 25*. If its nontrivial zeta zeros lie in $2/25 \leq \mathrm{Re}\, s \leq 23/25$ and $M$ has a real embedding, then $\mathrm{rd}(M) > 40.87041$.*

*Proof.* Retain the real-place term $(r_1/n)A(F)$ in Proposition 13.2. For a totally real field take $a = 1/2$, $b = 13/2$ and $n_0 = 120$. The resulting root discriminant lower bound is greater than 42.81537.

For the second assertion, normality over $K$ implies $r_1/n \in \{0, 1/2, 1\}$. A real embedding therefore gives $r_1/n \geq 1/2$. Take $a = 21/50$, $b = 10$, $n_0 = 1008$ and $(Q, \rho_Q) = (25, 1/8)$. Including $A(F)/2$ gives the certified lower bound 40.87041. Monotonicity in $n$ gives the assertion for every larger degree. □

All entries are obtained with ball arithmetic from the integrals above. The program `cert_three_fields.py` in [21] includes the origin enclosure, the exact archimedean tail, every prime-power term in the support, and directed decimal rounding. Its supplementary rows for the stronger region $\mathrm{Re}\, s > 4/5$ are used only in Corollary 29.12.

# 14 Arithmetic of the split-dyadic fields

Put $\mathcal{D} = \{17, 33, 41, 57, 65, 73, 89\}$. These fields have fundamental discriminant $d$ and two splits in each. In this section let $K = \mathbf{Q}(\sqrt{d})$ with $d \in \mathcal{D}$, and write $w_0, w_1$ for its dyadic places. We record the arithmetic needed in the scalar-residual field construction.

**Lemma 14.1.** *Let $S_2$ consist of the dyadic and real places of $K$. Every finite-order character $G_K \to \overline{\mathbf{F}}_2^{\times}$ unramified outside two is trivial. Moreover*

$$H^1(G_{K,S_2}, \mathbf{F}_2) = \langle [-1], [\epsilon], [2], [\alpha] \rangle_{\mathbf{F}_2}$$

*has dimension four, with the choices in Table 1.*

*Proof.* Minkowski's bound is $\sqrt{d}/2$. Except at 57 and 73, every prime ideal required by this bound is dyadic. At 41, 65 and 89, three is inert, so its prime ideal has norm nine and lies outside the bound. The primes above three in the two remaining cases are principal, since $N(15 + 2\sqrt{57}) = -3$, $\quad N(17 + 2\sqrt{73}) = -3$. For $d \neq 65$, the displayed $\alpha$ has norm $\pm 2$ and generates a dyadic prime. At 65, its norm is four and its trace is odd, so it generates the square of a dyadic prime. That prime is not principal: an element of norm $\pm 2$ would yield $x^2 - 65y^2 = \pm 8$, impossible modulo five. Thus $h_{65} = 2$ and the class group is generated by a

| $d$ | $\epsilon$ | $\alpha$ | $N\epsilon$ | $N\alpha$ | $h_K$ | $h_K^+$ |
|---|---|---|---|---|---|---|
| 17 | $4+\sqrt{17}$ | $(5+\sqrt{17})/2$ | $-1$ | 2 | 1 | 1 |
| 33 | $23+4\sqrt{33}$ | $(5+\sqrt{33})/2$ | 1 | $-2$ | 1 | 2 |
| 41 | $32+5\sqrt{41}$ | $(7+\sqrt{41})/2$ | $-1$ | 2 | 1 | 1 |
| 57 | $151+20\sqrt{57}$ | $(7+\sqrt{57})/2$ | 1 | $-2$ | 1 | 2 |
| 65 | $8+\sqrt{65}$ | $(9+\sqrt{65})/2$ | $-1$ | 4 | 2 | 2 |
| 73 | $1068+125\sqrt{73}$ | $(9+\sqrt{73})/2$ | $-1$ | 2 | 1 | 1 |
| 89 | $500+53\sqrt{89}$ | $(217+23\sqrt{89})/2$ | $-1$ | 2 | 1 | 1 |

Table 1: Units, dyadic generators, and ordinary and narrow class numbers.

dyadic prime. This proves the ordinary class numbers and shows that the $S_2$-ideal class group is trivial in every case.

The unit squareclass group has dimension two. If $N\epsilon = -1$, the mixed signs of $\epsilon$ show that its class and $[-1]$ are independent. For $d = 33, 57$ the unit is totally positive and is not a square: a square root would have integral trace whose square is, respectively, 44 or 48, or 300 or 304. Thus $[-1]$ and $[\epsilon]$ are again independent and form a basis. No fundamental-unit calculation is needed. The norm-minus-one units give the narrow class numbers in the other five cases; for $33, 57$, norm minus one is impossible modulo three, giving the remaining entries in the table.

Kummer theory identifies the group in the lemma with $\mathcal{O}_{K,S_2}^\times/(\mathcal{O}_{K,S_2}^\times)^2$. For $d \neq 65$, the valuation vectors of 2 and $\alpha$ form a basis of the dyadic divisor lattice. For 65, that lattice consists of pairs whose sum is even, and its basis is supplied by the vectors $(1,1)$ and $(2,0)$ of 2 and $\alpha$. Together with $-1$ and the displayed unit squareclass, this proves the asserted basis.

Finally, a characteristic-two finite character has odd-order image. At either dyadic completion $\mathbf{Q}_2$, it kills the pro-two unit group and is unramified. It also kills real sign groups. Thus it factors through the ordinary class group, whose order is a power of two. The character is trivial. □

For $i = 0, 1$ write $V_i(K) = V_{w_i}^+(K)$ for the space in (1). We verify its vanishing using the full quadratic Selmer group, including the ideal-class contribution.

**Lemma 14.2** (Base scalar Selmer vanishing)**.** *For every $d \in \mathcal{D}$ and either dyadic choice, $V_i(K) = 0$.*

*Proof.* The sign map on the basis of Lemma 14.1 gives

$$H^+ = \langle [2], [\beta] \rangle, \qquad \beta = \begin{cases} \alpha, & d = 17, 41, 65, 73, 89, \\ \epsilon, & d = 33, 57, \end{cases}$$

where $H^+$ denotes the totally positive subspace of $H^1(G_{K,S_2}, \mathbf{F}_2)$. It suffices to prove that the local class of $\beta$ lies outside $\langle [2] \rangle$ at both dyadic places.

For $d = 17, 41, 65, 73, 89$, put $N\beta = 2^e$, with $e = 1$ except $e = 2$ for $d = 65$, and put $t = \mathrm{Tr}(\beta)$. The odd trace shows that the two dyadic valuations of $\beta$ are $0, e$. Write $u$ for the unit value. Then $u + 2^e/u = t$, so

$$u \equiv (1 + 2^e)t \equiv 7, 5, 5, 3, 3 \pmod 8$$

in the order $d = 17, 41, 65, 73, 89$. Thus $u$ is nonsquare in $\mathbf{Q}_2$. The other local value is $2^e/u$, so both local classes lie outside $\langle [2] \rangle$. For $d = 33, 57$, we have $N\beta = 1$ and $\mathrm{Tr}(\beta) \equiv 14 \pmod{16}$. At either dyadic place, a square unit $u$ would satisfy $u \equiv 1 \pmod 8$ and $u + u^{-1} \equiv 2 \pmod{16}$, a contradiction. Again the local unit class lies outside $\langle [2] \rangle$.

Restriction of $H^+$ to either dyadic completion is consequently injective. Its kernel at $w_{1-i}$ is $V_i(K)$. □

For $d = 17, 41, 73, 89$, Table 1 gives narrow class number one. The totally positive element $\alpha$ of norm two has trace $5, 7, 9, 217$, respectively. At the other dyadic place its unit root satisfies $u + 2u^{-1} = \mathrm{Tr}(\alpha)$ and is congruent to $7, 5, 3, 3$ modulo eight. Thus these four fields satisfy condition (S) of [25, Lemma 9.1].

## 14.1 Local point fields and their discriminants

For a finite Galois extension $E/\mathbf{Q}_2$, let $d(E)$ be its normalized different exponent, with $v_2(2) = 1$. Put $E_2 = \mathbf{Q}_2(\sqrt{-1}, \sqrt{2}, \sqrt{5})$.

**Lemma 14.3.** *Let $r : G_{\mathbf{Q}_2} \to \mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^a$, have finite image and kernel field $E$. Then $d(E) \le 2$. More precisely, either $E \subset E_2$, or wild inertia is nontrivial of order $Q \le q$, all positive upper breaks are at most one and the local abelianization has odd order, or $E$ is unramified or tame with ramification index three. In the wild case outside $E_2$, $d(E) \le 2(1 - 1/Q)$ and $E \cap \mathbf{Q}_2(\sqrt{-1}, \sqrt{5}) = \mathbf{Q}_2$. In the tame ramified case the image is $S_3$ and its quadratic subfield is unramified.*

*Proof.* If wild inertia is trivial, its tame generator has eigencharacters $\theta, \theta^{-1}$, and Frobenius squares them. The normalizer acts by identity or inversion, so the nontrivial tame order is three; Frobenius is the Weyl involution and the image is $S_3$. Its different exponent is $2/3$.

Otherwise the wild subgroup fixes a unique line, stable under the whole decomposition group. The diagonal characters are $\theta, \theta^{-1}$ and are unramified, since $\mathbf{Z}_2^\times$ is pro-two. If $\theta = 1$, the image is elementary abelian unipotent and $E \subset E_2$. The seven quadratic characters of $E_2$ have conductors $0, 2, 2, 3, 3, 3, 3$, giving $d(E_2) = 2$. If $\theta \ne 1$, Frobenius acts on the translation subgroup $P$ by the nontrivial scalar $\theta(\mathrm{Frob}_2)^2$. The operator obtained by subtracting one is invertible, so $[P, \mathrm{Frob}_2] = P$, $P$ is wild inertia, and $E^P/\mathbf{Q}_2$ is unramified of odd degree. Its Kummer module has no Frobenius invariants. The equivariant valuation map to $\mathbf{Z}/2\mathbf{Z}$ is zero, so all its squareclasses are units and all its quadratic characters have conductor at most two. The different bound follows from conductor–discriminant. The commutator identity gives odd local abelianization and hence disjointness from the stated biquadratic field. □

**Lemma 14.4.** *Put $H = K(\sqrt{-1}, \sqrt{\epsilon})$. It is biquadratic over $K$, Galois over $\mathbf{Q}$ and unramified at odd places, and its dyadic completions lie in $\mathbf{Q}_2(\sqrt{-1}, \sqrt{5})$. If $L/K$ has group $\mathrm{SL}_2(\mathbf{F}_q)$, $q \ge 8$, and is unramified outside two, put $M = LH$. Let $k$ count the dyadic places whose local kernel fields lie in $E_2$. Then*

$$[M : \mathbf{Q}] = 8q(q^2 - 1), \qquad \mathrm{rd}(M) \le \sqrt{d}\, 2^{2 - 1/q + k/(2q)}. \tag{20}$$

*Each of these $k$ places supplies in the explicit formula the contribution of at least $[M : \mathbf{Q}]/16$ primes of norm four.*

*Proof.* The classes $[-1], [\epsilon]$ are independent by their real signs and nonsquareness. Conjugation sends $\sqrt{\epsilon}$ to $1/\sqrt{\epsilon}$ or $\sqrt{-1}/\sqrt{\epsilon}$, so $H/\mathbf{Q}$ is Galois. Its defining polynomials have discriminants supported at two, and every unit squareclass over $\mathbf{Q}_2$ is generated by $[-1], [5]$. Simplicity gives $L \cap H = K$.

At a dyadic place with $E \subset E_2$, the compositum still lies in $E_2$ and has different exponent at most two, ramification index at most four and residue degree at most two. Otherwise, in the wild case of Lemma 14.3, adjoining the ramified quadratic part of $H$ enlarges its elementary wild group from $P$ to at most $P \times C_2$. All characters still have conductor at most two. Its different exponent is at most $2(1 - 1/(2Q)) \le 2 - 1/q$. In the tame case the compositum has ramification index at most six, with only a possible quadratic positive interval of length one. Its

different exponent is at most $(1-1/6)+(1-1/2)=4/3<2-1/q$. Averaging the two dyadic exponents proves (20).

At a counted place, residue degree two gives at least $[M:\mathbf{Q}]/16$ norm-four primes. Residue degree one gives at least $[M:\mathbf{Q}]/8$ norm-two primes; keeping their even Euler powers supplies the same lower contribution. □

**Lemma 14.5.** *For $d\in\mathcal{D}$, an irreducible characteristic-two residue unramified outside two has determinant one. It is modular if its image is solvable or is $\mathrm{SL}_2(\mathbf{F}_4)$. The remaining images are $\mathrm{SL}_2(\mathbf{F}_q)$ for $q\geq 8$ a power of two.*

*Proof.* Lemma 14.1 gives trivial determinant. For a solvable image use [25, Proposition 10.3], since $K/\mathbf{Q}$ is abelian and two splits completely. For $\mathrm{SL}_2(\mathbf{F}_4)$ use [3, Theorem 1.2(1) and Lemma 2.1]. The classification of finite subgroups gives the remaining list. □

# Part II
# Fields without GRH

## 15 The field $\mathbf{Q}(\sqrt{2})$

Throughout this section put $K = \mathbf{Q}(\sqrt{2})$, $s = \sqrt{2}$ and $\epsilon = 1 + s$. The field has discriminant 8 and ordinary class number one by Minkowski's bound. Its units are generated by $-1, \epsilon$, and the norm of $\epsilon$ is $-1$, so the narrow class number is also one. The prime two ramifies, three is inert, and the two primes above seven are $(3 + s)$ and $(3 - s)$.

### 15.1 Local lifts at seven and the switch at three

Write $\omega_p = \bar{\chi}_p$. A special type includes its quadratic scalar character. The following twists are used to make that character unramified.

**Lemma 15.1.** *Put $\alpha_\pm = 5 \pm 4s$. The quadratic characters attached to $K(\sqrt{\alpha_\pm})/K$ ramify at exactly one of the two seven-primes, are unramified at two and three, and their product is the character of $K(\sqrt{-7})/K$, split at two.*

*Proof.* We have $\alpha_+ = \epsilon(3 + s)$ and $\alpha_- = \alpha_+^\sigma$, with $\alpha_+\alpha_- = -7$. Both are units away from seven. The integral element $(1 + \sqrt{\alpha_\pm})/2$ has polynomial $X^2 - X - (1 \pm s)$. Its reduction at the dyadic prime is $X^2 + X + 1$, so the quadratic extensions there are unramified. At three the defining radicands are units, hence the quadratic extensions are unramified. Their odd seven-adic valuations give exactly the asserted seven-ramification. Finally $-7$ is a square in $\mathbf{Q}_2$ because it is one modulo eight. □

**Lemma 15.2.** *Let $\mathcal{R}$ be a pure, irreducible, totally odd weight-two strongly compatible system with cyclotomic determinant, good outside seven, whose seven-types are good, quadratic-special, or principal series of order three after the corrections above. If its three-adic residue is reducible, the actual system is modular. If that residue is irreducible, it remains irreducible on $G_{K(\zeta_3)}$.*

*Proof.* Twist away the scalar quadratic inertia using Lemma 15.1. Modulo three the inertia at both seven-primes is unipotent. A reducible global residue therefore has characters unramified outside three. The prime-to-three narrow ray group modulo three is $C_2$: $\epsilon$ has order eight in $\mathbf{F}_9^\times$ and signs $(+, -)$, while $-1$ supplies the independent sign relation. Its characters are $1, \omega_3$. Thus the residual constituents are exactly $1, \omega_3$.

Since $e(K_3) = 1 < 3 - 1$, finite-flat uniqueness identifies the rank-one models with the etale and multiplicative models. A nonordinary Barsotti–Tate source would be biconnected, as would its finite-flat subquotients; these rank-one models contradict biconnectedness. The actual source is ordinary, with unramified quotient reducing to one. Skinner–Wiles [75, §4.5, Theorem A] applies: the ratio is locally nontrivial, totally odd, and its splitting field $K(\zeta_3)$ is abelian over $\mathbf{Q}$. The determinant is $\chi_3$ and the local quotient has weight zero. Undo the twists.

For the second assertion, a cyclotomic-reducible irreducible residue would be induced from a prime-to-three character $\theta$ of $H = K(\zeta_3)$. Its restriction to seven-inertia is trivial since that inertia has order dividing three. The character is therefore unramified outside three and tame there. The absolute discriminant of $H$ is $8 \cdot (-3) \cdot (-24) = 576$, and its degree-four Minkowski bound is $36/\pi^2 < 4$. There is no prime of norm two or three: their norms are four and nine. Hence $H$ has class number one. Its unique prime above three has residue field $\mathbf{F}_9$, and $\epsilon$ generates the residue units. The tame ray group is trivial. This makes $\theta$ trivial, contradicting irreducibility. □

**Lemma 15.3.** *Every local determinant-$\omega_7$ residue over $\mathbf{Q}_7$ has a determinant-$\chi_7$ weight-two lift that is good, special, or principal series of order three after a scalar quadratic twist.*

*Proof.* The central-trivial Serre weights are $\mathrm{Sym}^r \det^b$, with $r = 0, 2, 4, 6$ and $r + 2b = 0 \pmod 6$. The two principal-series types with characters $\widetilde{\omega}_7^t, \widetilde{\omega}_7^{-t}$ have reductions

| $t$ | first constituent | second constituent |
|---|---|---|
| 1 | $(2, 5)$ | $(4, 1)$ |
| 2 | $(4, 4)$ | $(2, 2)$. |

Together with the two scalar crystalline and two scalar semistable weights, these cover all eight weights. Some multiplicity is positive because a potentially Barsotti–Tate lift exists, as in the proof of Theorem 4.15. The fixed-determinant formula [84, Theorem 1.2] therefore gives a nonempty weight-two local ring in this menu for the actual residual extension. At a scalar semistable point the monodromy is either zero, giving the good case after twist, or nonzero, giving the special case. A scalar quadratic twist interchanges the principal-series rows and changes order six to order three. The independent global characters in Lemma 15.1 implement these choices at the two split seven-primes without disturbing good reduction at two or three. □

## 15.2 Weight-two systems with one possible special place

**Proposition 15.4.** *Let $\mathcal{R}$ be a weight-two system over $K$ of cyclotomic determinant. Suppose it is good outside one place $v$ above seven and is good or quadratic-special at $v$. Then $\mathcal{R}$ is modular.*

We first treat nonscalar dyadic residues. The special parameter has an ordinary filtration away from the coefficient prime. After the quadratic scalar character is removed, its local characters are unramified with Frobenius values $\pm 7, \pm 1$, with common sign. In characteristic two both become one, so tame inertia in the residue has order at most two, for any stable lattice. The quadratic scalar character also reduces to one.

**Lemma 15.5.** *Let $K = \mathbf{Q}(\sqrt{2})$ and let $\bar{r} : G_K \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ be finite flat at two, unramified away from two and seven, with inertia of order at most two at both seven-primes. If its image is solvable, then $\bar{r}^{\mathrm{ss}} = 1 \oplus 1$.*

*Proof.* Apply Lemma 11.16a with the two seven-places as $S$. In the reducible case the dyadic residue field $\mathbf{F}_2$ and narrow class number one make its odd ray characters trivial. Otherwise write the residue as $\mathrm{Ind}_{G_L}^{G_K} \theta$; $L/K$ is quadratic and unramified outside two and seven. Since $K$ has class number one and unit group $\langle -1, \epsilon \rangle$, its squareclass is generated by $-1, \epsilon, s, 3 + s, 3 - s$, where $s^2 = 2$. Fontaine's finite-flat bound makes the quadratic conductor at two at most four. A quadratic extension of $K_2$ defined by an odd valuation radicand has conductor $2e(K_2) + 1 = 5$. Thus the exponent of $s$ is zero, and there are precisely the fifteen nontrivial classes generated by $-1, \quad \epsilon, \quad 3 + s, \quad 3 - s$. It remains to compute the odd dyadic ray quotients of these inducing fields.

For one allowed seven-place, conjugate it to $(3 + s)$ and put $\pi = 3 + s$, $\beta = \epsilon\pi = 5 + 4s$. The relevant classes are generated by $-1, \epsilon, \pi$. The following direct calculation treats this subfamily.

The possible prime ideals of $L$ needed for Minkowski's bound are as follows.

| $a$ | $r_2(L)$ | upper bound for $\lvert\operatorname{disc}(L)\rvert$ | prime ideal norms to consider |
|---|---|---|---|
| $-1$ | 2 | 256 | 2 |
| $\pm\epsilon$ | 1 | 1024 | 2 |
| $\pi$ | 0 | 7168 | $2, 7$ |
| $-\pi$ | 2 | 7168 | $2, 7, 9$ |
| $\beta$ | 1 | 448 | none |
| $-\beta$ | 1 | 1792 | 2 |

The first four discriminant bounds follow from the polynomials $X^4+1$, $X^4-2X^2-1$ and $X^4\mp 6X^2+7$, respectively. The fields for $\pm\epsilon$ are conjugate. For $\pm\beta$, the relative discriminant has valuation one at $v$ and no other odd factor. At two, $\beta$ is unramified and $-\beta$ has conductor two, since $-1$ has conductor two and $\beta$ is unramified. This gives the two bounds after multiplying by $8^2$. The list of prime norms follows from Minkowski and the fact that $3, 5, 11$ are inert in $K$. For $\beta$ the dyadic prime has norm four.

In each of the other cases there is a unique prime $\mathfrak{P}$ above two with $(s)\mathcal{O}_L = \mathfrak{P}^2$. Indeed $-1$ and $-\beta$ are ramified there and $\pm\epsilon, \pm\pi$ are nonsquares modulo $2\mathcal{O}_{K_2}$. This proves that the classes of dyadic primes have order dividing two. For $a = \pm\pi$, the element $t = \sqrt{a}$ generates the prime over $v$ and has absolute norm seven. The conjugate seven-place is inert in $K(\sqrt{\pi})$ and split in $K(\sqrt{-\pi})$, since $\pi$ has residue six there. For $a = -\pi$, the two additional norm-seven primes are picked out by $t-1$ and $t+1$, both of absolute norm fourteen. The polynomial $X^4+6X^2+7$ factors modulo three as $(X^2+X+2)(X^2-X+2)$ with distinct irreducible factors, and its discriminant is prime to three. The corresponding norm-nine primes are picked out by $t^2+t+2$ and $t^2-t+2$, both of absolute norm eighteen. Thus all the required classes have order dividing two.

The odd part of $(\mathcal{O}_L/(2))^\times$ is trivial except for $a = \beta$, when it is the multiplicative group of the residue field $\mathbf{F}_4$. The unit $(1+\sqrt{\beta})/2$ has relative norm $-\epsilon$ and maps to a root of $X^2+X+1$ in that residue field, so its image generates the odd part. The ray exact sequence now gives the required vanishing.

For the full two-place support the odd ray quotient is trivial in all fifteen cases. Here is a complete finite certificate: use the bits of an integer $1 \le m \le 15$ to select the four displayed generators, form their product $A(y)$, and take $P_m(X) = \operatorname{Res}_y\left((X-y)^2 - A(y), y^2-2\right)$. The accompanying program `verify_d2_solvable_two.gp` computes the maximal order and certified class group of each quartic field, and its ray group modulo two. It applies `bnfcertify` in every row and asserts that every ray invariant is a power of two. The possible absolute discriminants and ray invariants are

| $\operatorname{disc}(L)$ | $\mathrm{Cl}_{(2)}(L)$ |
|---|---|
| 256 | 1 |
| $-1024$ | 1 |
| 7168 | 1 or $C_4 \times C_2$ |
| $-448$ | 1 |
| $-1792$ | 1 |
| 12544 | 1 |
| 3136 | $C_2^2$ |
| $-50176$ | $C_2^2$. |

By Lemma 11.16a, modulus (2) has the same odd quotient as the radical modulus. Thus $\theta$ is trivial, contrary to irreducibility. The exact calculation is supplied in [20, `ramified/d2`]. □

**Lemma 15.6.** *If a dyadic residue in Proposition 15.4 is nonsolvable, then the system is modular.*

*Proof.* The kernel field $M/K$ has group $\mathrm{SL}_2(\mathbf{F}_q)$ with $q$ a power of two and $q \geq 4$. Its absolute degree is $2q(q^2-1)$. At two the base contributes $2^{3/2}$ to its root discriminant, and the relative normalized different is less than two by Lemma 11.10. At $v$ the tame ramification index is at most two, so the contribution is at most $7^{1/4}$. Hence

$$\mathrm{rd}(M) < 2^{7/2}7^{1/4} < 18.403.$$

For $q \geq 8$ the absolute degree is at least 1008, and the classical inequality of Section 13 gives $\mathrm{rd}(M) > 20.57608$. For $q = 4$ it also excludes a totally real kernel of degree 120, since the real-place term gives $\mathrm{rd}(M) > 41.69$. The small-image residual theorem and the actual dyadic lifting argument of Lemma 2.4 now give modularity. □

**Lemma 15.7.** *An ordinary crystalline dyadic member in Proposition 15.4 cannot have scalar residual semisimplification.*

*Proof.* Apply Lemma 4.10 to the absolutely irreducible dyadic member with trivial residual semisimplification, using its ordinary line and the dyadic decomposition group. The resulting nonzero quadratic character is supported at two and $v$ and is split at two. With $v = (3+s)$ it must belong to $\langle [-1], [\epsilon], [3+s] \rangle$ by the conductor argument of Lemma 15.5. The three generators can be replaced by $-1, \epsilon, \beta$, where $\beta = 5 + 4s = \epsilon(3+s)$. Locally $\beta$ is the nontrivial unramified class by Lemma 15.1. The class $-1$ has conductor two, since the biquadratic discriminant formula gives $\mathrm{disc}(K(i)) = 256$. The unit $\epsilon$ is not a square modulo $2\mathcal{O}_{K_2}$, so its conductor exceeds two by Lemma 11.6. The three local classes are independent. This contradicts the existence of the split class. □

*Proof of Proposition 15.4.* By the preceding lemmas, it remains to consider the case in which every dyadic coefficient member is nonordinary and every residual semisimplification is scalar. Remove the ramified quadratic scalar factor, if any, at the possible special place using Lemma 15.1. This preserves the good conditions, ordinarity, determinant, and scalar residues. Use the simultaneous order $E_0$ of Section 11 and put $E = E_0/2E_0$ and $J = \operatorname{rad} E$.

For $\lambda \in (J/J^2)^\vee$, the map $g \mapsto \lambda(r(g) - 1)$ is additive and the map from functionals to such characters is injective since the group image spans $E$. The corresponding two-dimensional quotient of the left regular representation is finite flat and biconnected by Lemma 11.8. Lemma 11.5 bounds the dyadic conductor by two. With $v = (3+s)$, the permitted squareclasses are therefore spanned by $-1, \beta$ by the preceding local unit test. If the system is good everywhere, only $-1$ remains. A real conjugation $c$ then gives $A = r(c) - 1$ spanning $J/J^2$, so $A$ generates $E$ as an $\mathbf{F}_2$-algebra. Since $A^2 = 0$, $E$ is spanned by $1, A$, contradicting $\dim_{\mathbf{F}_2} E = 4[D : \mathbf{Q}] \geq 4$, where $D$ is the trace field.

Suppose the system is special at $v$. Choose $c$ at the embedding $s > 0$, and let $C = r(\tau) - 1$ for a tame generator at $v$. The evaluations of $c, \tau$ on $(-1, \beta)$ are $(1,0), (0,1)$, so $A, C$ span $J/J^2$ and generate $E$. We have $A^2 = C^2 = 0$.

The class of $-1$ is unramified nonsquare at $v$. Choose an arithmetic Frobenius lift whose value on $\beta$ is zero, by multiplying it by inertia if necessary. Its difference $B = r(\mathrm{Frob}_v) - 1$ has the same values as $A$ on the allowed first-layer characters, so $B - A \in J^2$. On the image of the monodromy operator, Frobenius acts by 7 times its unramified sign, the same odd integer in every coefficient realization. Thus $BC = 0$ modulo two in the actual order and $AC \in J^3$. The only possibly nonzero words of length three are $ACA$ and $CAC$, and both now lie in $J^4$. Thus $J^3 = J^4$. Nilpotence gives $J^3 = 0$, so $AC = 0$. It follows that $1, A, C, CA$ span $E$, so $4[D : \mathbf{Q}] = \dim_{\mathbf{F}_2} E \leq 4$. The trace field is therefore $\mathbf{Q}$.

Take a coefficient place over three. The trace field is rational, so an irreducible residue is realized over $\mathbf{F}_3$ by finite-field Brauer descent. More explicitly, the characteristic polynomials

have coefficients in $\mathbf{F}_3$, and Frobenius conjugation on coefficients preserves its isomorphism class by Brauer–Nesbitt. The finite-field Schur index is one. The small-image residual theorem [3, Theorem 2.5] and Lemma 15.2 give its modularity and cyclotomic irreducibility, respectively. With a weight-two witness as in Lemma 2.1, Lemma 2.2(ii) gives modularity of the actual system. If the residue is reducible, Lemma 15.2 directly gives modularity of the actual three-adic member. Undo the finite quadratic twist. □

### 15.3 The seven-adic terminal member

**Proposition 15.8.** *The seven-adic terminal member of Proposition 6.2 over* $\mathbf{Q}(\sqrt{2})$ *is modular.*

*Proof.* The reducible and irreducible solvable residues are covered by Proposition 6.5, using Lemma 6.8 in the reducible case. It remains to treat nonsolvable residue.

Write $v_+, v_-$ for the two marked seven-places. Use Lemma 15.3 at $v_+$ and retain the actual crystalline weight-two component at $v_-$. Prescribed lifting and potential modularity as in Lemma 2.6(i) give a weight-two comparison system of exact cyclotomic determinant, good away from $v_+$. Remove a ramified scalar quadratic factor at $v_+$ when needed by Lemma 15.1. This preserves good reduction at $v_-$, two and three. The comparison system is now good or quadratic-special or of principal-series inertial order three at $v_+$.

If its three-adic residue is reducible, the actual system is modular by Lemma 15.2. Otherwise that lemma gives irreducibility on the cyclotomic subgroup. In characteristic three retain the old crystalline weight-two component and the unramified condition at $v_-$. At $v_+$ apply Lemma 4.12 to the full residual representation. Lemma 2.6(i) gives the new system. If the prescribed component is special, purity gives nonzero monodromy. This system is modular by Proposition 15.4. Lemma 2.1 supplies the weight-two witness, and Lemma 2.2(ii) gives modularity of the old three-adic member. Untwisting and applying Lemma 2.2(i) to the original seven-adic member proves the assertion.

□

*Completion for $d = 2$ in Theorem 1.1.* Apply Proposition 2.7. In the nonsolvable case use Proposition 6.2 at seven and Proposition 15.8, then return through the congruences and reduce the initial lift. □

## 16 The field $\mathbf{Q}(\sqrt{5})$

Throughout this section put $K = \mathbf{Q}(\sqrt{5})$.

### 16.1 Arithmetic at eleven and two

Put $\phi = (1 + \sqrt{5})/2$, $\pi = 3 + \phi$, and $\pi' = 4 - \phi$. Then $\pi\pi' = 11$ and the primes over eleven are $\mathfrak{p} = (\pi)$ and $\mathfrak{p}' = (\pi')$. The Minkowski bound $\sqrt{5}/2$ gives class number one. The unit group is generated by $-1$ and the norm-minus-one unit $\phi$, so the narrow class number is also one. Two and three are inert, five ramifies, and the dyadic completion is unramified quadratic. At $\mathfrak{p}, \mathfrak{p}'$ the residues of $\phi$ are $8 = 2^3$ and $4 = 2^2$, respectively.

Set $\alpha = \phi\pi = 1 + 4\phi$. Its squareclass gives a character $\kappa_\alpha$ ramified at $\mathfrak{p}$ and unramified at all other finite places. Indeed $(1 + \sqrt{\alpha})/2$ satisfies $X^2 - X - \phi = 0$, and the reduction is irreducible over $\mathbf{F}_4$ since $\mathrm{Tr}_{\mathbf{F}_4/\mathbf{F}_2}(\bar{\phi}) = 1$. The conjugate character has the analogous property at $\mathfrak{p}'$. We use these characters as reversible twists of determinant one.

**Lemma 16.1.** *A quadratic character of $G_K$ unramified outside $2\mathfrak{p}$ and the real places and split at two is trivial.*

*Proof.* By class number one its squareclass is $(-1)^a\phi^b 2^c\pi^d$, with $a, b, c, d \in \{0, 1\}$. A dyadic square has even valuation, so $c = 0$. Its norm from $K_2$ to $\mathbf{Q}_2$ is a square, so $b = d = 0$ because $-1, 11, -11$ are all nonsquares in $\mathbf{Q}_2$. Finally $-1$ remains nonsquare over the unramified quadratic extension of $\mathbf{Q}_2$. The statement for $\mathfrak{p}'$ follows by conjugation. □

## 16.2 A dyadic obstruction with one odd place

**Proposition 16.2.** *Let $\bar{r} : G_K \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ be continuous and finite flat at two. Suppose it is unramified outside $2\mathfrak{p}$ and has inertia of order at most two at $\mathfrak{p}$. Then $\bar{r}^{\mathrm{ss}} = 1 \oplus 1$.*

*Proof.* A nonsolvable image is $\mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^f \geq 4$, by Dickson's classification. The kernel field $M/K$ then has absolute degree at least 120. Fontaine's finite-flat bound over the unramified dyadic base gives classical upper breaks at most one [39, Section 2.1, Theorem 1 and Section 1.2(i)]. The integral of the upper filtration gives normalized dyadic different exponent strictly below two. At the single bad eleven-place the relative tame exponent is at most $1/2$. Thus $\mathrm{rd}(M) < 4\sqrt{5}\,11^{1/4} < 16.289$. The classical lower bound in Section 13 gives $\mathrm{rd}(M) > 16.66$.

If the image is irreducible and solvable, it is odd dihedral. It is induced from an odd-order character of a quadratic field $L/K$. The quadratic quotient is unramified outside $2\mathfrak{p}$. Its dyadic conductor is at most two by Fontaine's bound. A class of odd dyadic valuation gives an Eisenstein quadratic with different exponent three over the unramified base. Therefore $L = K(\sqrt{a})$ for a nontrivial class in $\langle [-1], [\phi], [\pi] \rangle$. The inducing character is unramified above eleven because its inertia there has both odd order and order at most two. At two it is tame, so it factors through the ordinary ray class group modulo (2) of $L$.

Here are the relevant quartic data, with $r_2$ the number of complex pairs. The Minkowski bounds in the last column are strict upper bounds.

| $a$ | $\lvert D_L \rvert$ | $r_2$ | Minkowski bound |
|---|---|---|---|
| $-1$ | 400 | 2 | 3.1 |
| $\phi,\ -\phi$ | 400 | 1 | 2.4 |
| $\pi$ | 4400 | 0 | 6.3 |
| $-\pi$ | 4400 | 2 | 10.2 |
| $\phi\pi$ | 275 | 1 | 2 |
| $-\phi\pi$ | 4400 | 1 | 8 |

For the discriminants, $\{1, \sqrt{a}\}$ is a relative integral basis except in the row $a = \phi\pi$, where $\{1, (1 + \sqrt{a})/2\}$ is an integral basis. At eleven this follows from the valuation-one radicands. At two a unit admits the latter type of basis exactly when it is square modulo four. The $\phi\pi$ row is the only unramified dyadic case, by the discriminant bound or by testing squares modulo four. To see this directly, $-1, \phi, \pi, -\pi, -\phi, -\phi\pi$ fail the test: the norm rules out $\pm\phi$ and $\pm\pi$, whose norms are $-1$ or 11, and the remaining $-1, -\phi\pi$ reduce to $-1$ modulo four.

In the first, second and fifth rows no prime ideal lies below the Minkowski bound. In the $\pi$ row the possible generators have norms four and five. The unique dyadic ideal has square (2), and $N_{L/\mathbf{Q}}(\sqrt{\pi} - 1) = 5$ gives a principal norm-five prime; conjugation over $K$ gives the other when five splits there. In the $-\pi$ row three has residue degree four, since $-\pi$ is nonsquare in $\mathbf{F}_9$ (its norm to $\mathbf{F}_3$ is 11 mod 3). The norm $N_{L/\mathbf{Q}}(\sqrt{-\pi} - \phi) = 20$ identifies the norm-five classes with the dyadic class. In the last row the only possible generating prime is the dyadic ideal: $-\phi\pi$ is nonsquare at five, since $\phi \equiv 3$ and $\pi \equiv 1$ there. Thus all the ordinary class groups have two-power order.

When $L/K$ is dyadically ramified, its involution acts trivially on the residue units. Anti-invariance of the inducing character, which follows from determinant one, makes its odd tame inertia trivial. It is then trivial by the ordinary class calculation. For $L = K(\sqrt{\phi\pi})$ the integral unit $t = (1 + \sqrt{\phi\pi})/2$ reduces to a root of $X^4 + X + 1$ of order fifteen in $\mathbf{F}_{16}^{\times}$. It kills the entire tame ray quotient at two. Again no inducing character survives. The seven class and ray groups are independently checked by the supplied PARI/GP certificate.

In the reducible case the two characters have odd order, hence are unramified at eleven and tame at two. The image of $\phi$ generates $\mathbf{F}_4^{\times}$, so the ray group of $K$ modulo (2) is trivial. □

**Proposition 16.3.** *There is no absolutely irreducible 2-adic representation $r$ of $G_K$ of determinant $\chi_2$ which is crystalline of labelled weights $\{0, 1\}$ at two, unramified outside $2\mathfrak{p}$, and unipotent on inertia at $\mathfrak{p}$. The same holds with $\mathfrak{p}'$ in place of $\mathfrak{p}$.*

*Proof.* A stable lattice at the dyadic place is the Tate module of a Barsotti–Tate group with coefficient action [51, Theorem 4.1]. At $\mathfrak{p}$ its residue has inertia of order at most two, since unipotent inertia factors through a procyclic pro-two quotient. Proposition 16.2 makes the residual semisimplification trivial.

Apply Lemma 4.11 to obtain a nonzero additive residual extension split at two. A nonzero $\mathbf{F}_2$-coordinate of the residual extension now gives a quadratic character unramified outside $2\mathfrak{p}$ and split at two, contrary to Lemma 16.1. □

**Lemma 16.4.** *Let $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ be absolutely irreducible, unramified outside five and eleven. If its inertia image at both eleven-primes has order dividing five, then $\bar{r}|_{G_{K(\zeta_5)}}$ is absolutely irreducible.*

*Proof.* Otherwise Clifford theory gives induction from a prime-to-five character $\theta$ of $H = K(\zeta_5) = \mathbf{Q}(\zeta_5)$. The character is unramified at eleven, since its inertia there has both prime-to-five order and order dividing five. It is unramified away from five, and tame at the unique prime $\mathfrak{P}$ above five. The discriminant of $H$ is 125, and its degree-four Minkowski bound is $3\sqrt{125}/(2\pi^2) < 2$, so its class number is one. The image of $\phi$ modulo $\mathfrak{P}$ is 3, which generates $\mathbf{F}_5^{\times}$. The ray group modulo $\mathfrak{P}$ is therefore trivial. This makes $\theta$ trivial, contrary to the irreducibility of its induction. □

## 16.3 The switch from eleven to five

**Lemma 16.5.** *Every continuous residual representation of $G_{\mathbf{Q}_{11}}$ of determinant $\bar{\chi}_{11}$ has a weight-two lift of determinant $\chi_{11}$ which is scalar potentially crystalline, quadratic-special, or potentially crystalline of principal-series type. After a local quadratic twist the principal-series inertial characters have order five.*

*Proof.* Take a regular crystalline lift with determinant $\chi_{11}$ by Lemma 4.1. Its algebraic type has trivial scalar central character, since the two weights sum to one. Positivity in the fixed-determinant Breuil–Mézard formula [84, Theorem 1.2] gives a nonzero cycle for at least one central-trivial Serre weight of the actual residual representation. Write $W(r, b) = \mathrm{Sym}^r \mathbf{F}_{11}^2 \otimes \det^b$. The central-trivial weights have $0 \le r \le 10$, $b$ modulo ten, and $r + 2b = 0$ modulo ten. The four determinant-one principal-series types with characters of exponents $(t, -t)$ give

| $t$ | constituents of the compact type | |
|---|---|---|
| 1 | $W(2, 9)$ | $W(8, 1)$ |
| 2 | $W(4, 8)$ | $W(6, 2)$ |
| 3 | $W(4, 3)$ | $W(6, 7)$ |
| 4 | $W(2, 4)$ | $W(8, 6)$ |

using the inertial convention of [43]. For $0 < a - b < 10$, the Brauer character of the principal series with exponents $(a, b)$ is the sum of those of $W(a-b, b)$ and $W(10-a+b, a)$. Indeed the two symmetric-power traces sum to the character of the induced representation on split semisimple elements. On nonsplit semisimple elements with eigenvalues $x, x^{11}$ the sum is zero, using $x^{120} = 1$. For the four rows take $(a, b) = (10-t, t)$. The scalar crystalline types give $W(0,0), W(0,5)$ and the scalar semistable types give $W(10,0), W(10,5)$. This accounts for every central-trivial weight. Fixed-determinant positivity yields a point in the corresponding weight-two ring. In a scalar semistable ring, a point of zero monodromy is scalar potentially crystalline. Otherwise take its special component. A quadratic twist exchanges rows one and four, and rows two and three. The even rows have order five. □

**Lemma 16.6.** *Let $\bar{s}$ have determinant $\bar{\chi}_5$, be finite flat above five, and be unramified outside five and $\mathfrak{p}$. Suppose its inertia at $\mathfrak{p}$ has order dividing five. It is not exceptional in the sense of [35, Definition 4.1.1]. The assertion also holds with $\mathfrak{p}'$ in place of $\mathfrak{p}$.*

*Proof.* In the exceptional case the projective kernel field $M$ has group $S_5$ over $K$. The sign quotient cuts out $K(\zeta_5) = \mathbf{Q}(\zeta_5)$, ramified quadratic above five. At the five-adic place $e(K_v/\mathbf{Q}_5) = 2$ and the residue field is $\mathbf{F}_5$. The rank-one finite-flat tame weights in [65, Corollaries 3.3.6 and 3.4.4] lie in $\{0, 1, 2\}$. For a locally reducible representation the two exponents sum to two, so the projective tame ratio has order at most two. In the locally irreducible case the niveau-two exponent is $a + 5b$ with $a + b = 2$. Its projective tame order is three when the two characters are distinct. Here $a = b$ would give scalar inertia and local reducibility.

If wild projective inertia is nontrivial, it is $C_5$, so its decomposition group lies in the normalizer $C_5 \rtimes C_4$. The inertia group is contained in $C_5 \rtimes C_2$ by the tame bound. This subgroup lies in $A_5$, since inversion of a five-cycle is a double transposition. The sign quotient would be unramified above five. Therefore wild projective inertia is trivial and the tame projective inertia has order two, as its sign is nontrivial. The absolute ramification index at five is four, with different contribution 3/4.

At $\mathfrak{p}$ the relative inertia has order at most five, and $\mathfrak{p}'$ is unramified. The degree of $M$ over $\mathbf{Q}$ is 240 and its root discriminant satisfies

$$\mathrm{rd}(M) \le 5^{3/4} 11^{2/5} < 8.726.$$

Section 13 gives $\mathrm{rd}(M) > 16.66$. □

**Proposition 16.7.** *A parallel-weight-two system of determinant cyclotomic, good away from $\mathfrak{p}$, and with one of the local types in Lemma 16.5 at $\mathfrak{p}$, is modular. The same holds with $\mathfrak{p}'$ in place of $\mathfrak{p}$.*

*Proof.* The character $\kappa_\alpha$ of Section 16.1 removes a ramified quadratic scalar type, or the quadratic part of the principal-series characters, without introducing any new conductor. After the twist the system is good, special with unramified quadratic twist, or of principal-series inertial order five at $\mathfrak{p}$.

Take its five-adic member $r_5$, which is crystalline of weight two at the prime over five. The actual residual inertia at $\mathfrak{p}$ is trivial or cyclic unipotent of order five: the principal-series eigencharacters coalesce, and the special case has tame pro-five unipotent monodromy. If the global residue is reducible, its constituents factor through the narrow ray group modulo the prime over five. This group is $C_2$: the local residue units and signs form $\mathbf{F}_5^\times \times \{\pm 1\}^2$, and the units $-1, \phi$ have image of order eight there, with $\phi \bmod 5 = 3$. Thus the constituents are $1, \bar{\chi}_5$. The corresponding rank-one finite-flat models are etale and multiplicative by Raynaud's classification at $e = 2 < 5 - 1$. Closing a stable residual line in the torsion gives such a model. As in Proposition 16.3, connectedness and duality then force the actual source to be

ordinary. Its unramified quotient reduces to one, and the residual ratio $\bar{\chi}_5$ cuts out the abelian extension $\mathbf{Q}(\zeta_5)/K$, is odd and is locally nontrivial. Skinner–Wiles [75, Section 4.5, Theorem A] modularizes the actual source.

In the irreducible case Lemma 16.4 gives irreducibility over $K(\zeta_5)$.

Lemma 16.6 excludes the characteristic-five exception. At $\mathfrak{p}$ apply Lemma 4.12 with $(q, p) = (11, 5)$. Keep the old crystalline weight-two component at five and good conditions away from $\mathfrak{p}$. Lemma 2.6(i) gives a weight-two system. Purity preserves the special condition when it was imposed. Its dyadic member contradicts Proposition 16.3. This excludes the irreducible branch for the original normalized system. Undoing the twist proves the assertion. □

### 16.4 The terminal eleven-adic source

*Completion for $d = 5$ in Theorem 1.1.* Start with Proposition 2.7, including the exceptional initial lift at five when required. In the nonsolvable case apply Proposition 6.2 with $\ell = 11$ and $v_+ = \mathfrak{p}$. If a terminal source remains, its reducible and irreducible solvable residues are covered by Proposition 6.5, using Lemma 6.8 for the constituent quotient.

For nonsolvable residue, retain the crystalline weight-two component at $\mathfrak{p}'$ and choose a weight-two lift at $\mathfrak{p}$ from Lemma 16.5. Lemma 2.6(i) gives a comparison system good away from $\mathfrak{p}$. Proposition 16.7 proves its modularity. The common residual witness gives modularity of the old eleven-adic member by Proposition 6.5. The backward implications and reduction at the original place complete the proof. □

*Remark* 16.8. The residual character calculation allows a nonzero global extension with ratio $\bar{\chi}_{11}$ that is split at $\mathfrak{p}'$. For example, take the Kummer class of $\pi = 3 + \phi$. It is unramified away from eleven and nonzero by its valuation at $\mathfrak{p}$. At $\mathfrak{p}'$ one has $\phi \equiv 37 \pmod{121}$ by Hensel's lemma, and $\pi^{10} \equiv 40^{10} \equiv 1 \pmod{121}$. A unit of $\mathbf{Q}_{11}$ is an eleventh power exactly when its tenth power is one modulo 121, so the local class is zero. Thus the global residual character calculation permits local splitting. The mixed theorem used above allows the actual ordinary source to be split.

## 17 The field $\mathbf{Q}(\sqrt{7})$

Put

$$K = \mathbf{Q}(r), \quad r^2 = 7, \qquad \pi_2 = 3 - r, \quad \pi_3 = 2 + r, \quad \mathfrak{p}_2 = (\pi_2), \quad v = (\pi_3), \quad F = K(i).$$

Write $v'$ for the other prime above three. Thus $K_v = K_{v'} = \mathbf{Q}_3$ and $K_{\mathfrak{p}_2} \simeq \mathbf{Q}_2(i)$. For the dyadic completion put $\varpi = 1 + i$ and $U_j = 1 + \varpi^j \mathcal{O}_{K_{\mathfrak{p}_2}}$.

**Lemma 17.1.** *The ordinary class number of $K$ is one and its narrow class number is two. Its units are $\{\pm(8 + 3r)^n : n \in \mathbf{Z}\}$. The extension $F = K(i)$ is the unique nontrivial quadratic extension unramified at every finite place, and is inert at $v$ and $v'$. Moreover,*

$$\mathrm{Cl}_{v\infty}(K) \simeq C_2, \qquad \mathrm{Cl}_{3\infty}(K) \simeq C_2 \times C_2.$$

*Every finite character $G_K \to \overline{\mathbf{F}}_3^{\times}$ unramified outside three therefore has square one.*

*Proof.* The Minkowski bound is $\sqrt{7} < 3$. The prime above two is principal, generated by $3 + r$ of norm two, so the ordinary class number is one. The equation $x^2 - 7y^2 = -1$ has no solution modulo seven. The least positive solution of $x^2 - 7y^2 = 1$ with $y > 0$ is $(8, 3)$. Thus $8 + 3r$ generates the positive units and the narrow class number is two.

The discriminant of the biquadratic field $F$ is $28(-4)(-7) = 28^2$, so its relative discriminant over $K$ is trivial. Narrow class field theory gives uniqueness, and the extension is inert at $v$ and $v'$ because $-1$ is nonsquare in $\mathbf{F}_3$.

For the ray groups, use the exact sequence of units, residues and signs, together with $h_K = 1$. The group $(\mathcal{O}_K/3)^\times \times \{\pm 1\}^2$ has order sixteen and exponent two. The image of the units has order four: $-1$ has negative signs at both embeddings and residues $(-1,-1)$, whereas $8+3r$ has positive signs and the same residues. The quotient has order four and exponent two. For a single prime above three, the quotient has order two. A character valued in $\overline{\mathbf{F}}_3^\times$ has order prime to three, so its finite conductor at three is tame and divides 3. It factors through $\mathrm{Cl}_{3\infty}(K)$, proving the final assertion. □

## 17.1 Scalar dyadic sources

**Theorem 17.2.** *Let $E/\mathbf{Q}_2$ be a finite extension and let $\rho : G_K \to \mathrm{GL}_2(E)$ be continuous, absolutely irreducible and unramified outside a finite set. Suppose that:*

(i) $\det \rho = \chi_2$ *and* $\overline{\rho}^{\mathrm{ss}} = 1 \oplus 1$*;*
(ii) $\rho$ *is Barsotti–Tate at* $\mathfrak{p}_2$*;*
(iii) *inertia at every odd prime other than $v$ acts by scalars;*
(iv) $\rho|_{G_{K_v}}$ *has an invariant line after a finite extension of coefficients.*

*Then $\rho$ is ordinary at $\mathfrak{p}_2$ and*

$$\rho \simeq \mathrm{Ind}_{G_F}^{G_K} \theta$$

*for a geometric character $\theta$ of $G_F$. In particular, the monodromy of $\rho$ at $v$ is zero.*

**Corollary 17.3.** *A source satisfying Theorem 17.2 cannot be special at $v$, even after a local quadratic twist. If instead it has scalar inertia at every odd prime, it is induced from an algebraic Hecke character of $F$ and is modular.*

*Proof.* In the special case nonzero monodromy cannot disappear after a finite extension. In the scalar case an eigenline of Frobenius is preserved by the local decomposition group at $v$, so the theorem applies. The inducing character is de Rham at the dyadic places and unramified at almost all places. Class field theory identifies it with an algebraic Hecke character, and automorphic induction from the CM field $F$ gives modularity. □

## 17.2 Two injections of quadratic classes

### 17.2.1 Quadratic classes over $K$

Put $\epsilon = -8 - 3r$. Lemma 17.1 gives the class number and unit group of $K$.

Since $-7$ is a square in $\mathbf{Q}_2$, we have $K_{\mathfrak{p}_2} \simeq \mathbf{Q}_2(i)$. Moreover,

$$\epsilon = -2\left(\frac{3+r}{2}\right)^2. \tag{21}$$

In particular, $K(\sqrt{\epsilon}) = K(\sqrt{-2})$. The quadratic classes unramified outside $\{\mathfrak{p}_2, v\}$ and the real places form

$$W_K = \langle -1, \epsilon, \pi_2, \pi_3 \rangle \subset K^\times/K^{\times 2}.$$

**Lemma 17.4.** *The classes in $W_K$ whose localization at $\mathfrak{p}_2$ belongs to the image of $U_3$ are exactly*

$$C_K = \langle -1, \epsilon\pi_3 \rangle.$$

*The restriction $C_K \to K_v^\times/K_v^{\times 2}$ is injective.*

*Proof.* An odd power of $\pi_2$ gives odd valuation and does not belong to the image of $U_3$. The class of $-1$ is trivial in $\mathbf{Q}_2(i)$. By (21), the local class of $\epsilon$ is that of 2, which equals that of $i$, since $(1+i)^2 = 2i$. The extension $\mathbf{Q}_2(\zeta_8)/\mathbf{Q}_2(i)$ has conductor exponent four: the absolute discriminant exponents are eight and two, respectively, and the relative conductor–discriminant formula gives $8 - 2 \cdot 2 = 4$. Lemma 11.4 gives exponent at most two for classes in $U_3$. Thus this class is not in the image of $U_3$. On the other hand,

$$\epsilon\pi_3 - 1 = -80 + 14\pi_2, \qquad v_{\mathfrak{p}_2}(-80 + 14\pi_2) = 3,$$

since the two summands have valuations eight and three. Thus $\epsilon\pi_3$ belongs to $U_3$. Since the image of $U_3$ is a subgroup, this determines $C_K$ exactly. At $v$, the class of $-1$ is the nontrivial unramified quadratic class of $\mathbf{Q}_3$. The class $\epsilon\pi_3$ has odd valuation and gives a ramified extension. They are independent. □

### 17.2.2 Quadratic classes over $F$

The two dyadic completions of $F$ are $\mathbf{Q}_2(i)$. Its absolute discriminant is

$$\operatorname{disc}(F) = 28 \cdot (-4) \cdot (-7) = 784.$$

The elements

$$a = \frac{2 + r + i}{2}, \qquad b = \frac{2 + r - i}{2}$$

are algebraic integers of absolute norm two and satisfy $ab = 3 + r$. They generate the two dyadic primes $\mathfrak{P}_1, \mathfrak{P}_2$. Both satisfy the monic polynomial $X^4 - 4X^3 + 3X^2 + 2X + 2$. The two primes of $K$ above three are inert in $F/K$, because $-1$ is not a square in $\mathbf{F}_3$.

The Minkowski bound for $F$ is

$$\frac{4!}{4^4}\left(\frac{4}{\pi}\right)^2 \sqrt{784} = \frac{42}{\pi^2} < 5.$$

The only primes of norm less than five are the two principal dyadic primes. There are no primes of norm three or four. Hence $h_F = 1$.

The element

$$\eta = \frac{(1+i)(3+r)}{2}$$

is a unit and satisfies $\eta^2 = i(8 + 3r)$ and $N_{F/K}(\eta) = 8 + 3r$. Every unit of $F$ has totally positive norm in $K$. Dividing it by a power of $\eta$ leaves a unit of norm one, all of whose complex absolute values are one; it is a root of unity. The roots of unity in $F$ are $\mu_4$. Consequently,

$$\mathcal{O}_F^\times = \mu_4 \times \langle \eta \rangle.$$

If $\mathfrak{P}_v$ is the prime above $v$, we have

$$W_F = \langle i, \eta, a, b, \pi_3 \rangle$$

for the quadratic classes unramified outside $\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_v\}$. A class with conductor at most four at both dyadic primes cannot have odd valuation at either of them. It therefore belongs to the subspace

$$C_F = \langle i, \eta, \pi_3 \rangle.$$

**Lemma 17.5.** *Localization of $C_F$ at either dyadic prime is injective.*

*Proof.* It suffices to choose one prime; the other is obtained by conjugation, which preserves $C_F$. Identify the completion with $L = \mathbf{Q}_2(i)$ and write $r = it$, where $t^2 = -7$ and $t \equiv 5 \pmod{16}$. This choice is possible: choose the sign of the root with $t \equiv 5 \pmod 8$, and the equation modulo 32 forces the congruence modulo 16. Then

$$\eta = \frac{3 - t + i(3 + t)}{2} \equiv -1 + 4i \pmod 8.$$

Thus $-\eta \equiv 5 \pmod{(1+i)^5}$. By Hensel's lemma, $U_5 \subset L^{\times 2}$; since $-1$ is a square, $[\eta] = [5]$ in $L^\times / L^{\times 2}$.

The class [5] defines the unramified quadratic extension of $L$, whereas $[i]$ defines the ramified extension $\mathbf{Q}_2(\zeta_8)/L$. They are independent. Finally, using the *local norm*,

$$N_{L/\mathbf{Q}_2}(i) = N_{L/\mathbf{Q}_2}(\eta) = 1, \qquad N_{L/\mathbf{Q}_2}(\pi_3) = 4 + t^2 = -3.$$

Since $-3 \equiv 5 \pmod 8$ is not a square in $\mathbf{Q}_2$, no class with an odd power of $\pi_3$ can be a product of the other two modulo squares. The three classes are independent. □

## 17.3 Proof of the scalar theorem

*Proof of Theorem 17.2.* We may enlarge the coefficients whenever necessary. Suppose first that $\rho$ is not ordinary at $\mathfrak{p}_2$. The local line at $v$ and Lemma 4.10 give a lattice whose reduction is a nonsplit extension and is trivial on $G_{K_v}$. Extract the actual two-dimensional $\mathbf{F}_2$-coordinate subquotient of Lemma 4.10. At every odd prime other than $v$, the original inertia is scalar; its determinant is one, so the scalar is $\pm 1$. It acts trivially on every reduction modulo $\lambda$. The resulting quadratic class therefore belongs to $W_K$.

By Lemmas 11.2 and 11.3, at $\mathfrak{p}_2$ this order-four subquotient is finite flat and is an extension of $J$ by $J$. Lemma 11.4 places its class in the image of $U_3$. It therefore belongs to $C_K$. But it is trivial on $G_{K_v}$, contradicting the injectivity in Lemma 17.4. This proves that $\rho$ is ordinary.

Suppose now that $\rho|_{G_F}$ is absolutely irreducible. Its ordinary line at $\mathfrak{P}_1$ allows us to apply Lemma 4.10 again, this time with $G = G_F$ and $D = G_{F_{\mathfrak{P}_1}}$. A nonzero coordinate of the resulting residual extension splits at $\mathfrak{P}_1$ and belongs to $W_F$. The subquotient realizing it is finite flat and killed by two at both dyadic primes. By Lemma 11.4, its conductor at each is at most four. It cannot have odd valuation at either prime, so its class belongs to $C_F$. The injectivity in Lemma 17.5 gives another contradiction.

Consequently, after enlarging $E$, the restriction $\rho|_{G_F}$ is reducible. Since $F/K$ is normal quadratic and $\rho$ is absolutely irreducible, Clifford theory gives $\rho = \operatorname{Ind}_{G_F}^{G_K} \theta$. More explicitly, a $G_F$-stable line and its conjugate under the other coset are distinct, and their sum is the whole representation. The restriction to $G_F$ is a direct sum of characters.

Restriction to a subgroup of finite index does not kill nonzero monodromy: in a special representation, the unipotent image of inertia remains infinite upon restriction to an open subgroup. A sum of characters cannot have this unipotent image. Thus the monodromy at $v$ is zero. □

## 17.4 Solvable residues and discriminant bounds

**Lemma 17.6.** *Let $\bar{s} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ be a continuous representation of determinant one, finite flat at $\mathfrak{p}_2$, unramified outside $\{2, v\}$ and with inertia of order at most two at $v$. If its image is solvable, then $\bar{s}^{\mathrm{ss}} = 1 \oplus 1$.*

*Proof.* Lemma 11.16a, with $S = \{v\}$, reduces the reducible case to odd dyadic ray characters. The residue field at $\mathfrak{p}_2$ is $\mathbf{F}_2$ and $h_K = 1$, so they are trivial. In the irreducible case let $M/K$

be the quadratic inducing field provided by that lemma. Its dyadic conductor is at most four by Fontaine's bound, since $M$ is contained in the residual field. A class with odd valuation at $\mathfrak{p}_2$ would have conductor five. Moreover, $M$ is unramified at $v'$. Thus

$$M = K(\sqrt{d}), \qquad [d] \in \langle -1, -2, \pi_3 \rangle \setminus \{1\}.$$

Only the seven fields in the table need be considered. The final column records the odd part of the class group. Section 17.7 proves its triviality by Minkowski bounds and explicit principal ideals. The discriminants listed are field discriminants, not polynomial discriminants.

| $d$ | Absolute polynomial | $\operatorname{disc}(M)$ | $\operatorname{Cl}(M)_{\mathrm{odd}}$ |
|---|---|---|---|
| $-1$ | $X^4 - 12X^2 + 64$ | 784 | 1 |
| $-2$ | $X^4 - 10X^2 + 81$ | 12544 | 1 |
| 2 | $X^4 - 18X^2 + 25$ | 12544 | 1 |
| $\pi_3$ | $X^4 - 4X^2 - 3$ | $-37632$ | 1 |
| $-\pi_3$ | $X^4 + 4X^2 - 3$ | $-37632$ | 1 |
| $2\pi_3$ | $X^4 - 8X^2 - 12$ | $-9408$ | 1 |
| $-2\pi_3$ | $X^4 + 8X^2 - 12$ | $-9408$ | 1 |

All their dyadic primes have residue degree one, so the odd dyadic ray quotient is the odd class-group quotient. The table therefore excludes the inducing character.

For the three biquadratic fields, the odd part is determined by the quadratic subfields. For the other four fields, the explicit norm witnesses in Section 17.7 make every prime ideal within the Minkowski bound principal. The dyadic residue degrees are also proved there.

□

**Proposition 17.7.** *Let $\bar{s} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ have determinant one, be finite flat at $\mathfrak{p}_2$, and have nonsolvable image. Let $M$ be its residual field. If $\bar{s}$ is unramified outside two, no such representation exists. More precisely, if its image is $\mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^a \geq 4$, the dyadic contribution gives*

$$[M : \mathbf{Q}] = 2q(q^2 - 1), \qquad \operatorname{rd}(M) \leq 8\sqrt{7}\, 2^{-2/q}$$

*in the absence of odd ramification. If inertia of order at most two is allowed at $v$, the bound is multiplied by $3^{1/4}$.*

*Proof.* Dickson's classification, together with determinant one, identifies the image with a conjugate of $\mathrm{SL}_2(\mathbf{F}_q)$. Consider the decomposition group at $\mathfrak{p}_2$. If wild inertia $P$ is nontrivial, it fixes a unique line, which the whole decomposition group preserves. Its two diagonal characters have odd order and hence are unramified: $\mathbf{Q}_2(i)$ has residue field $\mathbf{F}_2$. Thus the whole inertia group is an elementary unipotent group $P$, with $|P| \leq q$.

In classical upper numbering, Fontaine's bound gives a maximum break of at most three. The different formula implies

$$\frac{d}{e} \leq 4\left(1 - \frac{1}{|P|}\right) \leq 4\left(1 - \frac{1}{q}\right).$$

If wild inertia is trivial, tame inertia has order one or three. Indeed, in the reducible case the diagonal characters are unramified; in the irreducible case Frobenius interchanges two characters $\alpha, \alpha^2$, and determinant one forces $\alpha^3 = 1$. Its contribution is smaller still.

Since $e(K_{\mathfrak{p}_2}/\mathbf{Q}_2) = 2$, we obtain

$$\operatorname{rd}(M) \leq \sqrt{28}\, 2^{2-2/q} = 8\sqrt{7}\, 2^{-2/q}.$$

Ramification of order two at the single prime $v$ is tame and contributes $N(v)^{(1-1/2)/[K:\mathbf{Q}]} = 3^{1/4}$.

The case without odd ramification is excluded by the three unconditional comparisons for $d = 7$ in Proposition 13.10.

□

## 17.5 Cyclotomic irreducibility in characteristic three

**Lemma 17.8.** *Let $K = \mathbf{Q}(\sqrt{7})$ and let $\bar{s} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ be a continuous, absolutely irreducible representation, unramified at every finite place not dividing three, with determinant $\omega_3$. Then $\bar{s}|_{G_{K(\zeta_3)}}$ is absolutely irreducible.*

*Proof.* Put $M = K(\sqrt{-3})$ and let $c$ be the nontrivial automorphism of $M/K$. If the restriction were reducible, Clifford theory would give

$$\bar{s} \simeq \mathrm{Ind}_{G_M}^{G_K} \theta, \qquad \theta^c \neq \theta.$$

The character $\theta$ has finite image of order prime to three. It is unramified outside the places of $M$ above three.

At each place $v \mid 3$ of $K$, we have $K_v = \mathbf{Q}_3$ and $M_w = \mathbf{Q}_3(\zeta_3)$, a totally and tamely ramified quadratic extension. The character $\theta$ kills wild inertia. Let $\tau$ lift a topological generator of the tame inertia of $K_v$. Then $\tau \notin G_{M_w}$, $\omega_3(\tau) = -1$, and the matrix of the induction satisfies

$$\det \bar{s}(\tau) = -\theta(\tau^2).$$

The prescribed determinant forces $\theta(\tau^2) = 1$. Since $\tau^2$ generates the tame inertia of $M_w$ and wild inertia already acts trivially, $\theta$ is unramified at $w$. This holds at both places above three. Thus $\theta$ is a character of $\mathrm{Cl}(M)$.

We check that $\mathrm{Cl}(M) \simeq C_2$ without GRH. The three quadratic subfields have class numbers $1, 1, 4$, respectively:

$$h(\mathbf{Q}(\sqrt{7})) = 1, \qquad h(\mathbf{Q}(\sqrt{-3})) = 1, \qquad h(\mathbf{Q}(\sqrt{-21})) = 4.$$

The first two values follow from Minkowski's theorem; in the first case the only required prime, of norm two, is principal. For discriminant $-84$, the reduced primitive forms are $(1, 0, 21)$, $(2, 2, 11)$, $(3, 0, 7)$ and $(5, 4, 5)$. Kuroda's formula [54, formula (1.5)] gives $h(M) = 2q(M)$, where

$$q(M) = [\mathcal{O}_M^\times : \mu_6\langle\varepsilon_+\rangle], \qquad \varepsilon_+ = 8 + 3\sqrt{7}.$$

Here $\mu(M) = \mu_6$ and $\varepsilon_+$ is the fundamental unit of $K$. For a unit $u$ of $M$, the quotient $u/c(u)$ is a root of unity and $u\,c(u)$ is a totally positive unit of $K$. Thus $u^2 = \zeta\varepsilon_+^n$, with $\zeta \in \mu_6$. If $n$ were odd, either $\varepsilon_+$ or $-\varepsilon_+$ would be a square in $M$, since the elements of $\mu_3$ are squares. But

$$\varepsilon_+ = 2\left(\frac{3+\sqrt{7}}{2}\right)^2, \qquad M^{\times 2} \cap K^\times = K^{\times 2} \cup (-3)K^{\times 2}.$$

This would force one of $2, -2, 6, -6$ to be a square in $K$. A rational number is a square in $K$ only if it belongs to $\mathbf{Q}^{\times 2} \cup 7\mathbf{Q}^{\times 2}$, which excludes these four values. Hence $n$ is even and $u \in \mu_6\langle\varepsilon_+\rangle$. It follows that $q(M) = 1$ and $h(M) = 2$.

Every automorphism of a group of order two is trivial. Consequently, $c$ acts trivially on $\mathrm{Cl}(M)$ and $\theta^c = \theta$, a contradiction. □

## 17.6 The congruence chain

The congruence chain uses the prescribed lifts and compatible systems of Lemma 2.6, the conductor reduction of Section 6, and Theorem 6.4 for reducible residues. Lemma 17.8 excludes the bad dihedral residues that could remain in characteristic three. The comparison systems are strictly compatible and absolutely irreducible, by Lemma 2.6 in the odd characteristic steps and by the dyadic construction below. In particular, their dyadic members satisfy the irreducibility required by Theorem 17.2. We retain the quadratic local characters introduced by weight reduction in characteristic three.

Write $\eta_3$ for the ramified quadratic character of $G_{\mathbf{Q}_3}$ whose reduction is $\omega_3$; we may take the character of $\mathbf{Q}_3(\sqrt{-3})$. For $a \in \{0,1\}$, write $\mathrm{CR}(a)$ for a weight-two Barsotti–Tate type twisted by $\eta_3^a$, and $\mathrm{SP}(a)$ for a weight-two special type twisted by $\eta_3^a$. In the second case the monodromy is nonzero. Possible unramified quadratic signs are included in this notation. These twists do not change the cyclotomic determinant. We write $C(2)$ for the tame cuspidal inertial type whose two characters on inertia are the order-four characters $i$ and $-i$ of the unramified quadratic extension of $\mathbf{Q}_3$, interchanged by Frobenius. Its determinant on inertia is one.

**Lemma 17.9** (The local menu in characteristic three)**.** *Let $\bar{s} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ be a representation of determinant $\omega_3$. Then it has a weight-two lift with determinant exactly cyclotomic of one of the types*

$$\mathrm{CR}(0), \quad \mathrm{CR}(1), \quad \mathrm{SP}(0), \quad \mathrm{SP}(1).$$

*If $\bar{s}$ comes from a potentially Barsotti–Tate lift of cuspidal type $C(2)$, one of the two types $\mathrm{CR}(a)$ suffices.*

*Proof.* Apply the quadratic twist construction of Lemma 4.23, recording the twist by $\eta_3^a$, to obtain the four stated types. For the second assertion, write $\sigma(r;b) = \mathrm{Sym}^r \overline{\mathbf{F}}_3^2 \otimes \det^b$, with $0 \le r \le 2$ and $b \in \{0,1\}$. The cuspidal type $C(2)$ has reduction

$$\overline{\sigma(C(2))}^{\mathrm{ss}} = \sigma(0;0) \oplus \sigma(0;1) = \mathbf{1} \oplus \det .$$

These are two distinct characters of $\mathrm{GL}_2(\mathbf{F}_3)$. The Breuil–Mézard formula for potentially Barsotti–Tate types, valid for $p > 2$ [43, Corollaries 4.5.6 and 4.5.9], shows that at least one of these weights belongs to the weight set of the residue. This gives one of the types $\mathrm{CR}(a)$, as asserted. □

*Remark* 17.10. The last assertion cannot be replaced by the existence of an untwisted crystalline lift. For example, an irreducible residual representation with inertia $\omega_2^5 \oplus \omega_2^7$ has determinant $\omega_3$, but is not finite flat over $\mathbf{Q}_3$. It does have a potentially Barsotti–Tate lift of type $C(2)$: induce from the unramified quadratic extension a crystalline character with labelled weights $(0,1)$ and reduction $\omega_2^3$, multiplied by a finite character of order four with reduction $\omega_2^2$.

**Lemma 17.11** (The local switch in characteristic two)**.** *Let $\bar{s}$ be the reduction modulo two of a special representation over $\mathbf{Q}_3$ with cyclotomic determinant, possibly twisted by a quadratic character. There is a lift of $\bar{s}$ with the same determinant and cuspidal type $C(2)$. The assertion preserves the full residual extension, including the values of unramified Frobenius.*

*Proof.* Wild inertia is trivial. In a suitable basis, a generator $\tau$ of pro-two inertia and an arithmetic Frobenius $\Phi$ act by

$$\bar{T} = I + xN, \qquad \bar{F} = I + yN, \qquad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with $x, y$ in a finite field of characteristic two. This includes $x = 0$ and the case where both values are zero.

Enlarge the coefficient field. If $x \neq 0$, choose a unit $u$ reducing to $x$. If $x = 0$, first choose a coefficient extension of absolute ramification index greater than two and then take $u$ to be a uniformizer, so that $v(2) > 2v(u)$. Let $w$ lift $y$. The polynomial

$$X^2 - \frac{2w}{u}X + \frac{2w^2}{u^2} + 3$$

is monic and integral and reduces to $(X+1)^2$. After a further finite extension it has an integral root $b$, which necessarily reduces to one. Define

$$T = \begin{pmatrix} 1 & u \\ -2/u & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} b & w \\ 2w/u^2 - 2b/u & -b \end{pmatrix}.$$

Both matrices are integral and reduce exactly to $\bar{T}, \bar{F}$. A direct calculation gives

$$T^2 = -I, \qquad F^2 = -3I, \qquad TF = -FT, \qquad \det T = 1, \quad \det F = 3.$$

In particular, $FTF^{-1} = T^3$, which is the relation in the tame Galois group of $\mathbf{Q}_3$. The integral matrix $F$ generates a compact procyclic group, so its powers extend continuously from $\mathbf{Z}$ to $\widehat{\mathbf{Z}}$. Together with the finite-order matrix $T$ and the tame relation, this defines a continuous representation of the tame quotient. Its determinant is cyclotomic. Inertia has characters $i, -i$, interchanged by Frobenius; these are the order-four characters of the unramified quadratic extension. Hence its type is $C(2)$. □

**Lemma 17.12** (A nontrivial real conjugation)**.** *Let $\rho$ be a dyadic source of weight two and determinant $\chi_2$, Barsotti–Tate at $\mathfrak{p}_2$, good away from $v$ and of type* SP$(a)$ *there. If its residue is nonsolvable, some real conjugation acts nontrivially on it.*

*Proof.* The residual field $M/K$ has group $\mathrm{SL}_2(\mathbf{F}_q)$, where $q \geq 4$ is a power of two, and absolute degree $2q(q^2-1) \geq 120$. The scalar quadratic factor at $v$ disappears upon reduction modulo two; residual inertia has order at most two. The dyadic bound in Proposition 17.7 gives

$$\mathrm{rd}(M) \leq 8\sqrt{7}\, 2^{-2/q}\, 3^{1/4} < 28.$$

If every real conjugation acted trivially, $M$ would be totally real. Lemma 13.13 gives a root discriminant greater than 42.81537 for a totally real field of degree at least 120, a contradiction. □

### 17.6.1 Preparation at the prime three

**Proposition 17.13.** *To prove modularity over $K$ in every odd residual characteristic, it suffices to modularize actual 3-adic members $r$ with determinant $\chi_3$, good outside three, crystalline of weight two at $v'$, and absolutely irreducible and regular de Rham at $v$.*

*Proof.* The conductor reduction in Section 6 and Lemma 6.1 apply with $\ell = 3$. Starting from Proposition 2.7, we use this protected preparation at three, retaining a protective split prime $q > 7$ until the last step.

The universal wild type of Theorem 4.15 is constructed for every odd prime, including three. Its compact type has reduction $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(\mathbf{F}_3)} \mathbf{1}$, which contains all weights with trivial central character. Lift existence uses the Gee–Kisin result for $p > 2$. In the local parameter, two distinct wild characters of order three are interchanged by quadratic inertia. Thus inertia remains irreducible after reduction in every characteristic $q > 3$. The protective prime may be chosen split and greater than seven. This preserves adequacy during preparation, protection in

characteristic $q$, transport of the weight-two label and final removal of the protective prime. None of these steps uses GRH.

More explicitly, while the protective place $\mathfrak{q}$ remains, prescribe the wild potentially Barsotti–Tate type at $v$ and a regular crystalline potentially diagonalizable lift at $v'$. The first local representation is potentially diagonalizable by [43, Lemma 4.4.1]; the protective type gives adequate cyclotomic residue in characteristic three. Apply Lemma 2.6(ii), with backward modularity supplied by [84, Theorem 4.7]. Enlarge the coefficient field to contain $K$. Choose a $q$-adic realization placing the weight-two label at the conjugate prime $\mathfrak{q}'$, where the system is good. Retain that Barsotti–Tate component and prescribe a crystalline potentially diagonalizable component at $\mathfrak{q}$. The retained wild inertia at $v$ is irreducible modulo $q$ and lies in $G_{K(\zeta_q)}$, since the cyclotomic extension is unramified at $v$. A solvable residue already modularizes the old source by its solvable Artin witness and Tung's theorem; otherwise $q > 7$ gives adequacy, and the same globalization and backward lifting results remove $\mathfrak{q}$. Finally choose a 3-adic realization placing the preserved weight-two label at $v'$. Coefficient conjugation preserves the good Weil–Deligne condition at each fixed finite place, so the resulting source has exactly the required local conditions. □

At the prepared endpoint, treat the residue of the actual member $r$ before constructing a comparison system:

1. If $\bar{r}$ is reducible, its characters have order prime to three and conductor dividing $3\infty$. The corresponding ray class group is $C_2 \times C_2$; thus both characters are quadratic and their ratio is $\omega_3$. At $v'$ the actual member is ordinary, since a nonordinary crystalline representation of weight two over $\mathbf{Q}_3$ has irreducible local residue. At $v$ the actual member is absolutely irreducible. Theorem 6.4, namely Theorem A of [31], applies: $\omega_3$ extends to $G_{\mathbf{Q}}$ and is nontrivial at both places above three.
2. If $\bar{r}$ is irreducible, Lemma 17.8 ensures that its cyclotomic restriction is irreducible. It suffices to construct a modular system with that residue: Lemma 2.2(i), using Tung's theorem [84, Theorem 4.7], gives the backward implication for the actual member $r$.

In the first branch the theorem is applied directly to the actual source; prescribed global lifts are constructed in the second branch, after cyclotomic irreducibility has been checked.

### 17.6.2 Completion of the proof

**Theorem 17.14.** *Every continuous, irreducible, totally odd representation $\bar{r} : G_{\mathbf{Q}(\sqrt{7})} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, with $p$ odd, is modular. No hypothesis on Dedekind zeta zeros is required.*

*Proof.* Proposition 17.13 and the preceding residual case distinction reduce the assertion to the irreducible residual case of the endpoint $r$. Apply Lemma 17.9 at $v$ and retain the actual Barsotti–Tate component at $v'$. The prescribed-lift theorem [35, Theorem 1.1.1, equivalent to Theorem 4.2.1], and the compatible system construction Lemma 2.6(i), give a weight-two system $S_1$, good at two and at $v'$, of type $\mathrm{CR}(a)$ or $\mathrm{SP}(a)$ at $v$. Its determinant is exactly cyclotomic.

Let $\rho_1$ be its dyadic member. If the type at $v$ is $\mathrm{CR}(a)$, inertia at $v$ is scalar and disappears modulo two. By Proposition 17.7, the residue cannot be nonsolvable; by Lemma 17.6, its semisimplification is trivial. The local representation at $v$ has a line, since inertia is scalar and one may choose an eigenline of Frobenius. Theorem 17.2 makes $\rho_1$ induced from $K(i)$, and hence modular.

Suppose the type is $\mathrm{SP}(a)$. If the residue is solvable, Lemma 17.6 and Theorem 17.2 contradict nonzero monodromy. It remains to consider nonsolvable dyadic residue.

Lemma 17.11 supplies a local point of type $C(2)$ at $v$ with the same residue. At the unique

dyadic place, retain a Barsotti–Tate component containing the actual source $\rho_1$. At the other places retain the unramified condition. Prescribed lifting in Lemma 2.6(iii) applies: the residue is nonsolvable, and the dyadic place need not be unramified in $K$. It gives an actual lift $\rho_2$. For its avoiding potential modularity, use [83, Proposition 6.7] and Lemma 17.12, which preserves a nontrivial residual real conjugation. The actual-source transfer, with the dual convention as in Lemma 2.4, then gives potential modularity. The Brauer construction in Lemma 2.6(iii) supplies its system $S_2$. If $S_2$ is modular, the same actual-source transfer applies to $\rho_1$; no additional residual ordinary ordering is required.

Now consider the actual 3-adic member of $S_2$. At $v$ it has cuspidal type $C(2)$ and is absolutely irreducible; at $v'$ it is Barsotti–Tate. If its residue is reducible, apply Theorem 6.4 directly, with residual ratio $\omega_3$, as before. If it is irreducible, Lemma 17.8 guarantees irreducibility over $K(\zeta_3)$: this member has determinant $\chi_3$ and is unramified outside three.

In the remaining case retain the Barsotti–Tate component at $v'$, and use Lemma 17.9 to choose at $v$ a lift of type $\mathrm{CR}(b)$, $b \in \{0, 1\}$. The prescribed-lift theorem and the compatible system construction give a weight-two system $S_3$, good away from $v$, with this scalar type at $v$. Modularity of $S_3$ implies modularity of $S_2$ by [84, Theorem 4.7].

The dyadic member of $S_3$ has residue unramified outside two: the scalar quadratic character at $v$ disappears upon reduction. Proposition 17.7 and Lemma 17.6 force trivial semisimplification. Since the local representation at $v$ has a line, Theorem 17.2 proves that this member is of CM type and is modular. Modularity propagates through the three preceding implications and through the initial preparation. □

## 17.7 The seven quadratic inducing fields

We prove the class group information used in Lemma 17.6. Only triviality of the odd part is needed; the four fields that are not biquadratic in fact have class number one.

### 17.7.1 The three biquadratic fields

The quadratic subfields of $K(i)$, $K(\sqrt{-2})$ and $K(\sqrt{2})$ have class numbers, respectively,

$$(1, 1, 1), \qquad (1, 1, 4), \qquad (1, 1, 1).$$

Here the only value not equal to one is $h(\mathbf{Q}(\sqrt{-14})) = 4$: the reduced primitive forms of discriminant $-56$ are

$$(1, 0, 14), \quad (2, 0, 7), \quad (3, -2, 5), \quad (3, 2, 5).$$

The imaginary fields $\mathbf{Q}(i)$, $\mathbf{Q}(\sqrt{-2})$ and $\mathbf{Q}(\sqrt{-7})$ have class number one by their Minkowski bounds. The same holds for $\mathbf{Q}(\sqrt{2})$. For $\mathbf{Q}(\sqrt{14})$, the bound is $\sqrt{14} < 4$; the prime of norm two is generated by $4 + \sqrt{14}$ and three is inert. The class number of $K$ was proved above.

Let $B/\mathbf{Q}$ be one of these biquadratic fields, with nontrivial automorphisms $\sigma, \tau, \sigma\tau$, and let $c$ be an odd-order ideal class of $B$. Its norm to each quadratic subfield is trivial. Extending these norm classes back to $B$ gives

$$(1 + \sigma)c = (1 + \tau)c = (1 + \sigma\tau)c = 0$$

in additive notation. The first two equalities imply $\sigma\tau c = c$, whereas the third gives $2c = 0$. Thus $c = 0$.

All dyadic residue degrees are one. For $K(i)$ this follows from $K_{\mathfrak{p}_2} = \mathbf{Q}_2(i)$, so the quadratic extension splits at $\mathfrak{p}_2$. For the other two fields, the dyadic completion is $\mathbf{Q}_2(i, \sqrt{2}) = \mathbf{Q}_2(\zeta_8)$, which is totally ramified.

### 17.7.2 The four remaining quartic fields

For $j = 1, 2, 3, 4$, let $B_j = \mathbf{Q}(\alpha_j)$, where $\alpha_j$ is a root of

$$f_1(X) = X^4 - 4X^2 - 3, \quad f_2(X) = X^4 + 4X^2 - 3,$$
$$f_3(X) = X^4 - 8X^2 - 12, \quad f_4(X) = X^4 + 8X^2 - 12.$$

They are the fields with radicands $\pi_3, -\pi_3, 2\pi_3, -2\pi_3$. Each has signature $(2, 1)$. Put

$$\beta_1 = \alpha_1^3 + 2\alpha_1^2 + 1, \qquad \beta_2 = \alpha_2 + 1, \qquad \beta_j = \frac{\alpha_j^2 + 2\alpha_j + 2}{4} \quad (j = 3, 4).$$

Their respective minimal polynomials are

$$g_1(X) = X^4 - 20X^3 - 6X^2 - 4X + 2,$$
$$g_2(X) = X^4 - 4X^3 + 10X^2 - 12X + 2,$$
$$g_3(X) = X^4 - 6X^3 + 8X^2 - 4X + 2,$$
$$g_4(X) = X^4 + 2X^3 - 8X + 2.$$

All four are Eisenstein at two. Thus each field has a unique dyadic prime, of residue degree one, generated by $\beta_j$, and $N_{B_j/\mathbf{Q}}(\beta_j) = 2$.

The polynomial discriminants of $f_1, f_2$ are $-37632$, and those of $f_3, f_4$ are $-2408448$. Every $B_j$ contains $K$, so $28^2$ divides its field discriminant. These facts exclude any odd divisor of $[\mathcal{O}_{B_j} : \mathbf{Z}[\alpha_j]]$. Since $g_j$ is Eisenstein, its root generates the ring of integers of the dyadic completion. The valuations at two of $\mathrm{disc}(g_1), \ldots, \mathrm{disc}(g_4)$ are $8, 8, 6, 6$. Consequently the field discriminants are exactly

$$D_1 = D_2 = -37632, \qquad D_3 = D_4 = -9408.$$

The Minkowski bounds are $42\sqrt{3}/\pi < 24$ for $B_1, B_2$ and $21\sqrt{3}/\pi < 12$ for $B_3, B_4$.

At odd primes, Dedekind's factorization theorem applies to $f_j$, because its index has no odd divisor. Factoring modulo the rational primes below these bounds gives precisely the following norms of prime ideals within the bounds:

| | |
|---|---|
| $B_1$ | $2, 3, 3, 3, 7, 7$ |
| $B_2$ | $2, 3, 9, 19, 19, 19, 19$ |
| $B_3$ | $2, 3, 9, 7, 7$ |
| $B_4$ | $2, 3, 3, 3.$ |

For completeness, at three the factorizations are $X^2(X-1)(X+1)$ for $f_1, f_4$ and $X^2(X^2+1)$ for $f_2, f_3$. At seven, $f_1$ and $f_3$ have double roots $\pm 3$ and $\pm 2$, respectively; the other two have an irreducible quadratic factor of multiplicity two. At nineteen, $f_2$ has roots $\pm 3, \pm 5$, whereas $f_1$ has two irreducible quadratic factors. For $f_1, f_2$, the remaining primes $5, 11, 17, 23$ are inert and 13 has two factors of degree two. For $f_3, f_4$, the remaining primes $5, 11$ are inert.

The following elements have the indicated absolute norms; the sign choices in a single expression are taken together.

| Field | Element | Absolute norm |
|---|---|---|
| $B_1$ | $\alpha_1,\ \alpha_1 \pm 2$ | 3 |
| | $\pm\alpha_1^3 + 2\alpha_1^2 \pm \alpha_1 + 1$ | 7 |
| $B_2$ | $\alpha_2$ | 3 |
| | $\alpha_2^2 + 4$ | 9 |
| | $\alpha_2^2 \pm \alpha_2 - 1,\ 2\alpha_2^2 \pm \alpha_2 - 2$ | 19 |
| $B_3$ | $\alpha_3 + 3$ | 3 |
| | $(\pm\alpha_3^3 + \alpha_3^2 + 2)/4$ | 7 |
| $B_4$ | $\alpha_4 \pm 1,\ (\alpha_4^3 - 2\alpha_4)/4$ | 3 |

All these elements are integral. For the entries with denominator four, use the integral elements $\beta_j$ and

$$\gamma_3 = \frac{\alpha_3^3 + 2\alpha_3}{4}, \qquad \gamma_4 = \frac{\alpha_4^3 - 2\alpha_4}{4},$$

which satisfy $X^4 - 74X^2 - 3$ and $X^4 + 74X^2 - 3$, respectively. The two norm-seven elements in $B_3$ are $\beta_3 + \gamma_3 - \alpha_3$ and $\beta_3 - \gamma_3$. The norm identities follow directly by taking resultants with $f_j$.

Reduction at the displayed roots distinguishes the prime ideals: the three norm-three elements in $B_1$ and $B_4$ vanish at the three distinct roots modulo three, the norm-seven pairs vanish at the two roots modulo seven, and the four norm-nineteen elements in $B_2$ vanish at its four distinct roots modulo nineteen. The norm-nine element in $B_2$ generates the prime for $X^2 + 1$ modulo three. In $B_3$, let $\mathfrak{a}$ be the ramified norm-three prime and $\mathfrak{b}$ the norm-nine prime. Since $\mathfrak{a} = (\alpha_3 + 3)$ and $(3) = \mathfrak{a}^2\mathfrak{b}$, the ideal $\mathfrak{b}$ is principal. An integral generator is

$$\frac{3}{(\alpha_3 + 3)^2} = -20\alpha_3^3 + 61\alpha_3^2 - 26\alpha_3 + 79.$$

Together with the dyadic generators $\beta_j$, these identities make every prime ideal within the corresponding Minkowski bound principal. Hence $h(B_j) = 1$ for all four fields.

# 18 The field $\mathbf{Q}(\sqrt{13})$

Throughout this section let $K = \mathbf{Q}(\sqrt{13})$.

The earlier proof for $K$ of Dieulefait and Pacetti [30] uses GRH in the residual base. We replace that base by a characteristic-three to characteristic-two comparison. The two places above three admit weight-two lifts that are quadratic twists of crystalline or semistable representations. In the two-special case, the dyadic point field has a guaranteed norm-nine contribution to the classical explicit formula. The scalar dyadic case uses ordinary lattice rigidity with the unramified ordinary quotient held fixed.

## 18.1 Quadratic arithmetic over $K$

Write

$$w = \frac{1 + \sqrt{13}}{2}, \qquad w' = 1 - w, \qquad \epsilon = w + 1.$$

Then $w^2 = w + 3$, $ww' = -3$, and $N_{K/\mathbf{Q}}(\epsilon) = -1$. The two triadic primes are $\mathfrak{p} = (w)$ and $\mathfrak{p}' = (w')$, and their completions are $\mathbf{Q}_3$. At two, $w^2 - w - 3$ is irreducible modulo two, so $K_2/\mathbf{Q}_2$ is unramified quadratic. Its residue field is $\mathbf{F}_4$, generated by $\bar{w}$, and $\bar{\epsilon}$ generates $\mathbf{F}_4^\times$. The ordinary class number is one by the Minkowski bound $\sqrt{13}/2 < 2$. The units $-1$ and $\epsilon$ realize all four real signatures, so the narrow class number is also one.

The unit $\epsilon$ generates the units modulo $\{\pm 1\}$. Indeed, for a unit $u = (a + b\sqrt{13})/2$ in the first real embedding with $1 < u < \epsilon$, its conjugate is $\pm u^{-1}$. Both $u - u^{-1}$ and $u + u^{-1}$ are strictly less than $\epsilon + \epsilon^{-1} = \sqrt{13}$. Thus $0 < b < 1$, which is impossible.

**Lemma 18.1.** *A characteristic-three character unramified at all finite places away from three has square one. For a reducible residue of determinant $\bar{\chi}_3$ and this support, the constituent ratio is $\bar{\chi}_3$.*

*Proof.* The character kills the principal units at the two triadic places. Since the ordinary class group is trivial, its ray quotient is a quotient of $(\mathbf{F}_3^\times)^2 \times \{\pm 1\}^2$, which has exponent two. For constituents $\theta$ and $\bar{\chi}_3\theta^{-1}$ the ratio is therefore $\bar{\chi}_3$. □

Let $V$ be the group of squareclasses of $K$ with even valuations at every finite place outside $\{\mathfrak{p}, \mathfrak{p}'\}$, including at two. We use additive notation for this elementary abelian group. By the class and unit calculations, a basis is $[-1], [\epsilon], [w], [w']$.

**Lemma 18.2.** *The kernel of localization $V \to K_2^\times/K_2^{\times 2}$ is $\langle[-3]\rangle$. The two classes $[-w]$ and $[-w']$ are unramified at two and ramify at the two triadic places independently. The simultaneous localization*

$$V \longrightarrow K_{\mathfrak{p}}^\times/K_{\mathfrak{p}}^{\times 2} \oplus K_{\mathfrak{p}'}^\times/K_{\mathfrak{p}'}^{\times 2}$$

*is injective.*

*Proof.* A dyadic unit $a$ defines a quadratic extension unramified or split at two exactly when $a^3 \equiv 1 \pmod{4\mathcal{O}_{K_2}}$. Indeed $a^3$ represents the same squareclass and is one modulo two. The unramified quadratic class has a representative $1 + 4u$, from the discriminant of a lift of an irreducible Artin–Schreier polynomial. A square of a unit congruent to one modulo two is one modulo four. Conversely a class represented by $1 + 4u$ is given by $X^2 - X - u$, which is separable modulo two. The map $a \mapsto (a^3 - 1)/2$ modulo two is additive on these squareclasses. Its values on $-1, \epsilon, w, w'$ are $1, \bar{w}, 1, 1$. Its kernel in $V$ is generated by $[-w], [-w']$.

The element $u = (\epsilon + \sqrt{-w})/2$ is integral, with polynomial $X^2 - \epsilon X + \epsilon$, since $\epsilon^2 + w = 4\epsilon$. Modulo two this polynomial is irreducible. After dividing the variable by $\bar{\epsilon}$, its Artin–Schreier constant is $\bar{\epsilon}^{-1} = \bar{w}$, of trace one. Thus $[-w]$ is the nontrivial unramified local class, and the same holds for $[-w']$ by conjugation. Their product is $[-3]$. Each has odd valuation just at its own triadic prime, proving the assertions at two and the independent ramification at three.

Finally, a class split at both triadic places lies in $\langle[-1], [\epsilon]\rangle$ by the two valuations. At the places where $w$ is $0, 1$ modulo three, the quadratic residue vectors of $-1, \epsilon$ are $(-,-)$ and $(+,-)$, respectively. These are independent. □

**Lemma 18.3.** *For every nonzero class $[a] \in V$, the ordinary ray class group of $K(\sqrt{a})$ modulo (2) has two-power order.*

*Proof.* Up to conjugation over $\mathbf{Q}$, the nontrivial classes are the nine in the following table. Here $t = \sqrt{a}$, $L = K(t)$, and the right column lists all prime-ideal norms at most the Minkowski bound. A bound on $|D_L|$ follows from the relative discriminant of $X^2 - a$, which divides $(4a)$. For $-w$ use the integral generator $u$ from Lemma 18.2, and for $-3$ use $\zeta_3$. The middle column rounds the Minkowski bound upwards.

| $a$ | $\lvert D_L\rvert$ at most | $r_2$ | Minkowski bound below | prime norms |
|---|---|---|---|---|
| $-1$ | 2704 | 2 | 8 | 4 |
| $\epsilon$ | 2704 | 1 | 7 | $3, 4$ |
| $w$ | 8112 | 1 | 11 | $3, 4$ |
| $-w$ | 507 | 1 | 3 | none |
| $\epsilon w$ | 8112 | 0 | 9 | $3, 4$ |
| $-\epsilon w$ | 8112 | 2 | 14 | $3, 4, 13$ |
| $-3$ | 1521 | 2 | 6 | $3, 4$ |
| 3 | 24336 | 0 | 15 | $3, 4, 13$ |
| $3\epsilon$ | 24336 | 1 | 19 | $3, 4, 17$ |

These norm lists follow by splitting in $K$ first, then testing $a$ in the residue fields. The primes $5, 7, 11$ are inert in $K$. At 13 the residue is $w = 7$, and at 17 the two residues are $w = 5, 13$. For $\epsilon w$ the other three-place is inert and its norm-nine prime exceeds the Minkowski bound. The fields $K(\sqrt{-w})$ and $K(\sqrt{-3})$ are the only representatives unramified at two by Lemma 18.2. The first is inert and the second split over $K_2$. In each other case the unique prime of norm four squares to the principal ideal $(2)$ in $L$.

A ramified prime of norm three has class of order dividing two because its prime below in $K$ is principal. For $a = \epsilon, w$ the split norm-three prime is generated by $t-1$, of absolute norm three, and conjugation gives the other. For $a = -\epsilon w$ the split norm-three primes have two-primary classes because $|N_{L/\mathbf{Q}}(t-1)| = 12$, with the norm-three factor at the split place. This gives a relation with the unique norm-four prime. The remaining prime classes are accounted for by the following elements, with the indicated absolute norms:

| $a$ | element | $\lvert N_{L/\mathbf{Q}}\rvert$ |
|---|---|---|
| $-\epsilon w$ | $2\epsilon + t$ | 39 |
| 3 | $2 + wt$ | 13 |
| $3\epsilon$ | $1 + t$ | 17 |
| $3\epsilon$ | $1 - w + t$ | 51. |

The last two elements lie over different primes of $K$: their relative norms are $-2 - 3w$ and $1 - 4w$, respectively. Conjugation over $K$ covers the other primes of each split pair. For $a = -3$ put $z = \zeta_3$. The integer $-3 - 2w - (2 + w)z$ has relative norm $4(4 + 3w)$ and absolute norm sixteen. It vanishes at exactly one dyadic prime, since modulo two the two possible residues of $z$ are $\bar{w}, 1 + \bar{w}$. Hence it generates the square of one norm-four prime and conjugation gives the other. These relations show that every ordinary class group in the table has two-power order.

In the dyadically ramified cases, the odd part of the residue units modulo (2) has order three and is generated by $\bar{\epsilon}$. For $a = -w$, the residue of $u$ generates $\mathbf{F}_{16}^{\times}$: its norm to $\mathbf{F}_4$ is $\bar{\epsilon}$, and its residue lies outside $\mathbf{F}_4$. For $a = -3$, the dyadic residue unit group is $(\mathbf{F}_4^{\times})^2$. The images of $\epsilon$ and $z$ generate it, since one is diagonal and the other consists of inverse primitive cube roots. Thus the ray exact sequence introduces no odd quotient. □

## 18.2 The characteristic-three base

We call a system *bounded* if its determinant is cyclotomic, its weights are $\{0, 1\}$ at both embeddings, and its conductor is supported above three. At each triadic place it is a quadratic twist of a crystalline or semistable representation, as in the first assertion of Lemma 4.23. In the semistable case with nonzero monodromy, the local parameter is special, with an exact quadratic scalar character. This follows from the determinant and the ratio of its Frobenius eigenvalues. We refer to the other local condition as twisted crystalline.

**Lemma 18.5.** *For any dyadic residue of a bounded system, the inertia at each triadic place has order at most two. At a twisted crystalline place it is trivial. At a special place the decomposition image is elementary abelian of order at most four.*

*Proof.* At a twisted crystalline place the common Weil–Deligne parameter has scalar quadratic inertia and zero monodromy. In characteristic two that scalar reduces to one. At a special place the characteristic-zero dyadic restriction has two characters $\xi\chi_2, \xi$, with $\xi$ quadratic, and is upper triangular in a basis adapted to the saturated special line of any stable lattice. Both diagonal characters reduce to one. Wild inertia is pro-three, so the residual inertia factors through the procyclic tame quotient and has order at most two. The decomposition image is generated by this inertia and one Frobenius. Both are unipotent in dimension two in characteristic two, and they commute. □

**Proposition 18.6.** *A bounded system with a nonsolvable dyadic residue is modular. A bounded system has no irreducible solvable dyadic residue, and every reducible dyadic residue has trivial semisimplification.*

*Proof.* In the nonsolvable case the image is $\mathrm{SL}_2(\mathbf{F}_q)$, with $q$ a power of two at least four, since the determinant is one. Let $M/K$ be the kernel field. The dyadic member is crystalline of

weight two. Its residual representation is obtained from a finite flat group scheme killed by two, by taking the coefficient-uniformizer torsion of the Barsotti–Tate group realizing its lattice [51, Theorem 4.1]. Fontaine's bound [39, Theorem A] over the unramified dyadic base gives no upper break greater than one. The normalized different at two is therefore less than two. There is no extra base-field different at two.

If at most one triadic place is special, the root discriminant is less than $4\sqrt{13}\,3^{1/4} < 18.98069$ by Lemma 18.5. If both are special, then $\mathrm{rd}(M) < 4\sqrt{39} < 24.98000$. At either special place the relative ramification index $e$ and residue degree $f$ are at most two. In the explicit formula, if $f = 2$ the norm-nine coefficient contributed by that place is $1/(2e) \geq 1/4$. If $f = 1$, retain the second power of the norm-three Euler factor, whose contribution is the same as that of a norm-nine term with coefficient $1/(2e)$. Thus the two special places supply an effective norm-nine coefficient of at least $1/2$.

For $q \geq 8$ the degree $[M : \mathbf{Q}] = 2q(q^2 - 1)$ is at least 1008. The classical bounds in Section 13 give $\mathrm{rd}(M) > 20.57608$ without prime terms, and $\mathrm{rd}(M) > 25.32625$ with the indicated norm-nine contribution. These exclude $q \geq 8$ in the two cases above. For $q = 4$, a totally real kernel field of degree 120 would have $\mathrm{rd}(M) > 41.69019$, again impossible. Thus at least one residual real conjugation is nontrivial. The small-field witness and Thorne's theorem apply by Lemma 2.4.

For an irreducible solvable residue apply Lemma 11.16a with the triadic support. Its quadratic inducing field $L/K$ has upper break at most one at two by Fontaine and compatibility with quotients. A radicand of odd dyadic valuation has conductor three, as seen from the discriminant of its Eisenstein quadratic equation over the unramified base. The inducing squareclass is therefore in $V$. Lemma 18.3 makes its odd dyadic ray quotient trivial, contradicting the nontrivial inducing character.

For a reducible residue, Lemma 11.16a reduces to the odd dyadic ray quotient of $K$. Its residue-unit group $\mathbf{F}_4^\times$ is generated by $\epsilon$, and $h_K = 1$, so the quotient is trivial. □

## 18.3 The scalar dyadic case

**Proposition 18.7.** *A bounded system with a scalar-residual dyadic member is modular.*

*Proof.* Let $r$ be the dyadic member, and let $\nu$ denote its unramified ordinary quotient at two. By Lemma 4.11, choose a stable lattice $T$ whose residue is a nonzero additive extension split at two. Its additive extension coordinates are quadratic characters split at two and unramified away from the two triadic primes. Lemma 18.2 makes the extension a nonzero multiple of the additive character $h : G_K \to \mathbf{F}_2$ attached to $K(\sqrt{-3})$.

The character $h$ ramifies at both triadic places. Both actual local types must therefore be special by Lemma 18.5. Twist by the characters attached to $[-w]$ and $[-w']$ as needed to remove the ramified parts of the two quadratic scalar characters. These corrections are unramified at two, preserve the determinant, and leave the dyadic residue unchanged. Work with the resulting system and its ordinary quotient $\nu$. At the two special places the exact quadratic characters $\xi_1, \xi_2$ are now unramified.

Fix a real conjugation $c$. For the cyclic lattice of the normalized source, the basis $((r(c)-1)v, v)$ is integral because $h(c) = 1$. It gives the markings

$$\bar{r}(g) = 1 + h(g)E_{12}, \qquad r(c) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{ordinary line } e_2.$$

Let $\sigma$ be the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$ and choose a lift at the dyadic place. It acts trivially by conjugation on the unramified quotient of $G_{K_2}$. Hence $r^\sigma$ has the same exact

ordinary quotient $\nu$. Normalize $r^\sigma$ using the same real conjugation. Its residue is the same, since $h$ comes from $\mathbf{Q}$. Its two special characters are $\xi_2, \xi_1$.

Compare these normalized representations over a common coefficient ring with uniformizer $\varpi$ and residue field $k$. For congruence modulo $\varpi^n$, use the first difference $X$ of Lemma 12.1a. At two the exact ordinary quotient gives $x = y = 0$, and $z$ is a unit Kummer class, hence has even dyadic valuation. At either triadic place, $h$ is ramified and the special lines have a unit upper entry. The lemma gives $z|_{G_{K_v}} = 0$ and $x|_{G_{K_v}} = t_v h|_{G_{K_v}}$. If the unramified special signs differ, the additional term at $n = e := v_\varpi(2)$ is $u_0 U_v$, with $u_0 = \overline{2/\varpi^e} \neq 0$; before that stage it is zero.

Thus $z$ belongs to $V \otimes_{\mathbf{F}_2} k$ and is split at both triadic places. Lemma 18.2 implies $z = 0$. Now $x, y$ are additive and vanish at two, hence are multiples of $h$. The fixed matrix at $c$ forces both to vanish. This proves the homogeneous comparison at every stage. In the inhomogeneous case the vanishing of $x$ would make the nontrivial unramified character $u_0 U_v$ a multiple of $h|_{G_{K_v}}$, which is ramified. This is impossible. Starting modulo $\varpi$, induction therefore gives equality of the signs and then $r^\sigma \simeq r$.

By compatibility the same isomorphism holds for an odd-adic companion, which we take at five. This companion extends to $G_{\mathbf{Q}}$: an intertwiner for a lift of $\sigma$ can be scaled, by Schur's lemma, so that its square is the matrix of the square of that lift. The extension is absolutely irreducible, finitely ramified and odd, and is regular de Rham since this condition descends through finite local extensions. Pan's Theorem 1.0.4 [61] makes it modular. Quadratic base change is cuspidal because its restriction is irreducible. This proves modularity of the normalized system and hence of its original quadratic twist. □

**Theorem 18.8.** *Every absolutely irreducible characteristic-three representation of determinant $\bar{\chi}_3$, unramified outside three, is modular over $K$.*

*Proof.* A solvable image has a regular modular witness. Otherwise the cyclotomic restriction is absolutely irreducible, since reducibility on an index-two subgroup would make the image dihedral up to scalars. At the two triadic places use the actual lifts in the first assertion of Lemma 4.23. With determinant $\chi_3$ and good conditions elsewhere, Lemma 2.6(i) gives a pure compatible system. On a selected special component purity ensures nonzero monodromy. Thus this is a bounded system. Propositions 18.6 and 18.7 give modularity of its dyadic member in every possible case, and compatibility gives modularity of the original characteristic-three residue. □

### 18.4 Completion of the conductor reduction

*Completion for $d = 13$ in Theorem 1.1.* The irreducible supported determinant-cyclotomic base is Theorem 18.8. By Lemma 18.1, the constituent quotient of a reducible residue is $\bar{\chi}_3$. It is totally odd and extends to $G_{\mathbf{Q}}$. Two is inert and three splits. Corollary 6.6 applies. In its conductor reduction both triadic types are retained; the final lifting steps are Zhang's Theorems 6.1.1 and 5.1.1 as specified in Section 6. □

## 19 The classical split-dyadic cases

### 19.1 The field $\mathbf{Q}(\sqrt{17})$

By Lemma 14.5, it suffices to consider a kernel field $L$ of group $\mathrm{SL}_2(\mathbf{F}_q)$ with $q \geq 8$. The local bound of Lemma 14.3 gives $\mathrm{rd}(L) \leq 4\sqrt{17}$. In degree at least 1008 the arbitrary-signature row of Lemma 13.4 gives $\mathrm{rd}(L) > 20.70004 > 4\sqrt{17}$, a contradiction.

## 19.2 The residual base over $\mathbf{Q}(\sqrt{41})$

In this subsection $d = 41$. A special dyadic place means an unramified quadratic twist of Steinberg with nonzero monodromy. We first treat systems special at both dyadic places, then construct the congruences to these systems.

**Lemma 19.1.** *Let $M/K$ have Galois group $\mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^a \geq 4$, and be unramified at finite places away from two. Both dyadic places have nontrivial wild inertia, and at most one of their decomposition groups is elementary abelian.*

*Proof.* Lemma 14.3 gives $\mathrm{rd}(M) \leq 4\sqrt{41}$. If wild inertia is trivial at one place, the upper bound improves to $\sqrt{41}\,2^{4/3}$. This contradicts the first comparison in Appendix C.2, since $[M : \mathbf{Q}] \geq 120$.

Suppose both groups are elementary abelian. Their local fields are contained in $E_2$, so $e \leq 4$ and $f \leq 2$. For $q = 4, 8$ use $L = M^U$ as in Lemma 13.9. Each place gives a norm-two coefficient $r_0$ and an effective norm-four coefficient at least $(1 - r_0)/8$, retaining even powers of the free block if its residue degree is one. The upper bound is $\sqrt{41}\,2^{2(1-r_0)}$. For $q \geq 16$, use $M$ with a total effective norm-four coefficient $1/4$. The three remaining comparisons in Appendix C.2 give the contradictions. □

**Lemma 19.2.** *Let $\mathcal{R}$ be a potentially modular weight-two system of cyclotomic determinant, good away from two and special at both dyadic places. Then $\mathcal{R}$ is modular. The same holds for a system special at one dyadic place and good elsewhere if some dyadic member has solvable residue.*

*Proof.* At a special place the actual dyadic member is a nonsplit extension of an unramified sign by its cyclotomic twist, as in Lemma 4.2. Both characters reduce to one, so the residual decomposition group is elementary abelian for any stable lattice. With two special places, Lemma 19.1 forces solvable residue.

For reducible residue, the residual semisimplification is $1 \oplus 1$ by Lemma 14.1. The trivial ratio extends to $G_{\mathbf{Q}}$, and condition (S) holds by Lemma 14.2 and Table 1. We use [25, Theorem 9.4(b)] in this case and [25, Theorem 10.6] for irreducible solvable residue. The actual source is absolutely irreducible and totally odd over the abelian field $K$, completely split at two. At each special place it is de Rham of distinct weights, reducible and nonsplit.

In the one-special case, suppose the actual restriction at the good dyadic place splits. Its two crystalline characters have weights zero and one. Pass to a totally real field of potential modularity. Compatibility identifies the chosen dyadic member there with the corresponding member of the modular system. Special monodromy survives restriction and, by local–global compatibility at an odd coefficient prime, gives a square-integrable finite component. This is the hypothesis of [92, Theorem 1] when the degree of the field is even; it provides an abelian variety of $\mathrm{GL}_2$-type through a Shimura curve. The same monodromy excludes CM, since an induced character has zero monodromy. At the good place, the form has unit Hecke trace by [74, Theorem 1], since the two crystalline slopes are zero and positive. Zhao's theorem [92, Theorems 1 and 5.6] gives indecomposability, contrary to the split restriction. Thus the actual good restriction is irreducible or nonsplit reducible, in the two local cases of [25]. The stated source theorems apply, and compatibility gives modularity of the system. □

**Lemma 19.3.** *Let $\mathcal{R}$ be a weight-two system of cyclotomic determinant good at every odd place. Fix a coefficient place of odd characteristic $\ell$ whose residual inertia at both dyadic places is unipotent. If its residue is reducible, the system is modular. If its residue is irreducible, it remains so on $G_{K(\zeta_\ell)}$.*

*Proof.* In the reducible case write the constituents as $\psi, \overline{\chi}_\ell\psi^{-1}$. They are unramified outside $\ell$ at finite places. Use reciprocity on the totally positive unit $\epsilon^2$, where $\epsilon$ is from Table 1 and $\epsilon^2 - 1 = 64\epsilon$. At a split $\ell$ the rank-one finite-flat exponents are zero or one. The two residues of $\epsilon^2$ are inverses, so unequal exponents would make $\epsilon^2$ trivial in the residue field. At an inert $\ell$ the mixed niveau-two exponents $1, \ell$ are excluded by the same identity. At $\ell = 41$, the niveau-one exponents are $0, 1, 2$, with cyclotomic exponent two. The exponent one is excluded by the unit relation. Thus $\psi$ has either trivial inertia or cyclotomic inertia at all $\ell$-places. The narrow class number is one, so the constituents are $1, \overline{\chi}_\ell$. By Lemma 4.3(i) the actual source is ordinary and its unramified quotient lifts the trivial constituent. The ratio $\overline{\chi}_\ell$ is odd and nontrivial at every coefficient place, and $K(\zeta_\ell)/\mathbf{Q}$ is abelian. These are the hypotheses of [75, Theorem A].

For an irreducible residue at an unramified $\ell > 3$ apply Lemma 4.6. At $\ell = 3$ or 41, a reducible cyclotomic restriction would give induction from the quadratic subextension of $K(\zeta_\ell)/K$, with image of order prime to $\ell$. At $\ell = 41$ the inducing extension is ramified. The projective tame inertia is cyclic and contains a reflection, so has order two. The rank-one exponents $0, 1, 2$ give projective order one or twenty. For a locally irreducible residue, the niveau-two digits lie in $\{0, 1, 2\}$ and sum to two by the determinant. They are unequal since scalar inertia would force local reducibility in the prime-to-41 image. The order is then twenty-one, again a contradiction.

At $\ell = 3$ the inducing field would be $H = K(\zeta_3)$. The inducing character and its conjugate agree on tame inertia at three, because $H/K$ is ramified quadratic there with unchanged residue field. The unipotent inertia at two dies in an image of order prime to three. Their quotient is unramified at all finite places of $H$. This field has discriminant 15129 and Minkowski bound $369/(2\pi^2) < 19$. The only prime ideals of norm at most eighteen are the two above two of norm four and the one above three of norm nine. Indeed $5, 11, 17$ are inert in $\mathbf{Q}(\zeta_3)$ and $7, 13$ are inert in $K$. The generators are $\alpha$, its conjugate and $1 - \zeta_3$. Hence $H$ has class number one. The inducing character equals its conjugate, a contradiction. □

**Lemma 19.4.** *In the setting of Lemma 19.3, suppose the actual residual representations at both dyadic places admit special lifts of determinant $\chi_\ell$. Then $\mathcal{R}$ is modular.*

*Proof.* The reducible case follows from Lemma 19.3. In the irreducible case it gives cyclotomic irreducibility. At $\ell = 5$ the exceptional projective $S_5$ field cannot contain $K(\zeta_5)$, of degree four over $K$, since $S_5$ has abelianization of order two. Prescribe the special components at two and keep the good components at the $\ell$-places. Lemma 2.6(i) gives a potentially modular weight-two system, good away from two. The prescribed Frobenius ratio two and purity keep the special monodromy nonzero at both places. Lemma 19.2 gives modularity of the new system. Its $\ell$-adic member supplies the weight-two witness in Lemma 2.2(ii) for the actual old member. □

**Proposition 19.5.** *A nonsolvable determinant-one dyadic residue over $\mathbf{Q}(\sqrt{41})$ unramified outside two is modular.*

*Proof.* Lemma 19.1 leaves at least one triangular local image with distinct unramified diagonal characters. The distinct-character calculation in Proposition 9.1 gives an ordinary crystalline weight-two lift of the actual local extension, of determinant $\chi_2$.

Suppose first that the characters are distinct at both places. Choose $\tau_0$ at both using Proposition 9.1 and apply Lemma 2.6(iii). This gives a weight-two system $\mathcal{R}$ lifting the residue, with the two prescribed potentially nonordinary Barsotti–Tate types and good at all odd places. In a three-adic member the two primitive order-three inertia characters at each dyadic place reduce to one, so the residual inertia is unipotent. Frobenius exchanges the inertia eigenlines and has trace zero. Thus Lemma 4.7(i) gives a special lift of each actual local residue, including the nontrivial unipotent extensions. Apply Lemma 19.4 in characteristic three.

Suppose now that one dyadic decomposition group is elementary abelian. The Kummer construction in the proof of Proposition 9.1 lifts its actual scalar extension to an ordinary semistable weight-two representation with nonzero valuation coordinate and trivial special sign. Choose the special component through this point and the ordinary crystalline component at the other place. Both are ordinary of weight two, so the semistable case of Lemma 2.6(iii) gives a potentially modular system $\mathcal{R}$ lifting the residue. It is good away from the marked special place, where purity keeps monodromy nonzero.

If some dyadic member of $\mathcal{R}$ has solvable residue, Lemma 19.2 gives modularity. Otherwise every dyadic member has a nonsolvable residue with elementary decomposition at the special place. By Lemma 19.1 the two residual diagonal characters at the good place are unramified and distinct at every dyadic coefficient prime. The actual source there is ordinary by Lemma 4.3(ii). For the common Frobenius trace $a$ the unit root $u$ satisfies $a = u + 2/u$ and $\bar{u} \neq 1$. Thus $a$ is a unit and $\bar{a} \neq 1$ at every dyadic coefficient prime. Integrality at odd coefficient primes follows from stable lattices. Lemma 4.7(ii) gives an odd coefficient prime and a special lift of the residual representation at the good dyadic place. Retain the existing special component at the other place. Since the residual inertia at both is unipotent, Lemma 19.4 applies. In all cases modularity of $\mathcal{R}$ gives modularity of the starting residue. □

*Completion for $d = 17, 41$ in Theorem 1.1.* The residual base at two follows from the preceding argument for $d = 17$ and Proposition 19.5 for $d = 41$, together with Lemma 14.5. The class group and condition (S) are verified in Section 14. Both radicands are prime. For condition (i) of Theorem 10.1, apply [25, Theorem 9.4(b)] with both dyadic places in $\Sigma_n$. The split-dyadic criterion gives the result. □

# 20 The field $\mathbf{Q}(\sqrt{3})$

Put $K = \mathbf{Q}(\sqrt{3})$, $r = \sqrt{3}$, $\epsilon = 2 + r$, $\pi = 1 + r$ and $g = 1 + 2r$. We use the split prime eleven, with $v_+ = (g)$ and $v_- = (1 - 2r)$, and write $\mathfrak{p}_2 = (\pi)$.

## 20.1 Arithmetic and quadratic classes

The class number of $K$ is one, its units are $\pm\epsilon^{\mathbf{Z}}$, and its narrow class group is $C_2$. The nontrivial narrow character $\eta$ corresponds to $K(i) = \mathbf{Q}(\zeta_{12})$ and is unramified at every finite place. The prime two ramifies, with completion $L = \mathbf{Q}_2(\sqrt{3})$, ramification index two and residue field $\mathbf{F}_2$. The prime five is inert. The residues of $\epsilon$ at $v_+, v_-$ are $7, 8$, inverse generators of $\mathbf{F}_{11}^\times$, and its residue in $\mathbf{F}_{25}^\times$ has order three.

**Lemma 20.1.** *The quadratic classes unramified outside $\{\mathfrak{p}_2, v_+\}$ form $W = \langle -1, \epsilon, \pi, g\rangle$. Their localization in $L^\times/L^{\times 2}$ is injective. Their dyadic conductor exponents are as follows:*

| $d$ | $-1$ | $g$ | $-g$ | $\epsilon$ | $-\epsilon$ | $\epsilon g$ | $-\epsilon g$ | *with* $\pi$ |
|---|---|---|---|---|---|---|---|---|
| *conductor* | 0 | 2 | 2 | 4 | 4 | 4 | 4 | 5. |

*Thus the classes of conductor at most two form $\langle -1, g\rangle$.*

*Proof.* Minkowski's bound is $\sqrt{12}/2 < 2$, so $h_K = 1$ and the indicated generators give $W$. The Pell equation gives the fundamental unit $2 + r$, and reduction modulo three excludes a unit of norm $-1$. Hence $h_K^+ = 2$. The discriminant of $K(i)$ is $144 = 12^2$, which identifies $\eta$.

Suppose $(-1)^a\epsilon^b\pi^c g^d$ is a local square, where the exponents lie in $\{0, 1\}$. Its norm $(-2)^c(-11)^d$ is a square in $\mathbf{Q}_2$. Valuation gives $c = 0$, and $-11 \equiv 5 \pmod 8$ gives $d = 0$. Since $\epsilon = \pi^2/2$, the

remaining nontrivial classes are $-1, 2, -2$. None belongs to the kernel $\langle 3\rangle$ of $\mathbf{Q}_2^\times/\mathbf{Q}_2^{\times 2} \to L^\times/L^{\times 2}$. This proves injectivity.

The extension $L(i)/L = L(\sqrt{-3})/L$ is unramified. Since $-g = 1 - 2\pi$, the element $(\sqrt{-g} - 1)/\pi$ satisfies

$$X^2 + (2/\pi)X + 2/\pi = 0.$$

This polynomial is Eisenstein and its discriminant has valuation two. Twisting by the unramified character leaves a positive conductor unchanged, so $g$ has the same conductor. For $d \in \{\epsilon, -\epsilon, \epsilon g, -\epsilon g\}$ one has $v_L(d-1) = 1$. The polynomial of $\sqrt{d}-1$ is Eisenstein, with discriminant $4d$ of valuation four. A class involving $\pi$ has odd valuation and admits an Eisenstein equation $X^2 - d$ with discriminant valuation five. Conjugation gives the same assertions for the other prime over eleven. □

## 20.2 The congruence at eleven and the change of type at five

Apply Propositions 2.7 and 6.2 with $\ell = 11$, and let $r_{11}$ be the terminal member. A residual character unramified outside eleven has inertia $\omega_{11}^a, \omega_{11}^b$ at $v_+, v_-$. Reciprocity on the totally positive unit $\epsilon$ gives $7^{a-b} = 1$, so $a \equiv b \pmod{10}$. Dividing by this cyclotomic power leaves a narrow class character. If $\bar{r}_{11}$ is reducible, the crystalline weight-two condition at $v_-$ forces the exponents zero and one, and the determinant makes the two narrow twists equal:

$$\bar{r}_{11}^{\mathrm{ss}} = \alpha \otimes (1 \oplus \omega_{11}), \qquad \alpha \in \{1, \eta\}.$$

The quotient is $\omega_{11}^{\pm 1}$, which extends to $G_{\mathbf{Q}}$ and is nontrivial at both places over eleven. Theorem 6.4 modularizes the actual member. In particular its ordinary local extension need not be nonsplit.

We may therefore suppose $\bar{r}_{11}$ irreducible. Lemma 4.6 gives irreducibility on $G_{K(\zeta_{11})}$. A modular witness will then modularize $r_{11}$ by Lemma 2.2(i).

Let $\widetilde{\omega}$ be the Teichmüller character of order ten and put $\eta_{11} = \widetilde{\omega}^5$. Write $\mathrm{CR}(b)$ for scalar inertia $\eta_{11}^b$ with zero monodromy and $\mathrm{SP}(b)$ for the same inertia with nonzero monodromy, where $b \in \{0, 1\}$. At coefficient characteristic eleven the first condition means a quadratic twist of a crystalline Barsotti–Tate lift. In other coefficient characteristics these are conditions on Weil–Deligne representations. Write $\mathrm{PS}(i)$ for the principal series type $\widetilde{\omega}^i \oplus \widetilde{\omega}^{-i}$, with $1 \le i \le 4$.

**Lemma 20.2.** *Every representation $\bar{s} : G_{\mathbf{Q}_{11}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_{11})$ of determinant $\omega_{11}$ has a weight-two lift of determinant $\chi_{11}$ and type $\mathrm{CR}(b)$, $\mathrm{SP}(b)$ or $\mathrm{PS}(i)$.*

*Proof.* Put $\sigma(r;t) = \mathrm{Sym}^r \mathbf{F}_{11}^2 \otimes \det^t$, where $0 \le r \le 10$ and $t$ is taken modulo ten. The central-character condition is $r + 2t \equiv 0 \pmod{10}$. All twelve possibilities occur in the following reductions of the compact types:

| type | Jordan–Hölder constituents |
|---|---|
| $\mathrm{CR}(0), \mathrm{CR}(1)$ | $\sigma(0;0),\ \sigma(0;5)$, respectively |
| $\mathrm{SP}(0), \mathrm{SP}(1)$ | $\sigma(10;0),\ \sigma(10;5)$, respectively |
| $\mathrm{PS}(1)$ | $\sigma(2;9),\ \sigma(8;1)$ |
| $\mathrm{PS}(2)$ | $\sigma(4;8),\ \sigma(6;2)$ |
| $\mathrm{PS}(3)$ | $\sigma(4;3),\ \sigma(6;7)$ |
| $\mathrm{PS}(4)$ | $\sigma(2;4),\ \sigma(8;6)$. |

The Breuil–Mézard formula for potentially Barsotti–Tate representations gives a lift of the actual residue, including its extension class, whenever one of the potentially crystalline types occurs; see [43, Corollary 4.5.6]. If only $\sigma(10;t)$ occurs, the corresponding twisted très ramifiée extension has a special Kummer lift. An unramified twist by the inverse square root of a

character congruent to one fixes the determinant exactly. The square root exists because the coefficient characteristic is odd. □

Globalize this lift by Lemma 2.6(i), retaining the existing crystalline component at $v_-$ and the good conditions elsewhere. We obtain a weight-two system $\mathcal{S}_1$ of cyclotomic determinant, good away from $v_+$, with one of these types at $v_+$.

**Lemma 20.3.** *Suppose $\mathcal{S}_1$ has type* PS($i$) *at $v_+$, and let $\bar{r}_5$ be the reduction of its five-adic member. Then its complete restriction at $v_+$ has a lift of the same determinant and type* CR($b$) *or* SP($b$)*, where $b \equiv i \pmod 2$.*

*Proof.* The order-five part of inertia disappears modulo five. Remove the quadratic character $\eta_{11}^i$. Tame inertia is now $\bar{T} = 1 + xN$, with $N^2 = 0$. If $x = 0$, lift the complete residual Frobenius to an integral matrix of determinant eleven and take trivial inertia. This gives an unramified lift even when Frobenius is not semisimple.

If $x \neq 0$, the tame relation makes Frobenius centralize $\bar{T}$, because $11 \equiv 1 \pmod 5$. Thus $\bar{\Phi} = m(1 + yN)$, with $m^2 = 11 = 1$ in the residue field. Choose $c \in \{1, -1\}$ lifting $m$, and lifts $u, w$ of the upper entries of $\bar{T}, \bar{\Phi}$. Then

$$T = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 11c & w \\ 0 & c \end{pmatrix}$$

satisfy $\Phi T \Phi^{-1} = T^{11}$ and $\det \Phi = 11$. Since $u \neq 0$, the lift is special. Restoring the quadratic twist preserves the determinant and gives the stated type. □

**Proposition 20.4.** *For a system $\mathcal{S}_1$ of type* PS($i$)*, either its five-adic residue is reducible and $\mathcal{S}_1$ is modular, or there is a weight-two system $\mathcal{S}_1'$ with the same residue modulo five, good away from $v_+$ and of type* CR($b$) *or* SP($b$) *there.*

*Proof.* At the inert five-place, a reducible finite flat local representation has Raynaud digits $a, b \in \{0, 1\}$. The quotient of its two inertia characters has exponent $2(a + 5b) - 6$ modulo 24, hence order four or six. In the locally irreducible case a niveau-four character has exponent

$$m = a + 5b + 25(1 - a) + 125(1 - b) \pmod{624},$$

and the ratio of the conjugate characters has exponent $(1 - 25)m$, of order thirteen. These are orders of semisimplified tame projective inertia. A globally irreducible residue reducible over $K(\zeta_5)$ would be induced from $K(\sqrt{5})$, which is ramified at five. Its image would have order prime to five, and its cyclic projective inertia would contain a reflection and have order two. The listed orders exclude this. Thus any globally irreducible residue remains irreducible over $K(\zeta_5)$.

Suppose instead that the global residue is reducible. A constituent has inertia $\omega_0^a \omega_1^b$ at five and $\eta_{11}^i$ at $v_+$. Reciprocity on $\epsilon$ gives

$$\bar{\epsilon}^{a+5b}(-1)^i = 1.$$

The two factors have orders dividing three and two, respectively. Therefore $i$ is even and $a = b$. Dividing the constituents by 1 or $\omega_5$ leaves narrow characters, and the determinant gives

$$\bar{r}_5^{\mathrm{ss}} = \alpha \otimes (1 \oplus \omega_5), \qquad \alpha \in \{1, \eta\}.$$

Lemma 4.3(i) proves that the actual crystalline source is ordinary. Since $K/\mathbf{Q}$ is abelian and five is unramified in $K$, Lemma 4.5 applies to this actual source and proves modularity of $\mathcal{S}_1$.

In the irreducible case apply Lemma 20.3 and Lemma 2.6(i), preserving the actual crystalline component at five. Cyclotomic irreducibility was proved above, and the characteristic-five exception is excluded as in Section 2. This gives $\mathcal{S}_1'$. □

## 20.3 The scalar and solvable residues

**Proposition 20.5.** *There is no continuous absolutely irreducible representation $\rho : G_K \to \mathrm{GL}_2(E)$, with $E/\mathbf{Q}_2$ finite, satisfying all of the following conditions:* $\det\rho = \chi_2$, $\bar\rho^{\mathrm{ss}} = 1 \oplus 1$, *$\rho$ is Barsotti–Tate at $\mathfrak{p}_2$, its inertia is scalar at every odd prime other than $v_+$, and $\rho|_{G_{K_{v_+}}}$ has a line after extending $E$.*

*Proof.* If $\rho|_{G_L}$ had a line after extending coefficients, apply Lemma 4.10 with $D = G_L$. Scalar inertia at the other odd primes has square one, since the determinant is cyclotomic, and becomes trivial modulo two. We obtain a nonzero quadratic class in $W$ which is a square in $L$, contradicting Lemma 20.1. Thus the actual dyadic source is absolutely irreducible.

Apply Lemma 4.10 again, now with $D = G_{K_{v_+}}$. The resulting nonzero binary class $d \in W$ is split at $v_+$. Every stable lattice of the Barsotti–Tate source is realized by a divisible group, and actual finite flat closures and quotients realize the binary subquotient; see Lemma 11.2. Local absolute irreducibility makes the group and its dual connected by Lemma 11.3. The two order-two factors of the binary subquotient are therefore the intermediate Oort–Tate model $J$. By Lemma 11.4, an extension of $J$ by $J$ killed by two has generic quadratic conductor at most two.

Consequently $d \in \langle -1, g\rangle$. Splitting at $v_+$ excludes $g$ and $-g$, which have odd valuation there. The remaining class $-1$ is nonsquare modulo eleven. This is again a contradiction. □

**Lemma 20.6.** *Let $\bar s : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ have determinant one and solvable image. Suppose it is finite flat at $\mathfrak{p}_2$, unramified outside $\{\mathfrak{p}_2, v_+\}$, and has inertia of order at most two at $v_+$. Then $\bar s^{\mathrm{ss}} = 1 \oplus 1$.*

*Proof.* Use Lemma 11.16a with $S = \{v_+\}$. For a reducible residue, the dyadic residue field $\mathbf{F}_2$ and class number one make its characters trivial. Otherwise write its quadratic inducing field as $M = K(\sqrt{d})$. Fontaine's conductor bound is at most four at the dyadic place, so Lemma 20.1 leaves precisely the seven nontrivial classes in $\langle -1, \epsilon, g\rangle$. Their class groups and radical dyadic ray groups are

| $d$ | $\mathrm{disc}(M)$ | $h_M$ | ray group |
|---|---|---|---|
| $-1$ | 144 | 1 | 1 |
| $g, -g$ | $-6336$ | 1 | 1 |
| $\epsilon$ | 2304 | 1 | 1 |
| $-\epsilon$ | 2304 | 2 | $C_2$ |
| $\epsilon g, -\epsilon g$ | $-25344$ | 1 | 1. |

These degree-four class computations are certified by `bnfcertify`. Except for $d = -1$, the dyadic residue fields are $\mathbf{F}_2$, so no tame ray factor is added. For $d = -1$, the factor $\mathbf{F}_4^\times$ is generated by the image of a cubic root of unity already in $\mathbf{Q}(\zeta_{12})$. None of the ray groups has a nontrivial odd quotient, excluding the rotation character. □

## 20.4 Nonsolvable residues and the final implication

**Proposition 20.7.** *Let $\rho$ be the dyadic member of a weight-two system of cyclotomic determinant, good away from $v_+$ and of type* $\mathrm{CR}(b)$ *or* $\mathrm{SP}(b)$ *there. Suppose its residue has image $\mathrm{SL}_2(\mathbf{F}_q)$, where $q = 2^m \ge 4$, and let $M$ be its point field. Then $[M : \mathbf{Q}] = 2q(q^2-1)$ and*

$$\mathrm{rd}(M) \le 8\sqrt{3}\, N^{1/4} 2^{-2/q}, \qquad N = \begin{cases} 1, & \bar\rho \text{ unramified at } v_+, \\ 11, & \bar\rho \text{ ramified at } v_+. \end{cases}$$

*In the ramified case the decomposition group at $v_+$ is $C_2$ or $V_4$, giving respectively $[M:\mathbf{Q}]/4$ primes of norm eleven or $[M:\mathbf{Q}]/8$ primes of norm 121.*

*Proof.* Let $D, I, P$ be the dyadic images of decomposition, inertia and wild inertia. If $P \neq 1$, it is elementary abelian of order at most $q$ and its unique fixed line is $D$-stable. The diagonal characters have odd order. As the residue field is $\mathbf{F}_2$, the tame Frobenius relation forces their inertia to be trivial. Thus $I = P$. Fontaine's bound is shifted upper index at most four, equivalently last classical upper break at most three, in this ramification index two situation [39, Theorem A]; see also [69, Section 2]. The conductor–discriminant formula gives

$$\frac{d}{e} \leq 4\left(1 - \frac{1}{|P|}\right) \leq 4\left(1 - \frac{1}{q}\right).$$

If $P = 1$, tame inertia is cyclic, with eigenvalues $t, t^{-1}$. Frobenius preserves this pair while squaring it, so $t = 1$ or $t^3 = 1$. The possible tame $C_3$ contributes $d/e = 2/3$, below the same bound. Since $\mathrm{rd}(K) = 2\sqrt{3}$, the dyadic contribution gives $8\sqrt{3}\,2^{-2/q}$.

At $v_+$ the residual inertia is trivial or $C_2$. Its centralizer in $\mathrm{SL}_2(\mathbf{F}_q)$ is the elementary abelian translation group. Since the unramified quotient of the decomposition group is cyclic, the ramified possibilities are $C_2$ and $V_4$. Their residue degrees are one and two. Counting primes above this single degree-one prime of $K$ gives the stated densities. Tame ramification of order two adds the factor $11^{1/4}$ to the root discriminant. □

**Theorem 20.8.** *Every continuous absolutely irreducible totally odd representation $\bar{r} : G_{\mathbf{Q}(\sqrt{3})} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, with $p$ odd, is modular.*

*Proof.* Theorem B.1 gives $\mathrm{Re}\,\rho \leq 2759/3000 < 23/25$ for every nontrivial zero of $\zeta_M$, for every ramified point field $M$ in Proposition 20.7. The explicit formula of Section 13 and the inequalities of Proposition 13.11 exclude all the fields in Proposition 20.7. The unramified comparisons use $a = 1/2$ and require no zero-free hypothesis; the ramified comparisons use $a = 21/50$.

A weight-two system good away from $v_+$ and of type $\mathrm{CR}(b)$ or $\mathrm{SP}(b)$ cannot exist. Its nonsolvable dyadic residue has just been excluded. A solvable residue has trivial semisimplification by Lemma 20.6, and Proposition 20.5 excludes its actual dyadic source. The required local line at $v_+$ is a Frobenius eigenline in the scalar case and the monodromy line in the special case.

It remains to modularize the system $\mathcal{S}_1$ constructed after Lemma 20.2. Its type must be principal series. If its five-adic residue were irreducible, Proposition 20.4 would produce an excluded system $\mathcal{S}_1'$. The residue is therefore reducible, and the same proposition modularizes $\mathcal{S}_1$ through its actual ordinary five-adic member. This supplies a modular witness for $\bar{r}_{11}$. Lemma 2.2(i) modularizes the original $r_{11}$, and the preparation of Propositions 2.7 and 6.2 gives modularity in every odd residual characteristic. □

# 21 The field $\mathbf{Q}(\sqrt{6})$ using an explicit zero-free region

Put $K = \mathbf{Q}(s)$, $s = \sqrt{6}$, and $\epsilon = 5 + 2s$. We write $\omega_\ell = \bar{\chi}_\ell$. A special local parameter is a scalar quadratic twist of the Steinberg parameter with nonzero monodromy. The dyadic place is ramified and five splits.

## 21.1 Reduction to a one-place source

**Proposition 21.1.** *Assume Theorem 6.4. To prove Theorem 1.1 it suffices to modularize parallel-weight-two systems of cyclotomic determinant special at just one place above five and good everywhere else. In any nonmodular system of this kind the three-adic and five-adic*

*residues are nonsolvable. At each dyadic coefficient place either the residue is scalar up to semisimplification and the source is nonordinary, or the residue is nonsolvable and the source is ordinary with residual Frobenius multiplier different from one. At least one dyadic coefficient place is of the latter kind.*

## 21.2 Local types and cubic classes

For $K = F_6$ or $F_{11}$, a bounded system is pure, totally odd, absolutely irreducible, of cyclotomic determinant and labelled weights $\{0, 1\}$, good outside five. Each five-adic type is quadratic-special, denoted SP, or tame determinant-one cuspidal, denoted SC, with inertial eigencharacters of order three or six. A special condition retains its exact quadratic character, including the unramified sign, and has nonzero monodromy. Its characters are $\alpha\chi_p, \alpha$ with $\alpha$ quadratic. At a coefficient-five place its monodromy-stable line gives the special filtration by weak admissibility. In the common arithmetic and ordinary argument below, a good place at five is allowed as well. Its residual inertia is trivial, so it satisfies the same character constraint. The later two-place source theorems use the stated SC, SP types.

### 21.2.1 Characters and cubic classes

Write

$$\epsilon = 5 + 2s, \qquad a = 1 + s, \qquad b = 1 - s, \qquad \pi = 3 + s, \qquad \gamma = 3 + 2s, \qquad H = K(\zeta_3).$$

Then $N(\epsilon) = 1$, $ab = -5$, $N(\pi) = 3$ and $N(\gamma) = -15$. Let $\sigma$ be the nontrivial automorphism of $K/\mathbf{Q}$ and put $\omega = \bar{\chi}_3$. We identify quadratic characters with squareclasses when convenient.

**Lemma 21.2.** *The class number of $K$ is one. The quadratic classes supported at three and five and unramified at two form the group generated by*

$$[-3] = [-\epsilon], \quad [5], \quad [\gamma].$$

*For a reducible $\bar{r}_3$ with determinant $\omega$ arising from a bounded source, its constituents are quadratic. Up to interchange and twisting by the character of $K(\sqrt{5})/K$, their pair is either $\{1, \omega\}$ or $\{\psi_\gamma, \psi_\gamma\omega\}$.*

*Proof.* The Minkowski bound is $\sqrt{24}/2 < 2.45$, and the prime of norm two is generated by $2 + s$, whose norm is $-2$. The unit $\epsilon$ is fundamental, by the least positive solution $(5, 2)$ of $x^2 - 6y^2 = 1$; norm $-1$ is impossible modulo three. The squareclasses supported at three and five have basis $[-1], [\epsilon], [\pi], [a], [b]$.

All five generators are dyadic units. Such a unit $u$ gives an unramified quadratic extension, possibly split, precisely when $u$ is a square modulo $4\mathcal{O}_{K_2}$. A square of a unit $A + Bs$ has coordinates $(1, 0)$ or $(3, 2)$ modulo four, according as $B$ is even or odd. Enumerating the thirty-two products gives exactly eight allowable classes, generated as stated. The identity $3/\epsilon = (3 - s)^2$ identifies $[-\epsilon] = [-3]$.

Let a residual constituent be $\psi$. At five, its square is unramified because semisimplified residual inertia of the bounded types is scalar quadratic. At three, a finite characteristic-three character kills principal units and the residue field is $\mathbf{F}_3$, so its square is unramified there too. It is unramified elsewhere. The narrow class group has order two; its nontrivial character is $\omega$, which is odd at both real places. A square character is even, so $\psi^2 = 1$. The preceding eight classes give the two pairs. Finally, $K_3 = \mathbf{Q}_3(\zeta_3)$ because $-2$ is a square in $\mathbf{Q}_3$. The first pair is locally unramified, while both characters of the second pair are ramified quadratic. □

Let $F = K_3$, normalize $v_F(s) = 1$, and put $U_i = 1 + \mathfrak{p}_F^i$.

**Lemma 21.3.** *The global $S$-unit cubeclass space for $S = \{v : v \mid 15\}$ has basis $[\epsilon], [\pi], [a], [b]$ and maps isomorphically to $F^\times/F^{\times 3}$. The preimage of $U_2/U_4$ is*

$$V = \langle q, d \rangle, \qquad q = [5], \qquad d = [5] + [\epsilon b].$$

*We have $\sigma(q) = q$, $\sigma(d) = -d$. In this basis the two valuations at five are $(1, 1)$ and $(1, -1)$, in some order.*

*Proof.* Since $\mu_3 \subset F$, the class space has dimension four. Moreover $U_1 = \mu_3 U_2$, and the logarithm identifies cubing $U_2 \to U_4$ with multiplication by three. Hence a unit is a cube exactly when it is congruent to 1 or $-1$ modulo $9\mathcal{O}_F$.

The $\pi$ exponent is detected by valuation. Of the twenty-seven products $\epsilon^i a^j b^k$, $0 \le i, j, k < 3$, only the trivial one is $\pm 1$ modulo nine. This finite calculation takes place in $(\mathbf{Z}/9\mathbf{Z})[s]/(s^2 - 6)$ and proves injectivity, hence the asserted isomorphism. Modulo three, $\epsilon = -(1 + s)$, $a = 1 + s$, $b = 1 - s$ and $s^2 = 0$. Membership in $\{\pm 1\}U_2$ is therefore the equation $i + j - k = 0$. This plane is generated by $[5] = [ab]$ and $w = [\epsilon b]$. Since $N(\epsilon b) = -5$, we have $\sigma(w) = q - w$. The assertions for $d = q + w$ and the valuations follow. □

**Lemma 21.4.** *The three-part of the ray class group modulo $\mathfrak{p}_3^3$, with or without the real places, is $C_3$. The two Frobenius classes at five coincide and generate it. The involution $\sigma$ acts trivially on its three-part.*

*Proof.* The unit group of $\mathcal{O}_K/\mathfrak{p}_3^3$ has order eighteen. The images of $-1$ and $\epsilon$ generate a subgroup of order six, giving $C_3$. Here $\mathfrak{p}_3^3 = (9, 3s)$. The residues of $a$ and $b$ belong to neither the image of the units nor its trivial class: the unit residues are $\pm 1, \pm(5 + 2s), \pm(4 + 2s)$ in this quotient. Thus both ray classes are nontrivial. The identity $a - \epsilon^2 b = 72 + 30s \in (9, 3s)$ makes their classes equal. The involution $\sigma$ exchanges the two primes and fixes their common generating class, hence acts trivially on the three-part. Infinite places add only two-primary data. □

**Lemma 21.5.** *No absolutely irreducible characteristic-three residue of a bounded source becomes reducible over $H$.*

*Proof.* The field $H = \mathbf{Q}(\sqrt{-2}, \sqrt{-3})$ has discriminant 576 and class number one. Indeed its Minkowski bound is $36/\pi^2 < 4$. There is no prime of norm two. If $A^2 = -2$ and $u^2 - u + 1 = 0$, the element $1 + Au$ has relative norm $-1 + A$ over $\mathbf{Q}(A)$, hence absolute norm three. Its conjugate gives the other prime of norm three.

A cyclotomic-reducible irreducible residue would be $\mathrm{Ind}_H^K \xi$ with $\xi^c = \xi^{-1}$. Its image has order prime to three. At five, its inertia is scalar quadratic, so $\xi^2$ is unramified. The two primes of $H$ over three have residue field $\mathbf{F}_3$, so $\xi^2$ is unramified there as well. It is unramified elsewhere, hence trivial by the class number. Then $\xi = \xi^c$, contradicting irreducibility of the induction. □

### 21.2.2 Ordinary sources and an unramified quotient

We treat the branch whose residual constituents are unramified at three. No local-distinctness assumption is imposed, since $\omega|_{G_F} = 1$. The argument permits distinct five-adic types and a good place paired with a ramified place.

**Proposition 21.6.** *Suppose that the residual constituents are unramified at three. Then the actual system is modular. After the quadratic normalization of Lemma 21.2, it is invariant under $\sigma$. Consequently neither a genuinely mixed* SC/SP *source nor a source ramified at just one five-place can have this residual branch. Two special conditions in this branch have the same exact character after normalization.*

*Proof.* By Lemma 21.2, a twist by $\psi_5$ makes the residual pseudorepresentation $1+\omega$. This quadratic character is invariant under $\sigma$ and unramified at three.

Lemma 4.3(ii) makes the crystalline source at $F$ ordinary. Write its unramified ordinary quotient as $\nu$; the subcharacter is $\chi_3\nu^{-1}$.

Put $A=\mathbf{Z}_3[[U]]$ and let $\psi_U$ be the unramified character of $G_F$ sending arithmetic Frobenius to $1+U$. Begin with the universal finite-flat-at-three fixed-determinant Cayley–Hamilton pair for $1+\omega$, unramified outside three and five, provided by [86, Theorems 2.5.3 and 2.5.5], and extend its coefficients to $A$. Impose, for every $g,h\in G_F$, the algebra relations

$$(g-\chi_3(g)\psi_U(g)^{-1})(h-\psi_U(h))=0. \tag{22}$$

No local type is imposed at five. To construct the Cayley–Hamilton quotient, take the closed two-sided ideal $I$ generated by these relations, then the scalar ideal generated by the nonconstant coefficients of $D(1-xt)$, $x\in I$, and use [86, Lemma 2.4.5]. Thus no coefficient-$A$ version of the printed representability theorem is assumed. Denote the scalar ring by $R$, its maximal ideal by $\mathfrak{m}$, and its GMA by $\left(\begin{smallmatrix}R & B\\ C & R\end{smallmatrix}\right)$. The actual source factors through it with $U=\nu(\Phi_3)-1$, since upper triangular multiplication proves (22).

The offdiagonal cotangent duals embed into the finite-flat global extension spaces by [86, Theorem 4.3.5]; scalar extension and further quotients retain these inclusions at $3=U=0$. Locally such an extension is unipotent. Fontaine's bound gives classical upper break at most two, hence cubic conductor at most three. A nonunit Kummer class instead has conductor four: after replacing the radicand by an inverse and a cube, $X^3-u$ is Eisenstein with $v_F(u)=1$, and its different exponent is eight. Thus both global extension spaces lie in $V_3=\langle[\epsilon],[a],[b]\rangle$. Their restriction to $G_F$ is injective by Lemma 21.3. For larger coefficient fields the same conclusion follows by extracting $\mathbf{F}_3$ coordinate extensions by pullback and quotient.

Let $J$ be the reducibility ideal and put

$$Q=\mathfrak{m}/(3R+UR+\mathfrak{m}^2),\qquad Q_{\mathrm{red}}=\mathfrak{m}/(3R+UR+\mathfrak{m}^2+J).$$

For $g,h\in G_F$ both residual diagonal characters equal one. The upper-left entry of (22), reduced in $Q$, is $b_gc_h=0$; its other term is a product of elements of $\mathfrak{m}$. Evaluations on all of $G_F$ span the dual of each allowed global extension space, by local injectivity. Hence every product in $(B/\mathfrak{m}B)\otimes(C/\mathfrak{m}C)$ has zero image in $Q$. These products generate the image of $J$, so $Q\simeq Q_{\mathrm{red}}$.

A diagonal tangent relative to $A$ has characters $1+\eta f$ and $\omega(1-\eta f)$, with $\eta^2=0$. Finite-flatness and Fontaine give conductor at most $\mathfrak{p}_3^3$ for $f$. It cannot ramify at five: wild inertia is pro-five and the tame unit quotient has order four. It is unramified elsewhere by construction. Therefore the dual of $Q_{\mathrm{red}}$ is a subspace of $\mathrm{Hom}(\mathrm{Cl}_{\mathfrak{p}_3^3}(K),\mathbf{F}_3)$. The involution $\sigma$ acts trivially on this space by Lemma 21.4. It consequently acts trivially on $Q$; the diagonal tangent need not itself vanish.

The involution $\sigma$ preserves the support, finite-flat condition, determinant and ordinary relations. It fixes $U$, because the unramified Frobenius quotient of $G_F$ is unchanged by the automorphism of $F/\mathbf{Q}_3$. Thus it acts on $R$ over $A$. Its action on $Q$ is the identity. Lemma 12.2, applied over $A$, makes the involution $\sigma$ trivial on $R$. Thus $r_3^{\sigma}\simeq r_3$.

Lemma 12.4, with $\eta=1$, now proves modularity of the original system.

Finally, $\sigma$ exchanges the two places above five, and invariance identifies their local Weil–Deligne representations. Their monodromy cannot be zero at one place and nonzero at the other. It also cannot have trivial inertia and monodromy at only one place. For two special types it also preserves the exact quadratic character. This proves the asserted exclusions. □

### 21.2.3 Ramified quadratic constituents

The other pair in Lemma 21.2 requires a different finite-flat extension space.

**Lemma 21.7.** *For either ramified quadratic character $\theta$ of $G_F$, the finite-flat selfextensions of $\theta$ form exactly the Kummer plane $U_2/U_4$.*

*Proof.* The rank-one model has Breuil–Kisin exponent one. An extension has Frobenius matrix

$$\begin{pmatrix} cu & h(u) \\ 0 & cu \end{pmatrix}.$$

The height-one condition imposes no further restriction. A change of upper lift changes $h$ by $cu(f(u^3) - f(u))$. Positive powers are removed recursively, while constants in $f$ disappear. The extension space has dimension at most two, represented by $1, u$. Schematic closure and flat quotient give the rank-one endpoints; the exact finite-flat description used here is [53, Theorem 1.1.3, Proposition 1.1.11 and Lemmas 1.2.4–1.2.5]. Coordinate pullback and quotient give the same bound for arbitrary coefficients.

The unramified selfextension is finite flat and gives $U_3/U_4$. A generator is $1+6s$: substituting $\alpha = 1 + sz$ in $\alpha^3 = 1 + 6s$ gives $z^3 + (s/2)z^2 + (1/2)z - 1 = 0$, whose reduction $z^3 - z - 1$ is irreducible and separable over $\mathbf{F}_3$.

For an independent class take $E : y^2 = x^3 - \frac{s}{8}x^2 + x$. Its discriminant is $-125/2$, and its reduction is supersingular. Its division polynomial factors as $\psi_3(x) = (x - 1/s)(3x^3 + 6x + s)$. At $x = 1/s$, the ordinate square is $s(5/12)^2$, so the stable subgroup has character $\theta_s$. Cardano's radicand for the other three lines is $2/(3s)$, whose product with $s^3$ is four. Hence its actual extension line is [2]. It lies in $U_2 \setminus U_3$ after multiplying by the cube $-1$. An unramified quadratic twist gives the other ramified character; specifically $\gamma/(2s) = 1 + s/4$ is a local square. These two lines exhaust $U_2/U_4$. □

**Proposition 21.8.** *A weight-two comparison system with just one five-place of type from Lemma 4.17 and good everywhere else is modular if its three-adic residue is reducible with ramified constituents at three.*

*Proof.* Write $a = 1 + s$, $b = 1 - s$, and $\gamma = 3 + 2s$. After conjugating $K$ if necessary, the two constituents are $\psi_\gamma, \psi_\gamma\omega_3$, where $\psi_\gamma$ is the quadratic character for $\gamma$. Indeed the other possible pair is its twist by $\psi_5$, which is also its quadratic conjugate up to interchange. The unique bad five-place is the one where $\psi_\gamma$ ramifies. The common local character at three is ramified because $K_3 = \mathbf{Q}_3(\zeta_3)$.

Let $V$ be the finite-flat global cubic extension space with both five-places allowed. By Lemmas 21.3 and 21.7 it is contained in the Kummer plane spanned by $[5]$ and $[5] + [\epsilon b]$. The two five-adic inertia evaluations are independent on this plane. Use the fixed-determinant finite-flat Cayley–Hamilton problem of Proposition 21.6, over $\mathbf{Z}_3$ without the parameter $U$. In this problem the two off-diagonal tangent duals embed into this plane. Imposing goodness at the other five-place puts both spaces in the kernel of its evaluation.

For a special source the exact special characters at the bad place forbid ramification in one of the two orientations, since the two residual Frobenius characters are distinct there. Thus one off-diagonal space also lies in the kernel of the second evaluation, and is zero. The generalized matrix algebra and Nakayama then make the characteristic-zero source reducible, a contradiction.

For a cuspidal source the bad place has inertial order six, since its residual quadratic scalar is ramified. After removing that fixed scalar locally, the relation $(T - 1)^2 = -3T$ holds for tame inertia and Frobenius has trace zero. In the relative pseudocotangent the inertia relation kills the product of the two off-diagonal evaluations at this place. Indeed, modulo three and the square of the maximal ideal, the diagonal entries of $(T - 1)^2$ reduce to that product. The evaluation at the bad place is nonzero on the line cut out by the good place. Thus the product

relation kills the entire image of the reducibility ideal in the relative cotangent. The diagonal quotient comes from the three-part of the ray modulo $\mathfrak{p}_3^3$, which by Lemma 21.4 is $C_3$ generated by a five-Frobenius. The trace-zero condition at the bad five-place kills this tangent, because the residual diagonal Frobenius values there are opposite. The resulting complete local scalar ring is a quotient of $\mathbf{Z}_3$ and has an actual characteristic-zero point, so it is $\mathbf{Z}_3$. This argument uses the actual finite-flat condition of Wake–Wang-Erickson [86, Theorems 2.5.3, 2.5.5 and 4.3.5].

The CM curve

$$E: \quad y^2 = x^3 + \gamma^3(7+3s)^2/4$$

gives a point of this same problem. It is the $\gamma$-twist of $y^2 + (7+3s)y = x^3$, with the displayed residual pair from the three-torsion line at $x = 0$. Its conductor is the square of the bad five-place, where it has type II$^*$, and it is good at two, three and the other five-place. These conductor and local reduction calculations are checked by the CM seed certificate. The determinant-one cuspidal type at the bad place is the unique order-six pair in the bounded menu. This fixes the inertia type, and the exact Frobenius conditions are trace zero and square $-5$. Thus its trace point agrees with the unknown trace point. It follows that the actual unknown system is modular.

□

### 21.2.4 Scalar dyadic residues

**Lemma 21.9.** *For a parallel-weight-two system of cyclotomic determinant, special at one five-place and good elsewhere, every solvable dyadic residue has trivial semisimplification. A scalar-residual ordinary dyadic member is impossible. If every dyadic residue is scalar up to semisimplification and every dyadic member is nonordinary, the system is modular.*

*Proof.* Consider a system special at just one five-place $v$. In characteristic two its tame inertia at $v$ has order at most two and its decomposition group lies in $C_2^2$. For a solvable irreducible dyadic residue the inducing field is quadratic and the odd inducing character is unramified at $v$, tame at two and unramified elsewhere. Fontaine's bound gives dyadic conductor at most four for the quadratic field. Thus its squareclass is one of the seven nontrivial classes in $\langle [-1], [\epsilon], [1+s] \rangle$ after conjugating the base if needed. All seven ray groups modulo (2) are two-primary by the quartic certificate in the source archive. No irreducible solvable residue occurs. Reducible dyadic characters of odd order are unramified everywhere because the dyadic residue field is $\mathbf{F}_2$ and the inertia at five has order at most two. Therefore a solvable dyadic residue is scalar.

If a dyadic member with scalar residual semisimplification is ordinary, apply Lemma 4.10 with the dyadic decomposition group and its ordinary line. It gives a nonzero quadratic character supported at two and $v$ and split at two. The kernel of localization on the two-five-place space $\langle [-1], [\epsilon], [1+s], [1-s] \rangle$ is generated by $[-15]$ by the local square test modulo $8\mathcal{O}_{K_2}$: the squares of odd units modulo eight have coordinates $(1,0), (1,4), (7,2), (7,6)$ in the basis $1, s$, and only $1, [-15]$ in the indicated span are locally squares. But $[-15]$ ramifies at both five-places. This excludes an ordinary scalar source with one special place.

It remains to consider a system whose dyadic residues are all scalar and whose dyadic members are all nonordinary. Use the simultaneous trace and regular orders $R_0, E_0$ of Section 11 and the proof of Lemma 11.9. Put $R = R_0/2R_0$, $E = E_0/2E_0$, and $J = \mathrm{rad}(E)$. A functional on $J/J^2$ gives a quadratic character with dyadic conductor at most two by the middle finite-flat model calculation in Lemma 11.5. For one five-place the allowed space has basis $\alpha = [-1], \beta = [\epsilon] = [2]$. Indeed an odd valuation at two has conductor five, and the unit square test modulo 2 rules out the exponent of the generator $1+s$ of $v$. The two allowed characters are unramified at both five-places.

Let $A$ be the difference of a real conjugation from one, and $B$ that of a Frobenius at $v$. Their evaluations on $(\alpha, \beta)$ are $(1, 0)$ and $(0, 1)$. Thus they generate $E$ as an $\mathbf{F}_2$-algebra. The real involution and the exact special parameter give

$$A^2 = B^2 = 0, \qquad AB + BA = X = T(AB),$$

with all identities in the actual order modulo rational two. For $B$, the special Frobenius eigenvalues have common sign and are $\pm 5, \pm 1$, so its trace and determinant have the required reductions even if its quadratic character is ramified. The normal words $1, A, B, AB$ over $\mathbf{F}_2[X]$ have traces $0, 0, 0, X$, so $R = \mathbf{F}_2[X]$. The image of $X$ at every coefficient residue is zero since the reductions are upper unipotent. The product of the dyadic coefficient integer rings is integral over $R_0$. Lying over therefore puts $X$ in the radical of the finite algebra $R$.

By Lemma 11.9 the determinant of the Frobenius generator $\Phi(u)$ is divisible by the Eisenstein polynomial. For a functional $\lambda$ on $J/J^2$ the algebra quotient $E \to \mathbf{F}_2[\delta]/(\delta^2)$ with $\delta$-coefficient $\lambda$ on $J$ gives a middle-model self-extension. In that quotient $\Phi(u)$ has diagonal term $u$ and off-diagonal term $h(u)$ as in Lemma 11.5. Its constant coefficient $h(0)$ records the ramified part of the restricted character. Both $\alpha$ and $\beta$ are ramified, and their sum $[-3]$ is unramified at two. Thus $\Phi(0)$ and $A + B$ agree under every $\lambda$, giving $\Phi(0) = A + B + Z$ with $Z \in J^2$ in the order modulo two.

Compute the trace and determinant values in $R/\mathfrak{m}_R^2$. The normal words put $J^2$ in the span of $AB$ and $\mathfrak{m}_R E$. The latter terms have zero norm and zero polar pairing because the traces lie in $\mathfrak{m}_R$. The term $AB$ has zero norm and zero polar pairing with $A+B$, since $(A+B)AB = XB$ and $T(B) = 0$. Hence the constant norm is $\mathcal{D}(A + B) = X$. It vanishes by divisibility. Since $R = \mathbf{F}_2[X]$ and $X \in \mathfrak{m}_R^2$, Nakayama gives $R = \mathbf{F}_2$. The global trace field is $\mathbf{Q}$.

At a coefficient place over five an irreducible residue with rational traces has a realization over $\mathbf{F}_5$ by finite-field descent. It is modular by [3, Theorem 2.5]. Its restriction to $G_{K(\zeta_5)}$ is irreducible by Lemma 4.6 at the good five-place. The actual weight-two source is then modular. If the five-adic residue is reducible, the source is ordinary at the special place and also at the good weight-two place by the finite-flat argument. [32, Corollary 1.2.2] applies to the dual with determinant $\chi_5^{-1}$, because $K/\mathbf{Q}$ is abelian and five is unramified. This argument also applies to any special one-place system with globally reducible five-adic residue. An irreducible solvable five-adic residue is handled by the same cyclotomic irreducibility check and its Artin witness.

□

**Lemma 21.10.** *For a nonmodular system as in Lemma 21.9, a nonsolvable dyadic residue has an ordinary source with residual Frobenius multiplier different from one.*

*Proof.* By Dickson's classification the kernel field $M/K$ of the residue has group $\mathrm{SL}_2(\mathbf{F}_q)$, where $q \geq 4$ is a power of two, and $[M : \mathbf{Q}] = 2q(q^2 - 1)$. At the special five-place inertia has order at most two, so its contribution to $\mathrm{rd}(M)$ is at most $5^{1/4}$. At two Lemma 11.10 gives normalized contribution less than two. In particular $\mathrm{rd}(M) < 4\sqrt{24}\, 5^{1/4} < 30$. If $q = 4$, the classical real-place lower bound of Section 13 shows that $M$ is not totally real. Lemma 2.4 proves modularity.

Suppose $q \geq 8$. The degree is at least 1008, so the classical lower bound of Section 13 is greater than 20. If the actual source is nonordinary, its normalized dyadic different is less than one. Indeed, a locally irreducible residue is tame of inertia order three by the argument of Lemma 4.8. In the reducible case its diagonal characters are unramified, since the dyadic residue field is $\mathbf{F}_2$. After unramified base change the residue is an additive extension of trivial characters. Its $\mathbf{F}_2$-coordinate extensions have biconnected finite-flat models by schematic closure and quotient. Lemma 11.5 bounds their conductor by two, giving the claimed different bound. Thus $\mathrm{rd}(M) < 15$, a contradiction.

For an ordinary source with trivial residual multiplier the unramified quotient character reduces to one, and the local residue is an elementary abelian group of translations. Its quadratic characters have conductor at most four by Lemma 11.10, so all correspond to unit squareclasses. In a ramified quadratic completion with residue field $\mathbf{F}_2$, these form a space of dimension three. The conductor-at-most-two subspace has dimension two, by Lemma 11.6. The unramified subspace has dimension one. The conductor–discriminant formula therefore gives normalized different at most $5/4$: among the three nontrivial ramified inertia characters, the largest possible conductors are $4, 4, 2$. It follows that $\mathrm{rd}(M) \le 4\sqrt{24}\,5^{1/4}2^{-3/4} < 20$, a contradiction. □

*Proof of Proposition 21.1.* Start with Proposition 2.7. In the nonsolvable case, Proposition 6.2 with $\ell = 5$ leaves a terminal member with marked places $v_+, v_-$. The reducible terminal constituent ratio is an odd cyclotomic power by Lemma 6.7, so the reducible and irreducible solvable branches are covered by Proposition 6.5.

For nonsolvable residue, apply Lemma 4.17 at $v_+$ and keep the crystalline weight-two component at $v_-$. Lemma 2.6(i) gives a weight-two comparison system, whose modularity returns to the actual old five-adic member by Lemma 2.2(i).

If the comparison system's residue at a place above three is reducible, Proposition 21.6 or Proposition 21.8 applies according to the local constituent characters. If it is irreducible, it remains so over $K(\zeta_3)$ by Lemma 21.5. In the solvable case an Artin witness and Lemma 2.2(ii) prove modularity. For a nonsolvable residue, Lemma 4.18 prescribes a genuine special lift at $v_+$, retaining the full residual local representation. Retain the crystalline weight-two component at three and the good condition at $v_-$. Lemma 2.6(i) gives a comparison system special at $v_+$ and good everywhere else. Purity forces nonzero monodromy on the special component as in Section 2. Lemma 2.2(ii) gives the backward implication.

For a special one-place system the same analysis applies to every coefficient place at three. At five a reducible global residue makes the actual source ordinary at both places, by the special filtration and the finite-flat argument. The dual source has determinant $\chi_5^{-1}$ and ordinary weight $(0, 0)$ in the convention of [32, Corollary 1.2.2], which applies over this totally real abelian field unramified at five. An irreducible solvable residue gives an Artin witness, and Lemma 4.6 at the good five-place gives the cyclotomic irreducibility needed by Lemma 2.2(i).

Finally Lemmas 21.9 and 21.10 give the stated alternatives at each dyadic coefficient place. If all such places had scalar semisimplification, they would all be nonordinary and the simultaneous-order calculation would prove modularity. □

## 21.3 The switch at two

Let $a_2$ be the common crystalline Frobenius trace at the dyadic place of a nonmodular system left by Proposition 21.1. At an ordinary dyadic coefficient place its quotient has unit-root value $u$ and $a_2 = u + 2/u$, with $\bar{u} \neq 1$ since the residual multiplier is nontrivial. At a nonordinary dyadic coefficient place both crystalline Frobenius eigenvalues have positive valuation, since a slope-zero factor would give an unramified quotient. Thus $\bar{a}_2 \neq 1$ at all dyadic coefficient places, and $a_2$ is a unit at an ordinary one. These local formulas also give integrality at dyadic coefficient primes. At odd coefficient primes integrality follows from the stable lattices. Lemma 4.7 gives an odd coefficient prime $\lambda$ dividing $9 - a_2^2$ and a special lift of its residue at two. Write $\ell$ for its residue characteristic.

At $\ell = 3, 5$ the original residue is nonsolvable. A reducible restriction to the cyclotomic subgroup would make the projective image solvable by Clifford theory and the abelian quotient. At five the exceptional case of [35, Definition 4.1.1] would require the projective $S_5$ field to contain $K(\zeta_5)$. The latter has degree four over $K$, whereas $S_5$ has abelianization of order two.

For $\ell > 5$ suppose the residue has constituents $\psi, \omega_\ell\psi^{-1}$. The character $\psi^2$ is unramified

outside $\ell$ at finite places since the special scalar character at $v$ is quadratic. The fundamental unit $\epsilon = 5 + 2s$ is totally positive and has norm one. At an inert $\ell$, a mixed pair of finite-flat digits would give tame exponent 1 or $\ell$ for a constituent. Reciprocity on $\epsilon$ for $\psi^2$ would then imply $\epsilon^2 \equiv 1 \pmod{\ell}$. This is impossible since $N_{K/\mathbf{Q}}(\epsilon^2 - 1) = -96$. At a split $\ell$ the rank-one finite-flat exponents are already the two endpoints. By Lemma 4.3(i) the actual source is ordinary at every $\ell$-place. Its dual satisfies Dotto–Thorne [32, Corollary 1.2.2] with determinant $\chi_\ell^{-1}$ and ordinary weight $(0, 0)$.

For an irreducible residue at $\ell > 5$, Lemma 4.6 gives cyclotomic irreducibility. Retain the original weight-two components at $\ell$ and the exact special condition at $v$, and choose the special lift at two. Impose the good conditions elsewhere. Lemma 2.6(i) supplies a weight-two system. When $\ell = 5$ we keep the component through the original potentially semistable weight-two point at $v$. Purity gives nonzero monodromy at the special places. The new system is special at two and $v$, with unramified special sign at two. If it is modular, its $\ell$-adic member supplies the weight-two residual witness in Lemma 2.2(ii) for the actual original member.

## 21.4 Sources special at two and at one five-place

After conjugating the labelling if necessary, let $v = (1 + s)$, and put $\pi = 2 + s$ and $a = 1 + s$. They generate the dyadic prime and $v$ since their rational norms are $-2$ and $-5$. The class number of $K$ is one by Minkowski. Its units are generated by $-1, \epsilon$. These facts follow from the continued fraction of $s$ and the Minkowski bound $\sqrt{24}/2$.

At two and $v$ the residual decomposition images of a dyadic member $r$ are elementary abelian groups of translations. In each case the actual special filtration has characters reducing to one. At $v$ the inertia subgroup has order at most two. All other finite places are unramified.

**Lemma 21.11.** *If the residual image of $r$ is solvable, then the system is modular.*

*Proof.* In the reducible case its constituents have odd order and are unramified at every finite place, so are trivial by class number one. The actual image is pro-two. The supported global squareclasses have basis

$$[-1],\ [\epsilon],\ [\pi],\ [a].$$

Localization at two is injective. Valuation parity removes $[\pi]$ from a local square. The norm of $a$ is $-5$, a nonsquare in $\mathbf{Q}_2$, while the norms of $-1, \epsilon$ are squares. Finally $\pi^2 = 2\epsilon$, and the three nontrivial rational classes $-1, 2, -2$ remain nonsquare in $\mathbf{Q}_2(\sqrt{6})$. Dyadic decomposition thus surjects onto every pro-two quotient with the prescribed support. Indeed, a proper image in a finite two-group lies in an index-two subgroup and gives a nontrivial quadratic class split at two. In particular the actual global image equals its dyadic decomposition image. The special line would then be globally invariant, contrary to absolute irreducibility.

In the irreducible solvable case the image is dihedral and the rotation character has odd order. It is unramified everywhere on the inducing quadratic field, and trivial on the primes over two and $v$, since the decomposition images there are two-groups. The fifteen quadratic fields are given by the nontrivial supported squareclasses above. The ideal-relation certificate of Appendix C.3 bounds the odd part of the class group after killing the classes above two and $v$. It is trivial except possibly for $K(\sqrt{-\pi a})$, where its order divides three. This inducing field is CM, since $\pi a$ is totally positive, and ramifies at two by odd valuation. Allen's Lemma 5.1.2 and Theorem 5.2.1 [2] now apply to the actual ordinary source. Its parallel weights are $\{0, 1\}$, its ordinary quotient is unramified and the inducing CM field is nonsplit at the dyadic place. Compatibility concludes. □

**Lemma 21.12.** *Assume the zero-free half-plane $\Re s > 0.9586$ for Dedekind zeta functions, apart from the pole at one. A source as in this section has solvable dyadic residue.*

*Proof.* If its residue is nonsolvable, its kernel field $M/K$ has group $\mathrm{SL}_2(\mathbf{F}_q)$ with $q = 2^j \geq 4$. Write $(e, f)$ for its relative dyadic ramification index and residue degree. The local point field lies in the full elementary quadratic extension of $K_2$. This has degree sixteen, residue degree two and ramification index eight. With $v(2) = 1$, the relative different contribution is at most $D_e$, where

$$(D_1, D_2, D_4, D_8) = (0, 5/4, 7/4, 15/8).$$

The sixteen local characters have conductors $0, 0, 2, 2, 4, 4, 4, 4$ and eight conductors five. Odd valuation gives conductor five, units have conductor at most four, and the conductor-at-most-two space is the character group of $K_2^\times/(K_2^{\times 2}(1 + 2\mathcal{O}_{K_2}))$, of order four. For $e = 2$ use conductor at most five. For $e = 4, f = 1$ at most two of the three nontrivial characters have odd valuation. For $e = 4, f = 2$ at most four characters have odd valuation and two of the other four are unramified. For $e = 8$ either the entire space occurs or the dimension-three subspace meets the conductor-at-most-two space in a nonzero class. These facts and the conductor–discriminant formula give the displayed bounds. At $v$ the contribution to the absolute root discriminant is at most $5^{1/4}$, since $e_v \leq 2$ and $v$ has degree one over five. Therefore

$$\mathrm{rd}(M) \leq \sqrt{24}\, 2^{D_e} 5^{1/4}.$$

We use the densities $\rho_P = N_P/n$ of Section 13 in Appendix C.1. Keeping only even powers of a term of norm $P$ gives a term of norm $P^2$ with half its density.

For the small $q$ ranges let $L = M^U$ and $r_0 = 1/(q+1)$ as in Lemma 13.9. This gives the real fraction and multiplies both relative different contributions by $1 - r_0$. At $v$ the fixed block gives $\rho_5 = r_0/2$ and the free block gives $\rho_{25} \geq (1-r_0)/8$. At two the terms are $\rho_2 = r_0/2+(1-r_0)/(2e)$ if $f = 1$, and $\rho_2 = r_0/2$, $\rho_4 = (1 - r_0)/(4e)$ if $f = 2$. Thus the rows for $L$ handle $q = 4, 8$, the case $q = 16$ with $e = 8, f = 1$, and $e = 8, f = 2$ for $16 \leq q \leq 256$.

For $q \geq 16$ and $e \leq 4$ use $MH$ for $H = K(\zeta_3)$. The extension $H/K$ is unramified at finite places and is unramified quadratic locally at two and at $v$. It is disjoint from $M$ by simplicity. Thus the degree doubles, the root discriminant is unchanged, and one may take $\rho_4 = 1/16$ and $\rho_{25} = 1/8$. For $q \geq 32$ with $f = 1$ use $M$ with $\rho_2 = 1/16$ and $\rho_{25} = 1/8$.

Only $q \geq 512$ with $e = 8, f = 2$ remains. If the bad five-place has residue degree one or has trivial inertia, the indicated classical comparisons on $M$ apply. Otherwise its local field is the full elementary quadratic extension of $K_v$. Use the elementary abelian extension

$$V = K(\sqrt{-1}, \sqrt{\epsilon}, \sqrt{\pi}, \sqrt{a}).$$

It has degree sixteen over $K$ by independence of the supported squareclasses. Its local extensions at two and $v$ are absorbed by the two full elementary extensions in $M$. Both fields are unramified at all other finite places. Simplicity of the group over $K$ gives $M \cap V = K$. Hence $MV$ has degree $16[M : \mathbf{Q}]$, root discriminant $\mathrm{rd}(M)$, and densities $\rho_4 = 1/32$, $\rho_{25} = 1/8$. At $b = 150$ the interval certificate with $a_0 = 0.4586$ gives $\mathrm{rd}(MV) > 26.87542$, against the upper bound 26.87073. The bound increases with degree, so this row handles the entire tail. □

*Completion for $d = 6$ in Theorem 1.1.* The half-plane in Lemma 21.12 follows from Theorem B.11 below. Lemma 21.11 then proves modularity of the comparison system. The backward implication of Section 21.3 and the one-place reduction give the assertion. □

## 22 The field $\mathbf{Q}(\sqrt{33})$

Lemma 14.5 leaves kernel fields $L$ with group $\mathrm{SL}_2(\mathbf{F}_q)$, $q \geq 8$. Use $M$ and $k$ as in Lemma 14.4.

For $d = 33$, put $b = 24$. If $k \geq 1$, retain just the first norm-four term from Lemma 14.4. At degree at least 4032, (12) gives $\mathrm{rd}(M) > 23.31$, whereas (20) gives $\mathrm{rd}(M) \leq 4\sqrt{33} < 22.979$.

If $q = 8$ and $k = 0$, the same formula without prime terms gives $\mathrm{rd}(M) > 21.75 > 21.07115$, contradicting (20). For $k = 0$ use the original kernel field $L$: Lemma 14.3 gives $[L : \mathbf{Q}] = 2q(q^2-1)$ and $\mathrm{rd}(L) \le \sqrt{33}\, 2^{2-2/q}$. The resulting unconditional comparisons are

| $q$ | $[L : \mathbf{Q}]$ | lower $\mathrm{rd}(L)$ | upper $\mathrm{rd}(L)$ |
|---|---|---|---|
| 16 | 8160 | $> 21.97$ | $< 21.07115$ |
| 32 | 65472 | $> 22.15$ | $< 22.00405$ |

Only $k = 0$, $q \ge 64$ remains without the additional zero-free input.

Assume now ZF(31/32), as in Theorem 1.1. The preceding unconditional comparisons leave only $k = 0$, $q \ge 64$. For these cases $n = [M : \mathbf{Q}] = 8q(q^2 - 1) \ge 32640$. Use $a = 15/32$, $b = 30$, Lemma 13.6 and (15) in Proposition 13.2, discarding every prime term. It gives

$$\mathrm{rd}(M) \ge 8\pi e^{\gamma} \exp\left(-\frac{13}{20} - \frac{3\pi^2}{4 \cdot 30^2} - \frac{4J_{30}(-1/32)}{n}\right) > 23.06000 > 4\sqrt{33}.$$

Degree monotonicity covers the whole remaining range, contradicting (20). The remaining irreducible residual cases are thus excluded under ZF(31/32). The class group and scalar Selmer calculations are in Lemmas 14.1 and 14.2. Theorem 1.4 supplies the scalar source, and Theorem 10.1 completes the proof for $d = 33$ in Theorem 1.1. The finite strip constants and comparisons are certified in [37, `analytic/all_q_discriminants.py`].

# Part III
# Fields under GRH

## 23 Dihedral sources over $\mathbf{Q}(\sqrt{10})$

Put $K = \mathbf{Q}(s)$, $s^2 = 10$.

**Theorem 23.1.** *Assume GRH. Every continuous irreducible representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ of determinant $\bar{\chi}_3$ unramified outside three is modular.*

The places $v_+, v_-$ above three correspond to $s \equiv 1, -1$; their completions are $\mathbf{Q}_3$. The dyadic completion $F = K_2$ has ramification index two and residue field $\mathbf{F}_2$. The local menu below reduces the nonsolvable dyadic case to $A_5$. For solvable residue, the CM inducing field splits dyadically, so we instead bound the global trace field across all its dyadic embeddings to modularize the actual source. At $v_+$ use the wild type $\tau_+$ of Lemma 4.22, with uniformizer $+3$ and its specified quadratic subfield. At $v_-$ use the special or quartic type of Lemma 4.23; both have the characteristic-two kernel envelope (6).

**Proposition 23.2.** *Let $\overline{\rho}$ satisfy the hypotheses of Theorem 23.1, and suppose its image is nonsolvable. It lifts to a totally odd, potentially modular, strongly compatible system $\mathcal{R} = (r_\lambda)$ with determinant $\chi$ and weights $\{0, 1\}$. The system is unramified away from three, has type $\tau_+$ at $v_+$, and has one of the two local conditions of Lemma 4.23 at $v_-$.*

*Proof.* Apply Lemma 2.6(i) with cyclotomic determinant, the local points of Lemmas 4.22 and 4.23, and unramified conditions elsewhere. Nonsolvability gives irreducibility on $G_{K(\zeta_3)}$. The dyadic members are crystalline, and Section 2 retains nonzero monodromy on each selected special component. □

Here and in Section 30, write $r_2$ for a chosen dyadic member, $\overline{r}_2$ for its reduction, and $M$ for the kernel field of $\overline{r}_2$. By Lemmas 4.22 and 4.23, $\overline{r}_2$ is irreducible of determinant one; its kernel field has completion $E_+$ at the wild place and a subfield of $E'$ at the other place above three. Here these places are $v_+$ and $v_-$, respectively.

### 23.1 The controlled characteristic-two image

**Lemma 23.3.** *The splitting field $A/K$ of $X^3 - 3X + 1 + s$ has Galois group $S_3$. It is unramified away from three, and its dyadic completion is the unramified cubic extension of $K_2$. Its completions at $v_+$ and $v_-$ are $E_+$ and $\mathbf{Q}_3(\sqrt{3})$, respectively.*

*Proof.* The cubic reduces to $X^3 + X + 1$ at the unique dyadic prime. This polynomial is irreducible and separable over $\mathbf{F}_2$, proving both global irreducibility and the dyadic assertion. Its discriminant is

$$-27(7 + 2s) = -6\left(3 + \frac{3}{2}s\right)^2, \qquad N_{K/\mathbf{Q}}\big(-27(7 + 2s)\big) = 3^8.$$

It follows that $\mathrm{Gal}(A/K) = S_3$, with quadratic subfield $K(\sqrt{-6})$, and that no finite prime away from three ramifies.

At $v_+$ put $d = s - 1$. Since $d(s + 1) = 9$, we have $v_3(d) = 2$. Writing $X = 1 + y$ gives $y^3 + 3y^2 + d = 0$, whose roots have valuation $2/3$. The element $z = y^2/3$ satisfies $z^3 - 3z^2 - \frac{2d}{3}z - \frac{d^2}{27} = 0$. This polynomial is Eisenstein. In the resulting cubic extension the three

derivative terms have valuations $5, 4, 3$, respectively, so its different exponent is three. The lower ramification groups of the Galois closure consequently have orders $6, 3, 1$, giving normalized different $(5+2)/6 = 7/6$. Its quadratic subfield is $\mathbf{Q}_3(\sqrt{-6}) = \mathbf{Q}_3(\sqrt{3})$, so the closure is $E_+$.

At $v_-$, the constant term $1+s$ has valuation two. The Newton polygon has slopes $-1$ and $-1/2$, of lengths one and two. There is one linear factor and one ramified quadratic factor. Its splitting field is the quadratic field defined by the discriminant, namely $\mathbf{Q}_3(\sqrt{3})$. □

**Proposition 23.4.** *There is a solvable Galois extension $N/K$ of degree 768, unramified outside two and three, whose dyadic completions have no upper break greater than three. Its completion at $v_+$ is $E_+$, and its completion at $v_-$ is contained in $E'$.*

*Proof.* We use an abelian ray extension of the field $A$ in Lemma 23.3. An absolute equation for $A$ is

$$P_A(X) = X^{12} - 2X^{11} + 5X^{10} + 19X^8 + 2X^7 + 31X^6 + 2X^5 + 19X^4 + 5X^2 - 2X + 1.$$

It is obtained by taking the compositum of $X^6 - 6X^4 + 2X^3 + 9X^2 - 6X - 9$ with $X^2 + 6$; if $u$ is the root of the degree-six polynomial, the embedding of $K$ is given explicitly by $s = -u^3 + 3u - 1$. The class number is four. There are two primes $\mathfrak{P}_1, \mathfrak{P}_2$ above two, each of absolute ramification index two and residue degree three. Above three there is one prime of absolute ramification index six and residue degree one, lying over $v_+$, and three primes of ramification index two and residue degree one, lying over $v_-$. The field has discriminant 241864704000000 and signature $(0, 6)$.

Let $R$ be the ray class group of modulus $\mathfrak{P}_1^4\mathfrak{P}_2^4$. Quotient $R$ by the class of the prime over $v_+$ and by twice the class of each prime over $v_-$. Exact ideal and unit arithmetic gives

$$R \simeq C_4^2 \times C_2^5, \qquad R/\langle [\mathfrak{q}_+], 2[\mathfrak{q}_-] \rangle \simeq C_4 \times C_2^5.$$

Here the bracket in the denominator includes all three primes over $v_-$. These are calculations in degree twelve. The accompanying PARI/GP certificate computes maximal orders and ray class groups, checks the quotient order 128 by Hermite and Smith normal forms, and returns `bnfcertify` $= 1$ for the class-group calculation. The exact conductor of the quotient has exponent four at both dyadic primes. The exact calculation is supplied in [20, `ramified/d6_d10_d11/d10`].

Let $N/A$ be the associated ray extension. The modulus and the relations are preserved by $\mathrm{Gal}(A/K)$, so $N/K$ is Galois. Its group is an extension of an abelian group by $S_3$, hence is solvable, and $[N : K] = 128 \cdot 6 = 768$. At $v_+$ the ray relation makes $N/A$ split, so its completion remains $E_+$. At each prime over $v_-$ it is unramified of degree at most two, giving a subfield of $E'$. At two, conductor exponent at most four gives upper break at most three over $A$. Since $A/K$ is unramified there, the same upper bound holds over $K_2$. □

**Theorem 23.5.** *Assume GRH for Dedekind zeta functions. There is no finite Galois extension $M/K$, unramified outside two and three, satisfying all the following conditions:*

1. $\mathrm{Gal}(M/K) \simeq \mathrm{SL}_2(\mathbf{F}_q)$ *for $q = 2^r \geq 8$;*
2. *its dyadic completions have no upper break greater than three;*
3. *its completion at $v_+$ is $E_+$ and its completion at $v_-$ is contained in $E'$.*

*Proof.* The group $\mathrm{SL}_2(\mathbf{F}_q)$ is nonabelian simple in characteristic two. Thus $M \cap N = K$ for the solvable Galois extension in Proposition 23.4. For $L = MN$ we obtain $[L : \mathbf{Q}] = 2 \cdot 768\, q(q^2 - 1) \geq 774144$. Upper numbering is compatible with quotients, so the compositum still has no dyadic upper break greater than three. Since $K_2$ has absolute ramification index

two, its normalized relative different is strictly less than two. The completions at three remain in the stated extensions. The global discriminant formula therefore gives $\mathrm{rd}(L) < 4\sqrt{40}\,3^{5/6} = 63.1962098567\ldots$. Apply the GRH explicit formula of Section 13 with retained densities $\rho_3 = 1/12$ and $\rho_9 = 1/8$. The $v_+$ primes have norm three. If the other completion has residue degree two, its norm-nine primes give the second coefficient. If it has residue degree one, retaining only the even powers of its norm-three Euler terms gives at least the same contribution.

At $b = 13.3$ and degree 774144, interval arithmetic gives $\mathrm{rd}(L) > 64.5683408714$. The lower expression increases with the degree. This contradicts the preceding upper bound and proves the theorem. The exact calculation is supplied in [20, `ramified/d6_d10_d11/shared`]. □

**Corollary 23.6.** *Assume GRH. Let $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be a continuous representation of determinant $\chi_2$, unramified outside finitely many primes and Barsotti–Tate at two. Suppose that its residual representation is irreducible and nonsolvable, is unramified outside two and three, and has the local extensions specified in Theorem 23.5. Then $r$ is modular.*

*Proof.* Dickson's classification and Theorem 23.5 leave only residual image $\mathrm{SL}_2(\mathbf{F}_4) \simeq A_5$. Lemma 11.12 applied to the original kernel field $M$ gives $\mathrm{rd}(M) \le 4\sqrt{40}\,3^{5/6}2^{-2/4} < 44.68647$. The selected norm-three primes have $\rho_3(M) \ge 1/12$. Corollary 13.5(ii), or directly the classical lower bound 47.16252 of Lemma 13.4, therefore excludes a totally real $M$. Lemma 2.4 now applies to the actual Barsotti–Tate source $r$. □

**Lemma 23.7.** *Every finite characteristic-three character $\psi$ of $G_K$, unramified outside three, has square equal to 1 or to the quadratic character $\nu$ of $K(\sqrt{2})/K$. In particular $\bar{\chi}_3\psi^{-2}$ is totally odd and extends to $G_{\mathbf{Q}}$.*

*Proof.* The Minkowski bound is $\sqrt{40}/2 = \sqrt{10} < 3.17$. The dyadic prime $\mathfrak{p}_2$ is nonprincipal: an element of norm $\pm 2$ would give $x^2 \equiv \pm 2 \pmod 5$. Its square is (2). The element $2 + s$ has norm $-6$, so each prime above three represents the same ideal class as $\mathfrak{p}_2$; the two triadic classes are inverse and have order two. Thus $h_K = 2$. The unit $3 + s$ has norm $-1$, so the narrow class number is also two.

The principal units at three are pro-three, and each residue field is $\mathbf{F}_3$. Consequently $\psi^2$ is unramified at every finite place. The narrow class-group calculation makes it 1 or $\nu$. The biquadratic field $K(\sqrt{2}) = \mathbf{Q}(\sqrt{2}, \sqrt{5})$ has discriminant $8 \cdot 5 \cdot 40 = 40^2$; hence it is the quadratic Hilbert class field of $K$. Its character is the restriction of the quadratic character of $\mathbf{Q}(\sqrt{2})/\mathbf{Q}$, which is totally even. Multiplication by $\bar{\chi}_3$ proves the last assertion. □

## 23.2 The residual field at two

Let $K = \mathbf{Q}(s)$, $s^2 = 10$, and label the places $v_+$ and $v_-$ above three by $s \equiv 1$ and $s \equiv -1$, respectively. Let the two-adic member of the chosen weight-two system have the binary dihedral type at $v_+$ and the special or quartic type at $v_-$. Its determinant is cyclotomic and it is crystalline of labelled weights $\{0, 1\}$ at the dyadic place. Throughout this section the residual point field is the field cut out by the actual representation on a stable lattice. The local group at $v_+$ has standard image $S_3$ in characteristic two. In particular the residual representation is absolutely irreducible.

For a nonsolvable residual image, Proposition 23.4 and Theorem 23.5 leave only $\mathrm{SL}_2(\mathbf{F}_4)$; Corollary 23.6 proves modularity of the actual source in that case. These statements apply to either choice of the mild type, including special monodromy.

In the solvable case Dickson's classification gives induction from an odd-order character $\theta$ over a quadratic field $H/K$. The character defining $H$ is unramified outside two and three, has

conductor exponent at most four at two by the finite-flat bound, and cuts out $\mathbf{Q}_3(\sqrt{3})$ at $v_+$. The squareclasses unramified away from this support have basis

$$-1, \qquad \epsilon = 3 + s, \qquad 2, \qquad 3, \qquad 2 + s.$$

Lemma 23.7 gives the class group of order two, generated by the dyadic class, and identifies both triadic classes with it. The fundamental unit is $\epsilon = 3 + s$. The valuations of $2, 3, 2 + s$ generate the admissible index-two lattice on these three primes, proving the displayed squareclass basis.

For a representative $a$ of each of the 31 nonzero classes, test $3a$ for squareness at $v_+$ and evaluate the dyadic valuation of the relative discriminant of $X^2 - a$. The complete list at conductor at most four is

| $a$ | $3$ | $3\epsilon$ | $-6$ | $-6\epsilon$ |
|---|---|---|---|---|
| dyadic conductor | 2 | 4 | 0 | 4. |

Only $L = K(\sqrt{-6})$ is CM. If the source is ordinary, Allen's residually dihedral theorem [2, Lemma 5.1.2 and Theorem 5.2.1] applies for each of the three non-CM choices, with its ordinary witness construction and without a dyadic nonsplitting hypothesis for these choices. If the source is nonordinary, its residual dyadic point field has upper breaks at most one, by Theorem 11.13. Both conductor-four quadratics are therefore impossible.

It remains to consider $H = K(\sqrt{d})$ for $d = 3, -6$. Every dyadic completion of either field has residue field $\mathbf{F}_2$. Its local units are pro-two, and hence $\theta$ is unramified there. At $v_+$ the relative quadratic different exponent is one and the residue degree is one. The conductor formula for induction gives $3 = 1 + a_H(\theta)$, so its conductor exponent is two. At the prime above $v_-$ the full local image is a two-group for either mild type; thus $\theta$ kills the entire decomposition group there. At real places an odd-order character is trivial.

Write $P_+, P_-$ for the two primes of $H$ above $v_+, v_-$. The table records the ordinary ray group for $P_+^2$, the class of $P_-$ in the indicated generator, and the quotient by $[P_-]$ and $1 + \sigma$, where $\sigma$ generates $\mathrm{Gal}(H/K)$:

| $d$ | $(e, f)$ above 2 | $\mathrm{Cl}_{P_+^2}(H)$ | $[P_-]$ | quotient |
|---|---|---|---|---|
| 3 | $(4, 1)$ | $C_2$ | 1 | 0 |
| $-6$ | $(2, 1), (2, 1)$ | $C_{12}$ | 9 | $C_3$. |

Here $1 + \sigma$ imposes inversion on the rotation character, since the determinant is trivial modulo two. These computations can be performed in the quartic model $z = s + \sqrt{d}$ with polynomial $z^4 - 2(10 + d)z^2 + (10 - d)^2$. The expression $s = (z + (10 - d)/z)/2$ identifies the two triadic primes; $z \mapsto 2s - z$ identifies $\sigma$. Compute the ray group and its automorphism by `bnrinit` and `bnrgaloismatrix`, adjoin the column `bnrisprincipal`$(P_-)$ and the columns of $1 + \sigma$ to the diagonal relation matrix, and take its integer Smith form. The class groups of $H$ are $C_2$ and $C_4$, respectively, and `bnfcertify` returns one in both fields. For $L$ either triadic prime has class 1 in the chosen generator of $\mathrm{Cl}(L) = C_4$. The exact calculation is supplied in [20, `ramified/d6_d10_d11/d10`].

There is consequently no such residual representation induced from $K(\sqrt{3})$. The two nontrivial characters of the cubic quotient over $L$ cut out a single field $A$. Indeed the splitting field of $X^3 - 3X + 1 + s$ has the requisite local data by Lemma 23.3, so it realizes this cubic quotient. In particular, once a conjugate with one of the modular images has been disposed of, every remaining two-adic residual representation has this same kernel and is the standard representation over $\mathbf{F}_2$.

### 23.2.1 The two normal ray modules

**Proposition 23.8.** *Let $V_n$ be the binary quotient of the ray group of $A$ of dyadic depth $n$ in which the positive triadic prime splits. For the standard $S_3$-module $W_0$ we have* $\mathrm{Hom}_{S_3}(V_2, W_0) = 0$. *At*

*depth four this space has dimension two, and projection onto either nontrivial $C_3$-eigencharacter after scalar extension to $\mathbf{F}_4$ is injective on local units at either dyadic prime.*

*Proof.* Write $W_0$ for the two-dimensional standard module over $\mathbf{F}_2$ for $\mathrm{Gal}(A/K) = S_3$. Let $\mathfrak{P}_1, \mathfrak{P}_2$ lie above two and $\mathfrak{q}_+$ above $v_+$. Set

$$D_n = \mathrm{Cl}_{\mathfrak{P}_1^n \mathfrak{P}_2^n}(A), \qquad V_n = D_n/(2D_n + \langle [\mathfrak{q}_+] \rangle).$$

There is no infinite modulus. The normal first-error module in the basis of the wild order-three element is $W_0$ after extension of residue scalars. A class unramified at the mild prime, split at $\mathfrak{q}_+$, and of dyadic conductor at most $n$ therefore belongs to $\mathrm{Hom}_{S_3}(V_n, W_0)$ after extending the coefficients. To see the identification on the global cohomology class, restrict to $G_A$. The kernel and cokernel terms in inflation–restriction vanish since $W_0$ restricts to the regular module for a Sylow two-subgroup of $S_3$ and is projective. The restricted class is an equivariant additive character.

We compute the dimensions through the cubic point field $B_0/K$, using Lemma 11.16. An absolute equation and its quadratic embedding are

$$P(X) = X^6 - 6X^4 + 2X^3 + 9X^2 - 6X - 9, \qquad s = -X^3 + 3X - 1.$$

The field $B_0$ has signature $(2, 2)$, discriminant 5184000 and class group $C_2$. Its unique dyadic prime $\mathfrak{P}$ has absolute $(e, f) = (2, 3)$, so $(B_0)_{\mathfrak{P}}/K_2$ is unramified cubic. Let $C_n(B_0)$ and $C_n(K)$ be the ray class groups for $\mathfrak{P}^n$ and $\mathfrak{p}_2^n$, respectively, including all real places. Exact arithmetic gives

| $n$ | $C_n(B_0)$ | $C_n(K)$ | $\dim_{\mathbf{F}_2} \mathrm{Hom}_{S_3}(V_n, W_0)$ |
|---|---|---|---|
| 1 | $C_2$ | $C_2$ | 0 |
| 2 | $C_2^2$ | $C_2^2$ | 0 |
| 3 | $C_2^2$ | $C_2^2$ | 0 |
| 4 | $C_2^5$ | $C_2^3$ | 2 |

Here the last column is the difference of the binary ranks in the middle columns. Indeed the three-point permutation module is $\mathbf{F}_2 \oplus W_0$, and restriction splits corestriction since $[B_0 : K] = 3$. At two this decomposition preserves each conductor bound by unramified base change. Away from two it preserves the unramified conditions. At $v_+$ the cubic inertia has no $W_0$-invariants, so the unramified local cohomology vanishes. Thus its class is split there, as required. At $v_-$ the cubic local algebra has one linear and one quadratic factor, and the local Shapiro decomposition identifies the unramified condition with the unramified characters of those factors. At a real place $W_0$ restricts to the regular binary $C_2$-module and has zero first cohomology. These observations identify the norm-kernel summand with the stated normal ray space. The degree-six class group is certified by `bnfcertify`, and the four ray computations are reproduced in [20, `audit_simplifications/d10_pointfield_certificate.gp`]. In particular $\mathrm{Hom}_{S_3}(V_2, W_0) = 0$.

The exact matrices and local unit columns in Appendix D give a two-dimensional space of equivariant maps. For $\alpha^2 + \alpha + 1 = 0$, the determinants of the two local evaluation matrices for either $C_3$-eigencharacter are $1 + \alpha$ and $\alpha$, up to Galois conjugation. Both are nonzero. Vanishing of either eigenprojection on local units at either dyadic prime therefore kills the global depth-four class. These statements concern the normal error in characteristic two; the tame inertia in the actual joint trace order is checked below. □

**Lemma 23.9.** *Let $R$ be a complete local order in a finite product of finite extensions of $\mathbf{Q}_2$, finite free over the integers of an unramified extension containing $\zeta_3$. Let $r : G_K \to \mathrm{GL}_2(R)$ be an integral realization of a tuple of the controlled weight-two systems of Proposition 23.2. Suppose its residue is the standard representation with kernel field $A$. At the dyadic place,*

*suppose either all constituents are nonordinary, or all are ordinary with the same residual quotient character. Let $J$ be a proper closed ideal of $R$, and assume that the image of inertia above $v_-$ in $G_A$ is scalar modulo $J$. Then the image of $r$ modulo $J$ is monomial, with component character $\eta = \chi_{-6}$, in the fixed eigenbasis of the order-three element at $v_+$.*

*Proof.* The order-three eigenprojectors are integral. In their basis the normal space to the diagonal torus is the standard $S_3$-module $W_0$, extended to the residue field; it has no invariants under the fixed order-three element $u$, and $H^i(S_3, W_0) = 0$ for $i > 0$. Write $\mathfrak{m}$ for the maximal ideal of $R/J$. Assume monomiality modulo $\mathfrak{m}^n$ and work modulo $\mathfrak{m}^{n+1}$, with square-zero ideal $I = \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Erasing the entries forbidden by $\eta(g)$ in each matrix gives a monomial representation, since products of two erased entries vanish. Its first error is a cocycle $\kappa$ with values in $W_0 \otimes I$, and $\kappa(u) = 0$.

At $v_+$ the actual source normalizes the $u$-eigenspaces, so the error vanishes on the full decomposition group. At $v_-$ the hypothesis on inertia makes $\kappa|_{G_A}$ unramified. Thus, once its dyadic conductor is bounded, inflation–restriction places the class in the normal ray module of Proposition 23.8.

Here the conductor bounds apply to actual finite-flat subquotients. At the dyadic place $L/K$ is split, so the local component character is trivial. Put $S = (R/J)/\mathfrak{m}^{n+1}$. The module $S^2$ is a finite 2-primary quotient of the stable $\mathbf{Z}_2$-lattice underlying $R^2$; schematic closure and quotient in its Barsotti–Tate group realize it as a finite-flat group. The submodule $Se_2 + Ie_1$ and its submodule $\mathfrak{m}Se_2$ are stable. Their quotient, followed by a residue-linear functional on $I$, extracts one offdiagonal extension; interchanging the indices extracts the other. Over the unramified cubic completion of $A$, its prime-field coordinates are quadratic extensions. In a nonordinary tuple they have conductor at most two by the middle-model calculation in the proof of Theorem 11.13. The depth-two vanishing in Proposition 23.8 therefore kills the global class.

For an ordinary tuple, the two dyadic residual Frobenius eigenvalues are distinct. The common ordinary line is consequently the same Frobenius projector in every constituent, and extends over $R$. Straightening it at first order makes one normal orientation a local coboundary, which vanishes on $G_A$. Fontaine's bound puts both quadratic coordinates at conductor at most four. The injectivity of the corresponding local eigenprojection in Proposition 23.8 again kills the global class. The unramified local class is zero because Frobenius minus one is invertible on the nontrivial unramified cubic character.

Thus $\kappa$ is a coboundary. Its value at $u$ forces the matrix defining that coboundary into $W_0^u \otimes I = 0$, so the error itself vanishes in the fixed basis. Induction and completeness prove the claim. □

## 23.3 The quartic mild type

**Proposition 23.10.** *Suppose that the controlled two-adic source has the residual field $A$ and the tame quartic supercuspidal type at $v_-$. Then it is induced from a geometric character of $G_L$, where $L = K(\sqrt{-6})$, and is modular.*

*Proof.* Choose a coefficient ring containing $\zeta_3$ and the integral eigenbasis of the actual order-three wild inertia at $v_+$. At $v_-$ the quartic tame generator has scalar square. Its projective quadratic character is $\eta = \chi_{-6}$, so inertia inside $G_A$ has scalar image. Apply Lemma 23.9 with $J = 0$ to this single source, using its actual ordinary or nonordinary condition at two. The image is monomial in characteristic zero.

The component character is $\eta$, so $r = \operatorname{Ind}_{G_L}^{G_K} \theta$ for a character $\theta$. It is de Rham and finitely ramified, as a summand of the restriction of $r$. By global class field theory it is attached

to an algebraic Hecke character [71, Chapter III]. Its conjugate is distinct, by the absolute irreducibility of $r$. Automorphic induction from the CM field $L$ is cuspidal and gives $r$, with its regular weights [4, Chapter 3]. □

## 23.4 The coefficient field of the special source

Let $\mathcal{R}$ be a weight-two strongly compatible system with determinant cyclotomic, good outside the two places above three. Suppose that its type at $v_+$ is the binary supercuspidal type of Lemma 4.22 and its type at $v_-$ is special, with nonzero monodromy and a fixed quadratic scalar character. Let $B$ be the number field generated by its traces at the good places. We consider the case in which all its residual representations at coefficient places above two are the standard representation of $\mathrm{Gal}(A/K)$. Put $k = \mathbf{F}_4$, let $\mathcal{W} = W(k)$, and fix a primitive cube root of unity $\zeta \in \mathcal{W}$.

**Proposition 23.11.** *Let $d_0$ be the number of nonordinary embeddings of $B$ in $\overline{\mathbf{Q}}_2$. Then $d_0 \leq 2$. For each of the two possible residual unramified quotient characters at the dyadic place, there is at most one ordinary embedding. Every coefficient completion associated to a nonordinary embedding is ramified over $\mathbf{Q}_2$. In particular $[B : \mathbf{Q}] \leq 4$.*

We first describe the integral representation to which we shall apply the finite-flat conditions. Partition the embeddings of $B$ into a nonordinary set and two ordinary sets, according to the eigenvalue $\bar{\zeta}$ or $\bar{\zeta}^{-1}$ of residual Frobenius on the unramified quotient. Each set is stable under the absolute Galois group of $\mathcal{W}[1/2]$. For any nonempty set of this partition, project $B \otimes_{\mathbf{Q}} \mathcal{W}[1/2]$ to the product of field factors corresponding to the set. Let $R$ be the closed $\mathcal{W}$-subalgebra of this product generated by the traces of the associated tuple of representations. Its fraction algebra is the entire product, since already the traces at the good places generate $B$ over $\mathbf{Q}$. Thus $R$ is finite free over $\mathcal{W}$ of rank the cardinality of the set. It is local with residue field $k$. Indeed all constituent reductions give the same residual trace, and every maximal ideal of this integral order lifts to a prime of its normalization.

The joint representation has a model on $R^2$. To see this, choose four group elements whose residual matrices form a $k$-basis of $M_2(k)$, using absolute irreducibility. Their trace-pairing Gram determinant is a unit. Pairing any group matrix with these four elements expresses it uniquely in their joint span, with coefficients in $R$; the same argument applies to their products. Their span is thus a free rank-four algebra with reduction $M_2(k)$. Lift a primitive matrix idempotent to this complete algebra and take the corresponding left ideal. This ideal is free of rank two over $R$, and the action on it identifies the algebra with $M_2(R)$ by reduction and Nakayama. Denote the resulting representation by $r$. Its rational constituents are those of the original tuple. In particular the stable $\mathbf{Z}_2$-lattice underlying $R^2$ in their sum is realized by an isogenous Barsotti–Tate group over $K_2$, by the lattice classification [51, Theorem 4.1]. All its generic subquotients killed by two acquire finite-flat models by schematic closure and quotient. For the nonordinary set their special fibres and their Cartier duals are connected, by the isogeny argument in the proof of Theorem 11.13.

Write $T = R/2R$ with maximal ideal $\mathfrak{n}$. Choose a wild inertia element $u$ at $v_+$ whose image generates the actual order-three subgroup and use its two integral eigenprojectors to put $r(u) = \mathrm{diag}(\zeta, \zeta^{-1})$. The standard residual representation in this basis is monomial with component character $\eta = \chi_{-6}$. Here $\chi_a$ denotes the quadratic character of $K(\sqrt{a})/K$. The offdiagonal matrices with residual conjugation action form the standard module $W \otimes_{\mathbf{F}_2} k$ of Proposition 23.8. In particular $W^u = 0$ and $H^i(S_3, W) = 0$ for $i > 0$, since $W$ is projective.

**Lemma 23.12.** *In the fixed $u$-eigenbasis the image of $r$ over $T$ is monomial with component character $\eta$.*

*Proof.* A tame generator $U$ at $v_-$ has the two characteristic-zero eigenvalues $\epsilon, \epsilon$, with one fixed quadratic sign $\epsilon = \pm 1$ across the tuple. Hence $U^2 = 2\epsilon U - 1$ in the joint order and $U^2 = 1$ modulo rational two. The inertia in $G_A$ is cut out by the ramified quadratic subfield, so it acts trivially modulo two; higher inertia already acts trivially in the special source. Lemma 23.9, with $J = 2R$, proves the assertion for the nonordinary block and for each ordinary orientation. □

**Lemma 23.13.** *For the nonordinary set* $\dim_k \mathfrak{n}/\mathfrak{n}^2 \leq 1$. *If this space is nonzero, its visible first-order scalar character is a nonzero multiple of* $\chi_{-2}$ *modulo* $k\eta$. *For either ordinary set* $\mathfrak{n} = 0$.

*Proof.* A $k$-linear functional on $\mathfrak{n}/\mathfrak{n}^2$ gives a specialization to $k[\delta]/(\delta^2)$. Lemma 23.12 makes the specialized representation monomial. Compare it with the constant residual matrices. Their determinant agrees, since the cyclotomic determinant is one in characteristic two. The diagonal infinitesimal error therefore has equal entries, even on an antidiagonal coset after multiplying by the inverse residual matrix. The error is $\delta c(g)$ times the identity for an additive character $c : G_K \to k$. Changing basis by an infinitesimal diagonal matrix can add a multiple of $\eta$; conjugation of such a matrix by an antidiagonal element interchanges its diagonal entries. Accordingly the class of $c$ modulo $k\eta$ depends linearly on the functional. This map is injective: if the character is a multiple of $\eta$, the representation is conjugate to its constant reduction, so every trace has zero infinitesimal part and the trace generators force the functional to vanish.

At $v_+$ choose an antidiagonal inertia element $t$ with $t^2 = -1$. Multiply an arithmetic Frobenius lift by $t$ if necessary so that it centralizes $u$. It takes $t$ to $-u^i t$ for some $i$, since it acts by cubing on the tame quotient of order four. Multiplying the lift by a power of $u$ makes its diagonal ratio exactly $-1$. Its eigenvalues are then $\lambda, -\lambda$ with $\lambda^2 = -3$. The trace after multiplication by $u$ is $\lambda(\zeta - \zeta^{-1}) = \pm 3$. It is rational and common to the tuple by compatibility of the local Weil representations at $v_+$. Thus $\lambda$ is one fixed value $\pm(\zeta - \zeta^{-1})$ in $\mathcal{W}$. The scalar error vanishes on this Frobenius and on $u$, so $c|_{D_{v_+}}$ is a multiple of $\eta$. At the dyadic place the nonordinary scalar errors have conductor at most two, by extracting the middle self-extension on the unramified cubic base as in the proof of Lemma 23.9. Unramified base change preserves positive breaks. In an ordinary set the exact unramified quotient shows instead that $c$ is unramified at two.

Use the supported squareclass basis $\langle -1, 3 + s, 2, 3, 2 + s \rangle$ of Section 23.2. The condition at $v_+$ and dyadic conductor at most two leave

| $a$ | 1 | $-2$ | 3 | $-6$ |
|---|---|---|---|---|
| dyadic conductor exponent | 0 | 2 | 2 | 0. |

For an explicit verification form the 32 products $a$ of the displayed basis, keep those with $a$ or $-6a$ a square at $v_+$, and compute the dyadic exponent of the relative discriminant of $x^2 - a$ (zero for the trivial class). Equivalently this is the calculation in `scalar_characters.gp`. Scalar extension from $\mathbf{F}_2$ gives the same kernel conditions over $k$. Modulo $\eta$ only $\chi_{-2}$ survives, and it is ramified at two. Hence the nonordinary cotangent dimension is at most one and an ordinary set has zero cotangent dimension. Nakayama's lemma gives $\mathfrak{n} = 0$ in the latter case. The exact calculation is supplied in [20, `ramified`]. □

*Proof of Proposition 23.11.* Lemma 23.13 shows that each nonempty ordinary set has cardinality $\dim_k T = 1$. For the nonordinary set we may assume the cotangent dimension is one. Then $T = k[z]/(z^h)$, where $h = \dim_k T$, and its nonzero tangent is represented by $\chi_{-2}$ modulo $k\eta$ after scaling $z$. Restrict to the unramified cubic extension $F/K_2$ killing the residual dyadic characters. Since $L/K$ splits at 2, Lemma 23.12 splits the local representation as two rank-one $T$-characters. The restriction of $\eta$ is trivial, and the first-order error of each character is ramified over $F$, since $\chi_{-2}$ has conductor two there.

Take one of the rank-one summands as a finite-flat subquotient of the regular lattice modulo two. It and its dual have connected special fibres. By Lemma 11.5, the generic coefficient action, the images of powers of $z$, and the resulting exact filtrations extend to the finite-flat model. Its Kisin module is free of rank one over $T[[u]]$ on every residue embedding factor after enlarging the finite constant field to accommodate the residue field of $F$. Indeed the $z$-filtration is exact, with rank-one successive quotients on each factor. The quotient by $z$ has rank one over the residue power-series ring, so lifting its generator gives a surjection by Nakayama. The ranks over that power-series ring are equal, giving the assertion. Contravariance interchanges the subobject and quotient filtrations and gives the same lengths. Frobenius permutes the residue embedding factors.

The Frobenius multiplier on each factor reduces modulo $z$ to $u$ times a unit. Weierstrass preparation writes it as a unit times $u - \lambda_i$ with $\lambda_i \in (z)$. Height one implies $u^2 \in (u - \lambda_i)$, since the Eisenstein polynomial of a uniformizer reduces modulo two to $u^2$. Evaluating at $\lambda_i$ gives $\lambda_i^2 = 0$. If all $\lambda_i$ vanished modulo $z^2$, the first-order multipliers would each be $u$ times a power-series unit. After the middle exponent is removed, the corresponding Artin–Schreier class has no pole on any residue factor and is unramified. This is the middle self-extension calculation in the proof of Theorem 11.13, including its comparison of breaks with the field of norms. It contradicts the ramification just proved. Thus one $\lambda_i$ is a unit times $z$, and $z^2 = 0$. This gives $h \leq 2$.

Finally suppose that the completion of $B$ belonging to a nonordinary embedding is unramified over $\mathbf{Q}_2$. The trace-order construction above, applied to a single completion before and after unramified coefficient extension, realizes its representation on an integral rank-two lattice there. The residual local representation splits into two distinct unramified lines after unramified coefficient extension. Over the unramified cubic base, Lemma 11.5 splits its finite-flat model as the direct sum of middle models. Therefore Frobenius at $u = 0$ on the integral Kisin module vanishes modulo two. This can be checked after unramified base extension by faithful flatness. Over the original base with residue field $\mathbf{F}_2$, the integral module has coefficient rank two: the underlying finite-free module with unramified coefficient action is free over the coefficient power-series ring, by reduction and Nakayama. Since two is a uniformizer, every entry of its Frobenius-at-zero matrix belongs to $2\mathcal{O}_E$, where $E$ is the enlarged unramified coefficient field. Its determinant would have valuation at least two. On the other hand the determinant of crystalline Frobenius has valuation one because the determinant character is cyclotomic [51, Proposition 4.2]. This contradiction proves the ramification assertion. Summing the three bounds proves the proposition. □

## 23.5 The small coefficient prime

Let $B$ be the minimal global trace field of the system with a special condition at $v_-$. We are in the case in which the residual representation at every embedding $B \hookrightarrow \overline{\mathbf{Q}}_2$ is the standard representation of the same $S_3$ extension $A/K$. Modularity at any one coefficient prime suffices for modularity of the entire system.

**Proposition 23.14.** *If all the dyadic embeddings are ordinary, or all are nonordinary, then $[B : \mathbf{Q}] \leq 2$. Otherwise $B$ is quartic, ramified at two, and is generated by the Frobenius trace $a_2$ at the dyadic prime. Its minimal polynomial is congruent to $X^2(X^2 + X + 1)$ modulo two.*

*Proof.* The trace order in each set bounds the number of embeddings in that set. Residue Frobenius interchanges the ordinary sets, so if they occur each has exactly one embedding. A nonordinary completion must ramify and its block has at most two embeddings. In the mixed case it has exactly two. At an ordinary embedding one crystalline Frobenius root is the unramified quotient and the other is divisible by the maximal ideal; $a_2$ thus reduces to the

two distinct primitive cube roots in the two orientations. At each nonordinary embedding it is in the maximal ideal by the Newton slope criterion. Strong compatibility supplies $a_2 \in B$ by taking a coefficient prime away from two. Each ordinary root occurs once, whereas restricting the embeddings to a proper subfield would repeat every root $[B : \mathbf{Q}(a_2)]$ times. This proves the proposition. □

**Lemma 23.15.** *Let $B$ be a mixed quartic field as in Proposition 23.14. Then at least one of the following holds:*

*(i) Every prime of $B$ above three has residue degree at most two.*
*(ii) There is a degree-one prime of $B$ above five or seven.*
*(iii) Up to replacing $a_2$ by $-a_2$, its minimal polynomial is*

$$X^4 - X^3 - 7X^2 + 8X + 2 \quad \text{or} \quad X^4 - X^3 - 5X^2 + 2X + 2.$$

*Proof.* Every conjugate of the algebraic integer $a_2$ lies in $[-\sqrt{8}, \sqrt{8}]$. Write its minimal polynomial as $P = X^4 + aX^3 + bX^2 + cX + d$. By Proposition 23.14, $a, b$ are odd and $c, d$ are even. The elementary symmetric functions and Cauchy's inequality give

$$|a| \le 11, \quad |b| \le 48, \quad |c| \le 90, \quad 0 < |d| \le 64, \quad 0 \le a^2 - 2b \le 32, \quad 8b \le 3a^2.$$

Replacing $a_2$ by $-a_2$, we take $a > 0$. Exact rational Sturm sequences on $[-283/100, 283/100]$, with the integral coefficient bounds above, leave the twelve irreducible polynomials in the following table. For a row marked $(p, r, e)$ we have $P(r) \equiv 0$ and $P'(r) \equiv e \not\equiv 0 \pmod{p}$.

| $a$ | $b$ | $c$ | $d$ | Local check |
|---|---|---|---|---|
| 1 | -9 | -6 | 14 | $(5, 2, 2)$ |
| 1 | -7 | -10 | -2 | one prime above 3, of residue degree 2 |
| 1 | -7 | -8 | 2 | $(3, 2, 2)$; first exceptional polynomial |
| 1 | -7 | -6 | 2 | $(7, 2, 3)$ |
| 1 | -7 | -6 | 4 | $(7, 1, 1)$ |
| 1 | -7 | -4 | 4 | $(5, 1, 4)$ |
| 1 | -7 | -4 | 8 | $(5, 4, 4)$ |
| 1 | -7 | -4 | 10 | $(5, 0, 1)$ |
| 1 | -5 | -4 | 2 | $(5, 1, 3)$ |
| 1 | -5 | -2 | 2 | $(3, 1, 1)$; second exceptional polynomial |
| 1 | -5 | -2 | 4 | odd field discriminant |
| 3 | -3 | -12 | -6 | Eisenstein at 3 |

The accompanying certificate uses only integer and rational arithmetic: it tests irreducibility by Gauss's lemma and counts real roots by Sturm's theorem. Thus this step requires no maximal-order computation.

A simple root modulo five or seven gives an embedding $B \hookrightarrow \mathbf{Q}_p$ by Hensel's lemma, proving (ii). For the second row, set $Q = X^2 - X - 1$. Then $P = Q^2 + 3(X^3 - 2X^2 - 4X - 1)$. The polynomial $Q$ is irreducible modulo three, and the remainder of the second factor modulo $Q$ is $-4X - 2$. Dedekind's index criterion shows that three does not divide $[\mathcal{O}_B : \mathbf{Z}[a_2]]$; hence three has one prime in $B$, of ramification index two and residue degree two. The final row is Eisenstein at three and has one prime of residue degree one. Both rows give (i).

For the penultimate row, put $\alpha = a_2$ and $\beta = (\alpha^3 + \alpha^2 + \alpha)/2$. Direct reduction modulo $P$ gives $\beta^4 - 29\beta^2 + 10\beta + 19 = 0$. Thus $\beta$ is integral. Since $\operatorname{disc} P = 8900$, the integral lattice spanned by $1, \alpha, \alpha^2, \beta$ has discriminant 2225. The field discriminant is therefore odd, contrary to Proposition 23.14. The two remaining rows are exactly the polynomials in (iii), after replacing $X$ by $-X$. □

**Lemma 23.16.** *If an irreducible residual representation of determinant $\bar{\chi}_3$, unramified outside three, is restricted to $G_{K(\zeta_3)}$, it remains irreducible.*

*Proof.* Otherwise the representation is induced from a character $\theta$ of $H = K(\zeta_3)$ with $\theta^\sigma = \theta^{-1}$. Such a character has order prime to three. Its ray conductor divides the product of the two primes above three, both of residue degree one and ramification index two. An absolute equation of $H$ is $X^4 + 10X^2 + 100$. Its class group is cyclic of order four, certified by `bnfcertify`; its ray group for this modulus is also $C_4$. If $S$ is the action of the nonidentity automorphism over $K$, the Smith invariant of the lattice generated by 4 and $1 + S$ is two. Equivalently this ray group modulo $1 + \sigma$ is $C_2$. Thus $\theta^2 = 1$ and $\theta = \theta^\sigma$, contradicting irreducibility. These data are reproduced by `cyclotomic_three_ray.gp`. □

*Proof of Theorem 23.1.* Soluble residues have solvable odd Artin lifts, which give modularity by Hecke induction or Langlands–Tunnell and regular weight raising. Assume therefore that the prescribed three-residue is nonsolvable. The bounded lift, the discriminant estimate and the soluble-image classification reduce the proof to the special source with the fixed $S_3$ residue, the quartic source having already been treated. Apply Proposition 23.14. The exact calculation is supplied in [20, `ramified`].

If the degree of $B$ is at most two, or every coefficient prime above three has degree at most two, choose the originally prescribed prime. Its residue is realizable over $\mathbf{F}_3$ or $\mathbf{F}_9$ by its traces, absolute irreducibility and the vanishing of the Brauer group of a finite field. The small-image theorem of Allen–Khare–Thorne [3, Theorem 2.5] gives the required cuspidal residual witness.

At a degree-one coefficient prime of characteristic $p = 5$ or 7, the binary inertia at $v_+$ remains irreducible modulo $p$; the order-three element has distinct eigenvalues and a tame element interchanges its eigenlines. The local cyclotomic extension at $v_+$ is unramified, so the residual representation is absolutely irreducible even on $G_{K(\zeta_p)}$. Theorems 2.5 and 2.3 of [3] give respectively modularity of its residue and modularity of the actual de Rham member with weights $\{0, 1\}$. This includes the ramified prime five. Strong compatibility gives modularity at the original prime.

The last two polynomials in Lemma 23.15 have a coefficient place above three with residue field $\mathbf{F}_3$. If its residue were reducible, write the characters as $\psi$ and $\bar{\chi}_3\psi^{-1}$. Lemma 23.7 gives $\psi^2 = 1$ or $\nu$, the Hilbert class character, which is nontrivial unramified at both triadic places. On $G_{K_{v_-}}$ a quadratic Steinberg twist has semisimplification $\alpha \oplus \alpha\chi_3$ with $\alpha$ quadratic, so $\psi^2$ is locally trivial. It follows that $\psi^2 = 1$. At the good dyadic place the residual trace would be $\psi(\mathrm{Frob}_2)(1 + 2) = 0$. For the selected coefficient prime $\lambda \mid 3$, this gives $a_2 \in \lambda$. Hence $3 \mid N_{B/\mathbf{Q}}(a_2)$. Both polynomials have constant term 2, so $N_{B/\mathbf{Q}}(a_2) = 2$, a contradiction. The residue is therefore irreducible. Lemma 23.16 and [3, Theorems 2.5 and 2.3] make the actual member modular. This proves the residual theorem. □

# 24 Quadratic symmetry over $\mathbf{Q}(\sqrt{11})$

Put $K = \mathbf{Q}(s)$, $s^2 = 11$.

## 24.1 Arithmetic and modular witnesses

We will use the elements

$$\epsilon = 10 + 3s, \qquad \pi = 3 + s, \qquad \alpha = 4 + s, \qquad \alpha' = 4 - s.$$

They satisfy $N(\epsilon) = 1$, $N(\pi) = -2$ and $N(\alpha) = N(\alpha') = 5$.

**Lemma 24.1.** *The class number of $K$ is one, and its units are generated by $-1$ and $\epsilon$. Its finite ray class groups of moduli* (3) *and* (5) *are both cyclic of order four. With both real places added to the modulus, the groups are $C_8 \times C_2$ at three and $C_4 \times C_2$ at five.*

*Proof.* Minkowski's bound is $\sqrt{44}/2 < 3.32$. Three is inert, and the dyadic prime is generated by $\pi$, so the class number is one. The least nontrivial positive solution of $x^2 - 11y^2 = 1$ is $(10, 3)$; checking $y = 1, 2$ proves that $\epsilon$ is fundamental. The negative Pell equation has no solution because $-1$ is not a square modulo eleven.

Modulo three the residue group is $\mathbf{F}_9^\times$ and $\epsilon = 1$. The image of the units is $\{\pm 1\}$, giving a cyclic quotient of order four. Adding the real signs gives $(C_8 \times C_2^2)/\langle(-1,-1,-1)\rangle \simeq C_8 \times C_2$. At the two five-places, ordered by $s = 1, -1 \bmod 5$, $\epsilon$ has residues $(3, 2)$, of order four, and its square is the image $(-1,-1)$ of $-1$. The product map $(u, v) \mapsto uv$ identifies the finite ray quotient with $\mathbf{F}_5^\times$. Adding the signs leaves one independent sign factor and gives $C_4 \times C_2$. □

## 24.2 Characteristic-three sources

**Proposition 24.3.** *If $\bar{r}_3^{\mathrm{ss}} = \psi \oplus \omega_3\psi^{-1}$ for a bounded system $\mathcal{R}$, then $\psi^2 = 1$ and the actual source $r_3$ is ordinary and modular.*

*Proof.* At a five-place the semisimplified residual inertia of our two types is scalar quadratic: the cubic part of the cuspidal inertia disappears modulo three, as does the unipotent part of the special condition. Hence $\psi^2$ is unramified there. A characteristic-three finite character kills principal units at three, so $\psi^2$ factors through the finite ray group of modulus (3) from Lemma 24.1.

Fontaine–Laffaille theory [40] over the unramified quadratic $K_3$ gives the possible inertia exponents $0, 1, 3, 4 \bmod 8$ of a rank-one constituent in weight two. The determinant has exponent four. Thus the reducible pairs are $\{0, 4\}$ and $\{1, 3\}$. In the second case $\psi^2$ is a faithful quartic ray character. Both five-primes generate that ray group: their positive generators $\alpha, \alpha'$ reduce to $1 + s, 1 - s$ in $\mathbf{F}_9$, and

$$(1+s)^2 = -s, \qquad (1+s)^4 = -1, \qquad 1 - s = (1+s)^3.$$

It follows that $\psi^2(\mathrm{Frob}_v)$ is a primitive fourth root at either five-place.

The quotient of the two residual local characters would therefore have Frobenius value of order four. For both of our five-adic types that quotient is instead $-1$ in characteristic three: the special eigenvalues have ratio five, while the cuspidal parameter has trace zero and determinant five. Scalar quadratic factors do not change the quotient. This excludes $\{1, 3\}$.

For the pair $\{0, 4\}$, $\psi^2$ is unramified also at three and is even at both real places. The ordinary class number is one, so $\psi^2 = 1$. Since $K_3/\mathbf{Q}_3$ is unramified, Lemma 4.3(i) makes the actual source ordinary, with unramified quotient. Its residual constituent ratio is $\omega_3$, which is totally odd, extends to $G_{\mathbf{Q}}$, and is nontrivial at $K_3$. The field $K(\zeta_3)$ is abelian over $\mathbf{Q}$. All the hypotheses of Skinner–Wiles [75, Theorem A] are satisfied, so $r_3$ is modular. □

**Proposition 24.4.** *Let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_3)$ be continuous, irreducible, totally odd and finitely ramified, with determinant $\chi_3$, and crystalline of weight two at three. If $\bar{r}$ is absolutely irreducible but becomes reducible on $G_{K(\zeta_3)}$, then $r$ is ordinary at three and modular.*

*Proof.* Put $H = K(\zeta_3)$. Clifford theory gives $\bar{r} = \mathrm{Ind}_{G_H}^{G_K} \xi$, where $\xi$ has finite order prime to three. Wild inertia at three is therefore trivial. The extension $H_3/K_3$ is ramified quadratic. A tame generator $t$ outside $G_{H_3}$ acts antidiagonally in an inducing basis, and $\det \bar{r}(t) = -\xi(t^2) = \omega_3(t) = -1$. Thus $\bar{r}(t)^2 = 1$. A lift of Frobenius can be chosen in $G_{H_3}$, and its diagonal matrix

commutes with $\bar{r}(t)$ because the residue cardinality is nine. Its two eigenvalues are consequently equal, and the determinant makes that common value $\pm 1$. The local pair is exactly $\beta, \beta\omega_3$ with $\beta$ unramified quadratic. Lemma 4.3(i) applies to the actual source, since $e(K_3/\mathbf{Q}_3) = 1 < 3 - 1$.

Induce the Teichmuller lift of $\xi$ to obtain a finite-image, irreducible, totally odd Artin representation. Automorphic induction supplies its weight-one Hilbert theta series. At three its Euler factor has degree one and unramified root $\pm 1$, so the form is ordinary and has the correct residual ordinary quotient. Wiles’s Hilbert ordinary family theorem [87, Theorem 1.4.1] gives a regular ordinary specialization with the same residual quotient. This application is also explicit in [33, pp. 1133–1134].

We have obtained the quotient-compatible modular lift required by [76, Theorem 5.1]. The actual source is ordinary and distinguished at three, its determinant is cyclotomic, and its remaining geometric and parity hypotheses are the stated ones. That theorem permits reducibility on the cyclotomic subgroup and proves modularity of $r$. □

## 24.3 Nonsolvable and dihedral dyadic residues

In this section the system is special at both five-primes. Let $M$ be the field of a semisimple dyadic residue. At five its full decomposition image is elementary abelian of exponent at most two: removing its quadratic sign leaves an upper unipotent group, generated by tame inertia and Frobenius. In particular every local field is contained in the maximal elementary quadratic extension of $\mathbf{Q}_5$.

**Lemma 24.5.** *Put $F = K(i, \sqrt{5})$ and let $H$ be its Hilbert class field. Then $[H : \mathbf{Q}] = 16$, its dyadic primes have absolute $(e, f) = (2, 4)$ and its four five-primes have $(e, f) = (2, 2)$. Its fourth-power dyadic ray, with all five-prime classes killed, has group $C_4^2 \times C_2^4$.*

*Proof.* The defining polynomials for $F, H$ are respectively

$$y^8 - 4y^7 + 24y^6 - 58y^5 + 141y^4 - 190y^3 + 186y^2 - 100y + 20,$$
$$y^{16} - 9y^{14} + 39y^{12} - 111y^{10} + 214y^8 - 189y^6 + 69y^4 - 6y^2 + 1.$$

The class groups are $C_2, 1$; the second field is the class field of the first and has its squared discriminant. Over $H$ the full ray invariants are $12, 4, 4, 2, 2, 2, 2, 2$. Adding the four reciprocity columns of the five-primes to the diagonal relation matrix gives the displayed quotient by Smith reduction. The certificate `sp_auxiliary.gp` checks the polynomial identification, both class groups with `bnfcertify`, the prime decompositions and the Smith factors. The exact calculation is supplied in [20, `ramified/d6_d10_d11/d11`]. □

**Proposition 24.6.** *Assume GRH. A two-special bounded system with a nonsolvable dyadic residue is modular.*

*Proof.* Its image is $\mathrm{SL}_2(q)$, $q = 2^f \geq 4$, by Dickson and determinant one. Let $B$ be the ray field of Lemma 24.5, of absolute degree 4096 and Galois over $K$ by invariance of the modulus and quotient. The dyadic breaks of $BM/K$ are at most three: this follows from Fontaine’s bound for the actual Barsotti–Tate lattice, and from the fourth-power ray over the unramified dyadic extension $H/K$. The two-adic contribution to the logarithmic relative different, in valuation $v_2(2) = 1$, is at most $4/2 = 2$. At five the field $H$ contains the ramified field $\mathbf{Q}_5(\sqrt{5})$ and its unramified quadratic extension. This is already the full elementary quadratic envelope. Consequently for $L = BM$ and also $L = BMM^{\sigma}$, $\mathrm{rd}(L) \leq 4\sqrt{44 \cdot 5} < 59.329588, \quad \rho_{25}(L) = 1/4$.

Apply Section 13 with $B_{b,100}$ and the retained prime powers specified below. Directed 256-bit evaluation gives

| | $n \geq$ | $b$ | powers of 25 | $\mathrm{rd}(L) >$ |
|---|---|---|---|---|
| $q \geq 16$ | $4096 \cdot 4080$ | 16.4 | 9 | 59.981834 |
| $q = 8,\ M \neq M^\sigma$ | $4096 \cdot 504^2$ | 20.5 | 10 | 62.360509 |

Simplicity gives disjointness from the solvable auxiliary. Distinct simple conjugate fields are disjoint over $K$, and their product still has no nontrivial solvable quotient. The first two lines exclude their respective cases. The exact calculation is supplied in [20, `ramified/d6_d10_d11/shared`].

A real conjugation is nontrivial in either case: Lemma 11.12 gives $\mathrm{rd}(M) \leq 4\sqrt{220}\,2^{-2/q}$, so Corollary 13.5(i) excludes a totally real kernel without an analytic hypothesis. If $q = 8$, the equality $M = M^\sigma$ gives residual modularity by Lemma 2.5. Lemma 2.4 applies in this case and for $q = 4$. □

**Proposition 24.7.** *A two-special system with irreducible solvable two-residue is modular. Every reducible two-residue has trivial semisimplification.*

*Proof.* An irreducible solvable subgroup in determinant one is odd dihedral. Let $L/K$ be its inducing quadratic field. The odd rotation character is unramified away from two, trivial at every five-decomposition group over $L$, and at two is tame. Its conductor divides the product of the dyadic primes of $L$. The quadratic squareclasses are supported on $-1, \epsilon, \pi, \alpha, \alpha'$ by class number one. Odd dyadic valuation gives conductor five by an Eisenstein square-root polynomial and its derivative, contradicting Fontaine's upper break three. Among the fifteen nontrivial classes in $\langle -1, \epsilon, \alpha, \alpha' \rangle$, the odd part of the ray modulo all dyadic primes with all five-primes split is trivial except at $-5\epsilon$, where it is $C_3$. The certificate `sp_dihedral_rays.gp` verifies these quartic ray quotients. Since $\pi^2 = 2\epsilon$ the survivor is $K(\sqrt{-10})$. Its dyadic relative conductor is four. Theorem 11.13 excludes a nonordinary actual source. The ordinary source has a nonsplit dyadic prime in its imaginary inducing field and the specified near-ordinary line; Allen's ordinary dihedral theorem [2, Theorem 5.2.1] applies. For a reducible residue the odd-order constituents are unramified at two and five and everywhere else, so class number one makes both trivial. The exact calculation is supplied in [20, `ramified/d6_d10_d11/d11`]. □

## 24.4 Ordinary scalar systems

**Proposition 24.8.** *A two-special system with a scalar semisimple two-residue and an ordinary member above two is modular.*

*Proof.* The sixteen supported classes $\langle -1, \epsilon, \alpha, \alpha' \rangle$ contain precisely $1, 5, -1, -5$ unramified at two and precisely $1, -5$ split there; a class of odd dyadic valuation cannot split. These calculations are recorded in `ordinary_squareclasses.gp`. Lemma 12.1, with $\chi = \chi_{-5}$, gives residue $1 + \chi E_{12}$, real matrix $C = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and ordinary line $e_2$. Use the marked ordinary crystalline functor of [52, Proposition 2.4.4, Corollary 2.4.5], pulled back to the matrix problem with this fixed real matrix, support and determinant. At five fix the quadratic characters on inertia, and allow the unramified special signs to vary. The special line is represented without an extra free parameter: for a tame generator normalized by the given sign, its difference from one has unit upper-right entry, so the coordinate of a fixed line in the $e_1$ chart is determined by its first row. Stability and the character equations on that line and its quotient, including square one of the latter, are then closed equations. The exact calculation is supplied in [20, `ramified`].

For a homogeneous tangent use Lemma 12.1a with $h = \chi$ and the marked finite flat ordinary functor. It gives additive $z$, $y = 0$ and unramified $x$ at two, with $z$ a unit Kummer class there. At each five-prime the special-line equation gives $z = 0$ on the whole decomposition group. The two valuations exclude $\alpha, \alpha'$; $\epsilon$ is a nonsquare at both places. Thus $z$ is a multiple of $\chi_{-1}$ and the fixed real matrix kills it. The remaining $x, y$ are additive. The split-at-two calculation gives $y \in k\chi_{-5}$, and $y(c) = 0$ kills it. The unramified-at-two calculation and $x(c) = 0$ leave $x \in k\chi_5$. This last direction is detected by the marked unramified quotient $\nu$ on dyadic Frobenius, since $\chi_5$ is locally nontrivial unramified. Cotangent Nakayama makes the complete ring a quotient of $W(k)[[\nu(\mathrm{Frob}_2) - 1]]$.

Choose $\eta = 1$ or $\chi_{-5}$ aligning the inertial characters of $r$ and $r^\sigma \otimes \eta$ at both five-primes. The twist is dyadically trivial. The two normalized points agree on the sole generator, since the local involution fixes unramified Frobenius. They agree as marked representations. Lemma 12.4 applies. □

## 24.5 The integral order of nonordinary scalar systems

Take the tuple at all dyadic coefficient places of the bounded systems under consideration. Every member is nonordinary with trivial residual semisimplification, and the systems have common quadratic inertial signs at each special place. Use the orders $R_0, E_0$ of Section 11. For $m$ distinct coefficient orbits of common trace-field degree $d$,

$$\mathrm{rank}_{\mathbf{Z}_2} R_0 = md, \qquad \mathrm{rank}_{\mathbf{Z}_2} E_0 = 4md.$$

Write $E = E_0/2E_0$; its radical has residue field $\mathbf{F}_2$.

**Lemma 24.9.** *The left regular lattice of $E_0$ is Barsotti–Tate at two and biconnected. For the simple factors of its torsion with trivial generic character, the middle order-two model is unique. If $J$ is the radical of $E$, the differences $A, B, C$ from one of a real involution and of the two normalized tame five-inertias generate $E$ as an $\mathbf{F}_2$-algebra, and all three have square zero.*

*Proof.* Apply Lemmas 11.8 and 11.5 to the regular lattice and its trivial-simple torsion subquotients. These lemmas also hold after the unramified base changes used below.

A radical functional gives a middle/middle extension by quotienting the regular model. At the dyadic place its quadratic conductor is at most two by the height-one Artin–Schreier calculation in the proof of Theorem 11.13. Corollary 11.7 gives their complete space, spanned by $[-1]$, $[\epsilon\alpha]$, $[\epsilon\alpha']$. Their dyadic discriminant exponents are respectively $0, 2, 2$. The latter two ramify at distinct five-primes and split at the opposite prime. Real conjugation and the two tame inertias evaluate in the three coordinate directions. Hence $A, B, C$ span $J/J^2$. Nilpotence gives algebra generation. The real involution squares to one and each normalized tame generator is unipotent with square-zero difference, proving the lemma. The arithmetic is certified in `scalar_regular_rays.gp`. The exact calculation is supplied in [20, `ramified`]. □

## 24.6 A quaternion obstruction over the Hilbert class field

The next result is used both for the original reduction and a central quotient of it. Suppose $E'$ is such an actual regular finite-flat quotient, with elementary abelian dyadic-characteristic decomposition image at both five-primes of exponent two. Let $M'$ be its point field, and $P = \mathrm{Gal}(M'H/H)$. All five-primes split in $M'H/H$ because $H$ contains the maximal elementary quadratic completion at five. Its dyadic primes have absolute ramification index two; Fontaine's bound gives upper breaks at most three.

**Lemma 24.10.** *Assume GRH. The group $P$ is elementary abelian of rank at most two. Its span in $E'$ has dimension at most four and at most two radical generators.*

*Proof.* Denote the dyadic primes of $H$ by $v_1, v_2$. The ray groups with all five-prime classes killed at dyadic conductor exponents $(4,0)$ and $(0,4)$ are trivial; the corresponding $(2,2)$ group is $C_2^2$. If either inertia image were proper in the finite two-group $P$, it would lie in an index-two subgroup. The associated quadratic extension would contradict one of the first two ray calculations. Thus inertia at either dyadic place is all of $P$.

Let $C_0$ be the span of $P$ in $E'$ and $J_0$ its radical. Its regular representation is the actual submodule $C_0 \subset E'$. At either dyadic completion it therefore has a biconnected finite-flat model. A functional on $J_0/J_0^2$ gives a middle/middle extension, whose conductor is at most two by the calculation in the proof of Theorem 11.13. The $(2,2)$ ray calculation gives $\dim J_0/J_0^2 \leq 2$. At each dyadic place the ramified Artin–Schreier coordinates of independent such functionals are independent over $\mathbf{F}_2$: a nonzero linear combination with zero ramified coordinate would be unramified, whereas inertia is all of $P$.

The right action of $C_0$ extends to its finite-flat model by Lemma 11.5. The quotient by the right radical is the unique middle model. Exactness, Nakayama's lemma and equality of ranks give a Kisin module free of rank one over $(k \otimes_{\mathbf{F}_2} C_0)[[u]]$, where $k = \mathbf{F}_{16}$. Normalize its Frobenius to $u + V(u)$, with $V(u) \in (k \otimes J_0)[[u]]$. Height one makes $u^2(u+V)^{-1}$ integral. Modulo $J_0^3$ this expression is $u - V + V^2/u$, so $V(0)^2 = 0$.

With one radical generator its nonzero ramified coefficient gives $J_0^2 = J_0^3$, hence $J_0^2 = 0$. With two generators write $V(0) = \alpha X + \beta Y \pmod{J_0^2}$. Here $\alpha, \beta \in \mathbf{F}_{16}$ are $\mathbf{F}_2$-independent. If $\beta/\alpha \notin \mathbf{F}_4$, then $1, \beta/\alpha, (\beta/\alpha)^2$ are independent; otherwise their unique relation is $1 + t + t^2 = 0$. Consequently the classes of $X^2, XY + YX, Y^2$ in $J_0^2/J_0^3$ are equal to a common element $z$, which is zero in the first case. If $z = 0$, the associated graded algebra is a quotient of $\mathbf{F}_2[X,Y]/(X^2, Y^2)$, so $J_0^3 = 0$ and $C_0$ is itself a quotient of this commutative algebra.

If $z \neq 0$, choose two group elements whose augmentation differences form a cotangent basis. In the quotient modulo $J_0^3$ their squares and commutator are $1 + z$. The kernel over the two-dimensional additive quotient is central elementary abelian. Quotienting it by a complement to $\langle 1 + z \rangle$ gives a quaternion quotient of $P$. Its three quadratic subfields are the subfields of the unique split-five conductor-$(2,2)$ biquadratic extension of $H$. For the selected quadratic subfield $H'/H$, the certificate `scalar_quaternion.gp` gives dyadic break one at both places and the split-five ray group $C_4 \times C_2^5$ at conductor $(6,6)$. Its quotient on which the involution of $H'/H$ acts by inversion is $C_2^6$; the latter assertion is the Smith reduction obtained by adding the columns of $S + I$ to the ray relation matrix. A quaternion quotient would supply an order-four character of this quotient: Herbrand's formula bounds its break by $1 + 2(3-1) = 5$, and conjugation inverts the rotation character. This is impossible. The class and unit calculation in degree thirty-two uses GRH. The exact calculation is supplied in [20, `ramified/d6_d10_d11/d11`].

We have proved that $C_0$ is a quotient of $\mathbf{F}_2[X,Y]/(X^2, Y^2)$, so $P$ is elementary abelian and $|P| \leq 8$. To exclude order eight, suppose it occurs. Then $C_0$ is exactly this four-dimensional algebra and $P$ is its group of augmentation-one units. At either dyadic place write the regular Frobenius as

$$u(1 + fX + gY + hXY), \qquad f, g, h \in u^{-1}k[[u]].$$

After removing the simple factor $u$, a fundamental solution $1 + xX + yY + wXY$ satisfies

$$x^2 - x = f, \qquad y^2 - y = g, \qquad (w + xy)^2 - (w + xy) = h + fg.$$

Multiplication by $1 + aX + bY + cXY \in P$ translates $(x, y, w + xy)$ by $(a, b, c + ab)$. These three additive coordinates therefore form a basis of the quadratic characters of $P$. The first two right sides have poles of order at most one, and the third has order at most two. A double pole is equivalent to a simple pole modulo Frobenius minus identity because $k$ is perfect. Every character of $P$, being an $\mathbf{F}_2$-linear combination of these three, thus has field-of-norms upper break at most one. Independently, the actual finite-flat point field gives classical upper break at

most three for every such character by Fontaine's bound. The first break of the uniformizer tower is four and all later transition breaks are larger. Herbrand base change preserves these original breaks, and the ramification comparison in the proof of Theorem 11.13 identifies them with the field-of-norms breaks. Every character consequently has conductor at most two over each $H_{v_i}$. Thus $P$ is a quotient of the split-five ray group of conductor $(2,2)$, which is $C_2^2$, a contradiction. □

## 24.7 Quadratic conjugation and the scalar theorem

**Lemma 24.11.** *In the setting of Lemma 24.10, if $E'$ is generated by the three real and tame elements above, then* $\dim_{\mathbf{F}_2} E' \le 24$.

*Proof.* The group $\mathrm{Gal}(H/K)$ is dihedral of order eight: it has the biquadratic quotient $\mathrm{Gal}(F/K)$, a dyadic Frobenius of order four, and involutions given by real conjugation and five-inertia with independent images in that quotient. The two five-inertias belong to the same reflection class.

Set $\Gamma = \mathrm{Gal}(M'H/K)$. Choose real and tame involutions $a, b, c$ so that $b, c$ have the same image in $\mathrm{Gal}(H/K)$, and put $r = ab$, $c = jb$ with $j \in P$. Write $R, B$ for conjugation by $r, b$ on $P$. Thus $brb = r^{-1}$ and $Bj = j$. The image of the two-group $\Gamma$ in $\mathrm{Aut}(P)$ is either trivial or generated by one transvection on a two-dimensional space. In particular $R^2 = 1$.

At the five-place belonging to $b$, a Frobenius lift, after multiplication by inertia if necessary, has the form $f = pr^2$, $p \in P$. The five-decomposition image in the compositum $M'H$ is elementary abelian, so $f^2 = 1$. Since $R^2 = 1$ and $P$ has exponent two, $f^2 = r^4$, and therefore $r^4 = 1$. Thus $\langle r, b\rangle$ maps isomorphically to the dihedral quotient of order eight. The subgroup $\Gamma_0 = \langle a, b, c\rangle$ is $P_0 \rtimes \langle r, b\rangle$, where $P_0 = \langle j, Rj\rangle$; indeed $B$ fixes $j$ and commutes with $R$. Its group span maps onto $E'$, even when $\Gamma_0$ is proper in $\Gamma$.

If $\dim P_0 \le 1$, this gives $\dim E' \le |\Gamma_0| \le 16$. Otherwise $P_0 = P$ has rank two, $R = U$ is a transvection, and $j$ does not belong to its fixed line. Since $B \in \{1, U\}$ fixes $j$, we have $B = 1$.

Now $\Gamma = \Gamma_0 = P \rtimes D_8$ has order thirty-two. Let $q = R(j)j$ in multiplicative notation. The elements $q$ and $r^2$ are independent central involutions. The Cayley–Hamilton identity with determinant one makes $r + r^{-1}$ central in $E'$. Commuting it with $j$ gives $(1+q)(1+r^2) = 0$ in $E'$. Indeed $rj = qjr$ and $r^{-1}j = qjr^{-1}$, and the other factors in this commutator are units. The group algebra $\mathbf{F}_2[\Gamma]$ is free of rank eight over the central subalgebra $\mathbf{F}_2[\langle q, r^2\rangle]$. In the latter, the ideal generated by $(1+q)(1+r^2)$ has dimension one. It therefore generates an eight-dimensional ideal of $\mathbf{F}_2[\Gamma]$, and $\dim E' \le 32 - 8 = 24$. □

**Theorem 24.12.** *Assume GRH. A two-special bounded system all of whose dyadic members are nonordinary and residually scalar is modular.*

*Proof.* Let $h = \chi_{-1}$ and choose $\eta = 1$ or $\chi_{-5}$ aligning the two exact quadratic *inertia* signs of $r$ and $r^\sigma \otimes \eta$. Both characters are invariant under $\sigma$; $h$ is unramified at two, trivial at five, and $\eta$ is trivial at two. The commuting operations $r \mapsto r \otimes h$ and $g : r \mapsto r^\sigma \otimes \eta$ give a packet of at most four coefficient orbits. Denote their number by $m$ and the common degree of their trace fields by $d$. Each member retains the biconnected dyadic hypothesis. Use their joint reduction $E$. If their full special signs agree at both places, the five-group already has exponent two. Otherwise at each place, with Frobenius lifts corresponding under $\sigma$, the signs on the $r, rh$ packet differ from those on the $g(r), g(r)h$ packet. The discrepancy is the same at both places, since the involution $\sigma$ exchanges their characters, $h$ splits at five, and $\eta$ is invariant under $\sigma$. A difference of rational signs separates the two coefficient packets.

Write $S$ for the tuple of $+1$ on the first packet and $-1$ on the second. At each five-place $T(F) = \pm 6S$, so $t = 2S \in R_0$. For normalized tame differences $B, C$ at the two places, the exact

special matrices give $BF = \pm SB$ or $CF = \pm SC$. In $E$ this yields $tB = tC = 0$ and $t^2 = 0$. The equation $F^2 - T(F)F + 5 = 0$ makes the Frobenius an involution in $E' = E/tE$. Tame inertia there is an involution commuting with Frobenius by the fifth-power relation. If all signs agree, take $E' = E$ instead. In both cases the five decomposition groups have exponent two. The stable central quotient inherits the finite-flat regular model at two, by closure and quotient of the actual right radical maps; its regular group span, or its restricted group span over $H$, is precisely the torsion representation used in Lemma 24.10. Lemma 24.9 gives generation of $E$ by $A, B, C$ independently of the Frobenius signs, and therefore generation of $E'$. Put $R = R_0/2R_0$ and let $\mathfrak{m}$ be its maximal ideal. The polarized identities identify the anticommutators of $A, B, C$ with the scalars $X = T(AB)$, $Y = T(AC)$, $Z = T(BC)$. Ordering words with these relations and the three zero squares leaves eight square-free monomials. Their traces show that $X, Y, Z$ and $W = T(ABC)$ generate $\mathfrak{m}$: the actual trace order is generated by traces of group elements, and each such element belongs to the algebra generated by $A, B, C$ over $\mathbf{F}_2$. Applying the original $R$-valued trace to $tAB, tAC, tBC, tABC$ gives $tX = tY = tZ = tW = 0$ in $R$, independently of faithfulness of its reduced scalar action. Thus $t\mathfrak{m} = 0$. Since $t$ is central and kills $B, C$, the ideal $tE$ is spanned over $\mathbf{F}_2$ by $t, tA$. Hence Lemma 24.11 gives $4md = \dim E \leq 24+2 = 26$. An actual selftwist by $h$ is impossible: it would give induction from $K(i)$, while this quadratic extension splits at five and the source has nonzero monodromy there. The bound $\dim E \leq 26 < 32$ and Theorem 1.5 now give either an actual twisted-conjugacy identity involving $\sigma$, in which case Lemma 12.4 proves modularity, or a trace field of degree at most two or biquadratic over $\mathbf{Q}$. Every prime above three then has residue degree at most two.

In the small-residue-field cases use the triadic source results above for a solvable residue. A nonsolvable residue stays absolutely irreducible over $K(\zeta_3)$ by Clifford theory. Its trace field realizes it over $\mathbf{F}_3$ or $\mathbf{F}_9$ by Brauer–Nesbitt and the vanishing of the Brauer group of a finite field. Theorems 2.5 and 2.3 of [3], with Lemma 2.1, give modularity of the actual weight-two member and hence of the system. □

## 24.8 The supported residual base at five

**Proposition 24.13.** *Assume GRH. Every bounded system is modular. Every absolutely irreducible $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ of determinant $\omega_5$, unramified outside five, is modular.*

*Proof.* For the two-special menu, Propositions 24.6, 24.7, 24.8 and Theorem 24.12 exhaust the dyadic reductions. Consider now any bounded system in characteristic three. The preceding triadic results handle reducible and cyclotomic-dihedral residues. For an irreducible solvable residue not of the latter form, solvable Artin modularity and weight raising supply a weight-zero witness, and [3, Theorem 2.3] applies. Otherwise the residue remains absolutely irreducible on $K(\zeta_3)$. At both five-places prescribe the genuine special lifts of Lemma 4.18, and retain the given crystalline weight-two component at the unique triadic place. The prescribed-component theorem [35, Theorem 4.2.1] and the weight-two construction in Lemma 2.6 give a pure compatible replacement. A point on the special component has Frobenius eigenvalue ratio five; with zero monodromy this would contradict purity. Thus the replacement is special at both places and modular. A common modular residual witness lifts the actual old weight-two triadic member by [3, Theorem 2.3].

Finally, an irreducible solvable supported five-residue has a regular cuspidal witness by solvable Artin modularity. In the nonsolvable case its restriction to $K(\zeta_5)$ is irreducible. The EGPZ exceptional case cannot occur: $[K(\zeta_5) : K] = 4$, whereas the abelianization of $\mathrm{PGL}_2(\mathbf{F}_5)$ has order two. Use Lemma 4.17 at the split five-places to fix bounded weight-two lifts of the actual local representations. EGPZ gives a global lift, unramified off five, and Lemma 2.6 gives a bounded strongly compatible system. Its modularity proves the supported residual base by reduction. □

# 25 Special dyadic sources over $\mathbf{Q}(\sqrt{14})$

Put $K = \mathbf{Q}(s)$, with $s^2 = 14$, and write $\sigma$ for its nontrivial automorphism. Write $\omega_\ell = \bar{\chi}_\ell$, and let $\kappa_a$ be the quadratic character of a squareclass $[a]$. A bounded five-source is a potentially modular weight-two system of cyclotomic determinant that is good outside five, with type SP or SC at both five-places. Type SP specifies a special parameter with nonzero monodromy and its exact quadratic character. Type SC is one of the two determinant-one tame cuspidal types

$$\widetilde{\omega}_2^4 \oplus \widetilde{\omega}_2^{20}, \qquad \widetilde{\omega}_2^8 \oplus \widetilde{\omega}_2^{16}, \tag{23}$$

where $\omega_2$ has order 24. A special source is good outside two and five, special at both five-places, and special at two with unramified quadratic character.

A congruence at a coefficient prime dividing $9 - a_2^2$ allows us to pass to a special source. Its full local residual fields at two and five are elementary quadratic. The repeated-root case $a_2^2 = 8$ uses the regular trace order of Section 11.

## 25.1 Quadratic arithmetic and discriminant comparisons

Put

$$\varepsilon = 15 + 4s, \qquad \varpi = 4 + s, \qquad a = 3 + s, \qquad H = K(\sqrt{-2}).$$

The Minkowski bound is $\sqrt{56}/2 < 4$. Three is inert and $\varpi$ generates the dyadic prime, so the class number is one. The congruence $x^2 \equiv -1 \pmod 7$ excludes units of norm $-1$, so the narrow class number is two. The least positive $y$ for which $x^2 - 14y^2 = 1$ is $y = 4$. Hence $\varepsilon$ is the fundamental unit, and $\varepsilon = \varpi^2/2$.

**Lemma 25.1.** *The squareclasses supported at two and five form*

$$W = \langle [-1], [2], [5], [\varpi], [a] \rangle.$$

*At the five-places $v : s \equiv 2$ and $w : s \equiv -2$, their local restrictions are*

| | $-1$ | $2$ | $5$ | $\varpi$ | $a$ |
|---|---|---|---|---|---|
| $v$ | 0 | $U$ | $R$ | 0 | $R$ |
| $w$ | 0 | $U$ | $R$ | $U$ | 0 |

*where $U$ is the unramified quadratic class and $R = [5]$. Consequently the kernel of restriction to both five-decomposition groups is $\langle [-1] \rangle$. The kernel at two is $\langle [-2] \rangle$, and the inverse image of the unramified dyadic line is $\langle [-2], [5] \rangle$. The dyadic conductor-at-most-two subspace is $\langle [-1], [2], [5] \rangle$.*

*Proof.* The class number, the fundamental unit and generators of the three supported primes give the first assertion and independence. The five-adic table follows by reduction modulo five. At the place where $a$ is a uniformizer, $a/5 = -1/(3 - s)$ is a square.

At two we have $K_2 = \mathbf{Q}_2(\sqrt{-2})$, since $-7$ is a square in $\mathbf{Q}_2$. Hence $[-1] = [2]$, and $[5]$ is the nontrivial unramified class. The biquadratic extension $\mathbf{Q}_2(\sqrt{-1}, \sqrt{2})$ has discriminant exponent $2 + 3 + 3 = 8$ over $\mathbf{Q}_2$, so the conductor of $[-1]$ over $K_2$ is $8 - 2 \cdot 3 = 2$. The Eisenstein polynomials $X^2 + 2X + 1 - a$ and $X^2 - \varpi$ have discriminant valuations four and five in $K_2$, respectively. They give the conductors of $[a]$ and $[\varpi]$. These successive conductors give four independent local classes and the asserted filtration on $W$. □

The biquadratic field $H = \mathbf{Q}(\sqrt{-2}, \sqrt{-7})$ has discriminant $56^2$. It is split at two over $K$ and unramified quadratic at both five-places. Its Minkowski bound is $84/\pi^2 < 9$. Only the two dyadic primes have norm below nine, since the primes over three, five and seven have residue

degree two. If $u = (1+\sqrt{-7})/2$, then $u^2+2 = u$, so $u+\sqrt{-2}$ has relative norm $u$ over $\mathbf{Q}(\sqrt{-7})$ and absolute norm two. This element and its conjugate generate the two dyadic primes. Hence $H$ has class number one.

Define

$$V = K(\sqrt{-1}, \sqrt{2}, \sqrt{5}, \sqrt{\varpi}, \sqrt{a}). \tag{24}$$

**Lemma 25.2.** *The extension $V/\mathbf{Q}$ is Galois of degree 64, and*

$$\operatorname{disc}(V) = 2^{216}5^{32}7^{32}, \qquad R_V := \operatorname{rd}(V) = \sqrt{280}\,2^{15/8}. \tag{25}$$

*Its dyadic completion is the maximal elementary quadratic extension of $K_2$, of absolute ramification index 16 and residue degree two. At either five-place its completion is the maximal elementary quadratic extension of $\mathbf{Q}_5$. There is an unramified cyclic cubic extension $T/L$, split at all primes above two and five, where*

$$L = K(\sqrt{-5\varpi}) \subset V.$$

*In particular $TV/V$ has degree three and the same unramified and splitting properties.*

*Proof.* Independence in Lemma 25.1 gives the degree. The identities

$$[4-s] = [2] + [\varpi], \qquad [3-s] = [-1] + [5] + [a]$$

show stability of the squareclass space under $\sigma$, proving normality over $\mathbf{Q}$. Its dyadic localization has rank four, the full rank of $K_2^\times/K_2^{\times 2}$; the five-adic table has rank two at each place. This proves the completion assertions.

In the dyadic conductor filtration the dimensions on $W$ are $2, 3, 4, 5$ at conductors $0, 2, 4, 5$. Thus the sum of the dyadic conductors is $4\cdot 2 + 8\cdot 4 + 16\cdot 5 = 120$. At each five-place exactly sixteen characters have conductor one. The relative conductor–discriminant formula gives

$$\operatorname{disc}(V) = 56^{32}2^{120}5^{32}.$$

An equation for $L$ is $X^4 + 40X^2 + 50$. Exact class and unit arithmetic gives

$$\operatorname{Cl}(L) = C_6 \times C_2, \qquad \operatorname{Cl}(L)/\langle [\mathfrak{p}] : \mathfrak{p} \mid 10\rangle = C_3.$$

The class group and units are fully certified. Class field theory gives $T$. Since $[V : L]$ is a power of two, $T \cap V = L$. The asserted properties persist under base change. The class group, classes of the primes over two and five, and the resulting quotient are computed in [59, `d14_support/2026-11-07/code/certify_endpoint_arithmetic.py`]. □

Use Proposition 13.3 with $B_{b,100}$. Use the densities $\rho_m = N_m/n$ of Section 13. When a residue degree is one, retain only the even Euler powers. Their contribution equals that of a norm with residue degree two, with half the prime density. For the totally real comparison take $r_1 = n$.

Table 2 gives the strict comparisons from outward interval arithmetic [59, `d14_support/2026-11-07/certificates/discriminant_intervals.json`]. In the first three rows use $\rho_4 = 1/32$ and $\rho_{25} = 1/4$.

## 25.2 The special source at two

**Proposition 25.3.** *Assume GRH. A special source with nonsolvable dyadic residue is modular.*

| Field condition | $n_{\min}$ | $b$ | lower bound $>$ | upper bound $<$ |
|---|---|---|---|---|
| special, $q \geq 16$, cubic auxiliary | 783360 | 67/5 | 61.6606783461 | 61.3776501696 |
| special, $q = 8$, noninvariant | 16257024 | 82/5 | 65.1320008831 | 61.3776501696 |
| special, totally real | 120 | 5 | 73.3308693122 | 61.3776501696 |
| nonordinary, $q \geq 8$ | 2016 | 147/20 | 41.44624885 | 33.46640107 |

Table 2: Discriminant comparisons. The nonordinary row retains $\rho_{25} = 1/4$.

*Proof.* Let $M/K$ be the point field of its dyadic residue. Dickson's classification and determinant one identify $\mathrm{Gal}(M/K)$ with $G = \mathrm{SL}_2(\mathbf{F}_q)$, where $q \geq 4$ is a power of two. At two the actual source has an ordinary filtration with unramified quadratic quotient and cyclotomic times that quotient as subcharacter. Its two residual diagonal characters are one. The entire residual decomposition image is therefore elementary unipotent. At either five-place the same assertion follows from the special filtration: the quadratic character and the cyclotomic character both reduce to one. Thus all three local point fields are elementary quadratic extensions.

The simple nonabelian group $G$ has no nontrivial solvable quotient, so $M \cap V = K$. Lemma 25.2 absorbs all the local point fields at two and five. Consequently $MV/V$ is unramified at finite places and split at every place above two and five. In particular

$$[MV : \mathbf{Q}] = 64q(q^2 - 1), \qquad \mathrm{rd}(MV) = R_V,$$

and its prime densities are $\rho_4 = 1/32$, $\rho_{25} = 1/4$.

The cyclic cubic extension $TV/V$ is disjoint from $MV/V$: a common nontrivial subextension would be a cyclic quotient of $G$. The compositum has degree $192q(q^2 - 1)$, unchanged root discriminant $R_V$, and the same prime densities. For $q \geq 16$ the first row of Table 2 is a contradiction.

If every residual real conjugation were trivial, $M$ would be totally real of degree at least 120. Since its local fields are subfields of those of $V$, $\mathrm{rd}(M) \leq R_V$. Retaining even Euler powers if a residue degree is one gives $\rho_4 \geq 1/32$, $\rho_{25} \geq 1/4$. The third row of the table excludes this possibility. Thus some residual real conjugation is nontrivial.

Suppose $q = 8$. If $M^\sigma \neq M$, simplicity and equality of degrees give $M \cap M^\sigma = K$. The group of their compositum over $K$ is $G \times G$ and has no nontrivial solvable quotient. It is disjoint from $V$, so

$$[MM^\sigma V : \mathbf{Q}] = 64 \cdot 504^2 = 16257024.$$

Its completion at each of the three supported places is still contained in the corresponding completion of $V$. Its root discriminant and prime densities are again $R_V$, 1/32, 1/4. The second row of the table is a contradiction. Hence $M^\sigma = M$ and $M/\mathbf{Q}$ is Galois.

Use Lemma 2.5 for $q = 8$ and [3, Theorem 2.5] for $q = 4$ to obtain a modular residual witness.

Apply [83, Theorem 6.2] to obtain an ordinary parallel-weight-two witness after solvable totally real base change. Its real-place hypothesis was checked above. The source stays ordinary, the residual image stays nonsolvable, and the residual ordinary characters at two are both trivial. Apply Lemma 2.3(i), followed by solvable descent. □

### 25.2.1 Ordinary sources with solvable residue

The ordinary argument applies to both semistable and crystalline sources at two.

**Lemma 25.4** (Marked ordinary rigidity)**.** *Let $\mathcal{R}$ be a weight-two system of cyclotomic determinant, good outside two and five and special at both places above five. Suppose its dyadic member*

*$r$ is ordinary with an unramified quotient and scalar residual semisimplification. Then*

$$r^\sigma \simeq r \otimes \eta$$

*for a quadratic character $\eta$ restricted from $G_{\mathbf{Q}}$.*

*Proof.* Take the orbit lattice of Lemma 4.10, with $v_0$ on the actual ordinary line. Its additive residue is nonzero and trivial at two; this uses only ordinarity and therefore covers both crystalline and semistable sources. Lemma 25.1 makes the extension a nonzero multiple of $h = \kappa_{-2}$. Since $h(c) = 1$ at a real conjugation $c$, the integral basis

$$((r(c) - 1)v_0, v_0)$$

puts its matrix and the ordinary mark in the form

$$C = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad e_2, \qquad \overline{r} = 1 + hE_{12}. \tag{26}$$

Let $\theta_v, \theta_w$ denote the exact quadratic special characters. The restrictions of $h$ and $\kappa_5$ give the full quadratic character group at either five-place. Choose a character $\eta \in \{1, h, \kappa_5, h\kappa_5\}$ whose local value at both places is $\theta_v\theta_w$. Thus $r$ and $r^\sigma \otimes \eta$ have the same exact special characters at five. The twist is unramified at two by Lemma 25.1, so both ordinary quotients are unramified there.

Normalize both representations as in (26). Let $B$ be the closed $\mathbf{Z}_2$-order generated by their simultaneous matrix entries in the product of their coefficient rings. It is finite over $\mathbf{Z}_2$. The product of coefficient rings is integral over $B$. The two residue maps contract to the same maximal ideal of $B$, with residue field $\mathbf{F}_2$, by the common reduction (26). Hence $B$ is local. The ordinary line is $e_2$. At a five-place let $\lambda_v(g)$ be the exact character on the special line. A local Frobenius $F_v$ has $(1, 2)$ entry a unit of $B$, since $h(F_v) = 1$. The slope of the special line is the element

$$\frac{\lambda_v(F_v) - r(F_v)_{11}}{r(F_v)_{12}} \in B$$

in the simultaneous realization. Here $\lambda_v(F_v)$ is a common rational 2-adic integer. Both special-line conditions therefore hold over $B$.

For a relative tangent of $B$, apply Lemma 12.1a to $1 + hE_{12}$. Its lower coordinate $z$ is additive. At two the marked ordinary equations give $y = 0$ and unramified $x$; no finite flat condition on $z$ is imposed. The fixed special characters at five give

$$z|_{G_v} = 0, \qquad x|_{G_v} = t_v h|_{G_v}.$$

The additive character $z$ is supported at two and five. The kernel of the two five-localizations on $W$ gives $z = A\kappa_{-1}$. The fixed real matrix makes $z(c) = 0$, so $A = 0$.

Now $x$ and $y$ are additive. The dyadic restrictions in Lemma 25.1 give

$$y = b_0 h, \qquad x = c_0 h + d_0 \kappa_5.$$

At five, $h = U$ and $\kappa_5 = R$; the exact line equation forces $d_0 = 0$. The fixed real matrix gives $x(c) = y(c) = 0$, hence $b_0 = c_0 = 0$. The relative cotangent of $B$ is zero. Nakayama gives $\mathfrak{m}_B = 2B$, and then $B = \mathbf{Z}_2$, since $B$ is finite over $\mathbf{Z}_2$ with $B/2B = \mathbf{F}_2$. The simultaneous normalized matrices are equal, so $r^\sigma \simeq r \otimes \eta$. □

**Proposition 25.5.** *Let $\mathcal{R}$ be a weight-two system of cyclotomic determinant that is good outside two and five and special at both five-places. Suppose its actual dyadic member is ordinary with unramified quotient and has solvable residual image. Then $\mathcal{R}$ is modular.*

*Proof.* An irreducible residue is induced from an odd-order character of a quadratic extension $L/K$ with class in $W$. If $L$ splits at two then Lemma 25.1 gives $L = H$. In this case the inducing character is unramified at two because the ordinary residual diagonal characters are unramified. It is unramified at five because the residual inertia there is a two-group. The class number one of $H$ excludes this case. The ordinary modular witness in [2, Lemma 5.1.2] has the required ordinary ordering. Theorem 5.2.1 there applies to the actual dyadic member because the inducing field is nonsplit at two.

In the reducible case both characters have odd order and are unramified at every finite place. The narrow class group of order two makes the semisimplification scalar. Apply Lemmas 25.4 and 12.4. □

**Theorem 25.6.** *Assume GRH. Every special source is modular.*

*Proof.* Use Proposition 25.3 for a nonsolvable dyadic residue and Proposition 25.5 for a solvable one. A special source is an actual ordinary source at two. □

## 25.3 Odd coefficient primes

The following lemma isolates the residual cases in which prescribed lifting with irreducible cyclotomic restriction is unavailable.

**Lemma 25.7.** *Let $\mathcal{R}$ be a system good outside five and special at both five-places. Let $\lambda \mid \ell$ be an odd coefficient place. Suppose*

$$a_2^2 \equiv 9 \pmod{\lambda}. \tag{27}$$

*If $\overline{r}_\lambda$ is reducible, or becomes reducible on $G_{K(\zeta_\ell)}$, then $\mathcal{R}$ is modular. For $\ell = 3$ the same conclusion holds without* (27)*, and with either of the types* (23) *also permitted at five.*

*Proof.* Suppose first that $\overline{r}_\lambda^{\mathrm{ss}} = \psi \oplus \omega_\ell \psi^{-1}$. At five the square $\psi^2$ is unramified away from the coefficient characteristic. For the cuspidal types in characteristic three the cubic part of inertia reduces to one. Away from five and $\ell$ it is unramified as well. We have

$$N_{K/\mathbf{Q}}(\varepsilon^2 - 1) = -896 = -2^7 \cdot 7. \tag{28}$$

If $\ell \neq 5, 7$, the field is unramified at $\ell$ and the source is crystalline of weights $\{0, 1\}$. The rank-one finite-flat inertia exponents have digits zero or one [65]. Let the two digits corresponding to the embeddings of $K$ be $a_1, a_2 \in \{0, 1\}$. Unit reciprocity for $\psi^2$ and $N(\varepsilon) = 1$ give

$$\varepsilon^{\,2(a_1 - a_2)} \equiv 1 \pmod{\ell},$$

with the evident interpretation in the quadratic residue field when $\ell$ is inert. By (28), $a_1 = a_2$. The two local characters are therefore the etale and multiplicative endpoints. Lemma 4.3(i), with $e = 1 < \ell - 1$, therefore forces actual ordinarity.

At $\ell = 5$ the source is already ordinary at both places. Removing the quadratic special characters, write its two orientation digits as $a_v, a_w \in \{0, 1\}$. The reductions of $\varepsilon$ at the five-places are 3 and 2, both of order four. Reciprocity for $\psi^2$ gives $a_v = a_w$.

At $\ell = 7$ the unique local field has absolute ramification index two, smaller than $\ell - 1$. Its rank-one finite-flat exponents are $j = 0, 1, 2$, and $\omega_7$ has exponent two. If $j = 1$, the character $\psi^2 \omega_7^{-1}$ is unramified at every finite place. Its order divides the narrow class number, which is two. On the other hand (27) says that the two Frobenius eigenvalues at two are $\delta, 2\delta$, for $\delta \in \{1, -1\}$. Their quotient is 2 or 4 in characteristic seven, of order three. This is the value of $\psi^2 \omega_7^{-1}$ at that Frobenius, a contradiction. Hence $j = 0$ or 2, and Lemma 4.3(i) gives actual ordinarity, now using $e = 2 < 7 - 1$.

In all remaining reducible cases, the common endpoint orientation is $a \in \{0, 1\}$. The character $\psi^2 \omega_\ell^{-2a}$ is everywhere unramified and totally even. Since $h_K = 1$, it is trivial. Thus the constituent quotient is $\omega_\ell$ or $\omega_\ell^{-1}$. It is totally odd, extends to $G_{\mathbf{Q}}$, and is nontrivial at every $\ell$-adic decomposition group. The actual ordinary quotient matches the same global constituent at each such place. The field $K(\zeta_\ell)$ is abelian over $\mathbf{Q}$. Consequently [75, Section 4.5, Theorem A] applies to the actual source. Its ordinary quotient divided by the Teichmüller lift of its residue is pro-$\ell$ and has finite inertia, as required.

Suppose next that the residue is irreducible but its cyclotomic restriction is reducible. Clifford theory writes it as induction from the unique quadratic subextension

$$L_\ell = K\big(\sqrt{(-1)^{(\ell-1)/2}\ell}\big) \subset K(\zeta_\ell).$$

The inducing character has order prime to $\ell$, and the residual image has order prime to $\ell$.

For $\ell \neq 5, 7$, $L_\ell/K$ is ramified quadratic at every $\ell$-place. Tame inertia is cyclic and its projective dihedral image contains a reflection, so that image has order at most two. An irreducible finite-flat local residue over an unramified field of degree one has projective inertia order $\ell + 1$ [65]; over degree two it has order $(\ell^2 + 1)/2$. Neither is at most two. The local residue is therefore split reducible. The quotient orders in the rank-one finite-flat list are $\ell - 1$ for the endpoint pair and, in residue degree two, $\ell + 1$ for the mixed pair. A quadratic quotient is possible only when $\ell = 3$, at the endpoints. The endpoint characters force the actual source to be ordinary by the same finite-flat uniqueness argument. The residual ordinary characters are distinct. Allen's [2, Lemma 5.1.2] supplies a modular ordinary witness with the required quotient. Skinner's corrected theorem, in [90, Theorem 4.2], applies because the unique three-place is ramified, not split, in the CM extension $K(\zeta_3)$. This modularizes the actual three-adic source.

For $\ell = 5$, special inertia has projective order four after reduction. But $L_5/K$ is ramified there and cyclic tame inertia containing a reflection has order at most two. This is impossible.

It remains to consider $\ell = 7$. Now $L_7 = H$. Write the residue as $\mathrm{Ind}_{G_H}^{G_K} \xi$. Its image has order prime to seven, so the special monodromy at five reduces to zero and its inertia acts by a quadratic scalar. Thus $\xi^2$ is unramified outside seven. For $u$ as above and its conjugate $\bar{u}$, the ideals in $H$ are $(u) = P^2$ and $(\bar{u}) = P'^2$, where $P, P'$ are the two dyadic primes. At the unique seven-place, $u/\bar{u}$ is a principal unit. The prime-to-seven character $\xi^2$ is trivial on that unit group. Global reciprocity therefore gives

$$\left(\frac{\xi(\mathrm{Frob}_P)}{\xi(\mathrm{Frob}_{P'})}\right)^4 = 1.$$

Since $H/K$ splits at two, the quotient in parentheses is the quotient of the two residual Frobenius eigenvalues there. By (27) it has order three, a contradiction. □

**Lemma 25.8.** *A bounded five-source whose trace field has degree at most two is modular.*

*Proof.* Take a coefficient place above three. Its trace residue field is $\mathbf{F}_3$ or $\mathbf{F}_9$. For an irreducible residue, finite-field trace and determinant descent realizes it over that field; the Brauer group of a finite field is zero. The residual modularity theorem [3, Theorems 1.1 and 2.5] applies. If the cyclotomic restriction is irreducible, apply Lemmas 2.1 and 2.2(ii). Lemma 25.7 handles the other two residual cases; its characteristic-three assertion does not require (27). □

## 25.4 Nonordinary crystalline sources

**Proposition 25.9.** *Assume GRH. A system good outside five and special at both five-places is modular if it has a nonordinary crystalline dyadic member with nonsolvable residue.*

*Proof.* Write its residual image as $\mathrm{SL}_2(\mathbf{F}_q)$, $q \geq 4$, and its point field as $M$. At both five-places the residual decomposition group is elementary abelian by the special filtration, with ramification and residue degrees at most two.

Theorem 11.13 bounds the normalized dyadic different contribution by $1 - 1/q$. In the wild case the inertia has order $e \leq q$ and upper breaks at most one, giving the bound $1 - 1/e$. Tame inertia of order three contributes $1/3$.

The fields $M$ and $H$ are disjoint over $K$. The extension $H/K$ is unramified, split at two and inert at five. Thus

$$[MH : \mathbf{Q}] = 4q(q^2 - 1), \qquad \mathrm{rd}(MH) \leq \sqrt{280}\, 2^{1-1/q}.$$

Over each five-place of $K$ the compositum $MH$ has residue degree two and ramification index at most two. Thus $\rho_{25} \geq 1/4$ and the nonordinary row of Table 2 excludes $q \geq 8$. For $q = 4$ the kernel field has degree 120 and root discriminant at most $\sqrt{280}\, 2^{3/4} < 28.15$. Lemma 13.4 excludes a totally real kernel field, so the actual crystalline member is modular by Lemma 2.4. □

**Lemma 25.10.** *For a system good outside five, special at both five-places, a nonordinary dyadic residue with irreducible solvable image is locally irreducible at two.*

*Proof.* Such a residue is induced from an odd-order character of a quadratic extension $L/K$. Its inducing quadratic character has conductor at most two at two. For a reducible local residue this follows from Lemma 11.5 after unramified base change; for an irreducible one, its odd dihedral inertia is tame of order three and its quadratic quotient is unramified. Thus $L$ lies in the seven nontrivial classes of $\langle [-1], [2], [5] \rangle$.

The odd inducing character is unramified at five and elsewhere away from two; its dyadic ramification is tame. The ray groups of these seven fields modulo (2) are

| $c$ | $-1$ | $2$ | $-2$ | $5$ | $-5$ | $10$ | $-10$ |
|---|---|---|---|---|---|---|---|
| $\mathrm{Cl}_{(2)}(K(\sqrt{c}))$ | $C_8$ | $1$ | $C_2^2$ | $C_2$ | $C_8 \times C_4$ | $C_2$ | $C_{12} \times C_2$. |

The class groups and ray invariants are computed in [59, `d14_support/2026-11-07/certificates/endpoint_arithmetic.json`]. Only $c = -10$ has an odd part, necessarily of order three. Its ordinary class group is $C_2$, so the cubic character is ramified at two. The extension $K(\sqrt{-10})/K$ is unramified quadratic and nonsplit at two; induction gives the irreducible local tame $S_3$ representation. □

**Proposition 25.11.** *Assume GRH. A system good outside five and special at both five-places is modular if $a_2^2 = 8$.*

*Proof.* Potential modularity and the special finite place give an abelian realization after a finite extension: apply [77, Lemma 9.4.3] to the weight-two modular form, using a coefficient prime different from five. The special parameter remains square-integrable after base change. The abelian variety has good reduction above two, by compatibility and the Néron–Ogg–Shafarevich criterion. Frobenius of an abelian variety over a finite field is semisimple. A power of the original crystalline Frobenius is therefore semisimple, and so is the original Frobenius in characteristic zero.

The arithmetic crystalline polynomial is $X^2 - a_2 X + 2$, with a repeated root. Hence its Frobenius is scalar, equal to $a_2/2$. In the covariant filtered module its Newton slope is $1/2$. After enlarging coefficients, there are two labelled filtration lines, one for each embedding of the ramified quadratic completion $K_2$. They cannot coincide: a common line would have Hodge number one and Newton number $1/2$, contradicting weak admissibility. Each of the two distinct lines has Hodge and Newton numbers $1/2$, and is a weakly admissible subobject. Thus the

actual local representation splits as two crystalline characters with mixed labelled weights $(1,0)$ and $(0,1)$. In particular it is nonordinary.

This applies at every dyadic coefficient embedding. An irreducible solvable global residue would be locally irreducible by Lemma 25.10, contradicting local splitting of the actual source. A nonsolvable residue is covered by Proposition 25.9. Otherwise all dyadic residual semisimplifications are scalar: their one-dimensional constituents have odd order and are unramified at two, have trivial inertia at five, and are unramified elsewhere, so the narrow class number two makes them trivial.

Apply Proposition G.1 to all dyadic coefficient realization of this last system. The rational simultaneous trace algebra is $D \otimes_{\mathbf{Q}} \mathbf{Q}_2$ by Section 11. Since $a_2$ generates this algebra, $[D : \mathbf{Q}] = [\mathbf{Q}(a_2) : \mathbf{Q}]$, giving

$$D = \mathbf{Q}(a_2) = \mathbf{Q}(\sqrt{2}).$$

Lemma 25.8 proves modularity. □

## 25.5 The bounded five-source theorem

**Theorem 25.12.** *Assume GRH. Every system good outside five and special at both five-places is modular.*

*Proof.* Put $B = 9 - a_2^2 \in \mathcal{O}_D$. Every conjugate of $B$ is at least one. If $N_{D/\mathbf{Q}}(B) = 1$, every conjugate equals one, so $a_2^2 = 8$ and Proposition 25.11 applies.

Otherwise choose a coefficient prime dividing $B$. Suppose first that it lies over an odd prime $\ell$. Lemma 25.7 handles a reducible residue and an irreducible residue with reducible cyclotomic restriction. In the other case, the actual residual representation at two is unramified, with Frobenius eigenvalues $\delta, 2\delta$, $\delta \in \{1, -1\}$. They are distinct. It has a genuine special lift with determinant $\chi_\ell$: use Frobenius $\delta\,\mathrm{diag}(2,1)$ and nonzero tame monodromy parameter divisible by the coefficient uniformizer. This reduces to the same actual unramified residual representation.

Retain the old weight-two components at $\ell$, the exact special characters at five and the unramified conditions elsewhere. Lemma 2.6(i) gives a pure compatible system, and purity retains the special monodromy (Section 2). Theorem 25.6 makes this system modular. Lemmas 2.1 and 2.2(ii) transfer modularity to the old weight-two member.

Suppose instead that the chosen coefficient prime is dyadic. Then $a_2$ is a unit. Its crystalline polynomial has a unique unit-root line. Weak admissibility and the nonnegative labelled Hodge numbers make this line have Hodge number zero at both embeddings; it supplies an unramified ordinary quotient of the original representation. Its two residual ordinary characters are trivial, because their values at Frobenius are $a_2 \bmod \lambda = 1$.

A solvable global residue is covered by Proposition 25.5. For a nonsolvable residue we replace the ordinary crystalline local source at two by a special one with the same actual residue. After normalizing the two residual characters to one, Kummer theory gives

$$H^1(K_2, \mathcal{O}_E(1)) \longrightarrow H^1(K_2, k_E)$$

surjectively: $K_2^\times \widehat{\otimes} \mathbf{Z}_2$ surjects onto $K_2^\times / K_2^{\times 2}$, and then one extends coefficients. Choose a lift of the given extension and make its valuation nonzero by adding a uniformizer Kummer class multiplied by the coefficient uniformizer. The resulting extension of 1 by $\chi_2$ is semistable noncrystalline, with exactly the original residual extension and nonzero monodromy.

Prescribe this special component, retaining the exact special components at five and unramifiedness elsewhere. Lemma 2.6(iii) applies to the nonsolvable residue and the ordinary semistable component of weight two. The resulting special source is modular by Theorem 25.6.

Both ordinary residual characters at two are trivial, so its dyadic member supplies a matching witness. Apply Lemma 2.3(i) to the original ordinary crystalline member. □

**Theorem 25.13.** *Assume GRH. Every bounded five-source is modular.*

*Proof.* Choose a coefficient place above three. Apply Lemma 25.7 if the cyclotomic restriction of the residue is reducible. Otherwise use Lemma 4.18 at the cuspidal five-places, retain the special places and the crystalline weight-two component at three, and impose unramified conditions elsewhere. Lemma 2.6(i) gives a pure system special at both five-places. Theorem 25.12 and Lemmas 2.1 and 2.2(ii) give modularity of the original system. □

**Theorem 25.14.** *Assume GRH. Every irreducible representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ of determinant $\omega_5$, unramified at finite places away from five, is modular.*

*Proof.* The determinant makes the representation totally odd. Solvable image has an odd solvable Artin witness. Otherwise the cyclotomic restriction is irreducible, and the characteristic-five exception to Lemma 2.6 is absent over $K$. Apply Lemma 4.17 at both five-places and prescribe unramifiedness elsewhere. Lemma 2.6(i) gives a bounded five-source. Theorem 25.13 gives its modularity, and reduction gives the residual assertion. □

# 26 A one-sided terminal source over $\mathbf{Q}(\sqrt{15})$

**Theorem 26.1.** *Assume GRH for Dedekind zeta functions. Let $K = \mathbf{Q}(\sqrt{15})$ and let $r$ be the seven-adic terminal member of Proposition 6.2, with determinant $\chi_7$. Then $r$ is modular.*

Write $v_+, v_-$ for the two seven-places as in Proposition 6.2. The actual member is absolutely irreducible at $v_+$ and crystalline of weights $\{0, 1\}$ at $v_-$.

## 26.1 Irreducible terminal residues

For an irreducible terminal residue, Proposition 6.5 gives cyclotomic irreducibility and handles the solvable case.

For a nonsolvable residue keep the actual crystalline component at $v_-$ and use Lemma H.1 at $v_+$. Write CR for scalar quadratic inertia with zero monodromy, SP for quadratic special type with nonzero monodromy, and C4, C8 for determinant-one tame cuspidal types of orders four and eight. Lemma 2.6(i) gives a parallel-weight-two comparison system with determinant $\chi$ and one of the profiles

$$\mathrm{CR/CR}, \qquad \mathrm{SP/CR}, \qquad \mathrm{C4/CR}, \qquad \mathrm{C8/CR}.$$

Each profile records the type at $v_+$ first. The systems are good away from seven. The following argument proves their modularity, and Lemma 2.2(i) then applies to the original seven-adic member.

## 26.2 Dyadic comparison systems

For a nonsolvable dyadic residue that is unramified at both seven-places, or for a nonordinary nonsolvable dyadic source in any of the four profiles, the comparison is immediate. Adjoin $K(i)$ to the residual point field. This quadratic extension is finite-unramified and disjoint from the residual point field. For $q \geq 8$ the compositum has degree at least 2016. Omit all finite prime terms in (11). At $b = 7.35$, $N = 100$, the lower bound is greater than 31.07694. If

both seven-places are unramified the upper bound is $4\sqrt{60} < 30.98387$. For a nonordinary member with one ramified seven-place the upper bound is $2\sqrt{60}\,7^{1/4} < 25.19882$. These bounds follow from Lemma 11.12 and Theorem 11.13, respectively, and exclude $q \geq 8$. For $q = 4$ the source is modular by Lemma 2.4. Indeed, throughout the four profiles Lemma 11.12 gives $\mathrm{rd}(M) \leq 4\sqrt{60}\,7^{1/4}2^{-2/q}$. Since $4\sqrt{60}\,7^{1/4} < 4\sqrt{220}$, Corollary 13.5(i) makes $M$ non-totally real.

For the solvable comparison, the supported quadratic classes with even dyadic valuation are generated by $-1, 2, 3, 7$. This follows from ideal-class parity at the two seven-places, which have the same nontrivial ordinary class, and the unit identity $4+\sqrt{15} = (3+\sqrt{15})^2/6$. Lemma 11.16a places an odd inducing character in the dyadic ray quotient. The finite-flat bound excludes odd dyadic valuation for its quadratic inducing field. The ray calculation modulo (2) over the fifteen quadratic fields leaves a single odd quotient $C_3$, for $K(\sqrt{-21}) = K(\sqrt{-35})$, by the fifteen ray computations in [59, `d15_support/terminal_arithmetic/d15_dihedral.gp`]. Its quadratic character ramifies at both seven-places, so no irreducible solvable residue occurs in a one-sided profile. For a reducible residue the diagonal characters are unramified at all finite places, since their orders are odd, and hence are trivial.

The scalar branch is proved in Appendix H: Proposition H.3 treats CR/CR, and Proposition H.4 treats nonordinary members. For the remaining systems all dyadic members are ordinary with trivial residual semisimplification; use Theorem H.5 for the cuspidal profiles and Proposition H.6 for the special profile.

It remains to treat an ordinary nonsolvable residue ramified at the strict seven-place, with kernel field $M$ and image $\mathrm{SL}_2(\mathbf{F}_q)$, and let $C = K(i, \sqrt{3})$. The strict special or cuspidal place gives residual local index two and residue degree at most two. The good crystalline place may have an unramified Frobenius of odd order. In particular the two-place Euler-term estimate from the strict menu does not apply.

The compositum $MC$ is finite-unramified over $M$. All the following auxiliary fields retain the tame ramification bound at seven and the dyadic upper break bound three, giving

$$\mathrm{rd} \leq 4\sqrt{60}\,7^{1/4} < 50.39764.$$

Appendix I.1 constructs the solvable auxiliaries and applies Proposition 13.3 with Lemma 11.14. The degree-1024 and degree-2048 auxiliaries over $K$ exclude $q \geq 512$ and $q = 256$, respectively. The latter ray class computation uses GRH. In the remaining finite range the directed comparisons leave the following rows. The dyadic wild inertia has order $q$ and the ramified seven-place has residue degree two in every row.

| $q$ | dyadic multiplier orders | good seven-Frobenius orders |
|---|---|---|
| 8 | 7 | 3, 7, 9 |
| 16 | 5, 15 | 3, 5, 15, 17 |
| 32 | 31 | 3, 11, 31, 33 |
| 64 | 9, 21, 63 | 5, 7, 9, 13, 21, 63, 65 |
| 128 | 127 | 43, 127, 129 |

In every surviving row the strict seven-place has residual decomposition group $V_4$. In Appendix I we compare the congruence fields of two prescribed order-four cuspidal lifts, distinguished by their scalar inertial signs at one place over eleven. The comparison excludes $q \geq 16$ and forces all remaining $\mathrm{SL}_2(\mathbf{F}_8)$ point fields with the same ramified seven-place to agree. Appendix J then compares their simultaneous trace orders and excludes a nonmodular residue. The actual dyadic member is modular by Lemma 2.4, using the preceding application of Corollary 13.5.

### 26.3 Reducible terminal residues

Lemma 6.7 gives an odd cyclotomic power as the constituent quotient. It extends to $G_{\mathbf{Q}}$ and is nontrivial at both seven-places. Proposition 6.5 applies to the actual terminal member. At $v_-$ its finite-flat characters give quotient $\omega_7^{\pm 1}$ on inertia.

## 27 Terminal sources over $\mathbf{Q}(\sqrt{19})$

Put $K = \mathbf{Q}(s)$ with $s^2 = 19$. Write $\kappa_b$ for the quadratic character of a squareclass $b$. The primes

$$\mathfrak{v}_1 = (4 - s), \qquad \mathfrak{v}_2 = (4 + s), \qquad \mathfrak{p}_2 = (13 + 3s)$$

satisfy $3\mathcal{O}_K = \mathfrak{v}_1\mathfrak{v}_2$ and $2\mathcal{O}_K = \mathfrak{p}_2^2$. The two three-adic completions are $\mathbf{Q}_3$. The dyadic completion is $\mathbf{Q}_2(\sqrt{3})$, since $19/3$ is a square in $\mathbf{Q}_2$.

A terminal system in this section has weight two, cyclotomic determinant and good reduction away from three. Its local parameters above three are quadratic twists of special (SP) or good crystalline (CR) parameters. We retain the exact quadratic characters in the special case. We will verify the supported base at three for Theorem 1.3 by bounding the simultaneous dyadic trace orders and comparing them with two modular systems.

**Theorem 27.1.** *Assume GRH. Every terminal system over $K$ is modular. Every absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ of determinant $\overline{\chi}_3$ unramified outside three is modular.*

### 27.1 Arithmetic and local lifts

Put

$$\epsilon = 170 + 39s, \quad \pi = 13 + 3s, \quad a = 4 + s, \quad a' = 4 - s.$$

The identities $\pi^2 = 2\epsilon$, $aa' = -3$ and $\mathrm{N}_{K/\mathbf{Q}}(\pi) = -2$ will be used throughout. The class number is one: the Minkowski bound is $\sqrt{19} < 4.36$, and the primes of norms two and three are principal. The continued fraction of $\sqrt{19}$ has period $(2, 1, 3, 1, 2, 8)$, so the fundamental unit is $\epsilon$, of norm one. In particular the narrow class group has order two.

Let $S$ consist of the places over two and three and the real places. We write a radicand and its quadratic character interchangeably when describing an $\mathbf{F}_2$-vector space. The supported squareclasses are

$$K(S, 2) = \langle -1, \epsilon, \pi, a, a' \rangle. \tag{29}$$

The even-dyadic-valuation subspace is $\langle -1, \epsilon, a, a' \rangle$. Its localization kernel at $K_{\mathfrak{p}_2}$ is $\langle 3 \rangle$, and the inverse image of the unramified local line is $\langle -1, 3 \rangle$. The even-valuation classes split at both three-primes form $\langle -\epsilon \rangle = \langle -2 \rangle$. Indeed the two odd-prime valuations eliminate the exponents of $a, a'$. Both $-1$ and $\epsilon$ are nonsquare units at both three-primes. The dyadic localization assertions also follow directly by reducing units modulo eight in $\mathbf{Z}_2[s]$.

**Lemma 27.2.** *The supported quadratic characters of dyadic conductor at most two form*

$$V_2 = \langle -1, 3, 2a \rangle.$$

*Write $U$ for the unramified quadratic character of $\mathbf{Q}_3$ and $R = \kappa_3$ for one of its two ramified characters. With additive character notation, the localization table is*

| | $\mathfrak{v}_1$ | $\mathfrak{v}_2$ | *negative real embedding* |
|---|---|---|---|
| $-1$ | $U$ | $U$ | 1 |
| 3 | $R$ | $R$ | 0 |
| $2a$ | 0 | $U + R$ | 1 |

*The ramified dyadic coordinate in this basis is* $(0, 0, 1)$.

*Proof.* The supported unit squareclasses at two have basis $\langle -1, 2, 3, 2a\rangle$. The class 3 is locally square, $-1$ is unramified, 2 has conductor four, and every odd-valuation class has conductor five. The class $2a$ has conductor two. For the last assertion, use $[\epsilon] = [2]$ and

$$\epsilon a = 1421 + 326s, \qquad \epsilon a - (s+2)^2 = 4 \cdot 269 + 2 \cdot 161(s+1).$$

The right side has dyadic valuation three, giving quadratic defect three. The local character table follows from the reductions modulo three and nine. At $\mathfrak{v}_2$, $2a/3$ is a nonsquare unit. □

In the first assertion of Lemma 4.23, nonzero monodromy gives SP, and zero monodromy gives CR.

## 27.2 Dyadic residual images

Write $R_0, E_0$ for the simultaneous trace and matrix orders of Section 11. We use the construction also for a subcollection of dyadic coefficient embeddings, with the corresponding product of local trace factors. The regular lattice and the exact comparison for biconnected groups with trivial generic simple factors are given by Lemmas 11.5 and 11.8. For this scalar category, Lemma 11.9 applies with the Eisenstein polynomial $u^2 - 26u - 2$ of $\pi$.

**Lemma 27.4.** *Every absolutely reducible dyadic residue of a terminal system has trivial semisimplification.*

*Proof.* Its constituent characters have odd order. At three, special inertia reduces to a unipotent group and crystalline scalar inertia reduces to the identity, so these characters are unramified. At two their abelian odd-order inertia is trivial, because wild inertia is pro-two and the residue field is $\mathbf{F}_2$. They are unramified at every other finite place and at infinity. The class number one assertion kills both characters. □

**Proposition 27.5.** *An ordinary irreducible solvable dyadic source of a terminal system is modular. A nonordinary irreducible solvable dyadic residue is the representation $V$ cut out by the $S_3$ splitting field of*

$$x^3 - 2x - 2. \tag{30}$$

*The latter field is unramified outside two over $K$, with local inertia $C_3$ and unramified quadratic quotient at two. At each three-prime its completion is unramified quadratic.*

*Proof.* Lemma 11.16a, with the two three-primes as $S$, gives a quadratic inducing field $H/K$ and a nontrivial character of its odd dyadic ray quotient. For an ordinary source, Fontaine's bound excludes odd dyadic valuation for the inducing radicand, since that conductor is five. The possible CM classes in $\langle -1, \epsilon, a, a'\rangle$ are $-1, -2, -3, -6$. None is square in $K_{\mathfrak{p}_2} = \mathbf{Q}_2(\sqrt{3})$: the first and third give the unramified quadratic extension, and the others are ramified. Thus every CM inducing field is nonsplit at the dyadic place. Allen's ordinary theorem [2, Theorem 5.2.1], with the ordinary witness of [2, Lemma 5.1.2], applies to the actual source. Its extra nonsplitting condition is imposed precisely in the CM-dihedral case.

Suppose now that the source is nonordinary. If the dyadic image contains a nontrivial rotation, its wild inertia is trivial, since a nontrivial order-two subgroup of a dihedral group with odd rotations cannot be normalized by a rotation. The tame inertia has odd order, since a tamely ramified extension of a dyadic field has odd ramification index. Its inducing quotient is therefore unramified. If its inducing quotient is ramified, the local residual image has order two, and the middle-model conductor bound applies to the quadratic quotient. In either case its

inducing quadratic character belongs to $V_2$. There are seven nontrivial inducing classes whose odd dyadic ray quotients must be computed. The four mixed-sign fields are represented, up to conjugation over $\mathbf{Q}$, by $K(\sqrt{2a})$ and $K(\sqrt{-2a})$. Both have class number one and dyadic residue field $\mathbf{F}_2$, so neither admits the required odd character. For completeness, the defining quartics, discriminants and class groups needed for all seven cases are

| polynomial | discriminant | Cl |
|---|---|---|
| $x^4 - 16x^2 - 12$ | $-69312$ | 1 |
| $x^4 + 16x^2 - 12$ | $-69312$ | 1 |
| $x^4 - 44x^2 + 256$ | 51984 | 1 |
| $x^4 - 32x^2 + 484$ | 51984 | $C_2$ |
| $x^4 - 36x^2 + 400$ | 5776 | 1 |

The last three fields are $K(\sqrt{3})$, $K(\sqrt{-3})$ and $K(i)$. The class groups are certified by the accompanying exact computation.

The dyadic residue fields in $K(\sqrt{3})$ are $\mathbf{F}_2$. In $K(\sqrt{-3})$, the unit $\zeta_3$ generates the odd part of the dyadic residue units. Thus these fields also have no required odd character. In $K(i)$, put

$$w = \pi(1+i)/2, \qquad w^2 = i\epsilon.$$

The Hasse unit index is two and the units are generated by $i, w$. Both reduce to one at the dyadic prime of $K(i)$, whose residue field is $\mathbf{F}_4$. The odd tame dyadic ray group is therefore $C_3$. It gives exactly one dihedral representation over $\overline{\mathbf{F}}_2$.

The polynomial (30) realizes it. Its discriminant is $-76$ and it is Eisenstein at two. Its quadratic resolvent over $K$ is $K(i)$. At nineteen its ramified quadratic part is absorbed by $K/\mathbf{Q}$, leaving an unramified extension. At three it factors as $(x-1)(x^2+x+2)$, with distinct factors. These facts give the stated local properties and identify the surviving representation. □

**Proposition 27.6.** *Under GRH, a terminal system with a nonsolvable dyadic residue is modular, for each of the four* SP/CR *menus.*

*Proof.* Let $M/K$ be the residual kernel field, with group $\mathrm{SL}_2(\mathbf{F}_q)$. The proof of Lemma 11.12 gives relative different contribution at most $2 - 2/w$ in the wild case, with elementary abelian inertia of order $w \le q$ and upper breaks at most three. In the tame case the inertia order is at most three.

For the SP/SP menu, adjoin $C = K(i, \sqrt{3})$. This degree-four extension is unramified at two, and its completion at either three-prime is the full elementary quadratic extension of $\mathbf{Q}_3$. This completion contains the corresponding residual field, since the special filtration has trivial residual diagonal characters. Simplicity gives $M \cap C = K$. Thus $[MC : \mathbf{Q}] = 8q(q^2-1)$, its local invariants at either three-prime are $(e, f) = (2, 2)$, and

$$\mathrm{rd}(MC) \le \sqrt{228}\, 2^{2-2/q}. \tag{31}$$

When at least one CR place occurs, let $A/K$ be the $S_3$ field in Proposition 27.5. Again $M \cap A = K$, and

$$[MA : \mathbf{Q}] = 12q(q^2-1).$$

Its dyadic inertia has order $3w$ in the wild case. Positive upper ramification groups inject into the $M$ factor, so their orders are at most $w$, and they vanish after break three. The normalized dyadic different contribution is at most

$$\frac{1}{2}\left(1 - \frac{1}{3w} + 3\left(1 - \frac{1}{w}\right)\right) = 2 - \frac{5}{3w} \le 2 - \frac{5}{3q}.$$

In the tame case the contribution is $1/3$, which satisfies the same bound. At a CR place, residual inertia is trivial. At an SP place, adjoining $A$ gives residue degree two and ramification index at most two. If $h \in \{0, 1\}$ denotes the number of special places,

$$\mathrm{rd}(MA) \le \sqrt{76}\, 3^{h/4} 2^{\,2-5/(3q)}. \tag{32}$$

In (11) take $B_b^* = B_{b,100}$. For $MC$ retain both three-place terms with $(e, f) = (2, 2)$. For $MA$ retain the term at the special place when $h = 1$. We obtain the following strict bounds.

| menu | range | $n \ge$ | $b$ | lower bound | upper bound |
|---|---|---|---|---|---|
| SP/SP | $q = 8$ | 4032 | 8.06 | 51.288 | 50.789031 |
| SP/SP | $q = 16$ | 32640 | 10.15 | 58.999 | 55.385834 |
| SP/SP | $q \ge 32$ | 261888 | 12.23 | 63.922 | 60.398676 |
| CR/CR | $q = 8$ | 6048 | 8.10 | 33.801518 | 30.182292 |
| CR/CR | $q \ge 16$ | 48960 | 10.18 | 37.181815 | 34.871192 |
| SP/CR | $q = 8$ | 6048 | 8.30 | 42.318706 | 39.722130 |
| SP/CR | $q \ge 16$ | 48960 | 10.39 | 47.260048 | 45.893069 |

The last two rows also cover CR/SP. The unbounded ranges use the limiting exponent two in (31) and (32). The calculations use directed intervals [20].

For $q = 4$, $\mathrm{rd}(M) \le \sqrt{228}\, 2^{3/2} < 42.709$. The unconditional lower bound 43.14954 in Lemma 13.4 excludes a totally real field of degree 120. Lemma 2.4 gives modularity, using the small-field witness. □

### 27.3 Scalar and dihedral trace orders

**Proposition 27.7.** *A terminal system with a nonordinary scalar-residual dyadic member is modular. Such a member can occur only for the* SP/SP *menu. For fixed exact special characters, the simultaneous trace order of any finite collection of these members has $\mathbf{Z}_2$-rank one.*

**Proposition 27.8.** *Fix the exact quadratic special characters at $\mathfrak{v}_1, \mathfrak{v}_2$. For any finite collection of ordinary scalar-residual dyadic members of terminal systems, their simultaneous trace order has rank at most two. An ordinary scalar-residual member cannot occur in a menu containing* CR*. In the special case its normalized simultaneous matrix-entry order is generated over $\mathbf{Z}_2$ by the crystalline Frobenius trace $a_2$.*

**Proposition 27.9.** *Fix the terminal menu and its exact quadratic local data. For a finite collection of nonordinary dyadic realizations with residue $V$ of Proposition 27.5, the simultaneous trace order has $\mathbf{Z}_2$-rank at most two.*

The proofs of the three bounds are given in Appendix K.

### 27.4 The three-adic endpoint

**Lemma 27.10.** *An absolutely irreducible determinant-$\bar{\chi}_3$ representation unramified outside three over $K$ remains absolutely irreducible over $K(\zeta_3)$.*

*Proof.* Write $H = K(\zeta_3)$. If the restriction is reducible, Clifford theory gives induction of a character $\theta$ of $G_H$. The character has order prime to three and is unramified outside the two primes $P_1, P_2$ of $H$ above $\mathfrak{v}_1, \mathfrak{v}_2$. It therefore factors through the ray group of modulus $P_1 P_2$.

This ray group has exponent two. In fact, the class group of $H$ has order two by the table in Proposition 27.5. The class of $P_1$ is nontrivial: a generator would have norm to $K$ generating

$\mathfrak{v}_1$, which is impossible since $4-s$ has opposite real signs and every unit of $K$ has equal real signs. Put $\gamma = \sqrt{-3}+4-s$. Its valuations at $(P_1, P_2)$ are $(1,0)$. The fractional ideal $P_1/(\gamma)$ is prime to the modulus, and its square is generated by $(4-s)/\gamma^2$. At $P_1$ this element has residue $(4+s)^{-1} = -1$, since $(4-s)(4+s) = -3$. At $P_2$ its residue is $(4-s)^{-1} = -1$. Multiplying the generator by $-1$ gives residue one at both places. Thus the nontrivial ordinary ideal class lifts to a ray class of order two, and the kernel of the map to the class group is a quotient of $\mathbf{F}_3^\times \times \mathbf{F}_3^\times$. This proves the exponent assertion.

The determinant is trivial on $G_H$, so $\theta^\sigma = \theta^{-1}$ for the nontrivial $\sigma \in \mathrm{Gal}(H/K)$. Since $\theta^2 = 1$, this makes the induction reducible, a contradiction. □

**Proposition 27.11.** *A terminal system whose global trace field has degree at most two is modular.*

*Proof.* At a coefficient prime above three, the residual trace field is contained in $\mathbf{F}_3$ or $\mathbf{F}_9$. An absolutely irreducible residue is realizable over its trace and determinant field by finite-field descent. The small-field theorem [3, Theorem 2.5] and Lemma 2.1 give a weight-two modular witness. By Lemma 27.10 the cyclotomic restriction is irreducible. Apply Lemma 2.2 to the actual three-adic source.

Suppose the residue is reducible. At a special place the source is nearly ordinary with characters $\theta\chi_3, \theta$. At a scalar crystalline place, remove the quadratic twist. Over $\mathbf{Q}_3$ a nonordinary Barsotti–Tate group and its finite-flat subquotients are biconnected. The two characters of the scalar-untwisted reducible residue have trivial and cyclotomic inertia. Their order-three models are étale and multiplicative by finite-flat uniqueness for $e = 1 < 3-1$. Thus the source must be ordinary. The ordinary line has weight one in our convention at both places. Its Hodge–Tate number is smaller than that of the quotient in Pan's convention. By Lemma 6.7, twisting by the quadratic lift of a global constituent gives residual trace $1+\bar{\chi}_3$ and preserves the cyclotomic determinant. The determinant is de Rham and totally odd, and its residue extends to $G_{\mathbf{Q}}$ and is nontrivial at both three-places. The field $K$ is abelian and three is split. Pan's fully ordinary theorem [61, Theorem 5.1.2(2)] therefore applies, including when the two local ordinary lines have different residual labels. Untwisting and strong compatibility give modularity of the system. □

## 27.5 Quadratic conjugation and modular comparison systems

Suppose a terminal system is not modular. By Propositions 27.5, 27.6 and 27.7, every dyadic coefficient embedding then belongs to one of two classes: ordinary with scalar residue, or nonordinary with residue $V$. If a crystalline place occurs, the first class is impossible by Proposition 27.8, so Proposition 27.9 bounds the global trace degree by two. Proposition 27.11 gives a contradiction. We may therefore assume the menu is SP/SP.

For this menu, each of the two simultaneous orders has rank at most two. For any finite collection of distinct coefficient orbits with the same exact special characters, all of whose dyadic embeddings lie in these two classes, we have

$$\sum_i [D_i : \mathbf{Q}] \le 4. \tag{33}$$

This is the sum of the bounds on the two subcollections, by Section 11.

**Lemma 27.12.** *Let $\mathcal{R}$ be a nonmodular* SP/SP *terminal system. Normalize its exact special characters to $(1,\delta)$ by a rational quadratic twist. Then $\delta \in \{U, UR\}$, and its trace field $D = \mathbf{Q}(a_2)$ has degree four.*

*Proof.* Proposition 27.11 and (33) give $3 \le n \le 4$. Both dyadic classes must occur, since either alone has rank at most two. The crystalline trace $a_2$ is a unit at the ordinary embeddings and a nonunit at the nonordinary ones. It is therefore nonrational. For $n = 3$ this already gives $D = \mathbf{Q}(a_2)$. For $n = 4$, a proper nonrational subfield $\mathbf{Q}(a_2)$ would be quadratic. It has distinct dyadic places containing the unit and nonunit conjugates, so both places have degree one. The ordinary part of $D \otimes \mathbf{Q}_2$ has dimension two, since both parts have dimension at most two. But $a_2$ belongs to $\mathbf{Q}_2$ on that part and cannot generate its dimension-two algebra, contrary to Proposition 27.8. Hence $D = \mathbf{Q}(a_2)$.

The rational quadratic characters $1, \kappa_{-1}, \kappa_3, \kappa_{-3}$ give all four local characters at three. They preserve good reduction at two: $\kappa_3$ is trivial on $K_{\mathfrak{p}_2}$ and $\kappa_{-1}$ is unramified there. We may therefore normalize the first special character to one. If $\eta$ is the rational quadratic character restricting to $\delta$ at three, the operation

$$C(\mathcal{R}) = \mathcal{R}^\sigma \otimes \eta$$

preserves the normalized local pair and both dyadic classes. It is an involution on isomorphism classes. If $C(\mathcal{R})$ were a different coefficient orbit, the pair of orbits would contribute $2n > 4$ to (33). Thus $C$ induces an automorphism $\tau$ of $D$ with $\tau^2 = 1$.

There is only one dyadic place of $K$. For $\delta = 1, R$, the dyadic value of $\eta$ is one, so $\tau(a_2) = a_2$ and $\tau = 1$. The twisted conjugacy (108) then gives modularity by Lemma 12.4. For $\delta = U, UR$, the value is minus one, so $\tau(a_2) = -a_2$. Since $a_2$ is nonzero, $\tau$ is a nontrivial involution. The degree $n$ must be even, and hence equals four. □

### 27.5.1 The unramified relative character

**Lemma 27.13.** *There is a modular terminal system with special pair $(1, U)$, coefficient field $\mathbf{Q}(\sqrt{2})$, and both dyadic coefficient dimensions in the nonordinary $V$ class.*

*Proof.* Construct the parallel-weight-two newspace of level (3) and trivial central character over $K$. The rational dimensions of its newform constituents are $2, 2, 5, 5$. On a two-dimensional constituent the Hecke matrices, with a choice of basis, include

$$T_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \qquad U_{\mathfrak{v}_1} = -I, \qquad U_{\mathfrak{v}_2} = I,$$

$$T_{\mathfrak{q}} = -I \pm 2T_2 \qquad \text{at a prime } \mathfrak{q} \mid 5. \tag{34}$$

The second constituent interchanges the two special signs. Appendix K.6 gives the exact computation. The characteristic polynomial of $T_2$ is $X^2 - 2$, so the Hecke field is $\mathbf{Q}(e)$, $e^2 = 2$, and $a_2 = e$. The exact new level has exponent one at both three-primes. Both local parameters are therefore special with opposite unramified quadratic characters. A permitted rational quadratic twist normalizes the pair to $(1, U)$.

The Hecke field has one dyadic completion, of degree two and residue field $\mathbf{F}_2$. The member is nonordinary because $a_2$ is nonunit. If its residue were absolutely reducible, the odd-order constituent argument of Lemma 27.4 would make its semisimplification trivial: the characters are unramified at three by special inertia, at two because the residue field is $\mathbf{F}_2$, and elsewhere by the level. Class number one then kills them. All residual traces would be zero. But (34) gives the odd trace $-1 \pm 2e$ at five. Thus the residue is absolutely irreducible. The only absolutely irreducible subgroup of $\mathrm{GL}_2(\mathbf{F}_2)$ is $S_3$. The subgroup $C_3$ becomes reducible over $\mathbf{F}_4$ and has already been excluded. Proposition 27.5 identifies this residue with $V$. □

This orbit contributes two dyadic dimensions to the $V$ class and saturates Proposition 27.9. Any distinct system with the same exact local pair and a $V$ embedding would exceed that

bound. A system lying entirely in the ordinary scalar class has trace degree at most two and is modular. Hence every terminal system with relative character $U$ is modular.

### 27.5.2 The ramified relative character

**Lemma 27.14.** *The elliptic curve*

$$E: \quad y^2 + sy = x^3 - (1+s)x^2 - (20+8s)x - 25 - 16s \tag{35}$$

*defines a modular terminal system with special pair* $(1, \kappa_{-3}) = (1, UR)$ *and nonordinary dyadic residue* $V$.

*Proof.* Its invariants are

$$c_4 = 1280 + 416s, \qquad c_6 = 70744 + 23296s,$$

$$\Delta = 67392s - 343035 = -(288 - 117s)^2.$$

Since $288 - 117s = 9(32 - 13s)$ and $\mathrm{N}(32 - 13s) = -3^7$, the discriminant is supported only at three. The curve has good reduction at two, where its equation is $y^2 + y = x^3 + 1$ over $\mathbf{F}_2$. It has three points over $\mathbf{F}_2$, so $a_2(E) = 0$ and the two-adic member is nonordinary.

The local valuations are

| | $v(\Delta)$ | $v(c_4)$ | $v(j)$ |
|---|---|---|---|
| $\mathfrak{v}_1$ | 4 | 0 | $-4$ |
| $\mathfrak{v}_2$ | 18 | 2 | $-12$ |

Thus the curve is potentially multiplicative at both places, and the monodromy is nonzero. Its local splitting character is the quadratic character of $-c_6$. At $\mathfrak{v}_1$, $c_6 \equiv 2 \pmod 3$, so $-c_6$ is square. At $\mathfrak{v}_2$, $s \equiv 71 \pmod{81}$ gives $c_6 \equiv 27 \pmod{81}$. Its squareclass is 3, and that of $-c_6$ is $-3$. The special pair is therefore exactly $(1, \kappa_{-3})$. The conductor is $\mathfrak{v}_1\mathfrak{v}_2^2$.

After replacing the two-division coordinate $x$ by $z/2$, the two-division polynomial is

$$F(z) = z^3 - 2(1+s)z^2 - (80+32s)z - 162 - 128s.$$

If $\alpha$ satisfies (30), direct substitution gives $F(z_0) = 0$ for

$$z_0 = 2 + 5\alpha - \alpha^2 + \frac{s}{19}(-14 + 8\alpha + 20\alpha^2). \tag{36}$$

The cubic (30) is Eisenstein at two over $\mathbf{Q}$ and remains irreducible over the quadratic field $K$. The element $z_0$ is not in $K$, so it generates $K(\alpha)$. Consequently the two-division splitting field is exactly the $S_3$ field of Proposition 27.5. This proves that $E[2] \simeq V$.

Finally every elliptic curve over a real quadratic field is modular by [41, Theorem 1]. Thus (35) gives the required rational modular compatible system. □

If a nonmodular terminal system had relative character $UR$, Lemma 27.12 would give coefficient degree four. The rational modular orbit of Lemma 27.14 is distinct from it and belongs to the same bounded $V$ class. Together they contribute five dimensions to (33), which is impossible. The cases $1, R, U, UR$ are now all settled. This proves the first assertion of Theorem 27.1.

*Proof of the second assertion of Theorem 27.1.* Lemma 27.10 gives absolute irreducibility on the cyclotomic restriction. At both three-places use the first assertion of Lemma 4.23 on the actual local residue. With determinant $\chi_3$ and unramified conditions elsewhere, Lemma 2.6(i) supplies a terminal weight-two system. Purity retains its special components by Section 2. The first assertion of the theorem now gives modularity of the lift and hence of the residue. □

# 28 Mixed sources and pseudodeformation rigidity over $\mathbf{Q}(\sqrt{21})$

Put

$$K = \mathbf{Q}(\sqrt{21}), \qquad a = \frac{1+\sqrt{21}}{2}, \qquad \epsilon = a+2, \qquad \gamma = a+1, \qquad H = K(\zeta_3).$$

Thus $a^2 - a - 5 = 0$, $\gamma^2 = 3\epsilon$, and $N(\gamma) = -3$.

We first prove the mixed-source theorem by the common discriminant bounds, finite-flat rigidity and a toric realization; the pure menus then reduce to the characteristic-three cases listed below. The conventions of Section 2 apply. Potential modularity gives a cuspidal realization after base change, so all companions used below are absolutely irreducible.

Let $\omega_2$ be a fundamental character of niveau two over $\mathbf{Q}_5$, of order 24. The bounded cuspidal types are

$$\tau_3 = \widetilde{\omega}_2^8 \oplus \widetilde{\omega}_2^{16}, \qquad \tau_6 = \widetilde{\omega}_2^4 \oplus \widetilde{\omega}_2^{20}.$$

The notation SC means one of these types, and SP means a fixed quadratic twist of Steinberg with nonzero monodromy. The exact quadratic character, including its unramified sign, is retained whenever a special deformation condition is imposed.

**Proposition 28.1** (Mixed source)**.** *Assume GRH. A potentially modular compatible system over $K$, of labelled weights $\{0,1\}$ and determinant cyclotomic, unramified away from five and crystalline at two, is modular if its types at the two places above five are* SC/SP *in either order. The same conclusion holds if potential modularity is replaced by the existence over $K$ of a* $\mathrm{GL}_2$*-type abelian variety realizing the compatible system.*

The proof of this proposition is given below, including both the nonsolvable and solvable dyadic branches. The special condition retains its actual nonzero monodromy.

For a pure special menu with reducible five-adic residue, Proposition 28.2 applies directly. For either pure menu, the three-adic member $r_3$ has four exhaustive alternatives:

| residual condition | source argument |
|---|---|
| $\bar{r}_3\|_{G_H}$ irreducible | Proposition 28.20 |
| $\bar{r}_3$ irreducible, $\bar{r}_3\|_{G_H}$ reducible | Propositions 28.29, 28.31 |
| $\bar{r}_3^{\mathrm{ss}}$ reducible, constituents unramified at three | Proposition 28.22 |
| $\bar{r}_3^{\mathrm{ss}}$ reducible, common ramified quadratic at three | Theorem 28.33. |

The last line is essential: good supersingular elliptic curves in the present conductor class realize it.

**Proposition 28.2.** *Let $\mathcal{R}$ be a weight-two system over $K$ with cyclotomic determinant. Suppose its parameters at both places above five are quadratic-special with nonzero monodromy. If a five-adic residual semisimplification is reducible, then $\mathcal{R}$ is modular.*

*Proof.* At either five-place, remove the quadratic scalar character and apply Lemma 4.2; nonzero monodromy gives an ordinary quotient. Restoring the scalar character preserves ordinarity. Lemma 4.5 applies because $K/\mathbf{Q}$ is abelian, five is unramified, the actual source has weight two and cyclotomic determinant, and its residue is reducible. Its Dotto–Thorne clause has no residual constituent-ratio restriction. □

The bounded five-adic menu is Lemma 4.17; Lemma 4.18, with $\ell = 5$, gives both replacements of the actual characteristic-three residue, with determinant fixed.

## 28.1 Controlled characteristic-two residues

Put $F = K_2$, the unramified quadratic extension of $\mathbf{Q}_2$. We distinguish the residual conditions $\mathrm{SC}_2, \mathrm{SP}_2$ from the characteristic-zero conditions SC, SP.

*Definition* 28.3. The two permitted conditions at a place $v \mid 5$ of $K$ are:

(i) $\mathrm{SC}_2$: the kernel field is the tame $S_3$ extension of $\mathbf{Q}_5$ with ramification index three and residue degree two;
(ii) $\mathrm{SP}_2$: the kernel field is contained in $E_5' = \mathbf{Q}_5(\sqrt{2}, \sqrt{5})$.

The $\mathrm{SC}_2$ field is unique in a fixed algebraic closure and is the splitting field of $X^3 - 5$. Over the unramified quadratic field, Frobenius has eigenvalues $+1$ and $-1$ on the uniformizer and unramified lines of cube classes. Only the first is a ramified stable line, and it gives the inversion action on cubic inertia.

**Theorem 28.4.** *Assume GRH for Dedekind zeta functions. Let*

$$\bar{r} : G_K \longrightarrow \mathrm{SL}_2(\overline{\mathbf{F}}_2)$$

*be continuous and absolutely irreducible. Suppose that $\bar{r}$ is unramified outside two and five, is the generic fibre of a finite flat group scheme killed by two at the dyadic place, and satisfies either condition of Definition 28.3 independently at each place above five. Then $\bar{r}$ is modular.*

The theorem covers the pairs $(\mathrm{SC}_2, \mathrm{SC}_2)$, $(\mathrm{SC}_2, \mathrm{SP}_2)$ and $(\mathrm{SP}_2, \mathrm{SP}_2)$. The first pair requires the unit extension constructed below. A quadratic extension suffices for the other two pairs.

**Corollary 28.5.** *Assume GRH. Let $\mathcal{R} = (r_\lambda)_\lambda$ be a totally odd, strongly compatible two-dimensional system over $K$, of determinant the cyclotomic character and labelled Hodge–Tate weights $\{0, 1\}$. Suppose it has all-place Weil–Deligne compatibility, is unramified at $v \nmid 5$ for members of residue characteristic different from that of $v$, has crystalline two-adic member at two, and at both places above five has either a quadratic twist of special type with nonzero monodromy or one of the two potentially crystalline types in Lemma 4.17. If its characteristic-two residual representation has nonsolvable image, then the given system $\mathcal{R}$ is modular.*

The distinction between these two assertions matters. Theorem 28.4 provides a modular witness for a residue. Corollary 28.5 uses an automorphy lifting theorem to identify the given characteristic-zero representation with a modular one.

**Lemma 28.6.** *In a determinant-cyclotomic compatible system with all-place Weil–Deligne compatibility, reduction at a place above two of a local member of quadratic-special type satisfies* $\mathrm{SP}_2$. *Reduction of either cuspidal type satisfies* $\mathrm{SC}_2$.

*Proof.* In the special case the diagonal characters are, up to a quadratic twist, $\chi_2\nu, \nu$ with $\nu$ unramified quadratic. They both reduce to one. The full residual image, including Frobenius, is therefore elementary abelian of exponent two. Local class field theory puts its field inside $E_5'$. Consequently $e \leq 2$, $f \leq 2$ and $d \leq 1/2$.

For either cuspidal type the reduced inertia has exact order three. A Frobenius lift swaps its two eigenlines. In characteristic zero its square is $-5I$, by its determinant and zero trace. After reduction its square is one. Thus the full residual decomposition group is $S_3$, with $e = 3$, $f = 2$ and $d = 2/3$. □

Lemma 11.12 gives $d(E/F) \leq 2 - 2/q$ for each finite-flat determinant-one residue over the unramified quadratic dyadic field $F$. In the nontrivial tame-diagonal wild case it gives $d(E/F) < 4/3$.

### 28.1.1 The solvable auxiliary fields

Let $H = K(\sqrt{-3}) = \mathbf{Q}(\sqrt{-3}, \sqrt{-7})$. It is quadratic over $K$, unramified at every finite place, split at two, and unramified quadratic at both places above five. The discriminant product $(-3)(-7)(21) = 21^2$ verifies the first assertion; the other two follow from the quadratic local square classes. This field alone will handle the pairs involving $\mathrm{SP}_2$.

Let $S$ be the splitting field over $\mathbf{Q}$ of

$$f(X) = X^3 - 3X^2 - 2X - 1,$$

and put $A = KS$. The polynomial is irreducible modulo two and has discriminant $-175$. Thus $S/\mathbf{Q}$ has group $S_3$, with quadratic subfield $\mathbf{Q}(\sqrt{-7})$, and $A/K$ has group $S_3$ and fixed quadratic field $H$. Write $\tau$ for a generator of its normal subgroup of order three.

**Lemma 28.7.** *The extension $A/K$ is unramified outside five. Its dyadic completions are unramified cubic over $F$; at both places above five its completion is the $\mathrm{SC}_2$ field.*

*Proof.* The splitting field $S$ ramifies only at five and seven. At five, $f(X+1) = X^3 - 5X - 5$ is Eisenstein, so the cubic ramification is tame. The discriminant has nonsquare unit part $-7$, yielding the unramified quadratic field and hence $S_3$. At seven its inertia is quadratic. Its ramified quadratic field is $\mathbf{Q}_7(\sqrt{-7}) = \mathbf{Q}_7(\sqrt{21})$, since $-3$ is a square in $\mathbf{Q}_7$. Base change to $K$ removes this ramification. At two the irreducible reduction of $f$ gives an unramified cubic; its unramified quadratic field already lies in $F$. □

We now specify four unit squareclasses in $A$ without using a class-group calculation or a fundamental-unit computation. Choose a root $u$ of $f$ and $z = \sqrt{-3}$ in $A$ and set

$$v = \frac{9 + 26u - 8u^2 + (2u - 7)z}{2}.$$

The choices of roots below are made compatibly with $\tau$. Direct substitution gives

$$v^3 + \frac{15z - 1}{2}v^2 + \frac{15z + 1}{2}v - 1 = 0.$$

In particular $u, v$ are integral units. Their minimal polynomials over $\mathbf{Q}$ are

$$f(X), \qquad X^6 - X^5 + 170X^4 + 335X^3 + 170X^2 - X + 1.$$

Both have norm one for $A/H$.

**Lemma 28.8.** *In $W = \mathcal{O}_A^\times/(\mathcal{O}_A^\times)^2$, the four classes*

$$[u],\ [\tau u],\ [v],\ [\tau v]$$

*are independent and span $U = (\tau - 1)W$. At either dyadic prime of $A$ their image has dimension two. At both five-adic primes their image is zero.*

*Proof.* Here $A$ has signature $(0, 6)$, so $\dim_{\mathbf{F}_2} W = 6$ by Dirichlet's unit theorem, including the torsion squareclass. Since $A/H$ has odd degree, the $\tau$-invariants in $W$ are the images of the unit squareclasses of $H$: the norm of an invariant class is its third power and hence that same squareclass. The latter space has dimension two. Thus $(\tau - 1)W$ has dimension four, and every norm-one class belongs to it.

For completeness we give a finite exact certificate of independence and of the dyadic ranks. In the ordered list $(u, \tau u, v, \tau v)$, encode a product by its four-bit mask $m = \sum_{j=0}^{3} m_j 2^j$. The masks that are squares in the two dyadic completions are respectively

| first dyadic prime | second dyadic prime |
|---|---|
| $0, 6, 11, 13$ | $0, 7, 9, 14$ |

Each kernel has order four and their intersection is zero. This proves local rank two and global rank four. The calculation can be performed by reduction modulo eight: over an unramified dyadic field a unit is a square precisely when its residue modulo eight has a square root. Appendix E specifies an exact number-field certificate for this table. The exact calculation is supplied in [20, `d21`].

Finally, inertia at five contains $\tau$. Every class of the form $[w/\tau w]$ for a global unit $w$ has residue one, and is a local square because the residue characteristic is odd. This proves the last assertion for all of $U$. □

Define $B = A(\sqrt{U})$. Conjugation in $\mathrm{Gal}(A/K)$ exchanges $\tau$ and $\tau^{-1}$, so $U$ is stable under this group. Lemma 28.8 gives

$$[B : K] = 96, \qquad B/K \text{ Galois and solvable.} \tag{37}$$

The field $B/A$ is split at five. At two it has ramified group $C_2^2$ with upper break one over the unramified cubic stage. Indeed its local defining classes are units, and their nontrivial $C_3$-module has no unramified quadratic constituent.

**Lemma 28.9.** *If $E/F$ is the kernel field of a finite-flat representation killed by two, then adjoining a dyadic completion of $B$ preserves Fontaine's bound $d < 2$. The same is true after composing two such finite-flat fields.*

*Proof.* A quadratic extension defined by a unit is the point field of a finite flat extension of $\mathbf{Z}/2\mathbf{Z}$ by $\mu_2$; the usual Kummer construction gives a model over the ring of integers. Direct sums handle a space of unit classes. Restriction of scalars from the unramified cubic extension preserves finite flatness. A finite étale representation accounts for that unramified stage. Taking direct sums with the original finite-flat group schemes realizes the compositum of their point fields, still killed by two. Fontaine's corollary now applies. □

### 28.1.2 The stable degree-four image at two

The following refinement is the only place where the local rank two in Lemma 28.8 is essential.

**Lemma 28.10.** *Let $\bar{\rho} : G_F \to \mathrm{SL}_2(\mathbf{F}_{16})$ be finite flat, with kernel field $E$. Suppose its action extends to a semilinear projective action of $G_{\mathbf{Q}_2}$ with nontrivial field automorphism $x \mapsto x^4$. Then the compositum with a dyadic completion of $B$ satisfies*

$$d(EB/F) \le 15/8.$$

*Proof.* Write $T/F$ for the unramified cubic extension. The quadratic classes contributed by $B$ over $T$ form a two-dimensional, nontrivial $\mathrm{Gal}(T/F)$-module of units.

If $\bar{\rho}$ is locally irreducible, its tame inertia has order five, as in Lemma 11.12. After tame base change of degree five the three nonzero quadratic characters from $B$ have conductor six. The different tower gives

$$d(EB/F) \le \frac{4}{5} + \frac{3 \cdot 6}{4 \cdot 5} = \frac{17}{10} < \frac{15}{8}.$$

If the tame diagonal order is three, let $P \le 16$ be the order of the original translation group. Its nontrivial quadratic characters have conductor two over the tame cubic stage by

Lemma 11.12. The $B$-characters there have conductor four and trivial tame action; they are independent of the first module. Counting characters gives

$$d(EB/F) \le \frac{2}{3} + \frac{3P \cdot 4 + (P-1) \cdot 2}{3 \cdot 4P} = \frac{11}{6} - \frac{1}{6P} < \frac{15}{8}.$$

It remains to consider unramified diagonal characters. If wild inertia is zero, $E/F$ is unramified and the contribution of $B$ alone is $3/2$. Otherwise its unique fixed line is preserved by the semilinear extension. Let $s$ lift the rational Frobenius. The diagonal ratio of $s^2$ is $aa^4 \in \mathbf{F}_4^{\times}$. Thus the unramified diagonal ratio over $F$ has order one or three.

In the order-three case, over $T$ the original Kummer module and the module from $B$ both lie in the nontrivial $\mathrm{Gal}(T/F)$-part of the unit squareclasses. That part has dimension four: the total unit squareclass dimension over $T$ is seven, and the invariant part, from $F$, has dimension three. All its nonzero ramified characters have conductor two. Hence the combined wild inertia has order at most sixteen and different at most $2(1 - 1/16) = 15/8$.

In the order-one case, the unit squareclasses from $F$ have at most two ramified dimensions, since their dimension is three and they include the unramified quadratic line. The module from $B$ adds at most two ramified dimensions. Again the ramification index is at most sixteen, and every ramified quadratic character has conductor two. The same bound follows. □

### 28.1.3 The discriminant comparisons and residual modularity

Use the common test function and explicit formula of Section 13, with the finite-tail bound $B_{b,N}$ and the real-place correction $A_{b,4}$.

For every nonsolvable kernel field considered here, the group is $G = \mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^f \ge 4$. Compositing with a solvable Galois extension is disjoint over $K$, since $G$ is nonabelian simple. At an $\mathrm{SC}_2$ place a degree-$n$ compositum preserving that completion has $n/12$ norm-25 primes. It contributes $\rho_{25} = 1/12$. At an $\mathrm{SP}_2$ place, adjoining $H$ makes its residue degree two and leaves ramification index at most two, giving $\rho_{25} \ge 1/8$. The resulting bounds are

| pair at five | upper bound for rd | retained $\rho_{25}$ |
|---|---|---|
| $\mathrm{SC}_2, \mathrm{SC}_2$ | $4\sqrt{21}\,5^{2/3}$ | $1/6$ |
| $\mathrm{SC}_2, \mathrm{SP}_2$ | $4\sqrt{21}\,5^{7/12}$ | $5/24$ |
| $\mathrm{SP}_2, \mathrm{SP}_2$ | $4\sqrt{21}\,5^{1/2}$ | $1/4$ |

The common dyadic factor four comes from Fontaine's strict $d_2 < 2$. In the mixed case it can be replaced by $2^{2-2/q}$ using Lemma 11.12. For the stable $q = 16$ case with two $\mathrm{SC}_2$ places, Lemma 28.10 replaces it by $2^{15/8}$ even after adjoining $B$.

Table 3 records rigorous evaluations of (11). In every row $N = 100$ and computations use 220-bit real-ball arithmetic. The lower bounds are rounded downwards and the upper bounds upwards at the shown precision. The column $m$ states how many positive norm-25 Euler powers are retained. Every row discards the real-place contribution.

*Proof of Theorem 28.4.* The solvable absolutely irreducible images in characteristic two are dihedral and are modular by the usual Hecke theta-series argument, as recalled in [3, proof of Theorem 2.5]. For nonsolvable images Dickson's classification gives, after conjugation, $G = \mathrm{SL}_2(\mathbf{F}_q)$ with $q = 2^f \ge 4$. The case $q = 4$ is modular by [3, Theorem 2.5]. We may assume $q \ge 8$. Let $M/K$ be the kernel field.

Suppose first that at least one place above five satisfies $\mathrm{SP}_2$. Adjoin $H$. It is disjoint from $M$, is split at two, and preserves the different bounds at five while supplying the norm-25

| case | $n \geq$ | $b$ | $m$ | GRH lower | local upper |
|---|---|---|---|---|---|
| two $\mathrm{SC}_2$, distinct conjugates | 48 771 072 | 17.34 | 8 | 53.60787465 | 53.59813048 |
| two $\mathrm{SC}_2$, $q \geq 64$ | 50 319 360 | 17.37 | 8 | 53.62432870 | 53.59813048 |
| two $\mathrm{SC}_2$, stable $q = 16$ | 783 360 | 13.21 | 8 | 50.46443530 | 49.14970236 |
| mixed, $q = 8$ | 2 016 | 7.31 | 4 | 39.50635209 | 39.41346729 |
| mixed, $q = 16$ | 16 320 | 9.40 | 5 | 46.57108547 | 42.98069085 |
| mixed, $q \geq 32$ | 130 944 | 11.48 | 7 | 51.01199785 | 46.87077572 |
| two $\mathrm{SP}_2$, $q \geq 8$ | 2 016 | 7.36 | 4 | 41.44651216 | 40.98780307 |

Table 3: Certified root-discriminant contradictions.

coefficients above. Thus $[MH : \mathbf{Q}] = 4q(q^2 - 1)$. For a mixed pair, the three mixed rows of Table 3 exclude $q = 8$, $q = 16$ and all $q \geq 32$. The two-$\mathrm{SP}_2$ row excludes every $q \geq 8$ in the other pair.

We are left with two $\mathrm{SC}_2$ places. Adjoin $B$. By (37), simplicity and Lemma 28.9,

$$[MB : \mathbf{Q}] = 192q(q^2 - 1), \qquad \mathrm{rd}(MB) < 4\sqrt{21}\,5^{2/3}.$$

Its five-adic completions remain exactly $S_3$, since both $M$ and $A$ have the unique $\mathrm{SC}_2$ field and $B/A$ is locally split. The second row of Table 3 excludes $q \geq 64$.

Let $\sigma$ be the nontrivial automorphism of $K/\mathbf{Q}$. If $M \neq M^\sigma$, simplicity gives $M \cap M^\sigma = K$. The group of their compositum over $K$ is $G \times G$, again disjoint from solvable $B$. Its dyadic field is still the point field of a finite-flat group killed by two, and its five-adic completions are unchanged. Its degree is $[MM^\sigma B : \mathbf{Q}] = 192|G|^2 \geq 48\,771\,072$. The first row of the table excludes this possibility. Hence $M = M^\sigma$ and $M/\mathbf{Q}$ is Galois.

For $q = 8, 16, 32$ the conjugation action defines $C_2 \longrightarrow \mathrm{Out}(\mathrm{SL}_2(\mathbf{F}_{2^f})) = C_f$. If this action is trivial, the centerless extension $\mathrm{Gal}(M/\mathbf{Q})$ is $G \times C_2$: correct a lift of the generator by an inner element; its square then centralizes $G$ and lies in its trivial centre. Projection onto $G$ extends the given residual representation to $G_{\mathbf{Q}}$. Serre's theorem over $\mathbf{Q}$ [49, 50] and quadratic base change [4] prove modularity over $K$. The base change is cuspidal, since its residual restriction is absolutely irreducible.

For $q = 8$ and 32, $f$ is odd and the outer action is automatically trivial. For $q = 16$ its only nontrivial possibility is the field involution $x \mapsto x^4$. Since two is inert in $K$, its dyadic action is precisely the semilinear situation of Lemma 28.10. We have

$$[MB : \mathbf{Q}] = 783\,360, \qquad \mathrm{rd}(MB) \leq \sqrt{21}\,2^{15/8}5^{2/3}.$$

The third row of the table contradicts this. All cases have therefore been excluded or proved modular. □

**Lemma 28.11.** *Under the local hypotheses of Theorem 28.4, without GRH, a nonsolvable image satisfies $\bar{r}(c_v) \neq 1$ for at least one real place $v$ of $K$.*

*Proof.* Lemma 11.12 bounds the original kernel by $\mathrm{rd}(M) \leq 4\sqrt{21}\,5^{2/3}2^{-2/q}$, using the larger of the prescribed five-adic contributions. Since $4\sqrt{21}\,5^{2/3} < 4\sqrt{220}$, Corollary 13.5(i) excludes a totally real kernel. Thus some complex conjugation acts nontrivially. □

*Proof of Corollary 28.5.* Choose the two-adic member of $\mathcal{R}$ and an invariant lattice. Its nonsolvable residue is absolutely irreducible, has determinant one, and is finite flat at two because the member is crystalline of weights $\{0, 1\}$. Here we use the integral Barsotti–Tate classification, including residue characteristic two, to pass from a stable lattice to a finite flat torsion model; see [51, Theorems 2.3(3), 4.1 and Corollary 4.3]. All-place compatibility and Lemma 28.6 give

the actual local fields required by Theorem 28.4. That theorem provides a regular algebraic cuspidal modular witness for the residue. Lemma 28.11 supplies a nontrivial residual complex conjugation at some real place.

Lemma 2.4 now gives modularity of the actual member and of the system. The determinant supplies the essential self-duality of the residual witness, and that lemma includes the dual convention required for the lifting theorem. □

## 28.2 Solvable residual representations and their fields

Reduction in characteristic two of either cuspidal type has the exact tame $S_3$ field with $e = 3, f = 2$ at five. Reduction of the special condition is contained in $\mathbf{Q}_5(\sqrt{2}, \sqrt{5})$, with elementary quadratic image. These statements concern the full decomposition image, not only inertia; see Lemma 28.6.

The cuspidal place makes every characteristic-two residue in Proposition 28.1 absolutely irreducible. Its nonsolvable case is already modular as an actual source, by Corollary 28.5. In that result the GRH comparisons exclude simple images $\mathrm{SL}_2(\mathbf{F}_q)$ with $q \geq 8$ in the mixed case, the $q = 4$ residue is modular, and a separate totally real discriminant comparison verifies the real-place hypothesis of Thorne's automorphy lifting theorem [83, Theorem 6.1]. We therefore concentrate on solvable residues.

Such an absolutely irreducible determinant-one image in characteristic two is dihedral $D_{2n}$ with $n$ odd. Let $L/K$ be its quadratic inducing field. We first dispose of the inducing fields nonsplit at two.

**Proposition 28.14.** *For a residue occurring in Proposition 28.1, either its given source is modular by the ordinary theorem of Allen, or its quadratic inducing field is $H$ and splits at two.*

*Proof.* If $L/K$ ramifies at two, inertia has nontrivial sign in the dihedral image. Its wild image is a normal two-subgroup, and tame inertia is cyclic. Since $n$ is odd, inertia must be a reflection $C_2$. Its normalizer in $D_{2n}$ is that same $C_2$, so the local decomposition image is also $C_2$. The actual finite-flat representation therefore cuts out $L_F/F$. Fontaine's upper-break bound gives conductor at most two [39, Theorem A]. Its Kummer class has even dyadic valuation.

The local image $C_2$ has trivial semisimplification in characteristic two. Since $K_2/\mathbf{Q}_2$ is unramified, Lemma 4.3(ii) makes the actual crystalline source ordinary. Allen's ordinary modular witness lemma [2, Lemma 5.1.2] supplies a quotient-compatible nearly ordinary cuspidal lift of the globally irreducible dihedral residue. It has no CM nonsplitting hypothesis. The actual-source theorem [2, Theorem 5.2.1] then applies: if $L$ is CM, its dyadic place is nonsplit; for a non-CM $L$ this extra condition is vacuous. Equal residual local characters are allowed in both applications.

It remains to identify the possible inducing fields that are split or unramified at two. Put $\alpha = (1 + \sqrt{21})/2$ and $\epsilon = (5 + \sqrt{21})/2$. The ordinary class number of $K$ is one: Minkowski's bound is $\sqrt{21}/2 < 2.3$, and two is inert. The unit $\epsilon$ is fundamental and has norm one. The squareclasses supported at two and five have basis $[-1]$, $[\epsilon]$, $[2]$, $[\alpha]$, $[\alpha^\sigma]$. Even dyadic valuation removes $[2]$ and gives the alternative basis $[-1], [\epsilon], [5], [\alpha]$. In $F$ we have $[5] = 1$ and $[\epsilon] = [-1]$, the latter because $((3 + \sqrt{21})/2)^2 = 3\epsilon$, and $-3$ is a square in $F$. If $w = \alpha \bmod 2 \in \mathbf{F}_4$, then $\alpha^3 \equiv 1 + 2\alpha \pmod 4$; the images of $[-1], [\alpha], [-\alpha]$ in $U_1/U_2$ are $1, w, 1 + w$. All are nonzero. Thus every nontrivial local class in their span is ramified of conductor two, and none is the unramified quadratic class. Consequently an allowable $L$ is either ramified or split at two, never unramified nonsplit.

The kernel of the local squareclass map is generated by $[5]$ and $[-\epsilon]$. The three nontrivial split possibilities are $5, -\epsilon, -5\epsilon$. At the cuspidal place above five the inducing quadratic field

must be unramified quadratic. The first and third candidates ramify there. Hence only $L = K(\sqrt{-\epsilon}) = K(\sqrt{-3}) = H$ remains. □

We next enumerate the odd inducing characters for $H$ by the ray class exact sequence. Put

$$u = \frac{\sqrt{-3} + \sqrt{-7}}{2}, \qquad \zeta = \frac{-1 + \sqrt{-3}}{2},$$

with $\sqrt{-3}\sqrt{-7} = \sqrt{21}$. Then $u^2 = -\epsilon^{-1}$ and $N_{H/K}(u) = \epsilon^{-1}$. Since $\epsilon$ is the fundamental totally positive unit of $K$, every unit of $H$ is a power of $u$ times a root of unity. Indeed, after multiplying by a power of $u$ its norm to $K$ is one, so all its complex conjugates have absolute value one and it is a root of unity. The roots of unity of $H$ are $\mu_6$. Also $h_H = 1$. Its Minkowski bound is $63/(2\pi^2) < 4$, whereas primes above two have norm four and the prime above three has norm nine.

An odd inducing character is tame at two and five, so its conductor divides (10). The two primes of $H$ above two have residue field $\mathbf{F}_4$, and those above five have residue field $\mathbf{F}_{25}$. Thus the odd part of the local residue-unit group modulo (10) is $\mathbf{F}_3^4$. At the five-primes, $u^4 = -1$ modulo five, so $u$ has order eight and trivial image in the odd quotient. The polynomial $X^2 - 5X + 1$ of $\epsilon$ is irreducible modulo two. Since $u^2 = -\epsilon^{-1}$, at either dyadic prime $u$ has order three. Choose generators so that the images of $u$ and $\zeta$ in the four odd factors are

$$(1, 1, 0, 0), \qquad (1, -1, 1, 1),$$

with the dyadic factors first. The two dyadic residues of $u$ agree under the identification induced by $K$, whereas those of $\zeta$ are inverse. The full unit group was determined above, so the odd ray quotient is exactly $\mathbf{F}_3^2$. The involution of $H/K$ swaps the first two factors and negates the last two. Modulo the first unit row it acts as $-1$ on this quotient.

A character is represented by a vector $(a, -a, c, d)$ with $c + d = a$. Its four projective lines give exactly four cubic quotients, all dihedral over $K$, with conductors

$$(5), \quad (10), \quad (2)\mathfrak{q}_1, \quad (2)\mathfrak{q}_2, \tag{38}$$

where $\mathfrak{q}_i$ lies above the corresponding five-prime of $K$. The first line has $a = 0$ and $c = -d \neq 0$. For $a \neq 0$, normalize $a = 1$ and use $(c, d) = (2, 2), (1, 0), (0, 1)$.

For the mixed orientation, the first two quotients ramify cubically at the special place and are impossible. Exactly one of the last two ramifies at the selected cuspidal place. Denote its field over $K$ by $A_{\mathrm{m}}$. It has image $S_3$ and is the unique residual representation for that orientation up to $\mathrm{GL}_2(\overline{\mathbf{F}}_2)$-conjugacy: the two nontrivial cubic characters are inverse and give conjugate induced representations. A determinant-one scalar twist in characteristic two is trivial.

The field $A_{\mathrm{m}}$ has local residual characters of order three over $\mathbf{F}_4$ at two, with the two finite-flat signatures $(1, 0)$ and $(0, 1)$. At the special place it is unramified quadratic, and Frobenius is a nontrivial reflection.

### 28.2.1 Finite-flat tangent extensions at two

We use the exact Honda-system description at the boundary $e(F) = 1 = p - 1$, reviewed in [15, pp. 4–6]. On biconnected objects the relevant categories are abelian, with the usual schematic kernels and quotients, and the generic-fibre functor is fully faithful. The integral classification and exactness at two can also be expressed using [51, Corollaries 4.3–4.4].

**Lemma 28.15.** *Let $k = \mathbf{F}_4$, let $R = k[\varepsilon]/(\varepsilon^2)$, and let $\theta : G_F \to k^\times$ have nontrivial tame inertia of order three and a finite-flat model. If $\theta(1+\varepsilon c)$ is finite flat as an underlying $\mathbf{F}_2$-representation, then $c$ is unramified.*

*Proof.* Every finite-flat model of $\theta$ or its selfextensions is biconnected. An étale quotient or a multiplicative subobject would have an unramified generic constituent, whereas every constituent has the nontrivial ramified character $\theta$; modulo two the cyclotomic character is trivial. Full faithfulness therefore extends the actual $R$-action to the model, and schematic closure and quotient give an exact selfextension of the rank-one model.

By the exact coefficient version of the classification [51, Corollaries 4.3–4.4 and Proposition 5.7.3], its two coefficient factors have exponents $r_0, r_1 \in \{0, 1\}$. The tame exponent is $r_0 + 2r_1$ modulo three. For the two nontrivial order-three characters the unique signatures are $(1, 0)$ and $(0, 1)$. The subobject and quotient therefore have the same rank-one model, including its unramified constants. Lemma 11.11, with $p = 2$, $e = 1$ and $f = 2$, makes the generic extension unramified after dividing by $\theta$, as required. □

**Lemma 28.16.** *For $A = A_{\mathrm{m}}$, every normal finite-flat dual-number deformation at two gives quadratic Kummer characters over the dyadic completions of $A$ of conductor at most four.*

*Proof.* Over $k = \mathbf{F}_4$ the residual local representation is $\theta \oplus \theta^{-1}$, with both characters nontrivial tame of order three. Its model is a direct sum of the unique biconnected models. A finite-flat $k[\varepsilon]$ deformation is an extension of this model by itself. Pullback along either constituent inclusion and pushout along either constituent projection in the abelian biconnected category extract its two offdiagonal extension classes. These are finite-flat extensions already over the original unramified field $F$.

Apply Fontaine's theorem there: classical upper breaks are at most one. Killing the two residual characters then makes a tame cubic base change, which multiplies positive upper breaks by three. Each resulting quadratic character has break at most three, hence conductor at most four. □

The order of these operations supplies the bound for every dual-number tangent point. A finite-flat group defined only over the ramified cubic field can instead have conductor six. No division by a dual-number parameter, and no characteristic-zero lift of the tangent point, is used in Lemma 28.16.

### 28.2.2 The normal Selmer calculations

Write $\bar{r} = \mathrm{Ind}_H^K \chi$ over $k = \mathbf{F}_4$. In an inducing basis we have

$$\mathrm{ad}^0(\bar{r}) = kI \oplus V, \qquad V = \mathrm{Ind}_H^K \chi^2. \tag{39}$$

The two offdiagonal entries give $V$, and the equal diagonal entries give the scalar summand. This is a direct calculation. The normal module is the scalar extension of the standard $\mathbf{F}_2$-representation of $S_3$.

Let $L_{\mathrm{m}}$ be the cubic point field over $K$ inside $A_{\mathrm{m}}$. An absolute equation is

$$f_{\mathrm{m}}(X) = X^6 - X^5 - 2X^4 + 3X^3 - 9X + 9.$$

It has signature $(2, 2)$, class number one and absolute discriminant 3704400. Its unique dyadic prime has absolute ramification index three and residue degree two. Over five the ramification indices and residue degrees are $(e, f) = (3, 1), (1, 1), (1, 2)$.

Apply Lemma 11.16 with $S$ the places above two and five together with the infinite places. The spaces $K(S, 2)$ and $L_{\mathrm{m}}(S, 2)$ have dimensions five and eight, respectively, since the class groups are trivial. Thus the normal cohomology space has dimension three over $\mathbf{F}_2$, and its computation requires only eight squareclasses.

The exact degree-six calculation in Appendix F gives the dyadic conductors

$$0, 2, 6, 6, 6, 6, 6, 6.$$

Here zero denotes the trivial class. The nonzero mixed class of conductor two is unramified and nonsplit at the degree-one prime above the special five-place. In particular its unramified quadratic character takes value one on Frobenius, in additive $\mathbf{F}_2$ notation.

To relate this to the special condition, realize the standard module as the vectors of coordinate sum zero in the permutation module on three letters. A residual special Frobenius is a transposition. Its fixed letter corresponds to the degree-one completion of $L_{\mathrm{m}}$. On the standard module the fixed coordinate is a nonzero functional on the coinvariants of that transposition. The trace derivative gives the same functional. Consequently a normal class unramified at that place has zero first variation of Frobenius trace exactly when its degree-one quadratic character is split. This excludes the nonzero mixed class surviving the dyadic bound.

**Proposition 28.17.** *For the mixed residue, a finite-flat normal tangent with homogeneous trace zero at the special place is a coboundary.*

*Proof.* Lemma 28.16 gives conductor at most four after killing the residual cubic characters. The point-field dyadic completion equals either residual-field completion, because $H/K$ splits at two. The displayed conductor list therefore leaves only the mixed line, which the special trace condition kills. Scalar extension gives the assertion for every finite coefficient field. The class groups are trivial, so no ideal-class contribution is omitted from these squareclass spaces. □

### 28.2.3 Closed deformation conditions and the full tangent

We now pass from the necessary normal cuts to the actual global deformation functor. Residual $S_3$ is absolutely irreducible over $\mathbf{F}_2$, so its universal global deformation ring over $\mathbf{Z}_2$ exists. We fix determinant $\chi_2$ and ramification outside two and five. The finite-flat condition at two is a closed deformation condition on Artin coefficient rings; see [15, Definition 1.1.1 and the following discussion]. Here its residual model is biconnected, so models and coefficient actions are unique. Local framed conditions give closed quotients without a local-centralizer restriction.

At a cuspidal five-adic place, require the fixed finite inertia kernel to act trivially and a chosen generator to have the characteristic polynomial for the selected $\tau_3$ or $\tau_6$. The residual cubic eigenvalues are distinct, so these equations specify the finite inertia representation up to conjugacy. They are closed matrix equations.

For the mixed case we impose a stronger special condition retaining the exact quadratic character. Fix $\alpha : G_{K_{v_{\mathrm{SP}}}} \to \{1, -1\}$ and an arithmetic Frobenius $\Phi$, and write $a = \alpha(\Phi) \in \{1, -1\}$. Let $M = r(\Phi) - aI$. The residual matrix $S - I$ has a unit pivot. Impose

$$\operatorname{tr} r(\Phi) = 6a, \qquad r(g)v = \chi_2(g)\alpha(g)v \quad (g \in G_{K_{v_{\mathrm{SP}}}}), \tag{40}$$

where $v$ is the pivot column of $M$. Since the determinant is already five, $\det M = 5 - a(6a) + a^2 = 0$. The unit pivot therefore makes $\operatorname{im} M$ a free rank-one direct summand generated by $v$. Cayley–Hamilton gives $(r(\Phi) - 5aI)M = 0$. The second equation in (40) imposes the exact character $\chi_2\alpha$ on that canonical line; the quotient character is $\alpha$ by the determinant. These are closed equations in the universal ring. Every actual special source with the chosen $\alpha$ satisfies them. This construction avoids any division by the eigenvalue difference four.

Let $D_{\mathrm{m}}$ denote this mixed quotient.

**Proposition 28.18.** *The map $\mathbf{Z}_2 \to D_{\mathrm{m}}$ is surjective. If this problem has a characteristic-zero point, its ring is $\mathbf{Z}_2$.*

*Proof.* We calculate the true relative tangent, after scalar extension to $k = \mathbf{F}_4$. Fixed determinant makes the cocycles take values in $\mathrm{ad}^0(\bar{r})$, but strict changes of basis use every matrix in $\mathrm{End}(\bar{r})$. The actual quotient is consequently

$$Z^1(\mathrm{ad}^0\,\bar{r})/d\,\mathrm{End}(\bar{r}), \tag{41}$$

with the stated local conditions. In characteristic two this need not equal $H^1(\mathrm{ad}^0\,\bar{r})$.

For the mixed problem, (40) gives homogeneous trace zero on tangent Frobenius, since $6a$ maps to zero in $k[\varepsilon]$. Proposition 28.17 kills the normal $V$-component in (39). After a strict global change of basis the tangent is therefore scalar, $cI$. Fixed finite cuspidal inertia makes $c$ unramified there. The marked exact character on the special line makes $c$ trivial on inertia at that place. At two its generic rank-one summands have finite-flat closures and quotients; applying Lemma 28.15 makes $c$ unramified. It is unramified elsewhere by the support condition.

The narrow class number of $K$ is two, whereas its ordinary class number is one. Hence the only everywhere-finite-unramified additive quadratic character, after extension to $k$, is the character $\delta_H$ of $H/K$. But $\delta_H I$ is a coboundary in the quotient (41). Indeed, for $X = \mathrm{diag}(1, 0)$, conjugation by $1 + \varepsilon X = \mathrm{diag}(1 + \varepsilon, 1)$ fixes the inducing subgroup and multiplies reflections by $1 + \varepsilon$. Its boundary is exactly $\delta_H I$. The matrix $X$ has trace one; restricting changes of basis to trace-zero matrices would incorrectly retain this tangent. The same strict conjugation transports the marked special line.

The special marking adds no tangent of its own: with representation and eigencharacter fixed, an infinitesimal line motion $w$ satisfies $(S - I)w = 0$ and thus lies in the original line. There is no motion in its quotient. The true relative tangent is zero.

All the data are defined over $\mathbf{Z}_2$; the $\mathbf{F}_4$ calculation is scalar extension of the $\mathbf{F}_2$ tangent. A complete Noetherian local $\mathbf{Z}_2$-algebra with residue $\mathbf{F}_2$ and zero relative tangent is a quotient of $\mathbf{Z}_2$, by Nakayama's lemma and completeness. Any characteristic-zero point rules out a nonzero ideal in this quotient, so the ring is $\mathbf{Z}_2$ itself. □

### 28.2.5 Rational traces and a toric place

We first isolate the rational endomorphism argument needed for the mixed source. It removes a possible Schur-index ambiguity in passing from rational traces to an elliptic curve.

**Proposition 28.19.** *Let $B/K'$ be an abelian variety of dimension $g$ over a number field, and let $\ell$ be prime. Suppose that $r$ is an absolutely irreducible two-dimensional $\mathbf{Q}_\ell$-valued representation such that $V_\ell(B) \simeq r^{\oplus g}$. If, at a finite place of residue characteristic different from $\ell$, the monodromy of $r$ has rank one after finite local base change, then $B$ is $K'$-isogenous to $E^g$ for an elliptic curve $E/K'$ with $V_\ell(E) \simeq r$.*

*Proof.* Make the reduction semistable and its torus split by a finite local extension. Monodromy on $V_\ell(B)$ has rank $g$, so $B$ has purely toric reduction, with torus $T$ of rank $g$. Put $C = \mathrm{End}_{K'}(B) \otimes \mathbf{Q}$. Faltings' endomorphism theorem [36] gives

$$C \otimes \mathbf{Q}_\ell = \mathrm{End}_{G_{K'}}(V_\ell B) = M_g(\mathbf{Q}_\ell),$$

so $\dim_{\mathbf{Q}} C = g^2$. Endomorphisms extend to the local Néron model and act on $X_*(T) \otimes \mathbf{Q}$.

Let $I$ be the kernel of this rational action. With a basis of the Tate twist fixed, write $N$ for monodromy on $V = V_\ell(B)$. Purely toric reduction gives $W = \mathrm{im}\,N = \ker N = V_\ell(T)$. Every $f \in I$ kills $W$. Since $fN = Nf$, its image lies in $\ker N = W$. Products of any two elements of $I$ therefore act trivially on $V$, and Faltings' faithfulness gives $I^2 = 0$. The algebra $C$ is semisimple by Poincaré complete reducibility, so $I = 0$.

Thus $C$ embeds in $\mathrm{End}_{\mathbf{Q}}(X_*(T)\otimes\mathbf{Q}) = M_g(\mathbf{Q})$. Equality of dimensions makes this an isomorphism. A primitive rational matrix idempotent cuts out an abelian subvariety with one copy of $r$ as its Tate module, hence an elliptic curve $E/K'$. The matrix units identify its $g$ factors up to $K'$-isogeny. □

*Proof of Proposition 28.1.* Assume that $\mathcal{R}$ is nonmodular. Modularity is preserved by coefficient conjugation. At every coefficient embedding into $\overline{\mathbf{Q}}_2$, the nonsolvable residual case is modular by Corollary 28.5, and all dihedral cases other than $H$ split at two are modular by Proposition 28.14. The ray calculation therefore puts every coefficient conjugate in the unique mixed residual class $A_{\mathrm{m}}$.

The cuspidal type is rational: coefficient conjugation exchanges its two inverse constituents, preserving $\tau_3$ and $\tau_6$ separately. The exact special character $\alpha$ takes values in $\{1,-1\}$. For a fixed arithmetic Frobenius $\Phi$ at the special place, $\operatorname{tr} r_2(\Phi) = 6\alpha(\Phi) \in \{6,-6\}$. Every coefficient embedding fixes this number, its sign and the quadratic inertia character. Thus every conjugate belongs to the same closed problem $D_{\mathrm{m}}$ of Section 28.2.3. The actual characteristic-zero point gives $D_{\mathrm{m}} = \mathbf{Z}_2$. All conjugates are consequently the same two-adic representation, defined over $\mathbf{Q}_2$. Every good Frobenius trace has only one $\mathbf{Q}$-conjugate and is rational. This uses all coefficient embeddings; a single $\mathbf{Q}_2$ realization would not give the conclusion.

Potential modularity and the special finite place give a $\mathrm{GL}_2$-type abelian variety $B/K$ realizing the system. More precisely, after potential modularity the square-integrable local component gives an abelian variety by [77, Lemma 9.4.3]; [77, Lemma 9.4.4] descends the realization to $K$. These lemmas and the realization argument in the proof of Proposition 9.4.1 use the given potential modularity, rather than imposing that proposition's odd-prime residual hypotheses on the present characteristic-two residue.

Let $g = \dim B$ and retain the $\mathbf{Q}_2$ model $r_2$. The realization may enlarge the coefficient field. For a good place, its trace in that enlarged field maps under one embedding to the original trace; every other embedding gives a $\mathbf{Q}$-conjugate of that same algebraic number. Since it is rational, the enlargement introduces no additional trace values. Chebotarev and Brauer–Nesbitt give

$$V_2(B)\otimes\overline{\mathbf{Q}}_2 \simeq (r_2\otimes\overline{\mathbf{Q}}_2)^{\oplus g}.$$

The space $\mathrm{Hom}_{G_K}(r_2, V_2(B))$ has $\mathbf{Q}_2$-dimension $g$ by scalar extension. Its evaluation map is an isomorphism after extension to $\overline{\mathbf{Q}}_2$, hence before extension. Therefore $V_2(B) \simeq r_2^{\oplus g}$.

At the special place monodromy on $r_2$ has rank one, after killing the quadratic twist. Proposition 28.19 supplies an elliptic curve $E/K$ with $V_2(E) \simeq r_2$. Every elliptic curve over a real quadratic field is modular [41, Theorem 1]. Thus $r_2$ and the compatible system are modular, a contradiction. □

## 28.3 Switching the pure menus to characteristic three

**Proposition 28.20** (Switchable sources)**.** *Assume GRH. Let $\mathcal{R}$ be a weight-two compatible system of determinant cyclotomic and support at five, with either pure bounded menu. If $\bar{r}_3|_{G_H}$ is irreducible, then $\mathcal{R}$ is modular.*

*Proof.* Apply Lemma 4.18, with $\ell = 5$, at one five-place to change its type and at the other to retain its type. At three retain the original crystalline weight-two component. Prescribe determinant $\chi_3$ and no ramification elsewhere. Lemma 2.6(i) applies because $\bar{r}_3|_{G_H}$ is irreducible; in characteristic three there is no exceptional-five exclusion. It supplies a potentially modular compatible replacement. The special component has Frobenius ratio five, so Section 2 retains its nonzero monodromy. Proposition 28.1 makes the replacement modular. Its residue has a weight-two witness; Lemma 2.2(ii) then applies to the actual old three-adic member and proves modularity of the original system. □

## 28.4 Reducible residual characters and the ordinary case

**Lemma 28.21.** *If the three-adic residue of a bounded-menu system of determinant cyclotomic and support at five is reducible, its semisimplification is $\psi \oplus \psi\bar{\chi}_3$, where $\psi$ is quadratic. Up to interchanging the constituents,*

$$\psi = 1, \quad \psi_5, \quad \psi_\gamma, \quad \psi_5\psi_\gamma.$$

*Here $\psi_x$ denotes the character of $K(\sqrt{x})/K$. The last two possibilities have common ramified quadratic inertia at three. Reducibility forces the same quadratic inertia parity at the two five-places.*

*Proof.* Write the constituents initially as $\psi$ and $\bar{\chi}_3\psi^{-1}$. At five their semisimplified inertia is scalar quadratic. At three a character with values in characteristic three kills principal units, and the residue field has multiplicative group of order two. Thus $\psi^2$ is unramified at every finite place.

The ordinary class number of $K$ is one: its Minkowski bound is $\sqrt{21}/2 < 2.3$ and two is inert. Its units are $\{\pm\epsilon^n\}$ and there is no norm-minus-one unit, since the norm-minus-one equation is impossible modulo three. The narrow class number is two, with class-field character $\eta_0 = \bar{\chi}_3$. Since $\psi(c)^2 = 1$ at complex conjugations and $\eta_0$ is totally odd, $\psi^2$ cannot equal $\eta_0$. Hence $\psi^2 = 1$.

The quadratic squareclasses supported at three and five have basis $[-1], [\epsilon], [\gamma], [5], [a]$. Their unramified-at-two subspace is $\langle [-\epsilon], [5], [\gamma] \rangle$. For an explicit check in the unramified quadratic dyadic ring, a unit class is unramified exactly when its cube is one modulo four. The identities $a^3 = 6a + 5, \quad \epsilon^3 = 24a + 43, \quad \gamma^3 = 12a + 21$ give this kernel: the ramified images of $-1$ and $a$ are independent, while $[\epsilon] = [-1]$. Both 5 and $-\epsilon$ are locally squares; $\gamma$ is unramified nonsplit, since the residue trace of $(\gamma^3 - 1)/4 = 3a + 5$ is one.

At three both 5 and $-\epsilon$ are nonsquare units, whereas $\gamma$ has odd valuation. At the five-places $a \equiv 0, 1$, their local squareclasses are

| | $a \equiv 0$ | $a \equiv 1$ |
|---|---|---|
| $-\epsilon$ | $U$ | $U$ |
| 5 | $R$ | $R$ |
| $\gamma$ | 1 | $U$, |

where $U$ is unramified quadratic and $R$ is the character of $\mathbf{Q}_5(\sqrt{5})$. Interchanging the constituents multiplies $\psi$ by $\eta_0$. The four stated representatives and the parity assertion follow. □

**Proposition 28.22.** *A bounded-menu compatible system whose three-adic residue has constituents unramified at three is modular.*

*Proof.* By Lemmas 4.3(ii) and 28.21, the actual crystalline weight-two three-adic source is ordinary, with constituent quotient $\bar{\chi}_3$. Skinner–Wiles, Theorem A of [75], applies: $K$ is totally real abelian, $K(\bar{\chi}_3) = H$ is abelian over $\mathbf{Q}$, the quotient is totally odd and is nontrivial on $G_{K_3}$, and the ordinary quotient is unramified. Removing the Teichmüller character of the ordinary quotient gives the required pro-three character with finite inertia. The determinant is $\chi_3$. The theorem imposes no splitting condition at three and proves modularity of the actual source. □

## 28.5 The cyclotomic-dihedral exception at three

Write $\omega = \bar{\chi}_3$ and let $\sigma$ be the nontrivial automorphism of $K/\mathbf{Q}$. The two places $v_1, v_2$ above five have completion $\mathbf{Q}_5$. For determinant $\chi_3$, the Weil representations of $\tau_3, \tau_6$ have Frobenius trace zero and square $-5I$.

### 28.5.1 The residual representation and the quartic fields

**Lemma 28.23.** *Let $\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ be absolutely irreducible, with determinant $\omega$, unramified outside three and five, and reducible on $G_H$. Suppose that its inertia at either place above five is scalar quadratic. Then*

$$\bar{r} \simeq \mathrm{Ind}_{G_H}^{G_K} \xi, \qquad \xi^c = \xi^{-1}, \qquad \mathrm{ord}(\xi) = 4,$$

*where $c$ is complex conjugation, and $\xi^2$ cuts out $H(\sqrt{\gamma})/H$. Its linear image is $D_8$, and it is defined over $\mathbf{F}_3$. If $\eta, \eta'$ are the quadratic characters of*

$$L = K(\sqrt{\gamma}), \qquad L' = K(\sqrt{-3\gamma}),$$

*then*

$$\mathrm{ad}^0\, \bar{r} = \omega \oplus \eta \oplus \eta', \qquad \eta' = \omega\eta. \tag{42}$$

*The summand normal to induction from $H$ is $\eta \oplus \eta'$. The two scalar quadratic inertia characters at five are different. For a specified ordered pair of these inertia characters there is at most one residual isomorphism class satisfying these hypotheses.*

*Proof.* Clifford theory gives the displayed induction. Restriction of the determinant to $G_H$ gives $\xi^c = \xi^{-1}$. The character $\xi^2$ is unramified outside three. At three every finite character with values in $\overline{\mathbf{F}}_3^\times$ is tame, so $\xi^2$ factors through the ray class group of $H$ with modulus its unique prime $\mathfrak{P}_3$ above three.

Here is an elementary calculation of this ray group. The field $H = \mathbf{Q}(\sqrt{-3}, \sqrt{-7})$ has discriminant 441. Its Minkowski bound is $63/(2\pi^2) < 4$. The prime ideals above two have norm four, and the unique prime above three has norm nine. Consequently $h_H = 1$. The algebraic integer $u = \frac{\sqrt{-3}+\sqrt{-7}}{2}$, $\quad u^4 + 5u^2 + 1 = 0$, is a unit. Its image at $\mathfrak{P}_3$ has order four in $\mathbf{F}_9^\times$: its square is $-1$. The ray class exact sequence therefore bounds $\mathrm{Cl}_H(\mathfrak{P}_3)$ by a group of order two.

The quadratic extension $H(\sqrt{\gamma})/H$ is nontrivial and tame of conductor $\mathfrak{P}_3$. Indeed $\gamma$ has odd valuation at three and is a unit at every other finite place. At two its squareclass is represented by $\gamma^3 \equiv 1 \pmod 4$, so the quadratic extension is unramified. At all other odd primes it is unramified as well. This proves $\mathrm{Cl}_H(\mathfrak{P}_3) \simeq C_2$. Thus $\xi$ has order at most four. Order at most two would give $\xi^c = \xi$ and make the induction reducible. Hence its order is four, and its square is the unique nontrivial ray character.

The relation $\gamma\gamma^\sigma = -3$ identifies $L'$ with the quadratic conjugate of $L$. The usual adjoint decomposition of this dihedral representation gives (42). Rotation and reflection may be represented over $\mathbf{F}_3$ by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At each five-prime of $H$, the unit $u$ has residue of order eight in $\mathbf{F}_{25}^\times$, since $u^4 = -1$ modulo five. An order-four tame character at three takes the value $-1$ on $u$. A nontrivial quadratic tame character at either five-prime also takes the value $-1$ on $u$. The global unit relation therefore requires an odd number of the two quadratic inertia characters at five to be nontrivial.

Finally the order-four inertia character at three has just two possibilities, inverse to one another. Once it and the two five-adic quadratic inertia characters are fixed, two possible global characters differ by an everywhere unramified character of $H$. Such a character is trivial because $h_H = 1$. Replacing all characters by their inverses leaves the quadratic characters at five unchanged and gives the same induced representation. This proves uniqueness. □

*Remark* 28.24. The scalar-inertia hypothesis in Lemma 28.23 is automatic for the cyclotomic-reducible residue of either source considered below. An induction from a character with values

in $\overline{\mathbf{F}}_3^\times$ has image of order prime to three. The order-three part of cuspidal inertia, or the unipotent part of special inertia, must therefore disappear in that image. The remaining inertia is scalar quadratic. This argument does not discard a possible nilpotent inertia direction in a deformation.

**Lemma 28.25.** *Let $\iota$ be the nontrivial automorphism of $L/K$, and put*

$$y = \sqrt{\gamma}, \qquad z = \frac{a+y}{2} = \frac{y^2+y-1}{2}, \qquad g = \frac{1-y^2-5y}{2} = 2a - 5z.$$

*Then $h_L = 1$, $\operatorname{disc}(L) = -1323$, and the minus eigenspace of the $S_{3,5}$-unit cubeclasses of $L$ has basis $[g], [z]$ over $\mathbf{F}_3$. There is a unique degree-two prime $\mathfrak{q}$ above five. The classes in that eigenspace which are cubes in $L_\mathfrak{q}$ are precisely*

$$1, \quad [g], \quad [g]^2.$$

*Each of the two nontrivial classes has cubic conductor six at three after adjoining $\zeta_3$.*

*Proof.* The identities

$$N_{L/K}(z) = 1, \qquad z + z^{-1} = a, \qquad N_{L/K}(g) = -a^3, \qquad g + g^\iota = -a \tag{43}$$

follow directly from $y^2 = \gamma$ and $a^2 = a + 5$. The element $z$ is an integral unit, with polynomial $X^4 - X^3 - 3X^2 - X + 1$. The relative discriminant of $\mathcal{O}_K[z]$ is $(a^2 - 4) = (\gamma)$, the prime above three. It is squarefree, so this order is maximal. Since $a = z + z^{-1}$, it follows that $\mathcal{O}_L = \mathbf{Z}[z]$ and $\operatorname{disc}(L) = 21^2 N_{K/\mathbf{Q}}(\gamma) = -1323$. Modulo two the polynomial of $z$ is $\Phi_5(X)$, irreducible over $\mathbf{F}_2$, so two is inert in $L$. The signature is $(2, 1)$ and Minkowski's bound is less than five. The only possible ideal norm between two and four is therefore three. The unique prime over three is principal, generated by $y$, whose rational norm is $-3$. Hence $h_L = 1$.

The unit ranks of $L$ and $K$ are two and one; their torsion groups have no three-primary part. Restriction and norm split the invariant part modulo cubes because two is invertible modulo three. Thus the minus part of the unit cubeclasses has dimension one. The prime above three is fixed by $\iota$. Of the two places of $K$ above five, one splits in $L$ and the other is inert: the values of $\gamma = a + 1$ modulo five are respectively one and two. The minus part of the $S_{3,5}$-divisor space has dimension one. The triviality of the class group consequently makes the minus part of the $S_{3,5}$-unit cubeclasses two-dimensional.

Both $g$ and $z$ belong to this minus part by (43). The element $g$ is integral because $g = 2a - 5z$. Its norm and trace show that its valuations at the two split five-primes are one and two in some order: their sum is three, while their unequal minimum is the valuation one of $-a$. In particular its divisor class modulo three is nonzero.

At $\mathfrak{q}$ one has $a = 1$, $y^2 = 2$ in $\mathbf{F}_{25}$ and $g = 2, \quad z = (1 + y)/2$. The element $2 \in \mathbf{F}_5^\times$ is a cube in $\mathbf{F}_{25}^\times$. The displayed residue of $z$ has cube $-1$ and is not $\pm 1$, so it has order six and is not a cube in the group of order twenty-four. Thus $[z]$ is a nonzero unit class, independent of $[g]$, and the local cube condition at $\mathfrak{q}$ kills exactly its coefficient. Units are cubes over this five-adic field exactly when their residues are cubes, since cubing is an automorphism of the principal units.

At three, $y$ is a uniformizer of $L_3$ and the absolute ramification index is four. The extension $M = L_3(\zeta_3)/L_3$ is unramified quadratic. Since $g \equiv -1 - y \pmod{y^2}$, the polynomial $(X+1)^3 + g$ is Eisenstein over $M$. At a root its derivative has valuation $v_E(3) = 12$, because $X + 1$ is a unit. An Eisenstein generator gives the full ring of integers. The cubic extension is cyclic because $\zeta_3$ belongs to $M$, and the conductor–discriminant formula gives conductor $12/(3 - 1) = 6$. For $g^2$, the polynomial $(X + 1)^3 - g^2$ is Eisenstein and gives the same calculation. □

**Corollary 28.26.** *A class in $H^1(K, \mathbf{F}_3(\eta'))$ which is unramified outside three and five, is zero at the five-place where $\eta'(\mathrm{Frob}_5) = 1$, and has cubic conductor at most five at three after killing its quadratic character, is zero. The analogous assertion holds for $\eta$.*

*Proof.* Since $\eta' = \omega\eta$, Kummer theory identifies

$$H^1(K, \mathbf{F}_3(\eta')) \simeq (L^\times / L^{\times 3})^-.$$

The support condition and $h_L = 1$ make such a class an $S_{3,5}$-unit class. At the specified five-place, $\eta(\mathrm{Frob}_5) = -1$, so $L/K$ is inert; local vanishing means being a cube at the degree-two prime $\mathfrak{q}$. Lemma 28.25 leaves only $1, g, g^2$, and its conductor calculation excludes the latter two. Quadratic conjugation exchanges $L$ and $L'$, proving the other assertion. □

### 28.5.2 The normal tangent space

**Lemma 28.27.** *Let a class in $H^1(K, \eta \oplus \eta')$ arise from a finite-flat tangent deformation of the residual representation of Lemma 28.23, with determinant fixed and ramification supported at three and five. If the first variation of the Frobenius trace is zero at both places above five, then this normal class is zero.*

*Proof.* We verify the conditions of Corollary 28.26 for each normal summand.

At a five-place where a normal character $\nu$ has Frobenius value $+1$, $H^1(K_v, \mathbf{F}_3(\nu))$ is purely unramified. The tame ramified direction would require $\nu(\mathrm{Frob}_5) = 5 = -1$ in $\mathbf{F}_3$. In the adjoint representation the commuting traceless line is spanned by the residual Frobenius matrix $\bar{\Phi}$, normalized by $\bar{\Phi}^2 = I$. If the unramified class takes the value $b\bar{\Phi}$, its variation of the trace is $b\,\mathrm{tr}(\bar{\Phi}^2) = 2b$. The trace condition therefore kills this unramified class. At the other five-place, where $\nu(\mathrm{Frob}_5) = -1$, we impose no cut on the ramified direction. In particular this proof allows first-order nilpotent inertia of order three.

At three, work first over $H_3$, an unramified quadratic extension of $K_3$ with absolute ramification index two. We may compute over $\mathbf{F}_3$, where the residue is defined. To obtain an actual $\mathbf{F}_9$-action on a finite-flat model, take a direct sum of two copies with the regular $\mathbf{F}_9$-action. Over $H_3$, schematic closures and quotients of the $\mathbf{F}_9$-linear generic constituents extract the upper and lower extensions. These operations preserve the existing scalar action. This construction avoids assuming that an arbitrary generic coefficient action extends to an arbitrary finite-flat model.

Fontaine's ramification bound, applied over this original field of absolute ramification index two, bounds the classical upper breaks of each extracted point field by

$$2\left(1 + \frac{1}{3-1}\right) - 1 = 2. \tag{44}$$

Here the shift by one comes from the convention $G^{(u)} = G^{u-1}$ in [39, Theorem A]. Pass to the projective kernel field, which is a quotient of the point field and retains the bound. The diagonal ratio has order two, although the original diagonal characters have order four. Killing the ratio by its tame quadratic extension multiplies positive upper breaks by two. The resulting cubic characters have upper break at most four, hence conductor at most five.

If extracting a quadratic-adjoint summand requires combining the two extensions, take the compositum of their point fields first. The direct sum of their finite-flat groups satisfies the same bound, and every projected character field is a quotient. The relevant field over $L_3$ acquires $\zeta_3$ by an unramified extension, which does not alter the conductor. Thus Corollary 28.26 applies to both summands. Scalar extension of its $\mathbf{F}_3$ calculation gives the same conclusion for any finite residual coefficient field. □

*Remark* 28.28. The projective quotient in this proof is essential. We apply Fontaine's bound to actual finite-flat extensions over $H_3$ before making any ramified base change. We do not twist a finite-flat representation by a ramified character and assert that it remains finite flat.

### 28.5.3 Two cuspidal places

**Theorem 28.29.** *Let $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_3)$ be continuous and totally odd, with determinant $\chi_3$, unramified outside three and five, and crystalline of labelled Hodge–Tate weights $\{0,1\}$ at three. Suppose its type at each place above five is $\tau_3$ or $\tau_6$. If its residue is absolutely irreducible but reducible over $H$, then $r$ is induced from a geometric character of $G_H$. In particular $r$ is modular.*

*Proof.* By Lemma 28.23, the residue is $\bar{r} = \mathrm{Ind}_H^K \xi$. Enlarge the residue field to contain $\mathbf{F}_9$ and choose an inducing basis. Put $J = \mathrm{diag}(1,-1)$. Then $J\bar{r}(g)J^{-1} = \omega(g)\bar{r}(g)$ and $J^2 = 1$.

Let $D$ be the universal fixed-determinant deformation ring of $\bar{r}$, finite flat at three, unramified outside three and five, and with the prescribed finite inertia at the two five-places and Frobenius trace zero there. These conditions are represented by a closed quotient of the universal deformation ring: finite-flatness is a stable condition, and the local finite-inertia and Frobenius relations are closed algebra relations; see [86, Corollary 3.3.5 and Section 5.2]. The actual source gives a characteristic-zero point of $D$.

The character $\omega$ is unramified at every finite place of $K$. Consequently the operation

$$r_A \longmapsto J^{-1}(r_A \otimes \omega)J$$

preserves these conditions, the determinant and the identified residue. Unramified twisting preserves finite-flatness by descent from the unramified extension killing the twist. At five, twisting preserves the prescribed inertia and the zero Frobenius trace. This operation defines an involution of $D$.

On $\mathrm{ad}^0\,\bar{r} = \omega \oplus \eta \oplus \eta'$ it acts as $+1$ on the diagonal summand and as $-1$ on the two normal summands. The anti-invariant projection of an admissible tangent is again admissible, since the involution preserves the deformation conditions and two is invertible. This normal tangent is zero by Lemma 28.27: diagonal variations have zero Frobenius trace at five, since Frobenius is offdiagonal in the inducing basis. Hence the involution is trivial on the entire relative tangent, and therefore on the relative cotangent. Lemma 12.2, with $m = 2$, makes it the identity on $D$.

It follows that the actual irreducible source satisfies $r \simeq r \otimes \omega$. An intertwiner squares to a scalar by Schur's lemma. After rescaling, its two eigenspaces are preserved by $G_H$ and interchanged by the other coset. Thus $r = \mathrm{Ind}_H^K \psi$. The character $\psi$ is de Rham and finitely ramified, hence algebraic by global class field theory [71, Chapter III]. Since $H/K$ is CM and the two characters are distinct, quadratic automorphic induction is regular algebraic and cuspidal [4, Chapter III]. □

### 28.5.4 A finite-flat selfextension calculation

**Lemma 28.30.** *Let $F = H_3$, and let $k$ be a finite coefficient field containing $\mathbf{F}_9$. Let $\theta : G_F \longrightarrow k^\times$ have the order-four tame inertia character occurring in Lemma 28.23. A finite-flat selfextension of $\theta$, equipped with an actual $k$-action on its finite-flat model, is unramified after dividing by the common character $\theta$ on its generic fibre.*

*Proof.* Take the schematic closure of the generic subline and the finite-flat quotient. The exact finite-flat/Kisin-module equivalence and its coefficient versions apply to this sequence [53, Theorem 1.1.3, Proposition 1.1.11 and Lemmas 1.2.4–1.2.5]. The two coefficient idempotents

give rank-one endpoints with Frobenius exponents $(r_0, r_1)$. Since $e(F) = 2$, one has $0 \le r_i \le 2$. The tame exponent is $r_0 + 3r_1$ modulo eight. The two order-four inertia characters have exponents two and six, whose unique possibilities are respectively $(2, 0), \quad (0, 2)$. The rank-one models of the two endpoints are consequently isomorphic, including their unramified constants.

Lemma 11.11 applies with $p = 3$, $e = 2$ and $f = 2$. Thus the only selfextensions are the unramified ones after dividing by $\theta$. This uses the exact rank-two $k[[u]]$ extension furnished above, without assuming freeness over $k[t]/(t^2)[[u]]$. □

### 28.5.5 Two special places

**Theorem 28.31.** *Let $\mathcal{R}$ be a totally odd, absolutely irreducible, strongly compatible two-dimensional system over $K$, with cyclotomic determinant and labelled Hodge–Tate weights $\{0, 1\}$. Assume all-place Weil–Deligne compatibility, support at five away from the coefficient characteristic, and a crystalline three-adic member at three. At $v_i \mid 5$ suppose it is special with nonzero monodromy and fixed quadratic twisting character $\alpha_i : G_{\mathbf{Q}_5} \longrightarrow \{\pm 1\}$, including the unramified sign. If the characteristic-three residue is absolutely irreducible but reducible over $H$, then $\mathcal{R}$ is modular.*

*Proof.* Again Lemma 28.23 applies. Fix its model over $\mathbf{F}_3$. Form the universal global deformation ring with determinant $\chi_3$, support at three and five, and the finite-flat condition at three, and impose the following closed local equations at $v_i$. For arithmetic Frobenius $\Phi_i$ put $b_i = \alpha_i(\Phi_i) \in \{\pm 1\}$, require

$$\operatorname{tr} r(\Phi_i) = 6b_i, \qquad r(g)|_{\ell_i} = \chi_3(g)\alpha_i(g), \tag{45}$$

where $\ell_i$ is the $5b_i$-eigenline of $r(\Phi_i)$. The Frobenius eigenvalues $5b_i, b_i$ have difference $4b_i$, a unit at three. Thus the eigenline is functorial and the equations are closed; the determinant supplies the quotient character $\alpha_i$.

For a precise construction of this quotient, impose the stable condition on finite-cardinality $\mathbf{Z}_3[G_{K,\{3,5\}}]$-modules consisting of finite-flatness at three and the algebra annihilator identities

$$(g - \chi_3(g)\alpha_i(g))(h - \alpha_i(h)) = 0 \qquad (g, h \in G_{K_{v_i}})$$

at each five-place. Finite-flatness is stable under subquotients and finite direct sums, as are these identities. Wake–Wang-Erickson's stable-condition theorem gives the universal pseudodeformation quotient [86, Theorem 2.5.5 and Section 5.2]; impose the fixed determinant $\chi_3$ as well. Since the residual representation is absolutely irreducible, pseudodeformations and ordinary deformations agree, and the universal Cayley–Hamilton algebra is $M_2(D)$ over the resulting ring $D$. The displayed annihilator identities are equivalent to (45), because the two residual Frobenius eigenvalues are distinct. This constructs the stated deformation condition without requiring uniqueness of finite-flat models at $e = p - 1$. Every actual special source with these characters gives a point of $D$.

We show that its relative tangent over $\mathbf{Z}_3$ is zero. Its normal projection belongs to $H^1(K, \eta \oplus \eta')$. Equation (45) has zero first trace variation in characteristic three. The tangent construction of Lemma 28.27, including its finite-flat extraction, kills both normal summands. After a strict conjugation, the tangent lies in the remaining diagonal summand $\omega$ of (42).

At a five-place, residual inertia is scalar and residual Frobenius is a reflection with distinct eigenlines. In an inducing basis, a traceless diagonal matrix takes either Frobenius eigenline to the other. The prescribed character on $\ell_i$ in (45) therefore forces the diagonal tangent to vanish on inertia. Movement of the line cannot compensate for this error, since the residual inertia matrix is scalar. The diagonal class is consequently unramified at both places above five.

At three, compute the tangent over $\mathbf{F}_3$ and equip a direct sum of two copies of its finite-flat model with the regular $\mathbf{F}_9$-action. Over $H_3$, take schematic closures and quotients of its $\mathbf{F}_9$-linear

generic constituents. These operations preserve the existing coefficient action and extract the selfextensions of the two order-four characters. Lemma 28.30 shows that the restriction of the diagonal class to $H_3$ is unramified. The character $\omega$ over $K_3$ is the nontrivial unramified quadratic character. Hence $H^1_{\mathrm{unr}}(K_3, \mathbf{F}_3(\omega)) = \mathbf{F}_3/(\omega(\mathrm{Frob}_3) - 1)\mathbf{F}_3 = 0$. The extension $H_3/K_3$ is unramified of degree two; restriction is injective and does not change inertia. The diagonal class is therefore locally zero at three.

Globally Kummer theory identifies $H^1(K, \mathbf{F}_3(\omega)) = K^\times/K^{\times 3}$. The class number and unit calculation of Lemma 28.21 applies. The support condition, unramifiedness at five, and the trivial class group leave just $[\epsilon]$ and $[\gamma]$ as possible global classes. A cube at three has valuation divisible by three, so the $\gamma$ coefficient is zero. Finally $\epsilon = 1 + \gamma$, while a unit cube with residue one is 1 modulo $\gamma^3$, since $v_\gamma(3) = 2$. Neither $\epsilon$ nor $\epsilon^2$ is such a cube. The diagonal tangent vanishes.

All deformation data are defined over $\mathbf{Z}_3$. The vanishing of the relative tangent and Nakayama's lemma make $D$ a quotient of $\mathbf{Z}_3$. An actual characteristic-zero source point forces

$$D = \mathbf{Z}_3. \tag{46}$$

Thus the source is the unique lift with its ordered pair of exact special characters.

It remains to compare this unique lift with its quadratic conjugate; this also avoids any need to control all other three-adic coefficient places. Let $\eta_0$ be the quadratic character of $H/K$ and let $\psi_5$ be that of $K(\sqrt{5})/K$. At both five-places $\eta_0$ is the unramified quadratic character and $\psi_5$ is the quadratic character of $\mathbf{Q}_5(\sqrt{5})$. They form a basis of the local quadratic characters, with the same restrictions at the two places. Put $\beta = \alpha_1\alpha_2$ and take the unique $\delta \in \{1, \eta_0, \psi_5, \eta_0\psi_5\}$ whose restriction at both places is $\beta$. The inertia parity in Lemma 28.23 makes $\beta$ ramified, so here $\delta = \psi_5 \quad \text{or} \quad \delta = \eta_0\psi_5$. Both characters are unramified at three. The extension $H/K$ is unramified at every finite place; at three the unit five is a nonsquare in $\mathbf{F}_3$ and gives an unramified quadratic extension as well.

The representation $r_3^\sigma \otimes \delta$ has the original ordered pair $(\alpha_1, \alpha_2)$, the same determinant and support, and the same crystalline condition. Its residue has the same restored ordered scalar inertia characters, hence is the same residual isomorphism class by Lemma 28.23. It is another point of $D$. Equation (46) gives

$$r_3^\sigma \simeq r_3 \otimes \delta. \tag{47}$$

Both possible characters $\delta$ are restricted from $G_{\mathbf{Q}}$: $\psi_5$ is cut out by $\mathbf{Q}(\sqrt{5})$ and $\eta_0\psi_5$ by $\mathbf{Q}(\sqrt{-15})$. Strong compatibility transports (47) to the whole system. Lemma 12.4, with $\eta = \delta$, gives its modularity directly. □

*Remark* 28.32. The class, unit and conductor calculations above are elementary and use no GRH. The corresponding PARI/GP certificates have, in the basis $[g], [z]$, strict-five masks $0, 1, 2$ and conductor-at-most-five masks $0, 4, 8$, with intersection $\{0\}$.

The exact calculation is supplied in [20, `d21`].

## 28.6 The nonordinary reducible characteristic-three source

This section proves a source modularity theorem for both pure local menus. Its input is an actual characteristic-zero compatible system. The proof uses finite-flat extensions at the ramified prime above 3, a universal pseudodeformation ring with explicit local conditions, and descent under quadratic conjugation. In particular, it does not replace source modularity by modularity of a semisimplified residue.

Let $\sigma$ be the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$, and write $\omega = \overline{\chi}_3|_{G_K}$. The character $\omega$ cuts out $K(\sqrt{-3})/K$. Let $\psi_\gamma$ cut out $K(\sqrt{\gamma})/K$. The norm identity gives

$$\psi_\gamma^\sigma = \psi_\gamma \omega. \tag{48}$$

**Theorem 28.33** (Nonordinary reducible source modularity)**.** *Let $\mathcal{R}$ be an odd, absolutely irreducible, strongly compatible system over $K$ of parallel weight two and cyclotomic determinant, good outside the two primes above 5. Assume its 5-adic companion is absolutely irreducible, as holds for the potentially modular systems constructed in the controlled reduction chain. Suppose its local types at the two primes over 5 are either both quadratic twists of Steinberg with nonzero monodromy, or both the tame dihedral supercuspidal types with inertia of order 3 or 6.*

*Suppose the semisimplification of its characteristic-three residue is globally reducible, and both constituent characters are ramified quadratic at the prime over 3. Then $\mathcal{R}$ is modular.*

*More precisely, after a quadratic twist unramified outside 5, the system has Frobenius traces invariant under quadratic conjugation. Its 5-adic companion descends to an odd geometric representation of $G_{\mathbf{Q}}$, and modularity follows from Pan's theorem over $\mathbf{Q}$ at the prime 5.*

### 28.6.1 Quadratic normalization

In the reducible characteristic-three case, each constituent has quadratic inertia at a prime over 5. The residues of the fundamental unit $\epsilon$ at the two primes over 5 are 2 and 3. The global unit relation therefore forces the two quadratic ramification parities to agree. If both are ramified, twist by the quadratic character $\psi_5$ of $K(\sqrt{5})/K$. This removes both ramification factors, preserves goodness at 3, and is trivial at the dyadic completion. After this normalization, the two types are either unramified quadratic twists of Steinberg at both places, or the SC types of inertia order 3 at both places.

Lemma 28.21 now gives, up to interchanging labels,

$$\chi_1 = \psi_\gamma, \qquad \chi_2 = \psi_\gamma \omega, \tag{49}$$

with distinct characters defined over $\mathbf{F}_3$. We henceforth work with the normalized system and the labels (49).

### 28.6.2 The finite-flat extension line at three

Let $F = K_{\mathfrak{p}_3} = \mathbf{Q}_3(\sqrt{21})$. Its ramification index is 2 and its residue field is $\mathbf{F}_3$. The character $\omega|_{G_F}$ is the nontrivial unramified quadratic character: over $F$, adjoining $\sqrt{-3}$ amounts to adjoining the square root of the nonsquare unit $-7$. Thus the two characters in (49) have the same ramified quadratic inertia and opposite unramified factors.

**Lemma 28.34** (The local finite-flat line)**.** *For either orientation of the pair* (49)*, the finite-flat subspace of $H^1(F, \mathbf{F}_3(\omega))$ has dimension one. Under the Kummer identification it is spanned by $[a\epsilon]$. For any finite $k/\mathbf{F}_3$, the finite-flat subspace with $k$ coefficients is the scalar extension of this line.*

*Proof.* First work over $\mathbf{F}_3$. A rank-one finite-flat model of either character has unique Breuil–Kisin exponent $r = 1$: the inertia exponent is odd, whereas height one permits only $0 \le r \le 2$. The two unramified factors differ by $-1$.

Any finite-flat extension gives an exact extension of these rank-one Kisin modules. Indeed, take the schematic closure of the generic rank-one subrepresentation and then the finite-flat quotient. The exact correspondence on objects killed by $p$ is supplied by [53, Theorem 1.1.3

and Proposition 1.1.11]; the coefficient versions are [53, Lemmas 1.2.4–1.2.5]. In suitable bases the Frobenius matrix is, up to a common unramified factor,

$$\Phi = \begin{pmatrix} u & h(u) \\ 0 & -u \end{pmatrix}, \qquad h \in \mathbf{F}_3[[u]]. \tag{50}$$

Every such matrix has height at most one, since $u^2\Phi^{-1}$ is integral. A change of the upper lift by $f(u)$ changes $h$ by a nonzero scalar multiple of $u(f(u) + f(u^3))$. The operator $f \mapsto f + f(u^3)$ is bijective on $\mathbf{F}_3[[u]]$: its constant term is multiplication by 2, and its other coefficients are solved recursively. Thus every extension is represented by a constant $h$ and the Kisin extension space is one-dimensional. This proves that the actual finite-flat Galois extension space has dimension at most one. It uses no injectivity assertion for restriction to $G_{F_\infty}$.

We identify a nonzero class using the curve with number field label `2.2.21.1-25.1-a4` in [55]. In generalized Weierstrass notation its model is

$$E = [a, a, a+1, -a-3, -a-2]. \tag{51}$$

Its conductor is (5) [55], so it has good reduction at 3; counting its points gives $a_{\mathfrak{p}_3}(E) = 0$. Its 3-division polynomial factors as

$$3\left(x - \frac{a+1}{3}\right)\left(x^3 + (2a+2)x^2 + (2a+3)x + \frac{4a+7}{3}\right). \tag{52}$$

The point above the rational root has squared doubled generalized $y$-coordinate $(61a + 109)/9$, and the identity

$$\frac{61a + 109}{9\gamma} = \left(\frac{2a+3}{3}\right)^2$$

identifies its squareclass as $\gamma$. The stable subgroup therefore has character $\chi_1 = \psi_\gamma$; the quotient has character $\chi_2 = \psi_\gamma\omega$.

Twisting $E[3]$ by $\chi_2$ gives an extension of the trivial character by $\omega$. Its action on the three projective lines other than the invariant line has affine form

$$z \longmapsto \omega(g)z + c(g), \qquad [c] \in H^1(K, \mathbf{F}_3(\omega)).$$

The cubic factor of (52) parametrizes those three lines. Depressing it gives

$$P = -2a - 5, \qquad Q = \frac{34a + 55}{9}, \qquad \frac{Q^2}{4} + \frac{P^3}{27} = \frac{50a}{9} + \frac{1075}{108} = \left(\frac{5(4a+7)}{18}\right)^2.$$

One Cardano radicand is $-3a - 5 = -a\epsilon$, and the other is its inverse times a cube. The splitting field of the cubic is therefore $K(\sqrt[3]{-a\epsilon}, \zeta_3)$. The affine action on these roots identifies the actual extension line with $\langle [a\epsilon] \rangle$; changing an affine coordinate can multiply the cocycle by $-1$, which does not change the line. This identifies the extension, rather than only its quadratic resolvent. The class $[a\epsilon]$ is locally nonzero at 3, as the unit calculation below proves. Since $E[3]$ is finite-flat at 3, the upper bound is attained. Conjugating the curve supplies the opposite orientation; the same calculation below shows that its local line is unchanged.

Finally let $k/\mathbf{F}_3$ be finite, and let $c$ be a $k$-valued extension whose underlying $\mathbf{Z}_3$-module is finite-flat. Pull back along the $\mathbf{F}_3$-line in its quotient. For every $\mathbf{F}_3$-linear functional $\lambda : k \to \mathbf{F}_3$, quotient the resulting kernel by $\ker(\lambda)\chi_1$. The resulting two-dimensional $\mathbf{F}_3$-extension is finite-flat, by stability under subquotients, and has class $\lambda \circ c$. Each coordinate class lies in the line just found; therefore $c$ lies in that line tensored with $k$. Conversely scalar extension of a finite-flat extension is finite-flat by direct sums and subquotients. This argument avoids assuming that arbitrary generic coefficient endomorphisms extend to a chosen finite-flat model. □

### 28.6.3 The global plane and local evaluations

Write classes in $K^\times/K^{\times 3}$ additively and put

$$b = [a\epsilon], \qquad s = [5], \qquad d = [5a\epsilon] = s + b.$$

**Lemma 28.35** (The global finite-flat extension plane)**.** *The global classes unramified outside* $\{3, 5\}$ *and finite-flat at* 3*, for either orientation of* (49)*, form*

$$V = \langle b, s\rangle = \langle s, d\rangle. \tag{53}$$

*Quadratic conjugation acts by* $\sigma(s) = s$*,* $\sigma(d) = -d$*. If* $\mathfrak{q}$ *is the prime over* 5 *at which* $a$ *has valuation one, the two local restriction maps, in the basis* $(s, d)$*, are*

$$\ell_{\mathfrak{q}} = (1, -1), \qquad \ell_{\bar{\mathfrak{q}}} = (1, 1). \tag{54}$$

*Moreover*

$$\chi_1(\mathrm{Frob}_{\mathfrak{q}}) = 1, \qquad \chi_1(\mathrm{Frob}_{\bar{\mathfrak{q}}}) = -1, \tag{55}$$

*and the values of* $\chi_2$ *are the opposite ones.*

*Proof.* The class and unit calculation in Lemma 28.21 shows that the global cubic classes with the indicated support have basis

$$[\epsilon], \quad [\gamma], \quad [a], \quad [\bar{a}], \qquad \bar{a} = 1 - a,$$

and dimension four. We compute their local-cube kernel at 3 elementarily. Normalize $v_F(\gamma) = 1$ and put $U_i = 1 + \mathfrak{p}_F^i$. The identity $\epsilon^3 - 5a\epsilon = 9\epsilon$ gives

$$\frac{5a\epsilon}{\epsilon^3} = 1 - \frac{9}{\epsilon^2} \in U_4.$$

The 3-adic logarithm identifies cubing $U_2 \to U_4$ with multiplication by 3, so this map is an isomorphism. Thus $d$ is locally a cube. On the other hand

$$\epsilon = 1 + \gamma \in U_1 \setminus U_2, \qquad -a\epsilon = 1 - 3\epsilon \in U_2 \setminus U_3.$$

Every unit cube lies in $\{\pm 1\}U_3$. Therefore $[\epsilon]$ and $[a\epsilon]$ are independent local cubic classes. The valuation of $\gamma$ gives a third independent class. Since $\mu_3 \not\subset F$, the dimension of $F^\times/F^{\times 3}$ is $[F : \mathbf{Q}_3] + 1 = 3$. The local-cube kernel on the four-dimensional global space is consequently exactly $\langle d\rangle$. Lemma 28.34 now gives (53).

The norm identity $N(a\epsilon) = -5$ gives $\sigma(b) = s - b$, and hence $\sigma(d) = -d$. In particular $\sigma(b)$ has the same local class as $b$, since $d$ is locally zero and $-2 = 1$ in $\mathbf{F}_3$. At either 5-adic place, units have no nonzero cubic Kummer classes because $3 \nmid 5 - 1$. The valuations of $s$ are $(1, 1)$ and those of $d$ are $(2, 1)$, giving (54). Finally $\gamma$ reduces to 1 at $\mathfrak{q}$ and to 2 at $\bar{\mathfrak{q}}$; their quadratic residue symbols give (55). □

### 28.6.4 A universal pseudoring with explicit local conditions

Fix a finite residue coefficient field $k/\mathbf{F}_3$ containing that of the actual source, and put $W = W(k)$. Let $G = G_{K,\{3,5\}}$. We impose a stable condition $\mathcal{C}$ on finite-cardinality $\mathbf{Z}_3[G]$-modules: finite-flatness at 3, and the local algebra conditions at 5 specified below. Here stability means closure under isomorphisms, subquotients, and finite direct sums. Finite-flatness is stable by [86, §5.2]; an algebra annihilator condition is stable by definition.

The universal pseudodeformation ring with residual pseudorepresentation $\chi_1 \oplus \chi_2$ and condition $\mathcal{C}$ exists by [86, Theorem 2.5.5]. It is a complete noetherian local $W$-algebra. Impose in addition the closed condition that the determinant equals $\chi_3$, and call the resulting ring $R$. Every normalized actual source defines a point of $R$. Indeed, all finite quotients of a stable lattice in its crystalline weight-two representation are finite-flat at 3, and the exact local types at 5 satisfy the algebra identities below.

Because the residual pseudorepresentation is multiplicity-free, its universal Cayley–Hamilton algebra has generalized-matrix-algebra form

$$\mathcal{E} = \begin{pmatrix} R & B \\ C & R \end{pmatrix}.$$

The reducibility ideal $J$ is the image of $B \otimes_R C \to R$; see [86, Proposition 4.2.2]. Writing $\mathfrak{m}$ for the maximal ideal of $R$, the extension interpretation [86, Theorem 4.3.5] gives

$$(B/\mathfrak{m}B)^\vee \subseteq \mathrm{Ext}^1_{\mathcal{C}}(\chi_2, \chi_1), \qquad (C/\mathfrak{m}C)^\vee \subseteq \mathrm{Ext}^1_{\mathcal{C}}(\chi_1, \chi_2). \tag{56}$$

Before the fixed-determinant quotient these are equalities; the inclusions suffice afterwards. The construction is functorial for quadratic conjugation whenever the local conditions are exchanged.

For a normalized SP type, let $\alpha_v$ be its exact unramified quadratic twisting character. We impose

$$\big(g - \chi_3(g)\alpha_v(g)\big)\big(h - \alpha_v(h)\big) = 0 \qquad (g, h \in G_{K_v}). \tag{57}$$

These are relations in the completed group algebra. They hold on every extension of $\alpha_v$ by $\chi_3\alpha_v$, hence on every stable lattice in the actual Steinberg representation. The two residual Frobenius eigenvalues are distinct. Thus the relations allow ramification only in the orientation whose invariant character is $\chi_3\alpha_v$, and they fix the local characters exactly. The universal condition is permitted to contain points of zero monodromy; the actual source has nonzero monodromy and is included.

For a normalized SC3 type, choose arithmetic Frobenius $\Phi$ and a generator $T$ of pro-3 tame inertia, so that $\Phi T \Phi^{-1} = T^5$. Kill wild inertia and the prime-to-3 tame inertia, and impose

$$T^2 + T + 1 = 0, \qquad \Phi^2 + 5 = 0. \tag{58}$$

The actual normalized SC representation has inertial eigenvalues $\zeta_3, \zeta_3^{-1}$, exchanged by Frobenius. Its cyclotomic determinant gives $\Phi^2 = -5$, so it satisfies these relations. Replacing Frobenius by an inertial multiple, or changing the tame generator, preserves the conditions on this quotient. Thus quadratic conjugation exchanges the same local condition at the two places. In characteristic 3, its first relation becomes

$$(T - 1)^2 = 0. \tag{59}$$

### 28.6.5 The diagonal pseudotangent

**Lemma 28.36** (Vanishing of the reducible tangent)**.** *For either pure local menu, the ring $R$ satisfies*

$$\mathfrak{m}/(3R + \mathfrak{m}^2 + J) = 0. \tag{60}$$

*Consequently multiplication induces a surjection*

$$(B/\mathfrak{m}B) \otimes_k (C/\mathfrak{m}C) \twoheadrightarrow \mathfrak{m}/(3R + \mathfrak{m}^2). \tag{61}$$

*Proof.* A tangent in the reducible quotient $R/J$ has constituent characters

$$\chi_1(1+\varepsilon f), \qquad \chi_2(1-\varepsilon f), \qquad f \in H^1(G,k),$$

where the opposite signs follow from the fixed determinant. The constituent character deformations are themselves finite-flat modules at 3, by the stability of $\mathcal{C}$ and [86, Proposition 4.3.4].

Fontaine's bound for a finite-flat module killed by $p^n$ is triviality of $G^{(u)}$ for $u > e(n+1/(p-1))$; the convention in the cited statement is $G^{(u)} = G^{u-1}$, where the latter is classical upper numbering [39, pp. 515–516, Theorem A]. For $p = 3$, $n = 1$, and $e = 2$, the classical upper break is at most 2. The abelian cubic character $f$ therefore has conductor at most $\mathfrak{p}_3^3$. Its tame quadratic factor does not change positive ramification groups. At a prime over 5, a cyclic cubic character is unramified because $3 \nmid 5-1$.

We need only the ray group of conductor $\mathfrak{p}_3^3$ and its Frobenius classes at five. These have the following elementary calculation. The group $(\mathcal{O}_K/\mathfrak{p}_3^3)^\times$ has order 18. The unit $\epsilon = 1+\gamma$ has exact order 3 modulo $\gamma^3$, so the image of $\langle -1, \epsilon\rangle$ has order 6. Class number one then gives ray group $C_3$. The class of $\mathfrak{q} = (a)$ is nontrivial: $-a = 1-\gamma$ differs from $1, \epsilon, \epsilon^2$ with valuations $1, 1, 2$, respectively, so it does not belong to the unit image modulo $\gamma^3$. Its conjugate also has nontrivial class because conjugation acts as an automorphism of the ray group. Infinite places contribute only 2-primary data and do not affect this cubic quotient.

In the SP case, (57) fixes the two Frobenius characters, hence forces $f(\Phi) = 0$. In the SC case, $\Phi^2 = -5$ gives the same conclusion, since the residual Frobenius eigenvalues are $\pm 1$ and 2 is invertible. Therefore $f = 0$, proving (60). The ideal $J$ is generated by products from $B$ and $C$, and $\mathfrak{m}J \subseteq \mathfrak{m}^2$; this proves (61). □

### 28.6.6 The SP/SP action of quadratic conjugation

**Lemma 28.37** (The two SP signs agree)**.** *An irreducible actual source in the situation of Theorem 28.33 has equal normalized unramified Steinberg signs at the two primes over* 5*. For these signs, quadratic conjugation acts trivially on the relative cotangent space of* $R$*.*

*Proof.* Suppose first that both signs are $-1$. The invariant monodromy character is $\chi_3\alpha_v$ and reduces to the Frobenius eigenvalue $-\alpha_v$. Using (55), the permitted global extension spaces in (56) are contained in

$$\ker \ell_{\bar{\mathfrak{q}}} = \langle b\rangle, \qquad \ker \ell_{\mathfrak{q}} = \langle \sigma(b)\rangle, \tag{62}$$

respectively. If both signs are $+1$, the roles of the lines are exchanged. If the signs are opposite, one orientation must be unramified at both 5-places. Its global extension space is then contained in $\ker \ell_{\mathfrak{q}} \cap \ker \ell_{\bar{\mathfrak{q}}} = 0$. Nakayama gives $B = 0$ or $C = 0$, so $J = 0$ and every point of $R$ is globally reducible. This contradicts the actual irreducible source. Equivalently, Ribet's lattice lemma would require nonsplit residual extensions in both orientations, while one of them has no permitted nonzero extension. The signs therefore agree.

Quadratic conjugation preserves the resulting local conditions, interchanges the two residual characters, and interchanges the two lines (62). Its action on their product is $+1$. One can check the sign directly from $b = d - s$ and $\sigma(b) = s - b = -(d+s)$. For a basis-independent verification, choose a lift $h \in G_{\mathbf{Q}}$ of $\sigma$ which fixes $\zeta_3$; it exists since $K \cap \mathbf{Q}(\zeta_3) = \mathbf{Q}$. The coefficient action on $\omega$ is then trivial, and $\chi_1(h^2) = \chi_2(h^2)$. Squared transport is inner by a residual scalar, so the two exchange factors multiply to $+1$. The surjection (61) now shows that quadratic conjugation fixes the entire relative cotangent space. □

### 28.6.7 The SC/SC action of quadratic conjugation

**Lemma 28.38** (The SC anti-invariant product vanishes)**.** *For the normalized SC/SC menu, quadratic conjugation acts trivially on the relative cotangent space of $R$.*

*Proof.* Both extension spaces in (56) are subspaces of $V \otimes_{\mathbf{F}_3} k$. Hence the tensor surjecting onto the relative cotangent space is a quotient of $V^\vee \otimes V^\vee$. Quadratic conjugation swaps the factors and acts on each by $s \mapsto s$, $d \mapsto -d$. The scalar in this statement is fixed by the same choice of $h$ as in the preceding proof. The local equation (59) kills the image of $\ell_v \otimes \ell_v$ in the cotangent space. Indeed, in the universal generalized matrix algebra write

$$T - 1 = \begin{pmatrix} x & b_T \\ c_T & y \end{pmatrix}, \qquad x, y \in \mathfrak{m}.$$

Its upper-left squared entry is $x^2 + b_T c_T$. The exact relation over $R$ is $(T-1)^2 = -3T$, so $b_T c_T \in \mathfrak{m}^2 + 3R$; therefore $b_T c_T$ is zero in $\mathfrak{m}/(3R + \mathfrak{m}^2)$. The classes of $b_T$ and $c_T$ are evaluation on local inertia, which are the functionals in (54), up to a harmless nonzero change of local generator.

The evaluations in (54) are $s^\vee - d^\vee$ and $s^\vee + d^\vee$. Thus Lemma 12.3 makes the action on the relative cotangent trivial. The local calculation above concerns products of opposite offdiagonal extensions, not only traces of representation tangents. □

Quadratic conjugation defines a $W(k)$-linear involution of $R$: its square is induced by an inner automorphism of $G_K$, which does not change a pseudorepresentation. Lemmas 28.37 and 28.38, followed by Lemma 12.2, show that this involution is the identity. Consequently every normalized actual source satisfies

$$\operatorname{tr} r_3(g) = \operatorname{tr} r_3(hgh^{-1}) \qquad (g \in G_K). \tag{63}$$

No assertion that $R$ is finite over $W(k)$, that its whole cotangent space is zero, or that its traces are rational is needed. The essential fact is that an involution of order prime to the residue characteristic cannot survive when its anti-invariant cotangent is zero.

*Completion of the proof of Theorem 28.33.* Strong compatibility transports (63), first at every good Frobenius, to all companions of the normalized system. Chebotarev identifies the semisimplifications of the system and its quadratic conjugate. In particular its absolutely irreducible 5-adic member is invariant under $\sigma$.

Lemma 12.4 makes the normalized system modular. Undoing the finite quadratic normalization proves the claim. □

*Remark* 28.39. The seed has $a_v = -1$ at both places over five and $a_{\mathfrak{p}_3} = 0$. The displayed identities determine its local Kummer class and the local-cube kernel explicitly. The source proof uses these finite-flat and pseudodeformation calculations and no dyadic modularity assumption.

The exact calculation is supplied in [20, `d21`].

## 28.7 Completion of the bounded source theorem

**Theorem 28.40.** *Assume GRH. Let $\mathcal{R}$ be a potentially modular compatible system over $K$, of determinant cyclotomic and labelled weights $\{0, 1\}$, with conductor supported at five. Suppose each five-place has one of the bounded types* SC *or* SP*. Then $\mathcal{R}$ is modular.*

*Proof.* The mixed case is Proposition 28.1. A pure special source with reducible five-adic residue is covered directly by Proposition 28.2. For the other pure sources, consider the three-adic member, which is crystalline at three. If its residue remains irreducible on $G_H$, Proposition 28.20 applies. If the residue is irreducible but its restriction is reducible, the dihedral source theorems of the preceding sections apply: induction from $H$ for the cuspidal pair, and fixed-character rigidity followed by descent under quadratic conjugation for the special pair.

Otherwise Lemma 28.21 gives the complete constituent list. The two unramified-at-three possibilities are modular by Proposition 28.22. In the remaining two, twist by $\psi_5$ when necessary to remove the common quadratic inertia at five. This twist is unramified at three and preserves the determinant and crystalline condition. The nonordinary reducible source theorem proved above gives invariance under quadratic conjugation and modularity. These alternatives are exhaustive for every finite coefficient field and every residual lattice. □

**Theorem 28.41** (Unrestricted residual base at five)**.** *Assume GRH. Every irreducible representation*

$$\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_5), \qquad \det \bar{r} = \bar{\chi}_5,$$

*unramified outside five is modular. No local finite-flatness, tameness, coefficient-field or conductor-exponent restriction is imposed.*

*Proof.* Solvable projective image is handled by an odd Artin lift and solvable Artin automorphy, with weight raising to obtain a regular residual witness. In the nonsolvable case the restriction to $G_{K(\zeta_5)}$ is irreducible. The exceptional projective-$\mathrm{PGL}_2(\mathbf{F}_5)$ condition cannot occur: $[K(\zeta_5) : K] = 4$, whereas the abelianization of $\mathrm{PGL}_2(\mathbf{F}_5)$ has order two.

Lemma 4.17 provides actual weight-two local lifts at both five-places with determinant $\chi_5$. Prescribe their components and no ramification elsewhere. Theorem 4.2.1 of [35] gives a global lift. Snowden's potential-modularity and compatible-system construction gives a potentially modular weight-two system with the selected menu. At a selected special component, the purity observation in Section 2 retains nonzero monodromy. The system is crystalline at two and three by all-place compatibility. Theorem 28.40 makes the actual system modular, and its five-adic reduction is the original $\bar{r}$. □

## 28.8 The arbitrary-conductor transfer

We verify the terminal character condition and apply Theorem 1.3 with reset prime five. Its terminal source is allowed arbitrary regular weights.

### 28.8.1 Characters supported at five

**Lemma 28.42.** *Every finite characteristic-five character of $G_K$ unramified outside five extends to $G_{\mathbf{Q}}$. If a representation of determinant $\bar{\chi}_5$ and support at five is reducible, its constituent quotient is exactly $\bar{\chi}_5$ or $\bar{\chi}_5^{-1}$.*

*Proof.* Lemma 6.7 gives the quotient and writes a supported character as a cyclotomic power times a narrow class character. The latter is invariant under quadratic conjugation because the narrow class group has order two. It extends across the index-two group by choosing a square root of its value on the square of a lift of the nontrivial automorphism. The extension has finite image. □

*Remark* 28.43. The generic mixed-case theorem of [24, Theorem A] excludes the local ratio $\bar{\chi}_5^{\pm 1}$. It therefore does not replace the residually reducible endpoint used here. At three its

complete-splitting hypothesis also fails. The terminal argument uses the locally irreducible source theorems of Zhang, not the generic mixed theorem.

*Completion for $d = 21$ in Theorem 1.2.* Two is inert and five splits. Theorem 28.41 supplies the supported irreducible residual base, and Lemma 28.42 gives the extending odd constituent quotient $\bar{\chi}_5^{\pm 1}$. Apply Corollary 6.6. □

# 29 The field $\mathbf{Q}(\sqrt{29})$

**Theorem 29.1.** *Assume GRH for Dedekind zeta functions, together with the compatible-system and lifting inputs of Section 2, including Theorem 6.4. Then Serre modularity holds over $\mathbf{Q}(\sqrt{29})$.*

The proof below also isolates the weaker analytic input. The region $\operatorname{Re} s > 23/25$ reduces the problem to ordinary dyadic sources with a totally imaginary residual point field and decomposition group $V_4$ at the remaining prime above five. The stronger region $\operatorname{Re} s > 4/5$ excludes that remaining case.

Put $K = \mathbf{Q}(\sqrt{29})$, $y = (1 + \sqrt{29})/2$ and $\epsilon = y + 2$. Thus $y^2 - y - 7 = 0$ and $\mathrm{N}(\epsilon) = -1$. We fix

$$v_+ = (y), \quad v_- = (y - 1), \qquad w_+ = (y + 1), \quad w_- = (y - 2).$$

These are the primes above seven and five, respectively. The primes two and three are inert, with completions $\mathbf{Q}_4$ and $\mathbf{Q}_9$. The class and narrow class numbers are both one. The unit $\epsilon$ has order three in $\mathbf{F}_4^\times$ and order eight in $\mathbf{F}_9^\times$. Its residues at $v_+, v_-$ are $2, 3$ and its residues at $w_+, w_-$ are $1, -1$.

## 29.1 Quadratic characters and the solvable residue

For $w = w_+$ put $g = y + 1$, and for $w = w_-$ put $g = y - 2$. In both orientations $\mathrm{N}(g) = -5$. The quadratic classes unramified outside $\{2, w\}$ form $W = \langle -1, \epsilon, 2, g \rangle$.

**Lemma 29.2.** *The localization $W \to K_2^\times / K_2^{\times 2}$ is injective. The classes containing a factor two have dyadic conductor three. Among the other classes, $[\epsilon g]$ is the unique nontrivial class of conductor zero; all the others have conductor two. The extension $K(\sqrt{\epsilon g})/K$ is ramified at $w$ only among finite primes, and is unramified and nonsplit at two.*

*Proof.* If $(-1)^a \epsilon^b 2^c g^d$ is a dyadic square, with exponents in $\{0, 1\}$, its valuation gives $c = 0$. Its rational norm is $(-1)^b (-5)^d$. The three nontrivial possibilities $-1, -5, 5$ are nonsquares in $\mathbf{Q}_2$, so $b = d = 0$. Finally $-1$ cannot be square in the unramified quadratic extension $K_2/\mathbf{Q}_2$, because $\mathbf{Q}_2(i)/\mathbf{Q}_2$ is ramified. This proves injectivity.

For $g = y + 1$ one has $\epsilon g = 9 + 4y = 1 + 4(y + 2)$, where $y + 2$ has trace one in $\mathbf{F}_4/\mathbf{F}_2$. For $g = y - 2$ one has $\epsilon g = y + 3 = y^2 - 4$, and $-1/y^2$ again has residue trace one. Both therefore give the unramified nonsplit dyadic quadratic extension. Their valuations at odd places give exactly the indicated prime of ramification. The quadratic conductor calculation over the unramified dyadic ring gives conductor two for the remaining unit classes and conductor three for a class of odd dyadic valuation. □

For later use, define the quadratic character $\xi_7 = \chi_{-y}$. Its only finite ramified prime is $v_+$, and it is trivial on $G_{K_2}$. Indeed,

$$(1 + 3y)^2 + y = 64 + 16y,$$

so $-y$ is a square modulo eight and hence a square in $K_2$ by Hensel's lemma. For the conjugate orientation the radicand is $y-1$. At $v_+$ its mod-seven inertia is $\omega_7^3$.

The following class and ray groups use the modulus which is the product of the dyadic primes of the indicated quartic field. The exact arithmetic certificate is supplied in [21]:

| $d$ | $\operatorname{disc}(K(\sqrt{d}))$ | class group | ray group |
|---|---|---|---|
| $-1$ | 13456 | $C_3$ | $C_3$ |
| $\epsilon, -\epsilon$ | $-13456$ | 1 | 1 |
| $g, -g$ | $-67280$ | $C_2$ | $C_2$ |
| $\epsilon g$ | 4205 | 1 | 1 |
| $-\epsilon g$ | 67280 | $C_2$ | $C_2$. |

Both choices of $g$ give these invariants. The degree-four class computations are certified by `bnfcertify`. In $H = K(i)$ the unique dyadic prime is $(1+i)$, of norm four, and is principal. The four primes above five have classes $1, 1, 2, 2$ in $\mathrm{Cl}(H) = C_3$.

Let $\theta$ be a nontrivial cubic class character of $H = K(i)$ and put

$$\bar{s}_3 = \operatorname{Ind}_H^K \theta.$$

Since $h_K = 1$, the nontrivial automorphism $\sigma$ of $H/K$ acts by inversion on $\mathrm{Cl}(H)$: for every ideal $\mathfrak{a}$ of $H$, the ideal $\mathfrak{a}\mathfrak{a}^\sigma = N_{H/K}(\mathfrak{a})\mathcal{O}_H$ is principal. Thus $\theta^\sigma = \theta^{-1} \neq \theta$, and $\bar{s}_3$ is absolutely irreducible with image $S_3$. The two choices of $\theta$ have isomorphic inductions.

## 29.2 Two local changes of type

**Lemma 29.3.** *Let $\bar{s} : G_{\mathbf{Q}_7} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ be the reduction, for any stable lattice, of a special representation of determinant $\chi_3$ with scalar quadratic inertial character $\kappa$. It has a lift of determinant $\chi_3$ and principal series type $\widetilde{\omega}^2 \oplus \widetilde{\omega}^4$ if $\kappa$ is unramified, and $\widetilde{\omega} \oplus \widetilde{\omega}^5$ if $\kappa$ is ramified. Here $\widetilde{\omega}$ has order six on tame inertia.*

*Proof.* Remove $\kappa$, put $\zeta = \zeta_3$ and normalize $v_3(3) = 1$. Residual inertia is trivial or unipotent, and Frobenius has characteristic polynomial $(X-\alpha)^2$, where $\alpha = \pm 1$. Suppose first that inertia is nontrivial. Write its matrix and that of Frobenius as $1 + \bar{t}N$ and $\alpha + \bar{u}N$, where $N^2 = 0$ and $\bar{t} \neq 0$. Choose integral lifts $t, u$, with $t$ a unit, and put $\delta = (\zeta^{-1} - \zeta)u/t$. By Hensel's lemma choose

$$c_2 - c_1 = \delta, \qquad c_1 c_2 = 7, \qquad c_1 \equiv c_2 \equiv \alpha.$$

Then

$$T = \begin{pmatrix} \zeta & t \\ 0 & \zeta^{-1} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} c_1 & u \\ 0 & c_2 \end{pmatrix}$$

commute, satisfy $T^3 = 1$ and $\det \Phi = 7$, and lift the given matrices. In particular $\Phi T \Phi^{-1} = T^7$.

If inertia is trivial and Frobenius is scalar, take $T = \operatorname{diag}(\zeta, \zeta^{-1})$ and $\Phi = cI$, where $c^2 = 7$ and $c \equiv \alpha$. If Frobenius has a nontrivial Jordan block, extend coefficients, choose $0 < v_3(\delta) < 1/2$, and let $u$ be a unit lifting the upper entry of Frobenius. Solve the same equations for $c_1, c_2$ and take

$$T = \begin{pmatrix} \zeta & (\zeta^{-1} - \zeta)u/\delta \\ 0 & \zeta^{-1} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} c_1 & u \\ 0 & c_2 \end{pmatrix}.$$

The upper entry of $T$ has positive valuation, so $\bar{T} = I$ while $\bar{\Phi}$ is the prescribed Jordan matrix. The same exact tame relation and determinant hold. Restoring $\kappa$ gives the asserted type. A ramified quadratic twist remains nontrivial modulo three, so the two output types correspond to the two input twists. □

Write $\sigma(r;b) = \operatorname{Sym}^r \mathbf{F}_p^2 \otimes \det^b$ for a Serre weight at a split coefficient prime. At seven the two types in Lemma 29.3 have reductions

$$\overline{\mathrm{PS}(2,4)}^{\mathrm{ss}} = \sigma(2;2) \oplus \sigma(4;4), \qquad \overline{\mathrm{PS}(1,5)}^{\mathrm{ss}} = \sigma(2;5) \oplus \sigma(4;1).$$

At five let $\tau_3$ be the cuspidal type with conjugate inertia characters of order three, and let $\tau_6$ be its ramified quadratic twist. Their reductions are

$$\overline{\sigma(\tau_3)}^{\mathrm{ss}} = \sigma(0;2) \oplus \sigma(2;3), \qquad \overline{\sigma(\tau_6)}^{\mathrm{ss}} = \sigma(0;0) \oplus \sigma(2;1).$$

Together with the genuine quadratic special types, these cover all central-trivial weights. For every actual residue of determinant $\omega_5$, Lemma 4.17 gives a weight-two lift of determinant $\chi_5$ and type $\tau_3$, $\tau_6$ or genuine special. In characteristic three, Lemma 4.18 replaces either cuspidal type at five by a genuine quadratic special type, preserving the complete residual representation and the determinant.

## 29.3 The residues in characteristic three

**Proposition 29.4.** *Let $\mathcal{S}$ be a weight-two system of cyclotomic determinant, good at three and away from one of the primes $v_+$ or $w_+$, with one of the local types just described. If its three-adic residue is reducible, or is irreducible but becomes reducible over $K(\zeta_3)$, then $\mathcal{S}$ is modular. If the residue remains irreducible over $K(\zeta_3)$, the same conclusion holds provided it has a parallel-weight-two modular witness.*

*Proof.* In the last case apply Lemma 2.2(ii) to the actual three-adic member.

Suppose the residue is reducible, with constituents $\psi_1, \psi_2$ and $\psi_1\psi_2 = \omega_3$. At the other prime their inertial characters have order at most two, since the order-three part of the type disappears on reduction. At three a constituent has inertia $\omega_2^a$, where $\omega_2$ has order eight. Reciprocity on $\epsilon$ has only signs from the other places. Since $\epsilon$ generates $\mathbf{F}_9^\times$, it forces $a = 0$ or $4$ modulo eight. The two residual inertias are therefore $1, \omega_3$. Lemma 4.3(i), with $e = 1 < 2$, makes the actual source ordinary. If $\psi_1/\psi_2$ is nontrivial on $G_{K(\zeta_3)}$, apply Lemma 4.5. Otherwise the quotient is exactly $\omega_3$: its local inertia excludes the trivial character. Its field is $K(\sqrt{-3})$, abelian over $\mathbf{Q}$; the quotient is odd and locally nontrivial at three. Skinner–Wiles [75, Theorem A] applies.

Finally suppose the global residue is irreducible but its cyclotomic restriction is reducible. Write it as $\operatorname{Ind}_M^K \theta$, where $M = K(\sqrt{-3})$. Locally $M/K$ is ramified quadratic over $K_3 = \mathbf{Q}_9$. The character $\theta$ has order prime to three and trivial wild inertia. For a tame generator $\tau$,

$$-\theta(\tau^2) = \det \operatorname{Ind}(\theta)(\tau) = \omega_3(\tau) = -1.$$

Thus $\theta$ is unramified locally and the residual restriction is $\lambda \oplus \lambda\omega_3$, with $\lambda$ unramified. Lemma 4.3(i) again makes the actual source ordinary. The two local residual characters are distinct and the CM inducing field is nonsplit at three. The ordinary witness with the required ordering is supplied by [90, Lemma 4.3(2)] or [2, Lemma 5.1.2]. Apply the corrected Skinner theorem as stated in [90, Theorem 4.2]. □

## 29.4 The chain through seven and five

**Proposition 29.5.** *To prove odd-characteristic Serre modularity over $K$, it suffices to modularize every weight-two system of cyclotomic determinant which is good away from $w_+$ and genuinely special there.*

*Proof.* Apply the initial lift of Proposition 2.7 and the one-sided preparation of Proposition 6.2 at seven. Consider the terminal member, and also every later seven-adic member good away from $v_+$ which retains crystalline weight two at $v_-$. If its global residue is reducible, a constituent has tame exponents $a, b$ at $v_+, v_-$. Since $2 = -3^{-1}$ in $\mathbf{F}_7$, reciprocity on $\epsilon$ gives $a \equiv b \pmod 3$. At $v_-$ finite flatness gives $b = 0$ or 1. The constituent quotient therefore has inertia $\omega_7^{\pm 1}$ with the same sign at both places. Dividing by that global cyclotomic character leaves a character unramified at all finite places. It is trivial because $h_K^+ = 1$. Thus the quotient is exactly $\omega_7^{\pm 1}$, and Theorem 6.4 modularizes the actual member. This also applies to the principal series sources below; their local restrictions are partitioned according to actual reducibility, without asserting that they are crystalline.

If the residue is irreducible, the retained weight-two place gives cyclotomic irreducibility by Lemma 4.6. Lemma 2.2(i) then supplies every backward implication at seven once a modular witness is available.

Choose a Serre weight $\sigma(r; b)$ at $v_+$. Its determinant gives $r + 2b \equiv 0 \pmod 6$, so $r \in \{0, 2, 4, 6\}$. If $r \leq 4$, the two possible values of $b$ differ by three. Twist the global residue by $\xi_7$ if necessary to obtain $b \equiv -r/2 \pmod 6$. The twist is ramified only at $v_+$ among finite places and does not change the determinant. The definition of this weight gives a crystalline lift at $v_+$ with weights

$$\{-r/2, 1 + r/2\}$$

and determinant $\chi_7$, after an unramified square-root correction. Retain the existing crystalline weight-two component at $v_-$. The Hodge gap is at most five, strictly below $7 - 1$, so these components are potentially diagonalizable. Cyclotomic irreducibility gives adequacy in characteristic seven. Lemma 2.6(ii) produces an everywhere-good system with weights two and at most six.

If $r = 6$, first twist by $\xi_7$ if necessary to get $b = 0$. Choose a semistable weight-two lift of trivial scalar type and globalize it by Lemma 2.6(i), retaining crystalline weight two at $v_-$ and the good conditions elsewhere. If the lift is crystalline, the required weight reduction is complete. Otherwise pass to its three-adic member. A reducible or cyclotomically reducible global residue gives modularity directly by Proposition 29.4. In the remaining case apply Lemma 29.3 at $v_+$ and globalize by Lemma 2.6(i), retaining the actual crystalline component at three. The resulting principal series types at seven have only $r = 2, 4$ among their Serre weights. Apply the preceding crystalline construction, unless a reducible seven-adic residue has already given modularity. The backward step at three uses Proposition 29.4; if the available witness first has regular weights, Lemma 2.1 makes its weight parallel two. Undo the finite twists after propagating modularity.

We have reached an everywhere-good system $\mathcal{S}_1$ of cyclotomic determinant and centered weights $(2, k)$, with $k \leq 6$. Choose a five-adic realization placing the weight two at $w_-$. Both local restrictions are crystalline. If the global residue is reducible, apply Lemma 4.4 after local cyclotomic twists: $k \leq 6 = 5 + 1$. The actual source is ordinary at both five-primes, so Lemma 4.5 applies over the abelian field $K$, with five unramified.

Otherwise Lemma 4.6 gives cyclotomic irreducibility. The five-adic menu stated above and Lemma 2.6(i) give a weight-two system of type $\tau_3$, $\tau_6$ or genuine quadratic special at $w_+$, good elsewhere. The characteristic-five exception is absent because $[K(\zeta_5) : K] = 4$.

If the type is cuspidal, pass to the three-adic member. The first two cases of Proposition 29.4 modularize it directly. In the remaining case prescribe the special lift of Lemma 4.18 at $w_+$ and retain the crystalline component at three. Lemma 2.6(i) supplies a system $\mathcal{S}_2$ with genuine special type at $w_+$ and weight two. Once $\mathcal{S}_2$ is modular, its weight-two witness gives the backward step at three. The backward step at five applies Lemma 2.2(i) to the crystalline member of $\mathcal{S}_1$. The earlier backward steps return to the original representation. □

## 29.5 The dyadic terminal

Let $\rho$ be the dyadic member of a weight-two system of cyclotomic determinant, good away from $w_+$, with scalar quadratic inertia and zero or nonzero monodromy there. The latter alternatives are denoted CR and SP, respectively. In particular $\rho$ is crystalline at the dyadic place, whose absolute ramification index is one. Its residual inertia at $w_+$ is trivial in the scalar case and has order at most two in the special case.

**Lemma 29.6.** *A solvable residue of $\rho$ has semisimplification $1 \oplus 1$ or is isomorphic to $\bar{s}_3$.*

*Proof.* Apply Lemma 11.16a with $S = \{w_+\}$. The odd dyadic ray quotient of $K$ is trivial, since $\epsilon$ generates $\mathbf{F}_4^\times$ and $h_K = 1$. Thus a reducible residue is scalar up to semisimplification. Otherwise its quadratic inducing field is $K(\sqrt{d})$. Fontaine's conductor bound for the quadratic quotient is at most two dyadically, so Lemma 29.2 excludes the factor two from $d$. The remaining seven classes lie in $\langle -1, \epsilon, g\rangle \setminus \{1\}$. Their odd dyadic ray quotients leave only $H = K(i)$ and its two inverse cubic class characters. Their inductions are isomorphic to $\bar{s}_3$. □

**Proposition 29.7.** *There is no continuous absolutely irreducible representation $\rho : G_K \to \mathrm{GL}_2(E)$, with $E/\mathbf{Q}_2$ finite, of determinant $\chi_2$, trivial residual semisimplification, crystalline weight two at the dyadic place, and scalar inertia at every odd prime other than $w_+$.*

*Proof.* Lemma 4.3(ii), with $p = 2$ and $e = 1$, makes the actual source ordinary at two. It only requires the two semisimplified residual characters to be unramified; the residual extension itself may be ramified. Apply Lemma 4.10 to the ordinary local line and $D = G_{K_2}$. We obtain a nonzero quadratic class split at two. At every other odd prime except $w_+$, scalar inertia has values $\pm 1$ by the determinant, hence reduces to the identity. The class therefore belongs to $W$. This contradicts the injective localization in Lemma 29.2. □

**Lemma 29.8.** *If the type at $w_+$ is genuinely special, the residue is not $\bar{s}_3$.*

*Proof.* The characteristic-zero Frobenius eigenvalues are $5\mu, \mu$ and become equal modulo two. For $\bar{s}_3$, Frobenius at $w_+$ has eigenvalues $\theta(P), \theta(P)^{-1}$, where $P$ is a prime of $H$ above $w_+$. Its nontrivial class in $\mathrm{Cl}(H)$ makes these distinct primitive cube roots. □

**Proposition 29.10.** *Suppose the residue of $\rho$ has image $\mathrm{SL}_2(\mathbf{F}_q)$, where $q = 2^m \geq 4$, and let $M$ be its point field. Then*

$$[M : \mathbf{Q}] = 2q(q^2 - 1), \qquad \mathrm{rd}(M) \leq 4\sqrt{29}\, N^{1/4} 2^{-2/q},$$

*where $N = 1$ if the residue is unramified at $w_+$ and $N = 5$ otherwise. In the ramified case the decomposition group at $w_+$ is $C_2$ or $V_4$, with prime densities $(5, 1/4)$ or $(25, 1/8)$. If $\rho$ is nonordinary at two, the stronger bound*

$$\mathrm{rd}(M) < 2^{4/3}\sqrt{29}\, N^{1/4}$$

*holds. If $q = 4$, the system is modular without an analytic hypothesis.*

*Proof.* Let $P$ be the image of dyadic wild inertia. If $P = 1$, tame inertia has order dividing three in the reducible local case and five in the irreducible local case, so its normalized different contribution is at most $4/5$. If inertia is trivial, the residual semisimplification is a sum of unramified characters, even when Frobenius has a nontrivial Jordan block. Lemma 4.3(ii) then makes the actual source ordinary.

If $P \neq 1$, its unique fixed line is stable under decomposition, and $|P| \leq q$. The tame diagonal has order one or three. For trivial tame diagonal, Fontaine's bound and the conductor–discriminant formula give contribution at most $2(1 - |P|^{-1})$. For diagonal order three, after

a tame extension of index three, the noninvariant Kummer grades are one and five, with conductors six and two. The first would contradict Fontaine's upper-break bound. Thus in this case the contribution is at most

$$\frac{2}{3}+\frac{2}{3}\left(1-\frac{1}{|P|}\right)<\frac{4}{3}\le 2-\frac{2}{q}.$$

These are the local cases of Lemma 11.12; see also [18, Section 1]. The trivial diagonal case also satisfies the uniform bound $2-2/q$. This proves the dyadic factor in the displayed inequality. At $w_+$ the residual inertia is trivial or $C_2$. In the latter case an arithmetic Frobenius commutes with its nontrivial involution, since the tame relation raises inertia to the fifth power. The centralizer of that involution in $\mathrm{SL}_2(\mathbf{F}_q)$ is its upper-unipotent subgroup, an elementary abelian two-group. The decomposition group, generated by inertia and a Frobenius, is therefore $C_2$ or $V_4$, with local indices $(e,f)=(2,1)$ or $(2,2)$. Normality over $K$ gives respectively $n/4$ primes of norm five or $n/8$ primes of norm twenty-five above this chosen place. The other prime above five may contribute additional primes of these norms. This proves the stated density lower bounds. Tame ramification of index two contributes the factor $5^{1/4}$.

If the tame diagonal is trivial and $P\neq 1$, both local residual constituents are unramified. Lemma 4.3(ii) then makes the actual source ordinary. Consequently a nonordinary source belongs to one of the other local cases, whose contribution is strictly below $4/3$, proving the stronger bound.

When $q=4$, the upper bound is less than 22.777, whereas the totally real lower bound in Lemma 13.13 is greater than 42.81537 in degree at least 120. Thus the point field is not totally real. The residual witness and actual-source dyadic lifting theorem in Lemma 2.4 prove modularity. This argument allows either ordinary or nonordinary dyadic sources. □

**Proposition 29.11.** *Suppose the ramified nonsolvable point fields just considered satisfy the zero-free hypothesis* $\operatorname{Re} s>23/25$ *used in Proposition 13.12. To prove odd-characteristic Serre modularity over* $\mathbf{Q}(\sqrt{29})$*, it then suffices to modularize systems with all the following properties:*

(i) *they have weight two and cyclotomic determinant, are good away from* $w_+$*, and are genuinely special at* $w_+$*;*
(ii) *their actual dyadic source is ordinary;*
(iii) *the dyadic residue has image* $\mathrm{SL}_2(\mathbf{F}_q)$ *with* $q\ge 8$*, its local semisimplified inertia at two is trivial, and its decomposition group at* $w_+$ *is* $V_4$*;*
(iv) *its residual point field is totally imaginary.*

*Proof.* Proposition 29.5 reduces the problem to the indicated special terminal. Lemma 29.6, Proposition 29.7 and Lemma 29.8 exclude the solvable residues. The case $q=4$ is modular by Proposition 29.10. The unramified nonsolvable fields are excluded by the unconditional comparisons in Proposition 13.12. Its zero-strip comparisons exclude the ramified $C_2$ case. Its unconditional nonordinary comparison excludes the local cases whose dyadic different contribution is less than $4/3$. In every remaining case $q\ge 8$, the group at $w_+$ is $V_4$ and the local diagonal inertia at two is trivial. Lemma 4.3(ii) proves ordinarity of the actual source.

The extension $M/K$ is Galois, although $M/\mathbf{Q}$ need not be. Above each real place of $K$, either all embeddings of $M$ are real or none is. Consequently $r_1(M)/[M:\mathbf{Q}]\in\{0,1/2,1\}$. If $M$ had a real embedding, Lemma 13.13, with $a=21/50$, $b=10$, degree at least 1008 and norm-25 density $1/8$, would give $\mathrm{rd}(M)>40.87041$. This exceeds the uniform upper bound $4\sqrt{29}\,5^{1/4}<32.21080$. Thus $M$ is totally imaginary. □

### 29.6 The remaining analytic condition

**Corollary 29.12.** *Assume the hypotheses of Proposition 29.11. Suppose, in addition, that every remaining point field in that proposition has no nontrivial zeta zeros with* $\operatorname{Re} s > 4/5$. *Then Serre modularity holds over* $\mathbf{Q}(\sqrt{29})$.

*Proof.* For a remaining field, Proposition 29.10 gives

$$[M : \mathbf{Q}] = n(q), \qquad \operatorname{rd}(M) \le T_{29} 2^{-2/q}, \qquad \#\{\mathfrak{P} : \mathrm{N}\mathfrak{P} = 25\} \ge n(q)/8, \quad q \ge 8.$$

Take $a = 3/10$ in Proposition 13.2. The following certified comparisons [21] exclude all powers of two $q \ge 8$:

| $q$ | $n_0$ | $b$ | $L$ | $U$ |
|---|---|---|---|---|
| 8 | 1008 | 8 | 27.78690 | 27.08595 |
| 16 | 8160 | 12 | 31.11364 | 29.53744 |
| $\ge 32$ | 65472 | 18 | 32.53978 | 32.21080 |

Proposition 29.11 completes the implication. □

*Proof of Theorem 29.1.* GRH implies both zero-free hypotheses in Proposition 29.11 and Corollary 29.12 for every residual point field to which they are applied. The corollary therefore proves Serre modularity. This implication does not use the zero-free theorems of Appendix B. □

This corollary is conditional on the stronger zero-free region. Appendix B proves the boundary 2759/3000 using the formally verified inputs; it does not prove the boundary 4/5.

## 30 The characteristic-three base over $\mathbf{Q}(\sqrt{37})$

Put $K = \mathbf{Q}(\sqrt{37})$. We use one wild type and one bounded quadratic or quartic type at the two three-primes. The compatible system thereby has a finite-flat characteristic-two member whose local fields fit inside a fixed solvable auxiliary extension.

**Theorem 30.1.** *Assume GRH. Every continuous irreducible representation* $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ *with determinant* $\overline{\chi}_3$, *unramified outside three, is modular.*

Write

$$w = \frac{1 + \sqrt{37}}{2}, \qquad \epsilon = 5 + 2w, \qquad \pi = 3 + w, \qquad \pi' = 4 - w.$$

Thus $w^2 - w - 9 = 0$, $N_{K/\mathbf{Q}}(\epsilon) = -1$ and $\pi\pi' = 3$. Put $\mathfrak{p} = (\pi)$ and $\mathfrak{p}' = (\pi')$. Both completions above three are $\mathbf{Q}_3$, and the unique dyadic completion is the unramified quadratic extension of $\mathbf{Q}_2$.

**Lemma 30.2.** *The ordinary and narrow class numbers of* $K$ *are one. Every finite character* $G_K \to \overline{\mathbf{F}}_3^{\times}$ *unramified outside three has square one.*

*Proof.* The Minkowski bound is $\sqrt{37}/2 < 3.05$. There is no ideal of norm two, and the two ideals of norm three are principal. Thus the ordinary class number is one. The unit $\epsilon$ has norm $-1$, which also gives narrow class number one. The character assertion follows from Lemma 4.20. □

**Proposition 30.3.** *Let $\overline{\rho}$ satisfy the hypotheses of Theorem 30.1, and suppose its image is nonsolvable. It lifts to a totally odd, potentially modular, strongly compatible system $\mathcal{R} = (r_\lambda)$ with determinant $\chi$ and weights $\{0,1\}$. The system is unramified away from three, has type $\tau_+$ at $\mathfrak{p}$, and has one of the two local conditions of Lemma 4.23 at $\mathfrak{p}'$.*

*Proof.* Use the construction in Proposition 23.2, with the local points of Lemmas 4.22 and 4.23 at $\mathfrak{p}$ and $\mathfrak{p}'$. Nonsolvability again gives the cyclotomic irreducibility required by Lemma 2.6(i); Section 2 retains special monodromy. □

Use the notation following Proposition 23.2, with wild place $\mathfrak{p}$ and other place above three $\mathfrak{p}'$.

## 30.1 Ramification at two

For a finite Galois extension $L/F$ of dyadic local fields, $d(L/F)$ denotes the valuation of its different, normalized by $v_F(2) = 1$. Thus $d(L/F)$ is the local contribution to the exponent of two in the root discriminant. We use Serre's upper numbering for ramification groups.

**Lemma 30.4.** *Let $F/\mathbf{Q}_2$ be unramified and let $J$ be a finite flat commutative group scheme over $\mathcal{O}_F$ killed by two. The field generated by $J(\overline{F})$ has no positive upper ramification break greater than one, and its normalized different is strictly less than two.*

*Proof.* Fontaine's Theorem A and its corollary [39, pp. 515–516] apply to finite flat group schemes of any rank. His notation is $G^{(u)} = G^{u-1}$, where the right side uses Serre's upper numbering. For a group scheme killed by $p^n$ over a field of absolute ramification index $e$, his bound is

$$G^{(u)} \text{ acts trivially if } u > e\left(n + \frac{1}{p-1}\right).$$

Here $e = n = 1$ and $p = 2$. Thus $G^v$ acts trivially for $v > 1$. The different bound is the cited corollary. □

**Corollary 30.5.** *A compositum of finite Galois extensions of $F$ having no upper break greater than one has the same property, and its normalized different is strictly less than two.*

*Proof.* Upper-numbering ramification is compatible with quotients. If all quotient fields have trivial groups above one, so does their compositum. The integral formula for the different then gives $d < 2$; more precisely, for tame inertia order $t$ and wild inertia order $Q$, one has $d \le 2 - (t+1)/(tQ)$. □

## 30.2 An explicit solvable auxiliary field

Put

$$Q(X) = X^4 - 2X^3 + (3-2w)X^2 + (4+2w)X - w - 2, \qquad C = K(\sqrt{\pi}, \sqrt{\pi'}, \sqrt{-\epsilon}),$$

and let $H$ be the splitting field of $Q$ over $K$. We use $N = CH$.

**Proposition 30.7.** *The extension $N/K$ is solvable Galois of degree 96, is unramified outside two and three, and has no dyadic upper break greater than one. Its completions at $\mathfrak{p}$ and $\mathfrak{p}'$ are $E_+$ and $E'$, respectively.*

*Proof.* The cubic resolvent and discriminant of $Q$ are

$$R(T) = T^3 + (2w-3)T^2 + 4 - 4w, \qquad \operatorname{disc}(Q) = \operatorname{disc}(R) = 144\delta, \quad \delta = 99 - 28w = -\epsilon^{-3}\pi^3.$$

The quartic is irreducible modulo the prime above eleven with $w \equiv 7$: its reduction is $X^4 - 2X^3 + 7X + 2$. This polynomial is relatively prime to $X^{121} - X$ and divides $X^{14641} - X$ over $\mathbf{F}_{11}$, so is irreducible. Hence $Q$ is irreducible over $K$.

At $\mathfrak{p}$ we have $w \equiv 18 \pmod{27}$. The shifted polynomial is

$$Q(Y-1) = Y^4 - 6Y^3 + (15 - 2w)Y^2 + (6w - 12)Y - 5w.$$

Its Newton polygon has segments of slopes $-1$ and $-1/3$, of lengths one and three. It therefore has one root in $\mathbf{Q}_3$ and an irreducible totally ramified cubic factor. The cubic roots have valuation $1/3$, whereas the linear root has valuation one. The resultant of these factors has valuation one. Since $v_{\mathfrak{p}}(\mathrm{disc}(Q)) = 5$, the cubic factor has discriminant exponent three. Its root is a uniformizer, so this is also the field discriminant exponent. Its discriminant square class is that of $-\epsilon\pi$, namely 3. Thus its splitting field is totally ramified $S_3$, with normalized different $7/6$ and quadratic subfield $\mathbf{Q}_3(\sqrt{3})$. By Lemma 4.22 it is $E_+$. The quartic's linear factor shows that its full local splitting field is the same field.

The global quartic group is transitive and contains this point-stabilizer $S_3$, so it is $S_4$. Its unique quadratic subfield is $D = K(\sqrt{-\epsilon\pi})$. Valuations at $\mathfrak{p}, \mathfrak{p}'$ and the mixed real signs of $-\epsilon$ give $[C : K] = 8$. The intersection $H \cap C$ is an elementary abelian two-extension and hence is $D$, the maximal such quotient field of $S_4$. Therefore $[N : K] = 24 \cdot 8/2 = 96$.

At $\mathfrak{p}'$ the reduction of $Q$ is $X^2(X-1)^2$. Hensel factorization gives two quadratic factors with unit resultant. Their discriminant valuations are each at least one and sum to $v_{\mathfrak{p}'}(\mathrm{disc}(Q)) = 2$, so each is one. Both define ramified quadratic extensions; the local splitting field lies in $E'$. The local square classes of $\pi, \pi', -\epsilon$ are respectively $3, 1, 1$ at $\mathfrak{p}$, and $1, 3, -1$ at $\mathfrak{p}'$. Thus adjoining $C$ leaves $E_+$ unchanged and makes the other completion exactly $E'$.

At two put $h = X^2 + X + 1 - w$. Direct calculation gives

$$Q = h^2 + 2g, \qquad g = -2X^3 + (2w+1)X - 6.$$

Over the residue field $\mathbf{F}_4$, the polynomial $\bar{h}$ is irreducible, since $\mathrm{Tr}_{\mathbf{F}_4/\mathbf{F}_2}(1 - \bar{w}) = 1$, and $\bar{g} = X$ is relatively prime to it. Dedekind's index criterion therefore shows that the quartic order is maximal at two. Its local field has ramification index two and residue degree two, and its relative discriminant exponent is four. These statements also follow by passing to the unramified quadratic extension defined by $h$ and using the Eisenstein factor of $Q$ near a root of $h$.

The reduction of $R$ is $T^2(T+1)$, so it has a local linear factor and its local splitting group has order at most two. The local splitting group of $Q$ is consequently contained in a subgroup of order eight of $S_4$, and is a two-group. Thus its lower ramification groups satisfy $G_0 = G_1$. In the permutation action on the four roots, inertia has two orbits, since the quartic field has residue degree two. The terms for $G_0$ and $G_1$ in the Artin conductor are therefore two each. They exhaust its discriminant exponent four, so faithfulness of the permutation action gives $G_2 = 1$. The largest upper break is at most one.

Finally, the radicands defining $C$ are dyadic units. Every corresponding quadratic character has conductor at most two and upper break at most one. Corollary 30.5 gives the same bound for $N$. The displayed relative discriminant and the radicands also show that there is no ramification away from two and three. □

## 30.3 The discriminant comparisons

Let $L/K$ be finite Galois and unramified outside two and three. If its dyadic completion has no upper break greater than one, its dyadic different is less than two. If its completion at $\mathfrak{p}$ is $E_+$ and its completion at $\mathfrak{p}'$ is contained in $E'$, the discriminant tower gives

$$\mathrm{rd}(L) < 4\sqrt{37}\,3^{5/6} < 60.78017. \tag{64}$$

Each three-adic exponent is multiplied by one half, since $[K : \mathbf{Q}] = 2$.

**Proposition 30.8.** *Assume GRH for the Dedekind zeta function of $L$. If its completions above $\mathfrak{p}, \mathfrak{p}'$ are $E_+, E'$ and $n = [L : \mathbf{Q}] \geq 96768$, then $\mathrm{rd}(L) > 60.80125$.*

*Proof.* Use the explicit formula of Section 13 with $B_b \leq B_b^*$ and the finite Euler sums. There are $n/12$ primes of norm three and $n/8$ of norm nine. Thus $\rho_3 = 1/12$ and $\rho_9 = 1/8$. Discard the real-place term. At $b = 56/5$ and $n = 96768$, the directed lower bound is 60.8012536506. It increases with $n$ because $C_b > 0$. □

**Proposition 30.9.** *Assume GRH. The image of $\overline{r}_2$ constructed in Proposition 30.3 is either solvable or conjugate to $\mathrm{SL}_2(\mathbf{F}_4) \simeq A_5$.*

*Proof.* Dickson's classification makes a nonsolvable finite subgroup of $\mathrm{SL}_2(\overline{\mathbf{F}}_2)$ conjugate to $\mathrm{SL}_2(\mathbf{F}_q)$, for $q = 2^r$, $r \geq 2$. Suppose $q \geq 8$. This group is nonabelian simple, so its kernel field $M$ is linearly disjoint over $K$ from the solvable extension $N$. The compositum $L = MN$ satisfies $[L : \mathbf{Q}] = 2 \cdot 96 \cdot q(q^2 - 1) \geq 96768$. By Lemma 30.4, Proposition 30.7 and Corollary 30.5, its dyadic different is less than two. Its completions at three are exactly $E_+$ and $E'$. Equation (64) contradicts Proposition 30.8. □

## 30.4 Modularity of the remaining images

The sign of the type $\tau_+$ gives a direct treatment of the dihedral case. We make the required quadratic calculation explicit.

**Lemma 30.10.** *Let $D/K$ be quadratic, unramified at two and at every finite place away from three. If it ramifies at $\mathfrak{p}$, its completion there is $\mathbf{Q}_3(\sqrt{-3})$.*

*Proof.* Class number one shows that $D = K(\sqrt{\alpha})$ with $\alpha$ a product of $-1, \epsilon, \pi, \pi'$ modulo squares. At the dyadic completion the residue field is $\mathbf{F}_4$. In the basis $1, \bar{w}$, Lemma 11.15 gives the columns

| | $-1$ | $\epsilon$ | $\pi$ | $\pi'$ |
|---|---|---|---|---|
| $1$ | 1 | 0 | 0 | 1 |
| $\bar{w}$ | 0 | 1 | 1 | 1 |

.

For example these are obtained by cubing the four units modulo four and dividing their differences from one by two. The kernel is generated by $[\epsilon\pi]$ and $[-\pi\pi']$. Thus the four global classes that are unramified at two are

$$1, \qquad \epsilon\pi, \qquad -3, \qquad -\epsilon\pi'.$$

At $\mathfrak{p}$ we have $w = 18 \pmod{27}$, hence $\pi/3 = 1 \pmod 3$ and $\epsilon = -1 \pmod 3$. The two ramified classes in this list, $\epsilon\pi$ and $-3$, therefore both have local square class $-3$. □

**Proposition 30.11.** *Assume GRH. The characteristic-two member $r_2$ of Proposition 30.3 is modular.*

*Proof.* Suppose first that the residual image is solvable. Since $\overline{r}_2$ is irreducible and has determinant one, its image is an odd dihedral group $D_{2m}$. Let $D/K$ be its quadratic inducing extension. Its local quadratic field at $\mathfrak{p}$ is $\mathbf{Q}_3(\sqrt{3})$, by Lemma 4.22. Lemma 30.10 shows that $D/K$ ramifies at the unique dyadic place.

The dyadic wild inertia image is a nontrivial normal two-subgroup of the local decomposition image in $D_{2m}$. It is generated by a reflection. The normalizer of a reflection in an odd dihedral group is its own subgroup, so the entire local residual image is $C_2$. Its semisimplification is trivial,

and $K_2/\mathbf{Q}_2$ is unramified, so Lemma 4.3(ii) makes the actual crystalline member $r_2$ ordinary. Allen's nearly ordinary residually dihedral theorem [2, Theorem 5.2.1 and Lemma 5.1.2] now gives modularity. The extra condition when the inducing quadratic field is CM is satisfied because the dyadic place ramifies in $D$; in particular it does not split. Allen's dihedral argument supplies the ordinary modular witness required in that theorem.

In the nonsolvable case Proposition 30.9 leaves $\mathrm{SL}_2(\mathbf{F}_4)$. The kernel field has degree 120 and

$$\mathrm{rd}(M) \le 4\sqrt{37}\,3^{5/6}2^{-2/4} < 42.97807$$

by Lemma 11.12. The classical lower bound 43.14954 in Lemma 13.4 excludes a totally real kernel without any prime term or zero-free hypothesis. Lemma 2.4 therefore applies to $r_2$, whether ordinary or nonordinary; no condition $K_2 = \mathbf{Q}_2$ is imposed. □

*Proof of Theorem 30.1.* Solvable projective images are modular by automorphic induction and Langlands–Tunnell, with the characteristic-three lifting and twisting argument used in [3, proof of Theorem 2.5]. We may therefore suppose that the image is nonsolvable. Proposition 30.3 constructs a compatible system lifting $\overline{\rho}$. Proposition 30.11 makes its characteristic-two member modular. Strong compatibility, Chebotarev and strong multiplicity one then identify the system with the system of that Hilbert modular form. Its characteristic-three residual representation is $\overline{\rho}$, which is consequently modular. □

## 31 Completion at the odd terminal primes

*Completion for $d = 10, 11, 14, 19, 37$ in Theorem 1.2.* For each row the first citation gives the irreducible supported base and the second gives the extending quotient in the reducible case. Two does not split in these fields, and the indicated terminal prime splits. Apply Corollary 6.6.

| $d$ | $\ell$ | Irreducible base | Constituent quotient |
|---|---|---|---|
| 10 | 3 | Theorem 23.1 | Lemma 23.7 |
| 11 | 5 | Proposition 24.13 | Lemma 6.7 |
| 14 | 5 | Theorem 25.14 | Lemma 6.7 |
| 19 | 3 | Theorem 27.1 | Lemma 6.7 |
| 37 | 3 | Theorem 30.1 | Lemma 4.20 |

For $d = 10$, the quotient is $\omega_3$ or $\omega_3\nu$ with $\nu$ the ordinary Hilbert class character, which is totally even. At the other places in the table, total oddness follows from the cyclotomic determinant. For $d = 15$ use Propositions 2.7 and 6.2 at seven and Theorem 26.1. The full reduction for $d = 21$ is in Section 28, and the result for $d = 29$ is Theorem 29.1. □

## 32 The four remaining split-dyadic fields

**Theorem 32.1.** *Assume GRH. For $d \in \{57, 65, 73, 89\}$, every absolutely irreducible characteristic-two representation of $G_K$ unramified outside two is modular.*

*Proof.* By Lemma 14.5, it suffices to treat $\mathrm{SL}_2(\mathbf{F}_q)$ with $q \ge 8$.

Use the functions $F_b, J_b, B_b, C_b$ of Section 13. Put $a = \pi/(2b)$ and let

$$\log \mathcal{L}(n,b,k) = \gamma + \log(8\pi) - \sum_{j=0}^{3}\left\{\frac{1}{j+1/2} - J_b(j+1/2)\right\} - \frac{2a^2}{49} - \frac{C_b}{n} + \frac{k\log 4}{8}\sum_{m\ge 1} 4^{-m/2}F_b(m\log 4).$$

The tail estimate in that theorem makes this a lower bound for the root discriminant, under GRH, when the indicated norm-four Euler contributions are present. It increases with $n$ at fixed $b, k$. We shall also write $B(G), C(G)$ for the corresponding integrals with an even test function $G$.

For $M$ in Lemma 14.4, the following strict comparisons hold with $n = 8q(q^2-1)$ and $d \le 89$:

| $q$ | $k$ | $b$ | $\mathcal{L}(n,b,k) >$ | upper bound $<$ |
|---|---|---|---|---|
| 8 | 1 | 7.8 | 38.07399 | 36.13605 |
| 16 | 0 | 9.8 | 36.64088 | 36.13605 |
| $q \ge 32$ | 0 | 12 | 38.91320 | 37.73593 |

The last row drops Euler terms and uses $n \ge 261888$ and $\mathrm{rd}(M) \le 4\sqrt{89}$. In the first two rows, raising $k$ by one increases the logarithmic lower bound by $\frac{\log 4}{8} \sum_m 4^{-m/2} F_b(m \log 4)$, exceeding $\log 2/(2q)$ already in the first summand since $F_b(\log 4) > 1/2$. Thus the table excludes every case except $q = 8, k = 0$. The exact calculation is supplied in [20, `split_dyadic`].

Let $\sigma$ be the nontrivial automorphism of $K/\mathbf{Q}$. If $L^\sigma \ne L$, simplicity and Goursat's lemma give $[LL^\sigma : \mathbf{Q}] = 2 \cdot 504^2 = 508032$. Each local constituent has upper breaks at most one, since $k = 0$. The same is true of the compositum by compatibility of upper numbering with quotients, so $\mathrm{rd}(LL^\sigma) \le 4\sqrt{89}$. The last table row, by degree monotonicity, contradicts this.

Hence $L/\mathbf{Q}$ is Galois. Its degree is 1008 and $\mathrm{rd}(L) \le 4\sqrt{89} < 55.46400$, so Lemma 13.4 excludes a totally real kernel. Some real conjugation therefore acts nontrivially, and Lemma 2.5 gives modularity over $K$. □

*Completion for $d = 57, 65, 73, 89$ in Theorem 1.2.* The class group and scalar Selmer calculations in Lemmas 14.1 and 14.2 give the scalar source by Theorem 1.4. With Theorem 32.1, apply Theorem 10.1. □

# A The initial exceptional lift over $\mathbf{Q}(\sqrt{5})$

**Corollary A.1** (A classical quotient of the prepared KT ring)**.** *In the exact setup of [47, §7], assume the fixed set $V_2$ of $D^{\mathrm{St(uni)}}$ places is nonempty. The quotient $R_{\mathcal{S}}^{\mathrm{cl}}$ of the global deformation ring at the classical weight-two augmentation of its ordinary Iwasawa algebra $\Lambda$ is finite over $\mathcal{O}$.*

*Proof.* In the notation of [47, §7], let $\mathbb{T}$ be the localized ordinary Hecke algebra at infinite ordinary level. The paragraph preceding their Theorem 7.2 makes $\mathbb{T}$ finite over $\Lambda$. Theorem 7.2 in the final version of [47] gives a surjection

$$R_{\mathcal{S}} \longrightarrow \mathbb{T}$$

with nilpotent kernel $I$, since $V_2 \neq \varnothing$. We use the final-version numbering of that reference. The complete deformation ring $R_{\mathcal{S}}$ is Noetherian. Thus each $I^j/I^{j+1}$ is a finite module over $R_{\mathcal{S}}/I = \mathbb{T}$, and hence finite over $\Lambda$. The finite filtration by powers of $I$ shows that $R_{\mathcal{S}}$ is finite over $\Lambda$. Tensoring along the classical augmentation $\Lambda \to \mathcal{O}$ gives the claimed finiteness. This argument includes the entire deformation ring, including possible characteristic-five components; it uses no assumed characteristic-zero point or flatness of $R_{\mathcal{S}}$. □

**Proposition A.2** (An initial exceptional lift)**.** *Let $K = \mathbf{Q}(\sqrt{5})$, and let $\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ be continuous, absolutely irreducible and totally odd. Assume that its projective image is $\mathrm{PGL}_2(\mathbf{F}_5)$ and that its projective kernel field contains $K(\zeta_5)$. Put*

$$\nu = [\det \bar{r}\, \bar{\chi}_5^{-1}],$$

*where brackets denote the Teichmuller lift. There is a continuous lift $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_5)$ with determinant $\nu\chi_5$, ramified at finitely many places and de Rham of parallel weight two. It is totally odd. The lift is potentially modular over totally real extensions with avoidance of any prescribed finite extension, and belongs to a strongly compatible system.*

*Proof. Fixed local rings over $K$.* Let $S$ contain the places above 5 and the ramification of $\bar{r}$. At each $v \in S$, choose a definite-type local lift by [77, Propositions 7.7.1 and 7.8.1], weight two when $v \mid 5$. The determinant correction of [77, §7.9] twists by the square root of a finite character reducing to 1; hence it does not change the residual representation or Hodge–Tate weights and gives determinant $\nu\chi_5$. Choose a corresponding nonzero, reduced, $\mathcal{O}$-flat local fixed-type quotient $R_v^{\dagger}$ from [77, Propositions 7.3.1 and 7.4.1]. Its framed dimension is 4 for $v \nmid 5$, and $4 + [K_v : \mathbf{Q}_5]$ for $v \mid 5$. At 5 the type may be $A$, $B$ or $C$; we do not require a potentially Barsotti–Tate lift when only type $C$ was chosen.

Fix finite Galois local extensions killing these fixed inertial types, the local residual representations, and their finite determinant factors. After these extensions, the local characteristic-zero points are unramified or in the Steinberg closure away from 5; at 5 they are ordinary semistable for types $A, C$, or nonordinary Barsotti–Tate for type $B$. A single finite extension works for every point of each chosen local ring: the finite inertial type is fixed, while its possible monodromy is retained. Let $K_\nu$ be the field cut out by $\nu$. It is totally real since $\nu$ is totally even.

*Preparation and the modular residual witness.* Real approximation and Krasner's lemma, followed by taking a totally real Galois closure, supply a totally real finite extension containing $K_\nu$, for which every completion at a controlled place contains the fixed local extension. This Galois closure is taken before constructing any modular witness. Apply the residual potential-modularity theorem [77, Theorem 5.1.1] over this field. This theorem is for any odd residual representation and has no exceptional-image exclusion. It gives a weight-two modular residual witness after a further totally real extension.

Let $R$ now denote the set of places away from 5 at which the restrictions of the original chosen local rings have Steinberg type. Let $\sigma$ denote the places above 5 lying over a local type $A$ or $C$. Apply [47, Lemma 9.2] to the known modular witness, this $R$, and this $\sigma$. After a soluble totally real extension it supplies a weight-two witness $\pi$ that is ordinary and Iwahori at $\sigma$, nonordinary and spherical at the other places above 5, Steinberg at the pullback of $R$, and unramified at every other finite place. Thus incidental bad primes of the first potential-modularity witness have disappeared. The residual representation is trivial at the places above 5 and $R$. The already imposed original local conditions remain of the stated type under all these restrictions.

Finally make a totally real quadratic extension split at the current places above 5 and $R$, and base change $\pi$. Denote the resulting field by $F$. Then $[F : K]$ and $\#R_F$ are even. Its central character is everywhere unramified. In the dual normalization its ratio with the required Teichmuller character $\psi = 1$ is a finite unramified character of 5-power order; twisting by the inverse square root makes the central character exactly 1, without changing residues or local types. The residual representation over $F$ is unramified away from 5 and trivial at 5 and $R_F$. At each $w \in R_F$, residual triviality and the cyclotomic determinant give $\mathrm{N}\, w \equiv 1 \pmod 5$, as required in [47, §7].

*A fresh fixed unipotent prime.* Choose the auxiliary prime only after $F$ has been fixed. Let $M/K$ be the projective kernel field and let $F^{\mathrm{g}}/K$ be the Galois closure of $F/K$. Since $F^{\mathrm{g}}$ is totally real, $M \cap F^{\mathrm{g}} = K$: the only proper nontrivial normal subextension of the $S_5$-extension $M/K$ is its imaginary quadratic subfield $K(\zeta_5)$, and $M$ itself is not totally real. Let $L/K$ be the full linear residual kernel field, put $G = \mathrm{Gal}(L/K)$, let $Z$ be its scalar subgroup, and set $N = \mathrm{Gal}(L/(L \cap F^{\mathrm{g}}))$. Projective disjointness gives $ZN = G$, so $G/N$ is a quotient of the prime-to-five group $Z$. Every element of order five in $G$ therefore lies in $N$ and is an actual nontrivial unipotent matrix. It is trivial on $K(\zeta_5)$.

Chebotarev therefore gives a prime $q \notin S$, split completely in $F^{\mathrm{g}}$, with actual residual Frobenius a nontrivial unipotent of order 5. Then $\mathrm{N}q \equiv 1 \pmod 5$ and $\nu(q) = 1$. Impose $D^{\mathrm{St(uni)}}$ at $q$ on the original $K$-deformation problem, as well as at every place above $q$ on its $F$-restriction. This is a nonzero formally smooth local problem of framed dimension 4 by [47, Proposition 6.6]. Write $Q_F$ for the places above $q$. Its cardinality is $[F : K]$, hence even and positive. Take $V_1 = \varnothing$ and $V_2 = Q_F$ in [47, §7]. No congruence modulo 25 is needed for this fixed local problem.

Choose the good-subgroup auxiliary place $a$ using [47, Lemma 7.1], away from the places above 5, $R_F$ and $Q_F$. The raising step can be carried out with the prescribed local conditions at five. Use the level-raising argument of [47, Lemma 5.2 and proof of Theorem 9.1], in the form of [82, Lemma 4.13] that keeps the levels and ordinarity at the coefficient places. At a place in $\sigma$, keep Iwahori level and the unit $U_v$ eigensummand. At a place above five outside $\sigma$, keep spherical level and localize at the zero reduction of $T_v$. Both conditions are present in the initial witness. In the latter case the spherical Hecke polynomial has constant term of positive five-adic valuation, so the zero reduction of $T_v$ forces both roots to have positive valuation. The degeneracy maps, trace maps and vanishing-cycle sequences at the raising places commute with these operators at five. The support argument therefore applies in the indicated eigensummands, which are obtained by exact localization of the finite Hecke modules. At the new unipotent places the congruence of the two eigenvalues leaves only the required choice of the Steinberg sign.

At the auxiliary place $a$ from [47, Lemma 7.1], the good level is $U_1^1(a)$. Use the trivial-character projector for $U_0(a)/U_1^1(a) \simeq (k_a^\times)^2$ on every level-raising module. Its denominator $(\mathrm{N}a - 1)^2$ is a five-adic unit. The initial witness is fixed by this projector, and the maps at all the raising places commute with it. Hence the resulting representation has Iwahori invariants at $a$. Its local representation there is spherical or an unramified twist of Steinberg. The Steinberg

possibility would give

$$\frac{\operatorname{tr}(\bar{r}(\mathrm{Frob}_a))^2}{\det(\bar{r}(\mathrm{Frob}_a))} = \frac{(1+\mathrm{N}a)^2}{\mathrm{N}a}$$

in the residue field, excluded by the choice of $a$. It follows that $a$ contributes no ramification. The new witness has unramified central character, whose reduction is the required Teichmüller character. If a determinant correction is needed, twist by the inverse square root of the unramified five-power ratio. This preserves the local types and all residues.

Enlarge the coefficient field to contain the images of the embeddings at the five-places and all residual eigenvalues. Flatness and the local dimensions are preserved by this coefficient extension. The resulting witness satisfies the conditions of [47, §7], with $V_1 = \varnothing$ and $V_2 = Q_F$.

*Restriction at the level of local rings.* Let $R^{\dagger}_{K,q}$ be the global fixed-determinant deformation ring over $K$, with the original local quotients $R^{\dagger}_v$, the new $D^{\mathrm{St(uni)}}$ condition at $q$ transported from the Khare–Thorne dual convention, and unramifiedness outside $S \cup \{q\}$. Let $R^{\mathrm{cl}}_F$ be the classical weight-two quotient of the prepared KT ring just constructed. There is a restriction homomorphism

$$R^{\mathrm{cl}}_F \longrightarrow R^{\dagger}_{K,q}.$$

Here it is essential to verify the type conditions before passing to the global ring. On each original reduced, $\mathcal{O}$-flat local ring $R^{\dagger}_v$, restriction kills the defining ideal of the corresponding prepared local condition: this holds on all finite characteristic-zero points by the choice of the fixed local extensions, and these points are dense in the reduced flat local generic fiber. Thus it holds on the whole local ring, as in [77, Lemma 6.4.1].

The ordinary local ring is the reduced, $\mathcal{O}$-torsion-free image quotient of the framed ring obtained from the flag scheme; it forgets the flag [82, §5.4.1 and Proposition 5.13]. Its classical characteristic-zero points require an ordinary line with trivial inertial character, without marking a particular line. After preparation and dualization each type $A$ or $C$ point has such a line. The same density argument therefore gives the ordinary quotient-ring map on the original local ring. Type $B$ maps to the nonordinary Barsotti–Tate problem. Away from 5, finite-inertia type maps to the unramified problem and Steinberg type to its Steinberg closure. The determinant is identical, and the unramified sign at residual-trivial Steinberg places is forced by the residue. At the new $q$ the local fields are unchanged because $q$ splits in $F$, so the local restriction map preserves $D^{\mathrm{St(uni)}}$ directly. The resulting local ring homomorphisms remain valid after tensoring with any global ring, including one with 5-torsion. No density of global characteristic-zero points has been assumed.

The projective image over $F$ remains $S_5$, so the residual restriction is absolutely irreducible. Restriction is finite by [77, Lemma 6.3.1]. Passing to the local quotients preserves finiteness, since their defining ideals have already been shown to vanish. Corollary A.1 therefore makes $R^{\dagger}_{K,q}$ finite over $\mathcal{O}$.

*A characteristic-zero point.* Put $n = \#(S \cup \{q\})$. The completed tensor product of the local rings has dimension $1 + [K : \mathbf{Q}] + 3n$. The framed presentation in the proof of [77, Proposition 6.2.1] uses $r + n - 1$ variables and $r + [K : \mathbf{Q}]$ relations, so its global quotient has dimension at least $4n$. Forgetting the frames has relative dimension $4n - 1$; hence $\dim R^{\dagger}_{K,q} \geq 1$. The invariant vanishing used in that calculation follows from absolute irreducibility over $K(\zeta_5)$ and Schur's lemma; the exceptional-image exclusion is not used. A finite $\mathcal{O}$-algebra of positive dimension has nonzero generic fibre. A closed generic point gives the required weight-two lift, and its fixed determinant makes it totally odd.

*Actual potential modularity and the compatible system.* For any prescribed finite extension $B/K$, apply the avoiding version of [77, Theorem 5.1.1] to $\bar{r}$, including the residual and cyclotomic kernel fields in the avoidance data. It gives a totally real Galois field $K'/K$, disjoint from the prescribed avoidance field, and a weight-two modular residual witness over $K'$. The

actual lift $r|_{G_{K'}}$ is finitely ramified and weight two, and its residual restriction to $G_{K'(\zeta_5)}$ is absolutely irreducible. Apply [47, Theorem 1.1], after dualizing, to obtain the automorphy of this actual lift.

The proof of [77, Proposition 9.2.1 and Remark 9.2.2], using solvable descent and Brauer induction, now constructs the strongly compatible system. In this use the proposition's old exceptional exclusion is replaced in its proof by the actual avoiding potential modularity just proved; the remaining argument uses only that input and absolute irreducibility, not an adequacy hypothesis. In particular this is an application of that proof with its modularity input supplied by KT, not an invocation of the original proposition outside its stated hypotheses. □

# B A zero-free region for Dedekind zeta functions

We prove a zero-free region by combining the arithmetic average of Cureton and Thorner with a mean value for its sextic frequency family. The precise construction results used from [17] are stated below. All implied constants may depend on the fixed number field and the fixed local data. No uniformity in these data is required.

## B.1 Analytic inputs and normalization

Let $F$ be a number field containing $\mathbf{Q}(\zeta_{12})$. We use the finite set $S$, the lattice $R \subset F_S$, the unit group $\Gamma$, the elements $\ell_T$ with $|\ell_T|_S = T$, and the weighted cubic theta coefficients $B_m(Z)$ of [17, §§5–10]. The profiles $W_0, W_1$ are fixed, and $M(z)$ denotes the Mellin transform associated with $W_0$ in that construction. Write $\zeta_F^S$ and $L^S$ for the corresponding incomplete functions. The analytic inputs are the following. In (CT1), $M_0 > 1$ is a fixed scale, specified separately in each zero-free theorem. The remaining inputs do not change when that scale changes.

**(CT1)** At the specified fixed scale $M_0 > 1$, for each fixed compact set $B \subset F_S^\times$, the theta coefficients satisfy

$$\sum_{m \in R \cap \ell_{Z^{M_0}} B} |B_m(Z)|^2 \ll_\varepsilon Z^{2M_0 - 1 + \varepsilon} \qquad (Z \geq 1).$$

The estimate is uniform when $B$ is multiplied by elements of the fixed compact subsets of $F_S^1$ used in the average.

**(CT2)** The additive large-sieve operator on $R \cap \ell_N B$, with reduced fractions of denominator norm at most $V$, has squared norm $O(N + V^2)$. The smooth separation of the weight in the denominator variable is that of [17, §10.3].

**(CT3)** The common average $I(X, Y, Z)$ of [17, (5.6) and (11.1)] has the Poisson expression

$$\begin{aligned} I(X,Y,Z) = \frac{1}{(2\pi i)^3} \iiint_{(4,4,2)} & X^{1/2-z} Z^{\xi+z-1} Y^{w-1} e^{(\xi+z-1)^2} M(z) \widehat{W}_1(w) \\ & \cdot \sum_{u \in \mathcal{U}} q_u^{-z} \frac{\zeta_F^S(6z) L^S(w, \chi_u)}{L^S(\xi, \chi_u)} H_u(\xi, w, z)\, d\xi\, dw\, dz. \end{aligned} \tag{76}$$

Here $(4, 4, 2)$ specifies the real parts of $(\xi, w, z)$. The correction $H_u$ is holomorphic in the domain

$$\begin{gathered} x, y, z_0 > 0, \quad 6z_0 + y > 1, \quad x + 6z_0 > 1, \quad x + y > 1, \\ x + y + 6z_0 > 2, \quad 6x + 6z_0 > 5, \quad x > \tfrac{1}{2}, \quad 3x > 2, \quad 3x + y > \tfrac{5}{2}, \end{gathered} \tag{77}$$

where $(x, y, z_0) = (\operatorname{Re}\xi, \operatorname{Re} w, \operatorname{Re} z)$ [17, (11.2)], and is $O_\varepsilon(q_u^\varepsilon)$ uniformly in height on every fixed compact subregion of the real coordinates in (77).

The family $\mathcal{U}$ is the family of sixth-power representatives of [17, Lemma 7.4]; $q_1 = 1$, and $u = 1$ is its only principal character. The inducing primitive characters $\omega_u$, $u \neq 1$, have bounded multiplicity. A dyad $U \leq q_u < 2U$ contains $O(U)$ frequencies, with primitive conductor norms $O(U)$. The exterior ideal of $u$ has norm $q_u$ and valuations in $\{0, \dots, 5\}$, and the local possibilities at $S$ form a fixed finite set. These are the properties of [17, Lemmas 7.2, 7.4 and §12.1].
The principal specialization $H(s) = H_1(s, 1, 1/6)$ is the Euler product over $\mathfrak{p} \notin S$ whose local factor, with $Q = \mathrm{N}\mathfrak{p}$, is

$$H_{\mathfrak{p}}(s) = (1 - Q^{-2}) \frac{1 - (1 - Q^{-1})Q^{-s} - \vartheta(\mathfrak{p})^6 Q^{2-6s}}{(1 - \vartheta(\mathfrak{p})^6 Q^{3-6s})(1 - Q^{-s})}, \qquad |\vartheta(\mathfrak{p})| = 1. \tag{78}$$

This is [17, (11.3)].

**(CT4)** The function $\widehat{W}_1$ is entire, $M$ is holomorphic for $\operatorname{Re} z > 0$, and both have arbitrarily rapid vertical decay on the fixed strips used below. Moreover $M(1/6)\widehat{W}_1(1) \neq 0$. The profiles remain fixed when the small losses in the estimates are varied [17, §§10.2 and 13.1].

**(CT5)** For a nonprincipal primitive Hecke character of conductor norm $Q$,

$$|L(a + it, \omega)| \ll_{\varepsilon, F} Q^{(1-a)/2+\varepsilon}(2 + |t|)^{A_F} \qquad (0 \leq a \leq 1). \tag{79}$$

On each fixed half-plane $\operatorname{Re} s \geq a_* > 0$, the omitted Euler factors and their reciprocals cost $O_{\varepsilon, a_*}(U^{\varepsilon})$ for a frequency dyad. The first estimate is conductor convexity, as used in [17, §12.1]; the second follows from the elementary bound for the number of distinct prime factors of the omitted modulus.

Each of the inputs (CT1)–(CT5) is a theorem of the Lean formalization accompanying [17] (repository `AxiomMath/DedekindZetaZeroFree`, commit `09afbb4`, toolchain `leanprover/lean4:v4.35.0-rc3`), in the generality used here (for (CT1), at every scale $M_0 > 1$, of which $M_0 = 201/200$ is a compiled instance); we compiled the corresponding statements and checked that they depend only on the standard axioms `propext`, `Classical.choice` and `Quot.sound`, while the correspondence between the Lean definitions and the objects of this appendix was checked by hand (Table B.1).

Proposition 9.1 of [17] states (CT1) at $M_0 = 419/400$. The arbitrary-scale version at $M_0 > 1$ is used below; at the level of the analytic argument the scale enters via

$$P \ll Z^{M_0}, \qquad L \ll Z^{2M_0-1}, \qquad DR_0 \ll Z^{3M_0-1+\delta}.$$

The statements above retain the original notation in [17]. The Lean counterparts use normalized arithmetic, frequency and profile data, so the dictionary records the relevant formal declarations and modules rather than asserting literal identity of their notation.

| Input | Objects and conclusions here | Lean counterparts at commit `09afbb4` |
|---|---|---|
| Data | $F, S, R, \Gamma, \ell_T$, compact and unit quotients | `SArithmeticData` and associated local and unit data in `SArithmeticData.lean`. |
| (CT1) | Theta coefficients $B_m(Z)$, compact windows and mean square estimate for each $M_0 > 1$ | `ReflectedMoment.separated_mean_square_bound` in `SeparatedMeanSquareBounds.lean`, with the passage to the actual coefficient source in `DyadicMomentSource.lean`. |
| (CT2) | Additive sieve, norm $O(N + V^2)$, smooth separation | `ProbeAdditiveGramBounds.probe_additive_row_box_sieve` in `AdditiveRowBoxesAndSchurBounds.lean`. |
| (CT3) | $I(X, Y, Z)$, frequency decomposition, Poisson and Euler correction data | `Quantitative.PoissonIdentity.Iscale_eq_principal_add_nonprincipal` in `TripleMellinIdentity.lean` and the initial integral setup in `InitialIntegralIdentity.lean`. |
| (CT4) | Mellin profiles, vertical decay and nonzero principal residue | `ProbeSheetVerticalDecay.fourier_mellin_rapid_on_strip` in `VerticalStripRapidDecay.lean` and the principal residue calculation in `ResidueNonvanishing.lean`. |
| (CT5) | Conductor convexity, frequency dyads and omitted Euler factors | `Quantitative.RawSharpConvexity31.plus_sharp_convexity_dyad` in `FrequencyConvexity.lean` and deletion estimates in `DeletionMaskBounds.lean`. |

Table B.1: Dictionary between the analytic inputs and their formalized constructions. The matching of normalization and notation was checked by hand, not within Lean.

The two fixed scales used in our proofs are summarized below.

| Result | $M_0$ | Zero-free boundary $\sigma$ |
|---|---|---|
| Theorem B.1 | 201/200 | 2759/3000 |
| Theorem B.11 | 419/400 | 0.9586 |

For each row the analytic construction is carried out over $F = K(\zeta_{12})$ and yields $\zeta_K(s) \neq 0$ for $\Re s > \sigma$, $s \neq 1$.

**Theorem B.1.** *Let $K$ be a number field. Then*

$$\zeta_K(s) \neq 0 \qquad \left(\operatorname{Re} s > \frac{2759}{3000},\ s \neq 1\right).$$

*Every nontrivial zero $\rho$ of $\zeta_K$ therefore satisfies*

$$\frac{241}{3000} \leq \operatorname{Re} \rho \leq \frac{2759}{3000}.$$

*In particular all such zeros lie in the closed strip $2/25 \leq \operatorname{Re} \rho \leq 23/25$.*

For any fixed $0 < \beta \leq \alpha < 1$ with $\alpha + \beta = M_0 > 1$, put

$$M = \alpha + \beta = M_0, \qquad h = 1 - \alpha, \qquad C = \frac{\alpha}{3} + \frac{1}{6}, \qquad X = Z^\alpha, \quad Y = Z^\beta, \quad I_Z = I(X, Y, Z).$$

For Theorem B.1 we use

$$\alpha = \frac{503}{1000}, \quad \beta = \frac{251}{500}, \quad M = \frac{201}{200}, \quad h = \frac{497}{1000}, \quad C = \frac{1003}{3000}. \tag{80}$$

The normalization and contour exponents below are valid for the general $\alpha, \beta$ as well. The final subsection treats the fixed scale 419/400.

In the theta expression for this average, the normalized additive coefficients have total square mass $O(X^{-1}Y^{-2})$ and support in $R \cap \ell_{XY} B'$ for a fixed compact $B'$. Indeed there are $O(Y)$ denominators of norm comparable to $Y$, at most $q_{\mathfrak{s}}$ residues for a denominator of norm $q_{\mathfrak{s}}$, and

each coefficient has size $O(Y^{-1}q_{\mathfrak{s}}^{-1}X^{-1/2})$. Consequently (CT2), the smooth separation, and $Y \le X$ give the squared norm

$$\ll (XY + Y^2)X^{-1}Y^{-2} \ll Y^{-1}.$$

Cauchy–Schwarz with (CT1) yields

$$|I_Z| \ll_\varepsilon Z^{M-1/2-\beta/2+\varepsilon} = Z^{C-S_\theta+\varepsilon}, \qquad S_\theta = \frac{2}{3} - \frac{M}{2} - \frac{\alpha}{6}. \tag{81}$$

At (80), $S_\theta = 241/3000$. For these parameters we shall isolate a principal term in (76) and show that all its remainders have saving strictly larger than $S_\theta$.

## B.2 The weighted large sieve and the frequency mean value

All ideals in the squarefree large sieve below are prime to a fixed finite set containing $S$ and the primes required by [10, Theorem 1.3]. Enlarging this set changes neither zeros in $\operatorname{Re} s > 0$ nor the boundedness of the local possibilities. We use that theorem with order six, over the fixed field $F$. For squarefree indices in both variables it gives the kernel

$$\mathcal{K}(A, T) = A + T + (AT)^{2/3}.$$

**Lemma B.2.** *Let $a > 0$ be fixed and suppose that $|c_{\mathfrak{n}}| \le (\mathrm{N}\mathfrak{n})^{-a}$ for $T < \mathrm{N}\mathfrak{n} \le 2T$. Then*

$$\sum_{\substack{\mathfrak{a}\ \text{squarefree}\\ \mathrm{N}\mathfrak{a}\le A}} \left| \sum_{T<\mathrm{N}\mathfrak{n}\le 2T} c_{\mathfrak{n}}\chi_{\mathfrak{a}}(\mathfrak{n}) \right|^2 \ll_{a,\varepsilon} (AT)^{\varepsilon}\mathcal{K}(A, T)T^{1-2a}. \tag{82}$$

*The estimate also holds for a maximal prefix of the inner interval, with an additional power of its logarithm. Fixed character twists, complex conjugation, and divisor bounds on the coefficients are permitted after changing the small loss.*

*Proof.* Write uniquely $\mathfrak{n} = \mathfrak{m}\mathfrak{d}$, with $\mathfrak{m}$ squarefree, $\mathfrak{d}$ squarefull, and disjoint supports. For fixed $\mathfrak{d}$, apply the squarefree large sieve of [10, Theorem 1.3] to $\mathfrak{m}$. Multiplication by $\chi_{\mathfrak{a}}(\mathfrak{d})$ contracts the outer $\ell^2$ norm. There are $O(T/\mathrm{N}\mathfrak{d})$ possible $\mathfrak{m}$, their coefficients have size $O(T^{-a})$, and $\mathcal{K}(A, 2T/\mathrm{N}\mathfrak{d}) \ll \mathcal{K}(A, T)$. The resulting outer norm is at most

$$(AT)^{\varepsilon}\mathcal{K}(A, T)^{1/2}T^{1/2-a}(\mathrm{N}\mathfrak{d})^{-1/2}.$$

A squarefull ideal has a unique expression $\mathfrak{r}^2\mathfrak{s}^3$ with $\mathfrak{s}$ squarefree. Ideal counting therefore gives

$$\sum_{\substack{\mathrm{N}\mathfrak{d}\le 2T\\ \mathfrak{d}\ \text{squarefull}}} (\mathrm{N}\mathfrak{d})^{-1/2} \ll \log(2T) :$$

the sum over $\mathfrak{r}$ is logarithmic and the sum over $\mathfrak{s}$ converges at exponent $3/2$. Minkowski's inequality proves (82).

For a maximal prefix, decompose a prefix of the ordered squarefree ideals into binary blocks. At each block length the coefficient square masses of the disjoint blocks sum to the original mass. Applying the squarefree large sieve to these blocks introduces only logarithms. Then use Minkowski over $\mathfrak{d}$ as before. The same proof allows the stated coefficient and twist variants. Factors supported on the fixed finite set can be split off first; the resulting geometric sums converge because $a > 0$. □

Summing dyads in the outer $\ell^2$ norm also gives

$$\sum_{\substack{\mathfrak{a}\ \text{squarefree}\\ \mathrm{N}\mathfrak{a}\le A}}\left|\sum_{\mathrm{N}\mathfrak{n}\le N} c_{\mathfrak{n}}\chi_{\mathfrak{a}}(\mathfrak{n})\right|^2 \ll_{a,\varepsilon} (AN)^{\varepsilon}\mathcal{K}(A,N)\max(1,N^{1-2a}) \tag{83}$$

when $|c_{\mathfrak{n}}| \le (\mathrm{N}\mathfrak{n})^{-a}$. At $a = 1/2$ the logarithms are included in the small loss.

Write the exterior ideal of a frequency as $\mathfrak{a}\mathfrak{b}$, with $\mathfrak{a}$ squarefree, $\mathfrak{b}$ squarefull, and disjoint supports. For fixed $\mathfrak{b}$, the exterior character is $\chi_{\mathfrak{a}}\chi_{\mathfrak{b}}$ times one of finitely many fixed twists, by [10, Lemma 2.1 and Appendix II] and [17, Lemma 7.4]. The exterior exponents are at most five, so no nontrivial exterior ramification is removed by a sixth power. On $\mathrm{N}\mathfrak{b} \asymp B$ there are $O(B^{1/2})$ possibilities for $\mathfrak{b}$, and $\mathrm{N}\mathfrak{a} \ll U/B$. The primitive conductor norm is

$$Q_u \ll \frac{U}{B^{1/2}}. \tag{84}$$

Indeed the exterior conductor divides the radical of $\mathfrak{a}\mathfrak{b}$, whose norm is at most $\mathrm{N}\mathfrak{a}\,\mathrm{N}\mathfrak{b}^{1/2}$; the factors at the fixed primes are bounded.

Summing (83) over the squarefull part gives

$$\sum_{B\ \text{dyadic}} B^{1/2}\mathcal{K}(U/B,N) \ll \log(2U)\{U + U^{1/2}N + U^{2/3}N^{2/3}\}.$$

For $N \le U^{1/2+\lambda}$ this is $O(U^{1+O(\lambda)+\varepsilon})$. In particular, uniformly for real $v$ and also for the conjugate family,

$$\sum_{\omega}\left|\sum_{\mathrm{N}\mathfrak{n}\le N}\frac{\omega(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{a+iv}}\right|^2 \ll_{a,\varepsilon} U^{1+O(\lambda)+\varepsilon}\max(1,N^{1-2a}). \tag{85}$$

Here the sum is over the distinct primitive nonprincipal characters of a frequency dyad; bounded multiplicities make no difference.

**Lemma B.3.** *For fixed $0 < y \le 1/2$, sufficiently small fixed $\rho > 0$, and $|t| \le 2U^{\rho}$, we have*

$$\sum_{\omega}|L(y+it,\omega)|^2 \ll_{y,\varepsilon} U^{2-2y+O_F(\rho)+\varepsilon}, \tag{86}$$

$$\sum_{\substack{u\ne 1\\ U\le q_u<2U}}|L^S(y+it,\chi_u)| \ll_{y,\varepsilon} U^{3/2-y+O_F(\rho)+\varepsilon}. \tag{87}$$

*Proof.* Since $F$ is totally complex, all finite-order characters have the same archimedean factor $\gamma_F(s) = \Gamma_{\mathbb{C}}(s)^{[F:\mathbf{Q}]/2}$. Put $\mathcal{Q}_\omega = |d_F|Q_\omega$ and $X_\omega = \mathcal{Q}_\omega^{1/2}$. For $s = y + it$, the primitive approximate functional equation [46, Theorem 5.3] reads

$$\begin{aligned} L(s,\omega) &= \sum_{\mathfrak{n}}\frac{\omega(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^s}V_s(\mathrm{N}\mathfrak{n}/X_\omega)\\ &\quad + W(\omega)\mathcal{Q}_\omega^{1/2-s}\frac{\gamma_F(1-s)}{\gamma_F(s)}\sum_{\mathfrak{n}}\frac{\overline{\omega}(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{1-s}}V_{1-s}(\mathrm{N}\mathfrak{n}/X_\omega), \end{aligned}$$

where $|W(\omega)| = 1$ and

$$V_s(x) = \frac{1}{2\pi i}\int_{(2)}\frac{\gamma_F(s+w)}{\gamma_F(s)}e^{w^2}x^{-w}\frac{dw}{w}.$$

For a small $\eta > 0$, rapid decay permits truncation at $N = U^{1/2+c_F\rho+\eta}$ with an error $O(U^{-10})$. Here $c_F$ is fixed and sufficiently large.

In the remaining finite sums move the Mellin line to $\operatorname{Re} w = \eta$. The Gaussian and gamma kernel have integrable modulus of total mass $U^{O_F(\rho)+O_F(\eta)}$, and $|X_\omega^w| \ll U^{\eta/2}$. Cauchy–Schwarz in this integral, followed by (85), separates the conductor-dependent weight from the Dirichlet coefficients. The first polynomial contributes at most

$$U^{1+O_F(\rho)+O_F(\eta)} N^{\max(1-2y,0)} \ll U^{3/2-y+O_F(\rho)+O_F(\eta)}.$$

The dual polynomial has exponent $1-y+\eta \geq 1/2$, so its mean square is at most $U^{1+O_F(\rho)+O_F(\eta)}$. Stirling's formula and $Q_\omega \ll U$ give

$$\left| \mathcal{Q}_\omega^{1/2-s} \frac{\gamma_F(1-s)}{\gamma_F(s)} \right|^2 \ll U^{1-2y+O_F(\rho)}.$$

Since $3/2 - y \leq 2 - 2y$, choosing $\eta$ in terms of the desired $\varepsilon$ proves (86). Cauchy–Schwarz, the $O(U)$ frequency count and (CT5) then prove (87). □

## B.3 Exceptional characters

**Lemma B.4.** *Let $1/2 < \sigma < 1$ and $1/2 < \nu < 1$. Suppose that $[0, 2(1-\nu)+1/100]$ is covered by finitely many intervals $[v_1, v_2]$, each supplied with constants $m = 1 - v_1/2$ and $0 < \kappa < r < r'$ such that, with $c = 1/100$,*

$$\kappa(1-c) + \frac{m(1-c)}{2} + r(c-\sigma) \leq -\frac{1}{500}, \tag{88}$$

*and*

$$\begin{aligned} \max\Big\{ & 1 - \frac{v_1}{2} + (1-2\sigma)\kappa, \quad \frac{v_2}{2} + (2-2\sigma)r', \\ & \frac{2}{3} - \frac{v_1}{6} + \left(\frac{5}{3} - 2\sigma\right) t_* \Big\} \leq \nu - \frac{1}{250}, \end{aligned} \tag{89}$$

*where $t_* = r'$ if $\sigma \leq 5/6$ and $t_* = \kappa$ otherwise. For sufficiently small fixed $\rho > 0$, the number of primitive characters in a frequency dyad with a zero in*

$$\operatorname{Re} s \geq \sigma, \qquad |\operatorname{Im} s| \leq 4U^\rho$$

*is $O(U^\nu)$. A single $\rho$ may be chosen for any fixed finite collection of interval data satisfying these inequalities.*

*Proof.* Use the squarefree–squarefull decomposition above and group the squarefull part by $\mathrm{N}\mathfrak{b} \asymp B = U^v$. For $v \in [v_1, v_2]$, (84) bounds the conductor by $O(U^m)$. Consider the mollifier

$$\mathcal{M}_U(s,\omega) = \sum_{\mathrm{N}\mathfrak{d} \leq U^\kappa} \frac{\mu_F(\mathfrak{d})\omega(\mathfrak{d})}{(\mathrm{N}\mathfrak{d})^s}, \qquad a_U(\mathfrak{n}) = \sum_{\substack{\mathfrak{d} \mid \mathfrak{n} \\ \mathrm{N}\mathfrak{d} \leq U^\kappa}} \mu_F(\mathfrak{d}).$$

Then $a_U(1) = 1$, $a_U(\mathfrak{n}) = 0$ for $1 < \mathrm{N}\mathfrak{n} \leq U^\kappa$, and $|a_U(\mathfrak{n})| \ll_\eta (\mathrm{N}\mathfrak{n})^\eta$.

If $\beta_0 + i\gamma$ is one of the stated zeros, integrate

$$\mathcal{M}_U(\beta_0 + i\gamma + s, \omega) L(\beta_0 + i\gamma + s, \omega) \Gamma(s) U^{rs}$$

on $\operatorname{Re} s = 2$ and move to $\operatorname{Re} s = c - \beta_0$. The residue at $s = 0$ vanishes. There is no other pole between these lines. Convexity bounds the shifted integral by

$$U^{\kappa(1-c)+m(1-c)/2+r(c-\sigma)+\eta+O_F(\rho)}.$$

It tends to zero by (88), after choosing the losses sufficiently small. On the original line Mellin inversion gives

$$e^{-1/U^r}+\sum_{\mathrm{N}\mathfrak{n}>U^\kappa}\frac{a_U(\mathfrak{n})\omega(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{\beta_0+i\gamma}}e^{-\mathrm{N}\mathfrak{n}/U^r}.$$

The sum therefore has modulus bounded below. Its tail beyond $U^{r'}$ is negligible. Dyadic subdivision and partial summation remove the smooth weight, the factor $(\mathrm{N}\mathfrak{n})^{\sigma-\beta_0}$, and the height oscillation. For some dyadic interval with $U^\kappa/2<N\le U^{r'}$, a prefix of

$$\sum_{N<\mathrm{N}\mathfrak{n}\le 2N}\frac{a_U(\mathfrak{n})\omega(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{\sigma}}$$

has modulus $\gg U^{-\rho}/\log U$.

For fixed $\mathfrak{b}$ and a fixed local twist, apply the maximal form of Lemma B.2. Its bound is

$$(AN)^{\varepsilon}\{A+N+(AN)^{2/3}\}N^{1-2\sigma+2\eta},\qquad A\asymp U/B.$$

After multiplying by the $O(B^{1/2})$ choices of $\mathfrak{b}$, the three exponents, before the small losses, are

$$1-\frac{v}{2}+(1-2\sigma)t,\qquad \frac{v}{2}+(2-2\sigma)t,\qquad \frac{2}{3}-\frac{v}{6}+\left(\frac{5}{3}-2\sigma\right)t,\quad t=\frac{\log N}{\log U}.$$

For $v\in[v_1,v_2]$ and $\kappa\le t\le r'$, their maxima are precisely bounded by the expressions in (89). The constant factor in the lower bound $N>U^\kappa/2$ changes only an implied constant. The margin $1/250$ absorbs all dyadic sums, divisor losses and the height loss $U^{2\rho}$. Finally, for $v\ge 2(1-\nu)$ the direct count $O(U^{1-v/2})$ is already $O(U^\nu)$. Choose the losses and $\rho$ after the finite interval data are fixed. □

Table 4 supplies the twelve levels used below. Every decimal in the table denotes an exact terminating rational number, and $m=1-v_1/2$ in each row. Direct substitution gives (88) and (89) for all the rows; the intervals in each group cover the interval required by Lemma B.4.

Table 4: Exact parameters for the exceptional-character estimate.

| $\sigma$ | $\nu$ | $v_1$ | $v_2$ | $\kappa$ | $r$ | $r'$ |
|---|---|---|---|---|---|---|
| 0.730 | 0.887 | 0.000 | 0.236 | 0.255 | 1.041 | 1.042 |
| 0.750 | 0.852 | 0.000 | 0.306 | 0.304 | 1.079 | 1.080 |
| 0.770 | 0.813 | 0.000 | 0.384 | 0.354 | 1.116 | 1.117 |
| 0.790 | 0.771 | 0.000 | 0.468 | 0.402 | 1.148 | 1.149 |
| 0.810 | 0.727 | 0.000 | 0.552 | 0.447 | 1.175 | 1.176 |
| 0.810 | 0.727 | 0.552 | 0.556 | 0.002 | 0.453 | 0.454 |
| 0.830 | 0.680 | 0.000 | 0.536 | 0.491 | 1.199 | 1.200 |
| 0.830 | 0.680 | 0.536 | 0.650 | 0.085 | 0.547 | 0.548 |
| 0.850 | 0.629 | 0.000 | 0.010 | 1.250 | 2.065 | 2.066 |
| 0.850 | 0.629 | 0.010 | 0.047 | 1.200 | 2.004 | 2.005 |
| 0.850 | 0.629 | 0.047 | 0.184 | 1.015 | 1.775 | 1.776 |
| 0.850 | 0.629 | 0.184 | 0.640 | 0.405 | 1.015 | 1.016 |
| 0.850 | 0.629 | 0.640 | 0.752 | 0.079 | 0.497 | 0.498 |
| 0.870 | 0.570 | 0.000 | 0.008 | 1.373 | 2.159 | 2.160 |
| 0.870 | 0.570 | 0.008 | 0.020 | 1.355 | 2.136 | 2.137 |

Continued on the next page

Exceptional-character parameters, continued

| $\sigma$ | $\nu$ | $v_1$ | $v_2$ | $\kappa$ | $r$ | $r'$ |
|---|---|---|---|---|---|---|
| 0.870 | 0.570 | 0.020 | 0.038 | 1.328 | 2.101 | 2.102 |
| 0.870 | 0.570 | 0.038 | 0.066 | 1.287 | 2.049 | 2.050 |
| 0.870 | 0.570 | 0.066 | 0.108 | 1.223 | 1.967 | 1.968 |
| 0.870 | 0.570 | 0.108 | 0.171 | 1.128 | 1.846 | 1.847 |
| 0.870 | 0.570 | 0.171 | 0.266 | 0.985 | 1.663 | 1.664 |
| 0.870 | 0.570 | 0.266 | 0.410 | 0.769 | 1.387 | 1.388 |
| 0.870 | 0.570 | 0.410 | 0.628 | 0.441 | 0.968 | 0.969 |
| 0.870 | 0.570 | 0.628 | 0.827 | 0.163 | 0.585 | 0.586 |
| 0.870 | 0.570 | 0.827 | 0.870 | 0.028 | 0.373 | 0.374 |
| 0.890 | 0.560 | 0.000 | 0.379 | 0.977 | 1.664 | 1.665 |
| 0.890 | 0.560 | 0.379 | 0.701 | 0.420 | 0.931 | 0.932 |
| 0.890 | 0.560 | 0.701 | 0.890 | 0.120 | 0.503 | 0.504 |
| 0.920 | 0.545 | 0.000 | 0.654 | 0.725 | 1.335 | 1.336 |
| 0.920 | 0.545 | 0.654 | 0.908 | 0.158 | 0.541 | 0.542 |
| 0.920 | 0.545 | 0.908 | 0.920 | 0.006 | 0.306 | 0.307 |
| 0.950 | 0.519 | 0.000 | 0.787 | 0.650 | 1.214 | 1.215 |
| 0.950 | 0.519 | 0.787 | 0.944 | 0.102 | 0.429 | 0.430 |
| 0.950 | 0.519 | 0.944 | 0.970 | 0.015 | 0.296 | 0.297 |
| 0.950 | 0.519 | 0.970 | 0.972 | 0.001 | 0.275 | 0.276 |
| 0.980 | 0.510 | 0.000 | 0.926 | 0.548 | 1.072 | 1.073 |
| 0.980 | 0.510 | 0.926 | 0.987 | 0.033 | 0.310 | 0.311 |
| 0.980 | 0.510 | 0.987 | 0.990 | 0.001 | 0.262 | 0.263 |

## B.4 Reciprocals and contours

Let $(\sigma_0, \dots, \sigma_{11})$ be the successive values of $\sigma$ in Table 4, and let $(\nu_0, \dots, \nu_{11})$ be the corresponding density exponents. Choose one sufficiently small $\rho > 0$ for all its rows. A nonprincipal frequency is called zero free at level $\sigma_j$ if its primitive $L$-function has no zero in $\operatorname{Re} s \ge \sigma_j$, $|\operatorname{Im} s| \le 4U^\rho$. Class 0 consists of the frequencies zero free at $\sigma_0$; for $1 \le j \le 11$, class $j$ consists of those zero free at $\sigma_j$ but not at $\sigma_{j-1}$. Class 12 consists of the remaining frequencies. Its contours will stay to the right of one. By Lemma B.4, class $j$, $1 \le j \le 12$, has $O(U^{\nu_{j-1}})$ elements.

**Lemma B.5.** *Put $x_j = \sigma_j + 1/50000$ for $0 \le j \le 11$. For a frequency zero free at level $\sigma_j$,*

$$\frac{1}{|L(x_j + it, \omega_u)|} \ll_\varepsilon U^\varepsilon \qquad (|t| \le 2U^\rho).$$

*The same estimate holds for the incomplete reciprocal.*

*Proof.* Around $2 + it$ take disks of radii

$$0.999 < 2 - x_j < 2 - x_j + 10^{-5} < 2 - x_j + 1.5 \cdot 10^{-5}.$$

For all sufficiently large $U$ the outer disk lies in the stated zero-free rectangle: its left edge is $\sigma_j + 0.5 \cdot 10^{-5}$ and its imaginary height is less than $4U^\rho$. Choose the analytic logarithm agreeing with the Euler product at the center. It is bounded on the innermost disk, which lies in $\operatorname{Re} s > 1$. Convexity and Borel–Carathéodory give $O(\log U)$ on the disk of radius $2 - x_j + 10^{-5}$. Hadamard's three circles theorem then gives $O((\log U)^{\theta_j})$, with $\theta_j < 1$, on the disk of radius $2 - x_j$. Exponentiating bounds the reciprocal by $U^\varepsilon$. The number of configurations is finite,

and (CT5) handles the incomplete factors. The finitely many smaller dyads are absorbed in the implied constant. □

For a frequency dyad put $\tau = \log U/\log Z$ and set

$$\delta = 10^{-7}, \qquad x_{\mathrm{abs}} = 1+\delta, \qquad z_0 = \frac{2}{5}, \qquad \tau_{\max} = h + 10^{-4} = \frac{4971}{10000}.$$

The exponent supplied by the external powers in (76) on a real corner $(x, y, z)$ is

$$E(x,y,z) = \frac{\alpha}{2} + x - 1 + hz + \beta(y-1). \tag{90}$$

For $0 < y \le 1$, a nonprincipal family of size $O(U^\nu)$ on which the reciprocal is $O_\varepsilon(U^\varepsilon)$ has contribution

$$\ll_\varepsilon Z^{E(x,y,z)} U^{\nu+(1-y)/2-z+O_F(\rho)+\varepsilon}. \tag{91}$$

For class 0, Lemma B.3 replaces $\nu + (1-y)/2$ by $3/2 - y$. Thus a row with frequency exponent $k$ and $\tau \in [\tau_1, \tau_2]$ has saving, before the small losses,

$$S = C - E(x,y,z) - \max_{\tau\in[\tau_1,\tau_2]} k\tau. \tag{92}$$

We use the following contours. The principal term has a $w$-remainder on $(x_{\mathrm{abs}}, 1/2, 1/6+\delta)$ and a $z$-remainder on $(x_{\mathrm{abs}}, 1, 1/1000)$, as described below. For $0 \le \tau \le 1/5$ all nonprincipal frequencies use $(x_{\mathrm{abs}}, 1/1000, 2/5)$ and convexity. For $1/5 \le \tau \le \tau_{\max}$, class 0 uses $(x_0, 8/25, 2/5)$ and its mean value; class $j$, $1 \le j \le 11$, uses $(x_j, 1-\sigma_j, 2/5)$ and convexity; and class 12 uses $(x_{\mathrm{abs}}, 1/1000, 2/5)$ and convexity. For $\tau \ge \tau_{\max}$ all frequencies use $(4, 4, 10^5)$.

Each corner belongs to (77). For nonprincipal intermediate frequencies, move successively from $(4,4,2)$ through $(x,4,2)$ and $(x,4,2/5)$ to $(x,y,2/5)$. The real segments stay in the domain because all its conditions are affine and strict. There is no numerator pole: the character is nonprincipal and $\operatorname{Re} z > 1/6$. The $\xi$ movement takes place in the rectangle controlled by Lemma B.5; to the right of one use the Euler product. The same argument applies uniformly to the horizontal connections.

For these intermediate frequencies $U \ge Z^{1/5}$. Outside the truncated height boxes, arbitrarily rapid decay in $U^\rho$ therefore absorbs every fixed power of $Z$. For large $|\operatorname{Im}\xi|$, either $|\operatorname{Im} z|$ is comparable, where $M(z)$ decays, or $|\operatorname{Im}(\xi+z)|$ is large, where the Gaussian decays. The decay of $\widehat{W}_1$ handles $w$. Small frequencies keep $\operatorname{Re}\xi = x_{\mathrm{abs}} > 1$ at every height, so their contour movement uses the global Euler product bound and requires no truncated zero-free rectangle. The large-frequency contour likewise lies in absolute convergence. Its frequency exponent is $1 - 10^5 < 0$, so its dyadic sum is geometric.

For the principal frequency, first move $\xi$ to $x_{\mathrm{abs}}$ and reach $(x_{\mathrm{abs}}, 4, 1/6+\delta)$. Moving $w$ to $1/2$ crosses its pole at $w = 1$. In the resulting residue move $z$ to $1/1000$, crossing $z = 1/6$. All these movements take place in (77). The two remaining integrals have the principal corners specified above. The double residue is $c_0 f(Z)$, where

$$\begin{aligned} f(Z) &= \frac{1}{2\pi i}\int_{(2)} Z^{s-(1-C)} e^{(s-5/6)^2} \frac{H(s)}{\zeta_F^S(s)}\, ds, \\ c_0 &= \frac{1}{6}\left(\operatorname*{Res}_{s=1}\zeta_F^S(s)\right)^2 M(1/6)\widehat{W}_1(1) \neq 0. \end{aligned} \tag{93}$$

The line in the residue can be moved from $x_{\mathrm{abs}}$ to 2, since this movement stays to the right of one.

The savings in (92) are displayed in Table 5. Its last column consists of rational lower bounds obtained by rounding down to six decimal places. For the principal rows $S = C - E$. For the

| Row | $\nu$ | Treatment | Lower bound for $S$ |
|---|---|---|---|
| Principal $w$-remainder | | $C-E$ | 0.250999 |
| Principal $z$-remainder | | $C-E$ | 0.082336 |
| Small frequencies | 1 | Convexity | 0.165631 |
| Class 0 | | Mean value | 0.107635 |
| Class 1 | 0.887 | Convexity | 0.082013 |
| Class 2 | 0.852 | Convexity | 0.084480 |
| Class 3 | 0.813 | Convexity | 0.088936 |
| Class 4 | 0.771 | Convexity | 0.094883 |
| Class 5 | 0.727 | Convexity | 0.101825 |
| Class 6 | 0.680 | Convexity | 0.110257 |
| Class 7 | 0.629 | Convexity | 0.120678 |
| Class 8 | 0.570 | Convexity | 0.135076 |
| Class 9 | 0.560 | Convexity | 0.117651 |
| Class 10 | 0.545 | Convexity | 0.102711 |
| Class 11 | 0.519 | Convexity | 0.093239 |
| Class 12 | 0.510 | Convexity | 0.082548 |
| Large frequencies | 1 | Absolute convergence | 5.079733 |

Table 5: Remainder savings for the stated contours.

small row, $k>0$ and the maximum is at $\tau=1/5$; for all the intermediate rows $k>0$ and the maximum is at $\tau=\tau_{\max}$. For the last row the maximum occurs at $\tau_{\max}$ because $k<0$.

Substitution of the exact parameters gives the minimum, attained by class 1,

$$S_P=\frac{1230197}{15000000},\qquad S_P-S_\theta=\frac{25197}{15000000}>0. \tag{94}$$

In particular the contour comparison has a fixed positive margin. The exact rational checks for Tables 4 and 5 are also supplied by `analytic_rational_checks.py` in [21].

## B.5 Mellin inversion

The Euler product in (78) shows that $H$ is holomorphic and nowhere zero in $\operatorname{Re} s>2/3$, and that both $H$ and $1/H$ are bounded on every smaller closed half-plane. For completeness, if $\sigma>1/2$, the nonconstant part of its local numerator has modulus at most

$$(1-Q^{-1})Q^{-\sigma}+Q^{2-6\sigma}<(1-Q^{-1})Q^{-1/2}+Q^{-1}<1.$$

Both local denominator factors are also nonzero there. Moreover, uniformly in height on a smaller closed half-plane,

$$H_{\mathfrak{p}}(s)-1\ll Q^{-2}+Q^{-1-\sigma}+Q^{3-6\sigma}.$$

The right side is summable over primes when $\sigma>2/3$, which proves the assertions about $H$ and its reciprocal.

**Lemma B.6** (Mellin endpoint)**.** *Use the principal term $f(Z)$ of (93), with the value of $C$ belonging to the chosen $\alpha,\beta$. Suppose that, for some $0<\delta_0<1/3$ and every $\varepsilon>0$,*

$$f(Z)\ll_\varepsilon Z^{C-\delta_0+\varepsilon}\qquad(Z\geq 1).$$

*Then $\zeta_F(s)$ has no zero in $\operatorname{Re} s>1-\delta_0$, away from its pole.*

*Proof.* For $0 < Z < 1$, move the line defining $f$ to any fixed $R > 2$. Absolute convergence and Gaussian decay give

$$f(Z) = O_R(Z^{R-1+C}). \tag{95}$$

It follows that

$$\int_0^\infty f(Z) Z^{-s+1-C} \, \frac{dZ}{Z}$$

is holomorphic for $\operatorname{Re} s > 1 - \delta_0$: on each compact subset choose $\varepsilon$ smaller than its distance from the boundary and then choose $R$ larger than its real parts. Mellin inversion, first on $\operatorname{Re} s = 2$, identifies this transform with

$$e^{(s-5/6)^2} \frac{H(s)}{\zeta_F^S(s)}.$$

Analytic continuation gives the identity throughout the half-plane. The numerator is nowhere zero there, so neither $\zeta_F^S$ nor $\zeta_F$ has a zero there. Omitted Euler factors have no zeros in the positive half-plane. □

*Proof of Theorem B.1.* By (94), choose $\rho$ and all fixed losses so that every Poisson remainder has saving strictly greater than $S_\theta = 241/3000$. This is compatible with all the fixed margins in Lemma B.4. The profiles and $f$ remain fixed as the small losses are varied. The theta estimate, Poisson identity and nonzero principal residue give, for every $\varepsilon > 0$,

$$f(Z) \ll_\varepsilon Z^{C - S_\theta + \varepsilon} \qquad (Z \geq 1). \tag{96}$$

Lemma B.6 gives the half-plane for $F$. The extension $F = K(\zeta_{12})$ is abelian over $K$; all factors other than $\zeta_K$ in the abelian factorization of $\zeta_F$ are entire Hecke $L$-functions. A zero of $\zeta_K$ away from its pole would therefore be a zero of $\zeta_F$. This descends the assertion to $K$. The functional equation gives the symmetric lower bound for nontrivial zeros, and $2759/3000 < 23/25$ gives the displayed weaker strip. □

## B.7 A comparison using the scale 419/400

The following version uses the verified (CT1) at the fixed scale displayed in [17, Proposition 9.1]. It supplies the zero-free region used for $d = 6$, independently of the other scale used in Theorem B.1.

**Lemma B.10.** *Put $\sigma_0 = 167/200$. For sufficiently small fixed $\rho > 0$, the number of characters in a frequency dyad with a zero in $\operatorname{Re} s \geq \sigma_0$, $|\operatorname{Im} s| \leq 4U^\rho$, is $O(U^{67/100})$.*

*Proof.* Apply Lemma B.4 with $\nu = 67/100$ and the following exact terminating rational parameters:

| $[v_1, v_2]$ | $m$ | $\kappa$ | $r$ | $r'$ | $\kappa(1-c) + m(1-c)/2 + r(c-\sigma_0)$ |
|---|---|---|---|---|---|
| $[0, 1/2]$ | 1 | 0.51 | 1.22 | 1.222 | −0.0066 |
| $[1/2, 0.67]$ | 3/4 | 0.15 | 0.65 | 0.652 | −0.0165 |

The intervals cover $[0, 2(1-\nu) + 1/100] = [0, 0.67]$. The three density exponents are at most 0.6583, 0.65326, and 0.664967 in the first row, and 0.6495, 0.55016, and 0.582834 in the second. They are all below $\nu - 1/250 = 0.666$; both detector exponents are below $-1/500$. Thus the lemma applies with its fixed margins. □

**Theorem B.11.** *Let $K$ be a number field. Then $\zeta_K(s)$ is nonzero in $\operatorname{Re} s > 0.9586$, apart from its pole at one.*

*Proof.* Take

$$\alpha = 0.608, \quad \beta = 0.4395, \quad h = 0.392, \quad C = \alpha/3 + 1/6.$$

Then $\alpha + \beta = 419/400$ and the general theta estimate (81) has saving $S_\theta = 499/12000$. Put $\delta = 10^{-7}$, $x_1 = 1+\delta$, $x_g = 0.83502$, and $u_{\max} = 0.39202$. For $U = Z^u$ use the following contours:

| contribution | $u$ range | $(x, y, z)$ |
|---|---|---|
| small frequencies | $[0, 1/5]$ | $(x_1, 0.001, 0.4)$ |
| exceptional frequencies | $[1/5, u_{\max}]$ | $(x_1, 0.001, 0.4)$ |
| good frequencies | $[1/5, u_{\max}]$ | $(x_g, 0.165, 0.4)$ |
| principal $w$ remainder | | $(x_1, 1/2, 1/6 + \delta)$ |
| principal $z$ remainder | | $(x_1, 1, 1/20)$ |
| large frequencies | $[u_{\max}, \infty)$ | $(4, 4, 500000)$ |

The frequency counts are $O(U)$ for the small and good rows, and $O(U^{0.67})$ for the exceptional row by Lemma B.10. The good reciprocal follows from the proof of Lemma B.5, using disks centered at $2 + it$ with radii $0.999 < 1.16498 < 1.16499 < 1.164995$. All corners and coordinate segments lie in (77); in particular $x_g + 0.165 = 1.00002$. The small and exceptional reciprocals remain in absolute convergence. For the good intermediate range $U \geq Z^{1/5}$, the height estimates in the common contour argument apply. The principal moves cross exactly $w = 1$ and $z = 1/6$, with residue $c_0 f(Z)$ as in (93). On the large contour the exponent at $u_{\max}$ is

$$E(4, 4, 500000) + u_{\max}(1 - 500000) = -4.98548;$$

the negative coefficient of $u$ makes the dyads summable. Equations (90) and (92) give the following lower bounds before small analytic losses:

| Contribution | Saving at least |
|---|---|
| Theta | 0.0415833 |
| Small | 0.1276937 |
| Exceptional intermediate | 0.0459343 |
| Good intermediate | 0.0416154 |
| Principal $w$ remainder | 0.2197498 |
| Principal $z$ remainder | 0.0457332 |
| Large | 5 |

All have a strict margin over 0.0414 after total losses smaller than $10^{-4}$. Hence $f(Z) \ll Z^{C-0.0414}$. Lemma B.6, followed by the same abelian descent as in Theorem B.1, proves the assertion. □

# C Numerical comparisons and finite certificates

## C.1 The comparisons for $\mathbf{Q}(\sqrt{6})$

We use (10) with the real-place term retained. For $G = F_b/\cosh(Ax)$, $0 < A \leq 1/2$, its contribution to $\log \mathrm{rd}$ is at least $(r_1/n)(1 - \pi^4/(24b^2))$. Indeed $G \geq F_b/\cosh(x/2)$, and the classical estimate in this Section 13 applies. The densities in Lemma 21.12 are $\rho_P = N_P/n$; programs using Euler coefficients $c_P$ take $c_P = 2\rho_P$. We keep up to twelve powers of each specified norm in the support of the test function.

For the narrower strip put $a = 2293/5000$. Use Lemma 13.7 with $(M, r, s) = (60, 10, 40)$. For $B_0$ use the finite expansion of (16) with $R = 500$ and final alternating index 80, evaluated at this $a$. The classical rows use $N = 100$ in (13). The displayed decimals are outward bounds. In

a row for $L$, decreasing $e$ at fixed $f$ lowers the upper bound and increases the retained Euler coefficients. In a range of $q$, the lower bound uses the smallest degree.

| Field | Range or conditions | $b$ | $a_0$ | Lower | Upper |
|---|---|---|---|---|---|
| $L$ | $q = 4,\ f = 2$ | 5 | 1/2 | 20.8677 | 13.5186 |
| $L$ | $q = 4,\ f = 1$ | 5 | 1/2 | 22.0885 | 17.8379 |
| $L$ | $q = 8,\ f = 2$ | 8 | 1/2 | 25.6785 | 20.5922 |
| $L$ | $q = 8,\ f = 1$ | 8 | 1/2 | 26.5040 | 22.2408 |
| $L$ | $q = 16,\ f = 1$ | 12 | 1/2 | 28.0247 | 24.3108 |
| $L$ | $q = 16,\ f = 2$ | 12 | 1/2 | 25.9359 | 24.3108 |
| $L$ | $q = 32,\ f = 2$ | 20 | 1/2 | 25.8819 | 25.5200 |
| $L$ | $q = 64,\ f = 2$ | 30 | 0.4586 | 27.1375 | 26.1763 |
| $L$ | $q = 128,\ f = 2$ | 45 | 0.4586 | 26.9654 | 26.5186 |
| $L$ | $q = 256,\ f = 2$ | 60 | 0.4586 | 26.8955 | 26.6934 |
| $MH$ | $q \geq 16,\ e \leq 4$ | 25 | 1/2 | 25.9059 | 24.6406 |
| $M$ | $q \geq 32,\ f = 1$ | 43 | 1/2 | 27.1019 | 26.8708 |
| $M$ | $q \geq 64,\ f_v = 1$ | 50 | 1/2 | 31.5370 | 26.8708 |
| $M$ | $q \geq 64,\ e_v = 1$ | 50 | 1/2 | 22.3292 | 17.9696 |
| $MV$ | $q \geq 512,\ e = 8,\ f = 2$ | 150 | 0.4586 | 26.8754 | 26.8708 |

## C.2 The comparisons for $\mathbf{Q}(\sqrt{41})$

For Lemma 19.1 all rows use $A = 1/2$ and the classical integrals in (14). The notation is $M$ for the kernel field and $L = M^U$ for the Sylow subfield. The first row retains no Euler or real-place terms. The next two use real fraction at least $r_0 = 1/(q+1)$, coefficients $2r_0$ and $(1 - r_0)/4$ at norms two and four, and six and five prime powers respectively. The last row retains six norm-four powers with coefficient $1/4$. For ranges of $q$ the lower bound is evaluated at the smallest degree.

| Case | Field | $b$ | Lower | Upper |
|---|---|---|---|---|
| Tame or unramified at one place, $q \geq 4$ | $M$ | 8 | 16.82 | 16.135 |
| Both elementary, $q = 4$ | $L$ | 9/2 | 21.785 | 19.411 |
| Both elementary, $q = 8$ | $L$ | 7 | 22.80 | 21.957 |
| Both elementary, $q \geq 16$ | $M$ | 30 | 26.66 | 25.613 |

The certificate uses ninety terms of (14) with the stated tail bound and directed intervals.

## C.3 Arithmetic and interval certificates

The ray arguments for $d = 2, 5, 13$ use Minkowski's bound and principal ideal relations. The same method checks the two families over $\mathbf{Q}(\sqrt{6})$, giving 51 quartic cases in all. The certificates in [60] give these checks and the arithmetic of the quadratic base fields. For each quartic the program enumerates the primes up to a rational upper bound for the Minkowski cutoff. Factoring principal ideals gives an integer relation lattice. In the $d = 6$ two-special-place calculation we also impose relations killing the prescribed prime classes. The resulting lattice index bounds the order of the class group quotient. Its odd part is trivial except that it can divide three for the field identified in Lemma 21.11. For the local term in the ray sequence, it computes the finite group $(\mathcal{O}_L/(2))^\times$ and the images of explicit units of absolute norm one. The quotient by these images has two-power order. This suffices for the ordinary ray group, and adding real places only introduces factors of order two.

A script verifies the displayed discriminant comparisons for $d = 2, 5, 13, 17, 33$ using exact rationals. It uses the alternating series for arctangent, logarithmic series with geometric tail bounds, and Taylor bounds for exponential, sine and cosine. The Euler constant is bounded below by $H_n - \log n - 1/(2n)$ with $n = 2048$, using the convex trapezoidal estimate for $1/x$. Root-discriminant upper bounds are checked by integer powers. The directed interval programs for $d = 6$ and $d = 41$ evaluate the bounds of Sections C.1 and C.2, respectively. The exponent inequalities in the proof of Theorem B.11 are checked by exact rational arithmetic. The same archive contains the certificate for the CM comparison at five.

# D Depth-four normal ray calculation

We give the matrices used in Proposition 23.8. The fields $K, A$, the standard module $W_0$ and the binary ray coordinates are those in its proof. For the depth-four calculation write $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on $W_0$. In the seven binary ray coordinates the class killed by $[\mathfrak{q}_+]$ is $v = (1, 0, 0, 1, 1, 0, 1)^t$. The source matrices for the actions of $S, U$ in those coordinates are

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The solution space to $Xv = 0$, $XC_2 = SX$, $XC_3 = UX$ has basis

$$X_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Chinese remainder lifts of local units, chosen to be one at the other dyadic prime, have binary ray logarithm columns

$$g = (0, 1, 1, 0, 0, 0, 0)^t, \quad h = (0, 0, 0, 1, 0, 0, 0)^t, \quad j = (0, 0, 0, 1, 0, 1, 1)^t.$$

The first prime supplies $g, h$ and the second supplies $g, j$. For $\alpha^2 + \alpha + 1 = 0$, evaluation of the two equivariant maps followed by the left $U$-eigenrow $(1, \alpha)$ gives, respectively,

$$\begin{pmatrix} 0 & 1+\alpha \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \alpha \\ 1 & 1+\alpha \end{pmatrix}.$$

Both determinants are nonzero, and the conjugate eigenrow has the same property. Hence vanishing on local units in either one of the two $C_3$ eigencharacters, at either prime, kills the global depth-four class. These conclusions concern the characteristic-two normal error; the order of tame inertia after reduction in the joint trace order is checked separately. The matrices and local logarithms are reproduced in [20, `ramified/d6_d10_d11/d10`].

# E Auxiliary fields over the field of discriminant twenty-one

The unit assertions involve only a number field of degree twelve. One defining polynomial for $A$ is

$$P(X) = X^{12} - 3X^{11} + 8X^{10} - 9X^9 + 9X^8 + 6X^7 + X^6 \\ + 6X^5 + 9X^4 - 9X^3 + 8X^2 - 3X + 1.$$

The field defined by $P$ has discriminant $21^6 5^8$ and signature $(0, 6)$. The certificate reconstructs the embeddings of $u$ and $z$, checks their equations and the norm-one relations, and tests the sixteen products at the two dyadic primes. Integrality and unit status follow directly from their monic minimal polynomials with constant coefficient $\pm 1$. It uses only `nfinit`, exact field arithmetic, Galois conjugates and local-power tests; no computation of a full unit group or class group is used in the proof. The exact calculation is supplied in [20, `d21`].

The arithmetic and interval certificates are:

- `check_small_unit_formulas.gp`: the two-orbit unit identities and local squareclass kernels in Lemma 28.8;
- `verify_split5_local.py`: the two compact-type Brauer-character identities, calculated in $\mathbf{Z}[\zeta_{24}]$;
- `certify_K21_SC_SC.py`: real-ball evaluations of (11), with each comparison asserted only when its lower and upper balls are disjoint;
- `K21_SC_SC_arb_certificate.json`: the full-precision interval enclosures for Table 3.

The smallest gap, in the distinct-conjugate row, is greater than 0.00974417509. The analytic calculation certifies the displayed inequalities; the group and local hypotheses are proved in the preceding sections. The exact calculation is supplied in [20, `d21/inherited_python`, `d21/split5`].

# F Normal Selmer calculations over the field of discriminant twenty-one

The certificate `d21_point_field_selmer.gp` in [20, `audit_simplifications`] computes the mixed normal space of Section 28.2.2 in the cubic point field $L_{\mathrm{m}}$. Every class and unit computation has absolute degree six. The script verifies the embedding into the residual field, certifies class number one, and constructs the full $S$-unit group for $S$ above two and five. Applying $[u] \mapsto [u/N_{L/K}(u)]$ gives the complete norm kernel, of rank three.

The script prints its generators and enumerates their eight products. Before computing a quadratic relative discriminant it multiplies the radicand by a rational square to make every polynomial coefficient integral. The dyadic discriminant exponent is the quadratic conductor. The conductor-at-most-four subspace has dimension one, and its nonzero element is unramified at the degree-one special prime with Hilbert symbol $(u, 5) = -1$. These are basis-independent assertions checked explicitly by the script.

The scripts in [20, `d21/arithmetic`] also verify the elementary quartic ray classification. The nonsolvable source branch uses the directed discriminant comparisons in `certify_K21_SC_SC.py` and `K21_SC_SC_arb_certificate.json` in [20, `d21/split5`]. The normal Selmer computations above have no analytic hypothesis.

# G The scalar trace generator over $\mathbf{Q}(\sqrt{14})$

**Proposition G.1.** *Let $\mathcal{R}$ be a system of weight two and cyclotomic determinant over $K = \mathbf{Q}(\sqrt{14})$, good outside five and special at both five-places. Suppose its dyadic constituents are nonordinary crystalline and have scalar residual semisimplification. In the simultaneous orders of Section 11 one has*

$$R_0 = \mathbf{Z}_2[a_2] = \mathbf{Z}_2[a_7], \qquad R_0/2R_0 \simeq \mathbf{F}_2[t]/(t^n), \quad n = \operatorname{rank}_{\mathbf{Z}_2} R_0, \quad t = \overline{a_2}.$$

*Proof.* Put $R = R_0/2R_0$, $E = E_0/2E_0$, and $J = \operatorname{rad}(E)$. Choose $c \in \mathbf{Q}_2$ with $c^2 = -7$, which

exists since $-7 \equiv 1 \pmod 8$. In $K_2$ the uniformizer $\pi = \sqrt{14}/c$ has Eisenstein polynomial $E_\pi(u) = u^2 + 2$. Lemma 11.9 therefore gives

$$\mathcal{D}(\Phi(u)) \in (u^2+2)R_0[[u]], \qquad T(\Phi(0)) = a_2.$$

Every functional on $J/J^2$ gives a middle selfextension of the regular group. Lemmas 11.5 and 25.1 put its character in $\langle \alpha, \beta, \gamma \rangle = \langle [-1], [2], [5] \rangle$. Choose a real involution, a Frobenius at one fixed five-place acting trivially on [5], and a tame inertia generator there. Their evaluation vectors on $(\alpha, \beta, \gamma)$ are

$$(1,0,0), \quad (0,1,0), \quad (0,0,1).$$

Their differences from one, denoted $A, B, C$, span $J/J^2$ and therefore generate $E$.

At the selected five-place the common exact special sign can be normalized simultaneously. The normalized local differences satisfy $L(g)L(h) = (\chi_2(g) - 1)L(h)$ in the actual integral order. Modulo two their products and traces are zero. The real involution and polarized Cayley–Hamilton give

$$\begin{gathered} A^2 = B^2 = C^2 = BC = CB = 0, \\ AB + BA = X = T(AB), \qquad AC + CA = Y = T(AC). \end{gathered} \tag{103}$$

Lemma 11.9a gives these six normal words and $\mathfrak{m}_R = (X, Y)$. Work in $R' = R/\mathfrak{m}_R^2$ with the original trace and determinant laws. At two the ramified functional on $(\alpha, \beta, \gamma)$ is $(1, 1, 0)$, while $\gamma = [5]$ is the unramified character. Consequently the regular Kisin generator has

$$\begin{gathered} \Phi = N + u(1 + M) + O(u^2), \qquad N = A + B + Z, \\ M = m_A A + m_B B + m_C C + Z', \qquad Z, Z' \in J^2, \qquad m_C = 1. \end{gathered}$$

These coefficients are obtained by restricting the ramified and unramified functionals on the full allowed character space; this also applies when the actual first layer is a proper quotient.

Lemma 11.9a, with $(a, b, c) = (1, 1, 0)$, gives the constant and linear coefficients

$$0 = \mathcal{D}(N) = X, \qquad a_2 = T(N) = (m_A + m_B)X + m_C Y = Y.$$

Nakayama makes $R$ a quotient of $\mathbf{F}_2[t]$ with radical generated by $\overline{a_2}$. Its dimension is $n$, whence $R = \mathbf{F}_2[t]/(t^n)$ and $R_0 = \mathbf{Z}_2[a_2]$.

For the second generator, write $g - 1 = aA + bB + cC \pmod{J^2}$, with $a, b, c \in \mathbf{F}_2$. Since $\det(g) = 1$ modulo two, the determinant identity gives $T(g) = \mathcal{D}(g - 1)$. The norm formula of Lemma 11.9a gives

$$T(g) = abX + acY = acY \quad \text{in } R'.$$

The unique prime above seven is a good place with residue field $\mathbf{F}_7$. At its Frobenius, the classes $(-1, 2, 5)$ evaluate as $(1, 0, 1)$. Hence $a_7 = Y = a_2$ in the relative cotangent, and Nakayama proves $R_0 = \mathbf{Z}_2[a_7]$. □

# H Local and scalar comparisons over $\mathbf{Q}(\sqrt{15})$

## H.1 Weight-two local replacements at seven

In this section write $\chi = \chi_7$ and $\omega = \overline{\chi}$, and use Hodge–Tate weights with $\mathrm{HT}(\chi) = 1$. Scalar crystalline means crystalline after at most a scalar quadratic twist. Quadratic special means special with nonzero monodromy, after at most such a twist. Write $C_4, C_8$ for the determinant-one tame cuspidal types whose inertial characters have order four and eight respectively.

**Lemma H.1.** *Let $\bar{r} : G_{\mathbf{Q}_7} \to \mathrm{GL}_2(\overline{\mathbf{F}}_7)$ have determinant $\omega$. It admits a potentially semistable lift of weights $\{0,1\}$ and determinant $\chi$ which is scalar crystalline, quadratic special, of type $C_4$, or of type $C_8$. The choices in the split-seven reduction retain the actual residual representation, including its extension class.*

*Proof.* For Serre weights put $W(a,b) = \mathrm{Sym}^a \mathbf{F}_7^2 \otimes \det^b$, with $b$ taken modulo six. In our determinant normalization the relevant weights have $a + 2b = 0 \bmod 6$, so these weights are self-dual. The tame potentially Barsotti–Tate criterion of Caraiani–Emerton–Gee–Savitt [13, Lemma A.4] gives a potentially crystalline weight-two lift when an appropriate Serre weight occurs in the compact type. Apply it to the dual residual representation, then dualize the resulting lift, since its convention has $\mathrm{HT}(\chi) = -1$ and Hodge type zero $\{0,1\}$. The determinant-one types are self-dual. For the weight calculation in this convention, a crystalline $W(a,b)$-lift of the dual has weights $\{-a-b, 1-b\}$. After dualizing and using our sign convention, shifting both weights by $a+2b$ gives $\{b, a+b+1\}$ and preserves the residual representation, since $6 \mid a+2b$. This identifies the weights computed below with those in the criterion. Equivalently one can use the Gee–Kisin Breuil–Mézard criterion [43, Corollary 4.5.6]. The determinant ratio of the lift to $\chi$ is unramified and reduces to one, since the type has determinant one. Its inverse square root reducing to one corrects the determinant.

For the tame cuspidal type parametrized by $\omega_2^{6t}$, where $\omega_2$ is a niveau-two fundamental character and $t = 1, 2, 3$, the compact reduction has factors

$$W(2t-2, 1-t), \qquad W(6-2t, t).$$

Indeed, on eigenvalues $x, y$, with $w = x/y$, their Brauer characters are $\sum_{j=1-t}^{t-1} w^j$ and $y^6 \sum_{j=t}^{6-t} w^j$. On a regular split element $y^6 = 1$, so the sum is zero. On a regular nonsplit element take $y = x^7$; then $y^6 = w$, and the sum is $-w^t - w^{-t}$. These are the cuspidal character values. On a scalar the sum is six, the dimension of the cuspidal representation. Thus $t = 2$ gives $C_4$ with factors $W(2,2), W(2,5)$. The choices $t = 1, 3$ give the two order-eight types, containing $W(4,1), W(4,4)$ respectively.

Suppose first that $\bar{r}$ is irreducible. Wild inertia has nonzero invariants and is normal, so it acts trivially. The two tame characters are $\omega_2^u, \omega_2^{7u}$, where $u = 1 \bmod 6$. The unordered pairs modulo 48 are

$$(1,7), \quad (25,31), \quad (13,43), \quad (19,37).$$

For $(1,7)$ induce a crystalline character of the unramified quadratic extension with labelled weights zero and one. The given residual unramified factor can be retained, and an unramified scalar corrects the determinant. Twisting by a ramified quadratic character gives $(25,31)$, since the twist adds 24 to both exponents. For the last two pairs, the exponents $a+1+8b$ of $W(2,5)$ and $W(2,2)$ are 43 and 19 respectively. Both pairs therefore have $C_4$ lifts.

For a reducible $\bar{r}$, the two inertial exponents sum to one modulo six. The unordered possibilities are $\{2,5\}$, $\{0,1\}$, and $\{3,4\}$. For $\{2,5\}$ the ordered exponents $(a+1+b, b)$ of $W(2,2)$ and $W(2,5)$ give the two orderings. The ratio on inertia is $\omega^3$, so there is no exceptional cyclotomic extension class. Thus the explicit Serre weights and the tame criterion give $C_4$ for the actual extension in either ordering.

For $\{0,1\}$ or $\{3,4\}$ remove a scalar quadratic twist if needed, leaving exponents zero and one. In the split case lift the characters with weights zero and one and with inverse unramified factors. In the nonsplit case with invariant line of exponent one, the residual characters have the form $\omega\bar{\alpha}, \bar{\alpha}^{-1}$ with $\bar{\alpha}$ unramified. If $\bar{\alpha}^2 \neq 1$, use the characters $\chi\alpha, \alpha^{-1}$ for an unramified lift $\alpha$. Local Tate duality gives $H^2(\mathbf{Q}_7, \overline{\mathbf{F}}_7(\omega\bar{\alpha}^2)) = 0$. The integral $H^2$ is then zero by reduction and Nakayama, so the given extension class lifts integrally. In characteristic zero the Bloch–Kato dimension formula gives

$$H^1_f(\mathbf{Q}_7, E(\chi\alpha^2)) = H^1(\mathbf{Q}_7, E(\chi\alpha^2)).$$

This yields the scalar crystalline choice. If $\bar{\alpha}^2 = 1$, take the common unramified quadratic lift and use the Kummer description of the resulting cyclotomic extension. A residual class supported on units lifts to the crystalline subspace. If the residual valuation coordinate is nonzero, lift the unit and valuation coordinates, giving a special lift with nonzero monodromy. A special lift of a unit class is also possible by giving its lifted valuation coordinate a nonzero multiple of seven.

Finally, if the invariant line has exponent zero and the quotient has exponent one, the relevant ordered Serre weight is $W(4,1)$. For the original pair $\{3,4\}$ it is $W(4,4)$. The ratio on inertia is inverse-cyclotomic, so again the exceptional cyclotomic extension case does not occur. Both weights lie in the indicated order-eight types. Restore any removed quadratic twist in the scalar crystalline and special cases. It preserves the determinant. □

## H.2 Marked ordinary lattices

For the scalar comparisons put $K = \mathbf{Q}(\sqrt{15})$, $s = \sqrt{15}$ and $H = K(i)$. Write $\chi_d$ for the quadratic character of $K(\sqrt{d})/K$, written additively in tangent formulas. Put $h = \chi_{-1}$ and write $\varepsilon_2$ for the two-adic cyclotomic character. The class number of $K$ is two and its dyadic prime represents the nontrivial class. Since the fundamental unit $4 + s = (3 + s)^2/6$, the squareclasses with even valuation at every odd prime are generated by $-1, 2, 3$, and all have even dyadic valuation. The finite-unramified quadratic characters are $\langle h, \chi_3 \rangle$, of which precisely $1, h$ split at the dyadic completion $\mathbf{Q}_2(i)$.

Apply Lemma 4.10 to the marked ordinary vector $v$. The resulting nonzero additive residue is trivial at two. We determine its character before normalizing a real conjugation $c$. The supported squareclasses of even dyadic valuation for the residual extension are generated by $-1, 2, 3, 7$, as in Section 26.2. Whenever one seven-place has scalar inertia, its residual inertia vanishes. The two seven-valuations have equal parity, so 7 is excluded. The dyadic splitting condition then leaves only $h$. Thus the normalized basis $((c-1)v, v)$ gives residue $1 + hE_{12}$, real matrix $\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and ordinary line $e_2$.

For a relative tangent to a marked entry order, write $(1 + \delta X)(1 + hE_{12})$. Lemma 12.1a gives $X = \left(\begin{smallmatrix} x & y \\ z & x \end{smallmatrix}\right)$, additive $z, x + y$, and $dx = dy = h \smile z$, with $X(c) = 0$. At two, $y = 0$ and $x$ is unramified. For the unit Kummer condition on $z$, the stable marked lattice has a multiplicative sublattice and an étale quotient. Passing to its torsion tangent quotient by schematic closure, the span of $e_1, \delta e_1, \delta e_2$ modulo $\delta e_1$ has étale quotient and multiplicative line, with extension coordinate $z$. The finite-flat Kummer classification gives the unit condition.

**Lemma H.2.** *If the tangent cochains are unramified at the odd places, then $z = y = 0$ and $x \in \mathbf{F}_2 \chi_3$.*

*Proof.* The possible nonzero lower coordinates are $\chi_2, \chi_3, \chi_6$. For $d = 2, 3, 6$, a solution to $dx = h \smile \chi_d$ restricts to $G_H$ as the Kummer class of $\alpha_d \beta$, for some $\beta \in K^\times / K^{\times 2}$, where

$$\alpha_2 = 1 + i, \qquad \alpha_3 = (1 + i)(3 + is)/4, \qquad \alpha_6 = \alpha_2 \alpha_3.$$

Indeed $\mathrm{N}_{H/K}(\alpha_d) = d$. Tracking the signs of square roots of $\alpha_d$ and its conjugate gives a cochain with coboundary $\chi_d \smile h$, and adding the pointwise product $h\chi_d$ gives the required ordering. All solutions differ by a quadratic character of $K$.

For $d = 2$, odd unramifiedness forces even odd valuations of $\beta$. It has even dyadic valuation, while $\alpha_2$ has odd valuation, contrary to unramifiedness of $x$ at two. For $d = 3, 6$, the only odd valuation of $\alpha_d$ is over three, where $H/K$ is unramified inert. Then $\beta = (3 + s)\beta'$ with $\beta'$ in the rational squareclass span of $-1, 2, 3$. At a dyadic completion the nontrivial automorphism $\tau$ over $\mathbf{Q}_2$ negates $s, i$. The ratios $\alpha_j^\tau / \alpha_j$ for $j = 2, 3$ have class $[i] = [2]$, and $(3 + s)^\tau / (3 + s)$

has class $[-6]$. Thus $(\alpha_d\beta)^\tau/(\alpha_d\beta)$ has class $[-3]$ or $[-6]$, respectively, and neither is trivial in $\mathbf{Q}_2(i)$. An unramified or split local class would be fixed by $\tau$. This excludes all three nonzero lower coordinates. Now $x, y$ are additive, unramified at finite places, and vanish at $c$. Since $y$ is split at two, the finite-unramified character calculation gives the assertion. □

## H.3 Two scalar crystalline seven-places

**Proposition H.3.** *Assume GRH for Dedekind zeta functions. Every such system of profile* CR/CR*, good away from seven, is modular, for either choice of the two scalar inertial signs.*

*Proof.* On every stable dyadic lattice the residual inertia at seven is trivial, since both exact inertia matrices are rational scalars of order dividing two. The unramified-terminal nonsolvable comparison in Section 26.2 applies. The solvable irreducible case is excluded by the dihedral ray calculation, since its unique candidate ramifies at both seven-places. In the reducible case the residual semisimplification is $1 \oplus 1$. For an ordinary member compare it with its $h$-twist in the marked basis. Their exact scalar inertial signs at each seven-place agree, so the tangent cochains are unramified away from two. By Lemma H.2 their simultaneous entry order is generated by the ordinary quotient value $\nu$ on dyadic Frobenius. Indeed, $\nu - 1$ detects the only possible tangent $\chi_3$, and Nakayama gives $\mathbf{Z}_2[\nu]$. The value $\nu$ is identical in both factors because $h$ is split at two. Hence the matrices agree, giving the quadratic self-twist and automorphic induction.

Suppose the dyadic source is nonordinary. Fix the two exact scalar characters on inertia, and take a simultaneous packet with this profile. Use the integral trace order $R_0$ and the actual regular matrix order $E_0$ of Section 11, restricted to the selected local factors. The left regular lattice is biconnected Barsotti–Tate by Lemma 11.8. Write $E = E_0/2E_0$, of dimension four times the trace rank over $\mathbf{F}_2$. Terminal inertia is trivial on $E$, since the fixed scalar signs are rational. Let $H = K(i, \sqrt{3})$, the finite-unramified genus field of degree four over $K$. Let $C \subset E$ be the span of the image of $G_H$ and $J$ its radical. The regular $G_H$-module $C$ is an actual generic submodule of $E$, so it has a biconnected finite-flat model at each dyadic place of $H$ by schematic closure.

Functionals on $J/J^2$ give actual middle/middle finite-flat subquotients of conductor at most two at those places, by the middle-model calculation of Lemma 11.5. They inject into the quadratic ray characters of $H$ of dyadic modulus $\prod_{w|2} w^2$. The class group of $H$ is trivial and the ray group at this modulus is $C_2$, as certified in [59, `d15_support/scalar_round3/nonordinary_scalar_cr_cr.gp`]. Hence $J/J^2$ has dimension at most one, and a nonzero character ramifies at some dyadic place. In this case $C$ is monogenic, with radical generated by $X$. At such a ramified dyadic place its regular Kisin module is free of rank one over $(k \otimes_{\mathbf{F}_2} C)[[u]]$, where $k$ is the local residue field. Indeed right multiplication extends by the middle-model full faithfulness, its radical quotient is the unique middle model, and exactness and Nakayama give freeness by equality of ranks. The Frobenius is $u + V(u)$, with $V(0) \equiv \alpha X \bmod J^2$, $\alpha \neq 0$, as seen from the ramified Artin–Schreier coordinate of that middle selfextension. Height one makes $u^2(u+V)^{-1}$ integral. Modulo $J^3$ its coefficient of $u^{-1}$ is $\alpha^2 X^2$, so $J^2 = J^3 = 0$. In either case $\dim_{\mathbf{F}_2} C \leq 2$.

The four translates of $C$ span $E$, giving trace rank at most two for the fixed-profile packet. Apply this bound to the four finite-unramified rational quadratic twists generated by $\chi_{-1}$ and $\chi_3$, where here the subscript denotes a quadratic radicand. These twists preserve both exact signs. Without an actual nontrivial self-twist, the stabilizer of each local coefficient orbit acts faithfully on its trace field. The sum of the local trace degrees would then be divisible by four, contrary to the bound of two. A quadratic self-twist gives automorphic induction from a de Rham character. Strong compatibility concludes. □

## H.4 Nonordinary scalar sources with one strict place

**Proposition H.4.** *A comparison system of profile* SP/CR, C4/CR *or* C8/CR *is modular if a dyadic member is nonordinary crystalline with trivial residual semisimplification.*

*Proof.* For these profiles, the first layer of the regular matrix order is detected by $\chi_{-1}, \chi_3$, using the exact scalar sign at the crystalline seven-place and the conductor bound of Lemma 11.5. For the fixed special or order-four cuspidal profile, let $A_1 = c - 1$ and $A_2 = F - 1$, with the exact special sign removed where applicable. The trace-order relations are

$$A_1^2 = A_2^2 = 0, \qquad A_1A_2 + A_2A_1 = tI.$$

If the trace rank is $s$, the four words $1, A_1, A_2, A_1A_2$ are free over $\mathbf{F}_2[t]/(t^s)$. For $s \geq 3$, sending $A_1$ to $E_{12}$ and $A_2$ to $tE_{21}$ modulo $t^3$ gives an abelian image on $G_{K(i,\sqrt{3})}$ containing the order-four element $(cF)^2$. The dyadic conductor is at most four and the strict local inertia is of exponent at most two, with the rational uniformizer killed after the unramified quadratic base change. The corresponding ray quotient has exponent two, a contradiction. Thus $s \leq 2$. In the special case, the local products $(r(g) - 1)(r(h) - 1)$ vanish modulo two after sign removal. If $s = 2$, the depth-two ray bound makes the kernel over $K(i, \sqrt{3})$ of order at most two. The group algebra of the resulting quotient of order at most eight must surject onto the regular algebra of dimension eight. Equality would make a product of strict inertia and Frobenius differences nonzero in the group algebra and zero in the regular algebra. Here the kernel has nontrivial strict inertia if it is nontrivial, by the equality of the depth-one and depth-two ray quotients and the class number one of $K(i, \sqrt{3})$. This gives $s \leq 1$ for the special profile.

For the order-eight profile the same four-word relations hold. Writing $T = a + bA_1 + cA_2 + dA_1A_2$ for the strict tame generator, the first layer and determinant give $\tau = \operatorname{tr} T = td \in (t^2)$. If $s \geq 5$, then $\tau^2 = 0$ implies $\tau \in (t^3)$. In the preceding quotient modulo $t^3$, the strict local Frobenius and inertia again generate an elementary abelian group. The same ray contradiction gives $s \leq 4$.

For the special profile, twisting by $\chi_{-3}$ preserves the full local profile and a packet of two exceeds the rank bound one unless there is an actual self-twist. For the order-four and order-eight profiles the four finite-unramified genus twists preserve the exact scalar inertia and the cuspidal orientation. In the order-four case the rank is at most two. In the order-eight case every local trace field contains $\mathbf{Q}_2(\sqrt{2})$, fixed by the twists, so the rank would be at least eight in the absence of a self-twist. Each nontrivial actual self-twist gives modularity by automorphic induction. The ray computations are given in [59, `d15_support/scalar_work/nonordinary_scalar_sp_cr_ray.gp`].

□

## H.5 Ordinary scalar cusp sources

**Theorem H.5.** *Assume GRH for Dedekind zeta functions. A system with profile* C4/CR *or* C8/CR *is modular if all its dyadic members are ordinary with trivial residual semisimplification.*

*Proof.* The twist by $\chi_{-7}$, if needed, makes the scalar crystalline place good and is split at two. Fix the cuspidal orientation and take the packet under the four finite-unramified twists $V = \langle \chi_{-1}, \chi_3 \rangle$. The marked ordinary lattice has residual representation $1 + \chi_{-1}E_{12}$ after fixing a real conjugation. Let $\nu$ be its ordinary dyadic Frobenius quotient. In the cyclic basis at the cusp, with tame generator $T$ and arithmetic Frobenius $F$, write

$$T = \begin{pmatrix} a & b \\ 7b & \tau - a \end{pmatrix}, \qquad F^2 = -7I, \qquad 7b^2 = a\tau - a^2 - 1.$$

Here $\tau = 0$ for C4 and $\tau^2 = 2$ for C8. In the marked tangent, the trace of $F$ and the inversion relation $FTF^{-1} = T^{-1}$ give $z(F) = z(T) = 0$. The supported classes and the good seven-place then force $z = 0$ or $\chi_2$, since $\chi_3$ is nonsplit at the cusp. The additive $x + y$ is unramified at two and at the good seven-place. Ideal-class parity makes it unramified at the cusp, so $x(T) = y(T)$. If $z = \chi_2$ and $x(T) = 0$, the cup-product calculation for $\alpha_2$ in Lemma H.2 contradicts dyadic unramifiedness. Thus $b$, whose tangent is $y(T)$ in the cyclic basis, detects the nonzero lower direction. When $z = 0$, parity makes $x, y$ unramified at the cusp, so only the $\chi_3$ direction remains. It is detected by $\nu$. Nakayama gives $B = \mathbf{Z}_2[\nu, b]$. The two generators transform by independent characters of $V$. With $A = \mathbf{Z}_2[\nu^2, b^2]$, the integral grading is $B = A \oplus \nu A \oplus bA \oplus \nu bA$. The trace $a_3$ at the prime above three has the degree of $b$, so belongs to $bA$. Its tangent is $h(\mathrm{Frob}_3)z(\mathrm{Frob}_3)$. Both $h$ and $\chi_2$ are unramified nonsplit at three, so this detects the same direction as $b$. Thus $a_3/b$ is a unit whenever the $b$-direction occurs. If it did not occur, the grading and Nakayama would give $b = 0$, which is excluded by the reciprocity calculation below.

Modulo $b$, the action is monomial over $H = K(i)$. Its diagonal character $\vartheta$ satisfies $\vartheta^\sigma \vartheta = \varepsilon_2$ for the nontrivial $\sigma \in \mathrm{Gal}(H/K)$. At the cusp the character takes the value $a$ on a tame generator. Its two dyadic factors at the principal element $1 + i$ have product one, since the marked ordinary quotient is unramified, $(1 + i)/(1 - i) = i$ is a unit, and the cyclotomic character on the local Artin image of $1 - i$ is trivial. The cusp logarithm of $1 + i$ is 2 mod 4 and the other odd factors are unramified. Thus reciprocity gives $a^2 = 1$ in $B/(b)$. The cusp equations give $2 = 0$ there. The anticyclotomic ray calculation, with both dyads unramified, cusp inertia of exponent at most two and the cusp uniformizer killed, gives $C_4$, by [59, `d15_support/audit15/ordinary_cusp7_ray.gp`]. Therefore $B/(b)$ has length at most four.

The length is the sum of $[D_j : \mathbf{Q}_2]v_2(a_{3,j})$ over the distinct local trace-field factors $D_j$. These are the marked entry fields because dyadic inertia has distinct rational eigenvalues and the marked ordinary line is defined over the trace field. The length formula is the index of multiplication by $b$. At exact level $\mathfrak{p}^2$ of norm 49, the computation in [59, `d15_support/audit15/ordinary_cusp7_seeds.m`] and the corresponding residual checks give an ordinary scalar C8 form of degree sixteen with $\mathrm{N}(a_3) = 452^2$, and an ordinary scalar C4 form of degree two with $a_2 = 1$ and $a_3^2 = 2$. The Atkin–Lehner signs distinguish these cuspidal types. The principal-series possibility is excluded by the residual checks in [59, `d15_support/audit15/ORDINARY_SCALAR_CUSP_SEVEN.md`], using the trivial dyadic quotient and the one-sided ray groups. For the degree-sixteen seed, the only nonsolvable possibility is $\mathrm{SL}_2(\mathbf{F}_4)$. Its scalar dyadic quotient bounds the local different by $5/4$ in Lemma 11.14. The compositum with $K(i, \sqrt{3})$ has absolute degree 480 and root discriminant less than 35.25, while the Euler terms at two, three, five and the good seven-place give a lower bound greater than 45.33 [59, `d15_support/audit15/ordinary_cusp7_seed_intervals.py`]. A residual inertia of order three at the cusp is excluded by [59, `d15_support/audit15/ordinary_cusp7_ps_ray.gp`], and the unique solvable irreducible residue with inertia of order at most two ramifies at both seven-places. Conjugation by the nontrivial automorphism of $K$ supplies the other orientation.

The degree-sixteen form contributes four, exhausting the length bound for C8. A nonmodular packet would contribute positive additional length. For C4, a putatively nonmodular four-twist packet has positive even length. If it has length two, the cusp inertia dies in its monomial quotient, as it is the order-two subgroup of the anti ray $C_4$. Hence $a - 1 \in b^2 A$, and the determinant equation makes $b^2$ associate to two. The packet then has total trace degree at most four. The orbit calculation gives a coefficient-three realization over $\mathbf{F}_3$ or $\mathbf{F}_9$, and the cusp protects cyclotomic irreducibility. The small-image criterion and weight-two lifting give modularity [3]. The other possible packet length is four, which the degree-two modular form and its twist exceed when adjoined. □

## H.6 Ordinary scalar special sources

**Proposition H.6.** *A comparison system of profile* SP/CR *is modular if all its dyadic members are ordinary crystalline with trivial residual semisimplification.*

*Proof.* Twist by $\chi_{-7}$, if needed, to make the scalar crystalline place good. Form a packet closed under $\chi_{-3}$, which is trivial at both seven-places. Use $h$ to fix the sign $\lambda_0 = \pm 1$ of the special quotient on arithmetic Frobenius $F$. Both exact scalar signs on inertia are allowed. Adjoin their tuple $\mu$ at a tame generator $T$ to the marked entry order $A$. Put $N = \mu^{-1} r(T) - I$, so $N^2 = 0$, $NF = \lambda_0 N$, and $FN = 7\lambda_0 N$. In the basis $(cv, v)$ put $n = N_{11}$.

The tangent equations in the real basis give $z(F) = 0$ because $F$ has constant trace. Its eigenline is generated by $(1, \ell)$ with $\ell$ in the maximal ideal, as $F - \lambda_0 I$ has unit upper-right entry. The tangent of $N$ can therefore have only an upper-right entry at $T$. Thus $z(T) = 0$, $x(T) = d\mu$, and $y(T)$ is that entry. The same parity and cup-product calculations as for Theorem H.5 give $z = 0$ or $\chi_2$, with $x(T) = y(T) = 1$ in the latter case. Changing to the basis $(cv, v)$ makes $n$ detect this latter direction, and $\nu$ detects the remaining $\chi_3$ direction. Hence $A = \mathbf{Z}_2[\nu, n]$. For a fixed inertial sign the tangent of $N$ vanishes and the order is $\mathbf{Z}_2[\nu]$.

Adjoin the $h$-twists to form $B$, still with $\mu$ adjoined. In the basis $(cv, v)$, an entry at $g$ has $h$-sign $h(g)$ on the diagonal and $h(g)h(c)$ off the diagonal. The generators of $A$ are even, so $B = A \oplus \lambda I$, with $\lambda = (\lambda_0, -\lambda_0)$ in the rational product and $I$ an ideal of $A$. Write $N_{12} = \lambda u$ and $F_{22} = \lambda d$. The relation $NF = \lambda N$ gives

$$n = u(1 - d)/F_{12}.$$

Here $F_{12}$ is a unit and $d$ is in the maximal ideal, since $F$ reduces to the swap matrix. The line equations give $N_{21} \in \lambda u A$ and $N_{22} = -n$. Thus the monodromy ideal is $(u, \lambda u)$. The element $u$ is nonzero on every characteristic-zero factor by the nonzero monodromy condition.

Modulo this ideal, the tangent at $T$ is scalar and the additive $x + y$ vanishes on inertia by parity. Hence all tangent cochains are unramified at the odd places. Lemma H.2 and Nakayama make the quotient generated by the even element $\nu$, so $I = (u)$. On $G_H$ the off-diagonal products in the swap basis vanish modulo $u^2$. Its first diagonal gives a character $\vartheta$ over $A/(u^2)$ with $\vartheta^\sigma \vartheta = \varepsilon_2$. Its inertia value is $\mu(1 + n)$. Reciprocity on $1 + i$ as above gives $2n = 0 \bmod u^2$, since the local logarithm is twice an odd number, $\mu^2 = 1$, and $n^2 = 0 \bmod u^2$. As $n/u$ is a unit and $u$ is a nonzerodivisor, $2 \in uA$.

Modulo $u$ the character is anticyclotomic, unramified at both dyadic places, of inertia exponent at most two at the strict place and trivial on its uniformizer over $H$. For the last assertion $F$ becomes antidiagonal with $F^2 = -7I = I$ in characteristic two. The $C_4$ ray quotient therefore bounds the length of $A/(u)$ by four. The level-seven forms in [59, `d15_support/certificates/ordinary_special7_seeds.m`] have trace field $\mathbf{Q}(\sqrt{3})$, $a_2^2 = 3$ and $a_3 = \pm 2$. Their opposite Atkin–Lehner signs give the two choices of $\lambda_0$. The real nonzero traces at two and three make the narrow central character trivial, and the conductor exponent one gives unramified special type. Their dyadic residue field is $\mathbf{F}_2$, and the solvable classification of Section 26.2 gives trivial residual semisimplification. For the seed with the matching $\lambda_0$, the fixed-sign entry order is $\mathbf{Z}_2[a_2] = \mathbf{Z}_2[\sqrt{3}]$. The monodromy entries have zero tangent, hence valuation at least one. The divisibility $2 \in uA$ gives $v_2(u) = 1$ there. Adjoining this seed makes $2/u$ a unit in the joint order. Its rank is therefore at most four. The seed contributes two, leaving at most two for a hypothetical nonmodular packet. Thus its global trace field $D$ is rational or quadratic. In the quadratic case the packet twist $\chi_{-3}$ gives the nontrivial coefficient automorphism, negating $a_2$. Hence $D = \mathbf{Q}(a_2)$ and $a_2^2 \in \{3, 5, 7\}$ by ordinarity and the Weil bound.

If a coefficient-five place has residue field $\mathbf{F}_5$, an irreducible residue is handled by [3, Theorems 1.2 and 2.3], or by [82, Theorem 7.5] when the cyclotomic restriction is reducible.

Its cyclotomic quadratic subfield is the real field $K(\sqrt{5})$. For a reducible residue write the constituents as $\psi, \omega_5\psi^{-1}$. The character $\psi^2$ is quadratic and unramified away from five. The prime above five has the nontrivial ideal class, so valuation parity also makes $\psi^2$ unramified at five. Its even character is either trivial or the character of $K(\sqrt{5})/K$. The possible finite-flat rank-one tame exponents at five are $0, 1, 2$. Triviality of $\psi^2$ on inertia excludes the middle exponent. The actual crystalline source is ordinary by the finite-flat uniqueness at absolute ramification index $2 < 5 - 1$. Its residual ratio is odd, extends to $\mathbf{Q}$, and is nontrivial at the unique five-place. Apply [75, Section 4.5, Theorem A], with the residual quotient labelled by the actual ordinary filtration.

If $D = \mathbf{Q}(\sqrt{3})$ or $\mathbf{Q}(\sqrt{7})$, the residue at five is over $\mathbf{F}_{25}$ with inner twist $\eta = \chi_{-3}$. For an irreducible residue, Lemma 12.5 gives projective descent to $\mathbf{F}_5$ with totally odd squareclass, since $\eta$ is totally odd and $-1$ is a square in $\mathbf{F}_5$. The preceding irreducible transfer applies. If the residue is reducible, the fifth-power coefficient Frobenius switches the two constituents after twisting by $\eta$, as seen at real conjugation. This gives $\psi^6 = \omega_5\eta$. The character $\psi$ has 2-power order by its local inertia and the class group, so it has order eight. The residual ratio is then $\psi^4\eta = \omega_5^2\eta = \chi_{-1}$. At the special seven-place this ratio takes value $-1$ on Frobenius, whereas the exact special eigenvalues give ratio 7 or $7^{-1}$. These are distinct from $-1$ modulo five. Thus this reducible case is impossible. This treats both scalar inertia signs at the special place. □

# I Congruence fields over $\mathbf{Q}(\sqrt{15})$

Retain the ordinary residual cases of Section 26.2 over $K = \mathbf{Q}(\sqrt{15})$. In particular, the residual group is $G = \mathrm{SL}_2(k)$, $k = \mathbf{F}_q$, $q \geq 8$. The dyadic wild inertia is the whole upper-unipotent subgroup of $G$, and its unramified multiplier is nontrivial. At the ramified seven-place, the residual inertia and a Frobenius generate a Klein four group. We exclude the rows with $q \geq 16$ and obtain a uniqueness statement when $q = 8$.

## I.1 Auxiliary fields and the ordinary comparison

Put $J = K(\sqrt{-3})$, $H = K(i)$ and $C = K(i, \sqrt{3})$. These fields are finite-unramified over $K$. Their dyadic residue degrees are $2, 1, 2$, respectively, and at both seven-places they are $1, 2, 2$. The composita with a nonsolvable point field preserve root discriminants, and their local residue degrees are the least common multiples of the individual degrees.

The class group of $H$ is $C_2$. Let $\mathfrak{P}$ be its prime over the strict seven-place. The ray group of modulus $\mathfrak{P}$ is $C_8$, its dyadic Artin classes generate, and both the local uniformizer 7 and the prime over the other seven-place have Artin order two. The cyclic quartic quotient defines $A/H$. Over $K$ this is a solvable Galois field of degree eight, by stability of the quotient and modulus. Its strict local indices are $(2, 2)$, and its residue degrees at the good seven-place and at two are two and four. The depth-four ray group of $A$ at both dyadic primes is $C_4 \times C_2^7$. Splitting both strict primes cuts the quotient to $C_2^7$. Its class field $A^+$ is solvable Galois of degree 1024 over $K$. Its dyadic upper breaks are at most three because the conductor over the unramified dyadic completion of $A$ is at most four. It retains indices $(2, 2)$ at the strict place.

The depth-four ray quotient of $A$ obtained by splitting the strict primes and one dyadic prime is $C_2^2$. The quadratic subfield $B_1/A$ specified by the ray certificate splits over $A$ at the strict primes and has absolute degree 32. Its three dyadic completions have absolute indices $(2, 4), (4, 4), (2, 4)$. The depth-four ray quotient at these three primes, with the strict primes split, is $C_2^7$ under GRH. Let $B_1^+$ be its class field, of degree 2048 over $K$. Its normal closure is solvable and all strict completions remain those of $A$. At a dyadic completion over $A$, both $B_1/A$ and $B_1^+/B_1$ have upper breaks at most three. The inverse Herbrand function of the first step satisfies $\psi(u) \geq u$; the upper filtration therefore vanishes above three in the compositum of

the local conjugates over $A$. The same upper bound holds over $K_2$ because $A/K$ is unramified at two. The arithmetic data for both orientations are given by [59, `d15_support/ordinary15/ordinary_one_cr_aux.gp` and `d15_support/ordinary15/ordinary_one_cr_b1.gp`].

A nonsolvable point field is disjoint from each solvable normal closure by simplicity. For $q \geq 512$ apply Proposition 13.3 to $MA^+$, retaining the strict term $(e, f) = (2, 2)$. At $b = 25.9$ the lower bound exceeds 50.70665 at $q = 512$ and increases with degree. For $q = 256$, $MB_1^+$ and $b = 24.5$ give a lower bound above 50.48934 with the same term. Both exceed the common upper bound 50.39764. For the finite rows use $M, MJ, MH, MC, MA$ with the local bounds of Lemma 11.14 and the available Euler terms at two and both seven-places. The 923 strict comparisons in [59, `d15_support/ordinary15/ordinary_one_cr_intervals.py`] leave precisely the rows in Section 26.2.

## I.2 Two order-four cuspidal lifts

The residual tame generator and Frobenius at the ramified seven-place are simultaneously conjugate to $1 + aE_{12}, 1 + bE_{12}$, with $a, b \in k^\times$ distinct. Choose coefficient lifts $a_0, b_0$, with $a_0$ a unit, and adjoin $i$. Set

$$T = \begin{pmatrix} i & a_0 \\ 0 & -i \end{pmatrix}, \qquad F = \begin{pmatrix} x & b_0 \\ -2ix/a_0 & -x \end{pmatrix}, \qquad x^2 - (2ib_0/a_0)x + 7 = 0.$$

A root $x$ is integral and reduces to one. The equations give $T^2 = -I$, $F^2 = -7I$, and $FTF^{-1} = T^{-1} = T^7$. They define a continuous tame local lift of the actual residual representation, with cyclotomic determinant and the order-four cuspidal Weil–Deligne parameter. At the other seven-place choose an unramified lift. This is possible since the residual representation is unramified there. At two retain the ordinary crystalline weight-two component.

The rational prime 11 splits in $K$. Fix the place $v$ generated by $\pi = 2+\sqrt{15}$, of norm $-11$. At $v$ prescribe either the unramified condition, or its twist by a local ramified quadratic character. Such a twist has scalar quadratic inertia, preserves the determinant and reduces to the original local representation. At the other place over 11 and at all remaining finite places away from two and seven impose unramified conditions. Lemma 2.6(iii) applies, with nonsolvable residual image and cyclotomic determinant, and globalizes each choice. Only existence of the two dyadic lifts on the selected components is needed.

For each of the two resulting sources take the packet under the four everywhere-finite-unramified twists $V = \langle \chi_{-1}, \chi_3 \rangle$, with one representative for each distinct dyadic coefficient orbit. No actual self-twist occurs because it would make the characteristic-zero projective image dihedral, contrary to the nonsolvable residue. If $D$ is one local trace field and $W \subset V$ its coefficient-orbit stabilizer, the contribution of the packet to the sum of local trace degrees is $4[D^W : \mathbf{Q}_2]$. The fixed field contains all squares of actual traces, whose reductions generate $k$, so $[D^W : \mathbf{Q}_2] \geq f = [k : \mathbf{F}_2]$. The two packets have distinct exact scalar inertia at $v$, hence disjoint trace orbits.

## I.3 The simultaneous lattice and its congruence group

Let $A$ be the closed simultaneous trace order, a subring of the product of local coefficient integer rings. It is finite free over $\mathbf{Z}_2$, local with residue field $k$. The rational trace product is the product of the local trace fields of the distinct orbits. Indeed, a field isomorphism matching the traces of two factors would identify their coefficient orbits. Consequently

$$\dim_{\mathbf{F}_2} R \geq 8f, \qquad R = A/2A.$$

The simultaneous representations are realized over $A$. For an elementary verification, choose a common residual element with distinct eigenvalues. Its characteristic polynomial over $A$ has two roots in $A$ by Hensel lifting. Diagonalize it in the product lattice and scale the two basis vectors so that the upper off-diagonal entry of a common element with both residual off-diagonal entries nonzero is one. The two projectors, and then $E_{12}$, are in the $A$-algebra generated by the matrices. The opposite off-diagonal entry is in $A$ by taking a trace against $E_{12}$, and is a unit, so the same algebra contains $E_{21}$. Taking traces against these matrix units recovers every entry in $A$.

By Lemmas 11.8 and 11.10, the stable lattice $A^2$ is Barsotti–Tate at two and its quotient modulo two has upper breaks at most three. At the dyadic place the ordinary line over $A$ is recovered from the simple residual eigenvalues of a Frobenius lift and Hensel projectors, since all ordinary lines reduce to the unique invariant line of the residual wild subgroup. Its diagonal characters are $\chi_2\nu^{-1}, \nu$ with $\nu$ unramified. Therefore the image of dyadic inertia in $\mathrm{SL}_2(R)$ is upper unipotent. Conjugation by Frobenius multiplies the upper-right coordinate by a unit $d$ with $d-1$ a unit. The commutator map on the finite additive inertia image is bijective. Thus every abelian quotient of the simultaneous image is unramified at two.

Write $L$ for the finite linear image in $\mathrm{SL}_2(R)$. Its residual quotient is $G$. At the cusp the exact $T, F$ identities are common to both packets, so modulo rational two the decomposition group remains a Klein four group. At the CR place the common rational scalar inertia disappears modulo rational two. At $v$ the inertia image is generated by one central scalar of order at most two, corresponding to the two possible exact signs.

## I.4 Noncentrality

If the kernel of $L \to G$ were scalar, it would be elementary abelian because its determinant is one in characteristic two. The central extension splits. Over a Sylow two-subgroup of $G$ each lift of a nontrivial transvection has scalar square. The characteristic polynomial identity and a unit off-diagonal entry force its trace to vanish and its square to be one. Thus the Sylow preimage is elementary abelian. Restriction followed by corestriction multiplies the extension class by the odd Sylow index, so the global extension splits. Its scalar factor would then be an elementary abelian global character, unramified at two by the inertia commutator calculation. At the cusp it is unramified by ideal-class parity: that prime is nonprincipal and $v$ is principal, while all other finite odd primes are unramified. It is also unramified at $v$. In fact any radicand with odd valuation exactly at $v$ and even valuation at the other finite places has rational norm squareclass $-11$. To see this, divide it by $\pi$; the norm of any element of even valuation everywhere is a rational square, since the class group has order two and its even-support squareclasses are generated by $-1, 2$ and the totally positive fundamental unit. But both the split and unramified quadratic squareclasses of $K_2 = \mathbf{Q}_2(i)$ have square norm to $\mathbf{Q}_2$, the latter being represented by 3. The class $-11$ is not a dyadic square. The scalar factor thus has rank at most two, by the narrow class group $C_2^2$.

Every trace of the section $G \to L$ lies in the canonical copy of $k$ in $R$. For an odd-order element this follows by splitting the separable root-of-unity polynomial over an unramified coefficient extension. Every other nonidentity element of $G$ is an involution, whose trace is zero by Cayley–Hamilton and determinant one. Thus $R$ would be generated by $k$ and at most two scalar units squaring to one, with binary dimension at most $4f$. This contradicts the packet bound.

## I.5 The perfect core and cancellation at eleven

Let $L^\infty$ be the last term of the derived series of $L$, and let $\mathfrak{m}$ be the maximal ideal of $R$. It maps onto $G$. At the first noncentral projective congruence layer choose a kernel element which is scalar modulo $\mathfrak{m}^n$ but not modulo $\mathfrak{m}^{n+1}$. The space $M_2(k)/kI$ has no $G$-fixed vector: scalar conjugacy differences would give an additive character of the perfect group $G$, and then the centralizer is scalar. Hence its commutator with a suitable element of $L^\infty$ has a noncentral leading matrix. This commutator belongs to $L^\infty$ and is identity modulo $\mathfrak{m}^n$. Its leading matrix is trace zero by determinant one.

Project this leading layer along a $k$-linear functional $\mathfrak{m}^n/\mathfrak{m}^{n+1} \to k$ detecting a nonzero off-diagonal entry. The resulting additive $G$-stable subgroup of $\mathfrak{sl}_2(k)$ is all of $\mathfrak{sl}_2(k)$. Indeed if a matrix has lower entry $c \neq 0$, upper-unipotent conjugation differences give the matrices with diagonal $tc$ and upper entry $t^2c$. Torus conjugation differences give all pure upper matrices. Subtraction then gives every scalar matrix, and swapping gives the lower matrices. A matrix initially with only upper entry is handled by swapping first. Therefore

$$|L^\infty| \geq q^3|G|.$$

If the scalar inertia at $v$ is nontrivial, adjoin $Q = K(\sqrt{\pi})$. This quadratic field ramifies at $v$ and at two by the norm calculation. Its dyadic radicand is a unit, so its dyadic upper break is at most three. It is disjoint from the linear point field since all quadratic quotients of $L$ are unramified at two. At $v$ the inertia of the compositum is the central diagonal involution formed by the scalar inertia and the nontrivial element of $\mathrm{Gal}(Q/K)$. Quotient by this involution. This kills the ramification at $v$ and leaves $L^\infty \times 1$ injected. If the scalar inertia was trivial, take the original point field. The resulting Galois field $N/K$ is unramified away from two and the cusp, has cusp indices $(2, 2)$, has dyadic upper breaks at most three, and its perfect core has order at least $q^3|G|$. Adjoining the unramified-at-seven quadratic $Q$ cannot enlarge the cusp residue degree beyond two.

## I.6 The directed comparison

Use the degree-2048 extension $B_1^+/K$ just constructed. Its solvable normal closure preserves the same local bounds, and any solvable quotient of $N/K$ kills its perfect core. Thus

$$[NB_1^+ : \mathbf{Q}] \geq 4096q^4(q^2 - 1), \qquad \mathrm{rd}(NB_1^+) \leq 4\sqrt{60}\,7^{1/4}.$$

At $q = 16$, retaining only the cusp Euler term with indices $(2, 2)$, the directed comparison at parameter 24.5 gives lower bound greater than 50.48869 and upper bound less than 50.39764. The degree lower bound is increasing in $q$, which excludes all $q \geq 16$.

## I.7 Uniqueness of a remaining $q = 8$ point field in each orientation

Fix which seven-place ramifies. There is at most one residual point field $M/K$ in the remaining $q = 8$ rows, even allowing both binary Frobenius types of the ordinary multiplier. Indeed, for two distinct point fields choose the corrected extensions $N_i/K$ above. The kernel of each $\mathrm{Gal}(N_i/K) \to \mathrm{Gal}(M_i/K)$ is a 2-group. A common nonsolvable quotient would therefore have $\mathrm{SL}_2(\mathbf{F}_8)$ as its nonabelian simple quotient, identifying $M_1$ and $M_2$. Thus the intersection is solvable and the degree of the compositum is at least twice the product of the two perfect-core orders, namely

$$2\big(8^3\,|\mathrm{SL}_2(\mathbf{F}_8)|\big)^2 = 133177540608.$$

The two strict local completions agree, since there is a unique tame elementary quadratic extension of $\mathbf{Q}_7$ with indices $(2, 2)$. The dyadic upper breaks are still at most three. The same

root-discriminant upper bound applies. At parameter 25.2 the strict-seven Euler term gives a directed lower bound greater than 50.59732, a contradiction.

# J The ordinary $\mathrm{SL}_2(\mathbf{F}_8)$ residue over $\mathbf{Q}(\sqrt{15})$

Put $k = \mathbf{F}_8$. We call an auxiliary profile unsigned if it is unramified at ſ, and signed if it has ramified scalar inertia there. In this section $A$ denotes the simultaneous trace order of any finite family of the prescribed C4 lifts with aligned residue, closed under the four genus twists, and $R = A/2A$. The local identities and the ordinary filtration hold for every such family. We first prove $\dim_k R \le 8$, then apply the bound to all the dyadic factors with residue field $k$.

## J.1 The marked basis and two trace generators

The ordinary line is defined over $A$ by the distinct dyadic Frobenius eigenspaces. Choose a generator $v$ of this free rank-one module and a conjugate of the strict decomposition group such that the residual line fixed by its tame inertia element $T$ differs from the marked dyadic line. This is possible by residual surjectivity onto $\mathrm{SL}_2(k)$. Then $(Tv, v)$ is a basis. Let $F$ be the strict arithmetic Frobenius and $\nu$ the ordinary unit-root unramified character evaluated at a dyadic arithmetic Frobenius $F_2$. In the marked basis the cusp relations give

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad a^2 + b^2 = -7.$$

Put $u = a + b$. It is a unit with residual value one and $u^2 = 1$ in $R$.

The literal tuple twists act on the trace order, multiplying each trace by its twist sign. This action respects the matrix entries in the marked basis. Indeed the corresponding two representations with equal traces are conjugate over $A$ by the absolutely irreducible trace realization. The conjugating map commutes with the common matrix of $T$ and preserves the marked line. In the cyclic basis it is therefore scalar. Consequently the entries of $r(g)$ have the degree given by evaluating both genus characters at $g$. The integral ring is graded by these four degrees, since its trace generators are homogeneous and the rational character eigenspaces are disjoint. Its grading survives reduction modulo two and any quotient by a homogeneous ideal. The canonical coefficient subring and subfield are of degree zero. The units $u, \nu$ have independent degrees, namely $(1, 1)$ at the strict Frobenius and $(0, 1)$ at the dyadic Frobenius in additive binary notation. Multiplication by them gives a strong grading.

In $R$, let $P = \nu^2 - \bar{\nu}^2$, with $\bar{\nu}$ represented in the canonical copy of $k$. The ordinary multiplier in $R/(P)$ is constant, namely $\bar{\nu}^{\pm 2}$ of order seven. Suppose its projective image grows. Pass to the first noncentral radical layer. Lemma 11.14 gives the absolute normalized dyadic different at most $2 - 1/q - 1/q^2$. To remove the inertia at eleven, quotient by its central subgroup, of order at most two. The dyadic bound is preserved and the perfect core loses a factor of at most two. The first-layer perfect core before this quotient has order at least $q^3|\mathrm{SL}_2(k)|$. The degree-eight ray auxiliary preserves the dyadic bound and the strict indices $(2, 2)$. At absolute degree 2,064,384, the strict-seven Euler term gives a directed lower bound greater than 46.57449, against the upper bound 45.71701.

It follows that over $R/(P)$ the image is scalar times the constant residual image, up to conjugacy. The scalar character $\lambda$ has exponent two and is unramified at two by the ordinary local commutator calculation. The ideal class and norm argument of Appendix I makes it a quotient of the narrow class group $C_2^2$. In a constant-residual realization the marked dyadic line is its residual line lifted constantly. The marking vector in that line differs from the constant vector by a unit, and $T$ is constant because $\lambda(T) = 1$. Thus in the marked basis $u = \lambda(F)$ and

$\nu/\bar{\nu} = \lambda(F_2)$ in this quotient. The indicated two Frobenii generate the genus group. Since traces generate the ring, $R/(P)$ is generated by $u, \nu$. Lifting this generation to $R$ uses $P$ nilpotent and already in the algebra they generate: for the subalgebra $B = k[u, \nu]$, the equality $R = B + PR$ implies $R = B$ by iteration.

The invariant degree-zero subring is therefore $k[P]$, because $u^2 = 1$, $\nu^2 = P + \bar{\nu}^2$, and the two degrees are independent. If $P$ has nilpotence index $n$, the four grading pieces are free rank one over $k[P]/(P^n)$, by their homogeneous units. Equivalently

$$R \simeq k[X, Y]/(X^{2n}, Y^2), \qquad X = \nu - \bar{\nu}, \quad Y = u - 1, \quad P = X^2.$$

## J.2 Excluding a second noncentral layer

If $n \geq 3$, the dyadic Frobenius $F_2^7$ is scalar modulo $\mathfrak{m}^2$ and nonscalar modulo $\mathfrak{m}^3$, where $\mathfrak{m} = (X, Y)$. Indeed its ratio in the ordinary eigenbasis is $(\bar{\nu}^2 + P)^{\pm 7}$, with nonzero first $P$-term. Its square $F_2^{14}$ is scalar modulo $\mathfrak{m}^4$ and nonscalar modulo $\mathfrak{m}^5$, since the first term is a nonzero multiple of $P^2 = X^4$. At each step, commuting with a suitable residual element produces a nonscalar element in the linear square-zero kernel. The preceding conjugation calculation puts a block of order $q^3$ in the perfect core at each step. The perfect core of the deeper image surjects onto that of the earlier image. The two contributions therefore multiply, giving perfect core order at least $q^6|\mathrm{SL}_2(k)|$.

Transfer the auxiliary scalar ramification as before. The resulting field has dyadic upper break at most three and strict indices $(2, 2)$. Compositum with $A^+/K$, of relative degree 1024, keeps those bounds. The absolute degree is at least $2048q^6|\mathrm{SL}_2(k)| = 270582939648$. The second directed row gives lower root discriminant above 50.70436 and upper below 50.39764. This excludes $n \geq 3$, and proves the stated dimension bound.

With the two opposite auxiliary signs of Appendix I, the integral rank is at least $8[k : \mathbf{F}_2]$. It is therefore exactly that rank, and $n = 2$. As a further consequence, in this saturated characteristic-two ring the entries of the dyadic inertia matrices lie in the invariant ring $k[P]/(P^2)$, since both genus characters are unramified at two. The ordinary unipotent inertia consequently has order at most $q^2$ even when the multiplier is not constant.

Suppose the residue is nonmodular. Its image is $\mathrm{SL}_2(\mathbf{F}_8)$, its dyadic wild inertia is the full translation group, and its multiplier generates $\mathbf{F}_8^\times$. Its strict seven-place has decomposition group $V_4$ with nontrivial inertia, and the other seven-place is unramified in the residue. Prescribe order-four cuspidal type at the strict place, good reduction at the other, and the actual ordinary crystalline parallel-weight-two component at two. The preceding local matrices lift the actual residue at the strict place. Make two choices at the principal prime $\mathfrak{l} = (2 + \sqrt{15})$ of norm eleven: the unramified type and the ramified quadratic scalar twist of that type. Lemma 2.6(iii) gives the two pure strongly compatible systems. Their dyadic residues at the chosen factors are the given nonmodular residue. Consequently neither system can be modular at any coefficient.

## J.3 The two fixed trace fields

At every dyadic factor the residue is unramified at eleven, because the exact auxiliary inertia is a rational scalar sign. The existing one-sided dyadic comparisons, and the congruence comparison leave only ordinary nonsolvable $q = 8$ factors or scalar semisimplifications. The solvable irreducible dyadic residue would require ramification at both seven-primes. A reducible residue has trivial diagonal characters, by the odd-order character and class-group calculation. For a nonordinary scalar factor use the fixed-profile rank comparison. The common auxiliary scalar sign vanishes on the simultaneous order modulo rational two, so it contributes no ramification to the middle-model ray calculation. The bound of two for the trace rank with one fixed cuspidal

orientation follows as in the nonordinary C4/CR comparison. Its packet of four finite-unramified genus twists preserves that exact profile. Without a self-twist its trace rank is divisible by four, a contradiction. A self-twist gives modularity by automorphic induction. Thus no nonordinary scalar factor occurs in either nonmodular system.

We next bound the unsigned ordinary scalar packet and exclude signed ordinary scalar factors. Use the fixed-orientation marked entry order and the notation $\nu, b, A$ of the ordinary cuspidal argument. For an unsigned scalar packet from a nonmodular system, adjoin the unsigned modular seed of degree two and its twist with opposite $a_2$. Their disjoint factors contribute at least two to the index of $b$. Hence any unsigned scalar packet from the nonmodular system has index at most two. Its index is positive and even by the four-twist argument. If present, it has index two, rank four, and $b^2$ is associated to two on this packet.

For a signed ordinary scalar packet, adjoin the unsigned degree-two seed and take the four-twist packets together. In the marked tangent calculation an auxiliary inertia matrix is scalar. At an auxiliary inertia generator its tangent is $xI$, with $y = z = 0$ in the notation of the ordinary scalar calculation. The additive class $x + y$ is unramified at two and the good seven-place. Among odd places it can ramify only at the cusp and the principal auxiliary $\mathfrak{l} = (2 + \sqrt{15})$. The cusp prime is nonprincipal, so the even-valuation ideal-class condition excludes ramification there. The rational norm obstruction of the preceding congruence argument then excludes ramification at $\mathfrak{l}$. Thus the auxiliary inertia has zero tangent and the joint marked entry order still has the form

$$B = A \oplus \nu A \oplus bA \oplus \nu bA, \qquad A = \mathbf{Z}_2[\nu^2, b^2].$$

Let $s_{\mathfrak{l}}$ be the exact scalar tuple on a generator of auxiliary inertia. It equals 1 on the unsigned seed and $-1$ on the signed factors. It belongs to the degree-zero suborder. Modulo $b$, apply reciprocity on $1 + i$ to the first diagonal character over $K(i)$ in the swap basis. The two dyadic factors have product one, as in the unsigned calculation. The cusp factor is $-1$, since the diagonal entry there has square $-1$. The auxiliary factor is $s_{\mathfrak{l}}$: the local norm of $1 + i$ at the inert eleven-place is 2, a nonsquare modulo eleven. Hence $1 + s_{\mathfrak{l}} \in bB \cap A = b^2 A$. Writing $1 + s_{\mathfrak{l}} = b^2 u_0$, the unsigned seed has $v_2(b^2) = 1$, so $u_0$ is a unit on that factor and in the local order $A$. Thus $b = 0$ on every signed factor. In characteristic zero the swap-basis representation on such a factor is monomial over $K(i)$, so its quadratic self-twist gives automorphic induction. This contradicts nonmodularity.

Include all the $q = 8$ dyadic factors of the two systems, with the four genus twists of each. Their residual point fields agree by uniqueness for the fixed strict orientation. Automorphisms of $\mathrm{SL}_2(\mathbf{F}_8)$ are inner up to coefficient Frobenius, so align the embeddings to have the same actual residual representation. Put $k = \mathbf{F}_8$ and $R = A/2A$, where $A$ is the resulting simultaneous trace order. For each of the two systems its $q = 8$ packet contributes at least $4[k : \mathbf{F}_2]$, by the orbit argument with squared traces. The two profiles are disjoint by the exact scalar sign at eleven. The rank bound above gives equality, with each of the two $q = 8$ contributions exactly twelve.

For either system write $D$ for the global trace field, $W \subset V_0 = \langle \chi_{-1}, \chi_3 \rangle$ for the global twist stabilizer, and $F = D^W$. There is no genuine self-twist, since the dyadic nonsolvable residue excludes quadratic induction. The sum of all local trace degrees of the global genus packet is $4[F : \mathbf{Q}]$. The signed system has no scalar dyadic factors, so its invariant field is cubic. The unsigned system has at most four additional trace degrees, so its invariant field is cubic or quartic. For both systems the $q = 8$ dyadic place of $F$ is unramified of degree three, since the squared traces reduce onto $k$. If the unsigned field is quartic, the other dyadic place of $F$ has degree one. The scalar packet bound and $a_3/b$ being a unit then give $v_2(U_3) = 1$ at that place.

Write $U_j = a_j^2$, where $a_j$ is the good-place trace at the unique prime of norm $j = 2, 3$, with the crystalline Frobenius trace for $j = 2$. These squares lie in $F$. The dyadic crystalline trace is $\nu + 2/\nu$, where $\nu$ is the ordinary unramified quotient value. The residue of $\nu$ has order seven.

Therefore $U_2$ generates $F$, since its residue already has degree three and $[F : \mathbf{Q}]$ is three or four. It is totally positive, integral, with all conjugates at most eight. The fields $D$ and $F$ are totally real by purity.

### J.4 Odd coefficient constraints

The extension $D/F$ is unramified at odd coefficient places. Otherwise a nontrivial coefficient inertia element gives a genuine residual quadratic self-twist by its inner twist in $W$. At every odd coefficient prime other than seven the order-four cusp remains locally absolutely irreducible, with projective dihedral decomposition. Thus the global residue is irreducible and the self-twist makes it dihedral. The inducing quadratic character from the inner twist is finite-unramified. At odd primes other than three, five and seven, the good or scalar-twisted-good finite-flat condition at a coefficient place over $p$ gives projective tame inertia of order greater than two, so the global projective group cannot be $V_4$. Its inducing quadratic character is then unique. The cyclotomic quadratic subfield is ramified at the coefficient prime, hence different. At three the cyclotomic restriction is already irreducible at the cusp, since seven splits in $K(\zeta_3)/K$. At five the cyclotomically reducible case uses Thorne's theorem for the real inducing field $K(\sqrt{5})$. The other cases use the odd dihedral witness and the weight-two lifting theorem.

At coefficient seven the global residue is also irreducible. Indeed a globally reducible residue has diagonal exponents zero and one at the good seven-place by finite flatness. At the strict seven-place an order-four cuspidal potentially Barsotti–Tate lift of a locally reducible residue requires exponents two and five, by the compact constituents $W(2,2), W(2,5)$. Reciprocity on the fundamental unit $4 + \sqrt{15}$, whose values at the two seven-places are inverse generators, makes the two exponents of each global constituent equal when there is no auxiliary inertia. If the scalar sign at eleven is ramified, the exponents instead differ by three, since the unit is nonsquare at $\mathfrak{l}$. Neither relation permits the strict exponents two and five. The restriction to $K(\zeta_7)$ is irreducible by the finite-flat good-seven argument: an inducing character from $K(\sqrt{-7})$ has projective tame inertia of order at most two at the good place, whereas the good-place finite-flat inertia has tame order six or eight. Therefore a residual self-twist at coefficient seven again gives modularity by the odd dihedral witness and weight-two lifting. This proves the unramified assertion in all cases.

It follows that $U_2$ has even valuation at every odd place of $F$. It is a unit at both possible dyadic places because every actual dyadic factor is ordinary. In particular its rational norm is a square. The same even-valuation condition away from two holds for nonzero $U_3$.

At a degree-one coefficient place of $F$ over three the projective residue is defined over $\mathbf{F}_3$, and at a degree-one place over five it is defined over $\mathbf{F}_5$. The former case is modular by the small-image criterion. In the latter case the projective determinant-squareclass has uniform real parity because it is the coefficient Frobenius inner twist times the determinant squareclass, and the inner twists are rational genus characters. Both parity cases of Allen–Khare–Thorne at five apply, since $[K(\zeta_5) : K] = 4$. The actual weight-two transfer, including the real residually dihedral theorem when needed, gives modularity. Thus $F$ has no degree-one prime at three or five. Likewise at a degree-one seven-place, an even coefficient Frobenius inner twist gives a totally odd projective squareclass, so Allen–Khare–Thorne at seven and weight-two transfer apply. For these projective descents use Lemma 12.5. The coefficient extension is unramified and the residual Frobenius twist comes from $W$, so its parity gives the descended determinant squareclass as in that lemma.

## J.5 Cubic coefficient fields

We first impose a local constraint at the strict seven-place. Fix an ordinary dyadic line vector and move it by a global group element to a vector $v$ outside the unique residual cusp line. This is possible by global residual irreducibility. With the order-four cusp inertia matrix $T$, the pair $(Tv, v)$ is a basis. Divide the cusp Frobenius by the rational square root of $-7$ in $\mathbf{Q}_2$, obtaining the involution

$$\begin{pmatrix} x & y \\ y & -x \end{pmatrix}, \qquad x^2 + y^2 = 1.$$

The residual cusp $V_4$ condition makes $x$ and $y$ units: if either vanished the involution would reduce to the identity or to $T$. Hence the unit $w = x + y$ satisfies $w \equiv 1$ in the residue and $v_2(w - 1) = 1/2$, using $(w - 1)(w + 1) = 2xy$.

For the local coefficient twist stabilizer this marked basis is compatible with its semilinear action. The ordinary line is unique, $v$ was obtained from it by a fixed global group element, and $T$ is fixed by every finite-unramified quadratic twist. Thus $w$ transforms by the sign of the twist on the cusp Frobenius. Since $F$ is unramified at its cubic dyadic place, that sign is nontrivial on the local stabilizer. The global quadratic eigenfield for the sign therefore has a totally positive squareclass, with even odd valuations and even valuation at this dyadic place. In degree three this is the only dyadic place. Its local quadratic extension is ramified, by the valuation $v_2(w - 1) = 1/2$. The other relevant sign is evaluation at dyadic Frobenius, under which the ordinary quotient $\nu$, and hence $a_2$, transform. With $h = \chi_{-1}$ and $\eta = \chi_3$, the cusp sign is $h\eta$ and the dyadic sign is $\eta$, where these symbols in sign-degree statements denote the two evaluation coordinates on $V_0$.

The cubic computation in [59, `d15_support/round4/ordinary/q8_weil_certificate.gp`] leaves ten candidates for $U_2$, in six reduced cubic fields of discriminants 49, 361, 1489, 3889, 5521, and 7249. Their ordinary class groups are trivial, so the supported positive squareclasses are generated by the positive unit classes and 2. The cusp constraint eliminates fields with no suitable positive unit class. At a degree-one seven-place where every supported positive quadratic class splits, the coefficient Frobenius inner twist is trivial, also eliminating the field. The surviving possibilities are discriminants 5521 and 7249. In each, the exact positive supported squareclasses are generated by 2 and $U_2$. Their local classes at the cubic dyadic place are independent, the first of odd valuation and the second a ramified unit class. Thus the full stabilizer $W = V_0$ is impossible: its local stabilizer would also be full, and the two independent sign eigenfields for cusp and dyad would both be generated by units. Indeed the corresponding generators can be taken to be $w$ and $\nu$ locally. The nontrivial dyadic sign leaves $W = \langle \chi_3 \rangle$ or $W = \langle \chi_{-3} \rangle$, and the nontrivial cusp sign excludes the latter. The degree-one seven-place of discriminant 5521 now eliminates that field because $\chi_3$ is even.

The remaining cubic case has

$$F = \mathbf{Q}(x), \quad x^3 - x^2 - 12x + 1 = 0, \qquad U_2 = 4 - x, \qquad P_0(X) = X^3 - 11X^2 + 28X - 1.$$

At the prime of norm three the stabilizer acts by its nontrivial sign. Every nonzero $U_3$ in a cubic candidate therefore belongs to the squareclass of $U_2$ in $F$. Exact enumeration of the integral elements of this class with all conjugates in $(0, 12]$ gives only $U_3 = U_2$.

In fact the common residual Frobenius at the prime of norm three has nonzero trace. If the trace were zero, its order would be at most two. The finite-unramified quadratic extension $J = K(\sqrt{-3})$ splits at three and at the cusp. Its compositum with the residual point field has absolute degree 2016 and root discriminant at most $2^{7/4}\sqrt{60}\,7^{1/4}$. Retain the cusp term with $(e, f) = (2, 2)$ and the term at three with $e \le 2$ and $f \le 2$. The term with $f = 2$ is a lower bound because $F_b$ is decreasing on its support. At $b = 7.4$ the directed lower bound is greater

than 42.91435, whereas the upper bound is less than 42.37919. Consequently $U_3 \neq 0$ in both systems.

## J.6 The final comparison

If the unsigned field is quartic, its exact coefficient sieve in Section J.7 gives $\overline{U}_3 \neq \overline{U}_2$ at the cubic dyadic place. This contradicts the signed cubic identity $U_3 = U_2$, since the two systems have the same aligned dyadic residue. Both fields are therefore cubic, so $U_3 = U_2$ holds throughout the simultaneous trace order.

The saturated order modulo two has

$$R = k[X,Y]/(X^4, Y^2), \qquad P = X^2, \qquad R_{\text{degree }0} = k[P]/(P^2), \qquad \nu^2 = \bar{\nu}^2 + P.$$

Here degree refers to the four genus twists and $P \neq 0$. Both $a_3$ and the marked cusp unit $w$ have the same degree, since the cusp Frobenius and the norm-three Frobenius have the same values on the genus characters. The unit $w$ satisfies $w^2 = 1$ in $R$. Hence $a_3/w$ has degree zero and its square lies in $k$. Thus $U_3$ is constant in $R$. On the other hand, $a_2 = \nu + 2/\nu$, so $U_2 = \bar{\nu}^2 + P$ in $R$. This contradicts $U_3 = U_2$ and excludes the ordinary order-eight residue.

## J.7 The quartic alternative in the order-eight comparison

We use the notation of the order-eight comparison. Let $D$ be the trace field of the source unramified at eleven, let $W$ be its genus-twist stabilizer, and put $F = D^W$. Write $U_j = a_j^2$ for $j = 2, 3$. Assume $[F : \mathbf{Q}] = 4$ and denote its degree-three and degree-one dyadic primes by $\mathfrak{p}_c$ and $\mathfrak{p}_s$, respectively. Both are unramified. The $\mathfrak{p}_s$ packet is ordinary and scalar residually. In its marked entry order the determinant index comparison with the modular order-four witness has length two. The cusp equation and the integral grading give

$$a - 1 \in b^2 A, \qquad a^2 + 7b^2 = -1, \qquad b^2 \sim 2.$$

Here $A$ is the degree-zero order and $a, b$ are the marked cusp coordinates used in the ordinary scalar comparison. The grading gives $a_3 \in bA$. The tangent calculation in the proof of Theorem H.5 shows that $a_3/b$ is a unit for this nonzero $b$ direction. On projecting to $F_{\mathfrak{p}_s} = \mathbf{Q}_2$, write $a = 1 + 2c$ with $c \in \mathbf{Z}_2$. It follows that

$$\frac{b^2}{2} = -\frac{1 + 2c(c+1)}{7} \equiv 1 \pmod 4, \qquad U_3 \equiv 2 \pmod 8 \quad \text{in } F_{\mathfrak{p}_s}.$$

In particular $a_3 \neq 0$.

At $\mathfrak{p}_c$, use the order-four comparison basis and its unit $w$ from the cusp, with $v_2(w-1) = 1/2$. The local coefficient stabilizer acts on $w$ by the same sign as on $a_3$, since the two Frobenius elements have the same evaluations on the two genus characters. Thus $a_3/w \in F_{\mathfrak{p}_c}^{\times}$. We have $w^2 \in F_{\mathfrak{p}_c}^{\times}$ and $v_2(1 - w^2) = 1$, since $w + 1 = (w - 1) + 2$ also has valuation $1/2$. Over the unramified local ring, the polynomial of $w - 1$ is Eisenstein with discriminant $4w^2$. Hence $U_3$ has even valuation at $\mathfrak{p}_c$, and the quadratic extension defined by its squareclass has discriminant exponent two.

The odd coefficient restrictions in the order-eight comparison give even valuations of $U_2, U_3$ at all odd primes of $F$. For a degree-one prime of $F$ over seven or a degree-two prime over three, let $\eta \in W$ be the Frobenius in $D/F$. Since this extension is unramified there, the two squareclass signs of $U_2, U_3$ are the eigenvalues of $\eta$ on $a_2, a_3$. If a valuation is positive, compute its sign after dividing by the corresponding even power of a local uniformizer. The signs agree exactly when $\eta$ is totally even. At a degree-one seven-prime this yields the projective $\mathbf{F}_7$ comparison

in the text. At a degree-two three-prime the projective representation descends to $\mathbf{F}_9$ by the same cocycle argument. Since $-1$ is a square in $\mathbf{F}_9$, an even inner twist makes its determinant squareclass totally even. The cusp gives absolute irreducibility on the cyclotomic restriction at three. Thus [3, Theorems 1.2(5) and 2.3 and Lemma 2.1] give residual modularity and its transfer to the source. Consequently the squareclass signs must differ at each prime of either kind.

The sieve in [59, `certificates/d15_quartic/q8_weil_certificate.sh`] first enumerates the twenty-six possibilities for $U_2$, using its Weil bound, square norm, and the absence of degree-one primes over three and five. It then enumerates the integral $U_3$ with all conjugates in $(0, 12]$ satisfying the conditions just proved. The interval proof of completeness for the integral basis boxes is the same as for the cubic sieve. The unique surviving row is

$$F = \mathbf{Q}(x), \quad x^4 - x^3 - 8x^2 + x + 8 = 0, \qquad U_2 = -x^2 + x + 7, \quad U_3 = x^2 - 2x + 2.$$

At $\mathfrak{p}_c$ the reduction of $x$ has minimal polynomial $x^3 + x^2 + 1$, so $U_3$ is nonzero modulo two and its residue differs from that of $U_2$.

The source with the ramified sign at eleven has cubic fixed field. Its reductions at the cubic dyadic prime agree with those of the source unramified at eleven, since the residual representations were aligned. Hence its $U_3$ is nonzero. The cubic calculation gives $U_3 = U_2$ for that source, contradicting the distinct residues in the quartic row. This excludes the quartic alternative.

The cubic and quartic sieves use the programs and certificates in [59, `d15_support/round4/ordinary/` and `certificates/d15_quartic/`]. The polynomial bounds come from the Weil house and square norm. Exact Sturm counts certify the real roots and the sign conditions, and rational brackets with interval interpolation bound the integral basis coordinates. The remaining class group, decomposition and valuation tests are exact.

# K Dyadic trace orders over $\mathbf{Q}(\sqrt{19})$

We use the notation of Section 27. The scalar order bounds come from the middle model and the ordinary filtration, respectively. For the residual $S_3$ representation we compare the simple models in a finite-flat filtration.

## K.1 The nonordinary scalar order

*Proof of Proposition 27.7.* Let $R_0, E_0$ be the actual orders. Put $R = R_0/2R_0$, $E = E_0/2E_0$ and $J = \mathrm{rad}(E)$. The algebra $E$ is local with residue $\mathbf{F}_2$ by Section 11. The biconnected regular lattice and Lemma 11.5 place the characters dual to $J/J^2$ in $V_2$.

Suppose first that $\mathfrak{v}_1$ is special. After removing its fixed quadratic character, a local matrix difference $L(g) = r(g) - 1$ has top row $(\chi_2(g) - 1, *)$ and bottom row zero. Thus

$$L(g)L(h) = (\chi_2(g) - 1)L(h). \tag{104}$$

Choose Frobenius and inertia differences $A, B$ whose evaluations on $V_2$ are $(1, 0, 0)$ and $(0, 1, 0)$. Modulo rational two,

$$A^2 = B^2 = AB = BA = 0, \qquad T(A) = T(B) = D(A) = D(B) = 0. \tag{105}$$

If $\mathfrak{v}_2$ is crystalline, exact scalar inertia cuts the first layer to $\langle -1, 6a \rangle$. The two evaluations above span its dual, so $A, B$ generate $E$. Equation (105) gives $\dim_{\mathbf{F}_2} E \le 3$, contrary to $\mathrm{rank}_{\mathbf{Z}_2} E_0 = 4\,\mathrm{rank}_{\mathbf{Z}_2} R_0 \ge 4$. For CR/SP the surviving space is $\langle -1, 2a \rangle$, and the two local

special evaluations at $\mathfrak{v}_2$ give the same contradiction. For CR/CR the first layer is contained in $\langle -1 \rangle$. A real involution generates $E$ and gives dimension at most two. This excludes every menu containing CR.

For SP/SP, choose in addition the difference $C = c - 1$ of a negative-real conjugation. Its evaluation is $(1, 0, 1)$, so $A, B, C$ generate $E$. This remains true when the actual first layer is a proper quotient of the one described by $V_2$. We have $C^2 = T(C) = D(C) = 0$. Polarized Cayley–Hamilton gives

$$AC + CA = X1, \qquad BC + CB = Y1, \qquad X = T(AC), \quad Y = T(BC).$$

Apply Lemma 11.9a with its ordered generators $(C, A, B)$. Its scalars are $T(CA) = X$ and $T(CB) = Y$; $CA = X + AC$ and $CB = Y + BC$ identify its normal span with

$$1, A, B, C, AC, BC. \tag{106}$$

It gives $\mathfrak{m} = (X, Y)$ and, over $R' = R/\mathfrak{m}^2$,

$$J^2 \subset \mathfrak{m}E + R'AC + R'BC, \qquad T(J^3) = 0, \qquad D(J^2) = 0.$$

Taking $(a, b, c) = (1, 1, 0)$ in its norm formula gives, for $Z \in J^2$,

$$D(A + C + Z) = X. \tag{107}$$

These are identities in the original determinant law over $R'$. At $\mathfrak{v}_2$, the Frobenius and inertia differences have first layers

$$A' = C \pmod{J^2}, \qquad B' = A + B + C \pmod{J^2}.$$

The special identity $T(A'B') = 0$ gives $X + Y = 0$ in $R'$.

Let $\Phi(u)$ be the regular Frobenius generator of Lemma 11.9. The Eisenstein polynomial $u^2 - 26u - 2$ reduces to $u^2$ modulo two. The ramified coordinate of the first-layer extensions in Lemma 27.2 gives

$$\Phi(0) = A + C + Z, \qquad Z \in J^2.$$

Its determinant has zero constant term modulo two. By (107), $X = 0$ in $R'$, and then $Y = 0$. Nakayama yields $\mathfrak{m} = 0$, hence $R = \mathbf{F}_2$ and $\operatorname{rank}_{\mathbf{Z}_2} R_0 = 1$. The argument applies without change to a simultaneous order of members with the same exact special characters.

Let $\sigma$ denote the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$. If the two special characters are $\theta_1, \theta_2$, choose a rational quadratic $\eta \in \{1, \kappa_{-1}, \kappa_3, \kappa_{-3}\}$ whose restriction at three is $\theta_1\theta_2$. It is unramified at $K_{\mathfrak{p}_2}$, and $r$ and $r^\sigma \otimes \eta$ have the same exact local types. Both are nonordinary scalar-residual. Their simultaneous trace order has rank one, so their traces agree. Thus

$$r^\sigma \simeq r \otimes \eta. \tag{108}$$

Apply Lemma 12.4. □

## K.2 The ordinary scalar order

**Lemma K.1.** *There is no supported continuous cochain $x : G_K \to \mathbf{F}_2$ satisfying*

$$dx = \kappa_3 \smile \kappa_{-2}, \qquad x|_{G_{K_{\mathfrak{p}_2}}} = 0, \qquad x|_{G_{K_{\mathfrak{v}}}} \in \mathbf{F}_2\kappa_3 \quad (\mathfrak{v} \mid 3).$$

*Proof.* Put $H = K(\sqrt{3})$ and $\alpha_0 = 1+\sqrt{3}$. Since $\mathrm{N}_{H/K}(\alpha_0) = -2$, the usual dihedral embedding construction gives a cochain $x_0$ with $dx_0 = \kappa_3 \smile \kappa_{-2}$ whose restriction to $G_H$ is the Kummer character of $\alpha_0$. All other supported solutions differ by a quadratic character $\chi_\beta$ of $K$. The asserted local conditions would make $\alpha_0\beta$ square at every $H$-prime over three and at a chosen $H$-prime over two.

At three, $\alpha_0$ is square in $\mathbf{Q}_3(\sqrt{3})$, since it is one modulo the uniformizer. At two it has odd valuation. Write

$$\beta = \pi(-1)^i \epsilon^j a^k (a')^l, \qquad i,j,k,l \in \mathbf{F}_2.$$

At $\mathfrak{v}_1$, $a$ is a nonsquare unit and $a'/3$ is a square unit. At $\mathfrak{v}_2$ their roles are reversed. Both $-1, \epsilon$ are nonsquare units, and $\pi$ is square at both places. The two conditions over three are $i+j+k = i+j+l = 0$. The four possibilities are

$$\beta \in \pi\{1, 3, -6, -2\} \quad \text{in } K^\times/K^{\times 2}.$$

Choose $\sqrt{3} = cs$ in $K_{\mathfrak{p}_2}$ with $c^2 = 3/19$ and $c \equiv 9 \pmod{16}$. In $\mathcal{O}_{K_{\mathfrak{p}_2}}/8$,

$$\alpha_0/\pi = (57c-13)/2 + (3-13c)s/2 = 2+7s.$$

Since 3 is square and $[\epsilon] = [2]$ locally, the four tests reduce to $2+7s$ and $(2+7s)(-\epsilon) = 1+4s$ modulo eight. The complete list of unit squares in $(\mathbf{Z}/8)[s]/(s^2-19)$ is

$$1, \quad 3, \quad 5+4s, \quad 7+4s.$$

Neither candidate occurs. □

*Proof of Proposition 27.8.* Lemma 27.4 gives trivial residual semisimplification. Apply Lemma 4.10 to a vector $v$ on the marked ordinary line. Its orbit lattice has nonzero additive residue and trivial restriction at two. The localization kernel in (29) makes the global residue, after scaling,

$$\bar{r} = I + \kappa_3 E_{12}. \tag{109}$$

This is ramified at both three-primes. It cannot occur at a CR place, whose actual inertia is the fixed scalar quadratic character and reduces identically to one.

Fix an inertia element $g$ at $\mathfrak{v}_1$ with $\kappa_3(g) = 1$, and write $\theta = \theta_1(g) \in \{\pm 1\}$. Special monodromy gives $(r(g)-\theta)^2 = 0$. The basis $((r(g)-\theta)v, v)$ is integral, because its residual vectors are independent, and in this basis

$$r(g) = \begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix}, \qquad \text{marked ordinary line} = \langle e_2 \rangle. \tag{110}$$

The basis is unchanged by rescaling $v$, up to a common scalar. The centralizer of this matrix intersected with the stabilizer of $e_2$ consists only of scalars. Hence this normalization removes the framing directions for every member of the collection.

Let $B$ be the closed $\mathbf{Z}_2$-algebra generated by all simultaneous matrix entries in these normalized bases. It is finite free and local, with residue exactly $\mathbf{F}_2$: every residual entry in (109) belongs to $\mathbf{F}_2$, including after passing to any factor of the normalization. The representation on $B^2$ is an actual stable lattice in ordinary crystalline constituents, hence is Barsotti–Tate. Its marked line is the multiplicative sublattice $Be_2$, and its quotient is etale. Coefficient actions extend to the Barsotti–Tate group by full faithfulness. For a quotient of $B$, close the generic coefficient kernel and take the finite-flat quotient in this ordinary group. The image in the etale quotient is finite etale, and the remaining subgroup is multiplicative. Thus the coefficient quotients retain the actual ordinary finite-flat condition.

Take a tangent $B \to \mathbf{F}_2[\delta]/(\delta^2)$ and write

$$r_\delta(h) = (I + \delta X(h))\bar{r}(h), \qquad X = \begin{pmatrix} x & y \\ z & x \end{pmatrix}.$$

Lemma 12.1a, with $h = \kappa_3$, gives the trace-zero shape and the cocycle equations

$$z \text{ additive}, \qquad dx = dy = \kappa_3 \smile z, \qquad q := x + y \text{ additive}. \tag{111}$$

The exact normalization gives $X(g) = 0$. At two, the marked line and the etale quotient imply

$$y = 0, \qquad x = \kappa U_2, \qquad \kappa \in \mathbf{F}_2,$$

where $U_2$ is the unramified quadratic character. The class $z$ is a unit Kummer class. At each three-prime the unit upper entry of special inertia recovers its line over $B$. The fixed-character clause of Lemma 12.1a gives

$$z|_{G_{K_\mathfrak{v}}} = 0, \qquad x|_{G_{K_\mathfrak{v}}} = t_\mathfrak{v}\kappa_3 \quad (\mathfrak{v} \mid 3).$$

The arithmetic preceding Lemma 27.2 now forces $z \in \mathbf{F}_2\kappa_{-2}$.

If $z = 0$, the additive $x$ is unramified at two and lies in $\langle \kappa_{-1}, \kappa_3 \rangle$. The absence of an unramified component at three kills the first coefficient, and $X(g) = 0$ kills the second. Thus $x = \kappa = 0$. Then $y = q$ is split at two, so it is a multiple of $\kappa_3$, again killed by $y(g) = 0$. Since the entries generate $B$, this tangent is zero. The cotangent dimension is therefore at most one.

For a nonzero tangent, $z = \kappa_{-2}$. Lemma K.1 shows $\kappa \neq 0$, hence $\kappa = 1$. The additive $q$ is unramified at two with value $U_2$. Since $q(g) = 0$, we have $q = \kappa_{-1}$. At a real conjugation $c$ we have $\kappa_3(c) = 0$ and $z(c) = q(c) = 1$, so

$$X(c) = \begin{pmatrix} x(c) & x(c)+1 \\ 1 & x(c) \end{pmatrix}, \qquad \det X(c) = x(c)^2 + x(c) + 1 = 1. \tag{112}$$

The last equality uses the residue field $\mathbf{F}_2$ of the actual normalized order.

Put $T = B/2B$. If its maximal ideal is nonzero, the cotangent bound gives $T = \mathbf{F}_2[t]/(t^n)$ with $n \geq 2$. At a real conjugation, in $M_2(T)$ we have $r(c) - I = tX(c) + t^2Y$. Total oddness gives the exact identity $\det(r(c) - I) = 0$ in $B$. Reducing it to $T$ and using (112), we obtain

$$0 = t^2 \det(X(c) + tY),$$

and the second factor is a unit. Hence $t^2 = 0$ and $\operatorname{rank}_{\mathbf{Z}_2} B = \dim_{\mathbf{F}_2} T \leq 2$. The trace order is a submodule of $B$ and has the same upper bound.

Finally let $\nu$ be the unramified ordinary character, in the crystalline normalization. We have $a_2 = \nu(\mathrm{Frob}_2) + 2/\nu(\mathrm{Frob}_2)$. In the unique nonzero tangent, $\nu(\mathrm{Frob}_2) = 1 + \delta$, so $a_2 - 1$ generates the cotangent. Nakayama and completeness show $B = \mathbf{Z}_2[a_2]$. This also holds when the cotangent is zero. □

## K.3 The global norm kernel for $S_3$

Let $L = K(\alpha)$, where $\alpha^3 - 2\alpha - 2 = 0$. This field has signature $(2, 2)$, discriminant $2^8 19^3 = 1755904$, and class number one. An absolute polynomial is

$$y^6 - 61y^4 + 4y^3 + 1087y^2 + 220y - 5487.$$

There is one dyadic prime, of absolute ramification index six and residue degree one, and four primes over three: a degree-one and a degree-two prime over each $\mathfrak{v}_i$. The degree-six arithmetic, including certification of the class group, is given in `s3_extension_layers.gp` of [20].

The three-letter permutation module is $\mathbf{F}_2 \oplus V$. Shapiro's lemma therefore gives

$$H^1(G_{K,S}, V) = \ker(L(S_L, 2) \xrightarrow{\mathrm{N}_{L/K}} K(S, 2)). \tag{113}$$

The two squareclass spaces have dimensions nine and five. The norm is surjective since $[L : K] = 3$, so the kernel has dimension four. Define

$$\beta = \pi/\alpha, \quad \tau = \frac{2s}{3\alpha^2 - 2}, \quad z = \alpha + \tau, \quad z_{19} = \alpha - 3 - s, \quad q = \alpha^2 - \alpha + 1,$$

and the squareclasses

$$e = \alpha + 1, \quad b = 3(\alpha - 1), \quad c = (5 + s)z, \quad d = -\frac{z_{19}\beta^8 q}{sz^2}. \tag{114}$$

They are supported at two, three and infinity. The square roots of their norms to $K$ can be chosen as

$$1, \quad 9, \quad 2(5 + s)^2, \quad \frac{(\pi^3/2)^4}{4},$$

respectively. In particular they lie in (113). The identities

$$(3 - \alpha)q^2 = \frac{19}{\alpha + 1}, \qquad (\alpha - 3)^2 - 19 = (\alpha - 3)(\alpha + 2)^2,$$

$$(\alpha^2 - 2)^2 = 2(2 - \alpha)(\alpha + 1), \qquad \alpha^4(\alpha - 1) = 2(\alpha + 2)$$

give the support and norm assertions directly. For the last class, denote the primes in the corresponding factorizations by $P_2, A_3, P_{19}, Q_{19}$. We have

$$(z_{19}) = P_2^2 A_3^2 P_{19}, \quad (q) = Q_{19}, \quad (s) = P_{19}Q_{19}, \quad (z) = P_2^5 A_3, \quad (\beta) = P_2.$$

thus $d$ is a unit.

At the degree-one primes over $(\mathfrak{v}_1, \mathfrak{v}_2)$, the unramified quadratic coordinates, and at the degree-two primes the ramification coordinates, are as follows:

| | degree-one Frobenius | | degree-two inertia | |
|---|---|---|---|---|
| | $\mathfrak{v}_1$ | $\mathfrak{v}_2$ | $\mathfrak{v}_1$ | $\mathfrak{v}_2$ |
| $e$ | 1 | 1 | 0 | 0 |
| $b$ | 0 | 0 | 1 | 1 |
| $c$ | 0 | 1 | 1 | 0 |
| $d$ | 0 | 1 | 0 | 0 |

(115)

These coordinates include the nonsquare-unit tests after removing even valuations. For example, at the degree-one prime over $\mathfrak{v}_1$ one may use $\alpha \equiv 4$ and $s \equiv 10 \pmod{27}$ to test $d$.

The dyadic conductors of all fifteen nonzero combinations are

| class | $e$ | $b$ | $eb$ | $c$ | $ec$ | $bc$ | $ebc$ | every class containing $d$ |
|---|---|---|---|---|---|---|---|---|
| conductor | 8 | 8 | 2 | 6 | 8 | 8 | 6 | 12 |

(116)

Every combination is locally nontrivial, so the four classes in (114) are independent and form the full global kernel. One way to compute the table is to normalize $v(\beta) = 1$, so $v(\alpha) = 2$ and $v(2) = 6$, and use

$$\alpha\beta = \pi, \qquad \alpha^2\beta^2 - 26\alpha\beta - 2 = 0, \qquad \alpha^3 = 2(\alpha + 1).$$

For instance, $\alpha = \beta^2 + \beta^4 - \beta^5 \pmod{\beta^6}$. The quadratic defect of $e$ and $b$ is five, that of $c$ and $ebc$ is seven, and that of $eb$ is eleven. Every class containing $d$ has defect one. The

conductor formula is 13 − defect. Equivalently these are the exact dyadic valuations of the relative discriminants of $X^2 - u$ for the fifteen explicit $u$, as computed by the accompanying script.

For a normal tangent in the $V$ summand of $\mathrm{ad}^0(V)$, the necessary special Frobenius-trace condition at $\mathfrak{v}_i$ is the vanishing of the degree-one coordinate in (115). A residual Frobenius is a transposition. In the three-letter permutation module its fixed letter is the degree-one completion, and the trace derivative is that coordinate. A crystalline scalar condition instead kills the degree-two inertia coordinate. If a class is written additively as $Ee + Bb + Cc + Dd$, the four conditions are

$$\mathrm{SP}_1 : E = 0, \quad \mathrm{SP}_2 : E + C + D = 0, \qquad \mathrm{CR}_1 : B + C = 0, \quad \mathrm{CR}_2 : B = 0.$$

Their kernels and the directions of smaller conductor are

| $(\mathfrak{v}_1, \mathfrak{v}_2)$ | candidate space | smaller-conductor line | upper break over $L_2$ |
|---|---|---|---|
| SP, SP | $\langle b, c + d\rangle$ | $b$ | 7 |
| SP, CR | $\langle c, d\rangle$ | $c$ | 5 |
| CR, SP | $\langle e + b + c, e + d\rangle$ | $e + b + c$ | 5 |
| CR, CR | $\langle e, d\rangle$ | $e$ | 7 |

(117)

In each row the other two nonzero elements have upper break eleven.

## K.4 The three local models

Over $\mathbf{F}_2((u))$, the local generic module for $V$ has Frobenius matrix $A_1$ below. Its height-one integral models, with their generic identifications, are exactly

$$A_1 = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ u^2 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & u^2 \\ u & 0 \end{pmatrix}. \tag{118}$$

To see exhaustiveness, take the apartment of the two coordinate lines in the Bruhat–Tits tree, with vertex $M_k = \mathbf{F}_2[[u]]u^k e_1 + \mathbf{F}_2[[u]]e_2$. Frobenius sends $k$ to $1 - 2k$ and doubles distances from the apartment. A vertex at distance $r$ from it, projecting to $k$, has distance $3r + |1 - 3k|$ from its Frobenius image. Height one at absolute ramification two requires this to be at most two. Thus $r = 0$ and $k = 0$ or 1. The permitted homotheties give exactly (118).

Write an extension matrix as $\left(\begin{smallmatrix} A_i & B \\ 0 & A_j \end{smallmatrix}\right)$, and write $a, d$ for the diagonal entries of $B$. If $A_i = \left(\begin{smallmatrix} 0 & u^{r_i} \\ u^{s_i} & 0 \end{smallmatrix}\right)$, then $(r_i, s_i) = (1, 0), (0, 2), (2, 1)$. Integrality of $u^2 A_i^{-1} B A_j^{-1}$ gives

$$v_u(a) \geq \max(0, r_i + s_j - 2), \qquad v_u(d) \geq \max(0, s_i + r_j - 2). \tag{119}$$

Put $u = t^3$. The diagonal trivializations have exponent pairs

$$P_1 = (-1, -2), \qquad P_2 = (-4, -2), \qquad P_3 = (-4, -5).$$

The raw pole orders in the $V$ summand are bounded by

$$P_{i1} - 2P_{j1} - 3v_u(a), \qquad P_{i2} - 2P_{j2} - 3v_u(d).$$

Further coefficient terms subtract three, and Artin–Schreier reduction divides every even pole by two. Substituting (119) gives the following maxima

| submodule/quotient | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| $A_1$ | 1 | 1 | 7 |
| $A_2$ | none | 1 | 1 |
| $A_3$ | none | none | 1 |

(120)

The $A_1/A_3$ entry also permits pole five: its second raw sequence begins $8, 5, 2$, whose reduced orders are $1, 5, 1$. Thus both breaks five and seven force submodule $A_1$ and quotient $A_3$, while break eleven is impossible. The offdiagonal entries of $B$ give the scalar summand. The same calculation, with bounds $r_i + r_j - 2$ and $s_i + s_j - 2$, gives maximum pole three after the tame cubic extension, hence scalar upper break at most one over $K_{\mathfrak{p}_2}$.

The field-of-norms comparison preserves the breaks just used. After the tame cubic extension, the first ramification break of the dyadic radical tower is twelve. The breaks five, seven and eleven in (116) are smaller. The usual ramification comparison for this tower therefore identifies them with the Artin–Schreier poles of the generic Kisin modules by [89, Proposition 3.3.2 and Corollary 3.3.6]. This is the same comparison as in Lemma 11.5, with the absolute ramification index multiplied by three.

## K.5 The cotangent and the length

*Proof of Proposition 27.9.* In characteristic two we have $\mathrm{ad}^0(V) = \mathbf{F}_2 I \oplus V$. The $V$ component of a permissible tangent belongs to the corresponding row of (117). Table (120) excludes the two break-eleven directions, so it lies in the single displayed line.

Suppose its $V$ component is zero. At a special place let $F$ be Frobenius, with eigenvalues $3\theta, \theta$, and let $N$ denote the scalar-normalized inertia difference. The exact source identities include

$$(F - 3\theta)N = 0, \qquad N(F - \theta) = 0.$$

The residue is unramified there, and $\bar{F}$ is a nonidentity transposition. Scalar infinitesimal inertia $\delta c I$ would give $c(\bar{F} - I) = 0$, hence $c = 0$. At a crystalline place the inertia is exactly the fixed scalar and gives the same unramifiedness directly. The scalar local pole bound and Lemma 27.2 now leave only $\kappa_{-1}$ globally. This is a strict change of basis. If $S$ is an order-three matrix in the residual image, then a transposition conjugates $S$ to $S^2 = S + I$, whereas rotations centralize it. Thus

$$gSg^{-1} - S = \kappa_{-1}(g) I.$$

Strict equivalence permits every infinitesimal $\mathrm{GL}_2$ change of basis. The fixed determinant does not restrict conjugating matrices to trace zero. The scalar direction therefore vanishes in the trace deformation problem.

Let $R_0$ be the simultaneous trace order. Absolute residual irreducibility reconstructs the image algebra as $M_2(R_0)$ and supplies the actual lattice $R_0^2$. Its dyadic Barsotti–Tate group has finite-flat quotient modulo rational two. The cotangent calculation gives

$$R_0/2R_0 = \mathbf{F}_2[t]/(t^n)$$

for some $n \geq 1$. If $n \geq 2$, its first-order $V$ class is the line of break five or seven in (117).

Suppose $n \geq 3$. Close the generic $t$-adic filtration inside this single finite-flat group and take its successive finite-flat quotients. Apply the exact Kisin anti-equivalence of [51, Corollary 4.3 and Proposition 5.7.3], reversing the filtration. Every consecutive two-factor quotient has the same nonzero generic first-order class, up to a nonzero scalar. By (120), its lower simple model is $A_1$ and its upper simple model is $A_3$. For three successive factors the middle object would therefore have to be $A_3$ as one quotient and $A_1$ as the next subobject. These models are not isomorphic. This contradiction proves $n \leq 2$ and $\mathrm{rank}_{\mathbf{Z}_2} R_0 \leq 2$. □

*Remark* K.2. The length bound uses the common middle object of two adjacent extensions. The three simple models in (118) are essential for the comparison of breaks five and seven.

### K.6 The Hilbert comparison form

The newspace of parallel-weight-two Hilbert cusp forms of level (3) and trivial central character has dimension 14, with constituent dimensions $2, 2, 5, 5$. On each two-dimensional constituent the operator $T_2$ has polynomial $X^2 - 2$, the operators at the two three-places are opposite scalar signs, and the operator at the tested prime above five is $-I \pm 2T_2$. The complete Magma program `hilbert_seed_certificate.m` in [20] constructs the space and checks these identities. The proof uses the two exact local signs and the odd trace at the tested prime.

## AI disclosure

During the preparation of this article, the authors used ChatGPT (OpenAI) as an assistant in mathematical research and manuscript preparation. Its use included exploring proof strategies, checking mathematical arguments, investigating possible simplifications, assisting with computations and verification scripts, checking references, and drafting and revising portions of the exposition. AI-generated suggestions and calculations were critically examined and revised where necessary. The authors take full responsibility for the mathematical content, proofs, computations, and conclusions of the article.

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