Serre modularity over real quadratic fields

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Abstract

We prove Serre's modularity conjecture, in every odd residual characteristic and with arbitrary conductor, over twenty real quadratic fields $K = \mathbf{Q}(\sqrt{d})$. The proofs for $d = 2, 5, 13, 17, 41$ use classical discriminant bounds. For $d = 33$ the proof uses the zero-free region $\Re s > 31/32$ of Cureton and Thorner, and for $d = 6$ the region $\Re s > 0.9586$, which we derive from specified analytic inputs. For the remaining thirteen fields we assume GRH for Dedekind zeta functions. The main tools are conductor reductions at an odd split prime and at two, a weight-reduction step by congruences, a scalar dyadic modularity criterion, and integral trace orders. We also use a modularity theorem for the mixed ordinary–nonordinary case and, for the seven fields in which two splits, dyadic propagation results.

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1 Introduction

Let K be a real quadratic field. We study Serre's conjecture for continuous absolutely irreducible totally odd representations $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, with p odd and with arbitrary conductor. Here modularity means occurrence in the reduction of the Galois representation of a regular algebraic cuspidal Hilbert modular form. We do not prescribe its weight or level.

Over $\mathbf{Q}$, the conjecture originated in Serre's questions about modular eigenvalues modulo a prime and was subsequently refined by his predictions for weight, level and character [62, Section 3, Problem 1], [64]. The lifting theorems of Wiles, Taylor–Wiles and Skinner–Wiles, together with potential modularity, laid the foundations for its proof [78, 71, 66, 70].

An essential step in this development was the construction of compatible systems containing a given representation. Dieulefait proved such an existence theorem over $\mathbf{Q}$, in its original crystalline range, using potential modularity and descent [18, Theorem 1.1]. This permits a change of residual characteristic while retaining a single characteristic-zero compatible system. The method was used to establish cases of the Fontaine–Mazur conjecture and small-level cases of Serre's conjecture [19, 42]. Khare and Wintenberger subsequently proved Serre's conjecture in full generality [43, 44], with Kisin's two-adic modularity lifting theorem providing the final dyadic input [46]. Later lifting theorems permit shorter versions of this proof, as in Dieulefait–Pacetti [26]. Over totally real fields, the corresponding strategy of connecting residual representations through compatible systems is formulated in the Rayuela conjecture [23].

Over a real quadratic field, discriminant bounds need not exclude all small-characteristic representations of small conductor. For example, the elliptic curve

$$y^2 + \sqrt{6}xy + (1+\sqrt{6})y = x^3 + (1-\sqrt{6})x^2 \\ - (1+3\sqrt{6})x - 1 - 2\sqrt{6}$$

has everywhere good reduction over $\mathbf{Q}(\sqrt{6})$; its label is `2.2.24.1-1.1-a1` in [48]. Thus the base cases must establish modularity of surviving representations, rather than only nonexistence. In addition, residual reducibility and ordinarity of a characteristic-zero source are different conditions. A congruence is useful only when modularity can be transferred back to that actual source.

Our starting point is the characteristic-three method of Dieulefait–Pacetti over $\mathbf{Q}(\sqrt{13})$ [27]. Successive discriminant comparisons restrict the residual images, and a conductor reduction relates arbitrary representations to the remaining base case. We retain this distinction between the residual base and the conductor argument, remove the restriction at two, and supply new source theorems where residual modularity alone is insufficient. Fontaine's local bounds, refined local ramification calculations, and global discriminant estimates provide the arithmetic input [34, 49, 50]; finite-flat groups and integral trace orders control the scalar and dihedral cases.

For several fields we also use a weight-reduction step. The source is made crystalline of weight two at one of the two places above a split prime ℓ, and a weight-two lift with an inertial type of small order is chosen at the other place. A single congruence modulo an auxiliary prime p with $\ell \equiv 1 \pmod{p}$ then replaces this type by an unramified or special one. Modularity is transferred back along the resulting congruences.

We use the lifting and compatible-system inputs specified in Section 2. When two splits, we also use the dyadic structural and propagation results listed in Section 5, including the cited preprints. The mixed theorem is Theorem A of [24], on the residually reducible mixed ordinary–nonordinary case; it is stated as Theorem 6.4. These inputs have separate roles from the hypotheses on zeta zeros.

In both theorems, the assertion of Serre modularity over K is that, for every odd prime p,

every continuous absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is modular, with arbitrary conductor. Write $K = \mathbf{Q}(\sqrt{d})$ with d squarefree.

Theorem 1.1 (Fields without GRH)**.** *Serre modularity holds for $d \in \{2, 5, 6, 13, 17, 33, 41\}$ under the following inputs.*

(i) For $d = 2, 5, 13, 17, 41$ the discriminant bounds use the classical critical strip. For $d = 33$ use $\mathrm{ZF}(31/32)$ from [15, Theorem 1.1]. For $d = 6$ use the analytic inputs of Appendix B, which give $\mathrm{ZF}(0.9586)$. Here $\mathrm{ZF}(\sigma)$ is the nonvanishing hypothesis defined in Section 13.

(ii) For $d = 2, 5, 6$ use the mixed theorem (Theorem 6.4). The terminal characteristic-three proof for $d = 13$ uses [81, Theorems 5.1.1 and 6.1.1]. For $d = 17, 33, 41$ use the split-dyadic lifting and source theorems of Section 5. The scalar endpoint for $d = 17, 41$ uses [21, Theorem 9.4(b)], and for $d = 33$ the general scalar propagation argument.

Theorem 1.2 (Fields under GRH)**.** *Assume GRH for Dedekind zeta functions. Serre modularity holds for*

$$d \in \{3, 7, 10, 11, 14, 15, 19, 21, 37, 57, 65, 73, 89\}.$$

For $d = 15$ use the mixed theorem. For $d = 57, 65, 73, 89$ use the split-dyadic inputs of Section 5.

The proofs are grouped by their analytic input. In Theorem 1.1, the zero-free hypotheses for $d = 6, 33$ are used in the final discriminant comparisons. The conductor reductions and the source-lifting implications are shared with the proof of Theorem 1.2.

1.1 Structural theorems

The field list is an application of three mechanisms: conductor reduction to a specified terminal characteristic, scalar modularity from a quadratic Selmer condition, and coefficient-field rigidity from simultaneous integral trace orders. We state these mechanisms separately, since their hypotheses have different roles.

Conductor reduction. A strongly compatible system, and the conventions for weight two, are specified in Section 2. The following theorem concerns a system already constructed in characteristic zero; it does not supply an initial lift of an arbitrary residual representation.

Theorem 1.3 (Conductor reduction at a split terminal prime)**.** *Let K be real quadratic, suppose two does not split in K, and let ℓ be an odd rational prime that splits in K. Assume:*

(i) every absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ of determinant $\bar{\chi}_\ell$ and unramified outside ℓ is modular;

(ii) for every reducible representation with that determinant and ramification support, its constituent quotient is totally odd and extends to $G_{\mathbf{Q}}$.

Then every nondihedral potentially modular weight-two strongly compatible system over K, whose determinant is cyclotomic times a finite totally even character, is modular.

The proof is given in Section 6. The quotient in hypothesis (ii) is automatically totally odd, since its two constituents have product $\bar{\chi}_\ell$. The theorem does not assume that a modular residual witness makes an arbitrary regular source modular: the weight resets and retained local types are part of its proof.

A scalar Selmer condition. Suppose two splits in K, with dyadic places w_0, w_1. Put

$$V^+_{w_0}(K) = \left\{ [a] \in K^\times/K^{\times 2} : \begin{array}{l} a \text{ is totally positive,} \\ v_{\mathfrak{q}}(a) \equiv 0 \pmod 2 \quad (\mathfrak{q} \nmid 2), \\ a \in K^{\times 2}_{w_1} \end{array} \right\}. \tag{1}$$

This finite squareclass space classifies totally real quadratic extensions unramified outside w_0 and split at w_1. Quadratic conjugation exchanges the two choices of dyadic place. The condition we use is

$$V_{w_0}^{+}(K) = 0. \tag{SV}$$

It is not an odd-narrow-class-number condition.

Theorem 1.4 (Scalar dyadic modularity)**.** *Let K be a real quadratic field in which two splits and* (SV) *holds. Under the dyadic structural, propagation and classicality inputs of Section 5, let $E/\mathbf{Q}_2$ be finite and let $r : G_K \to \mathrm{GL}_2(E)$ be continuous and absolutely irreducible, unramified outside finitely many places, with $\det r = \varepsilon_2$ and $\bar{r}^{\mathrm{ss}} = \mathbf{1} \oplus \mathbf{1}$. Suppose that at both dyadic places r is absolutely irreducible, de Rham and has distinct Hodge–Tate weights. Then r is automorphic.*

The theorem imposes no restriction on finite ramification or on the regular weights, and its proof uses no GRH. Sections 7–8 construct the auxiliary fields and prove the theorem. Together with a residual base at two and a two-power ordinary class number, it gives the split-dyadic Serre criterion of Theorem 10.1. The seven split-dyadic fields in the main theorem satisfy these arithmetic conditions. The Selmer vanishing is also valid when the narrow class number is even.

Integral trace orders and symmetry. Let $\mathcal{R}$ be a strongly compatible system of cyclotomic determinant over a real quadratic field K, with trace field D. For finitely many coefficient orbits, form their simultaneous integral matrix order E_0, including the trace scalars and every dyadic coefficient place, as in Section 11. Its rank is four times the sum of the trace-field degrees. The integral order, rather than a single residual two-dimensional realization, is essential here.

Theorem 1.5 (A trace-order bound and quadratic symmetry)**.** *Let $K/\mathbf{Q}$ be real quadratic with conjugation σ, and let $\mathcal{R}$ be an absolutely irreducible strongly compatible rank-two system with cyclotomic determinant and trace field D. Let h, η be restrictions of quadratic characters of $G_{\mathbf{Q}}$. Form the coefficient orbits generated by*

$$H(\mathcal{R}) = \mathcal{R} \otimes h, \qquad G(\mathcal{R}) = \mathcal{R}^{\sigma} \otimes \eta.$$

Suppose $\mathcal{R} \not\simeq \mathcal{R} \otimes h$ and their simultaneous integral order satisfies $\dim_{\mathbf{F}_2}(E_0/2E_0) < 32$. Then either

$$\mathcal{R}^{\sigma} \simeq \mathcal{R} \otimes \eta \quad \text{or} \quad \mathcal{R}^{\sigma} \simeq \mathcal{R} \otimes \eta h,$$

or $[D : \mathbf{Q}] \leq 2$ or $D/\mathbf{Q}$ is biquadratic. In the latter alternatives every coefficient prime has residue degree at most two. In the former alternatives, a totally odd regular system is modular by quadratic descent.

The arithmetic work in applying Theorem 1.5 lies in bounding the actual order. The finite-flat ramification theorem 11.10 and the regular-order construction in Section 11 supply different parts of this argument. The bound of twenty-six over $\mathbf{Q}(\sqrt{11})$ then gives the coefficient-field alternative directly; the bound of ten over $\mathbf{Q}(\sqrt{3})$ uses Corollary 12.6.

Two further source theorems have a separate role. Proposition A.2 constructs the initial lift in the exceptional characteristic-five case over $\mathbf{Q}(\sqrt{5})$, which the general prescribed-lift theorem excludes. Theorem 29.33 proves modularity of the nonordinary reducible characteristic-three source over $\mathbf{Q}(\sqrt{21})$. Its proof kills the anti-invariant part of the full pseudodeformation cotangent space and applies Lemma 12.2; no finiteness theorem for the whole pseudodeformation ring is needed. These are source theorems, not consequences of residual-image bounds alone.

1.2 Field applications

Analytic input	Radicands d	Terminal characteristics in the same order
Classical critical strip	$2, 5, 13, 17, 41$	$7, 11, 3, 2, 2$
Narrower zero-free strip	$6, 33$	$5, 2$
GRH	$3, 7, 10, 11, 14, 15, 19, 21, 37$	$13, 3, 3, 5, 5, 7, 3, 5, 3$
GRH	$57, 65, 73, 89$	2 in each field

Part I gives the common constructions, including the two conductor reductions and a single explicit formula for all three analytic regimes. Part II proves the fields of Theorem 1.1, beginning with the classical cases. Part III proves Theorem 1.2 and records stronger supported source results in $d = 2, 5, 6, 13$. In particular, the separate terminal-seven analysis for $d = 5$ leaves a nonordinary S_3 source in the indicated weight-reduction chain. The full result over $\mathbf{Q}(\sqrt{5})$ uses the eleven-adic route. The appendices give the longer analytic and finite calculations; their code is cited where used.

Buzzard–Diamond–Jarvis formulated the weight conjecture at unramified residual primes [10]; see also [37]. Our modularity results yield the weight part of Serre's conjecture under the image hypotheses of Gee–Liu–Savitt [39, Theorem 6.2.1]. Namely, suppose the residual representation remains absolutely irreducible after adjoining the roots of unity of order the residual characteristic and, in characteristic five, that the projective image of this restriction is not the alternating group on five letters. Then the occurring Serre weights are exactly those prescribed by the local crystalline-lift recipes, including at ramified residual primes and for all predicted companion weights. These consequences inherit the assumptions of the corresponding modularity theorem; the weight theorem introduces no further analytic hypothesis.

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Part I
Common methods

2 Compatible systems and congruences

We use d for the squarefree radicand, D for a fundamental discriminant, and q for the residual coefficient-field cardinality. The finite arithmetic and numerical checks are supplied in [52, 53]. The longer certificate descriptions appear in the appendices.

Throughout, $G_F = \mathrm{Gal}(\overline{F}/F)$ and the cyclotomic character χ_p has labelled Hodge–Tate weight one. Irreducibility means absolute irreducibility. A representation is totally odd if every real complex conjugation has determinant -1.

A system over a totally real field is a pure, regular, totally odd, absolutely irreducible, strongly compatible rank-two system. We require compatibility of the Frobenius-semisimplified Weil–Deligne representations at every finite place, including a place above the coefficient characteristic, and compatibility of labelled weights. Its conductor support consists of the places where the common Weil–Deligne representation is ramified or has nonzero monodromy. Outside this support its p-adic member is crystalline above p and unramified at the other places. Weight two means labelled weights $\{0, 1\}$. Regular weights may include negative integers.

On the closure of a special local component the Frobenius eigenvalues have ratio $\mathrm{N}(v)$. For a pure system, monodromy cannot vanish at such a place: otherwise both eigenvalues would have the same complex absolute value, contradicting that ratio.

We write $\mathcal{R} \rightsquigarrow \mathcal{R}'$ when a congruence, possibly followed by a finite character twist, has the property that modularity of $\mathcal{R}'$ implies modularity of $\mathcal{R}$. Every such implication below is an application of a lifting theorem to the actual old characteristic-zero member. Finite twists preserve modularity. Equality of good Frobenius polynomials and Chebotarev then propagate modularity from one member to the whole system.

2.1 Residual witnesses and lifting

Lemma 2.1. *Let F be totally real of even degree. An irreducible residual representation with a regular algebraic cuspidal Hilbert modular witness has a parallel-weight-two cuspidal witness, after allowing arbitrary level.*

Proof. Transfer the witness to the quaternion algebra over F ramified at every real place and split at every finite place. At sufficiently small level $U = U^p U_p$, reduction of a primitive integral eigenvector is nonzero, with coefficient module V over a finite field k. Its U_p-action factors through a finite quotient G; write U_p' for the kernel. The map $V \hookrightarrow \mathrm{Map}(G, V)$ given by $v \mapsto (g \mapsto gv)$ is equivariant for $(hf)(g) = f(gh)$. The target is a sum of regular representations. Pointwise application and induction in stages give the same away-from-p Hecke eigencharacter in scalar forms of level $U^p U_p'$. Their integral space is $\mathcal{O}^X$ for a finite double-coset set X, so reduction is surjective. Deligne–Serre's lifting lemma [17, Lemme 6.11] lifts the eigencharacter to characteristic zero. A norm character would have reducible residue; inverse Jacquet–Langlands therefore gives a parallel-weight-two cuspidal form.

Retaining both operators T_v, S_v, with polynomial $X^2 - T_v X + \mathrm{N}(v) S_v$, preserves the residual trace and determinant. Chebotarev and Brauer–Nesbitt identify the residue. No exact characteristic-zero central character or ordinary ordering is prescribed by this argument. □

Totally odd solvable residual representations have solvable Artin witnesses, followed by regular weight raising, in the scopes of automorphic induction and Langlands–Tunnell; see

[3, Theorem 1.2(1), Lemma 2.1 and proof of Theorem 2.5]. Lemma 2.1 gives weight two whenever it is needed. We use Dickson's classification and the rank-two adequacy criterion [6, Proposition A.2.1] throughout.

Lemma 2.2. *Let K be real quadratic, $p > 2$, and let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be finitely ramified and totally odd, with modular residue whose restriction to $G_{K(\zeta_p)}$ is irreducible. Then r is modular in either case:*

(i) p splits in K, and r is potentially semistable with distinct labelled weights at the places above p;
(ii) r is de Rham of labelled weights $\{0, 1\}$ above p, and its residue has a parallel-weight-two modular witness.

Proof. Apply respectively [74, Theorem 4.7] and [3, Theorem 2.3], dualizing to their inverse-cyclotomic normalization when required. In (ii), the parallel-weight-two witness gives the required weight-zero automorphic convention. □

Lemma 2.3. *Let F be a totally real extension of a real quadratic field, and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be finitely ramified and totally odd, with determinant $\chi_2\psi$, where ψ has finite order. Suppose its residue is nonsolvable. Then r is modular in either case:*

(i) r is nearly ordinary of regular weights at every dyadic place and its residue has a nearly ordinary regular modular witness;
(ii) r is potentially Barsotti–Tate and potentially nonordinary at every dyadic place and its residue has a regular modular witness.

Proof. The first case is [2, Theorem 5.2.1]. If the line and quotient have weights $b_\tau > a_\tau$ with sum one, its parameters are $w_\tau = a_\tau$ and $k_\tau = b_\tau - a_\tau + 1$, so $k_\tau + 2w_\tau = 2$. Nonsolvability excludes the CM-dihedral exception. The theorem does not require a marked residual ordinary ordering or distinct residual characters.

In the second case choose finite Galois extensions of the dyadic completions over which the source is Barsotti–Tate. The solvable local realization theorem [5, Chapter X, Section 2], in the form [14, Theorem 3.1 and Example 3.3], realizes them in a solvable totally real Galois extension. No degree is prescribed. The nonsolvable residual projective image is $\mathrm{PSL}_2(\mathbf{F}_{2^a})$, $a \geq 2$; its image after solvable Galois restriction is normal with solvable quotient and is unchanged. Solvable base change retains a cuspidal modular witness, put in weight two by Lemma 2.1. Apply [46, Theorem 3.3.5]. Its condition that a dyadic completion equal $\mathbf{Q}_2$ applies only at potentially ordinary places, and there are none.

For descent take a cyclic series. Strong multiplicity one makes the automorphic representation invariant at each step because the Galois representation extends downstairs. The descended Galois representation differs from the given one by a character of the cyclic quotient. Twisting and iterating proves modularity over F; see [7, Lemma 2.2.2]. □

Lemma 2.4. *Let a system over a real quadratic field have a crystalline weight-two dyadic member of cyclotomic determinant whose residual image is $\mathrm{SL}_2(\mathbf{F}_q)$, with $q \geq 4$. Suppose the residual point field is not totally real. If the residue is modular or if $q = 4$, then the system is modular.*

Proof. For $q = 4$, [3, Theorem 2.5] gives a regular cuspidal residual witness. The point-field hypothesis says that some real conjugation is nontrivial in the residue. Apply [73, Theorem 6.1] to the dual actual dyadic member, to match its weight convention, and then dualize back. Strong compatibility proves modularity of the system. □

Lemma 2.5. *Let K be real quadratic and $\bar{r} : G_K \to \mathrm{SL}_2(\mathbf{F}_8)$ have full image. Suppose its point field M is Galois over $\mathbf{Q}$, and some real conjugation acts nontrivially in $\bar{r}$. Then the actual residue extends to $G_{\mathbf{Q}}$ and is modular over K.*

Proof. The outer automorphism group of $\mathrm{SL}_2(\mathbf{F}_8)$ has order three. The conjugation action of $\mathrm{Gal}(M/\mathbf{Q})$ on $\mathrm{Gal}(M/K)$ is therefore inner, because it is inner on a subgroup of index two. The center is trivial, so the unique inner conjugating elements give a homomorphism $\mathrm{Gal}(M/\mathbf{Q}) \to \mathrm{Gal}(M/K)$ restricting to the identity on $\mathrm{Gal}(M/K)$. Compose with the given matrix realization to extend the actual residue to $G_{\mathbf{Q}}$. Rational Serre modularity [43, 44] and quadratic base change give a regular cuspidal witness over K. Its cuspidality follows from irreducibility of the restriction. □

2.2 Globalization and compatibility

Lemma 2.6. *Let K be real quadratic, fix a finite set S of places containing the coefficient places and all residual ramification, and let $\bar{r}$ be totally odd with irreducible cyclotomic restriction. At coefficient characteristic two assume nonsolvable image. Fix a totally odd global de Rham determinant lifting its determinant and unramified outside S, and nonempty compatible local components with regular weights at the coefficient places. Outside the exceptional characteristic-five case of [31, Definition 4.1.1], these components admit a global lift unramified outside S. Once such a lift has been constructed, it is potentially modular and belongs to a strongly compatible system in the following cases:*

- *(i) odd coefficient characteristic and parallel weight two, with irreducible cyclotomic residual restriction;*
- *(ii) odd coefficient characteristic, adequate cyclotomic residue, and regular potentially crystalline local components that are potentially diagonalizable;*
- *(iii) coefficient characteristic two, determinant χ_2, and either ordinary crystalline components of regular weights at all dyadic places, ordinary semistable components of weight two at all dyadic places, or potentially Barsotti–Tate and potentially nonordinary components of weight two at all dyadic places.*

Proof. Existence is [31, Theorem 4.2.1 and Remark 1.1.2]. Away from the coefficient prime, inertial conditions and conductor bounds specify unions of components. At that prime, take components through the supplied points; potential diagonalizability persists after a fixed sufficiently large local extension.

In (i), first use the avoiding weight-two residual witness of [68, Theorem 5.1.1], followed by the actual-source theorem [3, Theorem 2.3]. The potential-modularity field is chosen Galois and disjoint from the residual kernel and cyclotomic fields. Thus the actual source becomes modular after this extension. This argument starts with the weight-two point already constructed; it does not assume the existence of an exceptional initial characteristic-five point.

For (ii), choose type A at all coefficient places in Snowden's avoiding theorem [68, Theorem 5.1.1]. This gives an ordinary potentially crystalline weight-two residual witness over a totally real Galois field disjoint from the residual and cyclotomic fields. Choose its finite determinant factor to lift $\det \bar{r}/\bar{\chi}_p$. Its local representations are potentially diagonalizable by [7, Lemma 1.4.3(1)]. Pass to a CM quadratic extension disjoint from the same fields and apply [22, Theorem 1] to the actual source with polarization multiplier $\det r$. The residual polarized representation is automorphic, adequacy persists, and the CM field excludes ζ_p. Cyclic descent gives modularity over the totally real Galois field.

At two use the Hilbert–Blumenthal construction in the proof of [73, Proposition 6.7]. In its Moret–Bailly datum take each finite Galois dyadic field large enough to trivialize the local

residue and, in the potentially nonordinary case, to make the actual source Barsotti–Tate. The ordinary good-reduction neighbourhoods in that proof remain available after this enlargement. Put the global residual field and the prescribed avoidance field in the avoidance datum. This gives one totally real Galois extension of the original base, with the required local completions and with the nonsolvable residual image unchanged. The automorphic Hilbert–Blumenthal abelian variety in the proof already supplies a parallel-weight-two, ordinary residual witness there, before any further solvable base change to lower its level away from two. It suffices to take this witness, since the lifting theorem permits finite ramification away from two. Apply Lemma 2.3 over this Galois extension to the actual source. The ordinary case also applies to the ordinary weight-two semistable components. This gives the actual avoiding potential modularity used below.

The construction of compatible systems by potential modularity and descent originates in Dieulefait's work over $\mathbf{Q}$ [18, Theorem 1.1]. For the totally real fields and local conditions here, use the Brauer-induction construction of [68, Proposition 9.2.1 and Remark 9.2.2], retaining the finite characters in the Brauer identity. Mackey's formula and Frobenius reciprocity express the self-pairing of each virtual member in terms of pairings of restricted modular representations with finite character twists. Each restriction is absolutely irreducible: the subgroups in the Mackey formula contain the potential-modularity subgroup, on which these representations restrict to the cuspidal system, its conjugates or character twists. After extending coefficient scalars, each pairing is zero or one. Compatibility and Chebotarev decide equality by Frobenius polynomials, so these pairings are independent of the coefficient prime. At the original prime the self-pairing is one, so each virtual member is an irreducible or its negative. Its dimension two excludes the negative. Local–global compatibility for the modular systems [65, Theorem 1 and equation (1)] includes the coefficient prime and arbitrary cohomological weights. Restriction–induction identities give the common Weil representations; purity determines monodromy [7, Lemma 1.3.6(2)–(3)]. Labelled weights are compatible by the same restriction argument. □

2.3 Initial systems for odd residual characteristic

Proposition 2.7. *Let K be real quadratic and p odd. A continuous absolutely irreducible totally odd residual representation with solvable projective image is modular. If its projective image is nonsolvable, it is the residue of a nondihedral, potentially modular, strongly compatible weight-two system with determinant cyclotomic times a finite totally even character. For an exceptional characteristic-five representation over $\mathbf{Q}(\sqrt{5})$ use the lift of Appendix A.*

Proof. For solvable image, the odd solvable Artin witness and regular weight raising are as in Section 2. Otherwise the restriction to $G_{K(\zeta_p)}$ is absolutely irreducible: its projective image contains the nonsolvable derived subgroup. If $K \neq \mathbf{Q}(\sqrt{5})$, the characteristic-five exception in the prescribed-lift theorem would require the projective S_5 kernel field to contain $K(\zeta_5)$, a cyclic quartic extension, whereas S_5 has abelianization of order two. The exceptional condition for the initial lift in [3, Theorem 2.2] is also impossible, since the projective determinant-squareclass character has order at most two and $\bar{\chi}_5$ has order four over K.

In the nonexceptional cases use [3, Theorem 2.2] and the weight-two compatible-system construction of Lemma 2.6(i). Over $\mathbf{Q}(\sqrt{5})$ in the exceptional case, $\bar{\chi}_5$ is quadratic and cuts out $K(\zeta_5)$, so the two exceptional conditions agree. Use Proposition A.2, which also supplies avoiding potential modularity and compatibility. Nonsolvability of the given residue makes the system nondihedral. □

3 An auxiliary prime for congruences

In this section K is real quadratic. The finite set S may include any prescribed terminal reset primes.

Proposition 3.1. *Let $\mathcal{R}$ be a nondihedral weight-two system of determinant $\chi\psi$, where ψ has finite two-power order. Fix a finite set S of rational primes containing $2, 3$, all primes dividing the discriminant of K, and every conductor prime. There are a prime $N > \max(S \cup \{5\})$, a degree-one prime λ of K of norm $l > N$, and a congruent weight-two system $\mathcal{R}'$ such that $\mathcal{R} \rightsquigarrow \mathcal{R}'$ in characteristic N, its determinant remains $\chi\psi$, and its only new conductor prime is λ. At λ its type is induced from the unramified quadratic extension with inverse inertia characters of order N.*

While this type is retained and conductor is supported at places over $S \cup \{l\}$, the residual representations at every rational prime in S have nonsolvable image. At odd such characteristics their cyclotomic restrictions are adequate.

Proof. Let K_S/K be the maximal elementary quadratic extension unramified outside the places over S and infinity. It is finite. Write 2^a for the order of ψ. By [23, Proposition 4.6], choose $N > \max S$, $N > 5$, split in K and the coefficient field, with $N \equiv 1 \pmod{2^{\max\{2,a+1\}}}$, such that a residual member contains $\mathrm{SL}_2(\mathbf{F}_N)$. Its determinant is surjective on inertia above N, so its image is $\mathrm{GL}_2(\mathbf{F}_N)$.

Let R/K be the projective kernel field. Prescribe Frobenius to act trivially on K_S, by inversion on $K(\zeta_N)$, and as projective complex conjugation on R. The first two prescriptions agree on the intersection, since $K(\zeta_N)/K$ is totally ramified above N, whereas K_S/K is unramified there. The only nontrivial solvable quotient of $\mathrm{PGL}_2(\mathbf{F}_N)$ is its determinant-squareclass quotient. Our congruence makes $\bar{\psi}$ square-valued, so that quotient is cut out by $\overline{\chi}_N^{(N-1)/2}$ and lies in $K(\zeta_N)$. Inversion fixes it, and projective complex conjugation belongs to $\mathrm{PSL}_2(\mathbf{F}_N)$, since -1 is square modulo N. The prescriptions therefore agree on $R \cap K_S K(\zeta_N)$.

Chebotarev gives a positive-density set of such prime ideals. Those of residue degree greater than one have zero prime-ideal density. We may consequently choose λ of degree one and arbitrarily large norm l, outside the rational primes already used. Thus l splits in K, $l \equiv -1 \pmod{N}$, and the residual Frobenius eigenvalues are $a, -a$.

Put $\delta = \chi_N \psi$. Choose α lifting a with $\alpha^2 = -\delta(\mathrm{Frob}_\lambda)$, and define local inertia and Frobenius by

$$T = \begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_N^{-1} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}.$$

They satisfy $\Phi T \Phi^{-1} = T^l$ and have determinant δ; reduction gives the original unramified local representation. Retain the old crystalline weight-two lifts above N and compatible conditions elsewhere. Lemma 2.6 gives a global weight-two lift and its system. Lemma 2.2 at the split prime N gives backward modularity.

For a later coefficient characteristic $t \in S$, the two order-N inertia characters at λ remain distinct and are exchanged by Frobenius. The residual representation is therefore irreducible and its projective image contains an element of order $N > 5$, excluding A_4, S_4, A_5. If it were dihedral, the inducing quadratic extension would be unramified outside S: at λ its odd-order inertia lies in the rotations. But Frobenius at λ exchanges the rotation eigenspaces, making λ inert in that quadratic extension, contrary to its complete splitting in K_S. Dickson's classification forces nonsolvable projective image. Cyclotomic restriction retains its perfect derived subgroup, hence irreducibility. The order-N element excludes the inadequate characteristic-three and characteristic-five cases; for $t > 5$, the usual rank-two criterion applies. In characteristic two, nonsolvability is the required conclusion. □

4 Local lifts and persistent inertial types

Lemma 4.1. *Let $F/\mathbf{Q}_p$ be finite, with $p > 2$, and let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ have determinant $\overline{\chi}_p$. Then $\overline{r}$ has a regular crystalline, potentially diagonalizable lift with determinant exactly χ_p.*

Proof. Apply [9, Proposition 2.7] with prescribed determinant χ_p. In rank two, its construction also gives potential diagonalizability. An irreducible residual representation is induced from a character of the unramified quadratic extension; the constructed lift is induced from a crystalline character, so becomes a sum of crystalline characters after restriction. A reducible representation is lifted as a crystalline extension of crystalline characters, to which [7, Lemma 1.4.3(1)] applies. The unramified scalar correction in the fixed-determinant construction preserves both properties. □

Lemma 4.2. *Let $F/\mathbf{Q}_p$ be finite, and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be potentially semistable with labelled Hodge–Tate weights $\{0, 1\}$ and determinant χ_p times a finite-order character. If its monodromy is nonzero, then r is potentially ordinary. Thus a potentially nonordinary representation with these weights is potentially Barsotti–Tate.*

If $F = \mathbf{Q}_p$ and r is attached to a weight-two newform which is Steinberg at p, with determinant χ_p and $a_p = s \in \{1, -1\}$, its ordinary quotient is unramified with Frobenius value s.

Proof. Pass to a finite extension over which r is semistable and the finite determinant factor is trivial. Apply the filtered-module functor to $r^\vee$, with filtration jumps zero and one. If the residue degree is f and $N \neq 0$, the relation $N\varphi = p\varphi N$ gives eigenvalues $\alpha, p^f\alpha$ for φ^f, with the first eigenline equal to $\ker N$. Equality of the total Hodge and Newton numbers gives $v_p(\alpha) = 0$.

Every labelled Hodge number on this line is zero or one. Weak admissibility bounds their average by the Newton slope zero, so all are zero. The line is weakly admissible and corresponds to a crystalline weight-zero, hence unramified, subrepresentation of $r^\vee$. Its dual is an ordinary quotient of r. Consequently potential nonordinarity forces $N = 0$, and the resulting potentially crystalline representation with these weights is potentially Barsotti–Tate.

In the last assertion, local–global compatibility identifies the Frobenius eigenvalues on the dual module as s, ps [65, equation (1) and Theorem 1]. The same argument gives its unramified line with unit eigenvalue s; dualizing replaces the Frobenius value by $s^{-1} = s$. □

Lemma 4.3. *Let $F/\mathbf{Q}_p$ be finite with absolute ramification index e, and let $r : G_F \to \mathrm{GL}_2(E)$ be crystalline of labelled weights $\{0, 1\}$ and determinant χ_p. Suppose the residual semisimplification is a sum of characters. Then r is ordinary in either of the following cases.*

(i) $e < p - 1$, and the two characters are unramified twists of 1 and $\overline{\chi}_p$.
(ii) $p \in \{2, 3\}$, $e = p - 1$, and both characters are unramified.

Proof. A stable lattice is realized by a Barsotti–Tate group with coefficient action. A nonzero étale quotient has coefficient rank one by the weights and gives ordinarity. Otherwise the group is connected. Its Cartier dual is also connected because $r^\vee(1) \simeq r$ and connectedness is invariant under isogeny. The coefficient-uniformizer torsion is then biconnected.

After an unramified base change trivializing the unramified characters, the stable residual line contains a stable $\mathbf{F}_p$-line. Its schematic closure has order p. In (i) its generic character is trivial or cyclotomic. The rank-one model is étale or multiplicative by finite-flat uniqueness for $e < p - 1$ [57]. In (ii) its generic character is unramified. The Oort–Tate valuations lie between zero and $p - 1$ [84, Theorem 2]. The endpoints give étale and multiplicative models. The only possible interior valuation is one for $p = 3$, and it gives a ramified quadratic generic character. In either case the closure contradicts biconnectedness, since a multiplicative subgroup gives

an étale quotient of the dual. The canonical étale quotient descends over the unramified base change. □

Lemma 4.4. *Let K be real quadratic and let $\ell > 3$ be unramified in K. Suppose an absolutely irreducible characteristic-ℓ residue has crystalline lifts of weights $\{0, 1\}$ and determinant χ_ℓ at the places above ℓ. Then its restriction to $G_{K(\zeta_\ell)}$ is absolutely irreducible.*

Proof. A reducible cyclotomic restriction would give induction from the quadratic subextension of $K(\zeta_\ell)/K$. It is ramified at every ℓ-place. The image has order prime to ℓ, so projective inertia is cyclic and contains a reflection, hence has order two.

For a reducible finite-flat local residue the labelled digits are zero or one by [57]. The determinant makes the two digits at each embedding complementary. Projective inertia has order $\ell - 1$, or $\ell + 1$ for mixed digits at an inert place. In the locally irreducible case, if f is the residue degree the two niveau-$2f$ characters are conjugate by ℓ^f, and paired digits sum to one. The ratio has order $\ell + 1$ when $f = 1$. For $f = 2$ the exponent of the ratio, divided by $\ell^2 - 1$, is $\pm 1 \pm \ell$ modulo $\ell^2 + 1$, which has gcd two with the modulus. The order is $(\ell^2 + 1)/2$. These orders all exceed two. □

Lemma 4.5. *Let w have norm two.*

(i) In odd coefficient characteristic ℓ, a local residue at w of determinant $\overline{\chi}_\ell$, with unipotent inertia and Frobenius trace congruent to ± 3, admits a special lift of determinant χ_ℓ with unramified quadratic sign.

(ii) Let a weight-two system of cyclotomic determinant be good at w, with integral common Frobenius trace a. Suppose a is a unit at some dyadic coefficient place and that its reduction differs from one at every dyadic coefficient place. Then an odd coefficient prime dividing $9 - a^2$ gives a residue to which (i) applies.

Proof. For (i), wild inertia is trivial since the residue is unipotent on inertia and ℓ is odd. Frobenius has distinct eigenvalues $\delta, 2\delta$ for a sign $\delta = \pm 1$. If tame inertia is nontrivial, its invariant line is preserved by Frobenius and the tame relation makes the line eigenvalue twice the quotient eigenvalue. Thus we can choose the sign δ as the quotient value. If inertia is trivial we choose the eigenline with value 2δ. In this basis lift arithmetic Frobenius and a tame generator to

$$\begin{pmatrix} 2\delta & 0 \\ 0 & \delta \end{pmatrix}, \qquad \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$

with $c \neq 0$ lifting the residual parameter. These matrices satisfy the tame relation and have the required determinant. Their inertia factors through the pro-ℓ tame quotient. This is a special lift of the actual residual representation.

In (ii), the two Frobenius eigenvalues at every complex embedding have modulus $\sqrt{2}$ and product two. Thus a is totally real, $|a| \leq 2\sqrt{2}$ at every embedding, and $9 - a^2$ has positive integral norm. Every conjugate of $9 - a^2$ is at least one. Norm one would force $a^2 = 8$, contrary to the unit hypothesis. Also $9 - a^2$ is a unit at every dyadic coefficient place by the assumption on $\bar{a}$. It therefore has an odd prime divisor. The residue at that coefficient prime is unramified at w and has trace congruent to ± 3 there. □

Lemma 4.6. *Let $F/\mathbf{Q}_2$ be quadratic. Let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ have determinant one.*

(i) If $\overline{r}$ is reducible, it has a crystalline ordinary lift of determinant χ_2 with regular labelled weights. It may retain any specified invariant residual line. Its invariant line has the larger weight at every embedding, and the lift lies on an ordinary component of the fixed-determinant crystalline space. If F is ramified, the weights may be chosen to be $\{1 - n, n\}$ at both embeddings for some $n \geq 2$.

(ii) If $\overline{r}$ is irreducible, it has a crystalline lift of determinant χ_2 and weights $\{0,1\}$. Every potentially Barsotti–Tate lift with these weights is potentially nonordinary.

Proof. In the reducible case use the proof of [9, Proposition 2.7], starting with the given residual line and quotient. Choose crystalline character lifts with product χ_2. Multiplying the first by a sufficiently large positive power of χ_2 and the second by the inverse power preserves their reductions and product, since $\overline{\chi}_2 = 1$. Their weights are respectively positive and negative at every embedding. The crystalline extension constructed there lifts the given extension and has those labelled weights. Its unramified scalar determinant correction reduces to one and preserves the filtration. We obtain weights $a_\tau < b_\tau$, with $a_\tau + b_\tau = 1$, and the invariant line has weight b_τ.

For fixed regular weights, the ordinary crystalline deformation ring is a reduced quotient of the crystalline ring, with the same pure dimension, and both generic fibres are smooth [7, Section 1.4]. The ordinary locus is therefore a union of irreducible components of the crystalline generic fibre. Smoothness makes this union open and closed; its intersection with the fixed-determinant fibre is again a union of components. We choose a component through the constructed point.

For a ramified quadratic $F/\mathbf{Q}_2$, every odd-order character of $\mathcal{O}_F^\times$ is trivial, so the two residual diagonal characters are unramified. Starting with their Teichmuller lifts times χ_2^n and χ_2^{1-n} in the same construction gives the asserted constant labelled weights for large n.

Suppose that $\overline{r}$ is irreducible. Wild inertia acts trivially: its fixed space is nonzero and is stable under the whole local group. Tame inertia has two distinct characters exchanged by Frobenius.

If F is ramified, Frobenius acts by squaring. If one is θ, the other is θ^2, and determinant one gives $\theta^3 = 1$. Thus θ has order three. A Frobenius matrix exchanges the two eigenlines; determinant one makes its square the identity, and diagonal conjugation makes both off-diagonal entries one. Hence there is exactly one such representation up to isomorphism, with image $\mathrm{GL}_2(\mathbf{F}_2) \simeq S_3$.

The elliptic curve

$$A/\mathbf{Q}: \quad y^2 + y = x^3,$$

with Cremona label `27a3` (LMFDB label `27.a4`) [48], realizes it over $\mathbf{Q}_2$. Its discriminant is -27, so it has good reduction at two. Its nonzero two-torsion points have $y = -1/2$ and $4x^3 + 1 = 0$, giving $\mathbf{Q}_2(A[2]) = \mathbf{Q}_2(\zeta_3, \sqrt[3]{2})$. This is an S_3 extension, since $X^3 - 2$ is Eisenstein and $\mathbf{Q}_2(\zeta_3)/\mathbf{Q}_2$ is unramified quadratic. Its unique quadratic subfield is therefore unramified, so it is disjoint from the ramified quadratic extension F. Thus $A[2]|_{G_F}$ is the required residual representation, and $V_2(A)|_{G_F}$ is a crystalline Barsotti–Tate lift by good reduction, with determinant χ_2 by the Weil pairing [33, Theorem 6.2].

If F is unramified, then for the unramified quadratic extension E/F we may write $\overline{r} = \mathrm{Ind}_{G_E}^{G_F} \bar{\theta}$, with $\bar{\theta}|_I = \omega_4^e$, where ω_4 takes values in $\mathbf{F}_{16}^\times$. Frobenius over F acts by the fourth power. Determinant one gives $5e \equiv 0 \pmod{15}$, and irreducibility excludes $e = 0$, so $e \in \{3, 6, 9, 12\}$.

Index the four embeddings of E by $0, 1, 2, 3$, with i and $i+2$ restricting to the same embedding of F, and normalize ω_4 so a crystalline character of weights k_i reduces on inertia to exponent $\sum_i 2^i k_i$. The profiles $(k_0, k_1, k_2, k_3) = (a, b, 1-a, 1-b), \quad a, b \in \{0, 1\}$, have paired sums one and exponents $12 - 3a - 6b \in \{3, 6, 9, 12\}$. Lift $\bar{\theta}$ crystallinely with the appropriate profile and correct its unramified value. Induction along the unramified extension gives a crystalline lift with weights $\{0, 1\}$ at both embeddings of F. Its determinant differs from χ_2 by an unramified character δ reducing to one. An unramified character ν with $\nu^2 = \delta^{-1}$ exists after enlarging coefficients and reduces to one, because characteristic two has no other square root of one. Twisting by ν fixes the determinant.

Finally, a potentially ordinary lift of these weights has, over a finite Galois extension, a line of constant labelled weight one. This line is unique, since two such lines would give weight one twice at every embedding. Its Galois conjugates are the same line, so it descends to G_F. Its intersection with a stable lattice is saturated; reduction gives an invariant line in $\overline{r}$, a contradiction. □

Lemma 4.7. *For a finite extension $F/\mathbf{Q}_l$ with $l \neq 2$, every continuous $\overline{r} : G_F \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ has a lift of determinant χ_2, with finite inertia or Steinberg type. An unramified $\overline{r}$ has an unramified such lift.*

Proof. If $\overline{r}$ is reducible with diagonal characters α, α^{-1} and $\alpha \neq 1$, then $H^1(G_F, \alpha^2) = 0$: local Euler characteristic and duality away from two give this from the vanishing of $H^0(G_F, \alpha^2)$ and $H^0(G_F, \alpha^{-2}\overline{\chi}_2)$, since $\overline{\chi}_2 = 1$. Thus $\overline{r}$ is split, and $\widetilde{\alpha} \oplus \chi_2\widetilde{\alpha}^{-1}$ is the required lift.

If $\alpha = 1$, write $\overline{r} = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$. The additive character c kills wild inertia and odd-order tame inertia. If τ generates pro-two tame inertia and Φ is arithmetic Frobenius, with residue cardinality Q, their relation is $\Phi\tau\Phi^{-1} = \tau^Q$. Choose lifts n, x of $c(\tau), c(\Phi)$ and set

$$\tau \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad \Phi \longmapsto \begin{pmatrix} Q & x \\ 0 & 1 \end{pmatrix}.$$

This satisfies the relation and has determinant χ_2. Take $n = 0$ when $\overline{r}$ is unramified; otherwise the lift has nonzero rank-one monodromy.

If $\overline{r}$ is irreducible, its finite local image is solvable. Dickson's classification in characteristic two makes it odd dihedral; its scalar kernel is trivial because its determinant is one. Write $\overline{r} = \mathrm{Ind}_{G_E}^{G_F} \bar{\theta}$ for a quadratic extension E/F and an odd-order character $\bar{\theta}$. The induced-determinant formula gives $\bar{\theta}|_{F^\times} = 1$. Its Teichmüller lift θ_0 has the same restriction. Let $\epsilon_{E/F}$ be the quadratic character of $F^\times$. Choose a character ϕ of $E^\times$ reducing to one and restricting to $\chi_2\epsilon_{E/F}$ on $F^\times$ as follows.

For unramified E/F, take ϕ unramified with the required value on a common uniformizer. For ramified E/F, which is tamely quadratic, take the quadratic residue character on $\mathcal{O}_E^\times$. It restricts to $\epsilon_{E/F}$ on $\mathcal{O}_F^\times$. Write $\pi_F = u\pi_E^2$ and choose $\phi(\pi_E)$ to be a square root of $(\chi_2\epsilon_{E/F})(\pi_F)/\phi(u)$. All these values reduce to one. Then $\mathrm{Ind}_{G_E}^{G_F}(\theta_0\phi)$ has determinant $\epsilon_{E/F}(\theta_0\phi)|_{F^\times} = \chi_2$, the required reduction, and finite inertia. An irreducible representation over an algebraically closed field cannot be unramified, since an unramified group is procyclic. □

Lemma 4.8. *Let r be an absolutely irreducible dyadic representation over a totally real field with one dyadic place, whose completion is unramified over $\mathbf{Q}_2$. Assume determinant χ_2, crystalline weights $\{0, 1\}$ at two, and global residual semisimplification $1 \oplus 1$. At two, r is ordinary with an unramified quotient of rank one. There is a stable lattice whose reduction is a nonzero additive extension of 1 by 1 and is trivial at two.*

Proof. The dyadic restriction is ordinary by Lemma 4.3(ii). Its étale quotient has coefficient rank one, and the kernel is the ordinary line.

Choose a nonzero vector on that line and let its global orbit generate a lattice. Compactness and absolute irreducibility make the lattice full and stable. The vector is primitive and cyclic modulo the uniformizer by Nakayama's lemma, so the residue is a nonsplit extension of the two trivial characters. Its unique global invariant line differs from the reduction of the ordinary line. The two distinct lines are stable at two, where the residue therefore acts trivially. □

Lemma 4.9 (An unramified or special replacement)**.** *Let q and p be distinct rational primes with p odd and $q \equiv 1 \pmod{p}$. Let $\bar{s}$ be a representation of $G_{\mathbf{Q}_q}$ in characteristic p of cyclotomic determinant. Suppose wild inertia acts trivially and tame inertia is trivial or cyclic unipotent of*

order p. After extending coefficients, $\bar{s}$ has a cyclotomic-determinant lift which is unramified in the first case and special with unramified quadratic sign and nonzero monodromy in the second.

Proof. In the unramified case lift the whole Frobenius matrix and correct its determinant to q by a scalar square root reducing to one. In the other case a tame generator acts by $T = 1 + tE_{12}$ with $t \neq 0$. The relation $\bar{\Phi} T \bar{\Phi}^{-1} = T^q = T$ and determinant one give $\bar{\Phi} = aI + bE_{12}$ with $a = \pm 1$. Use lifts $\widetilde{t}, \widetilde{b}$ and set

$$U(x) = 1 + \widetilde{t} x E_{12}, \qquad \Phi = \begin{pmatrix} qa & \widetilde{b} \\ 0 & a \end{pmatrix}.$$

These matrices satisfy $\Phi U(x) \Phi^{-1} = U(qx)$ on the tame $\mathbf{Z}_p$ quotient. They have the required determinant and preserve both the residual inertia and Frobenius parameters. The tame parameter is nonzero, so the lift is genuinely special. □

4.1 A type containing every relevant Serre weight

Let $F/\mathbf{Q}_p$ be finite, where $p > 2$, with residue field k of size q. We give the cyclotomic character χ_p Hodge–Tate weight one. Put $K_0 = \mathrm{GL}_2(\mathcal{O}_F)$, choose a uniformizer π, put $K_1 = 1 + \pi M_2(\mathcal{O}_F)$, and write

$$I^+ = \left\{ \begin{pmatrix} 1 + \pi a & b \\ \pi c & 1 + \pi d \end{pmatrix} : a, b, c, d \in \mathcal{O}_F \right\}, \qquad J = \mathcal{O}_F^\times I^+.$$

Fix a nontrivial additive character $\psi : k \to \overline{\mathbf{Q}}_p^\times$. Define an affine generic character of J by

$$\theta \left(z \begin{pmatrix} 1 + \pi a & b \\ \pi c & 1 + \pi d \end{pmatrix} \right) = \psi(\bar{b} + \bar{c}), \qquad z \in \mathcal{O}_F^\times.$$

The ambiguity in z does not affect the two displayed residue coordinates. Let Z and U denote the scalar subgroup and the upper unipotent subgroup of $\mathrm{GL}_2(k)$.

Lemma 4.10. *Every irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(k)$ with trivial central character is a quotient of $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1$.*

Proof. Let V be such a representation. A finite p-group acting on a nonzero vector space in characteristic p has a nonzero fixed vector. Thus $V^U \neq 0$. Since Z acts trivially, $V^{ZU} \neq 0$. Frobenius reciprocity gives a nonzero map $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1 \to V$, which is surjective by irreducibility. □

Lemma 4.11. *The representation $\sigma = \mathrm{Ind}_J^{K_0} \theta$ is irreducible of dimension $q^2 - 1$. It is the compact type of a simple-supercuspidal inertial type τ with determinant one. Its natural integral lattice has reduction equal to the inflation of $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1$.*

Proof. On $K_1 = 1 + \pi M_2(\mathcal{O}_F)$, the character is $1 + \pi A \mapsto \psi(\bar{A}_{21})$. Under conjugation by $\mathrm{GL}_2(k)$, its stabilizer is ZU. Indeed the trace pairing on $M_2(k)$ is nondegenerate, including after composing with the nontrivial additive character of k, and the centralizer of a nonzero nilpotent matrix is ZU. The restriction of σ to K_1 is therefore a sum of $q^2 - 1$ distinct characters permuted transitively by K_0. This proves irreducibility and the dimension assertion.

The simple-supercuspidal construction identifies σ as the compact type of the representations obtained by extending θ to $\langle g \rangle F^\times I^+$, where $g = \left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$; see [1, Section 3.1]. To check the occurrence condition, let an irreducible smooth representation contain θ on J, and use its central character to extend θ to $F^\times I^+$. By Frobenius reciprocity the compact induction from

this latter group surjects onto the representation. That compact induction is the direct sum of the two irreducible simple-supercuspidal representations in [1, Section 3.1, p. 6]. Changing the extension on g, or its central character on a uniformizer, gives an unramified twist. Conversely each such twist contains σ. The occurrence characterization of inertial types therefore identifies σ with $\sigma(\tau)$, up to duality; compare [38, Theorem 1.1.3]. The Langlands parameter is the induced parameter of [1, Section 3.4 and Theorem 3.18]. The central character on $\mathcal{O}_F^\times$ is trivial, so $\det \tau = 1$.

All values of θ are p-power roots of unity and reduce to one. The coset basis of the induced lattice therefore reduces to the permutation representation on K_0/J. Since $K_1 \subset J$ and $J/K_1 = ZU$, this is the asserted inflation. Taking the dual does not change its Jordan–Hölder support. □

Theorem 4.12 (A universal wild inertial type)**.** *Let $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be continuous with determinant $\bar{\chi}_p$. Then $\bar{r}$ admits a potentially crystalline lift of Hodge–Tate weights $\{0, 1\}$ at every embedding, determinant exactly χ_p, and inertial type τ from Lemma 4.11. For $F = \mathbf{Q}_p$, write this type as $\tau_{p,w}$. Its inertia remains absolutely irreducible after reduction in every coefficient characteristic greater than p, for every stable lattice.*

Proof. We use the potentially Barsotti–Tate Breuil–Mézard theorem of Gee–Kisin, [38, Corollaries 3.5.3 and 3.5.6]. To match their conventions, apply their theorem to the dual residual representation, and dualize the resulting lift at the end. Their cyclotomic character has Hodge–Tate weight -1. The central-character restriction in Corollary 3.5.3 is therefore trivial in the present determinant situation.

At least one of the nonnegative multiplicities μ_V is positive. To see this, use the existence of some potentially Barsotti–Tate lift of any rank-two residual representation. This existence is explicitly justified in the proof of [38, Theorem A.2]: the irreducible case is contained in [68, Proposition 7.8.1], and the reducible case follows from the crystalline-extension construction cited there. The nonzero potentially Barsotti–Tate ring for that lift has positive Hilbert–Samuel multiplicity; the Breuil–Mézard formula makes at least one μ_V positive.

Every V with positive multiplicity has trivial central character. By Lemmas 4.10 and 4.11, it occurs in the reduction of the simple-supercuspidal type. The Breuil–Mézard formula for Hodge type zero and type τ consequently has positive right side. Its potentially crystalline lifting ring is nonzero and p-torsion free, hence has nonempty generic fibre. Dualizing supplies a lift r with the desired Hodge–Tate weights and type: a two-dimensional determinant-one type is self-dual.

Finally $\det r/\chi_p$ is unramified and reduces to one. It is unramified because it is potentially crystalline of weight zero with trivial inertial type. As p is odd, its inverse has an unramified square root reducing to one. Twisting r by this square root fixes its determinant without changing its reduction, weights or inertial type.

For $F = \mathbf{Q}_p$, the parameter is induced from a ramified quadratic extension. Its restriction to wild inertia has two distinct characters of order p, exchanged by a quadratic inertia element [1, Section 3.4 and Theorem 3.18]. In coefficient characteristic greater than p the two characters remain distinct, and the exchanging element prevents either eigenline from being stable. The finite inertial image has order divisible only by 2 and p, so semisimplicity makes this conclusion independent of the stable lattice. □

Remark 4.13. For $p = 3$ this theorem applies to both unramified and ramified quadratic completions. It gives a local lift of weight two. It makes no assertion that an arbitrary-weight global source is modular whenever such a replacement is modular. A backward modularity implication must be verified separately for the actual global source.

Lemma 4.14 (Bounded five-adic lifts)**.** *Let ω_2 have niveau two and order 24 over $\mathbf{Q}_5$, and put $\tau_3 = \widetilde{\omega}_2^8 \oplus \widetilde{\omega}_2^{16}$ and $\tau_6 = \widetilde{\omega}_2^4 \oplus \widetilde{\omega}_2^{20}$. Every continuous $\bar{r} : G_{\mathbf{Q}_5} \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ of determinant $\bar{\chi}_5$ has a weight-two lift of determinant χ_5 which has type τ_3 or τ_6, or is quadratic-special with nonzero monodromy.*

Proof. The weights with trivial scalar central character are

$$1, \quad \overset{2}{\det}, \quad \mathrm{Sym}^2 \det, \quad \mathrm{Sym}^2 \overset{3}{\det}, \quad \mathrm{Sym}^4, \quad \mathrm{Sym}^4 \overset{2}{\det}.$$

The cuspidal compact types satisfy

$$\overline{\sigma}(\tau_6)^{\mathrm{ss}} = 1 \oplus \mathrm{Sym}^2 \det, \qquad \overline{\sigma}(\tau_3)^{\mathrm{ss}} = \overset{2}{\det} \oplus \mathrm{Sym}^2 \overset{3}{\det}.$$

Indeed their Brauer character is zero on split regular elements, $-\theta(z) - \theta(z^5)$ on nonsplit regular elements, and $4\theta(a)$ on scalars. Substitution of the exponents four and eight proves both identities. In the full fixed-determinant Breuil–Mézard formula [74, Theorem 1.2], the multiplicities are nonnegative and at least one is positive, since a potentially Barsotti–Tate lift exists. The determinant confines them to the six displayed weights. The first four yield the cuspidal alternatives.

The last two occur in the semistable compact type for scalar quadratic inertia. A point with nonzero monodromy gives the special alternative. If every such point is crystalline, the corresponding crystalline compact type, 1 or $\det^2$, has positive multiplicity, so a cuspidal lift again exists. This applies to the actual residual extension over any finite coefficient field. The deformation rings have fixed determinant; equivalently, an unramified determinant error reducing to one is corrected by its square root. □

Lemma 4.15. *Let $\ell \neq 3$ be an odd prime with $\ell \equiv -1 \pmod 3$. Let $\bar{s}$ be the reduction, with any stable lattice, of a determinant-χ_3 representation of $G_{\mathbf{Q}_\ell}$ which is either a quadratic twist of special type or tame dihedral with inertia of order three or six. After a finite coefficient extension, $\bar{s}$ has lifts of both kinds with determinant exactly χ_3. The scalar quadratic inertial character determines whether the cuspidal lift has inertia of order three or six, and the special lift may be chosen with nonzero monodromy.*

Proof. Remove the scalar quadratic character. Wild inertia is trivial; tame inertia is trivial or unipotent of order three. Frobenius has characteristic polynomial $X^2 - 1$ modulo three: in the special case its eigenvalues are $\ell\alpha, \alpha$, $\alpha = \pm 1$, and in the cuspidal case its trace is zero and determinant is ℓ. The two eigenvalues are distinct. A nonzero inertial extension has one Frobenius eigenline as its invariant line and is normalized by diagonal conjugation. There are two nonsplit orientations and the split case.

For a cuspidal lift take

$$T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Phi = bS, \quad b^2 = -\ell, \quad b \equiv 1 \pmod 3.$$

Hensel's lemma supplies b. We have $T^3 = 1$, $STS = T^{-1}$ and $\det \Phi = \ell$. The standard lattice gives nontrivial unipotent inertia with eigenvalue -1 on its invariant line; replacing Φ by $-\Phi$ gives the other orientation. For trivial residual inertia use $T = \mathrm{diag}(\zeta_3, \zeta_3^{-1})$ over $\mathbf{Q}_3(\zeta_3)$ and the same exchanging Frobenius.

For a special lift use

$$\Phi = \mathrm{diag}(\ell, 1), \qquad U(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \Phi U(t) \Phi^{-1} = U(\ell t).$$

Choose its three-primary tame parameter to be a unit multiple when residual inertia is nontrivial, and three times a unit multiple otherwise. Both choices have nonzero monodromy. An unramified quadratic sign selects either orientation. Restoring the removed quadratic character preserves the determinant and gives the required inertial order. □

Lemma 4.16. *Let $p \in \{2,3\}$, let $F/\mathbf{Q}_p$ have absolute ramification index $p-1$, and let $r : G_F \to \mathrm{GL}_2(E)$ be crystalline of labelled weights $\{0,1\}$ and determinant χ_p. If both residual constituents are unramified, then r is ordinary.*

Proof. After a finite unramified extension the residual image is a p-group, so its underlying $\mathbf{F}_p$-representation has a fixed nonzero vector. Let G be a Barsotti–Tate group with coefficient action realizing a lattice. A nonzero etale quotient has coefficient rank one, by the weights, and proves ordinarity. Otherwise G is connected. Its Cartier dual is also connected, since $r^\vee(1) \simeq r$ and connectedness is invariant under isogeny. Coefficient-uniformizer torsion is therefore biconnected.

The schematic closure of the fixed $\mathbf{F}_p$-line has order p and unramified generic character. Its Oort–Tate parameter valuation lies between zero and $p-1$. The endpoints give etale and multiplicative models. For $p = 3$ the only other valuation is one, which gives a ramified quadratic generic character. Thus the closure is etale or multiplicative, both impossible in a biconnected group: the latter would give an etale quotient of its Cartier dual. The canonical maximal etale quotient descends through the unramified extension, proving ordinarity over F. □

4.2 Local constructions at a split terminal prime

We record the local characteristic-three calculations used in the three supported residual bases below. The universal existence statement is Theorem 4.12; here we also need the exact characteristic-two reduction of its type over $\mathbf{Q}_3$.

Lemma 4.17. *Let K be a real quadratic field of ordinary class number one in which three splits. Every finite character $G_K \to \overline{\mathbf{F}}_3^\times$ unramified outside three has square one.*

Proof. Such a character has order prime to three and kills the principal units at the two places above three. The ray-class sequence, with both real places included, makes its group a quotient of $(\mathbf{F}_3^\times)^2 \times \{\pm 1\}^2$, which has exponent two. □

For a finite Galois extension $L/\mathbf{Q}_3$, write $d(L) = v_3(\Delta(L/\mathbf{Q}_3))/[L : \mathbf{Q}_3]$, with $v_3(3) = 1$. Unramified base extension does not change this contribution.

Lemma 4.18. *Let $\overline{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ have determinant $\overline{\chi}_3$, and let L be its linear kernel field. If wild inertia is nontrivial, its fixed line makes $\overline{r}$ triangular, with diagonal quotient $\overline{\chi}_3\alpha$ for an unramified character α. Write Q for the wild inertia order.*

(i) If $\alpha \neq 1$, then $d(L) \le 3/2 - 1/Q < 3/2$.
(ii) If $\alpha = 1$, then $Q \in \{3,9\}$ and $d(L) \le 13/6 - 1/Q$. Its projective kernel field is totally ramified of degree $2Q$.
(iii) In the second case with $Q = 9$, the projective kernel field is

$$E_0 = \mathbf{Q}_3(\zeta_3, \sqrt[3]{3}, \sqrt[3]{4}), \qquad [E_0 : \mathbf{Q}_3] = 18, \qquad d(E_0) = 37/18. \tag{2}$$

In the tame case $d(L) < 1$. In all cases $d(L) \le 37/18$; the same upper bounds hold for projective subfields.

Proof. A nontrivial normal 3-subgroup has a unique fixed line, which the decomposition group preserves. Rank-one characters over $\mathbf{Q}_3$ have inertia of order dividing two. The diagonal inertia characters are therefore $1, \overline{\chi}_3$ in some order, and their quotient is $\overline{\chi}_3\alpha$ with α unramified.

The kernel P of the two diagonal characters in the finite image consists of translations. It equals wild inertia. Indeed, diagonalize a tame inertia involution, whose conjugation on translations is inversion. The commutator with any translation belongs to inertia and is twice that translation. Since 2 is invertible in characteristic three, every translation lies in wild inertia. In particular P is elementary abelian of order Q.

Put $T = L^P$. It contains μ_3 by the determinant condition and has ramification index two. Thus $T = U(\zeta_3)$ for an unramified extension $U/\mathbf{Q}_3$ of degree prime to three. Let k be its residue field and $\varpi = \zeta_3 - 1$. Local reciprocity gives an equivariant surjection $T^\times/(T^\times)^3 \to P$. Write $U_j = 1 + \varpi^j \mathcal{O}_T$. The logarithm gives $U_2^3 = U_4$, and

$$(1 + a\varpi)^3 \equiv 1 + (\bar{a}^3 - \bar{a})\varpi^3 \pmod{\varpi^4},$$

since $\varpi^2 = -3(1 + \varpi)$. Thus the possible unit grades modulo cubes are $1, 2, 3$, and the last is $k/\{a^3 - a : a \in k\}$, of dimension one over $\mathbf{F}_3$. The tame involution acts on grade j by $(-1)^j$, and on the valuation quotient trivially. Only grades one and three can map to P. A last nonzero grade j gives conductor $j + 1$; these are the usual local reciprocity and conductor conventions of [63, 49].

A Frobenius fixing ζ_3 acts trivially on the top quotient. If $\alpha \neq 1$, its action on P is multiplication by a nontrivial scalar; none of the associated $\mathbf{F}_3$-eigenvalues is one. The top grade cannot contribute, so every nontrivial character of P has conductor at most two. The tower and conductor–discriminant formulas give

$$d(L) = \frac{1}{2} + \frac{1}{2Q} \sum_{\xi \neq 1} a_T(\xi) \leq \frac{3}{2} - \frac{1}{Q}. \tag{3}$$

If $\alpha = 1$, divide by the lower diagonal character. The extension class belongs to

$$H^1(\mathbf{Q}_3, \overline{\mathbf{F}}_3(\overline{\chi}_3)) = (\mathbf{Q}_3^\times/(\mathbf{Q}_3^\times)^3) \otimes_{\mathbf{F}_3} \overline{\mathbf{F}}_3.$$

The rational Kummer space has basis the classes of 3 and 4, coming from valuation and principal units. The additive image therefore has rank at most two. At least $Q/3$ characters kill the image of the one-dimensional top unit grade. Their nontrivial members have conductor at most two; the others have conductor at most four. Hence

$$\sum_{\xi \neq 1} a_T(\xi) \leq 2(Q/3 - 1) + 4(2Q/3) = 10Q/3 - 2.$$

Equation (3) gives $d(L) \leq 13/6 - 1/Q \leq 37/18$.

In the projective representation, a Frobenius fixing ζ_3 is a translation. Its commutator with the tame inertia involution again shows that it belongs to inertia. Projective decomposition therefore equals inertia, of order $2Q$. When $Q = 9$, the two Kummer coordinates are independent, so their common kernel cuts out precisely E_0. Over $\mathbf{Q}_3(\zeta_3)$ its eight nontrivial characters have conductors two, twice, and four, six times. Thus

$$d(E_0) = \frac{1}{2} + \frac{2 \cdot 2 + 6 \cdot 4}{18} = \frac{37}{18}.$$

The tame formula is $1 - 1/e < 1$, and normalized differents cannot increase on passage to a subfield. □

The bound $Q \leq 9$ here is confined to an exactly cyclotomic diagonal quotient. For an arbitrary determinant, the bound we use is Moon–Taguchi's inequality

$$d(L) \leq \frac{13}{6} - \frac{1}{Q} \tag{4}$$

when the wild ramification index is $Q > 1$; tame extensions have $d(L) < 1$ [49, Theorem 3 and the following Remark (1)].

4.2.1 A wild type over $\mathbf{Q}_3$

Specialize the construction of Lemma 4.11 to $F = \mathbf{Q}_3$, with uniformizer $+3$ and additive character $\psi(x) = \zeta_3^x$. Retain its notation K_0, K_1, J and write ϑ for the affine generic character. The sign of this uniformizer fixes the quadratic subfield in the following lemma.

Lemma 4.19. *The representation $\sigma_+ := \operatorname{Ind}_J^{K_0} \vartheta$ is the irreducible eight-dimensional type of a supercuspidal inertial type τ_+ with determinant one. Its reduction satisfies*

$$\overline{\sigma}_+^{\mathrm{ss}} = 1 \oplus \det \oplus \operatorname{Sym}^2 \mathbf{F}_3^2 \oplus (\det \otimes \operatorname{Sym}^2 \mathbf{F}_3^2). \tag{5}$$

The inertia representation τ_+ is irreducible after reduction in every characteristic greater than three. For a determinant-cyclotomic compatible system of this type, its characteristic-two residual local kernel field is the unique totally ramified S_3 extension $E_+/\mathbf{Q}_3$ with quadratic subfield $\mathbf{Q}_3(\sqrt{3})$ and normalized different $7/6$. Every continuous $\bar{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with determinant $\bar{\chi}_3$ has a potentially crystalline lift of determinant χ_3, weights $\{0, 1\}$ and type τ_+.

Proof. The construction of Theorem 4.12 at $\ell = 3$ identifies σ_+ as the irreducible compact type of dimension eight, with determinant-one inertial parameter τ_+.

In the induced integral lattice, ϑ reduces to one, so $\overline{\sigma}_+ = \operatorname{Ind}_{ZU}^{\mathrm{GL}_2(\mathbf{F}_3)} 1$. If B is the upper triangular Borel, then B/ZU has order two, with characters $1, \det$. Induction in stages gives

$$\overline{\sigma}_+ = \operatorname{Ind}_B^{\mathrm{GL}_2(\mathbf{F}_3)} 1 \oplus \det \otimes \operatorname{Ind}_B^{\mathrm{GL}_2(\mathbf{F}_3)} 1.$$

The first summand is the permutation module on $\mathbf{P}^1(\mathbf{F}_3)$. Its constant line splits, since four is invertible in characteristic three, and its augmentation module is the Steinberg representation $\operatorname{Sym}^2 \mathbf{F}_3^2$. For example, evaluating a quadratic at the four projective points identifies it with that augmentation module. This proves (5); its four constituents are exactly the Serre weights with trivial scalar central character.

By [1, Section 3.4 and Theorem 3.18], the parameter is induced from a character of a ramified quadratic extension of $\mathbf{Q}_3$, of order three on the first principal-unit quotient, with conjugate the inverse there. Its inertia contains, in a suitable basis,

$$u = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The off-diagonal matrix has this form because inertia has determinant one. These matrices generate a group of order twelve. In characteristic $l > 3$, the two u-eigenlines remain distinct and v exchanges them. Thus reduction is irreducible; semisimplicity for this group of order prime to l makes the conclusion independent of the stable lattice. We check the quadratic field in the local parameter. In the parametrization of [1, Sections 3.1, 3.4 and Theorem 3.18], the affine-generic character $b + c$ uses uniformizer $+3$. The inducing quadratic extension is consequently $\mathbf{Q}_3(\sqrt{3})$. Reduction modulo two gives the irreducible group S_3. Its normalizer in $\mathrm{SL}_2(\overline{\mathbf{F}}_2)$ is itself: all automorphisms of S_3 are inner, and its centralizer consists of determinant-one scalars, hence is

trivial. Thus the entire local image equals its inertia image and the residual field has $e = 6$, $f = 1$.

The wild character has conductor two over the ramified quadratic field. The two-dimensional Artin conductor is three and remains three after reduction modulo two: the wild inertia has odd order and the inertia invariants remain zero. The lower groups of the residual field have orders $6, 3, 1$, giving normalized different $(5 + 2)/6 = 7/6$. Finally, over $\mathbf{Q}_3(\sqrt{3})$ the conductor-two anti-invariant character space has dimension one over $\mathbf{F}_3$. Its unique nonzero line gives a unique S_3 extension. This proves the stated uniqueness; the splitting field of $x^3 - 3x - 3$ is one realization.

The lifting assertion is Theorem 4.12 with $F = \mathbf{Q}_3$ and the affine-generic character fixed above. □

Lemma 4.20. *Let $\overline{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ have determinant $\overline{\chi}_3$. It has a lift of weights $\{0, 1\}$ and determinant χ_3 which is either a quadratic twist of a semistable representation with nonzero monodromy, or potentially crystalline of tame supercuspidal type*

$$\tau_4 = \widetilde{\omega}_2^2 \oplus \widetilde{\omega}_2^6,$$

where ω_2 is a fundamental character of niveau two. For either choice in a determinant-cyclotomic compatible system, the characteristic-two residual local kernel field is contained in

$$E' = \mathbf{Q}_3(\sqrt{3}, \sqrt{-1}), \qquad e(E'/\mathbf{Q}_3) = 2, \quad f(E'/\mathbf{Q}_3) = 2, \quad d(E'/\mathbf{Q}_3) = \tfrac{1}{2}. \tag{6}$$

Proof. We first obtain a quadratic twist of a semistable lift. If $\overline{r}$ is reducible, a ramified quadratic twist, when necessary, puts its invariant character in the form $\overline{\chi}_3\lambda$ and its quotient in the form λ^{-1}, with λ unramified. Let η be the unramified Teichmüller lift of λ. If $\lambda^2 \neq 1$, local duality gives $H^2(G_{\mathbf{Q}_3}, \mathcal{O}(\chi_3\eta^2)) = 0$. Thus the actual residual extension lifts to an extension of η^{-1} by $\chi_3\eta$; the one-dimensional characteristic-zero extension space is crystalline. If $\lambda^2 = 1$, take $\eta^2 = 1$. Kummer theory lifts both the unit and valuation directions of $\mathbf{Q}_3^\times/(\mathbf{Q}_3^\times)^3$; the former is crystalline and the latter is semistable with nonzero monodromy. If $\overline{r}$ is irreducible, its wild inertia is trivial and its tame exponents are $\{1, 3\}$ or $\{5, 7\}$ modulo eight. A ramified quadratic twist interchanges the pairs. The first lifts by induction from a crystalline character with labelled weights $(1, 0)$ over the unramified quadratic extension. An unramified scalar correction fixes the determinant. Twisting back gives the claimed semistable lift; compare [68, proof of Proposition 7.8.1].

Retain this lift if its monodromy is nonzero. Otherwise its inertial type is scalar quadratic, say $\kappa \oplus \kappa$. Its potentially crystalline compact type is $\kappa \circ \det$, with reduction 1 or det. The fixed-determinant formula of [74, Theorem 1.2], throughout with determinant χ_3, gives $\mu_1(\overline{r}) > 0$ or $\mu_{\det}(\overline{r}) > 0$.

The cuspidal type for τ_4 has dimension two and trivial central character. It is the two-dimensional representation of $\mathrm{PGL}_2(\mathbf{F}_3) \simeq S_4$ obtained from the standard representation of S_3. The integral augmentation lattice in $\mathbf{Z}_3^3$ reduces to a module with a constant line and sign quotient. Sign inflates to det, so its semisimplification is $1 \oplus \det$. Breuil–Mézard for Hodge type $(1, 0)$ and τ_4 has multiplicity $\mu_1(\overline{r}) + \mu_{\det}(\overline{r}) > 0$. Its three-torsion-free deformation ring gives the required potentially crystalline lift, with determinant exactly χ_3.

For the semistable choice, the characteristic-two local member is a quadratic twist of a special representation with diagonal characters $\chi_2\nu$ and ν, where ν is unramified quadratic. Both reduce to one, as does the quadratic twist. The full residual image is therefore an elementary abelian two-group. For the tame choice, inertia and arithmetic Frobenius matrices T, Φ satisfy

$$T^2 = -I, \qquad \Phi T \Phi^{-1} = T^{-1}, \qquad \Phi^2 = -3I.$$

The last equality follows from trace zero and determinant three. Modulo two these matrices become commuting involutions, again giving an elementary abelian two-group for every stable lattice. The field E' is the maximal elementary quadratic extension of $\mathbf{Q}_3$, so it contains both residual kernel fields. It is the compositum of the ramified and unramified quadratic extensions; its tame ramification index two gives the stated different. □

5 Dyadic automorphy and pro-modularity inputs

The local weight-two lifts and their support-preserving global construction are proved in Section 9, using the fixed-determinant Breuil–Mézard theorem [75, Corollary 8.0.2] and the prescribed-lift theorem [31, Theorem 4.2.1]. Potential modularity is established separately there. For backward modularity we use Kisin's and Tung's theorems with their local-source and residual-image hypotheses [46, Theorem 3.3.5], [75, Theorem 8.0.3].

The scalar argument uses the representation spaces, relative presentations and local geometry of [51, Section 3], and the completed-forms and local block comparisons of [51, Propositions 4.1, 4.2 and 4.4, Lemma 4.6]. The localized propagation theorem [51, Theorem 5.1] applies over totally real solvable fields of even degree completely split at two, to a potentially pro-modular characteristic-two curve whose generic representation is not virtually solvable and whose allowed odd local images are finite. In these conventions the completed-form central character is $(\det r)\varepsilon_2$ and the local block parameter is $r \otimes \varepsilon_2$. The regular de Rham locally irreducible parameters are treated by the local correspondence [12, Theorems 1.1 and 1.3].

For irreducible solvable residue we use [21, Proposition 10.3 and Theorem 10.6]. The application is over an abelian totally real field completely split at two, with every dyadic restriction of the actual source absolutely irreducible, so there are no locally reducible places. The auxiliary fields and scalar-source deduction needed here are proved in Sections 7 and 8.

6 Conductor reduction at a split odd prime

The local constructions above preserve the actual residual extension. The lifting implications in Section 2 apply to the old characteristic-zero source at each congruence.

Lemma 6.1. *Let K be real quadratic with two nonsplit, and let the odd prime ℓ split in K. A nondihedral potentially modular weight-two system with determinant cyclotomic times a finite totally even character admits a chain of congruences to a system of cyclotomic determinant, good outside the two ℓ-places and one auxiliary place $\mathfrak{q}$ of split norm $q > \max\{5, \ell\}$. The system has the wild type of Theorem 4.12 at both ℓ-places. The type at $\mathfrak{q}$ satisfies Proposition 3.1 for every subsequent coefficient characteristic in the preparation, including ℓ. Modularity of the last system implies modularity of the first.*

Proof. Twist by the inverse square root of the odd-order determinant factor, leaving determinant $\chi\psi$ with ψ of two-power order. Let S contain $2, 3, 5, \ell$, all primes dividing the discriminant of K, and all rational conductor primes. Apply Proposition 3.1; write $\mathfrak{q}$ for the auxiliary place, of split rational norm $q > N > \max S$. The order-N type gives nonsolvable residual images and adequate cyclotomic restrictions at all later coefficient characteristics in S.

In characteristic two, use Lemma 4.6 at the unique quadratic dyadic completion to choose a crystalline target of exact determinant χ_2. For a reducible local residue take an ordinary regular target, retaining the old ordinary filtration when present; for an irreducible local residue take a weight-two target, which is potentially nonordinary. Lemma 4.7 supplies determinant-χ_2 lifts at odd conductor places and keeps unramified places unramified. Retain the protective

type at $\mathfrak{q}$, correcting its unramified scalar. Lemma 2.6 gives the new system, good at two. For the old weight-two source, Lemma 4.2 and descent of the unique potentially ordinary line give the two cases of Lemma 2.3; in the ordinary case the target has matching residual ordering. Thus modularity returns to the old system. The new weights may be higher than two.

Reset in characteristic ℓ, imposing the type of Theorem 4.12 at both ℓ-places and retaining $\mathfrak{q}$ and all cleared conditions. Lemma 2.6 gives a weight-two system, with backward modularity by Lemma 2.2(i).

For each remaining odd conductor characteristic $t \neq \ell$, use the crystalline potentially diagonalizable components of Lemma 4.1, retaining the two wild ℓ-types, the protective place and all cleared conditions. If t is nonsplit, first reset at ℓ whenever the source is not of weight two; then apply Lemma 2.2(ii) backwards, using Lemma 2.1. If t splits, use Lemma 2.2(i) without a reset. In either case Lemma 2.6 supplies the replacement system. The protective order-N element ensures adequacy and excludes the exceptional characteristic-five image. Each step clears t, may change the weights, and introduces no conductor prime outside $S \cup \{q\}$. This gives the asserted system. □

Proof of Theorem 1.3. Apply Lemma 6.1. Work in coefficient characteristic $q > \max\{5, \ell\}$. The retained wild inertia at an ℓ-place remains irreducible modulo q, by Theorem 4.12. It lies in the cyclotomic subgroup, since $K(\zeta_q)/K$ is unramified there. Thus the residual cyclotomic restriction is absolutely irreducible and adequate. If the residual image is solvable, an odd solvable Artin witness and split-q lifting already prove modularity of the old system. Otherwise the crystalline replacements above q, Lemma 2.6 and split-q lifting remove $\mathfrak{q}$. The resulting system has exact cyclotomic determinant, arbitrary regular weights and conductor supported at ℓ.

Its actual ℓ-adic member is absolutely irreducible at both ℓ-places: a Galois line would give a Weil–Deligne line, contradicting the retained irreducible inertia type. If its residue is irreducible, (i) and [81, Theorem 6.1.1] give modularity. If it is reducible, apply (ii) and [81, Theorem 5.1.1]. Here K is totally real abelian, ℓ splits completely, and the constituent quotient is also nontrivial at both ℓ-places: a residual constituent has tame inertia ω_ℓ^a, so the quotient has inertia ω_ℓ^{1-2a}, an odd power modulo the even integer $\ell - 1$. Propagate modularity backwards and undo the finite twist. □

6.1 A terminal source with one crystalline place

Proposition 6.2. *Let K be real quadratic with two nonsplit, and let $\ell \geq 5$ be a rational prime that splits in K. Fix one of the two places above ℓ as v_+. To modularize every nondihedral potentially modular weight-two system with determinant cyclotomic times a finite totally even character, it suffices to modularize the following actual ℓ-adic members. Their systems have determinant χ_ℓ and conductor supported at one place $v_+ \mid \ell$. The member has regular labelled weights and is absolutely irreducible at v_+. At the other place $v_- \mid \ell$ it is crystalline of weight two. The congruence chain either proves modularity directly or has such a terminal member whose modularity implies modularity of the original system.*

Proof. Use Lemma 6.1 through the stage with possible conductor at ℓ and $\mathfrak{q}$. The conjugate $\mathfrak{q}'$ of the auxiliary place is still good. While the auxiliary type remains, switch in characteristic ℓ. At v_+ prescribe the wild potentially Barsotti–Tate type of Theorem 4.12, and at v_- prescribe a regular crystalline potentially diagonalizable lift by Lemma 4.1. The local representation at v_+ is potentially diagonalizable by [38, Lemma 3.4.1]. The auxiliary type guarantees adequacy and excludes the exceptional characteristic-five case. Lemma 2.6(ii) supplies the system and Lemma 2.2(i) gives the backward implication.

Enlarge its coefficient field E to contain K. The weight-two label is attached to one embedding $K \hookrightarrow E$. At a split coefficient prime, a choice of embedding of E into the local coefficient closure puts this label at either prime of K. The Weil–Deligne representation at a fixed finite place is conjugated on coefficients by this choice, so its good condition is preserved.

Choose a q-adic realization with the weight-two label at $\mathfrak{q}'$. There the actual member is crystalline of weight two, and hence potentially diagonalizable by [38, Lemma 3.4.1]. Retain that component and replace the component at $\mathfrak{q}$ by a regular crystalline potentially diagonalizable one. The retained wild type at v_+ protects irreducibility on the cyclotomic subgroup. The inertia group at v_+ is contained in the cyclotomic subgroup, since $K(\zeta_q)/K$ is unramified at v_+. Thus the restriction is irreducible. A solvable residue gives modularity of the old source by its Artin witness and split-q lifting. Otherwise the rank-two adequacy criterion applies since $q > 7$. Apply Lemma 2.6(ii) and use Lemma 2.2(i) backwards. The replacement is good away from v_+.

Finally choose an ℓ-adic realization placing the preserved weight-two label at v_-. Irreducible Weil inertia at v_+ precludes an actual local Galois line. This gives the asserted source, and all congruences have the required backward implications. □

Lemma 6.3. *Let $\ell \geq 5$ split in a real quadratic field K. Let $\bar{r}$ be an absolutely irreducible representation of G_K over $\overline{\mathbf{F}}_\ell$. If its restriction at one ℓ-place has a crystalline weight-two lift of determinant χ_ℓ, then $\bar{r}$ stays irreducible over $K(\zeta_\ell)$.*

Proof. Otherwise Clifford theory induces $\bar{r}$ from a character of the unique quadratic subfield of $K(\zeta_\ell)/K$. At the chosen place the quadratic extension of $\mathbf{Q}_\ell$ is ramified with unchanged residue field. The character has prime-to-ℓ order, so it and its conjugate agree on the inertia of the inducing field. Projective inertia has order at most two. On the other hand, the finite-flat classification over $\mathbf{Q}_\ell$ gives projective tame inertia of order $\ell - 1$ or $\ell + 1$, from the ordinary characters or the conjugate niveau-two characters respectively [57]. This is a contradiction. □

Theorem 6.4 ([24, Theorem A])**.** *Let F be a totally real field, abelian over $\mathbf{Q}$, and let p be an odd prime that splits completely in F. Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be continuous, absolutely irreducible, totally odd and unramified outside finitely many places. Let $\Sigma_p = \Sigma_{\mathrm{n}} \sqcup \Sigma_{\mathrm{o}}$ be a partition of the places above p, and suppose:*

(1) *for $v \in \Sigma_{\mathrm{n}}$, $\rho|_{G_{F_v}}$ is absolutely irreducible and de Rham with distinct Hodge–Tate weights;*
(2) *for $v \in \Sigma_{\mathrm{o}}$, $\rho|_{G_{F_v}}$ is reducible and de Rham with distinct Hodge–Tate weights;*
(3) *$\bar{\rho}^{\mathrm{ss}} = \bar{\chi}_1 \oplus \bar{\chi}_2$, the ratio $\bar{\chi}_1 \bar{\chi}_2^{-1}$ extends to a character of $G_{\mathbf{Q}}$, and it is nontrivial on G_{F_v} for every $v \mid p$.*

Then ρ is modular.

We call this result *the mixed theorem.*

Proposition 6.5 (The terminal member with one crystalline place)**.** *Let r be a terminal member of Proposition 6.2 with $\ell \geq 5$. If its global residue is irreducible, it remains irreducible on $G_{K(\zeta_\ell)}$, and a modular residual witness proves modularity of r. In particular this applies to solvable irreducible residues. If the global residue is reducible and its constituent ratio extends to $G_{\mathbf{Q}}$ and is nontrivial on both coefficient decomposition groups, the mixed theorem (Theorem 6.4) gives modularity of r.*

Proof. The irreducible case uses Lemma 6.3 at the crystalline place v_- and Lemma 2.2(i). For reducible residue, the local finite-flat characters at v_- are unramified twists of 1 and $\bar{\chi}_\ell$. Thus Lemma 4.3(i) makes the actual source ordinary there. The actual representation at v_+ is absolutely irreducible by its inertial type. Both restrictions are regular de Rham, the field is real abelian, and ℓ splits completely. The ratio is totally odd by the determinant. These are hypotheses (1)–(3) of Theorem 6.4, with $\Sigma_{\mathrm{n}} = \{v_+\}$ and $\Sigma_{\mathrm{o}} = \{v_-\}$. □

Corollary 6.6. *Under the hypotheses of Theorem 1.3, Serre modularity holds over K in odd residual characteristic with arbitrary conductor.*

Proof. Apply Proposition 2.7. In the nonsolvable case, apply Theorem 1.3 to the resulting system and reduce at the original coefficient place. □

6.2 Characters supported at the terminal prime

Lemma 6.7. *Let K be real quadratic and ℓ an odd split prime. Suppose that the narrow class group of K has exponent at most two and that a totally positive unit has inverse primitive residues at the two places above ℓ. If θ is a character over $\overline{\mathbf{F}}_\ell$ unramified outside ℓ at finite places, then*

$$\theta = \bar{\chi}_\ell^a \eta, \qquad \frac{\theta}{\bar{\chi}_\ell \theta^{-1}} = \bar{\chi}_\ell^{2a-1},$$

where η is a finite-unramified character of order at most two. In particular the quotient extends to $G_{\mathbf{Q}}$, is totally odd, and is nontrivial on inertia at both places above ℓ. If $\theta \oplus \bar{\chi}_\ell \theta^{-1}$ is finite flat at one such place, it is a quadratic twist of $1 \oplus \bar{\chi}_\ell$.

Proof. The character is tame at ℓ. Write its two inertial exponents relative to the cyclotomic character as a_1, a_2 modulo $\ell - 1$. Reciprocity on the unit gives $a_1 = a_2 = a$, so $\eta = \theta \bar{\chi}_\ell^{-a}$ is finite-unramified. It factors through the narrow class group. This gives the quotient formula. An odd power of the cyclotomic character has the asserted parity and has nontrivial inertia at a split prime. In the finite-flat case, the constituents over $\mathbf{Q}_\ell$ have inertia characters $1, \bar{\chi}_\ell$. Thus a is zero or one and we may interchange the two characters. □

Here are the applications used below. The two residues in the last column are taken in either order.

d	ℓ	$\mathrm{Cl}^+(K)$	positive unit	residues
6	5	C_2	$5 + 2\sqrt{6}$	$(2, 3)$
11	5	C_2	$10 + 3\sqrt{11}$	$(2, 3)$
14	5	C_2	$15 + 4\sqrt{14}$	$(2, 3)$
15	7	C_2^2	$4 + \sqrt{15}$	$(3, 5)$
19	3	C_2	$170 + 39\sqrt{19}$	$(2, 2)$
21	5	C_2	$(5 + \sqrt{21})/2$	$(2, 3)$

All displayed units have norm one. The residue assertions follow by evaluating them at the two roots of the defining quadratic polynomial. The ordinary class groups follow from the Minkowski bound $\sqrt{D_K}/2$. For $d = 6, 11, 14, 19$, dyadic ideals are principal by the elements $2 + \sqrt{6}$, $3 + \sqrt{11}$, $4 + \sqrt{14}$, $13 + 3\sqrt{19}$, respectively. At $d = 6$ the prime above three is principal by $3 + \sqrt{6}$, and at $d = 19$ by $4 + \sqrt{19}$. At $d = 11, 14$ three is inert. For $d = 21$ two is inert and the bound gives class number one. For $d = 15$ the primes above two and three represent the same class by $3 + \sqrt{15}$, of norm -6. The dyadic prime is nonprincipal since neither 2 nor -2 is a square modulo five. Its square is generated by the totally positive element two. In all six fields reduction at a ramified prime congruent to 3 mod 4 excludes units of norm -1. The sign kernel therefore has order two, giving the narrow class groups in the table. For $d = 15$ this kernel and the dyadic ideal class give independent classes of order two.

Write $F_d = \mathbf{Q}(\sqrt{d})$ for the field of radicand d.

Lemma 6.8. *Let $(K, \ell) = (F_2, 7), (F_3, 13)$ or $(F_5, 11)$. Let $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ have determinant ω_ℓ and reducible semisimplification, and be unramified outside ℓ. The quotient of its constituents is ω_ℓ^e for an odd e, up to inversion. In particular it is totally odd, extends to $G_{\mathbf{Q}}$, and is nontrivial at both ℓ-adic places.*

Proof. The three fields have ordinary class number one by the quadratic Minkowski bounds $\sqrt{8}/2, \sqrt{12}/2, \sqrt{5}/2 < 2$. Write the constituents as $\psi, \omega_\ell \psi^{-1}$. The finite character ψ has prime-to-ℓ order, hence kills wild inertia and factors through the narrow ray modulo ℓ and the two real places. Write a, b for its two tame exponents modulo $\ell - 1$, and $u, v \in \mathbf{Z}/2$ for its signs.

For $K = \mathbf{Q}(\sqrt{2})$ use the primitive root 3 modulo seven. At the two seven-primes the unit $1 + \sqrt{2}$, of signs $(+, -)$, has residues $4 = 3^4, 5 = 3^5$. The unit relations are

$$a + b + u + v \equiv 0 \pmod 2, \qquad 4a + 5b + 3v \equiv 0 \pmod 6.$$

The second relation modulo three gives $a \equiv b \pmod 3$, so $2a \equiv 2b \pmod 6$. For $K = F_5$, put $\phi = (1 + \sqrt{5})/2$. Its residues are $8 = 2^3, 4 = 2^2$ relative to the primitive root 2 modulo eleven, and its signs are $(+, -)$. The two unit relations give $a + b + u + v \equiv 0 \pmod 2$ and $3a + 2b + 5v \equiv 0 \pmod{10}$. Thus $a \equiv b \pmod 5$, $v \equiv a \pmod 2$ and $u \equiv b \pmod 2$. There are exactly twenty characters, and $2a \equiv 2b \pmod{10}$. For $K = F_3$ the positive unit $2 + \sqrt{3}$ has residues $6 = 2^5$ and $11 = 2^7$ modulo thirteen, with primitive root 2. Its relation $5a + 7b \equiv 0 \pmod{12}$ gives $a \equiv b \pmod{12}$. Thus in every case $\psi^2 \omega_\ell^{-2a}$ is unramified at all finite places and totally even. Ordinary class number one makes it trivial. The quotient is consequently ω_ℓ^{1-2a}, up to inversion. Its exponent is odd, so it is nontrivial at both split coefficient places and odd at every real place. □

7 Auxiliary fields with scalar Selmer vanishing

Let $K = \mathbf{Q}(\sqrt{d})$ be real quadratic, with $d > 1$ squarefree and $d \equiv 1 \pmod 8$. The space $V_{w_0}^+(K)$ and condition (SV) are defined in the introduction. No class-number hypothesis is imposed in this section.

Lemma 7.1 (Cyclic towers with bounded odd splitting)**.** *Let $K = \mathbf{Q}(\sqrt{d})$ be real quadratic with two split, let Σ be a finite set of odd rational primes, and let $n \geq 2$. There is an odd prime $r \notin \Sigma$ and a totally real cyclic field $L/\mathbf{Q}$ of degree 2^n such that:*

1. *L is ramified only at r, where it is totally ramified;*
2. *r is inert in K;*
3. *2 splits completely in L;*
4. *every $\ell \in \Sigma$ has at most two primes above it in KL.*

The prime r can avoid any additional prescribed finite set.

Proof. Let P be the union of Σ and the prime divisors of d, and put

$$B_n = \mathbf{Q}(\zeta_{2^{n+1}}, 2^{1/2^n}), \qquad C_n = B_n\big(t^{1/2^n} : t \in P\big).$$

The field B_n is Galois over $\mathbf{Q}$ and ramified only at two. At a prime of B_n above an odd rational prime t, the normalized valuation of t is one. These valuations show that the classes of the $t \in P$ generate a direct sum of copies of $\mathbf{Z}/2^n\mathbf{Z}$ in $B_n^\times/(B_n^\times)^{2^n}$: a product $\prod_{t \in P} t^{b_t}$ that is a 2^n-th power has $2^n \mid b_t$ for each t. Kummer theory consequently gives $\mathrm{Gal}(C_n/B_n) \simeq (\mathbf{Z}/2^n\mathbf{Z})^P$, where a vector (a_t) acts by $t^{1/2^n} \mapsto \zeta_{2^n}^{a_t} t^{1/2^n}$.

Let j be the number of prime factors of d. If j is odd, choose $a_t = 1$ for every $t \in P$. If j is even, choose $a_t = 2$ for exactly one prime dividing d, and $a_t = 1$ for every other $t \in P$. Then $\sum_{t \mid d} a_t$ is odd and each a_t has two-adic valuation at most one. The field C_n is Galois over $\mathbf{Q}$. Apply Chebotarev to the conjugacy class of the prescribed automorphism $h \in \mathrm{Gal}(C_n/B_n)$. We obtain a prime r split in B_n for which the Kummer exponents have the stated 2-adic valuations. Conjugating h can multiply all exponents by an odd integer, which does not change those valuations or their parity. Since $h(\sqrt{d}) = -\sqrt{d}$, the prime r is inert in K. Since r splits in B_n,

it satisfies $r \equiv 1 \pmod{2^{n+1}}$, $\quad 2 \in (\mathbf{F}_r^\times)^{2^n}$. Take L to be the unique subfield of $\mathbf{Q}(\zeta_r)$ of degree 2^n. The first congruence makes L totally real; the second makes two split completely. For $t \in P$, its Frobenius in $\mathrm{Gal}(L/\mathbf{Q})$ has order 2^n when a_t is odd, and order 2^{n-1} when $a_t = 2$. This follows from the 2^n-th power residue symbol at r. If $\ell \nmid d$, we chose $a_\ell = 1$, so L has one prime above ℓ and the quadratic extension KL/L yields at most two. If $\ell \mid d$, it ramifies in K and is unramified in L. The unramified local extension supplied by L and the ramified quadratic extension supplied by K are disjoint. Thus the number of primes over ℓ in KL equals their number in L, which is at most two. This proves all assertions. □

Lemma 7.2 (Propagation of one-dyadic scalar Selmer vanishing)**.** *Let K be a totally real field with two places w_1, w_2 above two, and suppose that the only totally real quadratic extension of K unramified outside w_1 and split at w_2 is the trivial extension. Let F/K be cyclic of degree 2^n, split at w_1, w_2, unramified outside a single finite prime $\mathfrak{r} \nmid 2$, and totally ramified there. Then the only totally real quadratic extension of F unramified outside the full fibre above w_1 and split at every place above w_2 is trivial.*

Proof. Let V be the corresponding subspace of $H^1(G_F, \mathbf{F}_2)$. It is stable under the 2-group $\Gamma = \mathrm{Gal}(F/K)$. If $V \neq 0$, the elementary invariant-vector theorem for representations of a finite 2-group in characteristic two gives a nonzero $c \in V^\Gamma$. Let H/F be the quadratic extension cut out by c. Then H/K is Galois, and its group G is a central extension of the cyclic group Γ by C_2. Every such central extension is abelian: a lift of a generator of Γ, together with the central kernel, generates G.

Because H/F is unramified at $\mathfrak{r}$ and F/K is totally ramified there, the inertia subgroup $I \subset G$ maps isomorphically onto Γ and intersects $\mathrm{Gal}(H/F)$ trivially. Consequently $G = I \times \mathrm{Gal}(H/F)$. The fixed field $E = H^I$ is quadratic over K, totally real, and unramified at $\mathfrak{r}$. At every finite place away from w_1 and $\mathfrak{r}$, both steps H/F and F/K are unramified, so E/K is unramified there. At w_2, both steps split completely, so E/K splits. This contradicts the assumed vanishing over K. □

Theorem 7.3 (Uniform scalar field preparation)**.** *Let K be real quadratic with two split and satisfying* (SV)*. Let Σ be a fixed finite set of odd rational primes, and fix integers $n > a \geq 1$. There are totally real abelian two-extensions $K \subset A \subset F$, split completely at two, and a dyadic place v_0 of A, with the following properties. Put*

$$D = [F : \mathbf{Q}], \qquad M = \{v \mid v_0 \text{ in } F\}, \qquad m = |M|,$$

and let S_A, S_F consist of the real places, the dyadic places, and the places above Σ.

1. *$D = 2^{n+1}$, $[F : A] = 2^{n-a}$, and $m/D = 2^{-a-1}$. The numbers u_A, u_F of allowed odd places are at most $2|\Sigma|$.*
2. *For every finite field k of characteristic two,*

$$\{c \in H^1(G_{F,S_F}, k) : \mathrm{loc}_v\, c = 0 \text{ for all } v \in S_F \setminus M\} = 0.$$

   Here and below k has trivial Galois action.
3. *Restriction $H^1(G_{A,S_A}, k) \longrightarrow H^1(G_{F,S_F}, k)$ is injective.*
4. *Let $H^1_{\mathrm{str}}(F, k)$ have zero local condition at M, full local condition at $S_F \setminus M$, including the real places, and unramified condition outside S_F. Then*

$$\dim_k H^1_{\mathrm{str}}(F, k) = 2D - 3m + u_F.$$

The degrees $[A : \mathbf{Q}]$ and $[F : A]$ can both be made arbitrarily large, while the allowed odd-place bound remains fixed.

Proof. Apply Lemma 7.1, take the fields $A = KL_a$ and $F = KL$ in Lemma 7.2, using (SV) as its base vanishing hypothesis. The first three assertions follow from the two preceding lemmas and complete splitting at two. For injectivity, a class killed under restriction factors through the cyclic group $\mathrm{Gal}(F/A)$. Its only nonzero quadratic character is ramified at the fresh conductor prime r, which is outside S_A. Such a class is excluded from $H^1(G_{A,S_A}, k)$, proving injectivity.

For the last assertion use the Selmer dimension formula from local and global Tate duality, with the real local conditions included [55, Theorem 4.2]. The scalar module k is self-dual with Tate twist, since the mod-two cyclotomic character is trivial. The dual local conditions are full at M, zero at $S_F \setminus M$, and unramified elsewhere. Their global Selmer group vanishes by the second assertion. The global invariant terms therefore cancel. Each unmarked dyadic place contributes $\dim H^1(G_{\mathbf{Q}_2}, k) - 1 = 2$, and each marked place contributes -1. A real place with full local condition contributes $1 - 1 = 0$. At every allowed odd finite place, local Kummer theory gives $\dim_k H^1(G_{F_v}, k) = 2$, so its contribution is one. Unramified local conditions contribute zero outside S_F. Consequently the strict Selmer dimension is $2(D - m) - m + u_F = 2D - 3m + u_F$, as claimed. Taking a and $n - a$ large proves the final assertion. □

When $\Sigma = \varnothing$, taking $n = a + 2$ in Theorem 7.3 gives $[F : A] = 4$ with $[A : \mathbf{Q}]$ arbitrarily large. There are no odd local Euler terms, so the same Selmer dimension formula gives $\dim_k H^1_{\mathrm{str}}(F, k) = 2[F : \mathbf{Q}] - 12$.

Remark 7.4. Condition (SV) is a genuine restriction. For $K = \mathbf{Q}(\sqrt{113})$, the totally positive element $\alpha = (11 + \sqrt{113})/2$ has norm two and valuation one at one dyadic prime. At the other its unit value u satisfies $u + 2/u = 11$, so $u \equiv 1 \pmod 8$ and is a square in $\mathbf{Q}_2$. Thus $[\alpha]$ is a nonzero element of the corresponding $V^+_{w_0}(K)$. This is a limitation of the criterion, not an obstruction to modularity over that field.

8 The scalar dyadic endpoint

We prove Theorem 1.4 using the field preparation of Section 7 and the dyadic inputs listed in Section 5. The argument treats arbitrary finite ramification and regular weights.

Proof of Theorem 1.4. First suppose r is virtually solvable, in the sense that the identity component of its algebraic monodromy group is solvable. An irreducible regular two-dimensional geometric representation with this property is induced from a character of a quadratic extension of K. Indeed, a nontrivial unipotent radical would have a unique invariant line preserved by the whole monodromy group, while scalar identity component would give finite projective image and, after finite extension, equal Hodge–Tate weights. The remaining torus case has two lines permuted by a quadratic quotient. The inducing quadratic field must be CM. The inducing character is algebraic. If its field had a real place, the absolute-value relation on units and Dirichlet's unit theorem would force all real algebraic exponents to agree. At the two real embeddings above any split real place of K, the induced weights would therefore coincide. This contradicts regularity and also excludes mixed signature. Its inducing character is de Rham and finitely ramified, hence is the 2-adic avatar of an algebraic Hecke character. Each dyadic place is nonsplit in the inducing extension: splitting would make the local induction a sum of characters, contrary to local absolute irreducibility. Quadratic automorphic induction gives a cuspidal automorphic representation with Galois representation r; its regular algebraic infinity type follows from the given regular Hodge–Tate weights. We henceforth assume that r is not virtually solvable. Its restriction to every finite extension is then absolutely irreducible.

The seed and the auxiliary fields. A cuspidal modular seed with scalar semisimple reduction is available over $\mathbb{Q}$: take the compatible system of Ramanujan's form Δ, twisted by ε_2^{-5}. Its determinant is ε_2, it is non-CM, and its residual semisimplification at 2 is $\mathbf{1} \oplus \mathbf{1}$. Indeed

Jacobi's identity gives $\Delta \equiv \sum_{j\geq 0} q^{(2j+1)^2} \pmod 2$, so the trace at every odd prime is zero and the residual determinant is one. Base change supplies a cuspidal seed over K and every abelian auxiliary field used below. Its determinant agrees exactly with the target; hence its quaternionic transfer has the required central character. It has tame level one, and the small uniform subgroups for completed forms are chosen at two.

Choose a finite set Σ of odd rational primes supporting the ramification of r and the seed, and enlarge it when necessary to include the prime factors of d. Theorem 7.3 supplies

$$K \subset A \subset F, \qquad D_A = [A : \mathbb{Q}], \quad D = [F : \mathbb{Q}], \qquad m = [F : A],$$

with $A/\mathbb{Q}$ and $F/\mathbb{Q}$ abelian, 2 completely split, and uniformly bounded numbers u_A, u_F of allowed odd places. The marked set M consists of the m places of F over one selected dyadic place of A. Both D_A and m may be made arbitrarily large, with $m/D = 1/D_A$. The same theorem gives injectivity of scalar residual extension classes under $A \subset F$, and gives

$$\{c \in H^1(G_{F,S_F}, k) : \mathrm{loc}_v\, c = 0 \text{ for all } v \in S_F \setminus M\} = 0. \tag{S1}$$

Here S_F consists of the real and dyadic places and the allowed odd places. The coefficient field k can be enlarged without changing the vanishing. The real local conditions in (S1) are zero conditions, so the assertion uses the totally positive squareclass condition (SV), not only an ordinary class-number condition.

For the strict Selmer group, Theorem 7.3(iv) gives

$$\dim_k H^1_{\mathrm{str}}(F, k) = 2D - 3m + u_F. \tag{S2}$$

Let c be a fixed generator bound for the augmentation ideal of the characteristic-two universal local pseudodeformation ring of $G_{\mathbf{Q}_2}$ with residual pseudorepresentation $\mathbf{1} \oplus \mathbf{1}$ and fixed determinant one. This bound is independent of A: every selected dyadic completion is $\mathbf{Q}_2$, and extending the residue field preserves a fixed bound. Choose

$$D_A > \max\{4, u_A + c + 3\}, \qquad m > u_F + 4. \tag{S3}$$

The uniform bound on u_A, u_F and independent growth of D_A, m make these choices possible. Since $D = mD_A$, they imply $D > u_F+4$ and $(2D-2m-4-u_F)-D = m(D_A-2)-(u_F+4) > 0$. They also give the dimension bound needed for entry from A below.

Connected transverse charts. Over the characteristic-two fixed-determinant pseudodeformation base for G_F, take the finite-type framed representation scheme and record an invariant projective line at each mark. In its residual fibre retain the open locus where no marked line is globally invariant. At such a point the residual representation at a mark is scalar: a nonzero unipotent extension has only one invariant line, whereas the global invariant line and the chosen transverse line are different. An upper extension with diagonal characters frozen must therefore be zero locally at every mark.

At a transverse point, the product of the real and marked dyadic local line problems has dimension $2D+5m$, by [51, Lemma 3.12]. The relative global presentation of [51, Proposition 3.6] has number of generators minus relations at least $-6m$. Every component of the completed chart consequently has dimension at least $2D-m$, and its components can be joined with intersections of dimension at least $2D - m - 1$, by complete-local connectedness [69, Lemma 52.14.1, Tag 0ECR].

The globally reducible locus has dimension at most $2D - 2m + u_F + 3$. Indeed, fixed-product diagonal characters contribute at most one parameter by Leopoldt for the abelian field F; moving the global invariant line and an upper coboundary contribute at most two; the strict extension class contributes at most $2D - 3m + u_F$ by (S2); and moving the m marked lines

contributes at most m. This is a relative generator bound after freezing the diagonal characters. Adding the global invariant line by proper incidence and then forgetting it bounds the entire reducible image. By (S3), this bound is strictly less than $2D - m - 1$.

Therefore a maximal-dimensional intersection in a component chain is generically globally irreducible. Forgetting the global frame and marked lines loses at most $3 + m$ dimensions. The remaining trace image has dimension at least $2D - 2m - 4$. Prescribing one Frobenius trace at each allowed odd place costs at most u_F equations. The consequence of (S3) displayed above leaves dimension greater than D. The dihedral-locus bound in [51, Lemma 3.8] and its curve-avoidance argument supply a non-virtually-solvable characteristic-two curve. The finite local-image conclusion is exactly [51, Lemma 3.9], applied after its fixed preliminary local extensions; the determinant on the characteristic-two curve is one. The prescribed residual Frobenius traces thus make all allowed odd local images finite. Theorem 5.1 of [51] then propagates potential pro-modularity across this intersection. Thus all components through a transverse residual point have the same potential pro-modularity status.

The transverse residual fibre is geometrically irreducible. Fixing the global line, its extensions form the nonzero vectors of the strict scalar cohomology space. This space contains the strict space with no odd ramification, of dimension $2D - 3m > 0$ by the same duality calculation. The complementary marked lines form $\mathbb{A}^m$. Conjugation gives a surjection from the product of these two spaces with GL_2, an irreducible parameter space. Since the representation scheme is noetherian, one irreducible component Y_0 contains this whole transverse residual fibre. Local chart propagation at the seed and target entry points, through Y_0, therefore connects their components. Completion is faithfully flat, so the completed-chart conclusion applies to components of the representation scheme.

Entering the transverse charts from the seed and target. We include this step because connecting deformation components for one fixed residual extension would not suffice. Work first over A. The seed gives a pro-modular component in the fixed-determinant pseudo-base of dimension at least $1 + 2D_A$, by [51, Proposition 4.4]. An unrestricted fixed-determinant framed component through $r|_{G_A}$ has trace image of dimension at least $2D_A - 2$, by the relative presentation over the real local rings and the loss of three frame parameters. Its generic representation is absolutely irreducible because it contains the given target.

In each component pass to the special fibre, prescribe the residual Frobenius traces at the allowed odd places, and prescribe the constant scalar local pseudorepresentation at the selected dyadic place of A. The last condition costs at most the c local generators fixed above. The total cost is at most $1 + u_A + c$ equations. Even for the smaller target component, the remaining dimension is at least $2D_A - 2 - (1 + u_A + c) = 2D_A - u_A - c - 3 > D_A$ by (S3). The bad-locus bound and curve avoidance therefore supply a non-virtually-solvable characteristic-two curve in each component. Lemma 3.9 of [51] makes its odd local images finite, as above.

At the selected dyadic place the curve representation has an invariant line after a finite coefficient extension. Choose a nonzero vector on that line and take its full G_A-orbit lattice over the curve's valuation ring. Compactness bounds it, and global irreducibility makes it full rank. The vector is primitive: otherwise all its translates lie in the maximal ideal times the orbit lattice, contradicting Nakayama's lemma. Its residual orbit spans the whole rank-two residual space. Consequently its residual line is not globally invariant. The residual representation is a nonzero additive extension. Injectivity of scalar extension classes under $A \subset F$ ensures that it remains nonzero over F. Transporting the local line to all marks gives transverse lines, because F/A is Galois. Thus both curves enter the transverse residual fibre over F.

The seed curve is potentially pro-modular by solvable base change. Propagation through the transverse charts, through the common component Y_0, makes the target curve potentially pro-modular. Applying [51, Theorem 5.1] back over A to the original target curve makes its

entire containing component potentially pro-modular. In particular, r becomes pro-modular over one finite solvable totally real extension of K, completely split at 2. Only finitely many component chains and propagation steps occur; their field extensions and tame levels can be replaced by one common Galois compositum and one common deeper tame level. This proves the scalar version of the pro-modularity input directly.

Classicality and descent. At every dyadic place of the resulting field the local representation is still the original absolutely irreducible regular de Rham representation over $\mathbb{Q}_2$, since the auxiliary extensions split completely above 2. Proposition 4.2 of [51], together with the all-prime local correspondence [12, Theorems 1.1 and 1.3], gives a nonzero locally algebraic vector in its Hecke eigenmodule, including when the residual local block is scalar. Proposition 4.1 identifies this vector with a regular algebraic definite-quaternionic eigensystem. Global irreducibility excludes reduced-norm characters, so Jacquet–Langlands produces a Hilbert cusp form. Local-global compatibility and Chebotarev identify its Galois representation with the restriction of r.

Finally descend through a tower of prime-degree cyclic extensions to K. At each step, Galois invariance of r and strong multiplicity one make the upstairs cuspidal representation invariant under the cyclic Galois group. Cyclic automorphic descent [4, Chapter III, Theorems 4.2(d) and 5.1] yields a cuspidal representation downstairs. Its Galois representation differs from the given one by a character of the cyclic quotient, because their restrictions are absolutely irreducible and isomorphic upstairs; twisting corrects that character. The archimedean types remain regular algebraic because every real place splits in a totally real extension. Iterating gives modularity of r over K. This is the locally irreducible part of the classicality and descent argument in [51, Section 8.8]; none of its ordinary dense-point arguments over $\mathbb{Q}$ is used here. □

9 Dyadic lifts and compatible systems

We construct two local types which suffice at every dyadic place and then globalize them without enlarging the prescribed ramification support. Steinberg lifts enter only in the local multiplicity calculation; the global lifts will all be potentially Barsotti–Tate and potentially nonordinary.

Let k be finite of characteristic two. For a continuous $\bar{s} : G_{\mathbf{Q}_2} \to \mathrm{GL}_2(k)$ put $\psi_{\bar{s}} = \varepsilon_2[\det \bar{s}]$, with brackets denoting Teichmüller lift. Choose $E/\mathbf{Q}_2$ with residue field k, enlarging both when necessary, and write $\mathcal{O}$ for its ring of integers and ϖ for a uniformizer. Let $\tau_0 = \widetilde{\omega}_2 \oplus \widetilde{\omega}_2^2$, where ω_2 has niveau two, and let τ_* be the inertial type of the elliptic curve

$$y^2 = x^3 - 2x,$$

with Cremona label `256b1` (LMFDB label `256.b1`) [48].

Proposition 9.1. *Every $\bar{s}$ has a potentially Barsotti–Tate lift of determinant $\psi_{\bar{s}}$ and labelled weights $\{0, 1\}$, with type τ_0 if it admits a crystalline lift of those weights, and τ_* otherwise. Both types force actual local irreducibility and potential nonordinarity. The inertia image of τ_* is Q_8.*

Proof. Every character $G_{\mathbf{Q}_2} \to k^\times$ is unramified: by local class field theory its inertial image is a quotient of the pro-two group $\mathbf{Z}_2^\times$, whereas $k^\times$ has odd order. If a crystalline weight-two lift exists, its determinant differs from $\psi_{\bar{s}}$ by an unramified character δ reducing to one. After enlarging coefficients, an unramified character ν with $\nu^2 = \delta^{-1}$ also reduces to one. Twisting by ν therefore fixes the determinant without changing the residue, weights or inertial type.

We first identify the residual representations for which no such crystalline lift exists. An irreducible $\bar{s}$ has trivial wild inertia, since the fixed space of this normal two-subgroup is nonzero and $G_{\mathbf{Q}_2}$-stable. After an unramified scalar twist its determinant is one; its two tame inertia

characters are exchanged by squaring and have order three. It is the unique such representation, realized by $A[2]$ for the good-reduction curve $A : y^2 + y = x^3$ used in the proof of Lemma 4.6. Thus it has a crystalline weight-two lift. A split residue $\bar{\alpha} \oplus \bar{\beta}$ lifts to $\varepsilon_2[\bar{\alpha}] \oplus [\bar{\beta}]$. For a nonsplit reducible residue, untwist its quotient and write its character ratio as $\bar{\eta}$. If $\bar{\eta} \neq 1$, local Euler characteristic and the Bloch–Kato formula [8, Corollary 3.8.4] give

$$\dim_E H^1(G_{\mathbf{Q}_2}, E(\varepsilon_2[\bar{\eta}])) = \dim_E H^1_f(G_{\mathbf{Q}_2}, E(\varepsilon_2[\bar{\eta}])) = 1.$$

Moreover $H^2(G_{\mathbf{Q}_2}, \mathcal{O}(\varepsilon_2[\bar{\eta}])) = 0$ by local Tate duality, since $[\bar{\eta}]^{-1} - 1$ takes a unit value. Reduction of H^1 is consequently surjective, and the actual residual extension lifts crystallinely. Hence failure of crystalline lifting is possible only for

$$\bar{s} = \bar{\alpha} \otimes \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \qquad 0 \neq c \in H^1(G_{\mathbf{Q}_2}, k),$$

with $\bar{\alpha}$ unramified.

For this last representation, Kummer theory identifies $H^1(G_{\mathbf{Q}_2}, \mathcal{O}(\varepsilon_2))$ with $\widehat{\mathbf{Q}_2^\times} \otimes_{\mathbf{Z}_2} \mathcal{O}$, where the hat denotes pro-two completion. Its reduction onto $H^1(G_{\mathbf{Q}_2}, k)$ is surjective. Choose a lift κ of c with nonzero valuation; adding a suitable $\varpi\mathcal{O}$-multiple of the Kummer class of two achieves this without changing its reduction. The extension of 1 by ε_2 defined by κ is semistable and noncrystalline: Kummer classes are semistable, and the crystalline subspace is the unit subspace. Twisting by $[\bar{\alpha}]$ gives determinant $\varepsilon_2[\bar{\alpha}]^2 = \psi_{\bar{s}}$ and preserves the actual residual extension.

We now apply the full fixed-determinant Breuil–Mézard formula [75, Conjectures 5.1.2–5.1.3 and Corollary 8.0.2]. The two Serre weights are $W_0 = 1$ and $W_1 = \mathrm{Sym}^1$. Write $\mu_0(\bar{s}), \mu_1(\bar{s})$ for their nonnegative multiplicities. For a compact type σ with semisimplified reduction containing W_i with multiplicity a_i, the Hilbert–Samuel multiplicity of the corresponding fixed-determinant weight-two ring modulo ϖ is $a_0\mu_0(\bar{s}) + a_1\mu_1(\bar{s})$. These deformation rings are taken with their reduced, $\mathcal{O}$-torsion-free structure; positivity is equivalent to a nonempty characteristic-zero fibre. The crystalline trivial type has reduction W_0, so $\mu_0 > 0$ exactly when the crystalline lift exists, with determinant fixed as above. The compact type of τ_0 is the inflation of the sign character of $\mathrm{GL}_2(\mathbf{F}_2) \simeq S_3$, whose reduction is also W_0. This proves the τ_0 assertion. Otherwise $\mu_0 = 0$, whereas the semistable lift just constructed gives $\mu_1 > 0$: the compact type for unrestricted monodromy and trivial inertia is the finite Steinberg representation, whose reduction is W_1. The determinant compatibility in both comparisons follows from $\det \tau_0 = 1$ and the unramifiedness of $[\det \bar{s}]$.

For τ_* we use one test representation. The two-torsion of the displayed curve has Kummer class two, which has odd valuation and is not finite flat [11, Definition A.3 and Lemma A.4]; in particular it has no crystalline weight-two lift. Its three-division polynomial is $3X^4 - 12X^2 - 4$. Choose $u^2 = -3$, $i^2 = -1$, $s = iu$. Its roots are $\pm\alpha, \pm\beta$, with $\alpha^2 = 2 + 4/s$, $\beta^2 = 2 - 4/s$, $\alpha\beta = 2u/3$. They recover s, u, so their splitting field is $L = \mathbf{Q}_2(u, i, \alpha)$. The field $U = \mathbf{Q}_2(u)$ is unramified quadratic, and $T = U(i)$ is ramified quadratic. For $\pi = 1 - i$, one has $\alpha^2/\pi^2 = i + 2/u$ and $v_\pi(i + 2/u - 1) = 1$. This unit is nonsquare even after unramified extension of T: a putative square root has residue one, while its square is one modulo π^2. Hence L/T is ramified quadratic, with $e(L/\mathbf{Q}_2) = 4$, $f(L/\mathbf{Q}_2) = 2$. Projective inertia on the four roots is $V_4 \subset A_4$. Its inverse image in $\mathrm{SL}_2(\mathbf{F}_3)$ is Q_8, with no proper subgroup surjecting onto V_4. The curve has potentially good reduction since $j = 1728$; its finite inertia on T_3 reduces injectively modulo three. Compatibility identifies its faithful irreducible Q_8 inertia with τ_*.

The Tate module of this curve gives a nonempty type-τ_* ring, while its residue has $\mu_0 = 0$. The multiplicity formula therefore forces $a_1(\tau_*) > 0$. For every $\bar{s}$ without a crystalline weight-two lift we already have $\mu_1(\bar{s}) > 0$, so the same formula gives a nonempty type-τ_* fibre. An

unramified scalar twist removes and restores $\bar{\alpha}$ without changing the compact type, giving the required determinant in general.

The two characters of τ_0 are exchanged by Frobenius, and τ_* is irreducible on inertia. Their Weil parameters are therefore irreducible, which forces Galois irreducibility. An ordinary line after finite Galois base change would be the unique line of labelled weight one and would descend, a contradiction. Thus neither type is potentially ordinary. □

Lemma 9.2. *Let F be totally real and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be finitely ramified, with nonsolvable residue, $\det r = \varepsilon_2\eta$ for a finite totally even character η, and potentially Barsotti–Tate and potentially nonordinary at every dyadic place. Then r is potentially modular.*

Proof. A finite character twist makes the residual determinant one. Apply [73, Proposition 6.7], avoiding its kernel field, to make the residue modular without shrinking its image. A further solvable totally real extension makes every dyadic source Barsotti–Tate. Nonsolvability and residual modularity persist. Kisin's theorem [46, Theorem 3.3.5] applies: its condition that a completion equal $\mathbf{Q}_2$ concerns only potentially ordinary places, and there are none. Restore the finite twist. □

Corollary 9.3 (Prescribed dyadic lifts in compatible systems)**.** *Let K be real quadratic with two split, and let $\bar{r} : G_K \to \mathrm{SL}_2(k)$ be continuous with nonsolvable image. Fix a finite set S containing the dyadic places and all residual ramification. At each odd place in S, choose a nonempty union of components of the generic fibre of the local lifting ring with determinant ε_2. Then $\bar{r}$ lifts to the two-adic member of a potentially modular weight-two strongly compatible system of cyclotomic determinant, unramified outside S, on the chosen odd local components, and with dyadic types prescribed by Proposition 9.1.*

Proof. Lemma 4.7 makes the odd local lifting rings nonempty and permits an unramified choice wherever the residue is unramified. At the dyadic places use Proposition 9.1. Choose a finite extension of each completion killing its type. In the corresponding potentially semistable generic fibre of weights $\{0, 1\}$, the type is locally constant, so the prescribed type selects a nonempty union of components. The associated Weil parameters are irreducible, so monodromy vanishes, and Proposition 9.1 gives potential nonordinarity.

The global determinant ε_2 lifts the residual determinant and is totally odd. Since $K(\zeta_2) = K$, the nonsolvable image gives the required absolute irreducibility. The prescribed-lift theorem [31, Theorem 4.2.1] therefore supplies a characteristic-zero point unramified outside S on these components. Lemma 9.2 makes this actual lift potentially modular. The compatible-system construction in the proof of Lemma 2.6 then gives the assertion, including the local conditions and ramification support. □

10 A Serre criterion when two splits

Theorem 10.1 (The split-dyadic criterion)**.** *Let K be real quadratic with two split. Use the lifting inputs of Sections 2 and 5 and the solvable dyadic source theorem cited there. Suppose:*

- *(i) every absolutely irreducible two-adic representation of G_K unramified outside two, with cyclotomic determinant, scalar residual semisimplification $1 \oplus 1$, and both dyadic restrictions absolutely irreducible and regular de Rham, is modular;*
- *(ii) the ordinary ideal class group $\mathrm{Cl}(K)$ has two-power order;*
- *(iii) every continuous absolutely irreducible representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ of determinant one, unramified outside two, is modular.*

Then every continuous absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, for every odd prime p and arbitrary conductor, is modular.

For (i), Theorem 1.4 applies whenever $V^+_{w_0}(K) = 0$ under its stated propagation inputs. Alternatively, when d is prime and condition (S) holds, [21, Theorem 9.4(b)] gives (i) with both dyadic places in Σ_n. Condition (iii) is a residual base case, separate from the scalar source theorem. The proof below has no analytic assumption beyond whatever is needed to establish that condition. Strong compatibility includes the coefficient prime, so a crystalline replacement clears that conductor from every companion.

Proof. **10.1 Initial lifts and the protective place**

Use the global construction and compatible systems of Lemma 2.6, the modular witnesses of Lemma 2.1, and the backward lifting statements in Lemma 2.2. For arbitrary regular dyadic sources at the split completions $\mathbf{Q}_2$, we use [75, Theorem 8.0.3]. Apply Proposition 2.7. In the nonsolvable case it supplies the initial weight-two compatible system. Two splits, so $K \neq \mathbf{Q}(\sqrt{5})$ and the characteristic-five exception is absent. The subsequent protective images exclude the exceptional small images.

The protective place in Proposition 3.1 can be chosen with the additional property that the order of two modulo its rational norm is odd. We give the simultaneous field conditions, since this property is needed when its own inertia ceases to protect the last congruence.

Write $K = \mathbf{Q}(\sqrt{d})$ with $d > 1$ squarefree. A reversible finite twist removes the odd-order part of its finite determinant, leaving $\det r_t = \varepsilon_t \psi_2$, with ψ_2 of two-power order and with no new rational conductor prime. Choose a fresh odd reset prime $s > 5$, above the needed prime bounds and split in K, by Chebotarev. Include it, two, three, five, every conductor prime and every prime dividing d in a finite set S before choosing the protective place.

Choose N and the projective residual field as in the proof of Proposition 3.1, with N large enough for the fixed weight bounds. Thus N splits in K and the coefficient field, the projective image is $\mathrm{PGL}_2(\mathbf{F}_N)$, and ψ_2 is square-valued modulo N.

Let K_S/K be the maximal elementary quadratic extension unramified outside S and infinity. Choose a degree-one place λ of rational norm ℓ with the standard good-dihedral Frobenius prescription and

$$\ell \equiv -1 \pmod{N}, \qquad \lambda \text{ split in } K_S, \qquad \ell \equiv 9 \pmod{16}, \qquad 2 \in (\mathbf{F}_\ell^\times)^8. \tag{7}$$

The last two conditions make $\mathrm{ord}_\ell(2)$ odd. We verify compatibility of the simultaneous conditions. The Galois group of $B = \mathbf{Q}(\zeta_8, 2^{1/8})$ has order sixteen. Writing its elements as (a, b) for the actions on $2^{1/8}$ and ζ_8, multiplication is $(a, b)(c, e) = (a + bc, be)$. The commutator of $(1, 3), (0, 7)$ is $(2, 1)$, generating $\mathrm{Gal}(B/\mathbf{Q}(\zeta_8))$. Thus its maximal abelian subfield is $\mathbf{Q}(\zeta_8)$. Since B is ramified only at two and K is real quadratic with two split, $K \cap B = \mathbf{Q}$. The same statement holds with $\mathbf{Q}(\zeta_{16})$ in place of B. The compositum with the elementary extension K_S/K has abelianization of exponent two: the same commutators survive with trivial K_S coordinate. Hence $K(\zeta_{16}) \not\subset BK_S$ and its intersection with BK_S is $K(\zeta_8)$. Splitting in BK_S is compatible with Frobenius nine on ζ_{16}. This two-extension is unramified at N; the only nontrivial solvable quotient of the projective residual field is its quadratic cyclotomic quotient, ramified at N. Its intersection with the compositum of the preceding two-extension and $K(\zeta_N)$ is exactly that quotient. Projective complex conjugation and inversion on ζ_N agree on it, since $N \equiv 1 \pmod 4$. Chebotarev therefore gives (7), with ℓ split in K and arbitrarily large.

Install the order-N induced type at λ and apply the forward construction and split-N backward lifting of Proposition 3.1. Its image argument makes every residue in S nonsolvable, with adequate cyclotomic restriction in odd characteristic. The additional conditions (7) do not alter that argument.

10.2 Dyadic preparation and odd conductor

Reduce the protected system modulo two. Its determinant is one, since ψ_2 has two-power order. At each odd ramified place use Lemma 4.7 to choose a component of determinant ε_2; its type need not equal the old type. At λ retain the order-N induction: take inertia $\mathrm{diag}(\zeta_N, \zeta_N^{-1})$ and antidiagonal Frobenius with off-diagonal product $-\ell$, giving determinant ℓ and the same residual local representation. Corollary 9.3 gives a potentially modular weight-two system of exact cyclotomic determinant, with dyadic types τ_0 or τ_* and no new conductor prime. For the backward arrow apply Tung over K itself, where the dyadic completions are $\mathbf{Q}_2$: the old regular source may have arbitrary dyadic conductor. Once the replacement is modular, the common nonsolvable residue is modular and the theorem applies.

Retain the dyadic types τ_0, τ_* in every subsequent odd-characteristic congruence. Remove all odd rational conductor primes except s, ℓ. Before removing a nonsplit prime t, change coefficient characteristic to s and choose a weight-two replacement, retaining the dyadic and protective types and every previously cleared place away from s. Weight-two local existence is [68, Proposition 7.8.1], followed by prescribed lifting. At s, the finite determinant correction reduces to one and has s-power order; its inverse square root fixes the determinant without changing the residue or weights [68, Section 7.9]. Unramified local components preserve cleared places, and the old components preserve retained types. Since s splits, Tung gives the backward implication for the actual old regular source.

In characteristic t, replace the source by regular crystalline potentially diagonalizable lifts at all places above t. This clears its conductor in the new compatible system. For nonsplit t, the old source has weight two by the reset, and [3, Theorem 2.3], with the witness from Lemma 2.1, gives the backward implication. For split t, use Tung without a weight restriction on the old source. The forward construction applies in characteristic three regardless of its splitting in K. All conditions away from t retain the dyadic and protective types. Finally clear s by a regular crystalline replacement in characteristic s and split-prime lifting. The weights are allowed to grow; no crystalline weight-two replacement has been asserted at a removal prime.

10.3 The last prime and the terminal source

Remove the protective place last, in characteristic ℓ. Its own order-N type can no longer protect that residue, so use the dyadic types. If one is τ_*, its Q_8 inertia stays absolutely irreducible modulo odd ℓ: the center acts as -1, while every one-dimensional character kills it. The local cyclotomic extension at two is unramified and preserves this inertia. If both are τ_0, the two order-three characters remain distinct modulo $\ell > 3$ and Frobenius interchanges them. By (7), the cyclotomic residue degree is odd, so that interchange survives restriction to $G_{K(\zeta_\ell)}$. The global residual cyclotomic restriction is therefore absolutely irreducible in both cases.

If the residue is solvable, its Artin lift and regular weight raising give a modular witness; Tung at the split prime ℓ already proves modularity of the old source, completing the backward chain. Otherwise ℓ is large enough for adequacy. Choose regular crystalline potentially diagonalizable lifts above ℓ, preserving all dyadic types and cleared places. Prescribed lifting and potential modularity give the system with λ removed. Tung gives the backward implication.

The terminal system has cyclotomic determinant, conductor supported above two, and both dyadic supercuspidal types. Its actual 2-adic member r_2 is locally absolutely irreducible, de Rham and regular, with no weight-two restriction. If its residue is irreducible and nonsolvable, hypothesis (iii) and Tung give modularity. If the residue is irreducible and solvable, apply [21, Theorem 10.6]: $K/\mathbf{Q}$ is abelian, two splits, all places belong to Σ_n, and $\Sigma_o = \varnothing$, so its nonsplitness requirement is vacuous. If the residual semisimplification is reducible, both characters have odd order. At the dyadic completions $\mathbf{Q}_2$ they kill the pro-two unit group, and

they also kill the real sign groups. They factor through the ordinary class group and are trivial by (ii). The actual member is therefore modular by (i).

In every case the actual terminal source is modular. Compatibility identifies its entire system with the automorphic system, and all backward implications return to the initial lift. Reducing proves Theorem 10.1. □

11 Integral orders and finite-flat groups

For the quadratic applications write $F_d = \mathbf{Q}(\sqrt{d})$. We first retain the full coefficient information in an integral order, then establish the finite-flat bounds used on its lattices.

11.1 Trace orders

We use simultaneous realizations at every dyadic coefficient place. This is essential when a local calculation bounds a global trace-field degree. Let $\mathcal{R}_1, \dots, \mathcal{R}_m$ be distinct coefficient orbits of absolutely irreducible strongly compatible rank-two systems, with trace fields D_i and cyclotomic determinants. Write $d_i = [D_i : \mathbf{Q}]$. Let R_0 be the closed $\mathbf{Z}_2$-algebra generated by their simultaneous traces and let E_0 be the actual order generated by their simultaneous group images and the trace scalars, in integral matrix realizations at all the coefficient places above two.

Lemma 11.1. *The rings R_0, E_0 are finite free $\mathbf{Z}_2$-modules, and*

$$R_0[1/2] = \prod_i (D_i \otimes_{\mathbf{Q}} \mathbf{Q}_2), \qquad \operatorname{rank}_{\mathbf{Z}_2} R_0 = \sum_i d_i, \qquad \operatorname{rank}_{\mathbf{Z}_2} E_0 = 4 \sum_i d_i.$$

The order E_0 is the closed $\mathbf{Z}_2$-span of the group image. Its trace and determinant laws take values in R_0. For a subcollection of dyadic coefficient orbits, the same assertions hold with the selected local trace-field factors and the sum of their degrees over $\mathbf{Q}_2$. If every residual semisimplification is scalar, $E_0/2E_0$ is a local algebra with residue field $\mathbf{F}_2$.

Proof. All the chosen matrices preserve integral lattices, so their span is contained in a finite free $\mathbf{Z}_2$-module. Cayley–Hamilton gives $T(g)1 = g + \det(g) g^{-1}$. Since $\det(g) \in \mathbf{Z}_2^{\times}$, the closed group span already contains every trace scalar and is an order. The determinant of a sum lies in R_0 by the polarization identity $\det(x+y) = \det(x) + \det(y) + T(x)T(y) - T(xy)$, first for linear combinations of group elements and then by continuity.

Good Frobenius traces generate each D_i. A rational subalgebra of $\prod_i D_i$ surjecting onto every factor identifies two simple factors only if an embedding matches their good traces. Brauer–Nesbitt would then identify their coefficient orbits. The rational simultaneous trace algebra is therefore the displayed product. The same argument with the selected local factors uses good traces and coefficient conjugacy over $\mathbf{Q}_2$. On each simple trace factor, absolute irreducibility makes the image algebra central simple of dimension four; it becomes M_2 after a coefficient extension. This proves both ranks.

In characteristic two a scalar residual semisimplification with cyclotomic determinant is trivial. Each actual residual image is unipotent, and the principal congruence kernels are pro-two. The simultaneous image is therefore pro-two. Its finite image in $E_0/2E_0$ spans that algebra, which is consequently a quotient of a finite two-group algebra over $\mathbf{F}_2$. Such a group algebra is local with residue field $\mathbf{F}_2$. □

The next finite-flat assertion is restricted to a category in which the order-two model is unique. Let $F/\mathbf{Q}_2$ have absolute ramification index two, and allow a finite unramified extension

of F. We call a finite flat group biconnected if its special fibre and the special fibre of its Cartier dual are connected.

Lemma 11.2. *Among biconnected finite flat groups killed by two whose generic Jordan–Hölder factors over $\mathbf{F}_2$ are trivial, every morphism of generic fibres extends uniquely. Generic subobjects and quotients have biconnected finite-flat models, and their exact sequences extend to those models. The simple model has Breuil–Kisin Frobenius u. Every self-extension in this category has generic quadratic characters of conductor at most two.*

Proof. Schematic closure and finite-flat quotient preserve biconnectedness. Indeed a closed subgroup of a connected finite group over the perfect residue field has local coordinate algebra and is connected; connectedness of quotients follows from surjectivity. Cartier duality gives the other two assertions.

For a group of order two, the height-one exponent is j with $0 \leq j \leq 2$. Biconnectedness excludes $j = 0, 2$, leaving $j = 1$. The unit factor is removed by change of generator when the generic character is trivial. This is the unique middle model, including its generic identification; see the connected classification in [45, Theorem 4.1, Proposition 4.2 and Corollaries 4.3–4.4].

A generic isomorphism between two objects extends as an isomorphism by induction on their common order. Close its graph in the product. For each projection, choose a generic subgroup of order two and close it in source and target. The restriction of the projection is an isomorphism by uniqueness of the middle model. Its induced map on the finite-flat quotients is an isomorphism by induction, hence the projection is an isomorphism. For an arbitrary generic morphism, apply this to the first projection of its graph closure. The second projection then extends the morphism. Flatness gives uniqueness. Taking schematic closures and finite-flat quotients gives the asserted exact sequences.

Let k_0 be the residue field. Kim's exact classification [45, Corollary 4.3 and Proposition 5.7.3] represents a self-extension of the simple model by

$$\Phi = \begin{pmatrix} u & h(u) \\ 0 & u \end{pmatrix}, \qquad h \in k_0[[u]].$$

Changing the upper lift replaces h by $h + u(\varphi(f) - f)$. The map $\varphi - 1$ is bijective on $uk_0[[u]]$, so we may write $h = h_0 + h_1 u$. Rescaling the endpoint bases by u^{-1} gives the étale φ-module class $h_0/u + h_1$, whose positive upper break is at most one over the field of norms.

To compare ramification, choose a uniformizer π of F and put $F_n = F(\pi^{1/2^n})$. The transition F_n/F_{n-1} has different exponent $2e(F_{n-1}) + 1$ and upper break $4 \cdot 2^{n-1}$. Fontaine's bound gives upper break at most three for the generic quadratic character of our finite flat group [34, Theorem A]; his numbering is shifted by one from classical upper numbering. Herbrand's theorem therefore preserves this character's break at each transition. Wintenberger's comparison [79, Proposition 3.3.2 and Corollary 3.3.6] identifies it with the field-of-norms break. Its conductor is consequently at most two. The argument is unchanged after finite unramified extension. □

Lemma 11.3. *Let $F/\mathbf{Q}_2$ have ramification index two and residue field $\mathbf{F}_2$. A unit $a \in \mathcal{O}_F^\times$ defines a quadratic character of conductor exponent at most two if and only if a is a square modulo $2\mathcal{O}_F$.*

Proof. Let π be a uniformizer. If $a \equiv x^2 \pmod{\pi^2}$, then $(x + \sqrt{a})/\pi$ is integral and the order it generates has discriminant valuation two. The quadratic field therefore has discriminant exponent at most two. Conversely, $\mathcal{O}_F[\sqrt{a}]$ has discriminant valuation four. If the conductor is at most two, the index of this order in the integral closure is divisible by π. A nonzero class in the π-torsion of the index quotient gives an integral element $(u + v\sqrt{a})/\pi$ with v a unit.

Its norm shows that $a \equiv (u/v)^2 \pmod{\pi^2}$. The assertion also holds for the trivial quadratic character. □

Corollary 11.4. *For $K = \mathbf{Q}(s)$ in the following table, let V be the group of squareclasses supported at the dyadic prime and the two primes above ℓ, whose quadratic characters have dyadic conductor at most two. Then the displayed three classes form a basis of V.*

s^2	ℓ	ϵ	a	b	*basis of V*
2	7	$1+s$	$3+s$	$3-s$	$[-1], [\epsilon a], [-7]$
3	13	$2+s$	$4+s$	$4-s$	$[-1], [\epsilon a], [\epsilon b]$
6	5	$5+2s$	$1+s$	$1-s$	$[-1], [\epsilon], [-15]$
11	5	$10+3s$	$4+s$	$4-s$	$[-1], [\epsilon a], [\epsilon b]$

Proof. These four fields have class number one. A class with odd dyadic valuation has quadratic conductor five. Every remaining supported class has a unique expression $[-1]^i[\epsilon]^j[a]^k[b]^l$. The three classes in each row are independent by the two odd-prime valuations and the independent unit classes. The local lemma imposes one binary linear equation. In the first, second and fourth rows, the images of ϵ, a, b in $(\mathcal{O}_K/2)^\times$ are the same nonidentity element, giving $j+k+l=0$. In the third row ϵ has image one and a, b have the same nonidentity image, giving $k+l=0$. These equations give exactly the displayed bases. For the third row use $3/\epsilon = (3-s)^2$ to identify $[\epsilon ab] = [-15]$. □

Lemma 11.5. *Let $F/\mathbf{Q}_2$ be finite, and let V be a direct sum of crystalline rank-two representations, with finite coefficient fields, labelled weights $\{0,1\}$ and determinant χ_2. Every stable $\mathbf{Z}_2$-lattice in V is Barsotti–Tate. If all the constituents are nonordinary, its Barsotti–Tate group and its finite-flat subquotients are biconnected. This applies, in particular, to the left regular lattice of E_0.*

Proof. The crystalline lattice classification gives a Barsotti–Tate group for each constituent. Every stable lattice in their sum is realized by an isogenous group, by schematic closure of the finite generic isogeny kernel. Connectedness and dual connectedness are invariant under isogeny.

The connected–étale decomposition is stable under coefficient actions. A nonzero rational étale factor in a rank-two constituent has coefficient rank one and gives an unramified quotient; the determinant and weights then give the ordinary filtration. Nonordinarity excludes this. Applying the same argument to $r^\vee(1) \simeq r$ excludes an étale factor of the dual. Subgroups and quotients preserve biconnectedness by the first paragraph of the proof of Lemma 11.2. After extending coefficients, the rational regular representation of E_0 is a sum of columns of the original constituents, so the same argument applies to it. □

Here the trace and determinant are those of the original integral order. A subsequent nonfaithful matrix quotient need not preserve the information they carry. In particular a vanishing matrix in such a quotient does not imply vanishing of its original trace.

Lemma 11.6. *Suppose every dyadic coefficient constituent is crystalline of labelled weights $\{0,1\}$ and determinant χ_2, is nonordinary, and has scalar residual semisimplification. Suppose the dyadic completion $F/\mathbf{Q}_2$ is ramified quadratic. Let $E_\pi(u)$ be the Eisenstein polynomial of a uniformizer π. The regular lattice E_0 is Barsotti–Tate. Its Breuil–Kisin module is free of rank one as a right $E_0[[u]]$-module. If its Frobenius generator is $\Phi(u)$, then*

$$\mathcal{D}(\Phi(u)) \in (E_\pi(u))R_0[[u]], \qquad T(\Phi(0)) = a_2 \in R_0.$$

The point field of $E_0/2E_0$ has upper breaks at most three over F.

Proof. The regular lattice is Barsotti–Tate and biconnected by Lemma 11.5. Lemma 11.2 gives exactness and full faithfulness for its torsion subquotients with trivial simple generic factors. We use the covariant Breuil–Kisin functor $\mathfrak{M}_{\mathrm{cov}}(G) = \mathfrak{M}_{\mathrm{contra}}(G^D)$. Cartier duality followed by Kim's anti-equivalence is covariant and exact [45, Corollary 4.3 and Proposition 5.7.3], so it preserves the right action and the quotient used below.

The commuting right action of E_0 extends by Tate full faithfulness. Write $J_0 = \mathrm{rad}(E_0)$ and let $\mathfrak{M}$ be the Breuil–Kisin module of the regular group. In the scheme killed by two take the sum of the right multiplication maps for a set of generators of $J_0/2E_0$. Its finite-flat quotient has generic fibre $E_0/J_0 = \mathbf{F}_2$ and is the unique middle model. Exactness of the covariant functor gives $\mathfrak{M}/\mathfrak{M}J_0 = \mathbf{F}_2[[u]]$ with Frobenius u. A generator lifts by Nakayama to a surjection $E_0[[u]] \to \mathfrak{M}$ of right modules. These have the same rank over $\mathbf{Z}_2[[u]]$, so this is an isomorphism.

Height one supplies Ψ with $\Phi\Psi = \Psi\Phi = E_\pi(u)$. In each integral matrix realization these two identities give a rank-two height-one matrix. Its rational étale Frobenius module is the corresponding column constituent of the regular representation, by its commuting right matrix action after extending scalars. The connected finite-flat comparison thus identifies its rank-two generic realization with the given crystalline member. Kisin's determinant comparison [46, Lemmas 2.3.2 and 2.3.4] gives divisibility of its determinant by $E_\pi(u)$. Division by this distinguished monic polynomial commutes with each actual coefficient embedding, hence its remainder already vanishes in $R_0[[u]]$. At $u = 0$ the covariant comparison gives crystalline Frobenius on $V(-1) = V^\vee$ for the rank-two member V. Its determinant is 2 and its trace is the arithmetic coefficient a_2, since $V^\vee \simeq V(-1)$ has the arithmetic Frobenius normalization. This fixes the convention in both identities. Finally Fontaine's finite-flat ramification bound for the group killed by two gives classical upper break at most three over F. □

Lemma 11.7. *Let $F/\mathbf{Q}_2$ be ramified quadratic and let M/F be the point field of a finite flat group scheme killed by two. Every positive upper break is at most three, and the normalized relative different contribution is strictly less than two, with $v_2(2) = 1$. In particular this applies to the reduction of any stable coefficient lattice of a crystalline rank-two source of weights $\{0, 1\}$.*

Proof. Fontaine's bound [34, Theorem A], converted to classical upper numbering, is $2e(F) - 1 = 3$. Integration of the upper filtration bounds the relative different exponent divided by the relative ramification index strictly below four. Dividing by $e(F/\mathbf{Q}_2) = 2$ gives the normalized bound. For the last assertion, realize the stable crystalline lattice by a Barsotti–Tate group. Its coefficient-uniformizer torsion is a finite-flat subgroup of rational two-torsion. □

11.2 Rank-one selfextensions at the endpoint exponents

Lemma 11.8. *Let $F/\mathbf{Q}_p$ have absolute ramification index e and residue degree f, and let k be a finite coefficient field containing its residue field. Consider a rank-one Breuil–Kisin module with coefficient factors indexed by $i \in \mathbf{Z}/f\mathbf{Z}$, whose Frobenius maps are $b_i u^{r_i}$, where $b_i \in k^\times$ and $r_i \in \{0, e\}$. On these factors Frobenius is k-linear and sends u to u^p. Every height-one selfextension, exact on the underlying free $k[[u]]$-modules, has Frobenius matrices*

$$A_i = b_i u^{r_i} \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}, \qquad a_i \in k,$$

after changing the lifts of the quotient generators. The extension classes form a one-dimensional k-space. Whenever the exact finite-flat comparison identifies the given models with these modules, their generic selfextensions become unramified after dividing by the common character.

Proof. Before changing lifts, write $A_i = \left(\begin{smallmatrix} b_i u^{r_i} & h_i \\ 0 & b_i u^{r_i} \end{smallmatrix}\right)$. The height-one condition makes $u^e A_i^{-1}$ integral. Its upper-right entry gives $u^e \mid h_i$ when $r_i = e$; when $r_i = 0$ no divisibility is needed.

Thus in either case $h_i = b_i u^{r_i} f_i(u)$ with $f_i \in k[[u]]$. Changing the quotient lift by μ_i changes f_i by $\varphi(\mu_{i-1}) - \mu_i$.

On $uk[[u]]$ this cyclic coboundary map is bijective: after one cycle its solution is obtained by inverting $1 - \varphi^f$, using the convergent series $\sum_{n \geq 0} \varphi^{fn}$. Hence all positive powers of u can be removed. On constants the coboundaries are $(\mu_{i-1} - \mu_i)_i$, whose cokernel is k, measured by $\sum_i a_i$. This proves the dimension assertion. Unramified selfextensions of the common character form a one-dimensional space and have finite-flat models: tensor the rank-one model with an unramified two-dimensional unipotent representation and descend from a finite unramified extension. The exact comparison identifies this nonzero line with the whole extension space, proving the generic assertion. □

The restriction $r_i \in \{0, e\}$ is used in the divisibility of h_i. The middle-exponent calculations below are separate.

11.3 Ramification of dyadic torsion

Lemma 11.9. *Let $F/\mathbf{Q}_2$ be quadratic, let $q = 2^f \geq 4$, and let $\bar{r} : G_F \to \mathrm{SL}_2(\mathbf{F}_q)$ be finite flat. If E/F is its kernel field, its normalized relative dyadic discriminant contribution, with $v_2(2) = 1$, satisfies*

$$d_2(E/F) := \frac{v_2(\mathfrak{d}_{E/F})}{[E : F]} \leq 2 - \frac{2}{q}.$$

If $F/\mathbf{Q}_2$ is unramified and both wild inertia and the tame diagonal characters are nontrivial, then $d_2(E/F) < 4/3$.

Proof. We use Fontaine's ramification theorem [34, Theorem A and its corollary, pp. 515–516]. His numbering is $G^{(u)} = G^{u-1}$ in Serre's upper numbering. Thus the upper breaks are at most $2e(F/\mathbf{Q}_2) - 1$.

Suppose first that $F/\mathbf{Q}_2$ is ramified. If wild inertia is nontrivial, its unique fixed line is defined over $\mathbf{F}_q$ and is preserved by the decomposition group. The diagonal characters are unramified, since their odd-order restrictions to units factor through $\mathbf{F}_2^\times$. Inertia is therefore a translation group of order $P \leq q$. Integrating the upper-break bound three and dividing by the absolute ramification index two gives $d_2 \leq 2(1 - 1/P) \leq 2 - 2/q$. If wild inertia is trivial, a nontrivial locally irreducible tame inertia has order three, since Frobenius squares and exchanges the determinant-one eigencharacters. Its contribution is $1/3$; in the reducible tame case inertia is trivial.

Now suppose that $F/\mathbf{Q}_2$ is unramified. If wild inertia is trivial, the reducible tame characters have order dividing three. In the irreducible case they are θ, θ^4, and determinant one gives $\theta^5 = 1$. The respective contributions are at most $2/3$ and $4/5$. If wild inertia is nontrivial, write the diagonal characters as η, η^{-1}. Killing them gives a field T over which the image is an elementary quadratic translation group of order $P \leq q$. If η is unramified, Fontaine's upper-break bound one makes every quadratic character of E/T have conductor at most two. The conductor–discriminant formula gives $d_2 \leq 2(1 - 1/P) \leq 2 - 2/q$.

Otherwise the tame diagonal order is three and $e(T/F) = 3$. Tame conjugation acts nontrivially on the translation module and its Kummer dual. The nontrivial tame grades of the quadratic unit filtration over T are one and five, giving conductors six and two; the grade-three and unramified directions are tame-invariant and cannot occur [16, §1]. A conductor-six character would have upper break $5/3$ over F, contrary to Fontaine's bound. Hence $d_2(E/F) \leq \frac{2}{3} + \frac{2}{3}(1 - 1/P) < \frac{4}{3}$. Unramified base changes do not affect any of these normalized bounds. □

We next prove a stronger local statement for a nonordinary source.

Theorem 11.10. *Let $F/\mathbf{Q}_2$ be ramified quadratic, and let $r_F : G_F \to \mathrm{GL}_2(E)$ be nonordinary, crystalline of labelled weights $\{0,1\}$ and determinant χ_2. For any stable lattice, the point field of its reduction has positive upper breaks at most one and normalized relative different contribution less than one, with $v_2(2) = 1$. If the reduction is locally absolutely irreducible, its full image is S_3 and its wild inertia is trivial.*

Proof. As in Lemma 11.5, the lattice is realized by a biconnected Barsotti–Tate group G. The same holds after isogeny, and $G[\varpi_E]$ and all its finite-flat subquotients are biconnected.

After enlarging the residue coefficient field, suppose first that the reduction is reducible. Its diagonal characters are unramified, since $\mathcal{O}_F^\times$ is pro-two. After an unramified extension it is $1 + c(g)E_{12}$, where c has values in a finite field k of characteristic two. For any $\mathbf{F}_2$-linear functional $\lambda : k \to \mathbf{F}_2$, pull the quotient line back to $\mathbf{F}_2$ and quotient the invariant line by $\ker\lambda$. This gives a biconnected self-extension of the simple model with generic character $\lambda \circ c$. Lemma 11.2 bounds its conductor by two. These characters separate the elementary abelian point-field group, so every positive upper break is at most one. If its ramification index is e, its different exponent is at most $2(e-1)$; division by $2e$ gives the asserted normalized bound. Unramified base change preserves these ramification bounds.

If the reduction is locally absolutely irreducible, the local classification in Lemma 4.6(ii) gives full image S_3 and trivial wild inertia. Tame index three contributes $1/3$ to the normalized relative different. □

Lemma 11.11. *Let $F/\mathbf{Q}_2$ be ramified quadratic. Consider an upper triangular representation of G_F over a finite local commutative $\mathbf{F}_2$-algebra, with unramified diagonal characters and point field of upper break at most three. Let t be the order of the Frobenius multiplier on the upper-right coordinate and assume that this order is odd. Write w for the order of inertia. If $t = 1$, the normalized relative different is at most $5/4$. For $t > 1$ put $s = 2^{\mathrm{ord}_t(2)}$. Then w is $1, s$, or s^2, with respective normalized different bounds*

$$0, \qquad 2 - 2/s, \qquad 2 - 1/s - 1/s^2.$$

In particular, for the residue of an ordinary crystalline weight-two representation of cyclotomic determinant, these bounds apply with multiplier $\bar\nu^{\pm 2}$, where $\bar\nu$ is the residual unramified quotient value at Frobenius.

Proof. After unramified extension of degree t, divide by the common diagonal character. The resulting translations define an elementary abelian extension with the same inertia and normalized different. The upper-break hypothesis excludes odd-valuation Kummer classes, which have conductor five. At absolute ramification index two, the ramified unit classes have conductors two and four. Indeed the square map from the first principal-unit grade fills the second, and the quadratic defect formula gives conductors four and two on the surviving first and third grades, respectively. Each is a subquotient of the additive residue field. Unramified classes do not contribute to the different.

For $t > 1$, conjugation by the original Frobenius acts on translations by multiplication by the multiplier. An odd-order unit in a finite local algebra has the same order as its residue and belongs to the canonical coefficient field. Thus the Kummer dual is isotypic for a nontrivial binary simple module of dimension $\mathrm{ord}_t(2)$. Frobenius acts on the two ramified grades as on the residue field of the unramified degree-t extension. This is the regular binary representation of the cyclic group of order t, so the given simple module occurs at most once in each grade. This proves the assertions on w. For $w = s^2$ there are at most $s^2 - s$ characters of conductor four, and the remaining $s - 1$ nontrivial characters have conductor two. The conductor–discriminant

formula, with normalization by $2w$, gives $2 - 1/s - 1/s^2$. For $w = s$ use the conductor-four bound on all $s - 1$ nontrivial characters.

When $t = 1$, each ramified grade has dimension at most one over $\mathbf{F}_2$. In the largest case the nontrivial conductors are $4, 4, 2$, giving $5/4$ by the same formula. Smaller groups satisfy this bound as well. The final assertion follows from the ordinary filtration and Lemma 11.7. □

Lemma 11.12. *Let $F/\mathbf{Q}_2$ be unramified with residue field $k = \mathbf{F}_q$. For $a \in \mathcal{O}_F^\times$ put*

$$\delta_F(a) = \frac{a^{q-1} - 1}{2} \pmod 2 \in k.$$

This induces a surjective $\mathbf{F}_2$-linear map $\mathcal{O}_F^\times/\mathcal{O}_F^{\times 2} \to k$. Its kernel is the one-dimensional unramified quadratic squareclass space. If $\delta_F(a) = 0$, set $b = a^{q/2}$ and $c = (a/b^2 - 1)/4$. Then a is a square in F if and only if $\mathrm{Tr}_{k/\mathbf{F}_2}(\bar{c}) = 0$.

Proof. Every unit satisfies $a^{q-1} \equiv 1 \pmod 2$. Multiplication modulo four gives $\delta_F(aa') = \delta_F(a) + \delta_F(a')$, and a square has zero image. For $a = 1 + 2x$ the image is $\bar{x}$, proving surjectivity. If $\delta_F(a) = 0$, then $b^2 = a^q \equiv a \pmod 4$, so $a/b^2 = 1 + 4c$. The polynomial $X^2 - X - c$ has unit derivative and discriminant $1 + 4c$. Its reduction is split precisely when $\mathrm{Tr}_{k/\mathbf{F}_2}(\bar{c}) = 0$, and otherwise gives the unramified quadratic extension. Conversely every unramified quadratic extension is obtained in this way. The unramified quadratic extension is unique, so the kernel consists of its squareclass and the trivial class. □

Lemma 11.13 (The cubic norm kernel)**.** *Let A/K be an S_3 extension, let L be the fixed field of a transposition, and let V be the standard two-dimensional $\mathbf{F}_2$-representation of $\mathrm{Gal}(A/K)$. Let S contain the dyadic places, the infinite places, and all places ramified in A/K. Write $L(S_L, 2)$ for the squareclasses with even valuation outside S_L. There is a natural isomorphism, compatible with localization,*

$$H^1(G_{K,S}, V) = \ker\big(L(S_L, 2) \xrightarrow{N_{L/K}} K(S, 2)\big).$$

The norm is surjective. Its kernel is the image of the projector $[u] \mapsto [u/N_{L/K}(u)]$, where the norm is viewed in L. If $\mathrm{Cl}(\mathcal{O}_{L,S_L})[2] = 0$, the entire calculation uses S_L-units modulo squares. The same statements hold after scalar extension to any field of characteristic two.

Proof. The permutation module on the three cosets is $\mathbf{F}_2^3 = \mathbf{F}_2 \oplus V$. The first summand is the constant line and V is the kernel of the sum of coordinates. The sum restricted to the constant line is the identity because $3 = 1$ in $\mathbf{F}_2$. Shapiro's lemma identifies its first cohomology with that of G_{L,S_L}, and the sum map with corestriction. Kummer theory identifies corestriction with the norm of squareclasses. Restriction followed by norm is multiplication by three, hence is the identity. This proves both the splitting and the projector formula. The Kummer exact sequence

$$0 \longrightarrow \mathcal{O}_{L,S_L}^\times/\mathcal{O}_{L,S_L}^{\times 2} \longrightarrow L(S_L, 2) \longrightarrow \mathrm{Cl}(\mathcal{O}_{L,S_L})[2] \longrightarrow 0$$

proves the last arithmetic assertion. Shapiro's decomposition commutes with restriction to decomposition groups, giving localization. □

11.4 Inert odd coefficient primes

Lemma 11.14. *Let (K, p) be one of $(F_2, 5), (F_3, 5), (F_3, 7)$. Let r be a member of a compatible system as above, crystalline of weight two at p and of cyclotomic determinant. Suppose its residue is globally reducible with constituents $\psi, \omega_p\psi^{-1}$ and ψ^2 is unramified at every finite place away from p. Then r is ordinary and modular.*

Proof. The completion K_p is unramified quadratic. A residual character is prime to p and therefore kills wild inertia. If a is the tame exponent of ψ modulo p^2-1, the rank-one finite-flat exponents and the unit relation for the totally even character ψ^2 give

K	p	$\mathrm{ord}(\epsilon \bmod p)$	a allowed by units	a finite flat
F_2	5	12	$6 \mid a$	$0, 1, 5, 6$
F_3	5	3	$3 \mid a$	$0, 1, 5, 6$
F_3	7	8	$4 \mid a$	$0, 1, 7, 8$

For the first order, $(1+\sqrt{2})^3 = 2$ and $(1+\sqrt{2})^6 = -1$ modulo five. For the second, $\epsilon_3^2+\epsilon_3+1 = 0$ modulo five. Modulo seven, $\epsilon_3^4 = -1$ and $\epsilon_3^8 = 1$. In each row only $a = 0, p+1$ survive: these are the etale and multiplicative endpoints. The tame exponent list is the rank-one finite-flat classification over the unramified quadratic base [57].

This implies actual ordinarity. Otherwise the Barsotti–Tate group of a stable coefficient lattice and its dual would be connected: a coefficient-stable etale factor would give the ordinary coefficient-rank-one filtration. Schematic closure and finite-flat quotient, after killing unramified character factors, produce a rank-one finite-flat subquotient with one of the two endpoint generic characters. At absolute ramification index one, finite-flat uniqueness [57, Theorem 3.3.3 and Corollary 3.3.6] identifies its model with the etale or multiplicative endpoint, contradicting biconnectedness. The unique ordinary filtration descends through the unramified extension. Apply [28, Corollary 1.2.2] to $r^\vee$: K is totally real abelian, p is unramified, the actual source is ordinary, its determinant is inverse cyclotomic, and its residue is reducible. This proves modularity of r. □

Lemma 11.15. *Let $p \geq 5$ be inert and unramified in a real quadratic field K, with $[K(\zeta_p) : K] = p-1$. An absolutely irreducible residue of an actual weight-two crystalline cyclotomic p-adic source remains absolutely irreducible over $K(\zeta_p)$.*

Proof. Otherwise Clifford theory induces it from the unique quadratic subfield of $K(\zeta_p)/K$. The image is dihedral of order prime to p, and that quadratic field is ramified at p. Cyclic tame projective inertia therefore contains a reflection and has order two. This contradicts the finite-flat inertia possibilities. Over the unramified quadratic completion the rank-one exponents are $0, 1, p, p+1$ modulo p^2-1. Cyclotomic determinant gives pairs $(0, p+1)$ or $(1, p)$, with projective orders $p-1$ or $p+1$. In the locally irreducible case the niveau-four exponents are $p+1, \quad p^2+p, \quad p^3+1, \quad p^3+p^2 \pmod{p^4-1}$. They are paired by multiplication by p^2. Their projective inertia order is $(p^2+1)/2$: subtracting the paired exponent reduces the computation to $\gcd(p^2+1, p \pm 1) = 2$. None of the displayed orders is two for $p \geq 5$. These are precisely the labelled finite-flat tame profiles [57], so the induced case is impossible. □

For either field at five, the cyclotomic degree is four; for F_3 at seven it is six. A small-field modular witness [3, Theorem 2.5], or the projective witness specified below, can therefore be lifted to the actual source by [3, Theorem 2.3] using Lemma 11.15. The prime three is treated separately, since its order $p-1 = 2$ would not give this exclusion.

11.5 A common cubic coefficient-field screen

Lemma 11.16. *Let D be a totally real number field of degree $d \leq 3$. If $d = 3$, suppose $D = \mathbf{Q}(\alpha)$, where α is an algebraic integer, every conjugate of α lies in $[-\sqrt{8}, \sqrt{8}]$, and its minimal polynomial is congruent to X^3 modulo two. Then D has a prime above three with residue field $\mathbf{F}_3$ or $\mathbf{F}_9$, or a prime above five with residue field $\mathbf{F}_5$.*

Proof. For $d \le 2$, any prime above three has the required residue field. In degree three write the minimal polynomial as $f(z) = z^3 + Az^2 + Bz + C$. Its nonleading coefficients are even, and purity gives $|A| \le 8$, $-8 \le B \le 24$, $|C| \le 22$. The bound on B follows by evaluating the pairwise-product sum at the vertices of the cube $[-\sqrt{8}, \sqrt{8}]^3$. Exact Sturm counting among these even coefficient triples leaves four irreducible polynomials and their conjugates $-f(-z)$:

$f(z)$	p	local certificate
$z^3 - 2z^2 - 4z + 6$	3	$f(0) \equiv 0, \quad f'(0) \not\equiv 0 \pmod 3$,
$z^3 - 6z - 2$	3	$f(z-1) = z^3 - 3z^2 - 3z + 3$ is Eisenstein,
$z^3 - 2z^2 - 2z + 2$	5	$f(3) \equiv 0, \quad f'(3) \not\equiv 0 \pmod 5$,
$z^3 - 4z - 2$	5	$f(1) \equiv 0, \quad f'(1) \not\equiv 0 \pmod 5$.

The enumeration discards $C = 0$ and reducible polynomials and counts roots on the indicated interval by Sturm's theorem; endpoint signs are evaluated exactly in $\mathbf{Q}(\sqrt{8})$. Hensel's lemma supplies a degree-one completion in the first, third and fourth rows. The second has a totally ramified cubic completion at three and hence residue field $\mathbf{F}_3$. The same conclusions hold for $-f(-z)$. The exact calculation is supplied in [52, `common`]. □

12 Symmetry and quadratic descent

The integral-order bounds can force either a small coefficient field or an isomorphism with a quadratic conjugate. We record the two rigidity mechanisms used below. The prime-to-characteristic cotangent argument is distinct from the dyadic order argument.

12.1 A marked ordinary lattice

Lemma 12.1. *Let K be real quadratic with one dyadic place w, and let S be a finite set of odd places. Let V be the squareclasses supported at $S \cup \{w\}$ with even valuation at w. Suppose $\ker(V \to K_w^\times/K_w^{\times 2}) = \langle [\delta] \rangle$, where δ is totally negative. Let $E/\mathbf{Q}_2$ be finite and $r : G_K \to \mathrm{GL}_2(E)$ absolutely irreducible, of cyclotomic determinant, unramified outside $S \cup \{w\}$, crystalline ordinary at w, and with scalar residual semisimplification. For any real conjugation c, there is a stable lattice with a basis in which*

$$\bar{r} = 1 + \bar{\chi}_\delta E_{12}, \qquad r(c) = C := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{ordinary line } e_2,$$

where $\bar{\chi}_\delta : G_K \to \mathbf{F}_2$ is the additive character of $K(\sqrt{\delta})/K$.

Proof. Choose a nonzero vector v on the ordinary line and let $\Lambda = \mathcal{O}_E[G_K]v$. It is a stable lattice by compactness and irreducibility, and v is primitive by Nakayama. Its reduction is cyclic. The scalar semisimplification is trivial because its determinant is one in characteristic two. Thus the reduction is a nonzero unipotent extension: a trivial cyclic module has dimension one. Its unique global invariant line is transverse to $\bar{v}$. Both lines are locally preserved at w, so the local residue is trivial. Every $\mathbf{F}_2$-coordinate of the additive extension therefore belongs to the displayed localization kernel. The extension is a nonzero multiple of $\bar{\chi}_\delta$, and $\bar{\chi}_\delta(c) = 1$. Consequently $((r(c) - 1)v, v)$ is an integral basis. The relation $r(c)^2 = 1$ gives the displayed real matrix and normalizes the multiple to one. □

12.2 Finite-order actions on complete local rings

Lemma 12.2 (Cotangent rigidity)**.** *Let $\mathcal{O}$ be a complete noetherian local ring, and let A be a complete noetherian local $\mathcal{O}$-algebra with the same residue field. Let a finite group Γ,*

with $|\Gamma| \in \mathcal{O}^\times$, act continuously by local $\mathcal{O}$-algebra automorphisms on A. If its action on $\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}}A)$ is trivial, its action on A is trivial.

Proof. Average lifts $x_1, \dots, x_n$ of a relative cotangent basis over Γ. The resulting invariant elements have the same cotangent classes. Nakayama and completeness give a surjection from $\mathcal{O}[[X_1, \dots, X_n]]$ to A sending its variables to these elements. The entire image is fixed. □

The invertibility hypothesis is essential: $T \mapsto -T$ is a nontrivial involution of $\mathbf{Z}_2[[T]]$ acting trivially on its relative cotangent space. In applications to pseudodeformations, the cotangent calculation must include products of opposite extensions, not only tangent vectors of representations.

Lemma 12.3. *Let k be a field of characteristic different from two, and let $V = ks \oplus kd$ carry the involution $\sigma(s) = s$, $\sigma(d) = -d$. On $V^\vee \otimes_k V^\vee$ put*

$$\iota(a \otimes b) = \sigma^\vee(b) \otimes \sigma^\vee(a), \qquad \ell_\pm = s^\vee \pm d^\vee.$$

If an ι-equivariant quotient kills both $\ell_+ \otimes \ell_+$ and $\ell_- \otimes \ell_-$, then ι acts trivially on that quotient.

Proof. The (-1)-eigenspace of ι is the line generated by $s^\vee \otimes d^\vee + d^\vee \otimes s^\vee$. It is killed because

$$\ell_+ \otimes \ell_+ - \ell_- \otimes \ell_- = 2(s^\vee \otimes d^\vee + d^\vee \otimes s^\vee).$$

Since two is invertible, the two eigenspaces split and the quotient has only the eigenvalue one. □

12.3 Descent from a twisted conjugacy

Lemma 12.4 (Quadratic descent)**.** *Let $K/\mathbf{Q}$ be real quadratic, with nontrivial automorphism σ. Let $\mathcal{R} = (r_\lambda)$ be a totally odd, absolutely irreducible strongly compatible rank-two system over K, with cyclotomic determinant and regular labelled Hodge–Tate weights. Suppose that, for one coefficient place,*

$$r_\lambda^\sigma \simeq r_\lambda \otimes \eta,$$

where η is the restriction of a quadratic character of $G_{\mathbf{Q}}$ (the trivial character is allowed). Then $\mathcal{R}$ is modular.

Proof. The identity of good Frobenius traces transfers the displayed isomorphism to a companion r in characteristic $p \geq 5$, by compatibility and Chebotarev. Write $H = G_K$ and choose $t \in G_{\mathbf{Q}} \setminus H$. There is an intertwiner A with

$$Ar(h)A^{-1} = \eta(h)r(tht^{-1}) \qquad (h \in H).$$

Since $\eta^\sigma\eta = 1$, Schur's lemma gives $A^2 r(t^2)^{-1}$ scalar. Thus $[r]$ and $[A]$ define a continuous projective representation of $G_{\mathbf{Q}}$.

After a finite coefficient extension, scale A so that $\det A = \chi_p(t)$. Define a continuous choice of matrices by $L(h) = r(h)$ and $L(ht) = r(h)A$ for $h \in H$. Then $\det L(g) = \chi_p(g)$ for every g, so its scalar cocycle

$$c(g_1, g_2) = L(g_1)L(g_2)L(g_1g_2)^{-1}$$

takes values in $\{\pm 1\}$. It is locally constant and is trivial on $H \times H$. Tate's theorem in [58, Theorem 6.3], for the *discrete trivial* $G_{\mathbf{Q}}$-module $\overline{\mathbf{Q}}^\times$, gives a continuous cochain $b : G_{\mathbf{Q}} \to \overline{\mathbf{Q}}^\times$ such that $c = \delta b$. Compactness makes the image of b finite. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ and enlarge the coefficient field by these finitely many algebraic values. Then $\widetilde{r} = b^{-1}L$ is

a continuous representation of $G_{\mathbf{Q}}$. Because $c|_{H\times H} = 1$, the restriction $b|_H$ is a finite-order character. In fact $c^2 = 1$ implies that b^2 is a finite-order character of $G_{\mathbf{Q}}$.

The representation $\widetilde{r}$ is absolutely irreducible, since its restriction to H is a finite twist of r. The locally constant function b factors through a finite Galois quotient. Outside the ramification of that quotient, $K/\mathbf{Q}$ and r, inertia lies in H and acts trivially, so $\widetilde{r}$ is finitely ramified. Finite twists preserve de Rham representations and their weights; de Rhamness descends through finite local extensions. Thus $\widetilde{r}$ is regular de Rham at p. Every complex conjugation belongs to H, and the square of its value under $b|_H$ is one, so $\widetilde{r}$ is odd. Apply [54, Theorem 1.0.4]. Cyclic base change to K is cuspidal because its Galois restriction is irreducible; untwisting by $b|_H$ proves modularity of r, and hence of the compatible system. □

12.4 Projective descent from coefficient inner twists

Lemma 12.5 (Finite-field projective descent)**.** *Let $k/\mathbf{F}_Q$ be a finite extension of odd characteristic, of degree n, and write $\phi : x \mapsto x^Q$. Let $\bar{r} : G \to \mathrm{GL}_2(k)$ be absolutely irreducible. Suppose a character $\eta : G \to \{\pm 1\}$ satisfies $\eta^n = 1$ and*

$$\bar{r}^{\phi} \simeq \bar{r} \otimes \eta.$$

Then its projectivization is conjugate to a representation with values in $\mathrm{PGL}_2(\mathbf{F}_Q)$. If $G = G_F$ for a totally real field and $\bar{r}$ is totally odd, the quadratic determinant character of the descended projective representation takes the value

$$(-1)^{(Q-1)/2}\eta(c_v)$$

at a real conjugation c_v.

Proof. Choose $A \in \mathrm{GL}_2(k)$ such that $\bar{r}^{\phi} = A\bar{r}A^{-1}\eta$. Since $\eta^n = 1$, absolute irreducibility gives $A^{\phi^{n-1}} \cdots A = cI$. Cyclically permuting the factors gives $c^{\phi} = c$. The norm $k^{\times} \to \mathbf{F}_Q^{\times}$ is surjective, so rescale A to make that product the identity. Matrix Hilbert 90 gives $A = B^{\phi}B^{-1}$. Then $\bar{r}' = B^{-1}\bar{r}B$ satisfies $(\bar{r}')^{\phi} = \eta\bar{r}'$. For each g, choose a scalar $u \in k^{\times}$ with $u^{\phi}/u = \eta(g)$. This makes $u\bar{r}'(g)$ a matrix over $\mathbf{F}_Q$, giving projective descent.

For a real conjugation, take $u = 1$ if $\eta(c_v) = 1$. If $\eta(c_v) = -1$, then $u^2 \in \mathbf{F}_Q^{\times}$ is a nonsquare: a square root in $\mathbf{F}_Q$ would force $u \in \mathbf{F}_Q$. The determinant of the chosen matrix is $-u^2$, proving the formula. □

12.5 Coefficient orbits and the integral-order bound

Proof of Theorem 1.5. The two operations commute and are involutions on isomorphism classes. Let $V = C_2^2$ be the resulting abstract group, let $m \in \{1, 2, 4\}$ be the number of coefficient orbits, and put $e = [D : \mathbf{Q}]$. Each operation preserves the trace-field degree, since it permutes the good primes and changes their traces only by rational signs. Section 11 gives $4me = \dim_{\mathbf{F}_2}(E_0/2E_0) < 32$.

The stabilizer V_0 of the coefficient orbit of $\mathcal{R}$ has order $4/m$. Its elements induce unique automorphisms of D, because the good traces generate D. These automorphisms form a homomorphism $V_0 \to \mathrm{Aut}(D)$ whose kernel consists of actual isomorphisms of systems. The element H is not in the kernel. Any nontrivial kernel therefore contains G or GH and gives one of the asserted twisted-conjugacy identities.

Otherwise V_0 embeds in $\mathrm{Aut}(D)$. The fixed-field theorem gives $4/m \mid e$, hence $4 \mid me$. Since $0 < me < 8$, we have $me = 4$. Thus $(m, e) = (4, 1), (2, 2), (1, 4)$. In the last case $D/\mathbf{Q}$ has a Klein four group of automorphisms and degree four, so is biquadratic. A residue-field Galois

group is then a cyclic quotient of a subgroup of C_2^2, and has order at most two. The other cases have degree at most two. The descent assertion is Lemma 12.4. □

Corollary 12.6 (The two-orbit alternative)**.** *Let $\mathcal{R}$ be an absolutely irreducible strongly compatible rank-two system over a real quadratic field, with cyclotomic determinant and trace field D. Let h be a rational quadratic character restricted to its base field. Suppose there is no actual h-selftwist. If the simultaneous order of the coefficient orbits generated by $\mathcal{R}$ and $\mathcal{R} \otimes h$ has reduction of dimension less than sixteen, then $D = \mathbf{Q}$, or D is quadratic and its nontrivial automorphism realizes the twist by h.*

Proof. Write $m \in \{1, 2\}$ for the number of orbits and $e = [D : \mathbf{Q}]$. The rank formula gives $me < 4$. If $m = 2$, then $e = 1$. If $m = 1$, the stabilizer C_2 acts faithfully on D, since there is no actual selftwist. Thus $2 \mid e$ and $e < 4$, so $e = 2$ and the nontrivial automorphism gives the stated identity. □

The bounds concern the actual simultaneous order, not one residual realization. The small-residue-degree alternative must still be combined with residual modularity and a lifting theorem for the actual source.

13 A common discriminant comparison

For $b > 0$ put $c_b = \pi/(2b)$ and define the even function

$$F_b(x) = \begin{cases} (1 - |x|/(2b)) \cos(c_b|x|) + \pi^{-1} \sin(c_b|x|), & |x| \le 2b, \\ 0, & |x| > 2b. \end{cases}$$

This is the normalized autocorrelation of $\cos(c_b t)$ on $[-b, b]$, extended by zero. Thus $F_b(0) = 1$, $0 \le F_b \le 1$, and its Fourier transform is nonnegative. Its Laplace transform is

$$J_b(s) = \int_0^{2b} F_b(x) e^{-sx}\, dx = \frac{s}{s^2 + c_b^2} + \frac{c_b^2(1 + e^{-2bs})}{b(s^2 + c_b^2)^2}. \tag{8}$$

Put

$$B_b = \int_0^\infty \frac{1 - F_b(x)}{2 \sinh(x/2)}\, dx, \qquad C_b = 2\{J_b(-1/2) + J_b(1/2)\}.$$

Expansion of $(2 \sinh(x/2))^{-1}$ gives $B_b = \sum_{j \ge 0}((j + 1/2)^{-1} - J_b(j + 1/2))$. Every summand lies between zero and $c_b^2(j + 1/2)^{-3}$. For $N \ge 1$ this gives

$$B_b \le B_{b,N} := \sum_{j=0}^{N-1} \Big((j + 1/2)^{-1} - J_b(j + 1/2)\Big) + \frac{c_b^2}{2(N - 1/2)^2}. \tag{9}$$

The midpoint inequality for x^{-3} also gives the convenient one-term bound $B_b \le 2 - J_b(1/2) + c_b^2/2$. The alternating four-term sum

$$A_{b,4} = J_b(1/2) - J_b(3/2) + J_b(5/2) - J_b(7/2)$$

is a lower bound for $\int_0^\infty F_b(x)/(2 \cosh(x/2))\, dx$.

Definition 13.1. For $1/2 \le \sigma < 1$, write $\mathrm{ZF}(\sigma)$ for the statement that every Dedekind zeta function is nonzero in $\Re s > \sigma$, away from its pole. The functional equation then places all nontrivial zeros in $|\Re s - 1/2| \le \sigma - 1/2$. GRH implies every such statement. Theorem B.4 proves $\mathrm{ZF}(4809/5000)$ from the analytic inputs specified in Appendix B.

For $0 \le a \le 1/2$ set $G_{a,b}(x) = F_b(x)/\cosh(ax)$.

Proposition 13.2. *Suppose all nontrivial zeros of ζ_L lie in $|\Re s - 1/2| \le a$. Let $n = [L : \mathbf{Q}]$, let r_1 be the number of real embeddings, and let N_Q count the prime ideals of norm Q. For $F = G_{a,b}$ put*

$$B(F) = \int_0^\infty \frac{1 - F(x)}{2\sinh(x/2)}\,dx, \quad C(F) = 4\int_0^\infty F(x)\cosh(x/2)\,dx,$$

$$A(F) = \int_0^\infty \frac{F(x)}{2\cosh(x/2)}\,dx.$$

Then

$$\begin{aligned}\log \mathrm{rd}(L) \ge \gamma + \log(8\pi) - B(F) - \frac{C(F)}{n} + \frac{r_1}{n}A(F) \\ + \sum_Q \frac{2N_Q}{n}\log Q \sum_{m\ge 1} Q^{-m/2}F(m\log Q).\end{aligned} \tag{10}$$

Any prime terms, and the real-place term, may be discarded. At fixed a, b, signature ratio and retained splitting densities, the lower bound increases with n.

Proof. For $a > 0$ and $|\theta| < a$, with $\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx}\,dx$, we have

$$\widehat{\frac{\cosh(\theta x)}{\cosh(ax)}}(t) = \frac{\pi}{a}\,\frac{\cos(\pi\theta/(2a))\cosh(\pi t/(2a))}{\sinh^2(\pi t/(2a)) + \cos^2(\pi\theta/(2a))} \ge 0.$$

At $|\theta| = a$ the quotient is one; when $a = 0$ this is the only case. Multiplication by the positive definite F_b preserves positive definiteness. If $\rho = 1/2 + \theta + i\gamma_0$, the real part of the explicit-formula transform is the Fourier transform of $F_b(x)\cosh(\theta x)/\cosh(ax)$ at γ_0, hence is nonnegative. The regularity of F justifies the explicit formula and convergence of its zero sum. Apply Poitou's formula [82, equation (7) and Proposition 4], [50, §2] and discard the nonnegative zero terms. The prime and real-place terms are nonnegative, and $C(F) > 0$. □

Proposition 13.3. *Assume GRH for the Dedekind zeta function of L. Let $n = [L : \mathbf{Q}]$, let r_1 be its number of real embeddings, and let N_Q count its prime ideals of norm Q. If B_b^* is either upper bound for B_b just given, then*

$$\log \mathrm{rd}(L) \ge \gamma + \log(8\pi) - B_b^* - \frac{C_b}{n} + \frac{r_1}{n}A_{b,4} + \sum_Q \frac{2N_Q}{n}\log Q \sum_{m\ge 1} Q^{-m/2}F_b(m\log Q). \tag{11}$$

Any subset of the prime terms may be retained. At fixed b, signature ratio and retained splitting densities, this lower bound increases with n.

Proof. Apply Proposition 13.2 with $a = 0$. Replace B_b by its upper bound and the real-place integral by its displayed lower bound. All omitted prime terms are nonnegative, and $C_b > 0$. The inner sums are finite because F_b has compact support. □

For comparisons without GRH, use $G_b(x) = F_b(x)/\cosh(x/2)$, and write $B(G_b), C(G_b)$ for the preceding integrals with F_b replaced by G_b. Proposition 13.2 applies with $a = 1/2$ without any hypothesis on zeros. The bound $1 - F_b(x) \le \pi^2x^2/(8b^2)$ gives

$$B(G_b) \le \log 2 + \frac{7\pi^2\zeta(3)}{16b^2}, \qquad C(G_b) = \frac{32b}{\pi^2}, \qquad \int_0^\infty \frac{F_b(x)}{2\cosh^2(x/2)}\,dx \ge 1 - \frac{\pi^4}{24b^2}.$$

Consequently, without any hypothesis on the zeros,

$$\log \mathrm{rd}(L) \geq \log(4\pi) + \gamma - \frac{7\pi^2\zeta(3)}{16b^2} - \frac{32b}{\pi^2 n} + \frac{r_1}{n}\left(1 - \frac{\pi^4}{24b^2}\right) + \sum_{Q} \frac{2N_Q}{n} \log Q \sum_{m\geq 1} \frac{2F_b(m\log Q)}{Q^m + 1}. \tag{12}$$

As before, one may discard prime terms or the real-place term. The numerical uses below take $\zeta(3) < 1.203$ and, when convenient, retain just the first prime-power term, using the same quadratic lower bound for F_b; their certificates are in [52, `common/verify_unconditional_refinements.py`].

The comparisons below specify the degree, signature, retained Euler densities and local discriminant upper bound. Their strict numerical inequalities include the tail in (9); no truncated infinite sum is used without a remainder bound.

13.1 Finite classical bounds and real places

The whole-strip kernel admits sharper finite estimates than the quadratic majorant. Put $T_b(s) = s^{-1} - J_b(s)$ for $s > 0$ and $D_b(j) = 1 - jJ_b(j)$ for positive integers j. Then

$$B(G_b) \leq \log 2 + 2\sum_{j=0}^{N-1} T_b(2j+1) + \frac{c_b^2}{2(2N-1)^2},$$
$$C(G_b) = \frac{32b}{\pi^2}, \qquad A(G_b) \geq 1 - 2\sum_{j=1}^{2H-1} (-1)^{j-1} D_b(j). \tag{13}$$

Indeed $0 \leq T_b(s) \leq c_b^2/s^3$, which bounds the tail in

$$B(G_b) = \log 2 + 2\sum_{j\geq 0} T_b(2j+1), \qquad C(G_b) = 32b/\pi^2. \tag{14}$$

Since F_b decreases on $[0, 2b]$, $D_b(j) = \int_0^{2b} (-F_b'(x)) e^{-jx}\,dx$ is positive and decreases with j. Integration by parts gives $A(G_b) = 1 - 2\sum_{j\geq 1}(-1)^{j-1} D_b(j)$, so odd truncation is a lower bound. All estimates are finite elementary expressions.

Lemma 13.4. *The following root-discriminant bounds require no hypothesis on zeros. In the rows marked $c_3 = 1/6$, assume that $2N_3/[L:\mathbf{Q}] \geq 1/6$.*

Signature	$[L:\mathbf{Q}] \geq$	b	c_3	$\mathrm{rd}(L) >$
arbitrary	*120*	*7*	*0*	*16.96717*
arbitrary	*1008*	*10*	*0*	*20.70004*
arbitrary	*8160*	*20*	*0*	*21.93341*
totally real	*120*	*7*	*0*	*43.14954*
totally real	*120*	*7*	1/6	*47.16252*
totally real	*1008*	*24*	*0*	*55.46400*
totally real	*8160*	*24*	*0*	*59.34662*
totally real	*8160*	*24*	1/6	*65.02119*

Proof. Use Proposition 13.2 with $a = 1/2$ and (13) with $N = 100$, $2H - 1 = 99$. For the marked rows retain only the first norm-three term, which is $(\log 3)F_b(\log 3)/12$. All other prime terms are discarded. The finite interval comparisons are supplied in [83, `analytic/all_q_discriminants.py`]. □

Corollary 13.5. *Let $q \geq 4$ be a power of two and let L be a number field satisfying*

$$[L : \mathbf{Q}] \geq 2q(q^2 - 1), \qquad \mathrm{rd}(L) \leq A\, 2^{-2/q}.$$

Then L is not totally real if either

(i) $A \leq 4\sqrt{220}$*; or*
(ii) $A \leq 4\sqrt{40}\, 3^{5/6}$ *and* $2N_3/[L : \mathbf{Q}] \geq 1/6$.

Proof. In (i), the upper bounds for $q = 4$, $q = 8$ and $q \geq 16$ are respectively less than 41.95236, 49.89004 and 59.32959. The corresponding lower bounds in Lemma 13.4 are 43.14954, 55.46400 and 59.34662. In (ii), the upper bounds are less than 44.68647, 53.14147 and 63.19621, whereas the lower bounds are 47.16252, 55.46400 and 65.02119. For $q = 8$ no prime term is needed. □

For the original finite-flat kernel fields appearing in the source arguments, Lemma 11.9 gives the factor $2^{-2/q}$. The remaining local bounds give the following values of A:

d	A	d	A
2	$8\sqrt{14}$	10	$4\sqrt{40}\, 3^{5/6}$
3	$8\sqrt{39}$	11	$4\sqrt{220}$
5	$4\sqrt{5}\, 11^{2/3}$	21	$4\sqrt{21}\, 5^{2/3}$
6	$4\sqrt{120}$	37	$4\sqrt{37}\, 3^{5/6}$

For $d = 10, 37$ the prescribed three-adic local field supplies $2N_3/[L : \mathbf{Q}] \geq 1/6$; see Theorem 24.5 and Lemma 4.19. Thus the real-place check in all eight source arguments is classical. In the $q = 4$ case for $d = 37$, even the prime term is unnecessary: the local upper bound is less than $42.97807 < 43.14954$.

13.2 Narrower zero-free strips

Put

$$B_\infty(a) = \int_0^\infty \frac{1 - \operatorname{sech}(ax)}{2\sinh(x/2)}\, dx, \qquad Q(a) = \int_0^\infty \frac{x^2}{2\sinh(x/2)\cosh(ax)}\, dx.$$

The derivative of F_b on $[0, 2b]$ gives $1 - F_b(x) \leq \pi^2 x^2/(8b^2)$ for every $x \geq 0$. Also $\cosh(x/2)/\cosh(ax) \leq e^{(1/2-a)x}$ for $a \leq 1/2$. Consequently

$$B(G_{a,b}) \leq B_\infty(a) + \frac{\pi^2 Q(a)}{8b^2}, \qquad C(G_{a,b}) \leq 4J_b(a - 1/2). \tag{15}$$

At $a = 1/2$, these specialize to $B_\infty = \log 2$, $Q = 7\zeta(3)/2$, and $C = 32b/\pi^2$.

Lemma 13.6. *For $a = 2309/5000 = 0.4618$,*

$$B_\infty(a) < \frac{63923}{100000}, \qquad Q(a) < \frac{241}{50}.$$

For $a = 15/32$, the corresponding bounds are $B_\infty(a) < 13/20$ and $Q(a) < 6$.

Proof. Write $t_a(s) = 2a^2/[s(s + a)(s + 2a)]$. The identity

$$1 - \operatorname{sech}(ax) = \frac{(1 - e^{-ax})^2}{1 + e^{-2ax}}$$

and geometric expansion give $B_\infty(a) = \sum_{m \geq 0} \sum_{j \geq 0} (-1)^j t_a(m + 1/2 + 2ja)$, with each inner sum interpreted as an alternating limit. Since t_a is positive and decreasing, the even partial sum

$P_{80}(s) = \sum_{j=0}^{80}(-1)^j t_a(s+2ja)$ is an upper bound for that limit and satisfies $0 < P_{80}(s) \le t_a(s)$. Thus

$$B_\infty(a) \le \sum_{m=0}^{499}\sum_{j=0}^{80}(-1)^j t_a(m+1/2+2ja) + \frac{a^2}{(499.5)^2} < \frac{63923}{100000}. \tag{16}$$

The tail uses $t_a(s) \le 2a^2/s^3$ and the integral test. For the finite rational sum, rounding every positive term and the tail upward and every subtracted term downward on the grid 10^{-20} gives the upper bound $\frac{63922728290911565369}{10^{20}}$. The three-term upper bound for the geometric expansion of sech similarly gives

$$Q(a) \le 4\sum_{m=0}^{3}\sum_{j=0}^{2}\frac{(-1)^j}{(m+1/2+(2j+1)a)^3} + \frac{2}{(3+1/2+a)^2} < 4.814654 < \frac{241}{50}. \tag{17}$$

The same finite sums at $a = 15/32$ give $B_\infty(a) < 0.64919 < 13/20$ and $Q(a) < 6$. These integer and rational checks are supplied in [83, `analytic/all_q_discriminants.py`]. □

Lemma 13.7. *Let $0 < a \le 1/2$, $M \ge 1$, and let $r, s \ge 0$ be even integers. If $B_0 \ge B_\infty(a)$, then*

$$B(G_{a,b}) \le B_0 + 2\sum_{m=0}^{M-1}\sum_{j=0}^{r}(-1)^j T_b(m+\tfrac12+(2j+1)a) + \frac{c_b^2}{(M-\frac12+a)^2},$$

$$C(G_{a,b}) \le 4\sum_{j=0}^{s}(-1)^j\left\{J_b((2j+1)a-\tfrac12) + J_b((2j+1)a+\tfrac12)\right\}. \tag{18}$$

Proof. Expand $(2\sinh(x/2))^{-1}$ and $\operatorname{sech}(ax)$. The difference $B(G_{a,b}) - B_\infty(a)$ is the corresponding alternating sum of $2T_b(m+1/2+(2j+1)a)$. The function T_b is positive and decreasing; its even partial sums are upper bounds and are at most the first term. The omitted m-tail is at most $2c_b^2\sum_{m\ge M}(m+1/2+a)^{-3}$, bounded by the displayed integral remainder. For the pole term, $J_b(u)$ is positive and decreasing for real u, so the even alternating partial sum is again an upper bound. The point $x = 0$ causes no difficulty by integration or Abel limits. □

Theorem 13.8. *Assume* ZF(4809/5000). *There is no number field L satisfying all of the following conditions, for a power of two $q \ge 4$:*

(i) *$L/\mathbf{Q}(\sqrt5)$ is Galois with group $\mathrm{SL}_2(\mathbf{F}_q)$;*
(ii) *at the prime above seven, inertia has order two and the relative decomposition group is C_2 or $C_2 \times C_2$;*
(iii) $\mathrm{rd}(L) \le U(q) := 4\sqrt{35}\,2^{-2/q}$.

For $q \le 16$ the exclusion is unconditional. For $q = 32, 64$ the weaker hypothesis ZF(31/32) *suffices. If the decomposition group in (ii) is C_2, the exclusion is unconditional for every q.*

Proof. The degree is $n(q) = 2q(q^2-1)$. For $q = 4, 8, 16$, the three arbitrary-signature bounds of Lemma 13.4 give respectively

q	lower $\mathrm{rd}(L)$	upper $U(q)$
4	> 16.96717	< 16.73321
8	> 20.70004	< 19.89925
16	> 21.93341	$< 21.70028.$

These comparisons use neither prime terms nor a zero-free hypothesis.

Since seven is inert in $K = \mathbf{Q}(\sqrt5)$, the relative residue degrees one and two in (ii) give absolute residue degrees two and four. In the latter case $N_{2401} = n/8$, and its first prime-power term is $(\log 7/49)G_{a,b}(4\log 7)$. In the former case $N_{49} = n/4$; its second prime-power term is

the same. Thus either case permits this common term. By (15) and Lemma 13.6, the new strip gives

$$\mathrm{rd}(L) \geq \mathcal{D}(n,b) := 8\pi \exp\Bigg(\gamma - \frac{63923}{100000} - \frac{241\pi^2}{400b^2} - \frac{4J_b(-191/5000)}{n} + \frac{\log 7}{49}\frac{F_b(4\log 7)}{\cosh((2309/1250)\log 7)}\Bigg). \tag{19}$$

The remaining comparisons are

q range	a	b	lower $\mathrm{rd}(L)$	upper comparison
$32, 64$	$15/32$	40	> 23.16457	$U(64) < 23.15725$
$128, 256, 512$	$2309/5000$	77	> 23.63669	$U(512) < 23.60034$
$q \geq 1024$	$2309/5000$	150	> 23.66539	$4\sqrt{35} < 23.66432.$

The first row uses $n \geq 65472$, $B_\infty < 13/20$, $Q < 6$, and no prime term. The other rows use (19) at $n(128) = 4194048$ and $n(1024) = 2147481600$. Degree monotonicity and the increase of $U(q)$ prove all the indicated ranges. The last comparison has margin greater than 10^{-3}.

Finally, if the relative decomposition group at seven is C_2, retain its first norm-49 term and use the whole-strip kernel with $b = 30$. For $q \geq 32$ and $n \geq 65472$, the finite classical bound (13), together with $(\log 7)F_{30}(2\log 7)/25$, gives $\mathrm{rd}(L) > 23.98628 > 4\sqrt{35}$. The smaller q were already excluded. All finite expressions and outward comparisons are certified in [83, `analytic/all_q_discriminants.py`]. □

For several one-place comparisons it is enough to use (12). At degree at least 1008, take $b = 10$. Dropping the real and prime terms gives $\mathrm{rd}(L) > 20.57608$; retaining the first norm-nine term with $2N_9/[L : \mathbf{Q}] \geq 1/2$ gives $\mathrm{rd}(L) > 25.32625$. At degree at least 120, take $b = 7$. The bound is $\mathrm{rd}(L) > 16.66$ in arbitrary signature and $\mathrm{rd}(L) > 41.69019$ in the totally real case. In particular $4\sqrt{5}\,11^{1/4} < 16.289$ and $5^{3/4}11^{2/5} < 8.726$. The exact rational checks are described in Appendix C.3.

13.3 Euler terms from a Sylow subfield

Lemma 13.9. *Let M/K be Galois with group $\mathrm{SL}_2(\mathbf{F}_q)$ for K real quadratic and $q \geq 4$ a power of two. For a Sylow two-subgroup U put $L = M^U$ and $r_0 = 1/(q+1)$. Then $[L : \mathbf{Q}] = 2(q^2-1)$ and the real fraction of L is at least r_0. At a place of K of norm P with elementary abelian decomposition group of ramification index e and residue degree f, its relative different contribution in $\log \mathrm{rd}(L)$ is $1 - r_0$ times that in $\log \mathrm{rd}(M)$. The Euler coefficients supplied by this place in L are r_0 at norm P and $(1-r_0)/(ef)$ at norm P^f, added when $f = 1$.*

Proof. A nontrivial two-subgroup is contained in a unique Sylow two-subgroup, the translations fixing its line on $\mathbf{P}^1$. Distinct Sylow two-subgroups intersect trivially. It therefore fixes $q-1$ cosets on $\mathrm{SL}_2(\mathbf{F}_q)/U$ and acts freely on the others. For a trivial subgroup we may partition the cosets into blocks of the same sizes. Applying the fixed-coset count to real conjugation gives the real fraction. For every nontrivial local ramification group J the normalized codimension of invariants is $(1-r_0)(1-1/|J|)$, so the conductor–discriminant formula gives the stated factor. The singleton and free orbits give the Euler coefficients. □

14 Arithmetic of the split-dyadic fields

Put $\mathcal{D} = \{17, 33, 41, 57, 65, 73, 89\}$. These fields have fundamental discriminant d and two splits in each. In this section let $K = \mathbf{Q}(\sqrt{d})$ with $d \in \mathcal{D}$, and write w_0, w_1 for its dyadic places. We record the arithmetic needed in the scalar-residual field construction.

Lemma 14.1. *Let S_2 consist of the dyadic and real places of K. Every finite-order character $G_K \to \overline{\mathbf{F}}_2^\times$ unramified outside two is trivial. Moreover*

$$H^1(G_{K,S_2}, \mathbf{F}_2) = \langle [-1], [\epsilon], [2], [\alpha] \rangle_{\mathbf{F}_2}$$

has dimension four, with the choices in Table 1.

d	ϵ	α	$N\epsilon$	$N\alpha$	h_K	h_K^+
17	$4+\sqrt{17}$	$(5+\sqrt{17})/2$	-1	2	1	1
33	$23+4\sqrt{33}$	$(5+\sqrt{33})/2$	1	-2	1	2
41	$32+5\sqrt{41}$	$(7+\sqrt{41})/2$	-1	2	1	1
57	$151+20\sqrt{57}$	$(7+\sqrt{57})/2$	1	-2	1	2
65	$8+\sqrt{65}$	$(9+\sqrt{65})/2$	-1	4	2	2
73	$1068+125\sqrt{73}$	$(9+\sqrt{73})/2$	-1	2	1	1
89	$500+53\sqrt{89}$	$(217+23\sqrt{89})/2$	-1	2	1	1

Table 1: Units, dyadic generators, and ordinary and narrow class numbers.

Proof. Minkowski's bound is $\sqrt{d}/2$. Except at 57 and 73, every prime ideal required by this bound is dyadic. At 41, 65 and 89, three is inert, so its prime ideal has norm nine and lies outside the bound. The primes above three in the two remaining cases are principal, since $N(15+2\sqrt{57}) = -3, \quad N(17+2\sqrt{73}) = -3$. For $d \neq 65$, the displayed α has norm ± 2 and generates a dyadic prime. At 65, its norm is four and its trace is odd, so it generates the square of a dyadic prime. That prime is not principal: an element of norm ± 2 would yield $x^2 - 65y^2 = \pm 8$, impossible modulo five. Thus $h_{65} = 2$ and the class group is generated by a dyadic prime. This proves the ordinary class numbers and shows that the S_2-ideal class group is trivial in every case.

The unit squareclass group has dimension two. If $N\epsilon = -1$, the mixed signs of ϵ show that its class and $[-1]$ are independent. For $d = 33, 57$ the unit is totally positive and is not a square: a square root would have integral trace whose square is, respectively, 44 or 48, or 300 or 304. Thus $[-1]$ and $[\epsilon]$ are again independent and form a basis. No fundamental-unit calculation is needed. The norm-minus-one units give the narrow class numbers in the other five cases; for $33, 57$, norm minus one is impossible modulo three, giving the remaining entries in the table.

Kummer theory identifies the group in the lemma with $\mathcal{O}_{K,S_2}^\times/(\mathcal{O}_{K,S_2}^\times)^2$. For $d \neq 65$, the valuation vectors of 2 and α form a basis of the dyadic divisor lattice. For 65, that lattice consists of pairs whose sum is even, and its basis is supplied by the vectors $(1,1)$ and $(2,0)$ of 2 and α. Together with -1 and the displayed unit squareclass, this proves the asserted basis.

Finally, a characteristic-two finite character has odd-order image. At either dyadic completion $\mathbf{Q}_2$, it kills the pro-two unit group and is unramified. It also kills real sign groups. Thus it factors through the ordinary class group, whose order is a power of two. The character is trivial. □

For $i = 0, 1$ write $V_i(K) = V_{w_i}^+(K)$ for the space in (1). We verify its vanishing using the full quadratic Selmer group, including the ideal-class contribution.

Lemma 14.2 (Base scalar Selmer vanishing)**.** *For every $d \in \mathcal{D}$ and either dyadic choice, $V_i(K) = 0$.*

Proof. The sign map on the basis of Lemma 14.1 gives

$$H^+ = \langle [2], [\beta] \rangle, \qquad \beta = \begin{cases} \alpha, & d = 17, 41, 65, 73, 89, \\ \epsilon, & d = 33, 57, \end{cases}$$

where H^+ denotes the totally positive subspace of $H^1(G_{K,S_2}, \mathbf{F}_2)$. It suffices to prove that the local class of β lies outside $\langle[2]\rangle$ at both dyadic places.

For $d = 17, 41, 65, 73, 89$, put $N\beta = 2^e$, with $e = 1$ except $e = 2$ for $d = 65$, and put $t = \mathrm{Tr}(\beta)$. The odd trace shows that the two dyadic valuations of β are $0, e$. Write u for the unit value. Then $u + 2^e/u = t$, so

$$u \equiv (1 + 2^e)t \equiv 7, 5, 5, 3, 3 \pmod 8$$

in the order $d = 17, 41, 65, 73, 89$. Thus u is nonsquare in $\mathbf{Q}_2$. The other local value is $2^e/u$, so both local classes lie outside $\langle[2]\rangle$. For $d = 33, 57$, we have $N\beta = 1$ and $\mathrm{Tr}(\beta) \equiv 14 \pmod{16}$. At either dyadic place, a square unit u would satisfy $u \equiv 1 \pmod 8$ and $u + u^{-1} \equiv 2 \pmod{16}$, a contradiction. Again the local unit class lies outside $\langle[2]\rangle$.

Restriction of H^+ to either dyadic completion is consequently injective. Its kernel at w_{1-i} is $V_i(K)$. □

For $d = 17, 41, 73, 89$, Table 1 gives narrow class number one. The totally positive element α of norm two has trace $5, 7, 9, 217$, respectively. At the other dyadic place its unit root satisfies $u + 2u^{-1} = \mathrm{Tr}(\alpha)$ and is congruent to $7, 5, 3, 3$ modulo eight. Thus these four fields satisfy condition (S) of [21, Lemma 9.1].

14.1 Local point fields and their discriminants

For a finite Galois extension $E/\mathbf{Q}_2$, let $d(E)$ be its normalized different exponent, with $v_2(2) = 1$. Put $E_2 = \mathbf{Q}_2(\sqrt{-1}, \sqrt{2}, \sqrt{5})$.

Lemma 14.3. *Let $r : G_{\mathbf{Q}_2} \to \mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^a$, have finite image and kernel field E. Then $d(E) \le 2$. More precisely, either $E \subset E_2$, or wild inertia is nontrivial of order $Q \le q$, all positive upper breaks are at most one and the local abelianization has odd order, or E is unramified or tame with ramification index three. In the wild case outside E_2, $d(E) \le 2(1 - 1/Q)$ and $E \cap \mathbf{Q}_2(\sqrt{-1}, \sqrt{5}) = \mathbf{Q}_2$. In the tame ramified case the image is S_3 and its quadratic subfield is unramified.*

Proof. If wild inertia is trivial, its tame generator has eigencharacters θ, θ^{-1}, and Frobenius squares them. The normalizer acts by identity or inversion, so the nontrivial tame order is three; Frobenius is the Weyl involution and the image is S_3. Its different exponent is $2/3$.

Otherwise the wild subgroup fixes a unique line, stable under the whole decomposition group. The diagonal characters are θ, θ^{-1} and are unramified, since $\mathbf{Z}_2^\times$ is pro-two. If $\theta = 1$, the image is elementary abelian unipotent and $E \subset E_2$. The seven quadratic characters of E_2 have conductors $0, 2, 2, 3, 3, 3, 3$, giving $d(E_2) = 2$. If $\theta \ne 1$, Frobenius acts on the translation subgroup P by the nontrivial scalar $\theta(\mathrm{Frob}_2)^2$. The operator obtained by subtracting one is invertible, so $[P, \mathrm{Frob}_2] = P$, P is wild inertia, and $E^P/\mathbf{Q}_2$ is unramified of odd degree. Its Kummer module has no Frobenius invariants. The equivariant valuation map to $\mathbf{Z}/2\mathbf{Z}$ is zero, so all its squareclasses are units and all its quadratic characters have conductor at most two. The different bound follows from conductor–discriminant. The commutator identity gives odd local abelianization and hence disjointness from the stated biquadratic field. □

Lemma 14.4. *Put $H = K(\sqrt{-1}, \sqrt{\epsilon})$. It is biquadratic over K, Galois over $\mathbf{Q}$ and unramified at odd places, and its dyadic completions lie in $\mathbf{Q}_2(\sqrt{-1}, \sqrt{5})$. If L/K has group $\mathrm{SL}_2(\mathbf{F}_q)$, $q \ge 8$, and is unramified outside two, put $M = LH$. Let k count the dyadic places whose local kernel fields lie in E_2. Then*

$$[M : \mathbf{Q}] = 8q(q^2 - 1), \qquad \mathrm{rd}(M) \le \sqrt{d}\, 2^{2-1/q+k/(2q)}. \tag{20}$$

Each of these k places supplies in the explicit formula the contribution of at least $[M : \mathbf{Q}]/16$ primes of norm four.

Proof. The classes $[-1], [\epsilon]$ are independent by their real signs and nonsquareness. Conjugation sends $\sqrt{\epsilon}$ to $1/\sqrt{\epsilon}$ or $\sqrt{-1}/\sqrt{\epsilon}$, so $H/\mathbf{Q}$ is Galois. Its defining polynomials have discriminants supported at two, and every unit squareclass over $\mathbf{Q}_2$ is generated by $[-1], [5]$. Simplicity gives $L \cap H = K$.

At a dyadic place with $E \subset E_2$, the compositum still lies in E_2 and has different exponent at most two, ramification index at most four and residue degree at most two. Otherwise, in the wild case of Lemma 14.3, adjoining the ramified quadratic part of H enlarges its elementary wild group from P to at most $P \times C_2$. All characters still have conductor at most two. Its different exponent is at most $2(1 - 1/(2Q)) \leq 2 - 1/q$. In the tame case the compositum has ramification index at most six, with only a possible quadratic positive interval of length one. Its different exponent is at most $(1 - 1/6) + (1 - 1/2) = 4/3 < 2 - 1/q$. Averaging the two dyadic exponents proves (20).

At a counted place, residue degree two gives at least $[M : \mathbf{Q}]/16$ norm-four primes. Residue degree one gives at least $[M : \mathbf{Q}]/8$ norm-two primes; keeping their even Euler powers supplies the same lower contribution. □

Lemma 14.5. *For $d \in \mathcal{D}$, an irreducible characteristic-two residue unramified outside two has determinant one. It is modular if its image is solvable or is $\mathrm{SL}_2(\mathbf{F}_4)$. The remaining images are $\mathrm{SL}_2(\mathbf{F}_q)$ for $q \geq 8$ a power of two.*

Proof. Lemma 14.1 gives trivial determinant. For a solvable image use [21, Proposition 10.3], since $K/\mathbf{Q}$ is abelian and two splits completely. For $\mathrm{SL}_2(\mathbf{F}_4)$ use [3, Theorem 1.2(1) and Lemma 2.1]. The classification of finite subgroups gives the remaining list. □

Part II
Fields without GRH

15 The field $\mathbf{Q}(\sqrt{2})$

Throughout this section put $K = \mathbf{Q}(\sqrt{2})$, $s = \sqrt{2}$ and $\epsilon = 1 + s$. The field has discriminant 8 and ordinary class number one by Minkowski's bound. Its units are generated by $-1, \epsilon$, and the norm of ϵ is -1, so the narrow class number is also one. The prime two ramifies, three is inert, and the two primes above seven are $(3 + s)$ and $(3 - s)$.

15.1 Local lifts at seven and the switch at three

Write $\omega_p = \bar{\chi}_p$. A special type includes its quadratic scalar character. The following twists are used to make that character unramified.

Lemma 15.1. *Put $\alpha_\pm = 5 \pm 4s$. The quadratic characters attached to $K(\sqrt{\alpha_\pm})/K$ ramify at exactly one of the two seven-primes, are unramified at two and three, and their product is the character of $K(\sqrt{-7})/K$, split at two.*

Proof. We have $\alpha_+ = \epsilon(3 + s)$ and $\alpha_- = \alpha_+^\sigma$, with $\alpha_+\alpha_- = -7$. Both are units away from seven. The integral element $(1 + \sqrt{\alpha_\pm})/2$ has polynomial $X^2 - X - (1 \pm s)$. Its reduction at the dyadic prime is $X^2 + X + 1$, so the quadratic extensions there are unramified. At three the defining radicands are units, hence the quadratic extensions are unramified. Their odd seven-adic valuations give exactly the asserted seven-ramification. Finally -7 is a square in $\mathbf{Q}_2$ because it is one modulo eight. □

Lemma 15.2. *Let $\mathcal{R}$ be a pure, irreducible, totally odd weight-two strongly compatible system with cyclotomic determinant, good outside seven, whose seven-types are good, quadratic-special, or principal series of order three after the corrections above. If its three-adic residue is reducible, the actual system is modular. If that residue is irreducible, it remains irreducible on $G_{K(\zeta_3)}$.*

Proof. Twist away the scalar quadratic inertia using Lemma 15.1. Modulo three the inertia at both seven-primes is unipotent. A reducible global residue therefore has characters unramified outside three. The prime-to-three narrow ray group modulo three is C_2: ϵ has order eight in $\mathbf{F}_9^\times$ and signs $(+, -)$, while -1 supplies the independent sign relation. Its characters are $1, \omega_3$. Thus the residual constituents are exactly $1, \omega_3$.

Since $e(K_3) = 1 < 3 - 1$, finite-flat uniqueness identifies the rank-one models with the etale and multiplicative models. A nonordinary Barsotti–Tate source would be biconnected, as would its finite-flat subquotients; these rank-one models contradict biconnectedness. The actual source is ordinary, with unramified quotient reducing to one. Skinner–Wiles [66, §4.5, Theorem A] applies: the ratio is locally nontrivial, totally odd, and its splitting field $K(\zeta_3)$ is abelian over $\mathbf{Q}$. The determinant is χ_3 and the local quotient has weight zero. Undo the twists.

For the second assertion, a cyclotomic-reducible irreducible residue would be induced from a prime-to-three character θ of $H = K(\zeta_3)$. Its restriction to seven-inertia is trivial since that inertia has order dividing three. The character is therefore unramified outside three and tame there. The absolute discriminant of H is $8 \cdot (-3) \cdot (-24) = 576$, and its degree-four Minkowski bound is $36/\pi^2 < 4$. There is no prime of norm two or three: their norms are four and nine. Hence H has class number one. Its unique prime above three has residue field $\mathbf{F}_9$, and ϵ generates the residue units. The tame ray group is trivial. This makes θ trivial, contradicting irreducibility. □

Lemma 15.3. *Every local determinant-ω_7 residue over $\mathbf{Q}_7$ has a determinant-χ_7 weight-two lift that is good, special, or principal series of order three after a scalar quadratic twist.*

Proof. The central-trivial Serre weights are $\mathrm{Sym}^r \det^b$, with $r = 0, 2, 4, 6$ and $r + 2b = 0 \pmod 6$. The two principal-series types with characters $\widetilde{\omega}_7^t, \widetilde{\omega}_7^{-t}$ have reductions

t	first constituent	second constituent
1	$(2, 5)$	$(4, 1)$
2	$(4, 4)$	$(2, 2)$.

Together with the two scalar crystalline and two scalar semistable weights, these cover all eight weights. Some multiplicity is positive because a potentially Barsotti–Tate lift exists, as in the proof of Theorem 4.12. The fixed-determinant formula [74, Theorem 1.2] therefore gives a nonempty weight-two local ring in this menu for the actual residual extension. At a scalar semistable point the monodromy is either zero, giving the good case after twist, or nonzero, giving the special case. A scalar quadratic twist interchanges the principal-series rows and changes order six to order three. The independent global characters in Lemma 15.1 implement these choices at the two split seven-primes without disturbing good reduction at two or three. □

15.2 Weight-two systems with one possible special place

Proposition 15.4. *Let $\mathcal{R}$ be a weight-two system over K of cyclotomic determinant. Suppose it is good outside one place v above seven and is good or quadratic-special at v. Then $\mathcal{R}$ is modular.*

We first treat nonscalar dyadic residues. The special parameter has an ordinary filtration away from the coefficient prime. After the quadratic scalar character is removed, its local characters are unramified with Frobenius values $\pm 7, \pm 1$, with common sign. In characteristic two both become one, so tame inertia in the residue has order at most two, for any stable lattice. The quadratic scalar character also reduces to one.

Lemma 15.5. *Let $K = \mathbf{Q}(\sqrt{2})$ and let $\bar{r} : G_K \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ be finite flat at two, unramified away from two and seven, with inertia of order at most two at both seven-primes. If its image is solvable, then $\bar{r}^{\mathrm{ss}} = 1 \oplus 1$.*

Proof. If the representation is reducible, its diagonal characters have odd order and are unramified at seven. Their inertia at two is trivial because the dyadic residue field is $\mathbf{F}_2$ and the wild inertia is pro-two. They are therefore everywhere unramified; narrow class number one makes them trivial.

Otherwise its image is an odd dihedral group, and it is induced from an odd-order character θ of a quadratic extension L/K. The quadratic quotient is unramified outside two and seven. Since K has class number one and unit group $\langle -1, \epsilon \rangle$, its squareclass is generated by $-1, \epsilon, s, 3+s, 3-s$, where $s^2 = 2$. Fontaine's finite-flat bound makes the quadratic conductor at two at most four. A quadratic extension of K_2 defined by an odd valuation radicand has conductor $2e(K_2) + 1 = 5$. Thus the exponent of s is zero, and there are precisely the fifteen nontrivial classes generated by $-1, \quad \epsilon, \quad 3 + s, \quad 3 - s$. The character θ is unramified at seven. If L/K ramifies there, its residual inertia is a reflection of order two and has trivial intersection with the odd rotation subgroup. If L/K is unramified there, inertia lies in that odd subgroup and has order dividing two, so is trivial. Its wild inertia at two is also trivial. Consequently θ factors through the odd quotient of the ray group of L modulo the primes above two.

For one allowed seven-place, conjugate it to $(3 + s)$ and put $\pi = 3 + s$, $\beta = \epsilon\pi = 5 + 4s$. The relevant classes are generated by $-1, \epsilon, \pi$. The following direct calculation treats this subfamily.

The possible prime ideals of L needed for Minkowski's bound are as follows.

a	$r_2(L)$	upper bound for $\lvert\operatorname{disc}(L)\rvert$	prime ideal norms to consider
-1	2	256	2
$\pm\epsilon$	1	1024	2
π	0	7168	$2, 7$
$-\pi$	2	7168	$2, 7, 9$
β	1	448	none
$-\beta$	1	1792	2

The first four discriminant bounds follow from the polynomials X^4+1, X^4-2X^2-1 and $X^4\mp 6X^2+7$, respectively. The fields for $\pm\epsilon$ are conjugate. For $\pm\beta$, the relative discriminant has valuation one at v and no other odd factor. At two, β is unramified and $-\beta$ has conductor two, since -1 has conductor two and β is unramified. This gives the two bounds after multiplying by 8^2. The list of prime norms follows from Minkowski and the fact that $3, 5, 11$ are inert in K. For β the dyadic prime has norm four.

In each of the other cases there is a unique prime $\mathfrak{P}$ above two with $(s)\mathcal{O}_L=\mathfrak{P}^2$. Indeed -1 and $-\beta$ are ramified there and $\pm\epsilon, \pm\pi$ are nonsquares modulo $2\mathcal{O}_{K_2}$. This proves that the classes of dyadic primes have order dividing two. For $a=\pm\pi$, the element $t=\sqrt{a}$ generates the prime over v and has absolute norm seven. The conjugate seven-place is inert in $K(\sqrt{\pi})$ and split in $K(\sqrt{-\pi})$, since π has residue six there. For $a=-\pi$, the two additional norm-seven primes are picked out by $t-1$ and $t+1$, both of absolute norm fourteen. The polynomial X^4+6X^2+7 factors modulo three as $(X^2+X+2)(X^2-X+2)$ with distinct irreducible factors, and its discriminant is prime to three. The corresponding norm-nine primes are picked out by t^2+t+2 and t^2-t+2, both of absolute norm eighteen. Thus all the required classes have order dividing two.

The odd part of $(\mathcal{O}_L/(2))^\times$ is trivial except for $a=\beta$, when it is the multiplicative group of the residue field $\mathbf{F}_4$. The unit $(1+\sqrt{\beta})/2$ has relative norm $-\epsilon$ and maps to a root of X^2+X+1 in that residue field, so its image generates the odd part. The ray exact sequence now gives the required vanishing.

For the full two-place support the odd ray quotient is trivial in all fifteen cases. Here is a complete finite certificate: use the bits of an integer $1\le m\le 15$ to select the four displayed generators, form their product $A(y)$, and take $P_m(X)=\operatorname{Res}_y\left((X-y)^2-A(y), y^2-2\right)$. The accompanying program `verify_d2_solvable_two.gp` computes the maximal order and certified class group of each quartic field, and its ray group modulo two. It applies `bnfcertify` in every row and asserts that every ray invariant is a power of two. The possible absolute discriminants and ray invariants are

$\operatorname{disc}(L)$	$\operatorname{Cl}_{(2)}(L)$
256	1
-1024	1
7168	1 or $C_4\times C_2$
-448	1
-1792	1
12544	1
3136	C_2^2
-50176	C_2^2.

The modulus (2) may contain higher dyadic powers; these add only two-primary ray factors, so its odd quotient equals that of the radical modulus required here. Infinite-place factors are also two-primary. Thus θ is trivial, contradicting the irreducibility of its induction. The exact calculation is supplied in [52, `ramified/d2`]. □

Lemma 15.6. *If a dyadic residue in Proposition 15.4 is nonsolvable, then the system is modular.*

Proof. The kernel field M/K has group $\mathrm{SL}_2(\mathbf{F}_q)$ with q a power of two and $q \geq 4$. Its absolute degree is $2q(q^2-1)$. At two the base contributes $2^{3/2}$ to its root discriminant, and the relative normalized different is less than two by Lemma 11.7. At v the tame ramification index is at most two, so the contribution is at most $7^{1/4}$. Hence

$$\mathrm{rd}(M) < 2^{7/2}7^{1/4} < 18.403.$$

For $q \geq 8$ the absolute degree is at least 1008, and the classical inequality of Section 13 gives $\mathrm{rd}(M) > 20.57608$. For $q = 4$ it also excludes a totally real kernel of degree 120, since the real-place term gives $\mathrm{rd}(M) > 41.69$. The small-image residual theorem and the actual dyadic lifting argument of Lemma 2.4 now give modularity. □

Lemma 15.7. *An ordinary crystalline dyadic member in Proposition 15.4 cannot have scalar residual semisimplification.*

Proof. Let v_0 be a nonzero vector on the local ordinary line, and take the stable lattice generated by its global Galois orbit. It is a lattice by compactness and absolute irreducibility. The vector v_0 is primitive by construction. The reduction is cyclic and has trivial semisimplification, so it is a nonsplit extension of the trivial character by itself. Its global invariant line is distinct from the reduction of the ordinary line, since the latter generates the module under the global group. Both lines are invariant at two, so the actual residual restriction at two is trivial.

A nonzero binary component of the global extension character then gives a supported quadratic class that is split at two. With $v = (3+s)$ it must belong to $\langle [-1], [\epsilon], [3+s] \rangle$ by the conductor argument of Lemma 15.5. The three generators can be replaced by $-1, \epsilon, \beta$, where $\beta = 5 + 4s = \epsilon(3+s)$. Locally β is the nontrivial unramified class by Lemma 15.1. The class -1 has conductor two, since the biquadratic discriminant formula gives $\mathrm{disc}(K(i)) = 256$. The unit ϵ is not a square modulo $2\mathcal{O}_{K_2}$, so its conductor exceeds two by Lemma 11.3. The three local classes are independent. This contradicts the existence of the split class. □

Proof of Proposition 15.4. By the preceding lemmas, it remains to consider the case in which every dyadic coefficient member is nonordinary and every residual semisimplification is scalar. Remove the ramified quadratic scalar factor, if any, at the possible special place using Lemma 15.1. This preserves the good conditions, ordinarity, determinant, and scalar residues. Use the simultaneous order E_0 of Section 11 and put $E = E_0/2E_0$ and $J = \operatorname{rad} E$.

For $\lambda \in (J/J^2)^\vee$, the map $g \mapsto \lambda(r(g) - 1)$ is additive and the map from functionals to such characters is injective since the group image spans E. The corresponding two-dimensional quotient of the left regular representation is finite flat and biconnected by Lemma 11.5. Lemma 11.2 bounds the dyadic conductor by two. With $v = (3+s)$, the permitted squareclasses are therefore spanned by $-1, \beta$ by the preceding local unit test. If the system is good everywhere, only -1 remains. A real conjugation c then gives $A = r(c) - 1$ spanning J/J^2, so A generates E as an $\mathbf{F}_2$-algebra. Since $A^2 = 0$, E is spanned by $1, A$, contradicting $\dim_{\mathbf{F}_2} E = 4[D : \mathbf{Q}] \geq 4$, where D is the trace field.

Suppose the system is special at v. Choose c at the embedding $s > 0$, and let $C = r(\tau) - 1$ for a tame generator at v. The evaluations of c, τ on $(-1, \beta)$ are $(1, 0), (0, 1)$, so A, C span J/J^2 and generate E. We have $A^2 = C^2 = 0$.

The class of -1 is unramified nonsquare at v. Choose an arithmetic Frobenius lift whose value on β is zero, by multiplying it by inertia if necessary. Its difference $B = r(\mathrm{Frob}_v) - 1$ has the same values as A on the allowed first-layer characters, so $B - A \in J^2$. On the image of the monodromy operator, Frobenius acts by 7 times its unramified sign, the same odd integer in every coefficient realization. Thus $BC = 0$ modulo two in the actual order and $AC \in J^3$. The only possibly nonzero words of length three are ACA and CAC, and both now lie in J^4.

Thus $J^3 = J^4$. Nilpotence gives $J^3 = 0$, so $AC = 0$. It follows that $1, A, C, CA$ span E, so $4[D : \mathbf{Q}] = \dim_{\mathbf{F}_2} E \leq 4$. The trace field is therefore $\mathbf{Q}$.

Take a coefficient place over three. The trace field is rational, so an irreducible residue is realized over $\mathbf{F}_3$ by finite-field Brauer descent. More explicitly, the characteristic polynomials have coefficients in $\mathbf{F}_3$, and Frobenius conjugation on coefficients preserves its isomorphism class by Brauer–Nesbitt. The finite-field Schur index is one. The small-image residual theorem [3, Theorem 2.5] and Lemma 15.2 give its modularity and cyclotomic irreducibility, respectively. With a weight-two witness as in Lemma 2.1, Lemma 2.2(ii) gives modularity of the actual system. If the residue is reducible, Lemma 15.2 directly gives modularity of the actual three-adic member. Undo the finite quadratic twist. □

15.3 The seven-adic terminal member

Proposition 15.8. *The seven-adic terminal member of Proposition 6.2 over* $\mathbf{Q}(\sqrt{2})$ *is modular.*

Proof. The reducible and irreducible solvable residues are covered by Proposition 6.5, using Lemma 6.8 in the reducible case. It remains to treat nonsolvable residue.

Write v_+, v_- for the two marked seven-places. Use Lemma 15.3 at v_+ and retain the actual crystalline weight-two component at v_-. Prescribed lifting and potential modularity as in Lemma 2.6(i) give a weight-two comparison system of exact cyclotomic determinant, good away from v_+. Remove a ramified scalar quadratic factor at v_+ when needed by Lemma 15.1. This preserves good reduction at v_-, two and three. The comparison system is now good or quadratic-special or of principal-series inertial order three at v_+.

If its three-adic residue is reducible, the actual system is modular by Lemma 15.2. Otherwise that lemma gives irreducibility on the cyclotomic subgroup. In characteristic three retain the old crystalline weight-two component and the unramified condition at v_-. At v_+ apply Lemma 4.9 to the full residual representation. Lemma 2.6(i) gives the new system. If the prescribed component is special, purity gives nonzero monodromy. This system is modular by Proposition 15.4. Lemma 2.1 supplies the weight-two witness, and Lemma 2.2(ii) gives modularity of the old three-adic member. Untwisting and applying Lemma 2.2(i) to the original seven-adic member proves the assertion.

□

Completion for $d = 2$ in Theorem 1.1. Apply Proposition 2.7. In the nonsolvable case use Proposition 6.2 at seven and Proposition 15.8, then return through the congruences and reduce the initial lift. □

16 The field $\mathbf{Q}(\sqrt{5})$

Throughout this section put $K = \mathbf{Q}(\sqrt{5})$.

16.1 Arithmetic at eleven and two

Put $\phi = (1 + \sqrt{5})/2$, $\pi = 3 + \phi$, and $\pi' = 4 - \phi$. Then $\pi\pi' = 11$ and the primes over eleven are $\mathfrak{p} = (\pi)$ and $\mathfrak{p}' = (\pi')$. The Minkowski bound $\sqrt{5}/2$ gives class number one. The unit group is generated by -1 and the norm-minus-one unit ϕ, so the narrow class number is also one. Two and three are inert, five ramifies, and the dyadic completion is unramified quadratic. At $\mathfrak{p}, \mathfrak{p}'$ the residues of ϕ are $8 = 2^3$ and $4 = 2^2$, respectively.

Set $\alpha = \phi\pi = 1 + 4\phi$. Its squareclass gives a character κ_α ramified at $\mathfrak{p}$ and unramified at all other finite places. Indeed $(1+\sqrt{\alpha})/2$ satisfies $X^2 - X - \phi = 0$, and the reduction is irreducible over $\mathbf{F}_4$ since $\mathrm{Tr}_{\mathbf{F}_4/\mathbf{F}_2}(\bar{\phi}) = 1$. The conjugate character has the analogous property at $\mathfrak{p}'$. We use these characters as reversible twists of determinant one.

Lemma 16.1. *A quadratic character of G_K unramified outside $2\mathfrak{p}$ and the real places and split at two is trivial.*

Proof. By class number one its squareclass is $(-1)^a\phi^b 2^c \pi^d$, with $a, b, c, d \in \{0, 1\}$. A dyadic square has even valuation, so $c = 0$. Its norm from K_2 to $\mathbf{Q}_2$ is a square, so $b = d = 0$ because $-1, 11, -11$ are all nonsquares in $\mathbf{Q}_2$. Finally -1 remains nonsquare over the unramified quadratic extension of $\mathbf{Q}_2$. The statement for $\mathfrak{p}'$ follows by conjugation. □

16.2 A dyadic obstruction with one odd place

Proposition 16.2. *Let $\bar{r} : G_K \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ be continuous and finite flat at two. Suppose it is unramified outside $2\mathfrak{p}$ and has inertia of order at most two at $\mathfrak{p}$. Then $\bar{r}^{\mathrm{ss}} = 1 \oplus 1$.*

Proof. A nonsolvable image is $\mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^f \geq 4$, by Dickson's classification. The kernel field M/K then has absolute degree at least 120. Fontaine's finite-flat bound over the unramified dyadic base gives classical upper breaks at most one [34, Section 2.1, Theorem 1 and Section 1.2(i)]. The integral of the upper filtration gives normalized dyadic different exponent strictly below two. At the single bad eleven-place the relative tame exponent is at most $1/2$. Thus $\mathrm{rd}(M) < 4\sqrt{5}\,11^{1/4} < 16.289$. The classical lower bound in Section 13 gives $\mathrm{rd}(M) > 16.66$.

If the image is irreducible and solvable, it is odd dihedral. It is induced from an odd-order character of a quadratic field L/K. The quadratic quotient is unramified outside $2\mathfrak{p}$. Its dyadic conductor is at most two by Fontaine's bound. A class of odd dyadic valuation gives an Eisenstein quadratic with different exponent three over the unramified base. Therefore $L = K(\sqrt{a})$ for a nontrivial class in $\langle [-1], [\phi], [\pi] \rangle$. The inducing character is unramified above eleven because its inertia there has both odd order and order at most two. At two it is tame, so it factors through the ordinary ray class group modulo (2) of L.

Here are the relevant quartic data, with r_2 the number of complex pairs. The Minkowski bounds in the last column are strict upper bounds.

a	$\lvert D_L \rvert$	r_2	Minkowski bound
-1	400	2	3.1
$\phi,\ -\phi$	400	1	2.4
π	4400	0	6.3
$-\pi$	4400	2	10.2
$\phi\pi$	275	1	2
$-\phi\pi$	4400	1	8

For the discriminants, $\{1, \sqrt{a}\}$ is a relative integral basis except in the row $a = \phi\pi$, where $\{1, (1+\sqrt{a})/2\}$ is an integral basis. At eleven this follows from the valuation-one radicands. At two a unit admits the latter type of basis exactly when it is square modulo four. The $\phi\pi$ row is the only unramified dyadic case, by the discriminant bound or by testing squares modulo four. To see this directly, $-1, \phi, \pi, -\pi, -\phi, -\phi\pi$ fail the test: the norm rules out $\pm\phi$ and $\pm\pi$, whose norms are -1 or 11, and the remaining $-1, -\phi\pi$ reduce to -1 modulo four.

In the first, second and fifth rows no prime ideal lies below the Minkowski bound. In the π row the possible generators have norms four and five. The unique dyadic ideal has square (2), and $N_{L/\mathbf{Q}}(\sqrt{\pi} - 1) = 5$ gives a principal norm-five prime; conjugation over K gives the other

when five splits there. In the $-\pi$ row three has residue degree four, since $-\pi$ is nonsquare in $\mathbf{F}_9$ (its norm to $\mathbf{F}_3$ is 11 mod 3). The norm $N_{L/\mathbf{Q}}(\sqrt{-\pi}-\phi)=20$ identifies the norm-five classes with the dyadic class. In the last row the only possible generating prime is the dyadic ideal: $-\phi\pi$ is nonsquare at five, since $\phi\equiv 3$ and $\pi\equiv 1$ there. Thus all the ordinary class groups have two-power order.

When L/K is dyadically ramified, its involution acts trivially on the residue units. Anti-invariance of the inducing character, which follows from determinant one, makes its odd tame inertia trivial. It is then trivial by the ordinary class calculation. For $L=K(\sqrt{\phi\pi})$ the integral unit $t=(1+\sqrt{\phi\pi})/2$ reduces to a root of X^4+X+1 of order fifteen in $\mathbf{F}_{16}^{\times}$. It kills the entire tame ray quotient at two. Again no inducing character survives. The seven class and ray groups are independently checked by the supplied PARI/GP certificate.

In the reducible case the two characters have odd order, hence are unramified at eleven and tame at two. The image of ϕ generates $\mathbf{F}_4^{\times}$, so the ray group of K modulo (2) is trivial. □

Proposition 16.3. *There is no absolutely irreducible 2-adic representation r of G_K of determinant χ_2 which is crystalline of labelled weights $\{0,1\}$ at two, unramified outside $2\mathfrak{p}$, and unipotent on inertia at $\mathfrak{p}$. The same holds with $\mathfrak{p}'$ in place of $\mathfrak{p}$.*

Proof. A stable lattice at the dyadic place is the Tate module of a Barsotti–Tate group with coefficient action [45, Theorem 4.1]. At $\mathfrak{p}$ its residue has inertia of order at most two, since unipotent inertia factors through a procyclic pro-two quotient. Proposition 16.2 makes the residual semisimplification trivial.

Apply Lemma 4.8 to obtain a nonzero additive residual extension split at two. A nonzero $\mathbf{F}_2$-coordinate of the residual extension now gives a quadratic character unramified outside $2\mathfrak{p}$ and split at two, contrary to Lemma 16.1. □

Lemma 16.4. *Let $\bar{r}:G_K\to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ be absolutely irreducible, unramified outside five and eleven. If its inertia image at both eleven-primes has order dividing five, then $\bar{r}|_{G_{K(\zeta_5)}}$ is absolutely irreducible.*

Proof. Otherwise Clifford theory gives induction from a prime-to-five character θ of $H=K(\zeta_5)=\mathbf{Q}(\zeta_5)$. The character is unramified at eleven, since its inertia there has both prime-to-five order and order dividing five. It is unramified away from five, and tame at the unique prime $\mathfrak{P}$ above five. The discriminant of H is 125, and its degree-four Minkowski bound is $3\sqrt{125}/(2\pi^2)<2$, so its class number is one. The image of ϕ modulo $\mathfrak{P}$ is 3, which generates $\mathbf{F}_5^{\times}$. The ray group modulo $\mathfrak{P}$ is therefore trivial. This makes θ trivial, contrary to the irreducibility of its induction. □

16.3 The switch from eleven to five

Lemma 16.5. *Every continuous residual representation of $G_{\mathbf{Q}_{11}}$ of determinant $\bar{\chi}_{11}$ has a weight-two lift of determinant χ_{11} which is scalar potentially crystalline, quadratic-special, or potentially crystalline of principal-series type. After a local quadratic twist the principal-series inertial characters have order five.*

Proof. Take a regular crystalline lift with determinant χ_{11} by Lemma 4.1. Its algebraic type has trivial scalar central character, since the two weights sum to one. Positivity in the fixed-determinant Breuil–Mézard formula [74, Theorem 1.2] gives a nonzero cycle for at least one central-trivial Serre weight of the actual residual representation. Write $W(r,b)=\mathrm{Sym}^r\,\mathbf{F}_{11}^2\otimes\det^b$. The central-trivial weights have $0\le r\le 10$, b modulo ten, and $r+2b=0$ modulo ten. The four

determinant-one principal-series types with characters of exponents $(t, -t)$ give

t	constituents of the compact type	
1	$W(2,9)$	$W(8,1)$
2	$W(4,8)$	$W(6,2)$
3	$W(4,3)$	$W(6,7)$
4	$W(2,4)$	$W(8,6)$

using the inertial convention of [38]. For $0 < a - b < 10$, the Brauer character of the principal series with exponents (a, b) is the sum of those of $W(a-b, b)$ and $W(10-a+b, a)$. Indeed the two symmetric-power traces sum to the character of the induced representation on split semisimple elements. On nonsplit semisimple elements with eigenvalues x, x^{11} the sum is zero, using $x^{120} = 1$. For the four rows take $(a, b) = (10 - t, t)$. The scalar crystalline types give $W(0,0), W(0,5)$ and the scalar semistable types give $W(10,0), W(10,5)$. This accounts for every central-trivial weight. Fixed-determinant positivity yields a point in the corresponding weight-two ring. In a scalar semistable ring, a point of zero monodromy is scalar potentially crystalline. Otherwise take its special component. A quadratic twist exchanges rows one and four, and rows two and three. The even rows have order five. □

Lemma 16.6. *Let $\bar{s}$ have determinant $\bar{\chi}_5$, be finite flat above five, and be unramified outside five and $\mathfrak{p}$. Suppose its inertia at $\mathfrak{p}$ has order dividing five. It is not exceptional in the sense of [31, Definition 4.1.1]. The assertion also holds with $\mathfrak{p}'$ in place of $\mathfrak{p}$.*

Proof. In the exceptional case the projective kernel field M has group S_5 over K. The sign quotient cuts out $K(\zeta_5) = \mathbf{Q}(\zeta_5)$, ramified quadratic above five. At the five-adic place $e(K_v/\mathbf{Q}_5) = 2$ and the residue field is $\mathbf{F}_5$. The rank-one finite-flat tame weights in [57, Corollaries 3.3.6 and 3.4.4] lie in $\{0, 1, 2\}$. For a locally reducible representation the two exponents sum to two, so the projective tame ratio has order at most two. In the locally irreducible case the niveau-two exponent is $a + 5b$ with $a + b = 2$. Its projective tame order is three when the two characters are distinct. Here $a = b$ would give scalar inertia and local reducibility.

If wild projective inertia is nontrivial, it is C_5, so its decomposition group lies in the normalizer $C_5 \rtimes C_4$. The inertia group is contained in $C_5 \rtimes C_2$ by the tame bound. This subgroup lies in A_5, since inversion of a five-cycle is a double transposition. The sign quotient would be unramified above five. Therefore wild projective inertia is trivial and the tame projective inertia has order two, as its sign is nontrivial. The absolute ramification index at five is four, with different contribution 3/4.

At $\mathfrak{p}$ the relative inertia has order at most five, and $\mathfrak{p}'$ is unramified. The degree of M over $\mathbf{Q}$ is 240 and its root discriminant satisfies

$$\mathrm{rd}(M) \le 5^{3/4} 11^{2/5} < 8.726.$$

Section 13 gives $\mathrm{rd}(M) > 16.66$. □

Proposition 16.7. *A parallel-weight-two system of determinant cyclotomic, good away from $\mathfrak{p}$, and with one of the local types in Lemma 16.5 at $\mathfrak{p}$, is modular. The same holds with $\mathfrak{p}'$ in place of $\mathfrak{p}$.*

Proof. The character κ_α of Section 16.1 removes a ramified quadratic scalar type, or the quadratic part of the principal-series characters, without introducing any new conductor. After the twist the system is good, special with unramified quadratic twist, or of principal-series inertial order five at $\mathfrak{p}$.

Take its five-adic member r_5, which is crystalline of weight two at the prime over five. The actual residual inertia at $\mathfrak{p}$ is trivial or cyclic unipotent of order five: the principal-series

eigencharacters coalesce, and the special case has tame pro-five unipotent monodromy. If the global residue is reducible, its constituents factor through the narrow ray group modulo the prime over five. This group is C_2: the local residue units and signs form $\mathbf{F}_5^{\times} \times \{\pm 1\}^2$, and the units $-1, \phi$ have image of order eight there, with $\phi \bmod 5 = 3$. Thus the constituents are $1, \bar{\chi}_5$. The corresponding rank-one finite-flat models are etale and multiplicative by Raynaud's classification at $e = 2 < 5 - 1$. Closing a stable residual line in the torsion gives such a model. As in Proposition 16.3, connectedness and duality then force the actual source to be ordinary. Its unramified quotient reduces to one, and the residual ratio $\bar{\chi}_5$ cuts out the abelian extension $\mathbf{Q}(\zeta_5)/K$, is odd and is locally nontrivial. Skinner–Wiles [66, Section 4.5, Theorem A] modularizes the actual source.

In the irreducible case Lemma 16.4 gives irreducibility over $K(\zeta_5)$.

Lemma 16.6 excludes the characteristic-five exception. At $\mathfrak{p}$ apply Lemma 4.9 with $(q, p) = (11, 5)$. Keep the old crystalline weight-two component at five and good conditions away from $\mathfrak{p}$. Lemma 2.6(i) gives a weight-two system. Purity preserves the special condition when it was imposed. Its dyadic member contradicts Proposition 16.3. This excludes the irreducible branch for the original normalized system. Undoing the twist proves the assertion. □

16.4 The terminal eleven-adic source

Completion for $d = 5$ in Theorem 1.1. Start with Proposition 2.7, including the exceptional initial lift at five when required. In the nonsolvable case apply Proposition 6.2 with $\ell = 11$ and $v_+ = \mathfrak{p}$. If a terminal source remains, its reducible and irreducible solvable residues are covered by Proposition 6.5, using Lemma 6.8 for the constituent quotient.

For nonsolvable residue, retain the crystalline weight-two component at $\mathfrak{p}'$ and choose a weight-two lift at $\mathfrak{p}$ from Lemma 16.5. Lemma 2.6(i) gives a comparison system good away from $\mathfrak{p}$. Proposition 16.7 proves its modularity. The common residual witness gives modularity of the old eleven-adic member by Proposition 6.5. The backward implications and reduction at the original place complete the proof. □

Remark 16.8. The residual character calculation allows a nonzero global extension with ratio $\bar{\chi}_{11}$ that is split at $\mathfrak{p}'$. For example, take the Kummer class of $\pi = 3 + \phi$. It is unramified away from eleven and nonzero by its valuation at $\mathfrak{p}$. At $\mathfrak{p}'$ one has $\phi \equiv 37 \pmod{121}$ by Hensel's lemma, and $\pi^{10} \equiv 40^{10} \equiv 1 \pmod{121}$. A unit of $\mathbf{Q}_{11}$ is an eleventh power exactly when its tenth power is one modulo 121, so the local class is zero. Thus the global residual character calculation permits local splitting. The mixed theorem used above allows the actual ordinary source to be split.

17 The field $\mathbf{Q}(\sqrt{13})$

Throughout this section let $K = \mathbf{Q}(\sqrt{13})$.

The earlier proof for K of Dieulefait and Pacetti [27] uses GRH in the residual base. We replace that base by a characteristic-three to characteristic-two comparison. The two places above three admit weight-two lifts that are quadratic twists of crystalline or semistable representations. In the two-special case, the dyadic point field has a guaranteed norm-nine contribution to the classical explicit formula. The scalar dyadic case uses ordinary lattice rigidity with the unramified ordinary quotient held fixed.

17.1 Quadratic arithmetic over K

Write

$$w = \frac{1+\sqrt{13}}{2}, \qquad w' = 1 - w, \qquad \epsilon = w + 1.$$

Then $w^2 = w + 3$, $ww' = -3$, and $N_{K/\mathbf{Q}}(\epsilon) = -1$. The two triadic primes are $\mathfrak{p} = (w)$ and $\mathfrak{p}' = (w')$, and their completions are $\mathbf{Q}_3$. At two, $w^2 - w - 3$ is irreducible modulo two, so $K_2/\mathbf{Q}_2$ is unramified quadratic. Its residue field is $\mathbf{F}_4$, generated by $\bar{w}$, and $\bar{\epsilon}$ generates $\mathbf{F}_4^\times$. The ordinary class number is one by the Minkowski bound $\sqrt{13}/2 < 2$. The units -1 and ϵ realize all four real signatures, so the narrow class number is also one.

The unit ϵ generates the units modulo $\{\pm 1\}$. Indeed, for a unit $u = (a + b\sqrt{13})/2$ in the first real embedding with $1 < u < \epsilon$, its conjugate is $\pm u^{-1}$. Both $u - u^{-1}$ and $u + u^{-1}$ are strictly less than $\epsilon + \epsilon^{-1} = \sqrt{13}$. Thus $0 < b < 1$, which is impossible.

Lemma 17.1. *A characteristic-three character unramified at all finite places away from three has square one. For a reducible residue of determinant $\bar{\chi}_3$ and this support, the constituent ratio is $\bar{\chi}_3$.*

Proof. The character kills the principal units at the two triadic places. Since the ordinary class group is trivial, its ray quotient is a quotient of $(\mathbf{F}_3^\times)^2 \times \{\pm 1\}^2$, which has exponent two. For constituents θ and $\bar{\chi}_3\theta^{-1}$ the ratio is therefore $\bar{\chi}_3$. □

Let V be the group of squareclasses of K with even valuations at every finite place outside $\{\mathfrak{p}, \mathfrak{p}'\}$, including at two. We use additive notation for this elementary abelian group. By the class and unit calculations, a basis is $[-1], [\epsilon], [w], [w']$.

Lemma 17.2. *The kernel of localization $V \to K_2^\times/K_2^{\times 2}$ is $\langle [-3] \rangle$. The two classes $[-w]$ and $[-w']$ are unramified at two and ramify at the two triadic places independently. The simultaneous localization*

$$V \longrightarrow K_{\mathfrak{p}}^\times/K_{\mathfrak{p}}^{\times 2} \oplus K_{\mathfrak{p}'}^\times/K_{\mathfrak{p}'}^{\times 2}$$

is injective.

Proof. A dyadic unit a defines a quadratic extension unramified or split at two exactly when $a^3 \equiv 1 \pmod{4\mathcal{O}_{K_2}}$. Indeed a^3 represents the same squareclass and is one modulo two. The unramified quadratic class has a representative $1 + 4u$, from the discriminant of a lift of an irreducible Artin–Schreier polynomial. A square of a unit congruent to one modulo two is one modulo four. Conversely a class represented by $1 + 4u$ is given by $X^2 - X - u$, which is separable modulo two. The map $a \mapsto (a^3 - 1)/2$ modulo two is additive on these squareclasses. Its values on $-1, \epsilon, w, w'$ are $1, \bar{w}, 1, 1$. Its kernel in V is generated by $[-w], [-w']$.

The element $u = (\epsilon + \sqrt{-w})/2$ is integral, with polynomial $X^2 - \epsilon X + \epsilon$, since $\epsilon^2 + w = 4\epsilon$. Modulo two this polynomial is irreducible. After dividing the variable by $\bar{\epsilon}$, its Artin–Schreier constant is $\bar{\epsilon}^{-1} = \bar{w}$, of trace one. Thus $[-w]$ is the nontrivial unramified local class, and the same holds for $[-w']$ by conjugation. Their product is $[-3]$. Each has odd valuation just at its own triadic prime, proving the assertions at two and the independent ramification at three.

Finally, a class split at both triadic places lies in $\langle [-1], [\epsilon] \rangle$ by the two valuations. At the places where w is $0, 1$ modulo three, the quadratic residue vectors of $-1, \epsilon$ are $(-,-)$ and $(+,-)$, respectively. These are independent. □

Lemma 17.3. *For every nonzero class $[a] \in V$, the ordinary ray class group of $K(\sqrt{a})$ modulo (2) has two-power order.*

Proof. Up to conjugation over $\mathbf{Q}$, the nontrivial classes are the nine in the following table. Here $t = \sqrt{a}$, $L = K(t)$, and the right column lists all prime-ideal norms at most the Minkowski bound. A bound on $|D_L|$ follows from the relative discriminant of $X^2 - a$, which divides $(4a)$. For $-w$ use the integral generator u from Lemma 17.2, and for -3 use ζ_3. The middle column rounds the Minkowski bound upwards.

a	$\lvert D_L\rvert$ at most	r_2	Minkowski bound below	prime norms
-1	2704	2	8	4
ϵ	2704	1	7	$3, 4$
w	8112	1	11	$3, 4$
$-w$	507	1	3	none
ϵw	8112	0	9	$3, 4$
$-\epsilon w$	8112	2	14	$3, 4, 13$
-3	1521	2	6	$3, 4$
3	24336	0	15	$3, 4, 13$
3ϵ	24336	1	19	$3, 4, 17$

These norm lists follow by splitting in K first, then testing a in the residue fields. The primes $5, 7, 11$ are inert in K. At 13 the residue is $w = 7$, and at 17 the two residues are $w = 5, 13$. For ϵw the other three-place is inert and its norm-nine prime exceeds the Minkowski bound. The fields $K(\sqrt{-w})$ and $K(\sqrt{-3})$ are the only representatives unramified at two by Lemma 17.2. The first is inert and the second split over K_2. In each other case the unique prime of norm four squares to the principal ideal (2) in L.

A ramified prime of norm three has class of order dividing two because its prime below in K is principal. For $a = \epsilon, w$ the split norm-three prime is generated by $t-1$, of absolute norm three, and conjugation gives the other. For $a = -\epsilon w$ the split norm-three primes have two-primary classes because $|N_{L/\mathbf{Q}}(t-1)| = 12$, with the norm-three factor at the split place. This gives a relation with the unique norm-four prime. The remaining prime classes are accounted for by the following elements, with the indicated absolute norms:

a	element	$\lvert N_{L/\mathbf{Q}}\rvert$
$-\epsilon w$	$2\epsilon + t$	39
3	$2 + wt$	13
3ϵ	$1 + t$	17
3ϵ	$1 - w + t$	51.

The last two elements lie over different primes of K: their relative norms are $-2 - 3w$ and $1 - 4w$, respectively. Conjugation over K covers the other primes of each split pair. For $a = -3$ put $z = \zeta_3$. The integer $-3 - 2w - (2 + w)z$ has relative norm $4(4 + 3w)$ and absolute norm sixteen. It vanishes at exactly one dyadic prime, since modulo two the two possible residues of z are $\bar{w}, 1 + \bar{w}$. Hence it generates the square of one norm-four prime and conjugation gives the other. These relations show that every ordinary class group in the table has two-power order.

In the dyadically ramified cases, the odd part of the residue units modulo (2) has order three and is generated by $\bar{\epsilon}$. For $a = -w$, the residue of u generates $\mathbf{F}_{16}^{\times}$: its norm to $\mathbf{F}_4$ is $\bar{\epsilon}$, and its residue lies outside $\mathbf{F}_4$. For $a = -3$, the dyadic residue unit group is $(\mathbf{F}_4^{\times})^2$. The images of ϵ and z generate it, since one is diagonal and the other consists of inverse primitive cube roots. Thus the ray exact sequence introduces no odd quotient. □

17.2 The characteristic-three base

Lemma 17.4. *Every continuous residual representation of $G_{\mathbf{Q}_3}$ with determinant $\bar{\chi}_3$ admits a lift of weights $\{0, 1\}$ and determinant χ_3 which is a quadratic twist of a crystalline or a semistable representation.*

Proof. In the reducible case the diagonal characters have tame inertia 1 and $\bar{\chi}_3$. Twist by the quadratic lift of $\bar{\chi}_3$ if necessary so that the given subcharacter is $\bar{\chi}_3\lambda$, with λ unramified, and the quotient is λ^{-1}. Let η be the unramified Teichmüller lift of λ. If $\lambda^2 \neq 1$, local duality gives $H^2(\mathbf{Q}_3, \mathcal{O}(\chi_3\eta^2)) = 0$, so the actual extension lifts to an integral extension with ratio $\chi_3\eta^2$. Over the coefficient field this extension is crystalline: H^1_f and H^1 each have dimension one by the local Euler characteristic and the crystalline extension formula. If $\lambda^2 = 1$, Kummer theory lifts both coordinates of the residual extension, in the unit and valuation directions. Their characteristic-zero lifts are semistable, with crystallinity equivalent to vanishing of the valuation coordinate. Twist back at the end.

If the residue is irreducible, its wild inertia is trivial and its tame exponents for the niveau-two fundamental character are $\{1, 3\}$ or $\{5, 7\}$ modulo eight. The determinant gives these pairs, since Frobenius multiplies an exponent by three and their sum is four modulo eight. A ramified quadratic twist interchanges the pairs. The first lifts by inducing a crystalline character of the unramified quadratic extension with labelled weights $(1, 0)$. Choose its unramified value to match the full residue. The determinant quotient is unramified and has trivial reduction, so an unramified scalar square root corrects the determinant. Undo the quadratic twist. □

We call a system *bounded* if its determinant is cyclotomic, its weights are $\{0, 1\}$ at both embeddings, and its conductor is supported above three. At each triadic place it has one of the two local conditions in Lemma 17.4. In the semistable case with nonzero monodromy, the local parameter is special, with an exact quadratic scalar character. This follows from the determinant and the ratio of its Frobenius eigenvalues. We refer to the other local condition as twisted crystalline.

Lemma 17.5. *For any dyadic residue of a bounded system, the inertia at each triadic place has order at most two. At a twisted crystalline place it is trivial. At a special place the decomposition image is elementary abelian of order at most four.*

Proof. At a twisted crystalline place the common Weil–Deligne parameter has scalar quadratic inertia and zero monodromy. In characteristic two that scalar reduces to one. At a special place the characteristic-zero dyadic restriction has two characters $\xi\chi_2, \xi$, with ξ quadratic, and is upper triangular in a basis adapted to the saturated special line of any stable lattice. Both diagonal characters reduce to one. Wild inertia is pro-three, so the residual inertia factors through the procyclic tame quotient and has order at most two. The decomposition image is generated by this inertia and one Frobenius. Both are unipotent in dimension two in characteristic two, and they commute. □

Proposition 17.6. *A bounded system with a nonsolvable dyadic residue is modular. A bounded system has no irreducible solvable dyadic residue, and every reducible dyadic residue has trivial semisimplification.*

Proof. In the nonsolvable case the image is $\mathrm{SL}_2(\mathbf{F}_q)$, with q a power of two at least four, since the determinant is one. Let M/K be the kernel field. The dyadic member is crystalline of weight two. Its residual representation is obtained from a finite flat group scheme killed by two, by taking the coefficient-uniformizer torsion of the Barsotti–Tate group realizing its lattice [45, Theorem 4.1]. Fontaine's bound [34, Theorem A] over the unramified dyadic base gives no upper break greater than one. The normalized different at two is therefore less than two. There is no extra base-field different at two.

If at most one triadic place is special, the root discriminant is less than $4\sqrt{13}\,3^{1/4} < 18.98069$ by Lemma 17.5. If both are special, then $\mathrm{rd}(M) < 4\sqrt{39} < 24.98000$. At either special place the relative ramification index e and residue degree f are at most two. In the explicit formula, if $f = 2$ the norm-nine coefficient contributed by that place is $1/(2e) \geq 1/4$. If $f = 1$, retain

the second power of the norm-three Euler factor, whose contribution is the same as that of a norm-nine term with coefficient $1/(2e)$. Thus the two special places supply an effective norm-nine coefficient of at least $1/2$.

For $q \geq 8$ the degree $[M : \mathbf{Q}] = 2q(q^2 - 1)$ is at least 1008. The classical bounds in Section 13 give $\mathrm{rd}(M) > 20.57608$ without prime terms, and $\mathrm{rd}(M) > 25.32625$ with the indicated norm-nine contribution. These exclude $q \geq 8$ in the two cases above. For $q = 4$, a totally real kernel field of degree 120 would have $\mathrm{rd}(M) > 41.69019$, again impossible. Thus at least one residual real conjugation is nontrivial. The small-field witness and Thorne's theorem apply by Lemma 2.4.

An irreducible solvable dyadic residue would be induced from an odd-order character of a quadratic field L/K. Its quadratic quotient at two has upper break at most one by Fontaine and compatibility with quotients. A radicand of odd dyadic valuation has conductor three, as seen from the discriminant of its Eisenstein quadratic equation over the unramified base. The inducing squareclass is therefore in V. The inducing character is unramified away from two: inertia at three has order at most two, and the character has odd order. It is at most tamely ramified at two, so factors through the ray class group modulo (2). By Lemma 17.3 it is trivial, a contradiction.

In the reducible case the two characters have odd order, and are unramified away from two by the local inertia bound. At two their tame inertia factors through $\mathbf{F}_4^\times$, which is generated by the image of the global unit ϵ. The ray sequence and class number one make both characters trivial. □

17.3 The scalar dyadic case

Proposition 17.7. *A bounded system with a scalar-residual dyadic member is modular.*

Proof. Let r be the dyadic member, and let ν denote its unramified ordinary quotient at two. By Lemma 4.8, choose a stable lattice T whose residue is a nonzero additive extension split at two. Its additive extension coordinates are quadratic characters split at two and unramified away from the two triadic primes. Lemma 17.2 makes the extension a nonzero multiple of the additive character $h : G_K \to \mathbf{F}_2$ attached to $K(\sqrt{-3})$.

The character h ramifies at both triadic places. Both actual local types must therefore be special by Lemma 17.5. Twist by the characters attached to $[-w]$ and $[-w']$ as needed to remove the ramified parts of the two quadratic scalar characters. These corrections are unramified at two, preserve the determinant, and leave the dyadic residue unchanged. Work with the resulting system and its ordinary quotient ν. At the two special places the exact quadratic characters ξ_1, ξ_2 are now unramified.

Fix a real conjugation c. For the cyclic lattice of the normalized source, the basis $((r(c)-1)v, v)$ is integral because $h(c) = 1$. It gives the markings

$$\bar{r}(g) = 1 + h(g)E_{12}, \qquad r(c) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{ordinary line } e_2.$$

Let σ be the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$ and choose a lift at the dyadic place. It acts trivially by conjugation on the unramified quotient of G_{K_2}. Hence r^σ has the same exact ordinary quotient ν. Normalize r^σ using the same real conjugation. Its residue is the same, since h comes from $\mathbf{Q}$. Its two special characters are ξ_2, ξ_1.

Compare these normalized representations r_1, r_2 over the same coefficient ring, with uni-

formizer ϖ and residue field k. If they agree modulo ϖ^n for $n \geq 1$, write

$$r_2(g) = (1 + \varpi^n X(g)) r_1(g) \pmod{\varpi^{n+1}}, \qquad X(g) = \begin{pmatrix} x(g) & y(g) \\ z(g) & x(g) \end{pmatrix}.$$

Equality of determinants gives the displayed trace-zero form. The adjoint action of $1 + h(g)E_{12}$ adds $h(g)z$ to both diagonal entries and the upper right entry, so z is additive. At two, the ordinary line and exact quotient give $x = y = 0$. The local class of z is finite flat. Indeed the difference of the two integral ordinary extension cocycles divided by ϖ^n remains integral and its rational class lies in the linear space of crystalline extensions of the same two characters. The resulting crystalline lattice of weights $\{0, 1\}$ is again Barsotti–Tate [45, Theorem 4.1]. Its reduction is the lower unipotent extension with coordinate z. Fontaine's bound on each quadratic coordinate therefore rules out odd dyadic valuation.

At either triadic place the special line is determined by the kernel of $r_i(T_v) - 1$ for a tame generator T_v with $h(T_v) = 1$. The upper right entry is a unit, so congruence of the matrices modulo ϖ^n gives congruence of the lines modulo ϖ^n. Write the divided variation of the line, normalized by its first coordinate, as $t_v e_2$. If the exact special characters agree modulo ϖ^{n+1}, their line equations yield

$$z|_{G_{K_v}} = 0, \qquad x|_{G_{K_v}} = t_v h|_{G_{K_v}}.$$

If their signs differ, these equations still hold for $n < v_\varpi(2)$. At $n = v_\varpi(2)$ the right side of the second equation has the additional term $u_0 U_v$, where U_v is the nontrivial unramified quadratic character in additive notation and $u_0 = \overline{2/\varpi^n} \neq 0$. This follows by subtracting the equations for the special eigenline with character $\chi_2 \xi_i$.

Thus z belongs to $V \otimes_{\mathbf{F}_2} k$ and is split at both triadic places. Lemma 17.2 implies $z = 0$. Now x, y are additive and vanish at two, hence are multiples of h. The fixed matrix at c forces both to vanish. This proves the homogeneous comparison at every stage. In the inhomogeneous case the vanishing of x would make the nontrivial unramified character $u_0 U_v$ a multiple of $h|_{G_{K_v}}$, which is ramified. This is impossible. Starting modulo ϖ, induction therefore gives equality of the signs and then $r^\sigma \simeq r$.

By compatibility the same isomorphism holds for an odd-adic companion, which we take at five. This companion extends to $G_{\mathbf{Q}}$: an intertwiner for a lift of σ can be scaled, by Schur's lemma, so that its square is the matrix of the square of that lift. The extension is absolutely irreducible, finitely ramified and odd, and is regular de Rham since this condition descends through finite local extensions. Pan's Theorem 1.0.4 [54] makes it modular. Quadratic base change is cuspidal because its restriction is irreducible. This proves modularity of the normalized system and hence of its original quadratic twist. □

Theorem 17.8. *Every absolutely irreducible characteristic-three representation of determinant $\bar{\chi}_3$, unramified outside three, is modular over K.*

Proof. A solvable image has a regular modular witness. Otherwise the cyclotomic restriction is absolutely irreducible, since reducibility on an index-two subgroup would make the image dihedral up to scalars. At the two triadic places use the actual lifts of Lemma 17.4. With determinant χ_3 and good conditions elsewhere, Lemma 2.6(i) gives a pure compatible system. On a selected special component purity ensures nonzero monodromy. Thus this is a bounded system. Propositions 17.6 and 17.7 give modularity of its dyadic member in every possible case, and compatibility gives modularity of the original characteristic-three residue. □

17.4 Completion of the conductor reduction

Completion for $d = 13$ in Theorem 1.1. The irreducible supported determinant-cyclotomic base is Theorem 17.8. By Lemma 17.1, the constituent quotient of a reducible residue is $\bar{\chi}_3$. It

is totally odd and extends to $G_{\mathbf{Q}}$. Two is inert and three splits. Corollary 6.6 applies. In its conductor reduction both triadic types are retained; the final lifting steps are Zhang's Theorems 6.1.1 and 5.1.1 as specified in Section 6. □

18 The classical split-dyadic cases

18.1 The field $\mathbf{Q}(\sqrt{17})$

By Lemma 14.5, it suffices to consider a kernel field L of group $\mathrm{SL}_2(\mathbf{F}_q)$ with $q \geq 8$. The local bound of Lemma 14.3 gives $\mathrm{rd}(L) \leq 4\sqrt{17}$. In degree at least 1008 the arbitrary-signature row of Lemma 13.4 gives $\mathrm{rd}(L) > 20.70004 > 4\sqrt{17}$, a contradiction.

18.2 The residual base over $\mathbf{Q}(\sqrt{41})$

In this subsection $d = 41$. A special dyadic place means an unramified quadratic twist of Steinberg with nonzero monodromy. We first treat systems special at both dyadic places, then construct the congruences to these systems.

Lemma 18.1. *Let M/K have Galois group $\mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^a \geq 4$, and be unramified at finite places away from two. Both dyadic places have nontrivial wild inertia, and at most one of their decomposition groups is elementary abelian.*

Proof. Lemma 14.3 gives $\mathrm{rd}(M) \leq 4\sqrt{41}$. If wild inertia is trivial at one place, the upper bound improves to $\sqrt{41}\,2^{4/3}$. This contradicts the first comparison in Appendix C.2, since $[M : \mathbf{Q}] \geq 120$.

Suppose both groups are elementary abelian. Their local fields are contained in E_2, so $e \leq 4$ and $f \leq 2$. For $q = 4, 8$ use $L = M^U$ as in Lemma 13.9. Each place gives a norm-two coefficient r_0 and an effective norm-four coefficient at least $(1 - r_0)/8$, retaining even powers of the free block if its residue degree is one. The upper bound is $\sqrt{41}\,2^{2(1-r_0)}$. For $q \geq 16$, use M with a total effective norm-four coefficient $1/4$. The three remaining comparisons in Appendix C.2 give the contradictions. □

Lemma 18.2. *Let $\mathcal{R}$ be a potentially modular weight-two system of cyclotomic determinant, good away from two and special at both dyadic places. Then $\mathcal{R}$ is modular. The same holds for a system special at one dyadic place and good elsewhere if some dyadic member has solvable residue.*

Proof. At a special place the actual dyadic member is a nonsplit extension of an unramified sign by its cyclotomic twist, as in Lemma 4.2. Both characters reduce to one, so the residual decomposition group is elementary abelian for any stable lattice. With two special places, Lemma 18.1 forces solvable residue.

For reducible residue, the residual semisimplification is $1 \oplus 1$ by Lemma 14.1. The trivial ratio extends to $G_{\mathbf{Q}}$, and condition (S) holds by Lemma 14.2 and Table 1. We use [21, Theorem 9.4(b)] in this case and [21, Theorem 10.6] for irreducible solvable residue. The actual source is absolutely irreducible and totally odd over the abelian field K, completely split at two. At each special place it is de Rham of distinct weights, reducible and nonsplit.

In the one-special case, suppose the actual restriction at the good dyadic place splits. Its two crystalline characters have weights zero and one. Pass to a totally real field of potential modularity. Compatibility identifies the chosen dyadic member there with the corresponding member of the modular system. Special monodromy survives restriction and, by local–global compatibility at an odd coefficient prime, gives a square-integrable finite component. This is

the hypothesis of [86, Theorem 1] when the degree of the field is even; it provides an abelian variety of GL_2-type through a Shimura curve. The same monodromy excludes CM, since an induced character has zero monodromy. At the good place, the form has unit Hecke trace by [65, Theorem 1], since the two crystalline slopes are zero and positive. Zhao's theorem [86, Theorems 1 and 5.6] gives indecomposability, contrary to the split restriction. Thus the actual good restriction is irreducible or nonsplit reducible, in the two local cases of [21]. The stated source theorems apply, and compatibility gives modularity of the system. □

Lemma 18.3. *Let $\mathcal{R}$ be a weight-two system of cyclotomic determinant good at every odd place. Fix a coefficient place of odd characteristic ℓ whose residual inertia at both dyadic places is unipotent. If its residue is reducible, the system is modular. If its residue is irreducible, it remains so on $G_{K(\zeta_\ell)}$.*

Proof. In the reducible case write the constituents as $\psi, \overline{\chi}_\ell \psi^{-1}$. They are unramified outside ℓ at finite places. Use reciprocity on the totally positive unit ϵ^2, where ϵ is from Table 1 and $\epsilon^2 - 1 = 64\epsilon$. At a split ℓ the rank-one finite-flat exponents are zero or one. The two residues of ϵ^2 are inverses, so unequal exponents would make ϵ^2 trivial in the residue field. At an inert ℓ the mixed niveau-two exponents $1, \ell$ are excluded by the same identity. At $\ell = 41$, the niveau-one exponents are $0, 1, 2$, with cyclotomic exponent two. The exponent one is excluded by the unit relation. Thus ψ has either trivial inertia or cyclotomic inertia at all ℓ-places. The narrow class number is one, so the constituents are $1, \overline{\chi}_\ell$. By Lemma 4.3(i) the actual source is ordinary and its unramified quotient lifts the trivial constituent. The ratio $\overline{\chi}_\ell$ is odd and nontrivial at every coefficient place, and $K(\zeta_\ell)/\mathbf{Q}$ is abelian. These are the hypotheses of [66, Theorem A].

For an irreducible residue at an unramified $\ell > 3$ apply Lemma 4.4. At $\ell = 3$ or 41, a reducible cyclotomic restriction would give induction from the quadratic subextension of $K(\zeta_\ell)/K$, with image of order prime to ℓ. At $\ell = 41$ the inducing extension is ramified. The projective tame inertia is cyclic and contains a reflection, so has order two. The rank-one exponents $0, 1, 2$ give projective order one or twenty. For a locally irreducible residue, the niveau-two digits lie in $\{0, 1, 2\}$ and sum to two by the determinant. They are unequal since scalar inertia would force local reducibility in the prime-to-41 image. The order is then twenty-one, again a contradiction.

At $\ell = 3$ the inducing field would be $H = K(\zeta_3)$. The inducing character and its conjugate agree on tame inertia at three, because H/K is ramified quadratic there with unchanged residue field. The unipotent inertia at two dies in an image of order prime to three. Their quotient is unramified at all finite places of H. This field has discriminant 15129 and Minkowski bound $369/(2\pi^2) < 19$. The only prime ideals of norm at most eighteen are the two above two of norm four and the one above three of norm nine. Indeed $5, 11, 17$ are inert in $\mathbf{Q}(\zeta_3)$ and $7, 13$ are inert in K. The generators are α, its conjugate and $1 - \zeta_3$. Hence H has class number one. The inducing character equals its conjugate, a contradiction. □

Lemma 18.4. *In the setting of Lemma 18.3, suppose the actual residual representations at both dyadic places admit special lifts of determinant χ_ℓ. Then $\mathcal{R}$ is modular.*

Proof. The reducible case follows from Lemma 18.3. In the irreducible case it gives cyclotomic irreducibility. At $\ell = 5$ the exceptional projective S_5 field cannot contain $K(\zeta_5)$, of degree four over K, since S_5 has abelianization of order two. Prescribe the special components at two and keep the good components at the ℓ-places. Lemma 2.6(i) gives a potentially modular weight-two system, good away from two. The prescribed Frobenius ratio two and purity keep the special monodromy nonzero at both places. Lemma 18.2 gives modularity of the new system. Its ℓ-adic member supplies the weight-two witness in Lemma 2.2(ii) for the actual old member. □

Proposition 18.5. *A nonsolvable determinant-one dyadic residue over $\mathbf{Q}(\sqrt{41})$ unramified outside two is modular.*

Proof. Lemma 18.1 leaves at least one triangular local image with distinct unramified diagonal characters. The distinct-character calculation in Proposition 9.1 gives an ordinary crystalline weight-two lift of the actual local extension, of determinant χ_2.

Suppose first that the characters are distinct at both places. Choose τ_0 at both using Proposition 9.1 and apply Lemma 2.6(iii). This gives a weight-two system $\mathcal{R}$ lifting the residue, with the two prescribed potentially nonordinary Barsotti–Tate types and good at all odd places. In a three-adic member the two primitive order-three inertia characters at each dyadic place reduce to one, so the residual inertia is unipotent. Frobenius exchanges the inertia eigenlines and has trace zero. Thus Lemma 4.5(i) gives a special lift of each actual local residue, including the nontrivial unipotent extensions. Apply Lemma 18.4 in characteristic three.

Suppose now that one dyadic decomposition group is elementary abelian. The Kummer construction in the proof of Proposition 9.1 lifts its actual scalar extension to an ordinary semistable weight-two representation with nonzero valuation coordinate and trivial special sign. Choose the special component through this point and the ordinary crystalline component at the other place. Both are ordinary of weight two, so the semistable case of Lemma 2.6(iii) gives a potentially modular system $\mathcal{R}$ lifting the residue. It is good away from the marked special place, where purity keeps monodromy nonzero.

If some dyadic member of $\mathcal{R}$ has solvable residue, Lemma 18.2 gives modularity. Otherwise every dyadic member has a nonsolvable residue with elementary decomposition at the special place. By Lemma 18.1 the two residual diagonal characters at the good place are unramified and distinct at every dyadic coefficient prime. The actual source there is ordinary by Lemma 4.3(ii). For the common Frobenius trace a the unit root u satisfies $a = u + 2/u$ and $\bar{u} \neq 1$. Thus a is a unit and $\bar{a} \neq 1$ at every dyadic coefficient prime. Integrality at odd coefficient primes follows from stable lattices. Lemma 4.5(ii) gives an odd coefficient prime and a special lift of the residual representation at the good dyadic place. Retain the existing special component at the other place. Since the residual inertia at both is unipotent, Lemma 18.4 applies. In all cases modularity of $\mathcal{R}$ gives modularity of the starting residue. □

Completion for $d = 17, 41$ in Theorem 1.1. The residual base at two follows from the preceding argument for $d = 17$ and Proposition 18.5 for $d = 41$, together with Lemma 14.5. The class group and condition (S) are verified in Section 14. Both radicands are prime. For condition (i) of Theorem 10.1, apply [21, Theorem 9.4(b)] with both dyadic places in Σ_n. The split-dyadic criterion gives the result. □

19 The field $\mathbf{Q}(\sqrt{6})$ with a zero-free input

Put $K = \mathbf{Q}(s)$, $s = \sqrt{6}$, and $\epsilon = 5 + 2s$. We write $\omega_\ell = \bar{\chi}_\ell$. A special local parameter is a scalar quadratic twist of the Steinberg parameter with nonzero monodromy. The dyadic place is ramified and five splits.

19.1 Reduction to a one-place source

Proposition 19.1. *Assume Theorem 6.4. To prove Theorem 1.1 it suffices to modularize parallel-weight-two systems of cyclotomic determinant special at just one place above five and good everywhere else. In any nonmodular system of this kind the three-adic and five-adic residues are nonsolvable. At each dyadic coefficient place either the residue is scalar up to semisimplification and the source is nonordinary, or the residue is nonsolvable and the source is ordinary with residual Frobenius multiplier different from one. At least one dyadic coefficient place is of the latter kind.*

19.2 Local types and cubic classes

For $K = F_6$ or F_{11}, a bounded system is pure, totally odd, absolutely irreducible, of cyclotomic determinant and labelled weights $\{0, 1\}$, good outside five. Each five-adic type is quadratic-special, denoted SP, or tame determinant-one cuspidal, denoted SC, with inertial eigencharacters of order three or six. A special condition retains its exact quadratic character, including the unramified sign, and has nonzero monodromy. Its characters are $\alpha\chi_p, \alpha$ with α quadratic. At a coefficient-five place its monodromy-stable line gives the special filtration by weak admissibility. In the common arithmetic and ordinary argument below, a good place at five is allowed as well. Its residual inertia is trivial, so it satisfies the same character constraint. The later two-place source theorems use the stated SC, SP types.

19.2.1 Characters and cubic classes

Write

$$\epsilon = 5 + 2s, \qquad a = 1 + s, \qquad b = 1 - s, \qquad \pi = 3 + s, \qquad \gamma = 3 + 2s, \qquad H = K(\zeta_3).$$

Then $N(\epsilon) = 1$, $ab = -5$, $N(\pi) = 3$ and $N(\gamma) = -15$. Let σ be the nontrivial automorphism of $K/\mathbf{Q}$ and put $\omega = \bar{\chi}_3$. We identify quadratic characters with squareclasses when convenient.

Lemma 19.2. *The class number of K is one. The quadratic classes supported at three and five and unramified at two form the group generated by*

$$[-3] = [-\epsilon], \quad [5], \quad [\gamma].$$

For a reducible $\bar{r}_3$ with determinant ω arising from a bounded source, its constituents are quadratic. Up to interchange and twisting by the character of $K(\sqrt{5})/K$, their pair is either $\{1, \omega\}$ or $\{\psi_\gamma, \psi_\gamma\omega\}$.

Proof. The Minkowski bound is $\sqrt{24}/2 < 2.45$, and the prime of norm two is generated by $2 + s$, whose norm is -2. The unit ϵ is fundamental, by the least positive solution $(5, 2)$ of $x^2 - 6y^2 = 1$; norm -1 is impossible modulo three. The squareclasses supported at three and five have basis $[-1], [\epsilon], [\pi], [a], [b]$.

All five generators are dyadic units. Such a unit u gives an unramified quadratic extension, possibly split, precisely when u is a square modulo $4\mathcal{O}_{K_2}$. A square of a unit $A + Bs$ has coordinates $(1, 0)$ or $(3, 2)$ modulo four, according as B is even or odd. Enumerating the thirty-two products gives exactly eight allowable classes, generated as stated. The identity $3/\epsilon = (3 - s)^2$ identifies $[-\epsilon] = [-3]$.

Let a residual constituent be ψ. At five, its square is unramified because semisimplified residual inertia of the bounded types is scalar quadratic. At three, a finite characteristic-three character kills principal units and the residue field is $\mathbf{F}_3$, so its square is unramified there too. It is unramified elsewhere. The narrow class group has order two; its nontrivial character is ω, which is odd at both real places. A square character is even, so $\psi^2 = 1$. The preceding eight classes give the two pairs. Finally, $K_3 = \mathbf{Q}_3(\zeta_3)$ because -2 is a square in $\mathbf{Q}_3$. The first pair is locally unramified, while both characters of the second pair are ramified quadratic. □

Let $F = K_3$, normalize $v_F(s) = 1$, and put $U_i = 1 + \mathfrak{p}_F^i$.

Lemma 19.3. *The global S-unit cubeclass space for $S = \{v : v \mid 15\}$ has basis $[\epsilon], [\pi], [a], [b]$ and maps isomorphically to $F^\times/F^{\times 3}$. The preimage of U_2/U_4 is*

$$V = \langle q, d \rangle, \qquad q = [5], \qquad d = [5] + [\epsilon b].$$

We have $\sigma(q) = q$, $\sigma(d) = -d$. In this basis the two valuations at five are $(1, 1)$ and $(1, -1)$, in some order.

Proof. Since $\mu_3 \subset F$, the class space has dimension four. Moreover $U_1 = \mu_3 U_2$, and the logarithm identifies cubing $U_2 \to U_4$ with multiplication by three. Hence a unit is a cube exactly when it is congruent to 1 or -1 modulo $9\mathcal{O}_F$.

The π exponent is detected by valuation. Of the twenty-seven products $\epsilon^i a^j b^k$, $0 \le i, j, k < 3$, only the trivial one is ± 1 modulo nine. This finite calculation takes place in $(\mathbf{Z}/9\mathbf{Z})[s]/(s^2 - 6)$ and proves injectivity, hence the asserted isomorphism. Modulo three, $\epsilon = -(1 + s)$, $a = 1 + s$, $b = 1 - s$ and $s^2 = 0$. Membership in $\{\pm 1\}U_2$ is therefore the equation $i + j - k = 0$. This plane is generated by $[5] = [ab]$ and $w = [\epsilon b]$. Since $N(\epsilon b) = -5$, we have $\sigma(w) = q - w$. The assertions for $d = q + w$ and the valuations follow. □

Lemma 19.4. *The three-part of the ray class group modulo $\mathfrak{p}_3^3$, with or without the real places, is C_3. The two Frobenius classes at five coincide and generate it. The involution σ acts trivially on its three-part.*

Proof. The unit group of $\mathcal{O}_K/\mathfrak{p}_3^3$ has order eighteen. The images of -1 and ϵ generate a subgroup of order six, giving C_3. Here $\mathfrak{p}_3^3 = (9, 3s)$. The residues of a and b belong to neither the image of the units nor its trivial class: the unit residues are $\pm 1, \pm(5 + 2s), \pm(4 + 2s)$ in this quotient. Thus both ray classes are nontrivial. The identity $a - \epsilon^2 b = 72 + 30s \in (9, 3s)$ makes their classes equal. The involution σ exchanges the two primes and fixes their common generating class, hence acts trivially on the three-part. Infinite places add only two-primary data. □

Lemma 19.5. *No absolutely irreducible characteristic-three residue of a bounded source becomes reducible over H.*

Proof. The field $H = \mathbf{Q}(\sqrt{-2}, \sqrt{-3})$ has discriminant 576 and class number one. Indeed its Minkowski bound is $36/\pi^2 < 4$. There is no prime of norm two. If $A^2 = -2$ and $u^2 - u + 1 = 0$, the element $1 + Au$ has relative norm $-1 + A$ over $\mathbf{Q}(A)$, hence absolute norm three. Its conjugate gives the other prime of norm three.

A cyclotomic-reducible irreducible residue would be $\mathrm{Ind}_H^K \xi$ with $\xi^c = \xi^{-1}$. Its image has order prime to three. At five, its inertia is scalar quadratic, so ξ^2 is unramified. The two primes of H over three have residue field $\mathbf{F}_3$, so ξ^2 is unramified there as well. It is unramified elsewhere, hence trivial by the class number. Then $\xi = \xi^c$, contradicting irreducibility of the induction. □

19.2.2 Ordinary sources and an unramified quotient

We prove the ordinary part of Theorem 33.25 without a local-distinctness assumption at three. Such an assumption would fail here because $\omega|_{G_F} = 1$. The argument permits distinct five-adic types and also detects a good place paired with a ramified place.

Proposition 19.6. *Suppose that the residual constituents are unramified at three. Then the actual system is modular. After the quadratic normalization of Lemma 19.2, it is invariant under σ. Consequently neither a genuinely mixed* SC/SP *source nor a source ramified at just one five-place can have this residual branch. Two special conditions in this branch have the same exact character after normalization.*

Proof. By Lemma 19.2, a twist by ψ_5 makes the residual pseudorepresentation $1 + \omega$. This quadratic character is invariant under σ and unramified at three.

Lemma 4.3(ii) makes the crystalline source at F ordinary. Write its unramified ordinary quotient as ν; the subcharacter is $\chi_3 \nu^{-1}$.

Put $A = \mathbf{Z}_3[[U]]$ and let ψ_U be the unramified character of G_F sending arithmetic Frobenius to $1 + U$. Begin with the universal finite-flat-at-three fixed-determinant Cayley–Hamilton pair for $1 + \omega$, unramified outside three and five, provided by [76, Theorems 2.5.3 and 2.5.5], and extend its coefficients to A. Impose, for every $g, h \in G_F$, the algebra relations

$$(g - \chi_3(g)\psi_U(g)^{-1})(h - \psi_U(h)) = 0. \tag{21}$$

No local type is imposed at five. To construct the Cayley–Hamilton quotient, take the closed two-sided ideal I generated by these relations, then the scalar ideal generated by the nonconstant coefficients of $D(1 - xt)$, $x \in I$, and use [76, Lemma 2.4.5]. Thus no coefficient-A version of the printed representability theorem is assumed. Denote the scalar ring by R, its maximal ideal by $\mathfrak{m}$, and its GMA by $\left(\begin{smallmatrix} R & B \\ C & R \end{smallmatrix}\right)$. The actual source factors through it with $U = \nu(\Phi_3) - 1$, since upper triangular multiplication proves (21).

The offdiagonal cotangent duals embed into the finite-flat global extension spaces by [76, Theorem 4.3.5]; scalar extension and further quotients retain these inclusions at $3 = U = 0$. Locally such an extension is unipotent. Fontaine's bound gives classical upper break at most two, hence cubic conductor at most three. A nonunit Kummer class instead has conductor four: after replacing the radicand by an inverse and a cube, $X^3 - u$ is Eisenstein with $v_F(u) = 1$, and its different exponent is eight. Thus both global extension spaces lie in $V_3 = \langle [\epsilon], [a], [b] \rangle$. Their restriction to G_F is injective by Lemma 19.3. For larger coefficient fields the same conclusion follows by extracting $\mathbf{F}_3$ coordinate extensions by pullback and quotient.

Let J be the reducibility ideal and put

$$Q = \mathfrak{m}/(3R + UR + \mathfrak{m}^2), \qquad Q_{\mathrm{red}} = \mathfrak{m}/(3R + UR + \mathfrak{m}^2 + J).$$

For $g, h \in G_F$ both residual diagonal characters equal one. The upper-left entry of (21), reduced in Q, is $b_g c_h = 0$; its other term is a product of elements of $\mathfrak{m}$. Evaluations on all of G_F span the dual of each allowed global extension space, by local injectivity. Hence every product in $(B/\mathfrak{m}B) \otimes (C/\mathfrak{m}C)$ has zero image in Q. These products generate the image of J, so $Q \simeq Q_{\mathrm{red}}$.

A diagonal tangent relative to A has characters $1 + \eta f$ and $\omega(1 - \eta f)$, with $\eta^2 = 0$. Finite-flatness and Fontaine give conductor at most $\mathfrak{p}_3^3$ for f. It cannot ramify at five: wild inertia is pro-five and the tame unit quotient has order four. It is unramified elsewhere by construction. Therefore the dual of Q_{red} is a subspace of $\mathrm{Hom}(\mathrm{Cl}_{\mathfrak{p}_3^3}(K), \mathbf{F}_3)$. The involution σ acts trivially on this space by Lemma 19.4. It consequently acts trivially on Q; the diagonal tangent need not itself vanish.

The involution σ preserves the support, finite-flat condition, determinant and ordinary relations. It fixes U, because the unramified Frobenius quotient of G_F is unchanged by the automorphism of $F/\mathbf{Q}_3$. Thus it acts on R over A. Its action on Q is the identity. Lemma 12.2, applied over A, makes the involution σ trivial on R. Thus $r_3^{\sigma} \simeq r_3$.

Lemma 12.4, with $\eta = 1$, now proves modularity of the original system.

Finally, σ exchanges the two places above five, and invariance identifies their local Weil–Deligne representations. Their monodromy cannot be zero at one place and nonzero at the other. It also cannot have trivial inertia and monodromy at only one place. For two special types it also preserves the exact quadratic character. This proves the asserted exclusions. □

19.2.3 Ramified quadratic constituents

The other pair in Lemma 19.2 requires a different finite-flat extension space.

Lemma 19.7. *For either ramified quadratic character θ of G_F, the finite-flat selfextensions of θ form exactly the Kummer plane U_2/U_4.*

Proof. The rank-one model has Breuil–Kisin exponent one. An extension has Frobenius matrix

$$\begin{pmatrix} cu & h(u) \\ 0 & cu \end{pmatrix}.$$

The height-one condition imposes no further restriction. A change of upper lift changes h by $cu(f(u^3) - f(u))$. Positive powers are removed recursively, while constants in f disappear. The extension space has dimension at most two, represented by $1, u$. Schematic closure and flat quotient give the rank-one endpoints; the exact finite-flat description used here is [47, Theorem 1.1.3, Proposition 1.1.11 and Lemmas 1.2.4–1.2.5]. Coordinate pullback and quotient give the same bound for arbitrary coefficients.

The unramified selfextension is finite flat and gives U_3/U_4. A generator is $1+6s$: substituting $\alpha = 1 + sz$ in $\alpha^3 = 1 + 6s$ gives $z^3 + (s/2)z^2 + (1/2)z - 1 = 0$, whose reduction $z^3 - z - 1$ is irreducible and separable over $\mathbf{F}_3$.

For an independent class take $E : y^2 = x^3 - \frac{s}{8}x^2 + x$. Its discriminant is $-125/2$, and its reduction is supersingular. Its division polynomial factors as $\psi_3(x) = (x - 1/s)(3x^3 + 6x + s)$. At $x = 1/s$, the ordinate square is $s(5/12)^2$, so the stable subgroup has character θ_s. Cardano's radicand for the other three lines is $2/(3s)$, whose product with s^3 is four. Hence its actual extension line is [2]. It lies in $U_2 \setminus U_3$ after multiplying by the cube -1. An unramified quadratic twist gives the other ramified character; specifically $\gamma/(2s) = 1 + s/4$ is a local square. These two lines exhaust U_2/U_4. □

Proposition 19.8. *A weight-two comparison system with just one five-place of type from Lemma 4.14 and good everywhere else is modular if its three-adic residue is reducible with ramified constituents at three.*

Proof. Write $a = 1 + s$, $b = 1 - s$, and $\gamma = 3 + 2s$. After conjugating K if necessary, the two constituents are $\psi_\gamma, \psi_\gamma\omega_3$, where ψ_γ is the quadratic character for γ. Indeed the other possible pair is its twist by ψ_5, which is also its quadratic conjugate up to interchange. The unique bad five-place is the one where ψ_γ ramifies. The common local character at three is ramified because $K_3 = \mathbf{Q}_3(\zeta_3)$.

Let V be the finite-flat global cubic extension space with both five-places allowed. By Lemmas 19.3 and 19.7 it is contained in the Kummer plane spanned by $[5]$ and $[5] + [\epsilon b]$. The two five-adic inertia evaluations are independent on this plane. Use the fixed-determinant finite-flat Cayley–Hamilton problem of Proposition 19.6, over $\mathbf{Z}_3$ without the parameter U. In this problem the two off-diagonal tangent duals embed into this plane. Imposing goodness at the other five-place puts both spaces in the kernel of its evaluation.

For a special source the exact special characters at the bad place forbid ramification in one of the two orientations, since the two residual Frobenius characters are distinct there. Thus one off-diagonal space also lies in the kernel of the second evaluation, and is zero. The generalized matrix algebra and Nakayama then make the characteristic-zero source reducible, a contradiction.

For a cuspidal source the bad place has inertial order six, since its residual quadratic scalar is ramified. After removing that fixed scalar locally, the relation $(T - 1)^2 = -3T$ holds for tame inertia and Frobenius has trace zero. In the relative pseudocotangent the inertia relation kills the product of the two off-diagonal evaluations at this place. Indeed, modulo three and the square of the maximal ideal, the diagonal entries of $(T - 1)^2$ reduce to that product. The evaluation at the bad place is nonzero on the line cut out by the good place. Thus the product relation kills the entire image of the reducibility ideal in the relative cotangent. The diagonal quotient comes from the three-part of the ray modulo $\mathfrak{p}_3^3$, which by Lemma 19.4 is C_3 generated by a five-Frobenius. The trace-zero condition at the bad five-place kills this tangent, because

the residual diagonal Frobenius values there are opposite. The resulting complete local scalar ring is a quotient of $\mathbf{Z}_3$ and has an actual characteristic-zero point, so it is $\mathbf{Z}_3$. This argument uses the actual finite-flat condition of Wake–Wang-Erickson [76, Theorems 2.5.3, 2.5.5 and 4.3.5].

The CM curve

$$E: \quad y^2 = x^3 + \gamma^3(7+3s)^2/4$$

gives a point of this same problem. It is the γ-twist of $y^2 + (7+3s)y = x^3$, with the displayed residual pair from the three-torsion line at $x = 0$. Its conductor is the square of the bad five-place, where it has type II*, and it is good at two, three and the other five-place. These conductor and local reduction calculations are checked by the CM seed certificate. The determinant-one cuspidal type at the bad place is the unique order-six pair in the bounded menu. This fixes the inertia type, and the exact Frobenius conditions are trace zero and square -5. Thus its trace point agrees with the unknown trace point. It follows that the actual unknown system is modular.

□

19.2.4 Scalar dyadic residues

Lemma 19.9. *For a parallel-weight-two system of cyclotomic determinant, special at one five-place and good elsewhere, every solvable dyadic residue has trivial semisimplification. A scalar-residual ordinary dyadic member is impossible. If every dyadic residue is scalar up to semisimplification and every dyadic member is nonordinary, the system is modular.*

Proof. Consider a system special at just one five-place v. In characteristic two its tame inertia at v has order at most two and its decomposition group lies in C_2^2. For a solvable irreducible dyadic residue the inducing field is quadratic and the odd inducing character is unramified at v, tame at two and unramified elsewhere. Fontaine's bound gives dyadic conductor at most four for the quadratic field. Thus its squareclass is one of the seven nontrivial classes in $\langle [-1], [\epsilon], [1+s] \rangle$ after conjugating the base if needed. All seven ray groups modulo (2) are two-primary by the quartic certificate in the source archive. No irreducible solvable residue occurs. Reducible dyadic characters of odd order are unramified everywhere because the dyadic residue field is $\mathbf{F}_2$ and the inertia at five has order at most two. Therefore a solvable dyadic residue is scalar.

If a scalar-residual dyadic member is ordinary, take a cyclic stable lattice generated by a vector on its ordinary line. The reduction is a nonsplit unipotent extension, with global invariant line different from the reduction of the ordinary line. Both lines are locally preserved at two, so the actual residual restriction there is trivial. Its quadratic extension coordinates must therefore be split at two. The kernel of localization on the two-five-place space $\langle [-1], [\epsilon], [1+s], [1-s] \rangle$ is generated by $[-15]$ by the local square test modulo $8\mathcal{O}_{K_2}$: the squares of odd units modulo eight have coordinates $(1,0), (1,4), (7,2), (7,6)$ in the basis $1, s$, and only $1, [-15]$ in the indicated span are locally squares. But $[-15]$ ramifies at both five-places. This excludes an ordinary scalar source with one special place.

It remains to consider a system whose dyadic residues are all scalar and whose dyadic members are all nonordinary. Use the simultaneous trace and regular orders R_0, E_0 of Section 11 and the proof of Lemma 11.6. Put $R = R_0/2R_0$, $E = E_0/2E_0$, and $J = \mathrm{rad}(E)$. A functional on J/J^2 gives a quadratic character with dyadic conductor at most two by the middle finite-flat model calculation in Lemma 11.2. For one five-place the allowed space has basis $\alpha = [-1], \beta = [\epsilon] = [2]$. Indeed an odd valuation at two has conductor five, and the unit square test modulo 2 rules out the exponent of the generator $1+s$ of v. The two allowed characters are unramified at both five-places.

Let A be the difference of a real conjugation from one, and B that of a Frobenius at v. Their evaluations on (α, β) are $(1, 0)$ and $(0, 1)$. Thus they generate E as an $\mathbf{F}_2$-algebra. The real involution and the exact special parameter give

$$A^2 = B^2 = 0, \qquad AB + BA = X = T(AB),$$

with all identities in the actual order modulo rational two. For B, the special Frobenius eigenvalues have common sign and are $\pm 5, \pm 1$, so its trace and determinant have the required reductions even if its quadratic character is ramified. The normal words $1, A, B, AB$ over $\mathbf{F}_2[X]$ have traces $0, 0, 0, X$, so $R = \mathbf{F}_2[X]$. The image of X at every coefficient residue is zero since the reductions are upper unipotent. The product of the dyadic coefficient integer rings is integral over R_0. Lying over therefore puts X in the radical of the finite algebra R.

By Lemma 11.6 the determinant of the Frobenius generator $\Phi(u)$ is divisible by the Eisenstein polynomial. For a functional λ on J/J^2 the algebra quotient $E \to \mathbf{F}_2[\delta]/(\delta^2)$ with δ-coefficient λ on J gives a middle-model self-extension. In that quotient $\Phi(u)$ has diagonal term u and off-diagonal term $h(u)$ as in Lemma 11.2. Its constant coefficient $h(0)$ records the ramified part of the restricted character. Both α and β are ramified, and their sum $[-3]$ is unramified at two. Thus $\Phi(0)$ and $A + B$ agree under every λ, giving $\Phi(0) = A + B + Z$ with $Z \in J^2$ in the order modulo two.

Compute the trace and determinant values in $R/\mathfrak{m}_R^2$. The normal words put J^2 in the span of AB and $\mathfrak{m}_R E$. The latter terms have zero norm and zero polar pairing because the traces lie in $\mathfrak{m}_R$. The term AB has zero norm and zero polar pairing with $A+B$, since $(A+B)AB = XB$ and $T(B) = 0$. Hence the constant norm is $\mathcal{D}(A + B) = X$. It vanishes by divisibility. Since $R = \mathbf{F}_2[X]$ and $X \in \mathfrak{m}_R^2$, Nakayama gives $R = \mathbf{F}_2$. The global trace field is $\mathbf{Q}$.

At a coefficient place over five an irreducible residue with rational traces has a realization over $\mathbf{F}_5$ by finite-field descent. It is modular by [3, Theorem 2.5]. Its restriction to $G_{K(\zeta_5)}$ is irreducible by Lemma 6.3 at the good five-place. The actual weight-two source is then modular. If the five-adic residue is reducible, the source is ordinary at the special place and also at the good weight-two place by the finite-flat argument. [28, Corollary 1.2.2] applies to the dual with determinant χ_5^{-1}, because $K/\mathbf{Q}$ is abelian and five is unramified. This argument also applies to any special one-place system with globally reducible five-adic residue. An irreducible solvable five-adic residue is handled by the same cyclotomic irreducibility check and its Artin witness.

□

Lemma 19.10. *For a nonmodular system as in Lemma 19.9, a nonsolvable dyadic residue has an ordinary source with residual Frobenius multiplier different from one.*

Proof. By Dickson's classification the kernel field M/K of the residue has group $\mathrm{SL}_2(\mathbf{F}_q)$, where $q \geq 4$ is a power of two, and $[M : \mathbf{Q}] = 2q(q^2 - 1)$. At the special five-place inertia has order at most two, so its contribution to $\mathrm{rd}(M)$ is at most $5^{1/4}$. At two Lemma 11.7 gives normalized contribution less than two. In particular $\mathrm{rd}(M) < 4\sqrt{24}\, 5^{1/4} < 30$. If $q = 4$, the classical real-place lower bound of Section 13 shows that M is not totally real. Lemma 2.4 proves modularity.

Suppose $q \geq 8$. The degree is at least 1008, so the classical lower bound of Section 13 is greater than 20. If the actual source is nonordinary, its normalized dyadic different is less than one. Indeed, a locally irreducible residue is tame of inertia order three by the argument of Lemma 4.6. In the reducible case its diagonal characters are unramified, since the dyadic residue field is $\mathbf{F}_2$. After unramified base change the residue is an additive extension of trivial characters. Its $\mathbf{F}_2$-coordinate extensions have biconnected finite-flat models by schematic closure and quotient. Lemma 11.2 bounds their conductor by two, giving the claimed different bound. Thus $\mathrm{rd}(M) < 15$, a contradiction.

For an ordinary source with trivial residual multiplier the unramified quotient character reduces to one, and the local residue is an elementary abelian group of translations. Its quadratic characters have conductor at most four by Lemma 11.7, so all correspond to unit squareclasses. In a ramified quadratic completion with residue field $\mathbf{F}_2$, these form a space of dimension three. The conductor-at-most-two subspace has dimension two, by Lemma 11.3. The unramified subspace has dimension one. The conductor–discriminant formula therefore gives normalized different at most $5/4$: among the three nontrivial ramified inertia characters, the largest possible conductors are $4, 4, 2$. It follows that $\mathrm{rd}(M) \le 4\sqrt{24}\,5^{1/4}2^{-3/4} < 20$, a contradiction. □

Proof of Proposition 19.1. Start with Proposition 2.7. In the nonsolvable case, Proposition 6.2 with $\ell = 5$ leaves a terminal member with marked places v_+, v_-. The reducible terminal constituent ratio is an odd cyclotomic power by Lemma 6.7, so the reducible and irreducible solvable branches are covered by Proposition 6.5.

For nonsolvable residue, apply Lemma 4.14 at v_+ and keep the crystalline weight-two component at v_-. Lemma 2.6(i) gives a weight-two comparison system, whose modularity returns to the actual old five-adic member by Lemma 2.2(i).

If the comparison system's residue at a place above three is reducible, Proposition 19.6 or Proposition 19.8 applies according to the local constituent characters. If it is irreducible, it remains so over $K(\zeta_3)$ by Lemma 19.5. In the solvable case an Artin witness and Lemma 2.2(ii) prove modularity. For a nonsolvable residue, Lemma 4.15 prescribes a genuine special lift at v_+, retaining the full residual local representation. Retain the crystalline weight-two component at three and the good condition at v_-. Lemma 2.6(i) gives a comparison system special at v_+ and good everywhere else. Purity forces nonzero monodromy on the special component as in Section 2. Lemma 2.2(ii) gives the backward implication.

For a special one-place system the same analysis applies to every coefficient place at three. At five a reducible global residue makes the actual source ordinary at both places, by the special filtration and the finite-flat argument. The dual source has determinant χ_5^{-1} and ordinary weight $(0, 0)$ in the convention of [28, Corollary 1.2.2], which applies over this totally real abelian field unramified at five. An irreducible solvable residue gives an Artin witness, and Lemma 6.3 at the good five-place gives the cyclotomic irreducibility needed by Lemma 2.2(i).

Finally Lemmas 19.9 and 19.10 give the stated alternatives at each dyadic coefficient place. If all such places had scalar semisimplification, they would all be nonordinary and the simultaneous-order calculation would prove modularity. □

19.3 The switch at two

Let a_2 be the common crystalline Frobenius trace at the dyadic place of a nonmodular system left by Proposition 19.1. At an ordinary dyadic coefficient place its quotient has unit-root value u and $a_2 = u + 2/u$, with $\bar{u} \neq 1$ since the residual multiplier is nontrivial. At a nonordinary dyadic coefficient place both crystalline Frobenius eigenvalues have positive valuation, since a slope-zero factor would give an unramified quotient. Thus $\bar{a}_2 \neq 1$ at all dyadic coefficient places, and a_2 is a unit at an ordinary one. These local formulas also give integrality at dyadic coefficient primes. At odd coefficient primes integrality follows from the stable lattices. Lemma 4.5 gives an odd coefficient prime λ dividing $9 - a_2^2$ and a special lift of its residue at two. Write ℓ for its residue characteristic.

At $\ell = 3, 5$ the original residue is nonsolvable. A reducible restriction to the cyclotomic subgroup would make the projective image solvable by Clifford theory and the abelian quotient. At five the exceptional case of [31, Definition 4.1.1] would require the projective S_5 field to contain $K(\zeta_5)$. The latter has degree four over K, whereas S_5 has abelianization of order two.

For $\ell > 5$ suppose the residue has constituents $\psi, \omega_\ell\psi^{-1}$. The character ψ^2 is unramified

outside ℓ at finite places since the special scalar character at v is quadratic. The fundamental unit $\epsilon = 5 + 2s$ is totally positive and has norm one. At an inert ℓ, a mixed pair of finite-flat digits would give tame exponent 1 or ℓ for a constituent. Reciprocity on ϵ for ψ^2 would then imply $\epsilon^2 \equiv 1 \pmod{\ell}$. This is impossible since $N_{K/\mathbf{Q}}(\epsilon^2 - 1) = -96$. At a split ℓ the rank-one finite-flat exponents are already the two endpoints. By Lemma 4.3(i) the actual source is ordinary at every ℓ-place. Its dual satisfies Dotto–Thorne [28, Corollary 1.2.2] with determinant χ_ℓ^{-1} and ordinary weight $(0, 0)$.

For an irreducible residue at $\ell > 5$, Lemma 4.4 gives cyclotomic irreducibility. Retain the original weight-two components at ℓ and the exact special condition at v, and choose the special lift at two. Impose the good conditions elsewhere. Lemma 2.6(i) supplies a weight-two system. When $\ell = 5$ we keep the component through the original potentially semistable weight-two point at v. Purity gives nonzero monodromy at the special places. The new system is special at two and v, with unramified special sign at two. If it is modular, its ℓ-adic member supplies the weight-two residual witness in Lemma 2.2(ii) for the actual original member.

19.4 Sources special at two and at one five-place

After conjugating the labelling if necessary, let $v = (1 + s)$, and put $\pi = 2 + s$ and $a = 1 + s$. They generate the dyadic prime and v since their rational norms are -2 and -5. The class number of K is one by Minkowski. Its units are generated by $-1, \epsilon$. These facts follow from the continued fraction of s and the Minkowski bound $\sqrt{24}/2$.

At two and v the residual decomposition images of a dyadic member r are elementary abelian groups of translations. In each case the actual special filtration has characters reducing to one. At v the inertia subgroup has order at most two. All other finite places are unramified.

Lemma 19.11. *If the residual image of r is solvable, then the system is modular.*

Proof. In the reducible case its constituents have odd order and are unramified at every finite place, so are trivial by class number one. The actual image is pro-two. The supported global squareclasses have basis

$$[-1],\ [\epsilon],\ [\pi],\ [a].$$

Localization at two is injective. Valuation parity removes $[\pi]$ from a local square. The norm of a is -5, a nonsquare in $\mathbf{Q}_2$, while the norms of $-1, \epsilon$ are squares. Finally $\pi^2 = 2\epsilon$, and the three nontrivial rational classes $-1, 2, -2$ remain nonsquare in $\mathbf{Q}_2(\sqrt{6})$. Dyadic decomposition thus surjects onto every pro-two quotient with the prescribed support. Indeed, a proper image in a finite two-group lies in an index-two subgroup and gives a nontrivial quadratic class split at two. In particular the actual global image equals its dyadic decomposition image. The special line would then be globally invariant, contrary to absolute irreducibility.

In the irreducible solvable case the image is dihedral and the rotation character has odd order. It is unramified everywhere on the inducing quadratic field, and trivial on the primes over two and v, since the decomposition images there are two-groups. The fifteen quadratic fields are given by the nontrivial supported squareclasses above. The ideal-relation certificate of Appendix C.3 bounds the odd part of the class group after killing the classes above two and v. It is trivial except possibly for $K(\sqrt{-\pi a})$, where its order divides three. This inducing field is CM, since πa is totally positive, and ramifies at two by odd valuation. Allen's Lemma 5.1.2 and Theorem 5.2.1 [2] now apply to the actual ordinary source. Its parallel weights are $\{0, 1\}$, its ordinary quotient is unramified and the inducing CM field is nonsplit at the dyadic place. Compatibility concludes. □

Lemma 19.12. *Assume the zero-free half-plane $\Re s > 0.9586$ for Dedekind zeta functions, apart from the pole at one. A source as in this section has solvable dyadic residue.*

Proof. If its residue is nonsolvable, its kernel field M/K has group $\mathrm{SL}_2(\mathbf{F}_q)$ with $q = 2^j \geq 4$. Write (e, f) for its relative dyadic ramification index and residue degree. The local point field lies in the full elementary quadratic extension of K_2. This has degree sixteen, residue degree two and ramification index eight. With $v(2) = 1$, the relative different contribution is at most D_e, where

$$(D_1, D_2, D_4, D_8) = (0, 5/4, 7/4, 15/8).$$

The sixteen local characters have conductors $0, 0, 2, 2, 4, 4, 4, 4$ and eight conductors five. Odd valuation gives conductor five, units have conductor at most four, and the conductor-at-most-two space is the character group of $K_2^\times/(K_2^{\times 2}(1 + 2\mathcal{O}_{K_2}))$, of order four. For $e = 2$ use conductor at most five. For $e = 4, f = 1$ at most two of the three nontrivial characters have odd valuation. For $e = 4, f = 2$ at most four characters have odd valuation and two of the other four are unramified. For $e = 8$ either the entire space occurs or the dimension-three subspace meets the conductor-at-most-two space in a nonzero class. These facts and the conductor–discriminant formula give the displayed bounds. At v the contribution to the absolute root discriminant is at most $5^{1/4}$, since $e_v \leq 2$ and v has degree one over five. Therefore

$$\mathrm{rd}(M) \leq \sqrt{24}\, 2^{D_e} 5^{1/4}.$$

We describe the Euler terms used in the comparisons in Appendix C.1. A term of norm P has coefficient $c_P = 2N_P/n$, where N_P counts primes of norm P and n is the absolute degree of the field being evaluated. A residue-degree-one prime may supply its even powers as a term of squared norm and half coefficient.

For the small q ranges let $L = M^U$ and $r_0 = 1/(q + 1)$ as in Lemma 13.9. This gives the real fraction and multiplies both relative different contributions by $1 - r_0$. At v the fixed block gives $c_5 = r_0$ and the free block gives $c_{25} \geq (1 - r_0)/4$. At two the terms are $c_2 = r_0 + (1 - r_0)/e$ if $f = 1$, and $c_2 = r_0$, $c_4 = (1 - r_0)/(2e)$ if $f = 2$. Thus the rows for L handle $q = 4, 8$, the case $q = 16$ with $e = 8, f = 1$, and $e = 8, f = 2$ for $16 \leq q \leq 256$.

For $q \geq 16$ and $e \leq 4$ use MH for $H = K(\zeta_3)$. The extension H/K is unramified at finite places and is unramified quadratic locally at two and at v. It is disjoint from M by simplicity. Thus the degree doubles, the root discriminant is unchanged, and one may take $c_4 = 1/8$ and $c_{25} = 1/4$. For $q \geq 32$ with $f = 1$ use M with $c_2 = 1/8$ and $c_{25} = 1/4$.

Only $q \geq 512$ with $e = 8, f = 2$ remains. If the bad five-place has residue degree one or has trivial inertia, the indicated classical comparisons on M apply. Otherwise its local field is the full elementary quadratic extension of K_v. Use the elementary abelian extension

$$V = K(\sqrt{-1}, \sqrt{\epsilon}, \sqrt{\pi}, \sqrt{a}).$$

It has degree sixteen over K by independence of the supported squareclasses. Its local extensions at two and v are absorbed by the two full elementary extensions in M. Both fields are unramified at all other finite places. Simplicity of the group over K gives $M \cap V = K$. Hence MV has degree $16[M : \mathbf{Q}]$, root discriminant $\mathrm{rd}(M)$, and coefficients $c_4 = 1/16, c_{25} = 1/4$. At $b = 150$ the interval certificate with $a_0 = 0.4586$ gives $\mathrm{rd}(MV) > 26.87542$, against the upper bound 26.87073. The bound increases with degree, so this row handles the entire tail. □

Completion for $d = 6$ in Theorem 1.1. The half-plane in Lemma 19.12 follows from Theorem B.6 below. Lemma 19.11 then proves modularity of the comparison system. The backward implication of Section 19.3 and the one-place reduction give the assertion. □

20 The field $\mathbf{Q}(\sqrt{33})$

Lemma 14.5 leaves kernel fields L with group $\mathrm{SL}_2(\mathbf{F}_q)$, $q \geq 8$. Use M and k as in Lemma 14.4.

For $d = 33$, put $b = 24$. If $k \geq 1$, retain just the first norm-four term from Lemma 14.4. At degree at least 4032, (12) gives $\mathrm{rd}(M) > 23.31$, whereas (20) gives $\mathrm{rd}(M) \leq 4\sqrt{33} < 22.979$. If $q = 8$ and $k = 0$, the same formula without prime terms gives $\mathrm{rd}(M) > 21.75 > 21.07115$, contradicting (20). For $k = 0$ use the original kernel field L: Lemma 14.3 gives $[L : \mathbf{Q}] = 2q(q^2-1)$ and $\mathrm{rd}(L) \leq \sqrt{33}\, 2^{2-2/q}$. The resulting unconditional comparisons are

q	$[L : \mathbf{Q}]$	lower $\mathrm{rd}(L)$	upper $\mathrm{rd}(L)$
16	8160	> 21.97	< 21.07115
32	65472	> 22.15	< 22.00405

Only $k = 0$, $q \geq 64$ remains without the additional zero-free input.

Assume now ZF(31/32), as in Theorem 1.1. The preceding unconditional comparisons leave only $k = 0$, $q \geq 64$. For these cases $n = [M : \mathbf{Q}] = 8q(q^2 - 1) \geq 32640$. Use $a = 15/32$, $b = 30$, Lemma 13.6 and (15) in Proposition 13.2, discarding every prime term. It gives

$$\mathrm{rd}(M) \geq 8\pi e^{\gamma} \exp\left(-\frac{13}{20} - \frac{3\pi^2}{4 \cdot 30^2} - \frac{4J_{30}(-1/32)}{n}\right) > 23.06000 > 4\sqrt{33}.$$

Degree monotonicity covers the whole remaining range, contradicting (20). The remaining irreducible residual cases are thus excluded under ZF(31/32). The class group and scalar Selmer calculations are in Lemmas 14.1 and 14.2. Theorem 1.4 supplies the scalar source, and Theorem 10.1 completes the proof for $d = 33$ in Theorem 1.1. The finite strip constants and comparisons are certified in [83, `analytic/all_q_discriminants.py`].

Part III
Fields under GRH

21 Nonsolvable dyadic sources

The following ray auxiliary is used only over F_3; over F_2 a quadratic auxiliary suffices. Its construction and local data are separated from the common source argument.

Proposition 21.1. *Let $K = \mathbf{Q}(\sqrt{3})$. Assume GRH for Dedekind zeta functions. There is a finite Galois solvable extension A/K with $[A : \mathbf{Q}] = 6144$ having the following properties:*

1. *A/K is unramified away from two and thirteen;*
2. *its dyadic classical upper ramification groups are trivial above three;*
3. *at both primes over thirteen its completion is the full tame biquadratic extension of $\mathbf{Q}_{13}$, with $(e, f) = (2, 2)$.*

In particular, if M/K is a Galois extension unramified outside two and thirteen, with dyadic upper breaks at most three and with each thirteen-adic completion contained in that biquadratic field, then $L = AM$ has

$$\mathrm{rd}(L) < 8\sqrt{39}, \qquad \frac{2N_{169}(L)}{[L : \mathbf{Q}]} = \frac{1}{2}.$$

Here $N_{169}(L)$ counts the primes of absolute norm 169.

Proof. Put $\epsilon = 2 + \sqrt{3}$, $\pi = 4 + \sqrt{3}$, $\pi' = 4 - \sqrt{3}$, and let $B = K(\sqrt{-1}, \sqrt{\epsilon}, \sqrt{\pi}, \sqrt{\pi'})$. The four squareclasses are independent: the valuations at the two thirteen-primes separate π, π', and the real signs separate -1 from the nonsquare positive unit ϵ. Alternatively the explicit successive compositа below verify the degree. Thus $[B : K] = 16$, and B/K is elementary abelian and unramified away from two and thirteen. We have $\epsilon = (1 + \sqrt{3})^2/2$ and $\pi\pi' = 13$.

There are two primes of B above two, each with absolute $(e, f) = (8, 2)$; there are eight primes above thirteen, each with $(e, f) = (2, 2)$. The latter completions are the full biquadratic extensions of $\mathbf{Q}_{13}$: ϵ has the squareclass of the nonsquare unit two, and one of π, π' is a uniformizer at each of the two split places. Thus the unramified and ramified quadratic directions both occur.

We record the dyadic filtration explicitly. In the sixteen-element squareclass group generated above, the counts of relative quadratic conductor exponents $0, 2, 4$ at K_2 are respectively $4, 4, 8$; exponent zero includes the locally trivial classes. The local kernel has order two, so the eight local characters have conductor multiplicities $2, 2, 4$. Thus the local extension has an unramified quadratic part, inertia of order four, and positive upper groups of orders four for $0 < u \le 1$ and two for $1 < u \le 3$. Its inverse Herbrand function satisfies $\psi_{B_w/K_2}(3) = 1 + 2(3 - 1) = 5$. These conductor counts follow directly by taking the dyadic valuation of the relative discriminant of $X^2 - a$ for the fifteen nontrivial products a. They are checked in the accompanying GP certificate; no finite-flat refinement is needed here. The exact calculation is supplied in [52, `ramified/d2`].

Let $\mathfrak{P}_1, \mathfrak{P}_2$ be the two dyadic primes of B and put $\mathfrak{m} = \mathfrak{P}_1^6\mathfrak{P}_2^6$. Take the full ray class field of B for $\mathfrak{m}$ and then its largest subfield A in which every prime over thirteen splits completely over B. There is no infinite part of the modulus, since B is totally imaginary. The modulus and the subgroup generated by all thirteen-prime classes are stable under $\mathrm{Gal}(B/K)$. Consequently A/K is Galois and solvable. It is unramified outside two and thirteen, and its thirteen-adic completions equal those of B.

The ray exponent six implies that the classical upper breaks of A/B at two are at most five. Herbrand's subgroup formula and the displayed inverse Herbrand value imply that the upper breaks of A/K at two are at most three. This step uses the general Fontaine-compatible bound three, not a stronger assertion for nonordinary joint trace orders.

Here are the exact arithmetic data defining the extension. An absolute equation for B is obtained by adjoining i, $\sqrt{2}$ and $\sqrt{13}$ successively to a root of $X^4 - 8X^2 + 13$, reducing the polynomial after each step. The result has degree 32, discriminant $2^{72}3^{16}13^{16}$, and class group C_{12}. The full ray group at $\mathfrak{m}$ is $C_{24} \times C_8 \times C_2^4$. Quotienting by the classes of all eight thirteen-primes gives $\mathrm{Gal}(A/B) \simeq C_{12} \times C_2^4$. Its order is 192, so $[A : \mathbf{Q}] = 32 \cdot 192 = 6144$. The program `verify_d3_aux6144.gp` constructs the degree-32 polynomial and computes these ray-class data under GRH. The exact calculation is supplied in [52, `ramified/d3/aux32`].

For the last assertion, upper-break bounds are preserved under compositum. At two, the relative different contribution over K_2, normalized by $v_2(2) = 1$, is strictly below $(1 + 3)/2 = 2$. Adding the base different contribution one gives an absolute contribution strictly below three. At three the extension is unramified over K, contributing the base exponent $1/2$. At thirteen the tame ramification index two contributes $1/2$. Thus $\mathrm{rd}(L) < 2^3\sqrt{3}\sqrt{13} = 8\sqrt{39}$. Every thirteen-adic completion has degree four over $\mathbf{Q}_{13}$ and residue degree two, so $N_{169}(L) = [L : \mathbf{Q}]/4$, as asserted. □

Proposition 21.2. *Assume GRH. Let (K, ℓ) be either $(F_2, 7)$ or $(F_3, 13)$, and let $\mathcal{R}$ be a cyclotomic weight-two compatible system, good outside ℓ. Over F_2 allow good or quadratic-special type at each seven-prime; over F_3 require quadratic-special type at both thirteen-primes. If some dyadic residue is nonsolvable, the actual system is modular. No dyadic ordinarity assumption is required.*

Proof. Let M/K be the point field of the actual dyadic residue. Its determinant is one, so Dickson's classification gives image $G = \mathrm{SL}_2(\mathbf{F}_q)$ with $q = 2^f \geq 4$. Its coefficient-uniformizer torsion is a finite-flat subgroup of rational two-torsion of the Barsotti–Tate group realizing the actual source. Fontaine's bound gives upper break at most three over K_2. Integration gives normalized relative different strictly below two, since $e(K_2) = 2$. The base dyadic contributions are $3/2$ over F_2 and 1 over F_3; thus the absolute contributions are strictly below $7/2$ and 3. At a special odd place the quadratic scalar disappears and inertia has order at most two. At a good place inertia is trivial. Hence

K	auxiliary A	$\mathrm{rd}(AM)$ upper bound	$2N_{169}(AM)/[AM : \mathbf{Q}]$
F_2	$K(i)$	$8\sqrt{14}$	not used
F_3	Proposition 21.1	$8\sqrt{39}$	$1/2$

For F_2, the biquadratic discriminant formula gives $\mathrm{disc}(K(i)) = 8(-4)(-8) = 256$. Its relative dyadic conductor is two, because $256/8^2 = 4$, and its positive upper break is one. The compositum preserves upper bound three and remains tame of index at most two at seven; no decomposition-group assertion at a good seven-place is needed. This proves the displayed upper bound.

For F_3, special Frobenius has trace ± 14 and determinant thirteen. It squares to one modulo two and commutes with the unipotent inertia. Thus the whole thirteen-decomposition image is a quotient of C_2^2, and its local field lies in the maximal biquadratic extension of $\mathbf{Q}_{13}$. The auxiliary in Proposition 21.1 contains precisely that local field, has dyadic upper bound three, and has absolute degree 6144. That proposition proves the displayed discriminant bound and the norm-169 density. Over F_2, only the inertia bound is used.

The group G is nonabelian simple, so M is disjoint from the solvable auxiliary over K. Apply the common GRH formula with $B_{b,100}$ and the positive compact test F_b. The directed 256-bit

Arb comparisons are

K	n lower bound	b	rd lower bound	prime terms
$F_2,\ q \geq 8$	$4 \cdot 504 = 2016$	7	31.19016847	0
$F_3,\ q \geq 16$	$6144 \cdot 4080 = 25067520$	16.8	50.07905369	N_{169}

These exceed respectively $8\sqrt{14} < 29.93325910$ and $8\sqrt{39} < 49.95998399$. At fixed b and prime density the lower expression increases with degree. They therefore exclude the full displayed ranges, without a nonordinary ramification refinement. The exact calculation is supplied in [52, `ramified/d2`].

For $K = F_3$ and $q = 8$, let σ be the nontrivial automorphism of $K/\mathbf{Q}$. If $M \neq M^\sigma$, their simple groups make the fields linearly disjoint over K, and their product, with group $G \times G$, is disjoint from the solvable auxiliary. The field MM^σ has degree $6144 \cdot 504^2 = 1560674304$, the same root-discriminant upper bound and norm-169 density. The common formula at $b = 20$ gives the directed lower bound 51.79576050, a contradiction. Thus $M = M^\sigma$ in the remaining $q = 8$ case.

In every remaining case some residual real conjugation is nontrivial. Indeed Lemma 11.9 bounds the original finite-flat kernel by $\mathrm{rd}(M) \leq A2^{-2/q}$, with $A = 8\sqrt{14}$ or $8\sqrt{39}$. Both are below $4\sqrt{220}$, so Corollary 13.5(i) excludes a totally real kernel without GRH. If $q = 8$, Lemma 2.5 gives the residual witness. Lemma 2.4 proves modularity in this case and for $q = 4$. □

22 Integral-order rigidity over $\mathbf{Q}(\sqrt{3})$

In this section $K = F_3$, $\ell = 13$ and $\epsilon = 2 + \sqrt{3}$. Every special type retains its exact quadratic scalar character. No global correction of the two signs is required.

22.1 Nonscalar dyadic sources

Proposition 22.1. *Let a pure cyclotomic weight-two system over K be good outside thirteen and quadratic-special at both thirteen-primes. If some dyadic residue is irreducible and solvable, its actual member is ordinary and modular.*

Proof. An irreducible solvable residue is necessarily odd dihedral. Write it as the induction of an odd-order character θ from a quadratic field L/K. Its quadratic quotient is unramified outside two and thirteen. The local finite-flat bound excludes odd dyadic valuation, whose quadratic conductor would be five. Since K has class number one, its possible quadratic squareclasses are in

$$\langle -1, \epsilon, a, b \rangle, \qquad \epsilon = 2 + \sqrt{3}, \quad a = 4 + \sqrt{3}, \quad b = 4 - \sqrt{3}, \quad ab = 13.$$

If the source is nonordinary, Theorem 11.10 makes the inducing dyadic conductor at most two. Corollary 11.4 gives the seven nontrivial possibilities, with masks $1, 6, 7, 10, 11, 12, 13$ in this displayed basis. Their ordinary ray groups modulo (2) are, respectively,

$$1,\ 1,\ C_8,\ 1,\ C_8,\ C_2,\ C_4 \times C_2.$$

The odd rotation character θ is unramified at thirteen because its inertia image there has order at most two, and at two it is tame. Its conductor divides the radical of (2), hence divides (2). Every displayed ray group is two-primary, so θ is trivial. This contradicts irreducibility. Therefore no solvable irreducible residue occurs in the nonordinary case.

For an ordinary source use the conductor-at-most-four list, which includes all fifteen nontrivial classes above. Their ray groups modulo (2) are two-primary except for mask fifteen,

$$L = K(\sqrt{-13\epsilon}) = K(\sqrt{-26}), \qquad \mathrm{Cl}_{(2)}(L) = C_{12} \times C_2 \times C_2.$$

Here the equality of fields follows from ϵ having squareclass 2 in K. This field is CM and is ramified at the dyadic place: its relative quadratic conductor there is four. In particular it does not split dyadically. Allen's ordinary residually dihedral theorem, including its ordinary residual-witness construction, applies to the actual source. This proves the second assertion.

The program `d3_quads13_all.gp` computes these fifteen ray groups, certifying each class group with `bnfcertify`. The exact calculation is supplied in [52, `ramified/d3`]. □

22.2 The scalar envelope

Lemma 22.2. *Put $K = \mathbf{Q}(\sqrt{3})$, $\epsilon = 2 + \sqrt{3}$, $\delta_1 = \epsilon(4 + \sqrt{3})$ and $\delta_2 = \epsilon(4 - \sqrt{3})$, and let*

$$F_0 = K(i, \sqrt{\delta_1}, \sqrt{\delta_2}).$$

This is a Galois extension of K of degree eight. Its absolute discriminant is $2^{24}3^8 13^8$, and its class group is C_2. Let F_1 be its Hilbert class field. Then F_1/K is Galois of degree sixteen. Write $\Gamma = \mathrm{Gal}(F_1/K)$. Its group has generators a, b, c, t with

$$a^2 = b^2 = c^2 = t^2 = 1, \qquad t \text{ central},$$
$$[a, b] = [a, c] = t, \qquad [b, c] = 1.$$

Here b and c can be inertia generators at the two thirteen-primes, and the corresponding decomposition groups are $\langle b, t\rangle$ and $\langle c, t\rangle$. Thus the group is $D_8 \times C_2$.

Let $\mathfrak{P}_1, \mathfrak{P}_2$ be the dyadic primes of F_0. The ray group for the dyadic modulus

$$\mathfrak{P}_1^6 \mathfrak{P}_2^6$$

is C_4. The class of every prime above thirteen is 3 in a common cyclic generator. Its quotient imposing Frobenius order at most two at all these primes is precisely the Hilbert quotient C_2.

Proof. We have $\delta_1\delta_2 = 13\epsilon^2$, and both δ_j are totally positive. Thus $F_0 = K(i, \sqrt{\delta_1}, \sqrt{13})$. The independent real sign and thirteen-adic valuation classes give $[F_0 : K] = 8$. Its dyadic completions have absolute $(e, f) = (4, 2)$; its eight thirteen-primes have $(e, f) = (2, 1)$. The discriminant and these local degrees give absolute dyadic different exponent 6. Subtracting twice the different exponent 2 of $K_2/\mathbf{Q}_2$ gives relative different exponent 2; the local ramification index is two, so its unique positive upper break is 1. Hence the local inverse Herbrand function has $\psi(3) = 1 + 2(3 - 1) = 5$.

The exact arithmetic computation starts from the displayed radical description of F_0. It verifies degree sixteen and absolute discriminant $2^{24}3^8 13^8$, and certifies its class group C_2 by `bnfcertify`=1. Computing the ray group at the stated dyadic modulus gives C_4, with all eight thirteen-prime classes equal to 3. Adding twice those classes to the relation lattice gives C_2, necessarily the Hilbert quotient.

The program `verify_d3_classfield_source.gp` constructs the Hilbert quadratic extension from the certified class field of F_0. Its absolute degree is 32 and its discriminant is $2^{48}3^{16}13^{16}$. Exact Galois computations, restricting through an embedding of $X^2 - 3$, give an order-sixteen group with derived subgroup $\langle t\rangle$. The thirteen-adic inertia generators b, c commute, and an involution a outside $\langle b, c, t\rangle$ has both commutators equal to t. The decomposition groups are $\langle b, t\rangle$ and $\langle c, t\rangle$. These computations use no class group of F_1. The exact calculation is supplied in [52, `ramified/d3`].

The group relation follows; putting $z = bc$ gives a central involution independent of t, with $\langle a, b\rangle \simeq D_8$ and $\Gamma = \langle a, b\rangle \times \langle z\rangle$. □

Lemma 22.3. *Let Γ be the order-sixteen group of Lemma 22.2. In its binary group algebra the quotient by the two-sided ideal generated by*

$$(1 + t)(1 + b), \qquad (1 + t)(1 + c)$$

has dimension ten over $\mathbf{F}_2$.

Proof. Put $z = bc$. The center is $\langle t, z\rangle \simeq C_2^2$, and $\mathbf{F}_2[\Gamma]$ is free of rank four over its central group algebra, with basis $1, a, b, ab$. Write $D = 1 + t$ and $Z = 1 + z$. Both are central and square to zero. The ideal in the statement is generated equivalently by $D(1 + b)$ and DZ, since $c = bz$. The central ideal generated by D has dimension eight. Quotienting it by DZ first leaves dimension four, with basis D, Da, Db, Dab. The remaining relation and its product with a identify $Db = D$ and $Dab = Da$, leaving dimension two. The quotient modulo D has dimension eight. The required total dimension is therefore $8 + 2 = 10$.

The multiplication-table computation in `verify_d3_group_algebra_independent.py` gives the same quotient directly. The exact calculation is supplied in [52, `ramified`]. □

22.3 The characteristic-five inner-twist endpoint

Proposition 22.4 (The quadratic inner-twist source at five)**.** *Put $K = \mathbf{Q}(\sqrt{3})$ and $h = \chi_{-1}$, the character of $K(i)/K$. Let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, parallel-weight-two strongly compatible system over K, with cyclotomic determinant. Assume it is good outside thirteen and has quadratic-special type, with nonzero monodromy, at the two thirteen-primes. Let D be its trace field. Suppose either $D = \mathbf{Q}$, or D is quadratic and its nontrivial automorphism τ satisfies*

$$\mathcal{R}^{\tau} \simeq \mathcal{R} \otimes h. \tag{$*$}$$

Then $\mathcal{R}$ is modular.

Proof. Choose a coefficient place above five and write r for the actual five-adic source and $\bar{r}$ for its semisimple residue. The source is crystalline of labelled weights $\{0, 1\}$ at five. Its residual trace field is contained in $\mathbf{F}_5$ or $\mathbf{F}_{25}$. If $\bar{r}$ is absolutely irreducible, finite-field Brauer descent realizes it over that field. If it is reducible, its two characters are individually defined over that field: coefficient Frobenius cannot interchange them because their values at a real conjugation are the distinct elements $1, -1$.

If $\bar{r}$ is globally reducible, the square of a constituent is unramified away from five: good places give no inertia and each special thirteen-place contributes at most a common quadratic scalar and unipotent inertia. Lemma 11.14 therefore proves actual-source modularity.

Now suppose $\bar{r}$ is absolutely irreducible. If $D = \mathbf{Q}$, or five splits or ramifies in the quadratic field D, its residual field is $\mathbf{F}_5$. The small-field modularity theorem [3, Theorem 2.5] supplies a modular witness. In the remaining case five is inert in D and τ induces the fifth-power automorphism of $\mathbf{F}_{25}$. Reducing $(*)$ gives $\bar{r}^{\phi} \simeq \bar{r} \otimes h$. Lemma 12.5 descends its projectivization to $\mathrm{PGL}_2(\mathbf{F}_5)$, with totally odd determinant squareclass. Moreover $[K(\zeta_5) : K] = 4$, since the unique quadratic subfield of $\mathbf{Q}(\zeta_5)$ is $\mathbf{Q}(\sqrt{5})$. The projective modularity theorem [3, Theorem 1.2(3)] applies. Its Lemma 2.1(2) returns a modular witness for the original linear representation $\bar{r}$, including its scalar character.

Lemma 11.15 proves the required absolute irreducibility over $K(\zeta_5)$. The actual weight-two automorphy lifting theorem [3, Theorem 2.3] now applies to r (or to its dual to match the cohomological normalization): it is finitely ramified, de Rham of weights $\{0, 1\}$ and has a modular, cyclotomically irreducible residue. This proves modularity of $\mathcal{R}$. □

22.4 All scalar dyadic residues: a ten-dimensional group algebra

Put $K = \mathbf{Q}(\sqrt{3})$ and let h be the character of $K(i)/K$. It is unramified at every finite place, nontrivial unramified at two, and trivial at both thirteen-places. Indeed the absolute discriminants of $K(i)$ and K are 144 and 12, and -1 is a square in $\mathbf{Q}_{13}$. A special condition includes its exact quadratic scalar character.

Proposition 22.5. *Let $\mathcal{R}$ be a system over K, with cyclotomic determinant and labelled weights $\{0,1\}$. Suppose it is good outside thirteen and special at both thirteen-places. If every dyadic coefficient member has scalar residual semisimplification, then the actual system is modular. No dyadic ordinarity assumption is needed.*

Proof. We first give a bound for a finite packet of such systems whose exact quadratic special characters agree at each thirteen-place. Apply Lemmas 11 and 11.5 to the whole simultaneous coefficient tuple. Write R_0 for its trace order, E_0 for its actual matrix order, and s for the sum of the degrees of the distinct coefficient orbits. Then $\operatorname{rank}_{\mathbf{Z}_2} R_0 = s, \quad \operatorname{rank}_{\mathbf{Z}_2} E_0 = 4s$. Put $E = E_0/2E_0$. Its joint Galois image G is a finite two-group and spans E, since every coefficient residue has scalar semisimplification. Let L/K be its point field. The left regular lattice is actual Barsotti–Tate. The general Fontaine bound on its rational-two torsion gives upper break at most three over K_2 for the whole field L; no coefficient-uniformizer improvement is used.

At a thirteen-place the integral special matrices have the form

$$\xi(\tau)\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}, \qquad \xi(F)\begin{pmatrix}13 & w\\ 0 & 1\end{pmatrix}, \quad \xi(\tau),\xi(F)\in\{\pm 1\}.$$

Normalize by the common rational signs. The inertia difference is in the actual order and squares to zero. The normalized Frobenius has polynomial $X^2 - 14X + 13$ and hence squares to one modulo two. It commutes with inertia there. Its whole local image is therefore elementary abelian of exponent at most two. Moreover

$$(F-1)(\tau-1) = (\tau-1)(F-1) = 0 \quad \text{in } E. \tag{22}$$

The integral normalized products are zero in one order and twelve times the inertia difference in the other. Exact common signs justify these normalizations in the actual joint order.

Put $\epsilon = 2+\sqrt{3}$, $\delta_1 = \epsilon(4+\sqrt{3})$, $\delta_2 = \epsilon(4-\sqrt{3})$, and $F_0 = K(i, \sqrt{\delta_1}, \sqrt{\delta_2})$. Lemma 22.2 gives $[F_0 : K] = 8$, dyadic ramified break one, and ray group C_4 for all dyadic primes to exponent six. Its quotient killing twice every thirteen-prime class is C_2. The group $H = \operatorname{Gal}(LF_0/F_0)$ is a finite two-group. The compositum has upper dyadic break at most three over K_2, since F_0/K has break one. Herbrand's formula $\psi_{F_0/K}(3) = 1 + 2(3-1) = 5$ bounds the conductors of characters of H^{ab} by six.

At thirteen the local field of L lies in the unique maximal elementary biquadratic extension of $\mathbf{Q}_{13}$. The field F_0 already has ramification index two, so LF_0/F_0 is unramified there and its Frobenii square to one. There is no other finite ramification. Thus H^{ab} is a quotient of the displayed C_2. By the finite-two-group Frattini theorem H itself is cyclic of order at most two. The unique possible quadratic ray extension is the Hilbert class field F_1/F_0, since $\mathrm{Cl}(F_0) = C_2$. Consequently $L \subset F_1$. This does not assume $F_0 \subset L$ or that F_0 contains all quadratic characters of G.

Lemma 22.2 identifies $\Gamma = \operatorname{Gal}(F_1/K)$, of order sixteen. Its thirteen-inertias are generated by b, c, and the corresponding decomposition groups are $\langle b, t\rangle, \langle c, t\rangle$ for one central involution t. Its Frobenius classes are t modulo inertia. Equation (22) makes E a quotient of

$$\mathbf{F}_2[\Gamma]/\big((1+t)(1+b), (1+t)(1+c)\big).$$

By Lemma 22.3 this algebra has dimension ten. We have proved the packet bound

$$4s \leq 10. \tag{23}$$

It bounds the actual matrix algebra and assumes no faithfulness of the scalar action $R_0/2 \longrightarrow E_0/2$.

Let D be the global trace field. Apply the bound also to $\{\mathcal{R}, \mathcal{R} \otimes h\}$. The twist preserves every exact thirteen-character and is unramified at two. An actual h-selftwist would give induction from $K(i)$, contrary to nonzero monodromy at the split thirteen-places. Corollary 12.6 therefore gives a rational trace field, or a quadratic trace field D with nontrivial automorphism ι satisfying

$$\mathcal{R}^{\iota} \simeq \mathcal{R} \otimes h. \tag{24}$$

The hypotheses of Proposition 22.4 are now satisfied: either $D = \mathbf{Q}$, or D is quadratic with the displayed inner twist. That proposition proves actual-source modularity in characteristic five and hence modularity of the system. □

Theorem 22.6. *Assume GRH. Every pure, totally odd, absolutely irreducible, cyclotomic-determinant weight-two strongly compatible system over K, good outside thirteen and quadratic-special at both thirteen-primes, is modular. The exact quadratic scalar signs are arbitrary.*

Proof. If a dyadic residue is nonsolvable, Proposition 21.2 applies. If it is irreducible and solvable, Proposition 22.1 applies. Otherwise every dyadic coefficient residue is reducible. Its characters have odd order and are unramified at thirteen because inertia has order at most two. At two wild inertia is pro-two and the residue unit group is $\mathbf{F}_2^{\times} = 1$, so those characters are unramified there too. They are unramified everywhere else and have trivial real-place action. Class number one makes them trivial. Thus every residue has scalar semisimplification, and Proposition 22.5 proves modularity. □

22.5 The supported residual base at thirteen

Lemma 22.7. *Every local residual representation of $G_{\mathbf{Q}_{13}}$ with cyclotomic determinant admits a weight-two lift of exact cyclotomic determinant whose type is quadratic special or determinant-one tame cuspidal with inertial order seven or fourteen.*

Proof. Write $W(r, b) = \mathrm{Sym}^r \mathbf{F}_{13}^2 \otimes \det^b$. There are fourteen central-trivial weights: $r + 2b = 0$ modulo twelve, with $r \in \{0, 2, 4, 6, 8, 10, 12\}$ and b modulo twelve. For the cuspidal type indexed by $t = 1, \ldots, 6$, with niveau-two character exponent $12t$, the compact-type reduction has constituents $W(2t-2, 1-t) \quad \text{and} \quad W(12-2t, t)$. These twelve weights are distinct and exhaust those with $r < 12$. The two remaining Steinberg weights occur in the scalar semistable compact types. The corresponding crystalline scalar compact types also occur in the cuspidal menu, so if a semistable component has only crystalline points one may choose a cuspidal lift instead. The full fixed-determinant formula [74, Theorem 1.2], with its nonnegative multiplicities therefore gives the required nonempty local lifting ring for the actual residual extension. The inertial orders are fourteen for odd t and seven for even t. No global quadratic correction of these signs is required. □

Lemma 22.8. *Let $\mathcal{R}$ be a determinant-cyclotomic weight-two strongly compatible system over $K = \mathbf{Q}(\sqrt{3})$, good outside thirteen, with the preceding cuspidal or special menu at thirteen. If a seven-adic residue is globally reducible, the actual seven-adic source is modular. If it is irreducible, its cyclotomic restriction remains absolutely irreducible.*

Proof. Suppose first that the constituents are ψ and $\omega_7\psi^{-1}$. At a thirteen-place both finite inertial eigencharacters reduce to the same quadratic sign; any surviving seventh-order inertia is unipotent. Consequently ψ^2 is unramified at thirteen. It is unramified at every other finite place away from seven as well. Lemma 11.14 with $(K, p) = (F_3, 7)$ proves actual ordinarity and modularity, without removing the scalar signs by a global twist. If the residue is irreducible, Lemma 11.15 gives cyclotomic irreducibility. □

Lemma 22.9. *At either thirteen-place, the actual reduction in characteristic seven of any member of the cuspidal/special menu admits a genuine quadratic-special lift of cyclotomic determinant. This includes the case of trivial residual inertia.*

Proof. Retain the exact quadratic scalar inertial character and first remove it locally. Cuspidal Frobenius has trace zero and determinant thirteen, so its residual eigenvalues are $1, -1$, which are distinct modulo seven. Special Frobenius has the same pair, up to an unramified sign. The relation with inertia, since $13 = -1$ modulo seven, places any nontrivial unipotent inertia in one offdiagonal orientation in this Frobenius eigenbasis. Choose $u = \pm 1$ for that orientation and take $T(x) = 1 + txE_{12}, \quad \Phi = \mathrm{diag}(13u, u)$, interchanging the basis vectors if needed. Choose $t \neq 0$ with the required residual value; if inertia was trivial choose t divisible by the coefficient uniformizer. These matrices have determinant one on inertia and thirteen on Frobenius and satisfy $\Phi T(x)\Phi^{-1} = T(13x)$. Restoring the quadratic character gives the asserted actual special lift, with nonzero monodromy. □

Proposition 22.10. *Assume GRH. Every irreducible cyclotomic-determinant residual representation unramified outside thirteen is modular.*

Proof. Solvable residual images have modular Artin witnesses. Otherwise the cyclotomic residual restriction in characteristic thirteen is irreducible. Use Lemma 22.7 and prescribed global lifting to construct a weight-two source, then establish its actual avoiding potential modularity and compatible system. In characteristic seven, a reducible residue gives modularity by Lemma 22.8. For an irreducible residue that lemma verifies the cyclotomic hypothesis. Prescribe the genuine special lifts of Lemma 22.9 at both thirteen-primes and retain the old Barsotti–Tate component above seven. Global lift existence in characteristic seven and actual potential modularity give the replacement system. Purity makes its special monodromy nonzero. Theorem 22.6 modularizes this actual target. Allen–Khare–Thorne then applies backwards to the old actual weight-two seven-adic source. Compatibility and reduction at thirteen return the required original residual witness. □

23 The characteristic-three base over $\mathbf{Q}(\sqrt{7})$

Throughout this section $K = \mathbf{Q}(\sqrt{7})$, $C = K(\zeta_3)$, and $\mathfrak{p}_1 = (2 + \sqrt{7})$, $\mathfrak{p}_2 = (2 - \sqrt{7})$.

Theorem 23.1. *Assume GRH. Every continuous irreducible representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with determinant $\overline{\chi}_3$, unramified outside 3, is modular.*

Lemma 23.2. *The ordinary class number of K is one and its narrow class number is two. The extension $K(i)/K$ is the unique nontrivial quadratic extension unramified at every finite place, and is inert at both places above 3. Every finite character $G_K \to \overline{\mathbf{F}}_3^{\times}$ unramified outside 3 has square one.*

Proof. The Minkowski bound is $\sqrt{7} < 3$. The prime above 2 is principal, generated by $3 + \sqrt{7}$ of norm two, so the ordinary class number is one. The equation $x^2 - 7y^2 = -1$ has no solution

modulo seven. The least positive solution of $x^2 - 7y^2 = 1$ with $y > 0$ is $(8, 3)$, so $\epsilon = 8 + 3\sqrt{7}$ generates the positive units and the narrow class number is two.

The discriminant of the biquadratic field $K(i)$ is $28(-4)(-7) = 28^2$. Thus its relative discriminant over K is trivial. Narrow class field theory gives uniqueness, and its completion at either place above three is unramified quadratic because -1 is nonsquare in $\mathbf{F}_3$.

The character assertion follows from Lemma 4.17. □

Lemma 23.3. *Assume GRH. Retain only a proved number N_3 of primes of norm three, and put $c = 2N_3/[L : \mathbf{Q}]$. The following lower bounds hold when the degree and c are at least the indicated values:*

b	$[L : \mathbf{Q}]$ *at least*	c	$\mathrm{rd}(L)$ *greater than*
21/2	60000	0	37.33078
15/2	2880	0	31.93013
9	19656	2/9	48.00416
23	20 000 000 000	1/9	50.67997.

In particular, every totally imaginary K-Galois field unramified outside three whose completions at both places above three are E_0 satisfies

$$[L : \mathbf{Q}] < 20\,000\,000\,000. \tag{25}$$

Proof. Use the explicit formula of Section 13, discarding the real-place term and retaining only the specified primes of norm three. Its one-term bound for B_b gives

$$\log \mathrm{rd}(L) \geq \gamma + \log(8\pi) - B_b^* - \frac{C_b}{n} + c \log 3 \sum_{m \geq 1} 3^{-m/2} F_b(m \log 3). \tag{26}$$

The prime sum is finite. Outward evaluation gives the four rows; the bound increases with n and c.

For the last assertion, the local degree is eighteen and residue degree one at both primes. Thus $N_3 = n/18$, $c = 1/9$, and the discriminant tower gives

$$\mathrm{rd}(L) = \sqrt{28}\, 3^{37/18} < 50.62072. \tag{27}$$

The last row contradicts this equality for degree at least twenty billion. □

Lemma 23.4. *Assume GRH. A representation as in Theorem 23.1 is modular unless its projective group is*

$$\mathrm{PSL}_2(\mathbf{F}_{3^{2e}}) \quad (e \geq 2), \qquad \textit{or} \qquad \mathrm{PGL}_2(\mathbf{F}_{3^{2e+1}}) \quad (e \geq 1).$$

In every remaining case its projective completion at each place above three is exactly E_0. In the projective general linear case the unique quadratic subfield over K is C.

Proof. Let L and M be the linear and projective kernel fields, with groups G and H over K, and let Z be the scalar subgroup of G. Both fields are totally imaginary, since complex conjugation has eigenvalues $1, -1$. Solvable images are modular. Dickson's classification leaves A_5, $\mathrm{PSL}_2(\mathbf{F}_q)$ and $\mathrm{PGL}_2(\mathbf{F}_q)$ with $q = 3^f \geq 9$.

We first consider $q \geq 27$. If H is perfect, then $|Z| \geq 4$: for $|Z| \leq 2$ the determinant would kill Z and factor through the perfect group H, contradicting oddness; scalar order is prime to three. For $H = \mathrm{PGL}_2(\mathbf{F}_q)$, Z is nontrivial. Otherwise its perfect subgroup $\mathrm{PSL}_2(\mathbf{F}_q)$ would embed in $\mathrm{SL}_2(\overline{\mathbf{F}}_3)$, which has only one nonidentity involution. Therefore $[L : \mathbf{Q}] \geq 78624$ in either case.

If one place above three is tame or has nontrivial unramified factor α, Lemma 4.18 gives $\mathrm{rd}(L) < \sqrt{28}\, 3^{(3/2+37/18)/2} = \sqrt{28}\, 3^{16/9} < 37.30737$, contrary to the first row of Lemma 23.3.

Both diagonal quotients are therefore cyclotomic with nontrivial wild inertia. If one wild rank is one, the projective field satisfies $\mathrm{rd}(M) \le \sqrt{28}\,3^{35/18} < 44.80380$. Its degree is at least $2|\,\mathrm{PSL}_2(\mathbf{F}_{27})| = 19656$, and its local degrees are at most 6 and 18, both totally ramified. Thus $c = 1/e_1 + 1/e_2 \ge 2/9$, contradicting the third row. Both wild ranks are two, so both local fields are E_0.

For odd f, the tame inertia involution has Borel torus coordinate -1, nonsquare in $\mathbf{F}_q$, and cannot belong to $\mathrm{PSL}_2(\mathbf{F}_q)$. For even f, a full $\mathrm{PGL}_2(\mathbf{F}_q)$ image would have a nontrivial squareclass quotient trivial on both local decomposition groups: these equal inertia, and -1 is a square. Its quadratic field would be unramified at every finite place and split at three, contrary to Lemma 23.2. This gives the asserted parity restrictions.

In the remaining projective general linear case, the squareclass quotient agrees with $\overline{\chi}_3$ on both decomposition groups at three. Their quotient is unramified at all finite places and split at three, so is trivial by the same lemma. The quadratic field is therefore C.

It remains to consider full $\mathrm{PGL}_2(\mathbf{F}_9)$. If its quadratic squareclass character ramifies at a place above three, projective wild inertia there must vanish: otherwise the decomposition group is in a Borel, and the tame Frobenius relation makes its inertia torus coordinates ± 1, both squares in $\mathbf{F}_9$. Linear wild inertia then vanishes as well, since a scalar in characteristic three has order prime to three. We obtain $\mathrm{rd}(L) < \sqrt{28}\,3^{55/36} < 28.34747$. If the squareclass character is unramified everywhere finite, it is the character of $K(i)$, nontrivial on Frobenius at both places above three. Neither local quotient can then be exactly cyclotomic with wild inertia, because its projective decomposition equals inertia. Hence $\mathrm{rd}(L) < \sqrt{28}\,3^{3/2} < 27.49546$. But $[L : \mathbf{Q}] \ge 2880$, contradicting the second row of Lemma 23.3.

The remaining small nonsolvable groups are A_5 and $\mathrm{PSL}_2(\mathbf{F}_9)$. Their embeddings in $\mathrm{PGL}_2(\mathbf{F}_9)$ have trivial squareclass character, since they are perfect. Theorem 1.2(5) and Lemma 2.1 of [3] give modularity of the projective representation and of its given linear lift. □

23.1 A classical form of level eighty-four

Let $\eta = \chi_{28}$ be the primitive quadratic Dirichlet character attached to $K/\mathbf{Q}$.

Lemma 23.5. *There is a normalized newform $f \in S_2(\Gamma_0(84), \eta)$ with coefficient field $F = \mathbf{Q}(\alpha)$ such that*

$$\alpha^4 + 14\alpha^2 + 32 = 0, \qquad a_5(f) = \alpha, \qquad a_{11}(f) = \frac{\alpha^3 + 10\alpha}{4}. \tag{28}$$

Moreover, $a_3(f) = 1$. The prime three is inert in F. Complex conjugation sends α to $-\alpha$, and its fixed field is $\mathbf{Q}(\sqrt{17})$, with $\sqrt{17} = \alpha^2 + 7$.

Proof. Take the normalized newform with LMFDB label `84.2.b.b` [48]. Its nebentypus is $\chi_{84}(55, \cdot)$, induced by η. The tabulated coefficient field is $\mathbf{Q}(\nu)$, where $\nu^4 - \nu^3 - 2\nu + 4 = 0$, and the coefficients are

$$a_3(f) = 1, \qquad a_5(f) = \nu^3 + \nu - 2, \qquad a_{11}(f) = \nu - \nu^2.$$

Put $\alpha = \nu^3 + \nu - 2$. Reduction modulo the polynomial of ν gives

$$\alpha^4 + 14\alpha^2 + 32 = 0, \qquad 4(\nu - \nu^2) = \alpha^3 + 10\alpha,$$

which proves (28).

Modulo three, $P(X) = X^4 + 14X^2 + 32$ becomes $X^4 - X^2 - 1$. If $y^2 = y + 1$ in $\mathbf{F}_9$, then $y^4 = -1$, so y has order eight and is not a square in $\mathbf{F}_9$. Thus P is irreducible modulo three and over $\mathbf{Q}$. It follows that $F = \mathbf{Q}(\nu) = \mathbf{Q}(\alpha)$. The polynomial discriminant is $2^{13}17^2$, prime to three, so three is inert in the full ring of integers of F. Finally $\alpha^2 = -7 \pm \sqrt{17}$ is negative at

both real embeddings. Hence F is CM, complex conjugation sends α to $-\alpha$, and its fixed field is $\mathbf{Q}(\sqrt{17})$. □

Proposition 23.6. *There exists a finite Galois extension B/K with*

$$\mathrm{Gal}(B/K) \simeq D := \{g \in \mathrm{GL}_2(\mathbf{F}_9) : \det g \in \{1, -1\}\}, \qquad [B : K] = 1440.$$

It is unramified outside three, and its two local completions at three are contained in E_0. Its maximal solvable quotient is C_2, with fixed field C.

Proof. Let λ be the unique prime of F above three. Restrict the representation attached to f to G_K, and let $\bar{r}$ be its semisimplified reduction at λ. Its determinant is $\overline{\chi}_3$, since the rational determinant is $\eta\chi_3$, and its initial coefficient field is $\mathbf{F}_{81}$.

Write $t = \bar{\alpha}$ and $x = t^2$, so $x^2 = x + 1$. The primes 5 and 11 are inert in K, and the rational residual determinants at both are one. Squaring their Frobenii gives elements of G_K with determinant one and traces $s_5 = x + 1, \quad s_{11} = 1 - x$. Here $\bar{a}_{11}^2 = -x$, from (28). Their projective invariants are

$$s_5^2 = 2, \qquad s_{11}^2 = -x - 1, \qquad (-x - 1)^2 = -1. \tag{29}$$

If z is squared trace divided by determinant, the eigenvalue ratio u satisfies $u + u^{-1} = z - 2$. The first equality gives projective order four. In characteristic three, $z^2 + 1 = 0$ is equivalent to $u^4 + u^3 + u^2 + u + 1 = 0$, so the second element has projective order five. Both have distinct eigenvalues.

All traces on G_K lie in $\mathbf{F}_9$. For a good rational prime q, the Petersson adjoint identity gives $\overline{a_q(f)} = \eta(q)a_q(f)$. At a split prime of K the trace is $a_q(f)$, and at an inert prime it is $a_q(f)^2 - 2\eta(q)q$. Both belong to the real field $F^+ = \mathbf{Q}(\sqrt{17})$, so their reductions lie in its residue field $\mathbf{F}_9$. Chebotarev gives the assertion for every element of G_K.

This also proves that $\bar{r}$ is irreducible. If its semisimplification were $\psi_1 \oplus \psi_2$, trace invariance under the ninth-power coefficient automorphism would either fix both characters or interchange them. In the first case their quotient has order dividing eight; in the second its ninth power is its inverse, so its order divides ten. Neither alternative permits both projective orders in (29).

Absolute irreducibility, together with the traces and determinant in $\mathbf{F}_9$ and the triviality of its Brauer group, gives a model of $\bar{r}$ over $\mathbf{F}_9$.

Both 1 and -1 are squares in $\mathbf{F}_9$, so its projective image lies in $\mathrm{PSL}_2(\mathbf{F}_9)$. By Dickson's classification, a proper subgroup cannot contain elements of both orders four and five: the Borel, subfield, A_4 and S_4 alternatives have no element of order five, the dihedral alternatives have orders dividing eight or ten, and A_5 has no element of order four. Hence its projective image is all of $\mathrm{PSL}_2(\mathbf{F}_9) \simeq A_6$.

The scalar subgroup of the linear image is contained in μ_4, since the determinant has order two. It cannot have order at most two: then the determinant would factor through the perfect projective group, contradicting total oddness. Thus it has order four. The linear image has order $4 \cdot 360 = 1440$ and equals D. Moreover $D = \mu_4 \mathrm{SL}_2(\mathbf{F}_9)$, with the scalar factor central and $\mathrm{SL}_2(\mathbf{F}_9)$ perfect. Therefore $D' = \mathrm{SL}_2(\mathbf{F}_9)$ and its maximal solvable quotient is its determinant quotient, which cuts out C.

We check the local kernels, not merely the projective conductor. At 2 and 7, local–global compatibility for the newform gives conductor exponents respectively two and one [65, equation (1)]. These equal the exponents of its determinant character η. For a two-dimensional Weil–Deligne representation with ramified determinant, equality $a(V, N) = a(\det V)$ forces its inertia to be $1 \oplus \det V$ and $N = 0$. Indeed, if $V^I = 0$, then

$$a(V, N) \geq 2 + \mathrm{Swan}(V) \geq 2 + \mathrm{Swan}(\det V) > a(\det V).$$

Thus V^I is a line. The other inertia character is the ramified determinant, and a commuting nilpotent operator is zero. Consequently the inertia at 2 and 7 is $1 \oplus \eta$, which becomes trivial over K. At every other prime away from three the representation is already unramified.

At three, $\eta|_{G_{\mathbf{Q}_3}} = 1$. Since $3 \| 84$ and $a_3(f) = 1$, the local representation is semistable of Steinberg type with unit eigenvalue one. Local–global compatibility and Lemma 4.2 give the trivial unramified ordinary quotient at both places above three. Its determinant is χ_3, so the reduction is an actual extension of 1 by $\overline{\chi}_3$.

Its extension class lies in $H^1(\mathbf{Q}_3, \overline{\mathbf{F}}_3(\overline{\chi}_3))$, the rational cube-class space tensored with $\overline{\mathbf{F}}_3$. That space is generated by 3 and 4, so its full linear local kernel field is contained in E_0. There is no additional unramified scalar factor. Taking $B = K^{\ker \bar{r}}$ proves the proposition. □

23.2 Auxiliary composita and the residual theorem

Proposition 23.7. *There exists a totally imaginary K-Galois extension $\mathcal{F}$ of degree* 38880 *over K, unramified outside three, with completion E_0 at both places above three. Its degree has prime factors only* $2, 3, 5$*, and it contains C.*

Proof. Put $\pi = 2 + \sqrt{7}$, $\pi' = 2 - \sqrt{7}$ and $\epsilon = 8 + 3\sqrt{7}$, and consider

$$E = K(\zeta_3, \sqrt[3]{\pi}, \sqrt[3]{\pi'}, \sqrt[3]{\epsilon}).$$

The three cube classes are independent over K: valuations at $\mathfrak{p}_1, \mathfrak{p}_2$ first give independence of the two prime generators, and the fundamental unit is not a cube. They remain independent over C, since taking norms across a degree-two extension is injective on cube classes from K. Thus $[E : C] = 27$ and $[E : K] = 54$, with solvable group $(C_3)^3 \rtimes C_2$, where the involution acts by inversion.

The radicands are units outside three, so their Kummer extensions are unramified there. At the two three-adic embeddings, $\sqrt{7}$ reduces to $4, 5$ modulo nine. At each place one of π, π' has valuation one and the other is a unit congruent to seven modulo nine, hence is not a cube. The first two radicands therefore span the rational local cube-class space, giving completion E_0 at both places. The fundamental unit enlarges the global degree without enlarging either completion.

Let B be the field of Proposition 23.6. Its maximal solvable quotient is C_2, whereas E/K is solvable and contains C. Therefore $B \cap E = C$. Set $\mathcal{F} = BE$. We have

$$[\mathcal{F} : K] = \frac{1440 \cdot 54}{2} = 38880 = 2^5 \cdot 3^5 \cdot 5. \tag{30}$$

Its local completions are still E_0, because those of B are contained in E_0 and those of E equal E_0. All relative extensions away from three are unramified, and $C \subset \mathcal{F}$ makes it totally imaginary. □

Whenever two K-Galois fields have completion E_0 at the same place, their local compositum is again E_0. This is equality of subfields of $\overline{\mathbf{Q}}_3$: the field in (2) is normal and is the full Kummer extension of the rational cube-class space. We shall repeatedly apply the degree bound (25) after taking such composita.

Proof of Theorem 23.1. By Lemma 23.4, it suffices to treat the two displayed families there, with projective field M locally equal to E_0. Let $H = \mathrm{Gal}(M/K)$ and use the auxiliary field $\mathcal{F}$ of Proposition 23.7.

Suppose $H = \mathrm{PSL}_2(\mathbf{F}_{3^{2e}})$, $e \geq 2$. Its simplicity makes $M \cap \mathcal{F}$ either K or M. The latter is impossible because $|H| \geq |\mathrm{PSL}_2(\mathbf{F}_{81})| = 265680 > 38880$. Hence $[M\mathcal{F} : \mathbf{Q}] \geq 2 \cdot 265680 \cdot 38880 = 20\,659\,276\,800$, contradicting (25).

Suppose $H = \mathrm{PGL}_2(\mathbf{F}_{3^{2e+1}})$ with $e \geq 2$. Its only proper nontrivial normal quotient is C_2, and its order exceeds 38880. Since both fields contain the unique quadratic subfield C, we have $M \cap \mathcal{F} = C$. Therefore $[M\mathcal{F} : \mathbf{Q}] \geq |\mathrm{PGL}_2(\mathbf{F}_{243})| \cdot 38880 = 557\,876\,056\,320$, again impossible.

It remains to consider $H = \mathrm{PGL}_2(\mathbf{F}_{27})$, of order $h = 19656$. Let σ be the nontrivial automorphism of $K/\mathbf{Q}$. If $M \neq M^\sigma$, the two fields intersect exactly in C: their only proper nontrivial common quotient is quadratic, and C is invariant under rational conjugation. For $P = MM^\sigma$ we have

$$\mathrm{Gal}(P/K) \simeq H \times_{C_2} H, \qquad [P : K] = h^2/2.$$

Its derived subgroup is $\mathrm{PSL}_2(\mathbf{F}_{27})^2$, which is perfect. Any quotient that does not kill this subgroup has a composition factor of order divisible by 7 and 13. Such a quotient cannot be a quotient of $\mathrm{Gal}(\mathcal{F}/K)$, whose order has prime factors only $2, 3, 5$. Thus $P \cap \mathcal{F} = C$, and $[P\mathcal{F} : \mathbf{Q}] = h^2 \cdot 19440 = 7\,510\,806\,051\,840$, contradicting (25). Consequently $M = M^\sigma$ and $M/\mathbf{Q}$ is Galois.

We now prove modularity rather than nonexistence. Write $\Gamma = \mathrm{Gal}(M/\mathbf{Q})$. Conjugation induces a map $\Gamma \to \mathrm{Aut}(H)$. Its outer image is a quotient of $\Gamma/H = C_2$, whereas $\mathrm{Out}(\mathrm{PGL}_2(\mathbf{F}_{27})) \simeq C_3$ consists of field automorphisms [40, Section 13, Lemma 52]. The outer action is trivial. As H is centreless, the conjugation map lands in $\mathrm{Inn}(H) \simeq H$. Composing it with $G_{\mathbf{Q}} \to \Gamma$ gives a two-dimensional projective representation of $G_{\mathbf{Q}}$ extending the given projective representation over K.

Its complex conjugations are nontrivial projective involutions. Tate's lifting theorem gives a finite linear lift in characteristic three, necessarily odd and irreducible [3, Section 2.1]. Serre modularity over $\mathbf{Q}$ [43, 44] and cyclic base change give residual modularity over K. The base change is cuspidal: otherwise its Galois representation restricted to K, and hence its residual semisimplification, would be reducible. Finally, the given linear representation differs from this restricted lift by a finite character, which preserves modularity. This proves the residual theorem. □

24 Dihedral sources over $\mathbf{Q}(\sqrt{10})$

Put $K = \mathbf{Q}(s)$, $s^2 = 10$.

Theorem 24.1. *Assume GRH. Every continuous irreducible representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ of determinant $\bar{\chi}_3$ unramified outside three is modular.*

The places v_+, v_- above three correspond to $s \equiv 1, -1$; their completions are $\mathbf{Q}_3$. The dyadic completion $F = K_2$ has ramification index two and residue field $\mathbf{F}_2$. The local menu below reduces the nonsolvable dyadic case to A_5. For solvable residue, the CM inducing field splits dyadically, so we instead bound the global trace field across all its dyadic embeddings to modularize the actual source. At v_+ use the wild type τ_+ of Lemma 4.19, with uniformizer $+3$ and its specified quadratic subfield. At v_- use the special or quartic type of Lemma 4.20; both have the characteristic-two kernel envelope (6).

Proposition 24.2. *Let $\overline{\rho}$ satisfy the hypotheses of Theorem 24.1, and suppose its image is nonsolvable. It lifts to a totally odd, potentially modular, strongly compatible system $\mathcal{R} = (r_\lambda)$ with determinant χ and weights $\{0, 1\}$. The system is unramified away from three, has type τ_+ at v_+, and has one of the two local conditions of Lemma 4.20 at v_-.*

Proof. Lemma 4.19 and Lemma 4.20 give the two local weight-two points with determinant χ_3. Choose their components, and unramified conditions elsewhere. Nonsolvability excludes induction from $K(\zeta_3)$, so the cyclotomic restriction is irreducible. Lemma 2.6(i) gives the lift

and strong compatibility, including crystallinity at two. At a selected special component, the Frobenius ratio is three; Section 2 therefore retains nonzero monodromy. The construction requires no global quadratic character. □

Choose a place above two in the coefficient field and write r_2 for this member and $\overline{r}_2$ for its reduction. Lemma 4.19 makes $\overline{r}_2$ irreducible, with determinant one. Let M be its kernel field. Its completion at v_+ is E_+, while Lemma 4.20 puts its completion at v_- inside E'.

24.1 The controlled characteristic-two image

Lemma 24.3. *The splitting field A/K of $X^3-3X+1+s$ has Galois group S_3. It is unramified away from three, and its dyadic completion is the unramified cubic extension of K_2. Its completions at v_+ and v_- are E_+ and $\mathbb{Q}_3(\sqrt{3})$, respectively.*

Proof. The cubic reduces to X^3+X+1 at the unique dyadic prime. This polynomial is irreducible and separable over $\mathbb{F}_2$, proving both global irreducibility and the dyadic assertion. Its discriminant is

$$-27(7+2s) = -6\left(3+\frac{3}{2}s\right)^2, \qquad N_{K/\mathbb{Q}}(-27(7+2s)) = 3^8.$$

It follows that $\mathrm{Gal}(A/K) = S_3$, with quadratic subfield $K(\sqrt{-6})$, and that no finite prime away from three ramifies.

At v_+ put $d = s-1$. Since $d(s+1) = 9$, we have $v_3(d) = 2$. Writing $X = 1+y$ gives $y^3+3y^2+d = 0$, whose roots have valuation $2/3$. The element $z = y^2/3$ satisfies $z^3-3z^2-\frac{2d}{3}z-\frac{d^2}{27} = 0$. This polynomial is Eisenstein. In the resulting cubic extension the three derivative terms have valuations $5,4,3$, respectively, so its different exponent is three. The lower ramification groups of the Galois closure consequently have orders $6,3,1$, giving normalized different $(5+2)/6 = 7/6$. Its quadratic subfield is $\mathbb{Q}_3(\sqrt{-6}) = \mathbb{Q}_3(\sqrt{3})$, so the closure is E_+.

At v_-, the constant term $1+s$ has valuation two. The Newton polygon has slopes -1 and $-1/2$, of lengths one and two. There is one linear factor and one ramified quadratic factor. Its splitting field is the quadratic field defined by the discriminant, namely $\mathbb{Q}_3(\sqrt{3})$. □

Proposition 24.4. *There is a solvable Galois extension N/K of degree 768, unramified outside two and three, whose dyadic completions have no upper break greater than three. Its completion at v_+ is E_+, and its completion at v_- is contained in E'.*

Proof. We use an abelian ray extension of the field A in Lemma 24.3. An absolute equation for A is

$$P_A(X) = X^{12}-2X^{11}+5X^{10}+19X^8+2X^7+31X^6 \\ +2X^5+19X^4+5X^2-2X+1.$$

It is obtained by taking the compositum of $X^6-6X^4+2X^3+9X^2-6X-9$ with X^2+6; if u is the root of the degree-six polynomial, the embedding of K is given explicitly by $s = -u^3+3u-1$. The class number is four. There are two primes $\mathfrak{P}_1, \mathfrak{P}_2$ above two, each of absolute ramification index two and residue degree three. Above three there is one prime of absolute ramification index six and residue degree one, lying over v_+, and three primes of ramification index two and residue degree one, lying over v_-. The field has discriminant 241864704000000 and signature $(0,6)$.

Let R be the ray class group of modulus $\mathfrak{P}_1^4\mathfrak{P}_2^4$. Quotient R by the class of the prime over v_+ and by twice the class of each prime over v_-. Exact ideal and unit arithmetic gives

$$R \simeq C_4^2 \times C_2^5, \qquad R/\langle [\mathfrak{q}_+], 2[\mathfrak{q}_-]\rangle \simeq C_4 \times C_2^5.$$

Here the bracket in the denominator includes all three primes over v_-. These are calculations in degree twelve. The accompanying PARI/GP certificate computes maximal orders and ray class groups, checks the quotient order 128 by Hermite and Smith normal forms, and returns `bnfcertify` $= 1$ for the class-group calculation. The exact conductor of the quotient has exponent four at both dyadic primes. The exact calculation is supplied in [52, `ramified/d6_d10_d11/d10`].

Let N/A be the associated ray extension. The modulus and the relations are preserved by $\mathrm{Gal}(A/K)$, so N/K is Galois. Its group is an extension of an abelian group by S_3, hence is solvable, and $[N : K] = 128 \cdot 6 = 768$. At v_+ the ray relation makes N/A split, so its completion remains E_+. At each prime over v_- it is unramified of degree at most two, giving a subfield of E'. At two, conductor exponent at most four gives upper break at most three over A. Since A/K is unramified there, the same upper bound holds over K_2. □

Theorem 24.5. *Assume GRH for Dedekind zeta functions. There is no finite Galois extension M/K, unramified outside two and three, satisfying all the following conditions:*

1. $\mathrm{Gal}(M/K) \simeq \mathrm{SL}_2(\mathbb{F}_q)$ *for* $q = 2^r \geq 8$;
2. *its dyadic completions have no upper break greater than three;*
3. *its completion at v_+ is E_+ and its completion at v_- is contained in E'.*

Proof. The group $\mathrm{SL}_2(\mathbb{F}_q)$ is nonabelian simple in characteristic two. Thus $M \cap N = K$ for the solvable Galois extension in Proposition 24.4. For $L = MN$ we obtain $[L : \mathbb{Q}] = 2 \cdot 768\, q(q^2 - 1) \geq 774144$. Upper numbering is compatible with quotients, so the compositum still has no dyadic upper break greater than three. Since K_2 has absolute ramification index two, its normalized relative different is strictly less than two. The completions at three remain in the stated extensions. The global discriminant formula therefore gives $\mathrm{rd}(L) < 4\sqrt{40}\, 3^{5/6} = 63.1962098567\ldots$. Apply the GRH explicit formula of Section 13 with the retained coefficients $c_3 = 1/6$ and $c_9 = 1/4$. The v_+ primes have norm three. If the other completion has residue degree two, its norm-nine primes give the second coefficient. If it has residue degree one, retaining only the even powers of its norm-three Euler terms gives at least the same contribution.

At $b = 13.3$ and degree 774144, interval arithmetic gives $\mathrm{rd}(L) > 64.5683408714$. The lower expression increases with the degree. This contradicts the preceding upper bound and proves the theorem. The exact calculation is supplied in [52, `ramified/d6_d10_d11/shared`]. □

Corollary 24.6. *Assume GRH. Let $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ be a continuous representation of determinant χ_2, unramified outside finitely many primes and Barsotti–Tate at two. Suppose that its residual representation is irreducible and nonsolvable, is unramified outside two and three, and has the local extensions specified in Theorem 24.5. Then r is modular.*

Proof. Dickson's classification and Theorem 24.5 leave only residual image $\mathrm{SL}_2(\mathbb{F}_4) \simeq A_5$. The characteristic-two A_5 modularity theorem [3, Theorem 2.5] supplies residual modularity. Lemma 11.9 applied to the original kernel field M gives $\mathrm{rd}(M) \leq 4\sqrt{40}\, 3^{5/6} 2^{-2/4} < 44.68647$. The selected norm-three primes have $2N_3/[M : \mathbf{Q}] \geq 1/6$. Corollary 13.5(ii), or directly the classical lower bound 47.16252 of Lemma 13.4, therefore excludes a totally real M. Thus some real complex conjugation has nontrivial residual image. All the hypotheses of [73, Theorem 6.1] are satisfied, so r is modular. □

Lemma 24.7. *Every finite characteristic-three character ψ of G_K, unramified outside three, has square equal to 1 or to the quadratic character ν of $K(\sqrt{2})/K$. In particular $\bar{\chi}_3\psi^{-2}$ is totally odd and extends to $G_{\mathbb{Q}}$.*

Proof. The Minkowski bound is $\sqrt{40}/2 = \sqrt{10} < 3.17$. The dyadic prime $\mathfrak{p}_2$ is nonprincipal: an element of norm ± 2 would give $x^2 \equiv \pm 2 \pmod 5$. Its square is (2). The element $2 + s$ has

norm -6, so each prime above three represents the same ideal class as $\mathfrak{p}_2$; the two triadic classes are inverse and have order two. Thus $h_K = 2$. The unit $3 + s$ has norm -1, so the narrow class number is also two.

The principal units at three are pro-three, and each residue field is $\mathbb{F}_3$. Consequently ψ^2 is unramified at every finite place. The narrow class-group calculation makes it 1 or ν. The biquadratic field $K(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ has discriminant $8 \cdot 5 \cdot 40 = 40^2$; hence it is the quadratic Hilbert class field of K. Its character is the restriction of the quadratic character of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, which is totally even. Multiplication by $\bar{\chi}_3$ proves the last assertion. □

24.2 The residual field at two

Let $K = \mathbf{Q}(s)$, $s^2 = 10$, and label the places v_+ and v_- above three by $s \equiv 1$ and $s \equiv -1$, respectively. Let the two-adic member of the chosen weight-two system have the binary dihedral type at v_+ and the special or quartic type at v_-. Its determinant is cyclotomic and it is crystalline of labelled weights $\{0, 1\}$ at the dyadic place. Throughout this section the residual point field is the field cut out by the actual representation on a stable lattice. The local group at v_+ has standard image S_3 in characteristic two. In particular the residual representation is absolutely irreducible.

For a nonsolvable residual image, Proposition 24.4 and Theorem 24.5 leave only $\mathrm{SL}_2(\mathbf{F}_4)$; Corollary 24.6 proves modularity of the actual source in that case. These statements apply to either choice of the mild type, including special monodromy.

In the solvable case Dickson's classification gives induction from an odd-order character θ over a quadratic field H/K. The character defining H is unramified outside two and three, has conductor exponent at most four at two by the finite-flat bound, and cuts out $\mathbf{Q}_3(\sqrt{3})$ at v_+. The squareclasses unramified away from this support have basis

$$-1, \qquad \epsilon = 3 + s, \qquad 2, \qquad 3, \qquad 2 + s.$$

Lemma 24.7 gives the class group of order two, generated by the dyadic class, and identifies both triadic classes with it. The fundamental unit is $\epsilon = 3 + s$. The valuations of $2, 3, 2 + s$ generate the admissible index-two lattice on these three primes, proving the displayed squareclass basis.

For a representative a of each of the 31 nonzero classes, test $3a$ for squareness at v_+ and evaluate the dyadic valuation of the relative discriminant of $X^2 - a$. The complete list at conductor at most four is

a	3	3ϵ	-6	-6ϵ
dyadic conductor	2	4	0	4.

Only $L = K(\sqrt{-6})$ is CM. If the source is ordinary, Allen's residually dihedral theorem [2, Lemma 5.1.2 and Theorem 5.2.1] applies for each of the three non-CM choices, with its ordinary witness construction and without a dyadic nonsplitting hypothesis for these choices. If the source is nonordinary, its residual dyadic point field has upper breaks at most one, by Theorem 11.10. Both conductor-four quadratics are therefore impossible.

It remains to consider $H = K(\sqrt{d})$ for $d = 3, -6$. Every dyadic completion of either field has residue field $\mathbf{F}_2$. Its local units are pro-two, and hence θ is unramified there. At v_+ the relative quadratic different exponent is one and the residue degree is one. The conductor formula for induction gives $3 = 1 + a_H(\theta)$, so its conductor exponent is two. At the prime above v_- the full local image is a two-group for either mild type; thus θ kills the entire decomposition group there. At real places an odd-order character is trivial.

Write P_+, P_- for the two primes of H above v_+, v_-. The table records the ordinary ray group for P_+^2, the class of P_- in the indicated generator, and the quotient by $[P_-]$ and $1 + \sigma$,

where σ generates $\mathrm{Gal}(H/K)$:

d	(e,f) above 2	$\mathrm{Cl}_{P^2_+}(H)$	$[P_-]$	quotient
3	$(4,1)$	C_2	1	0
-6	$(2,1),(2,1)$	C_{12}	9	C_3.

Here $1+\sigma$ imposes inversion on the rotation character, since the determinant is trivial modulo two. These computations can be performed in the quartic model $z = s + \sqrt{d}$ with polynomial $z^4 - 2(10+d)z^2 + (10-d)^2$. The expression $s = (z + (10-d)/z)/2$ identifies the two triadic primes; $z \mapsto 2s - z$ identifies σ. Compute the ray group and its automorphism by `bnrinit` and `bnrgaloismatrix`, adjoin the column `bnrisprincipal`(P_-) and the columns of $1+\sigma$ to the diagonal relation matrix, and take its integer Smith form. The class groups of H are C_2 and C_4, respectively, and `bnfcertify` returns one in both fields. For L either triadic prime has class 1 in the chosen generator of $\mathrm{Cl}(L) = C_4$. The exact calculation is supplied in [52, `ramified/d6_d10_d11/d10`].

There is consequently no such residual representation induced from $K(\sqrt{3})$. The two nontrivial characters of the cubic quotient over L cut out a single field A. Indeed the splitting field of $X^3 - 3X + 1 + s$ has the requisite local data by Lemma 24.3, so it realizes this cubic quotient. In particular, once a conjugate with one of the modular images has been disposed of, every remaining two-adic residual representation has this same kernel and is the standard representation over $\mathbf{F}_2$.

24.2.1 The two normal ray modules

Proposition 24.8. *Let V_n be the binary quotient of the ray group of A of dyadic depth n in which the positive triadic prime splits. For the standard S_3-module W_0 we have $\mathrm{Hom}_{S_3}(V_2, W_0) = 0$. At depth four this space has dimension two, and projection onto either nontrivial C_3-eigencharacter after scalar extension to $\mathbf{F}_4$ is injective on local units at either dyadic prime.*

Proof. Write W_0 for the two-dimensional standard module over $\mathbf{F}_2$ for $\mathrm{Gal}(A/K) = S_3$. Let $\mathfrak{P}_1, \mathfrak{P}_2$ lie above two and $\mathfrak{q}_+$ above v_+. Set

$$D_n = \mathrm{Cl}_{\mathfrak{P}_1^n \mathfrak{P}_2^n}(A), \qquad V_n = D_n / (2D_n + \langle [\mathfrak{q}_+] \rangle).$$

There is no infinite modulus. The normal first-error module in the basis of the wild order-three element is W_0 after extension of residue scalars. A class unramified at the mild prime, split at $\mathfrak{q}_+$, and of dyadic conductor at most n therefore belongs to $\mathrm{Hom}_{S_3}(V_n, W_0)$ after extending the coefficients. To see the identification on the global cohomology class, restrict to G_A. The kernel and cokernel terms in inflation–restriction vanish since W_0 restricts to the regular module for a Sylow two-subgroup of S_3 and is projective. The restricted class is an equivariant additive character.

We compute the dimensions through the cubic point field B_0/K, using Lemma 11.13. An absolute equation and its quadratic embedding are

$$P(X) = X^6 - 6X^4 + 2X^3 + 9X^2 - 6X - 9, \qquad s = -X^3 + 3X - 1.$$

The field B_0 has signature $(2,2)$, discriminant 5184000 and class group C_2. Its unique dyadic prime $\mathfrak{P}$ has absolute $(e,f) = (2,3)$, so $(B_0)_{\mathfrak{P}}/K_2$ is unramified cubic. Let $C_n(B_0)$ and $C_n(K)$ be the ray class groups for $\mathfrak{P}^n$ and $\mathfrak{p}_2^n$, respectively, including all real places. Exact arithmetic gives

n	$C_n(B_0)$	$C_n(K)$	$\dim_{\mathbf{F}_2} \mathrm{Hom}_{S_3}(V_n, W_0)$
1	C_2	C_2	0
2	C_2^2	C_2^2	0
3	C_2^2	C_2^2	0
4	C_2^5	C_2^3	2

Here the last column is the difference of the binary ranks in the middle columns. Indeed the three-point permutation module is $\mathbf{F}_2 \oplus W_0$, and restriction splits corestriction since $[B_0 : K] = 3$. At two this decomposition preserves each conductor bound by unramified base change. Away from two it preserves the unramified conditions. At v_+ the cubic inertia has no W_0-invariants, so the unramified local cohomology vanishes. Thus its class is split there, as required. At v_- the cubic local algebra has one linear and one quadratic factor, and the local Shapiro decomposition identifies the unramified condition with the unramified characters of those factors. At a real place W_0 restricts to the regular binary C_2-module and has zero first cohomology. These observations identify the norm-kernel summand with the stated normal ray space. The degree-six class group is certified by `bnfcertify`, and the four ray computations are reproduced in [52, `audit_simplifications/d10_pointfield_certificate.gp`]. In particular $\mathrm{Hom}_{S_3}(V_2, W_0) = 0$.

The exact matrices and local unit columns in Appendix D give a two-dimensional space of equivariant maps. For $\alpha^2 + \alpha + 1 = 0$, the determinants of the two local evaluation matrices for either C_3-eigencharacter are $1 + \alpha$ and α, up to Galois conjugation. Both are nonzero. Vanishing of either eigenprojection on local units at either dyadic prime therefore kills the global depth-four class. These statements concern the normal error in characteristic two; the tame inertia in the actual joint trace order is checked below. □

Lemma 24.9. *Let R be a complete local order in a finite product of finite extensions of $\mathbf{Q}_2$, finite free over the integers of an unramified extension containing ζ_3. Let $r : G_K \to \mathrm{GL}_2(R)$ be an integral realization of a tuple of the controlled weight-two systems of Proposition 24.2. Suppose its residue is the standard representation with kernel field A. At the dyadic place, suppose either all constituents are nonordinary, or all are ordinary with the same residual quotient character. Let J be a proper closed ideal of R, and assume that the image of inertia above v_- in G_A is scalar modulo J. Then the image of r modulo J is monomial, with component character $\eta = \chi_{-6}$, in the fixed eigenbasis of the order-three element at v_+.*

Proof. The order-three eigenprojectors are integral. In their basis the normal space to the diagonal torus is the standard S_3-module W_0, extended to the residue field; it has no invariants under the fixed order-three element u, and $H^i(S_3, W_0) = 0$ for $i > 0$. Write $\mathfrak{m}$ for the maximal ideal of R/J. Assume monomiality modulo $\mathfrak{m}^n$ and work modulo $\mathfrak{m}^{n+1}$, with square-zero ideal $I = \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Erasing the entries forbidden by $\eta(g)$ in each matrix gives a monomial representation, since products of two erased entries vanish. Its first error is a cocycle κ with values in $W_0 \otimes I$, and $\kappa(u) = 0$.

At v_+ the actual source normalizes the u-eigenspaces, so the error vanishes on the full decomposition group. At v_- the hypothesis on inertia makes $\kappa|_{G_A}$ unramified. Thus, once its dyadic conductor is bounded, inflation–restriction places the class in the normal ray module of Proposition 24.8.

Here the conductor bounds apply to actual finite-flat subquotients. At the dyadic place L/K is split, so the local component character is trivial. Put $S = (R/J)/\mathfrak{m}^{n+1}$. The module S^2 is a finite 2-primary quotient of the stable $\mathbf{Z}_2$-lattice underlying R^2; schematic closure and quotient in its Barsotti–Tate group realize it as a finite-flat group. The submodule $Se_2 + Ie_1$ and its submodule $\mathfrak{m}Se_2$ are stable. Their quotient, followed by a residue-linear functional on I, extracts one offdiagonal extension; interchanging the indices extracts the other. Over the unramified cubic completion of A, its prime-field coordinates are quadratic extensions. In a nonordinary tuple they have conductor at most two by the middle-model calculation in the proof of Theorem 11.10. The depth-two vanishing in Proposition 24.8 therefore kills the global class.

For an ordinary tuple, the two dyadic residual Frobenius eigenvalues are distinct. The common ordinary line is consequently the same Frobenius projector in every constituent, and

extends over R. Straightening it at first order makes one normal orientation a local coboundary, which vanishes on G_A. Fontaine's bound puts both quadratic coordinates at conductor at most four. The injectivity of the corresponding local eigenprojection in Proposition 24.8 again kills the global class. The unramified local class is zero because Frobenius minus one is invertible on the nontrivial unramified cubic character.

Thus κ is a coboundary. Its value at u forces the matrix defining that coboundary into $W_0^u \otimes I = 0$, so the error itself vanishes in the fixed basis. Induction and completeness prove the claim. □

24.3 The quartic mild type

Proposition 24.10. *Suppose that the controlled two-adic source has the residual field A and the tame quartic supercuspidal type at v_-. Then it is induced from a geometric character of G_L, where $L = K(\sqrt{-6})$, and is modular.*

Proof. Choose a coefficient ring containing ζ_3 and the integral eigenbasis of the actual order-three wild inertia at v_+. At v_- the quartic tame generator has scalar square. Its projective quadratic character is $\eta = \chi_{-6}$, so inertia inside G_A has scalar image. Apply Lemma 24.9 with $J = 0$ to this single source, using its actual ordinary or nonordinary condition at two. The image is monomial in characteristic zero.

The component character is η, so $r = \mathrm{Ind}_{G_L}^{G_K} \theta$ for a character θ. It is de Rham and finitely ramified, as a summand of the restriction of r. By global class field theory it is attached to an algebraic Hecke character [61, Chapter III]. Its conjugate is distinct, by the absolute irreducibility of r. Automorphic induction from the CM field L is cuspidal and gives r, with its regular weights [4, Chapter 3]. □

24.4 The coefficient field of the special source

Let $\mathcal{R}$ be a weight-two strongly compatible system with determinant cyclotomic, good outside the two places above three. Suppose that its type at v_+ is the binary supercuspidal type of Lemma 4.19 and its type at v_- is special, with nonzero monodromy and a fixed quadratic scalar character. Let B be the number field generated by its traces at the good places. We consider the case in which all its residual representations at coefficient places above two are the standard representation of $\mathrm{Gal}(A/K)$. Put $k = \mathbf{F}_4$, let $\mathcal{W} = W(k)$, and fix a primitive cube root of unity $\zeta \in \mathcal{W}$.

Proposition 24.11. *Let d_0 be the number of nonordinary embeddings of B in $\overline{\mathbf{Q}}_2$. Then $d_0 \le 2$. For each of the two possible residual unramified quotient characters at the dyadic place, there is at most one ordinary embedding. Every coefficient completion associated to a nonordinary embedding is ramified over $\mathbf{Q}_2$. In particular $[B : \mathbf{Q}] \le 4$.*

We first describe the integral representation to which we shall apply the finite-flat conditions. Partition the embeddings of B into a nonordinary set and two ordinary sets, according to the eigenvalue $\bar{\zeta}$ or $\bar{\zeta}^{-1}$ of residual Frobenius on the unramified quotient. Each set is stable under the absolute Galois group of $\mathcal{W}[1/2]$. For any nonempty set of this partition, project $B \otimes_{\mathbf{Q}} \mathcal{W}[1/2]$ to the product of field factors corresponding to the set. Let R be the closed $\mathcal{W}$-subalgebra of this product generated by the traces of the associated tuple of representations. Its fraction algebra is the entire product, since already the traces at the good places generate B over $\mathbf{Q}$. Thus R is finite free over $\mathcal{W}$ of rank the cardinality of the set. It is local with residue field k. Indeed all constituent reductions give the same residual trace, and every maximal ideal of this integral order lifts to a prime of its normalization.

The joint representation has a model on R^2. To see this, choose four group elements whose residual matrices form a k-basis of $M_2(k)$, using absolute irreducibility. Their trace-pairing Gram determinant is a unit. Pairing any group matrix with these four elements expresses it uniquely in their joint span, with coefficients in R; the same argument applies to their products. Their span is thus a free rank-four algebra with reduction $M_2(k)$. Lift a primitive matrix idempotent to this complete algebra and take the corresponding left ideal. This ideal is free of rank two over R, and the action on it identifies the algebra with $M_2(R)$ by reduction and Nakayama. Denote the resulting representation by r. Its rational constituents are those of the original tuple. In particular the stable $\mathbf{Z}_2$-lattice underlying R^2 in their sum is realized by an isogenous Barsotti–Tate group over K_2, by the lattice classification [45, Theorem 4.1]. All its generic subquotients killed by two acquire finite-flat models by schematic closure and quotient. For the nonordinary set their special fibres and their Cartier duals are connected, by the isogeny argument in the proof of Theorem 11.10.

Write $T = R/2R$ with maximal ideal $\mathfrak{n}$. Choose a wild inertia element u at v_+ whose image generates the actual order-three subgroup and use its two integral eigenprojectors to put $r(u) = \operatorname{diag}(\zeta, \zeta^{-1})$. The standard residual representation in this basis is monomial with component character $\eta = \chi_{-6}$. Here χ_a denotes the quadratic character of $K(\sqrt{a})/K$. The offdiagonal matrices with residual conjugation action form the standard module $W \otimes_{\mathbf{F}_2} k$ of Proposition 24.8. In particular $W^u = 0$ and $H^i(S_3, W) = 0$ for $i > 0$, since W is projective.

Lemma 24.12. *In the fixed u-eigenbasis the image of r over T is monomial with component character η.*

Proof. A tame generator U at v_- has the two characteristic-zero eigenvalues ϵ, ϵ, with one fixed quadratic sign $\epsilon = \pm 1$ across the tuple. Hence $U^2 = 2\epsilon U - 1$ in the joint order and $U^2 = 1$ modulo rational two. The inertia in G_A is cut out by the ramified quadratic subfield, so it acts trivially modulo two; higher inertia already acts trivially in the special source. Lemma 24.9, with $J = 2R$, proves the assertion for the nonordinary block and for each ordinary orientation. □

Lemma 24.13. *For the nonordinary set $\dim_k \mathfrak{n}/\mathfrak{n}^2 \le 1$. If this space is nonzero, its visible first-order scalar character is a nonzero multiple of χ_{-2} modulo $k\eta$. For either ordinary set $\mathfrak{n} = 0$.*

Proof. A k-linear functional on $\mathfrak{n}/\mathfrak{n}^2$ gives a specialization to $k[\delta]/(\delta^2)$. Lemma 24.12 makes the specialized representation monomial. Compare it with the constant residual matrices. Their determinant agrees, since the cyclotomic determinant is one in characteristic two. The diagonal infinitesimal error therefore has equal entries, even on an antidiagonal coset after multiplying by the inverse residual matrix. The error is $\delta c(g)$ times the identity for an additive character $c : G_K \to k$. Changing basis by an infinitesimal diagonal matrix can add a multiple of η; conjugation of such a matrix by an antidiagonal element interchanges its diagonal entries. Accordingly the class of c modulo $k\eta$ depends linearly on the functional. This map is injective: if the character is a multiple of η, the representation is conjugate to its constant reduction, so every trace has zero infinitesimal part and the trace generators force the functional to vanish.

At v_+ choose an antidiagonal inertia element t with $t^2 = -1$. Multiply an arithmetic Frobenius lift by t if necessary so that it centralizes u. It takes t to $-u^i t$ for some i, since it acts by cubing on the tame quotient of order four. Multiplying the lift by a power of u makes its diagonal ratio exactly -1. Its eigenvalues are then $\lambda, -\lambda$ with $\lambda^2 = -3$. The trace after multiplication by u is $\lambda(\zeta - \zeta^{-1}) = \pm 3$. It is rational and common to the tuple by compatibility of the local Weil representations at v_+. Thus λ is one fixed value $\pm(\zeta - \zeta^{-1})$ in $\mathcal{W}$. The scalar error vanishes on this Frobenius and on u, so $c|_{D_{v_+}}$ is a multiple of η. At the dyadic place the nonordinary scalar errors have conductor at most two, by extracting the middle self-extension on the unramified cubic base as in the proof of Lemma 24.9. Unramified base

change preserves positive breaks. In an ordinary set the exact unramified quotient shows instead that c is unramified at two.

Use the supported squareclass basis $\langle -1, 3+s, 2, 3, 2+s\rangle$ of Section 24.2. The condition at v_+ and dyadic conductor at most two leave

a	1	-2	3	-6
dyadic conductor exponent	0	2	2	$0.$

For an explicit verification form the 32 products a of the displayed basis, keep those with a or $-6a$ a square at v_+, and compute the dyadic exponent of the relative discriminant of x^2-a (zero for the trivial class). Equivalently this is the calculation in `scalar_characters.gp`. Scalar extension from $\mathbf{F}_2$ gives the same kernel conditions over k. Modulo η only χ_{-2} survives, and it is ramified at two. Hence the nonordinary cotangent dimension is at most one and an ordinary set has zero cotangent dimension. Nakayama's lemma gives $\mathfrak{n}=0$ in the latter case. The exact calculation is supplied in [52, `ramified`]. □

Proof of Proposition 24.11. Lemma 24.13 shows that each nonempty ordinary set has cardinality $\dim_k T = 1$. For the nonordinary set we may assume the cotangent dimension is one. Then $T = k[z]/(z^h)$, where $h = \dim_k T$, and its nonzero tangent is represented by χ_{-2} modulo $k\eta$ after scaling z. Restrict to the unramified cubic extension F/K_2 killing the residual dyadic characters. Since L/K splits at 2, Lemma 24.12 splits the local representation as two rank-one T-characters. The restriction of η is trivial, and the first-order error of each character is ramified over F, since χ_{-2} has conductor two there.

Take one of the rank-one summands as a finite-flat subquotient of the regular lattice modulo two. It and its dual have connected special fibres. By Lemma 11.2, the generic coefficient action, the images of powers of z, and the resulting exact filtrations extend to the finite-flat model. Its Kisin module is free of rank one over $T[[u]]$ on every residue embedding factor after enlarging the finite constant field to accommodate the residue field of F. Indeed the z-filtration is exact, with rank-one successive quotients on each factor. The quotient by z has rank one over the residue power-series ring, so lifting its generator gives a surjection by Nakayama. The ranks over that power-series ring are equal, giving the assertion. Contravariance interchanges the subobject and quotient filtrations and gives the same lengths. Frobenius permutes the residue embedding factors.

The Frobenius multiplier on each factor reduces modulo z to u times a unit. Weierstrass preparation writes it as a unit times $u-\lambda_i$ with $\lambda_i \in (z)$. Height one implies $u^2 \in (u-\lambda_i)$, since the Eisenstein polynomial of a uniformizer reduces modulo two to u^2. Evaluating at λ_i gives $\lambda_i^2 = 0$. If all λ_i vanished modulo z^2, the first-order multipliers would each be u times a power-series unit. After the middle exponent is removed, the corresponding Artin–Schreier class has no pole on any residue factor and is unramified. This is the middle self-extension calculation in the proof of Theorem 11.10, including its comparison of breaks with the field of norms. It contradicts the ramification just proved. Thus one λ_i is a unit times z, and $z^2 = 0$. This gives $h \le 2$.

Finally suppose that the completion of B belonging to a nonordinary embedding is unramified over $\mathbf{Q}_2$. The trace-order construction above, applied to a single completion before and after unramified coefficient extension, realizes its representation on an integral rank-two lattice there. The residual local representation splits into two distinct unramified lines after unramified coefficient extension. Over the unramified cubic base, Lemma 11.2 splits its finite-flat model as the direct sum of middle models. Therefore Frobenius at $u=0$ on the integral Kisin module vanishes modulo two. This can be checked after unramified base extension by faithful flatness. Over the original base with residue field $\mathbf{F}_2$, the integral module has coefficient rank two: the underlying finite-free module with unramified coefficient action is free over the coefficient

power-series ring, by reduction and Nakayama. Since two is a uniformizer, every entry of its Frobenius-at-zero matrix belongs to $2\mathcal{O}_E$, where E is the enlarged unramified coefficient field. Its determinant would have valuation at least two. On the other hand the determinant of crystalline Frobenius has valuation one because the determinant character is cyclotomic [45, Proposition 4.2]. This contradiction proves the ramification assertion. Summing the three bounds proves the proposition. □

24.5 The small coefficient prime

Let B be the minimal global trace field of the system with a special condition at v_-. We are in the case in which the residual representation at every embedding $B \hookrightarrow \overline{\mathbf{Q}}_2$ is the standard representation of the same S_3 extension A/K. Modularity at any one coefficient prime suffices for modularity of the entire system.

Proposition 24.14. *If all the dyadic embeddings are ordinary, or all are nonordinary, then $[B : \mathbf{Q}] \leq 2$. Otherwise B is quartic, ramified at two, and is generated by the Frobenius trace a_2 at the dyadic prime. Its minimal polynomial is congruent to $X^2(X^2 + X + 1)$ modulo two.*

Proof. The trace order in each set bounds the number of embeddings in that set. Residue Frobenius interchanges the ordinary sets, so if they occur each has exactly one embedding. A nonordinary completion must ramify and its block has at most two embeddings. In the mixed case it has exactly two. At an ordinary embedding one crystalline Frobenius root is the unramified quotient and the other is divisible by the maximal ideal; a_2 thus reduces to the two distinct primitive cube roots in the two orientations. At each nonordinary embedding it is in the maximal ideal by the Newton slope criterion. Strong compatibility supplies $a_2 \in B$ by taking a coefficient prime away from two. Each ordinary root occurs once, whereas restricting the embeddings to a proper subfield would repeat every root $[B : \mathbf{Q}(a_2)]$ times. This proves the proposition. □

Lemma 24.15. *Let B be a mixed quartic field as in Proposition 24.14. Then at least one of the following holds:*

(i) Every prime of B above three has residue degree at most two.
(ii) There is a degree-one prime of B above five or seven.
(iii) Up to replacing a_2 by $-a_2$, its minimal polynomial is

$$X^4 - X^3 - 7X^2 + 8X + 2 \quad \text{or} \quad X^4 - X^3 - 5X^2 + 2X + 2.$$

Proof. Every conjugate of the algebraic integer a_2 lies in $[-\sqrt{8}, \sqrt{8}]$. Write its minimal polynomial as $P = X^4 + aX^3 + bX^2 + cX + d$. By Proposition 24.14, a, b are odd and c, d are even. The elementary symmetric functions and Cauchy's inequality give

$$|a| \leq 11, \quad |b| \leq 48, \quad |c| \leq 90, \quad 0 < |d| \leq 64, \quad 0 \leq a^2 - 2b \leq 32, \quad 8b \leq 3a^2.$$

Replacing a_2 by $-a_2$, we take $a > 0$. Exact rational Sturm sequences on $[-283/100, 283/100]$, with the integral coefficient bounds above, leave the twelve irreducible polynomials in the following table. For a row marked (p, r, e) we have $P(r) \equiv 0$ and $P'(r) \equiv e \neq 0 \pmod{p}$.

a	b	c	d	Local check
1	-9	-6	14	$(5,2,2)$
1	-7	-10	-2	one prime above 3, of residue degree 2
1	-7	-8	2	$(3,2,2)$; first exceptional polynomial
1	-7	-6	2	$(7,2,3)$
1	-7	-6	4	$(7,1,1)$
1	-7	-4	4	$(5,1,4)$
1	-7	-4	8	$(5,4,4)$
1	-7	-4	10	$(5,0,1)$
1	-5	-4	2	$(5,1,3)$
1	-5	-2	2	$(3,1,1)$; second exceptional polynomial
1	-5	-2	4	odd field discriminant
3	-3	-12	-6	Eisenstein at 3

The accompanying certificate uses only integer and rational arithmetic: it tests irreducibility by Gauss's lemma and counts real roots by Sturm's theorem. Thus this step requires no maximal-order computation.

A simple root modulo five or seven gives an embedding $B \hookrightarrow \mathbf{Q}_p$ by Hensel's lemma, proving (ii). For the second row, set $Q = X^2 - X - 1$. Then $P = Q^2 + 3(X^3 - 2X^2 - 4X - 1)$. The polynomial Q is irreducible modulo three, and the remainder of the second factor modulo Q is $-4X - 2$. Dedekind's index criterion shows that three does not divide $[\mathcal{O}_B : \mathbf{Z}[a_2]]$; hence three has one prime in B, of ramification index two and residue degree two. The final row is Eisenstein at three and has one prime of residue degree one. Both rows give (i).

For the penultimate row, put $\alpha = a_2$ and $\beta = (\alpha^3 + \alpha^2 + \alpha)/2$. Direct reduction modulo P gives $\beta^4 - 29\beta^2 + 10\beta + 19 = 0$. Thus β is integral. Since $\operatorname{disc} P = 8900$, the integral lattice spanned by $1, \alpha, \alpha^2, \beta$ has discriminant 2225. The field discriminant is therefore odd, contrary to Proposition 24.14. The two remaining rows are exactly the polynomials in (iii), after replacing X by $-X$. □

Lemma 24.16. *If an irreducible residual representation of determinant $\bar{\chi}_3$, unramified outside three, is restricted to $G_{K(\zeta_3)}$, it remains irreducible.*

Proof. Otherwise the representation is induced from a character θ of $H = K(\zeta_3)$ with $\theta^\sigma = \theta^{-1}$. Such a character has order prime to three. Its ray conductor divides the product of the two primes above three, both of residue degree one and ramification index two. An absolute equation of H is $X^4 + 10X^2 + 100$. Its class group is cyclic of order four, certified by `bnfcertify`; its ray group for this modulus is also C_4. If S is the action of the nonidentity automorphism over K, the Smith invariant of the lattice generated by 4 and $1 + S$ is two. Equivalently this ray group modulo $1 + \sigma$ is C_2. Thus $\theta^2 = 1$ and $\theta = \theta^\sigma$, contradicting irreducibility. These data are reproduced by `cyclotomic_three_ray.gp`. □

Proof of Theorem 24.1. Soluble residues have solvable odd Artin lifts, which give modularity by Hecke induction or Langlands–Tunnell and regular weight raising. Assume therefore that the prescribed three-residue is nonsolvable. The bounded lift, the discriminant estimate and the soluble-image classification reduce the proof to the special source with the fixed S_3 residue, the quartic source having already been treated. Apply Proposition 24.14. The exact calculation is supplied in [52, `ramified`].

If the degree of B is at most two, or every coefficient prime above three has degree at most two, choose the originally prescribed prime. Its residue is realizable over $\mathbf{F}_3$ or $\mathbf{F}_9$ by its traces, absolute irreducibility and the vanishing of the Brauer group of a finite field. The small-image theorem of Allen–Khare–Thorne [3, Theorem 2.5] gives the required cuspidal residual witness.

At a degree-one coefficient prime of characteristic $p = 5$ or 7, the binary inertia at v_+ remains irreducible modulo p; the order-three element has distinct eigenvalues and a tame element interchanges its eigenlines. The local cyclotomic extension at v_+ is unramified, so the residual representation is absolutely irreducible even on $G_{K(\zeta_p)}$. Theorems 2.5 and 2.3 of [3] give respectively modularity of its residue and modularity of the actual de Rham member with weights $\{0, 1\}$. This includes the ramified prime five. Strong compatibility gives modularity at the original prime.

The last two polynomials in Lemma 24.15 have a coefficient place above three with residue field $\mathbf{F}_3$. If its residue were reducible, write the characters as ψ and $\bar{\chi}_3\psi^{-1}$. Lemma 24.7 gives $\psi^2 = 1$ or ν, the Hilbert class character, which is nontrivial unramified at both triadic places. On $G_{K_{v_-}}$ a quadratic Steinberg twist has semisimplification $\alpha \oplus \alpha\chi_3$ with α quadratic, so ψ^2 is locally trivial. It follows that $\psi^2 = 1$. At the good dyadic place the residual trace would be $\psi(\mathrm{Frob}_2)(1 + 2) = 0$. For the selected coefficient prime $\lambda \mid 3$, this gives $a_2 \in \lambda$. Hence $3 \mid N_{B/\mathbf{Q}}(a_2)$. Both polynomials have constant term 2, so $N_{B/\mathbf{Q}}(a_2) = 2$, a contradiction. The residue is therefore irreducible. Lemma 24.16 and [3, Theorems 2.5 and 2.3] make the actual member modular. This proves the residual theorem. □

25 Quadratic symmetry over $\mathbf{Q}(\sqrt{11})$

Put $K = \mathbf{Q}(s)$, $s^2 = 11$.

25.1 Arithmetic and modular witnesses

We will use the elements

$$\epsilon = 10 + 3s, \qquad \pi = 3 + s, \qquad \alpha = 4 + s, \qquad \alpha' = 4 - s.$$

They satisfy $N(\epsilon) = 1$, $N(\pi) = -2$ and $N(\alpha) = N(\alpha') = 5$.

Lemma 25.1. *The class number of K is one, and its units are generated by -1 and ϵ. Its finite ray class groups of moduli* (3) *and* (5) *are both cyclic of order four. With both real places added to the modulus, the groups are $C_8 \times C_2$ at three and $C_4 \times C_2$ at five.*

Proof. Minkowski's bound is $\sqrt{44}/2 < 3.32$. Three is inert, and the dyadic prime is generated by π, so the class number is one. The least nontrivial positive solution of $x^2 - 11y^2 = 1$ is $(10, 3)$; checking $y = 1, 2$ proves that ϵ is fundamental. The negative Pell equation has no solution because -1 is not a square modulo eleven.

Modulo three the residue group is $\mathbf{F}_9^\times$ and $\epsilon = 1$. The image of the units is $\{\pm 1\}$, giving a cyclic quotient of order four. Adding the real signs gives $(C_8 \times C_2^2)/\langle(-1, -1, -1)\rangle \simeq C_8 \times C_2$. At the two five-places, ordered by $s = 1, -1 \bmod 5$, ϵ has residues $(3, 2)$, of order four, and its square is the image $(-1, -1)$ of -1. The product map $(u, v) \mapsto uv$ identifies the finite ray quotient with $\mathbf{F}_5^\times$. Adding the signs leaves one independent sign factor and gives $C_4 \times C_2$. □

25.2 Characteristic-three sources

Lemma 25.2. *Let $F/\mathbf{Q}_3$ be unramified quadratic, and let r be crystalline of labelled weights $\{0, 1\}$ with determinant χ_3. If its residual constituents have inertia characters $1, \omega_3$, then r is ordinary.*

Proof. Since $e(F) = 1 < 3 - 1$, the rank-one finite-flat models of these characters are uniquely etale and multiplicative. Choose a Barsotti–Tate group with coefficient action realizing a stable

lattice. If it has a nonzero etale rational quotient, the coefficient rank is one, and the determinant gives the ordinary filtration. Otherwise the group and its Cartier dual are connected, using $r^\vee(1) \simeq r$ and invariance of connectedness under isogeny. The closure of a residual invariant line is etale or multiplicative. The former contradicts connectedness, and the latter gives an etale quotient of the connected dual. Thus the source is ordinary. □

Proposition 25.3. *If $\bar{r}_3^{\mathrm{ss}} = \psi \oplus \omega_3\psi^{-1}$ for a bounded system $\mathcal{R}$, then $\psi^2 = 1$ and the actual source r_3 is ordinary and modular.*

Proof. At a five-place the semisimplified residual inertia of our two types is scalar quadratic: the cubic part of the cuspidal inertia disappears modulo three, as does the unipotent part of the special condition. Hence ψ^2 is unramified there. A characteristic-three finite character kills principal units at three, so ψ^2 factors through the finite ray group of modulus (3) from Lemma 25.1.

Fontaine–Laffaille theory [35] over the unramified quadratic K_3 gives the possible inertia exponents $0, 1, 3, 4$ mod 8 of a rank-one constituent in weight two. The determinant has exponent four. Thus the reducible pairs are $\{0, 4\}$ and $\{1, 3\}$. In the second case ψ^2 is a faithful quartic ray character. Both five-primes generate that ray group: their positive generators α, α' reduce to $1 + s, 1 - s$ in $\mathbf{F}_9$, and

$$(1 + s)^2 = -s, \qquad (1 + s)^4 = -1, \qquad 1 - s = (1 + s)^3.$$

It follows that $\psi^2(\mathrm{Frob}_v)$ is a primitive fourth root at either five-place.

The quotient of the two residual local characters would therefore have Frobenius value of order four. For both of our five-adic types that quotient is instead -1 in characteristic three: the special eigenvalues have ratio five, while the cuspidal parameter has trace zero and determinant five. Scalar quadratic factors do not change the quotient. This excludes $\{1, 3\}$.

For the pair $\{0, 4\}$, ψ^2 is unramified also at three and is even at both real places. The ordinary class number is one, so $\psi^2 = 1$. Lemma 25.2 makes the actual source ordinary, with unramified quotient. Its residual constituent ratio is ω_3, which is totally odd, extends to $G_{\mathbf{Q}}$, and is nontrivial at K_3. The field $K(\zeta_3)$ is abelian over $\mathbf{Q}$. All the hypotheses of Skinner–Wiles [66, Theorem A] are satisfied, so r_3 is modular. □

Proposition 25.4. *Let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_3)$ be continuous, irreducible, totally odd and finitely ramified, with determinant χ_3, and crystalline of weight two at three. If $\bar{r}$ is absolutely irreducible but becomes reducible on $G_{K(\zeta_3)}$, then r is ordinary at three and modular.*

Proof. Put $H = K(\zeta_3)$. Clifford theory gives $\bar{r} = \mathrm{Ind}_{G_H}^{G_K} \xi$, where ξ has finite order prime to three. Wild inertia at three is therefore trivial. The extension H_3/K_3 is ramified quadratic. A tame generator t outside G_{H_3} acts antidiagonally in an inducing basis, and $\det \bar{r}(t) = -\xi(t^2) = \omega_3(t) = -1$. Thus $\bar{r}(t)^2 = 1$. A lift of Frobenius can be chosen in G_{H_3}, and its diagonal matrix commutes with $\bar{r}(t)$ because the residue cardinality is nine. Its two eigenvalues are consequently equal, and the determinant makes that common value ± 1. The local pair is exactly $\beta, \beta\omega_3$ with β unramified quadratic. Lemma 25.2 applies to the actual source.

Induce the Teichmuller lift of ξ to obtain a finite-image, irreducible, totally odd Artin representation. Automorphic induction supplies its weight-one Hilbert theta series. At three its Euler factor has degree one and unramified root ± 1, so the form is ordinary and has the correct residual ordinary quotient. Wiles's Hilbert ordinary family theorem [77, Theorem 1.4.1] gives a regular ordinary specialization with the same residual quotient. This application is also explicit in [29, pp. 1133–1134].

We have obtained the quotient-compatible modular lift required by [67, Theorem 5.1]. The actual source is ordinary and distinguished at three, its determinant is cyclotomic, and

its remaining geometric and parity hypotheses are the stated ones. That theorem permits reducibility on the cyclotomic subgroup and proves modularity of r. □

25.3 Nonsolvable and dihedral dyadic residues

In this section the system is special at both five-primes. Let M be the field of a semisimple dyadic residue. At five its full decomposition image is elementary abelian of exponent at most two: removing its quadratic sign leaves an upper unipotent group, generated by tame inertia and Frobenius. In particular every local field is contained in the maximal elementary quadratic extension of $\mathbf{Q}_5$.

Lemma 25.5. *Put $F = K(i, \sqrt{5})$ and let H be its Hilbert class field. Then $[H : \mathbf{Q}] = 16$, its dyadic primes have absolute $(e, f) = (2, 4)$ and its four five-primes have $(e, f) = (2, 2)$. Its fourth-power dyadic ray, with all five-prime classes killed, has group $C_4^2 \times C_2^4$.*

Proof. The defining polynomials for F, H are respectively

$$y^8 - 4y^7 + 24y^6 - 58y^5 + 141y^4 - 190y^3 + 186y^2 - 100y + 20,$$
$$y^{16} - 9y^{14} + 39y^{12} - 111y^{10} + 214y^8 - 189y^6 + 69y^4 - 6y^2 + 1.$$

The class groups are $C_2, 1$; the second field is the class field of the first and has its squared discriminant. Over H the full ray invariants are $12, 4, 4, 2, 2, 2, 2, 2$. Adding the four reciprocity columns of the five-primes to the diagonal relation matrix gives the displayed quotient by Smith reduction. The certificate `sp_auxiliary.gp` checks the polynomial identification, both class groups with `bnfcertify`, the prime decompositions and the Smith factors. The exact calculation is supplied in [52, `ramified/d6_d10_d11/d11`]. □

Proposition 25.6. *Assume GRH. A two-special bounded system with a nonsolvable dyadic residue is modular.*

Proof. Its image is $\mathrm{SL}_2(q)$, $q = 2^f \geq 4$, by Dickson and determinant one. Let B be the ray field of Lemma 25.5, of absolute degree 4096 and Galois over K by invariance of the modulus and quotient. The dyadic breaks of BM/K are at most three: this follows from Fontaine's bound for the actual Barsotti–Tate lattice, and from the fourth-power ray over the unramified dyadic extension H/K. The two-adic contribution to the logarithmic relative different, in valuation $v_2(2) = 1$, is at most $4/2 = 2$. At five the field H contains the ramified field $\mathbf{Q}_5(\sqrt{5})$ and its unramified quadratic extension. This is already the full elementary quadratic envelope. Consequently for $L = BM$ and also $L = BMM^\sigma$, $\mathrm{rd}(L) \leq 4\sqrt{44 \cdot 5} < 59.329588, \quad 2N_{25}(L)/[L : \mathbf{Q}] = 1/2$. Here N_Q counts primes of norm Q.

Apply Section 13 with $B_{b,100}$ and the retained prime powers specified below. Directed 256-bit evaluation gives

	$n \geq$	b	powers of 25	$\mathrm{rd}(L) >$
$q \geq 16$	$4096 \cdot 4080$	16.4	9	59.981834
$q = 8,\ M \neq M^\sigma$	$4096 \cdot 504^2$	20.5	10	62.360509

Simplicity gives disjointness from the solvable auxiliary. Distinct simple conjugate fields are disjoint over K, and their product still has no nontrivial solvable quotient. The first two lines exclude their respective cases. The exact calculation is supplied in [52, `ramified/d6_d10_d11/shared`].

A real conjugation is nontrivial in either case: Lemma 11.9 gives $\mathrm{rd}(M) \leq 4\sqrt{220}\, 2^{-2/q}$, so Corollary 13.5(i) excludes a totally real kernel without an analytic hypothesis. If $q = 8$, the equality $M = M^\sigma$ gives residual modularity by Lemma 2.5. Lemma 2.4 applies in this case and for $q = 4$. □

Proposition 25.7. *A two-special system with irreducible solvable two-residue is modular. Every reducible two-residue has trivial semisimplification.*

Proof. An irreducible solvable subgroup in determinant one is odd dihedral. Let L/K be its inducing quadratic field. The odd rotation character is unramified away from two, trivial at every five-decomposition group over L, and at two is tame. Its conductor divides the product of the dyadic primes of L. The quadratic squareclasses are supported on $-1, \epsilon, \pi, \alpha, \alpha'$ by class number one. Odd dyadic valuation gives conductor five by an Eisenstein square-root polynomial and its derivative, contradicting Fontaine's upper break three. Among the fifteen nontrivial classes in $\langle -1, \epsilon, \alpha, \alpha' \rangle$, the odd part of the ray modulo all dyadic primes with all five-primes split is trivial except at -5ϵ, where it is C_3. The certificate `sp_dihedral_rays.gp` verifies these quartic ray quotients. Since $\pi^2 = 2\epsilon$ the survivor is $K(\sqrt{-10})$. Its dyadic relative conductor is four. Theorem 11.10 excludes a nonordinary actual source. The ordinary source has a nonsplit dyadic prime in its imaginary inducing field and the specified near-ordinary line; Allen's ordinary dihedral theorem [2, Theorem 5.2.1] applies. For a reducible residue the odd-order constituents are unramified at two and five and everywhere else, so class number one makes both trivial. The exact calculation is supplied in [52, `ramified/d6_d10_d11/d11`]. □

25.4 Ordinary scalar systems

Proposition 25.8. *A two-special system with a scalar semisimple two-residue and an ordinary member above two is modular.*

Proof. The sixteen supported classes $\langle -1, \epsilon, \alpha, \alpha' \rangle$ contain precisely $1, 5, -1, -5$ unramified at two and precisely $1, -5$ split there; a class of odd dyadic valuation cannot split. These calculations are recorded in `ordinary_squareclasses.gp`. Lemma 12.1, with $\chi = \chi_{-5}$, gives residue $1 + \chi E_{12}$, real matrix $C = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and ordinary line e_2. Use the marked ordinary crystalline functor of [46, Proposition 2.4.4, Corollary 2.4.5], pulled back to the matrix problem with this fixed real matrix, support and determinant. At five fix the quadratic characters on inertia, and allow the unramified special signs to vary. The special line is represented without an extra free parameter: for a tame generator normalized by the given sign, its difference from one has unit upper-right entry, so the coordinate of a fixed line in the e_1 chart is determined by its first row. Stability and the character equations on that line and its quotient, including square one of the latter, are then closed equations. The exact calculation is supplied in [52, `ramified`].

Write a homogeneous tangent as $(1 + tX(g))\bar{r}(g)$, where $X(g) = \left(\begin{smallmatrix} x(g) & y(g) \\ z(g) & x(g) \end{smallmatrix}\right)$. Conjugation adds $\chi z(1 + E_{12})$, so z is additive. At two, $y = 0$, x is unramified and z is a unit Kummer class by the finite-flat ordinary extension condition. At both five-primes the special line has the form $e_1 + tt_w e_2$, whence $z = 0$ on the whole decomposition group. The two valuations exclude α, α'; ϵ is a nonsquare at both places. Thus z is a multiple of χ_{-1} and the fixed real matrix kills it. The remaining x, y are additive. The split-at-two calculation gives $y \in k\chi_{-5}$, and $y(c) = 0$ kills it. The unramified-at-two calculation and $x(c) = 0$ leave $x \in k\chi_5$. This last direction is detected by the marked unramified quotient ν on dyadic Frobenius, since χ_5 is locally nontrivial unramified. Cotangent Nakayama makes the complete ring a quotient of $W(k)[[\nu(\mathrm{Frob}_2) - 1]]$.

Choose $\eta = 1$ or χ_{-5} aligning the inertial characters of r and $r^\sigma \otimes \eta$ at both five-primes. The twist is dyadically trivial. The two normalized points agree on the sole generator, since the local involution fixes unramified Frobenius. They agree as marked representations. Lemma 12.4 applies. □

25.5 The integral order of nonordinary scalar systems

Take the tuple at all dyadic coefficient places of the bounded systems under consideration. Every member is nonordinary with trivial residual semisimplification, and the systems have common quadratic inertial signs at each special place. Use the orders R_0, E_0 of Section 11. For m distinct coefficient orbits of common trace-field degree d,

$$\operatorname{rank}_{\mathbf{Z}_2} R_0 = md, \qquad \operatorname{rank}_{\mathbf{Z}_2} E_0 = 4md.$$

Write $E = E_0/2E_0$; its radical has residue field $\mathbf{F}_2$.

Lemma 25.9. *The left regular lattice of E_0 is Barsotti–Tate at two and biconnected. For the simple factors of its torsion with trivial generic character, the middle order-two model is unique. If J is the radical of E, the differences A, B, C from one of a real involution and of the two normalized tame five-inertias generate E as an $\mathbf{F}_2$-algebra, and all three have square zero.*

Proof. Apply Lemmas 11.5 and 11.2 to the regular lattice and its trivial-simple torsion subquotients. These lemmas also hold after the unramified base changes used below.

A radical functional gives a middle/middle extension by quotienting the regular model. At the dyadic place its quadratic conductor is at most two by the height-one Artin–Schreier calculation in the proof of Theorem 11.10. Corollary 11.4 gives their complete space, spanned by $[-1]$, $[\epsilon\alpha]$, $[\epsilon\alpha']$. Their dyadic discriminant exponents are respectively $0, 2, 2$. The latter two ramify at distinct five-primes and split at the opposite prime. Real conjugation and the two tame inertias evaluate in the three coordinate directions. Hence A, B, C span J/J^2. Nilpotence gives algebra generation. The real involution squares to one and each normalized tame generator is unipotent with square-zero difference, proving the lemma. The arithmetic is certified in `scalar_regular_rays.gp`. The exact calculation is supplied in [52, `ramified`]. □

25.6 A quaternion obstruction over the Hilbert class field

The next result is used both for the original reduction and a central quotient of it. Suppose E' is such an actual regular finite-flat quotient, with elementary abelian dyadic-characteristic decomposition image at both five-primes of exponent two. Let M' be its point field, and $P = \operatorname{Gal}(M'H/H)$. All five-primes split in $M'H/H$ because H contains the maximal elementary quadratic completion at five. Its dyadic primes have absolute ramification index two; Fontaine's bound gives upper breaks at most three.

Lemma 25.10. *Assume GRH. The group P is elementary abelian of rank at most two. Its span in E' has dimension at most four and at most two radical generators.*

Proof. Denote the dyadic primes of H by v_1, v_2. The ray groups with all five-prime classes killed at dyadic conductor exponents $(4, 0)$ and $(0, 4)$ are trivial; the corresponding $(2, 2)$ group is C_2^2. If either inertia image were proper in the finite two-group P, it would lie in an index-two subgroup. The associated quadratic extension would contradict one of the first two ray calculations. Thus inertia at either dyadic place is all of P.

Let C_0 be the span of P in E' and J_0 its radical. Its regular representation is the actual submodule $C_0 \subset E'$. At either dyadic completion it therefore has a biconnected finite-flat model. A functional on J_0/J_0^2 gives a middle/middle extension, whose conductor is at most two by the calculation in the proof of Theorem 11.10. The $(2, 2)$ ray calculation gives $\dim J_0/J_0^2 \le 2$. At each dyadic place the ramified Artin–Schreier coordinates of independent such functionals are independent over $\mathbf{F}_2$: a nonzero linear combination with zero ramified coordinate would be unramified, whereas inertia is all of P.

The right action of C_0 extends to its finite-flat model by Lemma 11.2. The quotient by the right radical is the unique middle model. Exactness, Nakayama's lemma and equality of ranks give a Kisin module free of rank one over $(k \otimes_{\mathbf{F}_2} C_0)[[u]]$, where $k = \mathbf{F}_{16}$. Normalize its Frobenius to $u + V(u)$, with $V(u) \in (k \otimes J_0)[[u]]$. Height one makes $u^2(u+V)^{-1}$ integral. Modulo J_0^3 this expression is $u - V + V^2/u$, so $V(0)^2 = 0$.

With one radical generator its nonzero ramified coefficient gives $J_0^2 = J_0^3$, hence $J_0^2 = 0$. With two generators write $V(0) = \alpha X + \beta Y \pmod{J_0^2}$. Here $\alpha, \beta \in \mathbf{F}_{16}$ are $\mathbf{F}_2$-independent. If $\beta/\alpha \notin \mathbf{F}_4$, then $1, \beta/\alpha, (\beta/\alpha)^2$ are independent; otherwise their unique relation is $1 + t + t^2 = 0$. Consequently the classes of $X^2, XY + YX, Y^2$ in J_0^2/J_0^3 are equal to a common element z, which is zero in the first case. If $z = 0$, the associated graded algebra is a quotient of $\mathbf{F}_2[X,Y]/(X^2, Y^2)$, so $J_0^3 = 0$ and C_0 is itself a quotient of this commutative algebra.

If $z \neq 0$, choose two group elements whose augmentation differences form a cotangent basis. In the quotient modulo J_0^3 their squares and commutator are $1 + z$. The kernel over the two-dimensional additive quotient is central elementary abelian. Quotienting it by a complement to $\langle 1 + z \rangle$ gives a quaternion quotient of P. Its three quadratic subfields are the subfields of the unique split-five conductor-$(2,2)$ biquadratic extension of H. For the selected quadratic subfield H'/H, the certificate `scalar_quaternion.gp` gives dyadic break one at both places and the split-five ray group $C_4 \times C_2^5$ at conductor $(6,6)$. Its quotient on which the involution of H'/H acts by inversion is C_2^6; the latter assertion is the Smith reduction obtained by adding the columns of $S + I$ to the ray relation matrix. A quaternion quotient would supply an order-four character of this quotient: Herbrand's formula bounds its break by $1 + 2(3-1) = 5$, and conjugation inverts the rotation character. This is impossible. The class and unit calculation in degree thirty-two uses GRH. The exact calculation is supplied in [52, `ramified/d6_d10_d11/d11`].

We have proved that C_0 is a quotient of $\mathbf{F}_2[X,Y]/(X^2, Y^2)$, so P is elementary abelian and $|P| \leq 8$. To exclude order eight, suppose it occurs. Then C_0 is exactly this four-dimensional algebra and P is its group of augmentation-one units. At either dyadic place write the regular Frobenius as

$$u(1 + fX + gY + hXY), \qquad f, g, h \in u^{-1}k[[u]].$$

After removing the simple factor u, a fundamental solution $1 + xX + yY + wXY$ satisfies

$$x^2 - x = f, \qquad y^2 - y = g, \qquad (w + xy)^2 - (w + xy) = h + fg.$$

Multiplication by $1 + aX + bY + cXY \in P$ translates $(x, y, w + xy)$ by $(a, b, c + ab)$. These three additive coordinates therefore form a basis of the quadratic characters of P. The first two right sides have poles of order at most one, and the third has order at most two. A double pole is equivalent to a simple pole modulo Frobenius minus identity because k is perfect. Every character of P, being an $\mathbf{F}_2$-linear combination of these three, thus has field-of-norms upper break at most one. Independently, the actual finite-flat point field gives classical upper break at most three for every such character by Fontaine's bound. The first break of the uniformizer tower is four and all later transition breaks are larger. Herbrand base change preserves these original breaks, and the ramification comparison in the proof of Theorem 11.10 identifies them with the field-of-norms breaks. Every character consequently has conductor at most two over each H_{v_i}. Thus P is a quotient of the split-five ray group of conductor $(2,2)$, which is C_2^2, a contradiction. □

25.7 Quadratic conjugation and the scalar theorem

Lemma 25.11. *In the setting of Lemma 25.10, if E' is generated by the three real and tame elements above, then* $\dim_{\mathbf{F}_2} E' \leq 24$.

Proof. The group $\mathrm{Gal}(H/K)$ is dihedral of order eight: it has the biquadratic quotient $\mathrm{Gal}(F/K)$, a dyadic Frobenius of order four, and involutions given by real conjugation and five-inertia with independent images in that quotient. The two five-inertias belong to the same reflection class.

Set $\Gamma = \mathrm{Gal}(M'H/K)$. Choose real and tame involutions a, b, c so that b, c have the same image in $\mathrm{Gal}(H/K)$, and put $r = ab$, $c = jb$ with $j \in P$. Write R, B for conjugation by r, b on P. Thus $brb = r^{-1}$ and $Bj = j$. The image of the two-group Γ in $\mathrm{Aut}(P)$ is either trivial or generated by one transvection on a two-dimensional space. In particular $R^2 = 1$.

At the five-place belonging to b, a Frobenius lift, after multiplication by inertia if necessary, has the form $f = pr^2$, $p \in P$. The five-decomposition image in the compositum $M'H$ is elementary abelian, so $f^2 = 1$. Since $R^2 = 1$ and P has exponent two, $f^2 = r^4$, and therefore $r^4 = 1$. Thus $\langle r, b\rangle$ maps isomorphically to the dihedral quotient of order eight. The subgroup $\Gamma_0 = \langle a, b, c\rangle$ is $P_0 \rtimes \langle r, b\rangle$, where $P_0 = \langle j, Rj\rangle$; indeed B fixes j and commutes with R. Its group span maps onto E', even when Γ_0 is proper in Γ.

If $\dim P_0 \le 1$, this gives $\dim E' \le |\Gamma_0| \le 16$. Otherwise $P_0 = P$ has rank two, $R = U$ is a transvection, and j does not belong to its fixed line. Since $B \in \{1, U\}$ fixes j, we have $B = 1$.

Now $\Gamma = \Gamma_0 = P \rtimes D_8$ has order thirty-two. Let $q = R(j)j$ in multiplicative notation. The elements q and r^2 are independent central involutions. The Cayley–Hamilton identity with determinant one makes $r + r^{-1}$ central in E'. Commuting it with j gives $(1+q)(1+r^2) = 0 \quad \text{in } E'$. Indeed $rj = qjr$ and $r^{-1}j = qjr^{-1}$, and the other factors in this commutator are units. The group algebra $\mathbf{F}_2[\Gamma]$ is free of rank eight over the central subalgebra $\mathbf{F}_2[\langle q, r^2\rangle]$. In the latter, the ideal generated by $(1+q)(1+r^2)$ has dimension one. It therefore generates an eight-dimensional ideal of $\mathbf{F}_2[\Gamma]$, and $\dim E' \le 32 - 8 = 24$. □

Theorem 25.12. *Assume GRH. A two-special bounded system all of whose dyadic members are nonordinary and residually scalar is modular.*

Proof. Let $h = \chi_{-1}$ and choose $\eta = 1$ or χ_{-5} aligning the two exact quadratic *inertia* signs of r and $r^\sigma \otimes \eta$. Both characters are invariant under σ; h is unramified at two, trivial at five, and η is trivial at two. The commuting operations $r \mapsto r \otimes h$ and $g : r \mapsto r^\sigma \otimes \eta$ give a packet of at most four coefficient orbits. Denote their number by m and the common degree of their trace fields by d. Each member retains the biconnected dyadic hypothesis. Use their joint reduction E. If their full special signs agree at both places, the five-group already has exponent two. Otherwise at each place, with Frobenius lifts corresponding under σ, the signs on the r, rh packet differ from those on the $g(r), g(r)h$ packet. The discrepancy is the same at both places, since the involution σ exchanges their characters, h splits at five, and η is invariant under σ. A difference of rational signs separates the two coefficient packets.

Write S for the tuple of $+1$ on the first packet and -1 on the second. At each five-place $T(F) = \pm 6S$, so $t = 2S \in R_0$. For normalized tame differences B, C at the two places, the exact special matrices give $BF = \pm SB$ or $CF = \pm SC$. In E this yields $tB = tC = 0$ and $t^2 = 0$. The equation $F^2 - T(F)F + 5 = 0$ makes the Frobenius an involution in $E' = E/tE$. Tame inertia there is an involution commuting with Frobenius by the fifth-power relation. If all signs agree, take $E' = E$ instead. In both cases the five decomposition groups have exponent two. The stable central quotient inherits the finite-flat regular model at two, by closure and quotient of the actual right radical maps; its regular group span, or its restricted group span over H, is precisely the torsion representation used in Lemma 25.10. Lemma 25.9 gives generation of E by A, B, C independently of the Frobenius signs, and therefore generation of E'. Put $R = R_0/2R_0$ and let $\mathfrak{m}$ be its maximal ideal. The polarized identities identify the anticommutators of A, B, C with the scalars $X = T(AB)$, $Y = T(AC)$, $Z = T(BC)$. Ordering words with these relations and the three zero squares leaves eight square-free monomials. Their traces show that X, Y, Z and $W = T(ABC)$ generate $\mathfrak{m}$: the actual trace order is generated by traces of group elements, and

each such element belongs to the algebra generated by A, B, C over $\mathbf{F}_2$. Applying the original R-valued trace to $tAB, tAC, tBC, tABC$ gives $tX = tY = tZ = tW = 0$ in R, independently of faithfulness of its reduced scalar action. Thus $t\mathfrak{m} = 0$. Since t is central and kills B, C, the ideal tE is spanned over $\mathbf{F}_2$ by t, tA. Hence Lemma 25.11 gives $4md = \dim E \leq 24+2 = 26$. An actual selftwist by h is impossible: it would give induction from $K(i)$, while this quadratic extension splits at five and the source has nonzero monodromy there. The bound $\dim E \leq 26 < 32$ and Theorem 1.5 now give either an actual twisted-conjugacy identity involving σ, in which case Lemma 12.4 proves modularity, or a trace field of degree at most two or biquadratic over $\mathbf{Q}$. Every prime above three then has residue degree at most two.

In the small-residue-field cases use the triadic source results above for a solvable residue. A nonsolvable residue stays absolutely irreducible over $K(\zeta_3)$ by Clifford theory. Its trace field realizes it over $\mathbf{F}_3$ or $\mathbf{F}_9$ by Brauer–Nesbitt and the vanishing of the Brauer group of a finite field. Theorems 2.5 and 2.3 of [3], with Lemma 2.1, give modularity of the actual weight-two member and hence of the system. □

25.8 The supported residual base at five

Proposition 25.13. *Assume GRH. Every bounded system is modular. Every absolutely irreducible $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ of determinant ω_5, unramified outside five, is modular.*

Proof. For the two-special menu, Propositions 25.6, 25.7, 25.8 and Theorem 25.12 exhaust the dyadic reductions. Consider now any bounded system in characteristic three. The preceding triadic results handle reducible and cyclotomic-dihedral residues. For an irreducible solvable residue not of the latter form, solvable Artin modularity and weight raising supply a weight-zero witness, and [3, Theorem 2.3] applies. Otherwise the residue remains absolutely irreducible on $K(\zeta_3)$. At both five-places prescribe the genuine special lifts of Lemma 4.15, and retain the given crystalline weight-two component at the unique triadic place. The prescribed-component theorem [31, Theorem 4.2.1] and the weight-two construction in Lemma 2.6 give a pure compatible replacement. A point on the special component has Frobenius eigenvalue ratio five; with zero monodromy this would contradict purity. Thus the replacement is special at both places and modular. A common modular residual witness lifts the actual old weight-two triadic member by [3, Theorem 2.3].

Finally, an irreducible solvable supported five-residue has a regular cuspidal witness by solvable Artin modularity. In the nonsolvable case its restriction to $K(\zeta_5)$ is irreducible. The EGPZ exceptional case cannot occur: $[K(\zeta_5) : K] = 4$, whereas the abelianization of $\mathrm{PGL}_2(\mathbf{F}_5)$ has order two. Use Lemma 4.14 at the split five-places to fix bounded weight-two lifts of the actual local representations. EGPZ gives a global lift, unramified off five, and Lemma 2.6 gives a bounded strongly compatible system. Its modularity proves the supported residual base by reduction. □

26 Special dyadic sources over $\mathbf{Q}(\sqrt{14})$

Put $K = \mathbf{Q}(s)$, with $s^2 = 14$, and write σ for its nontrivial automorphism. Write $\omega_\ell = \bar{\chi}_\ell$, and let κ_a be the quadratic character of a squareclass $[a]$. A bounded five-source is a potentially modular weight-two system of cyclotomic determinant that is good outside five, with type SP or SC at both five-places. Type SP specifies a special parameter with nonzero monodromy and its exact quadratic character. Type SC is one of the two determinant-one tame cuspidal types

$$\widetilde{\omega}_2^{4} \oplus \widetilde{\omega}_2^{20}, \qquad \widetilde{\omega}_2^{8} \oplus \widetilde{\omega}_2^{16}, \tag{31}$$

where ω_2 has order 24. A special source is good outside two and five, special at both five-places, and special at two with unramified quadratic character.

A congruence at a coefficient prime dividing $9 - a_2^2$ allows us to pass to a special source. Its full local residual fields at two and five are elementary quadratic. The repeated-root case $a_2^2 = 8$ uses the regular trace order of Section 11.

26.1 Quadratic arithmetic and discriminant comparisons

Put

$$\varepsilon = 15 + 4s, \qquad \varpi = 4 + s, \qquad a = 3 + s, \qquad H = K(\sqrt{-2}).$$

The Minkowski bound is $\sqrt{56}/2 < 4$. Three is inert and ϖ generates the dyadic prime, so the class number is one. The congruence $x^2 \equiv -1 \pmod 7$ excludes units of norm -1, so the narrow class number is two. The least positive y for which $x^2 - 14y^2 = 1$ is $y = 4$. Hence ε is the fundamental unit, and $\varepsilon = \varpi^2/2$.

Lemma 26.1. *The squareclasses supported at two and five form*

$$W = \langle [-1], [2], [5], [\varpi], [a] \rangle.$$

At the five-places $v : s \equiv 2$ and $w : s \equiv -2$, their local restrictions are

	-1	2	5	ϖ	a
v	0	U	R	0	R
w	0	U	R	U	0

where U is the unramified quadratic class and $R = [5]$. Consequently the kernel of restriction to both five-decomposition groups is $\langle [-1] \rangle$. The kernel at two is $\langle [-2] \rangle$, and the inverse image of the unramified dyadic line is $\langle [-2], [5] \rangle$. The dyadic conductor-at-most-two subspace is $\langle [-1], [2], [5] \rangle$.

Proof. The class number, the fundamental unit and generators of the three supported primes give the first assertion and independence. The five-adic table follows by reduction modulo five. At the place where a is a uniformizer, $a/5 = -1/(3 - s)$ is a square.

At two we have $K_2 = \mathbf{Q}_2(\sqrt{-2})$, since -7 is a square in $\mathbf{Q}_2$. Hence $[-1] = [2]$, and $[5]$ is the nontrivial unramified class. The biquadratic extension $\mathbf{Q}_2(\sqrt{-1}, \sqrt{2})$ has discriminant exponent $2 + 3 + 3 = 8$ over $\mathbf{Q}_2$, so the conductor of $[-1]$ over K_2 is $8 - 2 \cdot 3 = 2$. The Eisenstein polynomials $X^2 + 2X + 1 - a$ and $X^2 - \varpi$ have discriminant valuations four and five in K_2, respectively. They give the conductors of $[a]$ and $[\varpi]$. These successive conductors give four independent local classes and the asserted filtration on W. □

The biquadratic field $H = \mathbf{Q}(\sqrt{-2}, \sqrt{-7})$ has discriminant 56^2. It is split at two over K and unramified quadratic at both five-places. Its Minkowski bound is $84/\pi^2 < 9$. Only the two dyadic primes have norm below nine, since the primes over three, five and seven have residue degree two. If $u = (1 + \sqrt{-7})/2$, then $u^2 + 2 = u$, so $u + \sqrt{-2}$ has relative norm u over $\mathbf{Q}(\sqrt{-7})$ and absolute norm two. This element and its conjugate generate the two dyadic primes. Hence H has class number one.

Define

$$V = K(\sqrt{-1}, \sqrt{2}, \sqrt{5}, \sqrt{\varpi}, \sqrt{a}). \tag{32}$$

Lemma 26.2. *The extension $V/\mathbf{Q}$ is Galois of degree 64, and*

$$\operatorname{disc}(V) = 2^{216} 5^{32} 7^{32}, \qquad R_V := \operatorname{rd}(V) = \sqrt{280}\, 2^{15/8}. \tag{33}$$

Its dyadic completion is the maximal elementary quadratic extension of K_2, of absolute ramification index 16 and residue degree two. At either five-place its completion is the maximal elementary quadratic extension of $\mathbf{Q}_5$. There is an unramified cyclic cubic extension T/L, split at all primes above two and five, where

$$L = K(\sqrt{-5\varpi}) \subset V.$$

In particular TV/V has degree three and the same unramified and splitting properties.

Proof. Independence in Lemma 26.1 gives the degree. The identities

$$[4 - s] = [2] + [\varpi], \qquad [3 - s] = [-1] + [5] + [a]$$

show stability of the squareclass space under σ, proving normality over $\mathbf{Q}$. Its dyadic localization has rank four, the full rank of $K_2^\times/K_2^{\times 2}$; the five-adic table has rank two at each place. This proves the completion assertions.

In the dyadic conductor filtration the dimensions on W are $2, 3, 4, 5$ at conductors $0, 2, 4, 5$. Thus the sum of the dyadic conductors is $4 \cdot 2 + 8 \cdot 4 + 16 \cdot 5 = 120$. At each five-place exactly sixteen characters have conductor one. The relative conductor–discriminant formula gives

$$\operatorname{disc}(V) = 56^{32}2^{120}5^{32}.$$

An equation for L is $X^4 + 40X^2 + 50$. Exact class and unit arithmetic gives

$$\operatorname{Cl}(L) = C_6 \times C_2, \qquad \operatorname{Cl}(L)/\langle [\mathfrak{p}] : \mathfrak{p} \mid 10 \rangle = C_3.$$

The class group and units are fully certified. Class field theory gives T. Since $[V : L]$ is a power of two, $T \cap V = L$. The asserted properties persist under base change. The class group, classes of the primes over two and five, and the resulting quotient are computed in [53, `d14_support/2026-11-07/code/certify_endpoint_arithmetic.py`]. □

Use Proposition 13.3 with $B_{b,100}$. For a norm m, write c_m for the lower bound on the number of primes of norm m divided by the absolute degree. When a residue degree is one, retain only the even Euler powers. Their contribution equals that of a norm with residue degree two, with half the prime density. For the totally real comparison take $r_1 = n$.

Table 2 gives the strict comparisons from outward interval arithmetic [53, `d14_support/2026-11-07/certificates/discriminant_intervals.json`]. In the first three rows use $c_4 = 1/32$ and $c_{25} = 1/4$.

Field condition	$n_{\min}$	b	lower bound $>$	upper bound $<$
special, $q \geq 16$, cubic auxiliary	783360	67/5	61.6606783461	61.3776501696
special, $q = 8$, noninvariant	16257024	82/5	65.1320008831	61.3776501696
special, totally real	120	5	73.3308693122	61.3776501696
nonordinary, $q \geq 8$	2016	147/20	41.44624885	33.46640107

Table 2: Discriminant comparisons. The nonordinary row retains $c_{25} = 1/4$.

26.2 The special source at two

Proposition 26.3. *Assume GRH. A special source with nonsolvable dyadic residue is modular.*

Proof. Let M/K be the point field of its dyadic residue. Dickson's classification and determinant one identify $\mathrm{Gal}(M/K)$ with $G = \mathrm{SL}_2(\mathbf{F}_q)$, where $q \geq 4$ is a power of two. At two the actual source has an ordinary filtration with unramified quadratic quotient and cyclotomic times that quotient as subcharacter. Its two residual diagonal characters are one. The entire residual decomposition image is therefore elementary unipotent. At either five-place the same assertion follows from the special filtration: the quadratic character and the cyclotomic character both reduce to one. Thus all three local point fields are elementary quadratic extensions.

The simple nonabelian group G has no nontrivial solvable quotient, so $M \cap V = K$. Lemma 26.2 absorbs all the local point fields at two and five. Consequently MV/V is unramified at finite places and split at every place above two and five. In particular

$$[MV : \mathbf{Q}] = 64q(q^2 - 1), \qquad \mathrm{rd}(MV) = R_V,$$

and its prime densities are $c_4 = 1/32$, $c_{25} = 1/4$.

The cyclic cubic extension TV/V is disjoint from MV/V: a common nontrivial subextension would be a cyclic quotient of G. The compositum has degree $192q(q^2 - 1)$, unchanged root discriminant R_V, and the same prime densities. For $q \geq 16$ the first row of Table 2 is a contradiction.

If every residual real conjugation were trivial, M would be totally real of degree at least 120. Since its local fields are subfields of those of V, $\mathrm{rd}(M) \leq R_V$. Retaining even Euler powers if a residue degree is one gives $c_4 \geq 1/32$, $c_{25} \geq 1/4$. The third row of the table excludes this possibility. Thus some residual real conjugation is nontrivial.

Suppose $q = 8$. If $M^\sigma \neq M$, simplicity and equality of degrees give $M \cap M^\sigma = K$. The group of their compositum over K is $G \times G$ and has no nontrivial solvable quotient. It is disjoint from V, so

$$[MM^\sigma V : \mathbf{Q}] = 64 \cdot 504^2 = 16257024.$$

Its completion at each of the three supported places is still contained in the corresponding completion of V. Its root discriminant and prime densities are again R_V, $1/32$, $1/4$. The second row of the table is a contradiction. Hence $M^\sigma = M$ and $M/\mathbf{Q}$ is Galois.

Use Lemma 2.5 for $q = 8$ and [3, Theorem 2.5] for $q = 4$ to obtain a modular residual witness.

Apply [73, Theorem 6.2] to obtain an ordinary parallel-weight-two witness after solvable totally real base change. Its real-place hypothesis was checked above. The source stays ordinary, the residual image stays nonsolvable, and the residual ordinary characters at two are both trivial. Apply Lemma 2.3(i), followed by solvable descent. □

26.2.1 Ordinary sources with solvable residue

The ordinary argument applies to both semistable and crystalline sources at two.

Lemma 26.4 (Marked ordinary rigidity)**.** *Let $\mathcal{R}$ be a weight-two system of cyclotomic determinant, good outside two and five and special at both places above five. Suppose its dyadic member r is ordinary with an unramified quotient and scalar residual semisimplification. Then*

$$r^\sigma \simeq r \otimes \eta$$

for a quadratic character η restricted from $G_{\mathbf{Q}}$.

Proof. Let v_0 span the actual ordinary line. Take the lattice generated by its global orbit, scaling v_0 to be primitive. Its residual vector is cyclic. The scalar semisimplification is trivial, so the residue is a nontrivial unipotent extension. Its unique global invariant line is transverse to the reduction of the marked ordinary line; otherwise the cyclic vector could not generate

the residue. Locally at two these two lines are invariant, so the residue is trivial. Lemma 26.1 makes its extension character a nonzero multiple of $h = \kappa_{-2}$. Normalize that multiple to one. For a real conjugation c, the integral basis

$$((r(c) - 1)v_0, v_0)$$

puts its matrix and the ordinary mark in the form

$$C = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad e_2, \qquad \overline{r} = 1 + hE_{12}. \tag{34}$$

Let θ_v, θ_w denote the exact quadratic special characters. The restrictions of h and κ_5 give the full quadratic character group at either five-place. Choose a character $\eta \in \{1, h, \kappa_5, h\kappa_5\}$ whose local value at both places is $\theta_v\theta_w$. Thus r and $r^\sigma \otimes \eta$ have the same exact special characters at five. The twist is unramified at two by Lemma 26.1, so both ordinary quotients are unramified there.

Normalize both representations as in (34). Let B be the closed $\mathbf{Z}_2$-order generated by their simultaneous matrix entries in the product of their coefficient rings. It is finite over $\mathbf{Z}_2$. The product of coefficient rings is integral over B. The two residue maps contract to the same maximal ideal of B, with residue field $\mathbf{F}_2$, by the common reduction (34). Hence B is local. The ordinary line is e_2. At a five-place let $\lambda_v(g)$ be the exact character on the special line. A local Frobenius F_v has $(1, 2)$ entry a unit of B, since $h(F_v) = 1$. The slope of the special line is the element

$$\frac{\lambda_v(F_v) - r(F_v)_{11}}{r(F_v)_{12}} \in B$$

in the simultaneous realization. Here $\lambda_v(F_v)$ is a common rational 2-adic integer. Both special-line conditions therefore hold over B.

For a relative tangent of B to $\mathbf{F}_2[\varepsilon]/(\varepsilon^2)$ write $r(g) = (1 + \varepsilon X(g))\bar{r}(g)$, with

$$X(g) = \begin{pmatrix} x(g) & y(g) \\ z(g) & x(g) \end{pmatrix}.$$

Conjugation by $1 + h(g)E_{12}$ adds $h(g)z(g)(I + E_{12})$. Thus z is additive. At two the marked ordinary equations give $y = 0$ and x unramified; no finite-flat condition is imposed on z. At a five-place, write the special line as $e_1 + \varepsilon t_v e_2$. Its exact character equation gives

$$z|_{G_v} = 0, \qquad x|_{G_v} = t_v h|_{G_v}.$$

The additive character z is supported at two and five. The kernel of the two five-localizations on W gives $z = A\kappa_{-1}$. The fixed real matrix makes $z(c) = 0$, so $A = 0$.

Now x and y are additive. The dyadic restrictions in Lemma 26.1 give

$$y = b_0 h, \qquad x = c_0 h + d_0 \kappa_5.$$

At five, $h = U$ and $\kappa_5 = R$; the exact line equation forces $d_0 = 0$. The fixed real matrix gives $x(c) = y(c) = 0$, hence $b_0 = c_0 = 0$. The relative cotangent of B is zero. Nakayama gives $\mathfrak{m}_B = 2B$, and then $B = \mathbf{Z}_2$, since B is finite over $\mathbf{Z}_2$ with $B/2B = \mathbf{F}_2$. The simultaneous normalized matrices are equal, so $r^\sigma \simeq r \otimes \eta$. □

Proposition 26.5. *Let $\mathcal{R}$ be a weight-two system of cyclotomic determinant that is good outside two and five and special at both five-places. Suppose its actual dyadic member is ordinary with unramified quotient and has solvable residual image. Then $\mathcal{R}$ is modular.*

Proof. An irreducible residue is induced from an odd-order character of a quadratic extension L/K with class in W. If L splits at two then Lemma 26.1 gives $L = H$. In this case the inducing character is unramified at two because the ordinary residual diagonal characters are unramified. It is unramified at five because the residual inertia there is a two-group. The class number one of H excludes this case. The ordinary modular witness in [2, Lemma 5.1.2] has the required ordinary ordering. Theorem 5.2.1 there applies to the actual dyadic member because the inducing field is nonsplit at two.

In the reducible case both characters have odd order and are unramified at every finite place. The narrow class group of order two makes the semisimplification scalar. Apply Lemmas 26.4 and 12.4. □

Theorem 26.6. *Assume GRH. Every special source is modular.*

Proof. Use Proposition 26.3 for a nonsolvable dyadic residue and Proposition 26.5 for a solvable one. A special source is an actual ordinary source at two. □

26.3 Odd coefficient primes

The following lemma isolates the residual cases in which prescribed lifting with irreducible cyclotomic restriction is unavailable.

Lemma 26.7. *Let $\mathcal{R}$ be a system good outside five and special at both five-places. Let $\lambda \mid \ell$ be an odd coefficient place. Suppose*

$$a_2^2 \equiv 9 \pmod{\lambda}. \tag{35}$$

If $\overline{r}_\lambda$ is reducible, or becomes reducible on $G_{K(\zeta_\ell)}$, then $\mathcal{R}$ is modular. For $\ell = 3$ the same conclusion holds without (35)*, and with either of the types* (31) *also permitted at five.*

Proof. Suppose first that $\overline{r}_\lambda^{\mathrm{ss}} = \psi \oplus \omega_\ell \psi^{-1}$. At five the square ψ^2 is unramified away from the coefficient characteristic. For the cuspidal types in characteristic three the cubic part of inertia reduces to one. Away from five and ℓ it is unramified as well. We have

$$N_{K/\mathbf{Q}}(\varepsilon^2 - 1) = -896 = -2^7 \cdot 7. \tag{36}$$

If $\ell \neq 5, 7$, the field is unramified at ℓ and the source is crystalline of weights $\{0, 1\}$. The rank-one finite-flat inertia exponents have digits zero or one [57]. Let the two digits corresponding to the embeddings of K be $a_1, a_2 \in \{0, 1\}$. Unit reciprocity for ψ^2 and $N(\varepsilon) = 1$ give

$$\varepsilon^{\,2(a_1 - a_2)} \equiv 1 \pmod{\ell},$$

with the evident interpretation in the quadratic residue field when ℓ is inert. By (36), $a_1 = a_2$. The two local characters are therefore the etale and multiplicative endpoints. Finite-flat uniqueness in absolute ramification index one forces the actual source to be ordinary: a nonordinary Barsotti–Tate group and its dual would both be connected, and could have no such endpoint subquotient.

At $\ell = 5$ the source is already ordinary at both places. Removing the quadratic special characters, write its two orientation digits as $a_v, a_w \in \{0, 1\}$. The reductions of ε at the five-places are 3 and 2, both of order four. Reciprocity for ψ^2 gives $a_v = a_w$.

At $\ell = 7$ the unique local field has absolute ramification index two, smaller than $\ell - 1$. Its rank-one finite-flat exponents are $j = 0, 1, 2$, and ω_7 has exponent two. If $j = 1$, the character $\psi^2 \omega_7^{-1}$ is unramified at every finite place. Its order divides the narrow class number, which is two. On the other hand (35) says that the two Frobenius eigenvalues at two are $\delta, 2\delta$, for $\delta \in \{1, -1\}$. Their quotient is 2 or 4 in characteristic seven, of order three. This is the value of

$\psi^2\omega_7^{-1}$ at that Frobenius, a contradiction. Hence $j = 0$ or 2, and the actual source is ordinary by finite-flat uniqueness, now using $2 < 7 - 1$.

In all remaining reducible cases, the common endpoint orientation is $a \in \{0, 1\}$. The character $\psi^2\omega_\ell^{-2a}$ is everywhere unramified and totally even. Since $h_K = 1$, it is trivial. Thus the constituent quotient is ω_ℓ or ω_ℓ^{-1}. It is totally odd, extends to $G_{\mathbf{Q}}$, and is nontrivial at every ℓ-adic decomposition group. The actual ordinary quotient matches the same global constituent at each such place. The field $K(\zeta_\ell)$ is abelian over $\mathbf{Q}$. Consequently [66, Section 4.5, Theorem A] applies to the actual source. Its ordinary quotient divided by the Teichmüller lift of its residue is pro-ℓ and has finite inertia, as required.

Suppose next that the residue is irreducible but its cyclotomic restriction is reducible. Clifford theory writes it as induction from the unique quadratic subextension

$$L_\ell = K\big(\sqrt{(-1)^{(\ell-1)/2}\ell}\big) \subset K(\zeta_\ell).$$

The inducing character has order prime to ℓ, and the residual image has order prime to ℓ.

For $\ell \neq 5, 7$, L_ℓ/K is ramified quadratic at every ℓ-place. Tame inertia is cyclic and its projective dihedral image contains a reflection, so that image has order at most two. An irreducible finite-flat local residue over an unramified field of degree one has projective inertia order $\ell + 1$ [57]; over degree two it has order $(\ell^2 + 1)/2$. Neither is at most two. The local residue is therefore split reducible. The quotient orders in the rank-one finite-flat list are $\ell - 1$ for the endpoint pair and, in residue degree two, $\ell + 1$ for the mixed pair. A quadratic quotient is possible only when $\ell = 3$, at the endpoints. The endpoint characters force the actual source to be ordinary by the same finite-flat uniqueness argument. The residual ordinary characters are distinct. Allen's [2, Lemma 5.1.2] supplies a modular ordinary witness with the required quotient. Skinner's corrected theorem, in [80, Theorem 4.2], applies because the unique three-place is ramified, not split, in the CM extension $K(\zeta_3)$. This modularizes the actual three-adic source.

For $\ell = 5$, special inertia has projective order four after reduction. But L_5/K is ramified there and cyclic tame inertia containing a reflection has order at most two. This is impossible.

It remains to consider $\ell = 7$. Now $L_7 = H$. Write the residue as $\mathrm{Ind}_{G_H}^{G_K}\xi$. Its image has order prime to seven, so the special monodromy at five reduces to zero and its inertia acts by a quadratic scalar. Thus ξ^2 is unramified outside seven. For u as above and its conjugate $\bar{u}$, the ideals in H are $(u) = P^2$ and $(\bar{u}) = P'^2$, where P, P' are the two dyadic primes. At the unique seven-place, $u/\bar{u}$ is a principal unit. The prime-to-seven character ξ^2 is trivial on that unit group. Global reciprocity therefore gives

$$\left(\frac{\xi(\mathrm{Frob}_P)}{\xi(\mathrm{Frob}_{P'})}\right)^4 = 1.$$

Since H/K splits at two, the quotient in parentheses is the quotient of the two residual Frobenius eigenvalues there. By (35) it has order three, a contradiction. □

Lemma 26.8. *A bounded five-source whose trace field has degree at most two is modular.*

Proof. Take a coefficient place above three. Its trace residue field is $\mathbf{F}_3$ or $\mathbf{F}_9$. For an irreducible residue, finite-field trace and determinant descent realizes it over that field; the Brauer group of a finite field is zero. The residual modularity theorem [3, Theorems 1.1 and 2.5] applies. If the cyclotomic restriction is irreducible, apply Lemmas 2.1 and 2.2(ii). Lemma 26.7 handles the other two residual cases; its characteristic-three assertion does not require (35). □

26.4 Nonordinary crystalline sources

Proposition 26.9. *Assume GRH. A system good outside five and special at both five-places is modular if it has a nonordinary crystalline dyadic member with nonsolvable residue.*

Proof. Write its residual image as $\mathrm{SL}_2(\mathbf{F}_q)$, $q \geq 4$, and its point field as M. At both five-places the residual decomposition group is elementary abelian by the special filtration, with ramification and residue degrees at most two.

Theorem 11.10 bounds the normalized dyadic different contribution by $1 - 1/q$. In the wild case the inertia has order $e \leq q$ and upper breaks at most one, giving the bound $1 - 1/e$. Tame inertia of order three contributes $1/3$.

The fields M and H are disjoint over K. The extension H/K is unramified, split at two and inert at five. Thus

$$[MH : \mathbf{Q}] = 4q(q^2 - 1), \qquad \mathrm{rd}(MH) \leq \sqrt{280}\, 2^{1-1/q}.$$

Over each five-place of K the compositum MH has residue degree two and ramification index at most two. Thus $c_{25} \geq 1/4$ and the nonordinary row of Table 2 excludes $q \geq 8$. For $q = 4$ the kernel field has degree 120 and root discriminant at most $\sqrt{280}\, 2^{3/4} < 28.15$. Lemma 13.4 excludes a totally real kernel field, so the actual crystalline member is modular by Lemma 2.4. □

Lemma 26.10. *For a system good outside five, special at both five-places, a nonordinary dyadic residue with irreducible solvable image is locally irreducible at two.*

Proof. Such a residue is induced from an odd-order character of a quadratic extension L/K. Its inducing quadratic character has conductor at most two at two. For a reducible local residue this follows from Lemma 11.2 after unramified base change; for an irreducible one, its odd dihedral inertia is tame of order three and its quadratic quotient is unramified. Thus L lies in the seven nontrivial classes of $\langle [-1], [2], [5] \rangle$.

The odd inducing character is unramified at five and elsewhere away from two; its dyadic ramification is tame. The ray groups of these seven fields modulo (2) are

c	-1	2	-2	5	-5	10	-10
$\mathrm{Cl}_{(2)}(K(\sqrt{c}))$	C_8	1	C_2^2	C_2	$C_8 \times C_4$	C_2	$C_{12} \times C_2$.

The class groups and ray invariants are computed in [53, `d14_support/2026-11-07/certificates/endpoint_arithmetic.json`]. Only $c = -10$ has an odd part, necessarily of order three. Its ordinary class group is C_2, so the cubic character is ramified at two. The extension $K(\sqrt{-10})/K$ is unramified quadratic and nonsplit at two; induction gives the irreducible local tame S_3 representation. □

Proposition 26.11. *Assume GRH. A system good outside five and special at both five-places is modular if $a_2^2 = 8$.*

Proof. Potential modularity and the special finite place give an abelian realization after a finite extension: apply [68, Lemma 9.4.3] to the weight-two modular form, using a coefficient prime different from five. The special parameter remains square-integrable after base change. The abelian variety has good reduction above two, by compatibility and the Néron–Ogg–Shafarevich criterion. Frobenius of an abelian variety over a finite field is semisimple. A power of the original crystalline Frobenius is therefore semisimple, and so is the original Frobenius in characteristic zero.

The arithmetic crystalline polynomial is $X^2 - a_2 X + 2$, with a repeated root. Hence its Frobenius is scalar, equal to $a_2/2$. In the covariant filtered module its Newton slope is $1/2$.

After enlarging coefficients, there are two labelled filtration lines, one for each embedding of the ramified quadratic completion K_2. They cannot coincide: a common line would have Hodge number one and Newton number $1/2$, contradicting weak admissibility. Each of the two distinct lines has Hodge and Newton numbers $1/2$, and is a weakly admissible subobject. Thus the actual local representation splits as two crystalline characters with mixed labelled weights $(1,0)$ and $(0,1)$. In particular it is nonordinary.

This applies at every dyadic coefficient embedding. An irreducible solvable global residue would be locally irreducible by Lemma 26.10, contradicting local splitting of the actual source. A nonsolvable residue is covered by Proposition 26.9. Otherwise all dyadic residual semisimplifications are scalar: their one-dimensional constituents have odd order and are unramified at two, have trivial inertia at five, and are unramified elsewhere, so the narrow class number two makes them trivial.

Apply Proposition G.1 to all dyadic coefficient realization of this last system. The rational simultaneous trace algebra is $D \otimes_{\mathbf{Q}} \mathbf{Q}_2$ by Section 11. Since a_2 generates this algebra, $[D : \mathbf{Q}] = [\mathbf{Q}(a_2) : \mathbf{Q}]$, giving

$$D = \mathbf{Q}(a_2) = \mathbf{Q}(\sqrt{2}).$$

Lemma 26.8 proves modularity. □

26.5 The bounded five-source theorem

Theorem 26.12. *Assume GRH. Every system good outside five and special at both five-places is modular.*

Proof. Put $B = 9 - a_2^2 \in \mathcal{O}_D$. Every conjugate of B is at least one. If $N_{D/\mathbf{Q}}(B) = 1$, every conjugate equals one, so $a_2^2 = 8$ and Proposition 26.11 applies.

Otherwise choose a coefficient prime dividing B. Suppose first that it lies over an odd prime ℓ. Lemma 26.7 handles a reducible residue and an irreducible residue with reducible cyclotomic restriction. In the other case, the actual residual representation at two is unramified, with Frobenius eigenvalues $\delta, 2\delta$, $\delta \in \{1, -1\}$. They are distinct. It has a genuine special lift with determinant χ_ℓ: use Frobenius $\delta\,\mathrm{diag}(2,1)$ and nonzero tame monodromy parameter divisible by the coefficient uniformizer. This reduces to the same actual unramified residual representation.

Retain the old weight-two components at ℓ, the exact special characters at five and the unramified conditions elsewhere. Lemma 2.6(i) gives a pure compatible system, and purity retains the special monodromy (Section 2). Theorem 26.6 makes this system modular. Lemmas 2.1 and 2.2(ii) transfer modularity to the old weight-two member.

Suppose instead that the chosen coefficient prime is dyadic. Then a_2 is a unit. Its crystalline polynomial has a unique unit-root line. Weak admissibility and the nonnegative labelled Hodge numbers make this line have Hodge number zero at both embeddings; it supplies an unramified ordinary quotient of the original representation. Its two residual ordinary characters are trivial, because their values at Frobenius are $a_2 \bmod \lambda = 1$.

A solvable global residue is covered by Proposition 26.5. For a nonsolvable residue we replace the ordinary crystalline local source at two by a special one with the same actual residue. After normalizing the two residual characters to one, Kummer theory gives

$$H^1(K_2, \mathcal{O}_E(1)) \longrightarrow H^1(K_2, k_E)$$

surjectively: $K_2^\times \widehat{\otimes} \mathbf{Z}_2$ surjects onto $K_2^\times / K_2^{\times 2}$, and then one extends coefficients. Choose a lift of the given extension and make its valuation nonzero by adding a uniformizer Kummer class multiplied by the coefficient uniformizer. The resulting extension of 1 by χ_2 is semistable noncrystalline, with exactly the original residual extension and nonzero monodromy.

Prescribe this special component, retaining the exact special components at five and unramifiedness elsewhere. Lemma 2.6(iii) applies to the nonsolvable residue and the ordinary semistable component of weight two. The resulting special source is modular by Theorem 26.6. Both ordinary residual characters at two are trivial, so its dyadic member supplies a matching witness. Apply Lemma 2.3(i) to the original ordinary crystalline member. □

Theorem 26.13. *Assume GRH. Every bounded five-source is modular.*

Proof. Choose a coefficient place above three. Apply Lemma 26.7 if the cyclotomic restriction of the residue is reducible. Otherwise use Lemma 4.15 at the cuspidal five-places, retain the special places and the crystalline weight-two component at three, and impose unramified conditions elsewhere. Lemma 2.6(i) gives a pure system special at both five-places. Theorem 26.12 and Lemmas 2.1 and 2.2(ii) give modularity of the original system. □

Theorem 26.14. *Assume GRH. Every irreducible representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ of determinant ω_5, unramified at finite places away from five, is modular.*

Proof. The determinant makes the representation totally odd. Solvable image has an odd solvable Artin witness. Otherwise the cyclotomic restriction is irreducible, and the characteristic-five exception to Lemma 2.6 is absent over K. Apply Lemma 4.14 at both five-places and prescribe unramifiedness elsewhere. Lemma 2.6(i) gives a bounded five-source. Theorem 26.13 gives its modularity, and reduction gives the residual assertion. □

27 A one-sided terminal source over $\mathbf{Q}(\sqrt{15})$

Theorem 27.1. *Assume GRH for Dedekind zeta functions. Let $K = \mathbf{Q}(\sqrt{15})$ and let r be the seven-adic terminal member of Proposition 6.2, with determinant χ_7. Then r is modular.*

Write v_+, v_- for the two seven-places as in Proposition 6.2. The actual member is absolutely irreducible at v_+ and crystalline of weights $\{0, 1\}$ at v_-.

27.1 Irreducible terminal residues

For an irreducible terminal residue, Proposition 6.5 gives cyclotomic irreducibility and handles the solvable case.

For a nonsolvable residue keep the actual crystalline component at v_- and use Lemma H.1 at v_+. Write CR for scalar quadratic inertia with zero monodromy, SP for quadratic special type with nonzero monodromy, and C4, C8 for determinant-one tame cuspidal types of orders four and eight. Lemma 2.6(i) gives a parallel-weight-two comparison system with determinant χ and one of the profiles

$$\mathrm{CR/CR}, \qquad \mathrm{SP/CR}, \qquad \mathrm{C4/CR}, \qquad \mathrm{C8/CR}.$$

Each profile records the type at v_+ first. The systems are good away from seven. The following argument proves their modularity, and Lemma 2.2(i) then applies to the original seven-adic member.

27.2 Dyadic comparison systems

For a nonsolvable dyadic residue that is unramified at both seven-places, or for a nonordinary nonsolvable dyadic source in any of the four profiles, the comparison is immediate. Adjoin

$K(i)$ to the residual point field. This quadratic extension is finite-unramified and disjoint from the residual point field. For $q \geq 8$ the compositum has degree at least 2016. Omit all finite prime terms in (11). At $b = 7.35$, $N = 100$, the lower bound is greater than 31.07694. If both seven-places are unramified the upper bound is $4\sqrt{60} < 30.98387$. For a nonordinary member with one ramified seven-place the upper bound is $2\sqrt{60}\,7^{1/4} < 25.19882$. These bounds follow from Lemma 11.9 and Theorem 11.10, respectively, and exclude $q \geq 8$. For $q = 4$ the source is modular by Lemma 2.4. Indeed, throughout the four profiles Lemma 11.9 gives $\mathrm{rd}(M) \leq 4\sqrt{60}\,7^{1/4}2^{-2/q}$. Since $4\sqrt{60}\,7^{1/4} < 4\sqrt{220}$, Corollary 13.5(i) makes M non-totally real.

For the solvable comparison, the supported quadratic classes with even dyadic valuation are generated by $-1, 2, 3, 7$. This follows from ideal-class parity at the two seven-places, which have the same nontrivial ordinary class, and the unit identity $4 + \sqrt{15} = (3 + \sqrt{15})^2/6$. An odd inducing character of a dihedral residue is unramified at seven, where the residual inertia has order at most two. At two it is at most tame, and the dyadic inducing quadratic cannot have odd valuation by the finite-flat bound. The ray calculation modulo (2) over the fifteen quadratic fields leaves a single odd quotient C_3, for $K(\sqrt{-21}) = K(\sqrt{-35})$, by the fifteen ray computations in [53, `d15_support/terminal_arithmetic/d15_dihedral.gp`]. Its quadratic character ramifies at both seven-places, so no irreducible solvable residue occurs in a one-sided profile. For a reducible residue the diagonal characters are unramified at all finite places, since their orders are odd, and hence are trivial.

When the dyadic semisimplification is scalar, the nonordinary cases and the ordinary CR/CR case follow from Propositions H.3 and H.4. For the other ordinary scalar cases it suffices to consider systems all of whose dyadic members have trivial residual semisimplification and are ordinary. The two cuspidal profiles are covered by Theorem H.5, and the special profile by Proposition H.6.

It remains to treat an ordinary nonsolvable residue ramified at the strict seven-place, with kernel field M and image $\mathrm{SL}_2(\mathbf{F}_q)$, and let $C = K(i, \sqrt{3})$. The strict special or cuspidal place gives residual local index two and residue degree at most two. The good crystalline place may have an unramified Frobenius of odd order. In particular the two-place Euler-term estimate from the strict menu does not apply.

The compositum MC is finite-unramified over M. All the following auxiliary fields retain the tame ramification bound at seven and the dyadic upper break bound three, giving

$$\mathrm{rd} \leq 4\sqrt{60}\,7^{1/4} < 50.39764.$$

Appendix I.1 constructs the solvable auxiliaries and applies Proposition 13.3 with Lemma 11.11. The degree-1024 and degree-2048 auxiliaries over K exclude $q \geq 512$ and $q = 256$, respectively. The latter ray class computation uses GRH. In the remaining finite range the directed comparisons leave the following rows. The dyadic wild inertia has order q and the ramified seven-place has residue degree two in every row.

q	dyadic multiplier orders	good seven-Frobenius orders
8	7	3, 7, 9
16	5, 15	3, 5, 15, 17
32	31	3, 11, 31, 33
64	9, 21, 63	5, 7, 9, 13, 21, 63, 65
128	127	43, 127, 129

In every surviving row the strict seven-place has residual decomposition group V_4. In Appendix I we compare the congruence fields of two prescribed order-four cuspidal lifts, distinguished by their scalar inertial signs at one place over eleven. The comparison excludes

$q \geq 16$ and forces all remaining $\mathrm{SL}_2(\mathbf{F}_8)$ point fields with the same ramified seven-place to agree. Appendix J then compares their simultaneous trace orders and excludes a nonmodular residue. The actual dyadic member is modular by Lemma 2.4, using the preceding application of Corollary 13.5.

27.3 Reducible terminal residues

Lemma 6.7 gives an odd cyclotomic power as the constituent quotient. It extends to $G_{\mathbf{Q}}$ and is nontrivial at both seven-places. Proposition 6.5 applies to the actual terminal member. At v_- its finite-flat characters give quotient $\omega_7^{\pm 1}$ on inertia.

28 Terminal sources over $\mathbf{Q}(\sqrt{19})$

Put $K = \mathbf{Q}(s)$ with $s^2 = 19$. Write κ_b for the quadratic character of a squareclass b. The primes

$$\mathfrak{v}_1 = (4 - s), \qquad \mathfrak{v}_2 = (4 + s), \qquad \mathfrak{p}_2 = (13 + 3s)$$

satisfy $3\mathcal{O}_K = \mathfrak{v}_1\mathfrak{v}_2$ and $2\mathcal{O}_K = \mathfrak{p}_2^2$. The two three-adic completions are $\mathbf{Q}_3$. The dyadic completion is $\mathbf{Q}_2(\sqrt{3})$, since $19/3$ is a square in $\mathbf{Q}_2$.

A terminal system in this section has weight two, cyclotomic determinant and good reduction away from three. Its local parameters above three are quadratic twists of special (SP) or good crystalline (CR) parameters. We retain the exact quadratic characters in the special case. We will verify the supported base at three for Theorem 1.3 by bounding the simultaneous dyadic trace orders and comparing them with two modular systems.

Theorem 28.1. *Assume GRH. Every terminal system over K is modular. Every absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ of determinant $\overline{\chi}_3$ unramified outside three is modular.*

28.1 Arithmetic and local lifts

Put

$$\epsilon = 170 + 39s, \quad \pi = 13 + 3s, \quad a = 4 + s, \quad a' = 4 - s.$$

The identities $\pi^2 = 2\epsilon$, $aa' = -3$ and $\mathrm{N}_{K/\mathbf{Q}}(\pi) = -2$ will be used throughout. The class number is one: the Minkowski bound is $\sqrt{19} < 4.36$, and the primes of norms two and three are principal. The continued fraction of $\sqrt{19}$ has period $(2, 1, 3, 1, 2, 8)$, so the fundamental unit is ϵ, of norm one. In particular the narrow class group has order two.

Let S consist of the places over two and three and the real places. We write a radicand and its quadratic character interchangeably when describing an $\mathbf{F}_2$-vector space. The supported squareclasses are

$$K(S, 2) = \langle -1, \epsilon, \pi, a, a' \rangle. \tag{37}$$

The even-dyadic-valuation subspace is $\langle -1, \epsilon, a, a' \rangle$. Its localization kernel at $K_{\mathfrak{p}_2}$ is $\langle 3 \rangle$, and the inverse image of the unramified local line is $\langle -1, 3 \rangle$. The even-valuation classes split at both three-primes form $\langle -\epsilon \rangle = \langle -2 \rangle$. Indeed the two odd-prime valuations eliminate the exponents of a, a'. Both -1 and ϵ are nonsquare units at both three-primes. The dyadic localization assertions also follow directly by reducing units modulo eight in $\mathbf{Z}_2[s]$.

Lemma 28.2. *The supported quadratic characters of dyadic conductor at most two form*

$$V_2 = \langle -1, 3, 2a \rangle.$$

Write U for the unramified quadratic character of $\mathbf{Q}_3$ and $R = \kappa_3$ for one of its two ramified characters. With additive character notation, the localization table is

	$\mathfrak{v}_1$	$\mathfrak{v}_2$	*negative real embedding*
-1	U	U	1
3	R	R	0
$2a$	0	$U + R$	1

The ramified dyadic coordinate in this basis is $(0, 0, 1)$.

Proof. The supported unit squareclasses at two have basis $\langle -1, 2, 3, 2a\rangle$. The class 3 is locally square, -1 is unramified, 2 has conductor four, and every odd-valuation class has conductor five. The class $2a$ has conductor two. For the last assertion, use $[\epsilon] = [2]$ and

$$\epsilon a = 1421 + 326s, \qquad \epsilon a - (s+2)^2 = 4 \cdot 269 + 2 \cdot 161(s+1).$$

The right side has dyadic valuation three, giving quadratic defect three. The local character table follows from the reductions modulo three and nine. At $\mathfrak{v}_2$, $2a/3$ is a nonsquare unit. □

Lemma 28.3. *Every continuous representation $\bar{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ of determinant $\overline{\chi}_3$ has a determinant-χ_3 lift of labelled weights $\{0, 1\}$ which becomes semistable after a quadratic twist. The lift preserves the actual residual extension. It therefore has one of the types* SP *or* CR.

Proof. If $\bar{r}$ is reducible, a quadratic twist puts its invariant character and quotient in the order $\overline{\chi}_3\lambda, \lambda^{-1}$, with λ unramified. Let η be its Teichmüller lift. If $\lambda^2 \neq 1$, local duality gives $H^2(G_{\mathbf{Q}_3}, \mathcal{O}(\chi_3\eta^2)) = 0$. The given residual extension therefore lifts integrally. The characteristic-zero extension space is one-dimensional and crystalline. If $\lambda^2 = 1$, choose $\eta^2 = 1$. Kummer theory lifts both the unit and the valuation directions in $\mathbf{Q}_3^\times/\mathbf{Q}_3^{\times 3}$. These are semistable extensions of 1 by χ_3. The unit direction is crystalline, and a nonzero valuation direction has nonzero monodromy. Twisting back preserves the determinant.

If $\bar{r}$ is irreducible, wild inertia is trivial. Its tame inertia exponents are $\{1, 3\}$ or $\{5, 7\}$ modulo eight, and a ramified quadratic twist interchanges them. Induce a crystalline character of weights $(1, 0)$ from the unramified quadratic extension, matching the residual Frobenius by its unramified character. Its determinant divided by χ_3 is crystalline of weight zero, hence unramified, and reduces to one. Twisting by the inverse square root, chosen to reduce to one, gives the required determinant. □

28.2 Dyadic residual images

Write R_0, E_0 for the simultaneous trace and matrix orders of Section 11. We use the construction also for a subcollection of dyadic coefficient embeddings, with the corresponding product of local trace factors. The regular lattice and the exact comparison for biconnected groups with trivial generic simple factors are given by Lemmas 11.2 and 11.5. For this scalar category, Lemma 11.6 applies with the Eisenstein polynomial $u^2 - 26u - 2$ of π.

Lemma 28.4. *Every absolutely reducible dyadic residue of a terminal system has trivial semisimplification.*

Proof. Its constituent characters have odd order. At three, special inertia reduces to a unipotent group and crystalline scalar inertia reduces to the identity, so these characters are unramified. At two their abelian odd-order inertia is trivial, because wild inertia is pro-two and the residue field is $\mathbf{F}_2$. They are unramified at every other finite place and at infinity. The class number one assertion kills both characters. □

Proposition 28.5. *An ordinary irreducible solvable dyadic source of a terminal system is modular. A nonordinary irreducible solvable dyadic residue is the representation V cut out by the S_3 splitting field of*

$$x^3 - 2x - 2. \tag{38}$$

The latter field is unramified outside two over K, with local inertia C_3 and unramified quadratic quotient at two. At each three-prime its completion is unramified quadratic.

Proof. An absolutely irreducible solvable subgroup in characteristic two is dihedral, induced from an odd-order character of a quadratic field H/K. For an ordinary source, Fontaine's bound excludes odd dyadic valuation for the inducing radicand, since that conductor is five. The possible CM classes in $\langle -1, \epsilon, a, a' \rangle$ are $-1, -2, -3, -6$. None is square in $K_{\mathfrak{p}_2} = \mathbf{Q}_2(\sqrt{3})$: the first and third give the unramified quadratic extension, and the others are ramified. Thus every CM inducing field is nonsplit at the dyadic place. Allen's ordinary theorem [2, Theorem 5.2.1], with the ordinary witness of [2, Lemma 5.1.2], applies to the actual source. Its extra nonsplitting condition is imposed precisely in the CM-dihedral case.

Suppose now that the source is nonordinary. If the dyadic image contains a nontrivial rotation, its wild inertia is trivial, since a nontrivial order-two subgroup of a dihedral group with odd rotations cannot be normalized by a rotation. The tame inertia has odd order, since a tamely ramified extension of a dyadic field has odd ramification index. Its inducing quotient is therefore unramified. If its inducing quotient is ramified, the local residual image has order two, and the middle-model conductor bound applies to the quadratic quotient. In either case its inducing quadratic character belongs to V_2. The odd inducing character is unramified at three and tame at two. There are seven nontrivial inducing classes. The four mixed-sign fields are represented, up to conjugation over $\mathbf{Q}$, by $K(\sqrt{2a})$ and $K(\sqrt{-2a})$. Both have class number one and dyadic residue field $\mathbf{F}_2$, so neither admits the required odd character. For completeness, the defining quartics, discriminants and class groups needed for all seven cases are

polynomial	discriminant	Cl
$x^4 - 16x^2 - 12$	-69312	1
$x^4 + 16x^2 - 12$	-69312	1
$x^4 - 44x^2 + 256$	51984	1
$x^4 - 32x^2 + 484$	51984	C_2
$x^4 - 36x^2 + 400$	5776	1

The last three fields are $K(\sqrt{3})$, $K(\sqrt{-3})$ and $K(i)$. The class groups are certified by the accompanying exact computation.

The dyadic residue fields in $K(\sqrt{3})$ are $\mathbf{F}_2$. In $K(\sqrt{-3})$, the unit ζ_3 generates the odd part of the dyadic residue units. Thus these fields also have no required odd character. In $K(i)$, put

$$w = \pi(1 + i)/2, \qquad w^2 = i\epsilon.$$

The Hasse unit index is two and the units are generated by i, w. Both reduce to one at the dyadic prime of $K(i)$, whose residue field is $\mathbf{F}_4$. The odd tame dyadic ray group is therefore C_3. It gives exactly one dihedral representation over $\overline{\mathbf{F}}_2$.

The polynomial (38) realizes it. Its discriminant is -76 and it is Eisenstein at two. Its quadratic resolvent over K is $K(i)$. At nineteen its ramified quadratic part is absorbed by $K/\mathbf{Q}$, leaving an unramified extension. At three it factors as $(x - 1)(x^2 + x + 2)$, with distinct factors. These facts give the stated local properties and identify the surviving representation. □

Proposition 28.6. *Under GRH, a terminal system with a nonsolvable dyadic residue is modular, for each of the four* SP/CR *menus.*

Proof. Let M/K be the residual kernel field, with group $\mathrm{SL}_2(\mathbf{F}_q)$. The proof of Lemma 11.9 gives relative different contribution at most $2 - 2/w$ in the wild case, with elementary abelian inertia of order $w \le q$ and upper breaks at most three. In the tame case the inertia order is at most three.

For the SP/SP menu, adjoin $C = K(i, \sqrt{3})$. This degree-four extension is unramified at two, and its completion at either three-prime is the full elementary quadratic extension of $\mathbf{Q}_3$. This completion contains the corresponding residual field, since the special filtration has trivial residual diagonal characters. Simplicity gives $M \cap C = K$. Thus $[MC : \mathbf{Q}] = 8q(q^2 - 1)$, its local invariants at either three-prime are $(e, f) = (2, 2)$, and

$$\mathrm{rd}(MC) \le \sqrt{228}\, 2^{2-2/q}. \tag{39}$$

When at least one CR place occurs, let A/K be the S_3 field in Proposition 28.5. Again $M \cap A = K$, and

$$[MA : \mathbf{Q}] = 12q(q^2 - 1).$$

Its dyadic inertia has order $3w$ in the wild case. Positive upper ramification groups inject into the M factor, so their orders are at most w, and they vanish after break three. The normalized dyadic different contribution is at most

$$\frac{1}{2}\left(1 - \frac{1}{3w} + 3\left(1 - \frac{1}{w}\right)\right) = 2 - \frac{5}{3w} \le 2 - \frac{5}{3q}.$$

In the tame case the contribution is $1/3$, which satisfies the same bound. At a CR place, residual inertia is trivial. At an SP place, adjoining A gives residue degree two and ramification index at most two. If $h \in \{0, 1\}$ denotes the number of special places,

$$\mathrm{rd}(MA) \le \sqrt{76}\, 3^{h/4} 2^{\,2-5/(3q)}. \tag{40}$$

In (11) take $B_b^* = B_{b,100}$. For MC retain both three-place terms with $(e, f) = (2, 2)$. For MA retain the term at the special place when $h = 1$. We obtain the following strict bounds.

menu	range	$n \ge$	b	lower bound	upper bound
SP/SP	$q = 8$	4032	8.06	51.288	50.789031
SP/SP	$q = 16$	32640	10.15	58.999	55.385834
SP/SP	$q \ge 32$	261888	12.23	63.922	60.398676
CR/CR	$q = 8$	6048	8.10	33.801518	30.182292
CR/CR	$q \ge 16$	48960	10.18	37.181815	34.871192
SP/CR	$q = 8$	6048	8.30	42.318706	39.722130
SP/CR	$q \ge 16$	48960	10.39	47.260048	45.893069

The last two rows also cover CR/SP. The unbounded ranges use the limiting exponent two in (39) and (40). The calculations use directed intervals [52].

For $q = 4$, $\mathrm{rd}(M) \le \sqrt{228}\, 2^{3/2} < 42.709$. The unconditional lower bound 43.14954 in Lemma 13.4 excludes a totally real field of degree 120. Lemma 2.4 gives modularity, using the small-field witness. □

28.3 Scalar and dihedral trace orders

Proposition 28.7. *A terminal system with a nonordinary scalar-residual dyadic member is modular. Such a member can occur only for the* SP/SP *menu. For fixed exact special characters, the simultaneous trace order of any finite collection of these members has $\mathbf{Z}_2$-rank one.*

Proposition 28.8. *Fix the exact quadratic special characters at $\mathfrak{v}_1, \mathfrak{v}_2$. For any finite collection of ordinary scalar-residual dyadic members of terminal systems, their simultaneous trace order has rank at most two. An ordinary scalar-residual member cannot occur in a menu containing* CR*. In the special case its normalized simultaneous matrix-entry order is generated over $\mathbf{Z}_2$ by the crystalline Frobenius trace a_2.*

Proposition 28.9. *Fix the terminal menu and its exact quadratic local data. For a finite collection of nonordinary dyadic realizations with residue V of Proposition 28.5, the simultaneous trace order has $\mathbf{Z}_2$-rank at most two.*

The proofs of the three bounds are given in Appendix K.

28.4 The three-adic endpoint

Lemma 28.10. *An absolutely irreducible determinant-$\bar{\chi}_3$ representation unramified outside three over K remains absolutely irreducible over $K(\zeta_3)$.*

Proof. Write $H = K(\zeta_3)$. If the restriction is reducible, Clifford theory gives induction of a character θ of G_H. The character has order prime to three and is unramified outside the two primes P_1, P_2 of H above $\mathfrak{v}_1, \mathfrak{v}_2$. It therefore factors through the ray group of modulus P_1P_2.

This ray group has exponent two. In fact, the class group of H has order two by the table in Proposition 28.5. The class of P_1 is nontrivial: a generator would have norm to K generating $\mathfrak{v}_1$, which is impossible since $4 - s$ has opposite real signs and every unit of K has equal real signs. Put $\gamma = \sqrt{-3} + 4 - s$. Its valuations at (P_1, P_2) are $(1, 0)$. The fractional ideal $P_1/(\gamma)$ is prime to the modulus, and its square is generated by $(4 - s)/\gamma^2$. At P_1 this element has residue $(4 + s)^{-1} = -1$, since $(4 - s)(4 + s) = -3$. At P_2 its residue is $(4 - s)^{-1} = -1$. Multiplying the generator by -1 gives residue one at both places. Thus the nontrivial ordinary ideal class lifts to a ray class of order two, and the kernel of the map to the class group is a quotient of $\mathbf{F}_3^\times \times \mathbf{F}_3^\times$. This proves the exponent assertion.

The determinant is trivial on G_H, so $\theta^\sigma = \theta^{-1}$ for the nontrivial $\sigma \in \mathrm{Gal}(H/K)$. Since $\theta^2 = 1$, this makes the induction reducible, a contradiction. □

Proposition 28.11. *A terminal system whose global trace field has degree at most two is modular.*

Proof. At a coefficient prime above three, the residual trace field is contained in $\mathbf{F}_3$ or $\mathbf{F}_9$. An absolutely irreducible residue is realizable over its trace and determinant field by finite-field descent. The small-field theorem [3, Theorem 2.5] and Lemma 2.1 give a weight-two modular witness. By Lemma 28.10 the cyclotomic restriction is irreducible. Apply Lemma 2.2 to the actual three-adic source.

Suppose the residue is reducible. At a special place the source is nearly ordinary with characters $\theta\chi_3, \theta$. At a scalar crystalline place, remove the quadratic twist. Over $\mathbf{Q}_3$ a nonordinary Barsotti–Tate group and its finite-flat subquotients are biconnected. The two characters of the scalar-untwisted reducible residue have trivial and cyclotomic inertia. Their order-three models are étale and multiplicative by finite-flat uniqueness for $e = 1 < 3 - 1$. Thus the source must be ordinary. The ordinary line has weight one in our convention at both places. Its Hodge–Tate number is smaller than that of the quotient in Pan's convention. By Lemma 6.7, twisting by the quadratic lift of a global constituent gives residual trace $1 + \bar{\chi}_3$ and preserves the cyclotomic determinant. The determinant is de Rham and totally odd, and its residue extends to $G_{\mathbf{Q}}$ and is nontrivial at both three-places. The field K is abelian and three is split. Pan's fully ordinary theorem [54, Theorem 5.1.2(2)] therefore applies, including when the two

local ordinary lines have different residual labels. Untwisting and strong compatibility give modularity of the system. □

28.5 Quadratic conjugation and modular comparison systems

Suppose a terminal system is not modular. By Propositions 28.5, 28.6 and 28.7, every dyadic coefficient embedding then belongs to one of two classes: ordinary with scalar residue, or nonordinary with residue V. If a crystalline place occurs, the first class is impossible by Proposition 28.8, so Proposition 28.9 bounds the global trace degree by two. Proposition 28.11 gives a contradiction. We may therefore assume the menu is SP/SP.

For this menu, each of the two simultaneous orders has rank at most two. For any finite collection of distinct coefficient orbits with the same exact special characters, all of whose dyadic embeddings lie in these two classes, we have

$$\sum_i [D_i : \mathbf{Q}] \leq 4. \tag{41}$$

This is the sum of the bounds on the two subcollections, by Section 11.

Lemma 28.12. *Let $\mathcal{R}$ be a nonmodular* SP/SP *terminal system. Normalize its exact special characters to $(1, \delta)$ by a rational quadratic twist. Then $\delta \in \{U, UR\}$, and its trace field $D = \mathbf{Q}(a_2)$ has degree four.*

Proof. Proposition 28.11 and (41) give $3 \leq n \leq 4$. Both dyadic classes must occur, since either alone has rank at most two. The crystalline trace a_2 is a unit at the ordinary embeddings and a nonunit at the nonordinary ones. It is therefore nonrational. For $n = 3$ this already gives $D = \mathbf{Q}(a_2)$. For $n = 4$, a proper nonrational subfield $\mathbf{Q}(a_2)$ would be quadratic. It has distinct dyadic places containing the unit and nonunit conjugates, so both places have degree one. The ordinary part of $D \otimes \mathbf{Q}_2$ has dimension two, since both parts have dimension at most two. But a_2 belongs to $\mathbf{Q}_2$ on that part and cannot generate its dimension-two algebra, contrary to Proposition 28.8. Hence $D = \mathbf{Q}(a_2)$.

The rational quadratic characters $1, \kappa_{-1}, \kappa_3, \kappa_{-3}$ give all four local characters at three. They preserve good reduction at two: κ_3 is trivial on $K_{\mathfrak{p}_2}$ and κ_{-1} is unramified there. We may therefore normalize the first special character to one. If η is the rational quadratic character restricting to δ at three, the operation

$$C(\mathcal{R}) = \mathcal{R}^\sigma \otimes \eta$$

preserves the normalized local pair and both dyadic classes. It is an involution on isomorphism classes. If $C(\mathcal{R})$ were a different coefficient orbit, the pair of orbits would contribute $2n > 4$ to (41). Thus C induces an automorphism τ of D with $\tau^2 = 1$.

There is only one dyadic place of K. For $\delta = 1, R$, the dyadic value of η is one, so $\tau(a_2) = a_2$ and $\tau = 1$. The twisted conjugacy (95) then gives modularity by Lemma 12.4. For $\delta = U, UR$, the value is minus one, so $\tau(a_2) = -a_2$. Since a_2 is nonzero, τ is a nontrivial involution. The degree n must be even, and hence equals four. □

28.5.1 The unramified relative character

Lemma 28.13. *There is a modular terminal system with special pair $(1, U)$, coefficient field $\mathbf{Q}(\sqrt{2})$, and both dyadic coefficient dimensions in the nonordinary V class.*

Proof. Construct the parallel-weight-two newspace of level (3) and trivial central character over K. The rational dimensions of its newform constituents are $2, 2, 5, 5$. On a two-dimensional constituent the Hecke matrices, with a choice of basis, include

$$T_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \qquad U_{\mathfrak{v}_1} = -I, \qquad U_{\mathfrak{v}_2} = I,$$

$$T_{\mathfrak{q}} = -I \pm 2T_2 \qquad \text{at a prime } \mathfrak{q} \mid 5. \tag{42}$$

The second constituent interchanges the two special signs. Appendix K.6 gives the exact computation. The characteristic polynomial of T_2 is $X^2 - 2$, so the Hecke field is $\mathbf{Q}(e)$, $e^2 = 2$, and $a_2 = e$. The exact new level has exponent one at both three-primes. Both local parameters are therefore special with opposite unramified quadratic characters. A permitted rational quadratic twist normalizes the pair to $(1, U)$.

The Hecke field has one dyadic completion, of degree two and residue field $\mathbf{F}_2$. The member is nonordinary because a_2 is nonunit. If its residue were absolutely reducible, the odd-order constituent argument of Lemma 28.4 would make its semisimplification trivial: the characters are unramified at three by special inertia, at two because the residue field is $\mathbf{F}_2$, and elsewhere by the level. Class number one then kills them. All residual traces would be zero. But (42) gives the odd trace $-1 \pm 2e$ at five. Thus the residue is absolutely irreducible. The only absolutely irreducible subgroup of $\mathrm{GL}_2(\mathbf{F}_2)$ is S_3. The subgroup C_3 becomes reducible over $\mathbf{F}_4$ and has already been excluded. Proposition 28.5 identifies this residue with V. □

This orbit contributes two dyadic dimensions to the V class and saturates Proposition 28.9. Any distinct system with the same exact local pair and a V embedding would exceed that bound. A system lying entirely in the ordinary scalar class has trace degree at most two and is modular. Hence every terminal system with relative character U is modular.

28.5.2 The ramified relative character

Lemma 28.14. *The elliptic curve*

$$E: \quad y^2 + sy = x^3 - (1+s)x^2 - (20+8s)x - 25 - 16s \tag{43}$$

defines a modular terminal system with special pair $(1, \kappa_{-3}) = (1, UR)$ *and nonordinary dyadic residue* V.

Proof. Its invariants are

$$c_4 = 1280 + 416s, \qquad c_6 = 70744 + 23296s,$$

$$\Delta = 67392s - 343035 = -(288 - 117s)^2.$$

Since $288 - 117s = 9(32 - 13s)$ and $\mathrm{N}(32 - 13s) = -3^7$, the discriminant is supported only at three. The curve has good reduction at two, where its equation is $y^2 + y = x^3 + 1$ over $\mathbf{F}_2$. It has three points over $\mathbf{F}_2$, so $a_2(E) = 0$ and the two-adic member is nonordinary.

The local valuations are

	$v(\Delta)$	$v(c_4)$	$v(j)$
$\mathfrak{v}_1$	4	0	-4
$\mathfrak{v}_2$	18	2	-12

Thus the curve is potentially multiplicative at both places, and the monodromy is nonzero. Its local splitting character is the quadratic character of $-c_6$. At $\mathfrak{v}_1$, $c_6 \equiv 2 \pmod 3$, so $-c_6$ is

square. At $\mathfrak{v}_2$, $s \equiv 71 \pmod{81}$ gives $c_6 \equiv 27 \pmod{81}$. Its squareclass is 3, and that of $-c_6$ is -3. The special pair is therefore exactly $(1, \kappa_{-3})$. The conductor is $\mathfrak{v}_1\mathfrak{v}_2^2$.

After replacing the two-division coordinate x by $z/2$, the two-division polynomial is

$$F(z) = z^3 - 2(1+s)z^2 - (80+32s)z - 162 - 128s.$$

If α satisfies (38), direct substitution gives $F(z_0) = 0$ for

$$z_0 = 2 + 5\alpha - \alpha^2 + \frac{s}{19}(-14 + 8\alpha + 20\alpha^2). \tag{44}$$

The cubic (38) is Eisenstein at two over $\mathbf{Q}$ and remains irreducible over the quadratic field K. The element z_0 is not in K, so it generates $K(\alpha)$. Consequently the two-division splitting field is exactly the S_3 field of Proposition 28.5. This proves that $E[2] \simeq V$.

Finally every elliptic curve over a real quadratic field is modular by [36, Theorem 1]. Thus (43) gives the required rational modular compatible system. □

If a nonmodular terminal system had relative character UR, Lemma 28.12 would give coefficient degree four. The rational modular orbit of Lemma 28.14 is distinct from it and belongs to the same bounded V class. Together they contribute five dimensions to (41), which is impossible. The cases $1, R, U, UR$ are now all settled. This proves the first assertion of Theorem 28.1.

Proof of the second assertion of Theorem 28.1. Lemma 28.10 gives absolute irreducibility on the cyclotomic restriction. At both three-places use Lemma 28.3 on the actual local residue. With determinant χ_3 and unramified conditions elsewhere, Lemma 2.6(i) supplies a terminal weight-two system. Purity retains its special components by Section 2. The first assertion of the theorem now gives modularity of the lift and hence of the residue. □

29 Mixed sources and pseudodeformation rigidity over $\mathbf{Q}(\sqrt{21})$

Put

$$K = \mathbf{Q}(\sqrt{21}), \qquad a = \frac{1+\sqrt{21}}{2}, \qquad \epsilon = a + 2, \qquad \gamma = a + 1, \qquad H = K(\zeta_3).$$

Thus $a^2 - a - 5 = 0$, $\gamma^2 = 3\epsilon$, and $N(\gamma) = -3$.

We first prove the mixed-source theorem by the common discriminant bounds, finite-flat rigidity and a toric realization; the pure menus then reduce to the characteristic-three cases listed below. The conventions of Section 2 apply. Potential modularity gives a cuspidal realization after base change, so all companions used below are absolutely irreducible.

Let ω_2 be a fundamental character of niveau two over $\mathbf{Q}_5$, of order 24. The bounded cuspidal types are

$$\tau_3 = \widetilde{\omega}_2^8 \oplus \widetilde{\omega}_2^{16}, \qquad \tau_6 = \widetilde{\omega}_2^4 \oplus \widetilde{\omega}_2^{20}.$$

The notation SC means one of these types, and SP means a fixed quadratic twist of Steinberg with nonzero monodromy. The exact quadratic character, including its unramified sign, is retained whenever a special deformation condition is imposed.

Proposition 29.1 (Mixed source)**.** *Assume GRH. A potentially modular compatible system over K, of labelled weights $\{0,1\}$ and determinant cyclotomic, unramified away from five and crystalline at two, is modular if its types at the two places above five are* SC/SP *in either order. The same conclusion holds if potential modularity is replaced by the existence over K of a* GL_2*-type abelian variety realizing the compatible system.*

The proof of this proposition is given below, including both the nonsolvable and solvable dyadic branches. The special condition retains its actual nonzero monodromy.

For a pure special menu with reducible five-adic residue, Proposition 29.2 applies directly. For either pure menu, the three-adic member r_3 has four exhaustive alternatives:

residual condition	source argument
$\bar{r}_3\vert_{G_H}$ irreducible	switch one five-type to the mixed menu
$\bar{r}_3$ irreducible, $\bar{r}_3\vert_{G_H}$ reducible	dihedral order-eight rigidity
$\bar{r}_3^{\mathrm{ss}}$ reducible, constituents unramified at three	ordinarity and Skinner–Wiles
$\bar{r}_3^{\mathrm{ss}}$ reducible, common ramified quadratic at three	quadratic descent.

The last line is essential: good supersingular elliptic curves in the present conductor class realize it.

Proposition 29.2. *Let $\mathcal{R}$ be an weight-two system over K with cyclotomic determinant. Suppose its parameters at both places above five are quadratic-special with nonzero monodromy. If a five-adic residual semisimplification is reducible, then $\mathcal{R}$ is modular.*

Proof. After removing the scalar quadratic character, the filtered-module argument of Lemma 4.2 gives an ordinary quotient at each five-place. Restoring that character preserves ordinarity. Apply [28, Corollary 1.2.2] to the dual of the actual five-adic source: $K/\mathbf{Q}$ is totally real abelian, five is unramified in K, the determinant is χ_5^{-1}, and the source is ordinary at both five-places with reducible residual semisimplification. The corollary has no residual constituent-ratio restriction. Dualizing back proves modularity of $\mathcal{R}$. □

The bounded five-adic menu is Lemma 4.14; Lemma 4.15, with $\ell = 5$, gives both replacements of the actual characteristic-three residue, with determinant fixed.

29.1 Controlled characteristic-two residues

Put $F = K_2$, the unramified quadratic extension of $\mathbf{Q}_2$. We distinguish the residual conditions $\mathrm{SC}_2, \mathrm{SP}_2$ from the characteristic-zero conditions SC, SP.

Definition 29.3. The two permitted conditions at a place $v \mid 5$ of K are:

(i) SC_2: the kernel field is the tame S_3 extension of $\mathbf{Q}_5$ with ramification index three and residue degree two;
(ii) SP_2: the kernel field is contained in $E_5' = \mathbf{Q}_5(\sqrt{2}, \sqrt{5})$.

The SC_2 field is unique in a fixed algebraic closure and is the splitting field of $X^3 - 5$. Over the unramified quadratic field, Frobenius has eigenvalues $+1$ and -1 on the uniformizer and unramified lines of cube classes. Only the first is a ramified stable line, and it gives the inversion action on cubic inertia.

Theorem 29.4. *Assume GRH for Dedekind zeta functions. Let*

$$\bar{r} : G_K \longrightarrow \mathrm{SL}_2(\overline{\mathbf{F}}_2)$$

be continuous and absolutely irreducible. Suppose that $\bar{r}$ is unramified outside two and five, is the generic fibre of a finite flat group scheme killed by two at the dyadic place, and satisfies either condition of Definition 29.3 independently at each place above five. Then $\bar{r}$ is modular.

The theorem covers the pairs $(\mathrm{SC}_2, \mathrm{SC}_2)$, $(\mathrm{SC}_2, \mathrm{SP}_2)$ and $(\mathrm{SP}_2, \mathrm{SP}_2)$. The first pair requires the unit extension constructed below. A quadratic extension suffices for the other two pairs.

Corollary 29.5. *Assume GRH. Let $\mathcal{R} = (r_\lambda)_\lambda$ be a totally odd, strongly compatible two-dimensional system over K, of determinant the cyclotomic character and labelled Hodge–Tate weights $\{0, 1\}$. Suppose it has all-place Weil–Deligne compatibility, is unramified at $v \nmid 5$ for members of residue characteristic different from that of v, has crystalline two-adic member at two, and at both places above five has either a quadratic twist of special type with nonzero monodromy or one of the two potentially crystalline types in Lemma 4.14. If its characteristic-two residual representation has nonsolvable image, then the given system $\mathcal{R}$ is modular.*

The distinction between these two assertions matters. Theorem 29.4 provides a modular witness for a residue. Corollary 29.5 uses an automorphy lifting theorem to identify the given characteristic-zero representation with a modular one.

Lemma 29.6. *In a determinant-cyclotomic compatible system with all-place Weil–Deligne compatibility, reduction at a place above two of a local member of quadratic-special type satisfies* SP_2. *Reduction of either cuspidal type satisfies* SC_2.

Proof. In the special case the diagonal characters are, up to a quadratic twist, $\chi_2\nu, \nu$ with ν unramified quadratic. They both reduce to one. The full residual image, including Frobenius, is therefore elementary abelian of exponent two. Local class field theory puts its field inside E_5'. Consequently $e \leq 2$, $f \leq 2$ and $d \leq 1/2$.

For either cuspidal type the reduced inertia has exact order three. A Frobenius lift swaps its two eigenlines. In characteristic zero its square is $-5I$, by its determinant and zero trace. After reduction its square is one. Thus the full residual decomposition group is S_3, with $e = 3$, $f = 2$ and $d = 2/3$. □

Lemma 11.9 gives $d(E/F) \leq 2 - 2/q$ for each finite-flat determinant-one residue over the unramified quadratic dyadic field F. In the nontrivial tame-diagonal wild case it gives $d(E/F) < 4/3$.

29.1.1 The solvable auxiliary fields

Let $H = K(\sqrt{-3}) = \mathbf{Q}(\sqrt{-3}, \sqrt{-7})$. It is quadratic over K, unramified at every finite place, split at two, and unramified quadratic at both places above five. The discriminant product $(-3)(-7)(21) = 21^2$ verifies the first assertion; the other two follow from the quadratic local square classes. This field alone will handle the pairs involving SP_2.

Let S be the splitting field over $\mathbf{Q}$ of

$$f(X) = X^3 - 3X^2 - 2X - 1,$$

and put $A = KS$. The polynomial is irreducible modulo two and has discriminant -175. Thus $S/\mathbf{Q}$ has group S_3, with quadratic subfield $\mathbf{Q}(\sqrt{-7})$, and A/K has group S_3 and fixed quadratic field H. Write τ for a generator of its normal subgroup of order three.

Lemma 29.7. *The extension A/K is unramified outside five. Its dyadic completions are unramified cubic over F; at both places above five its completion is the* SC_2 *field.*

Proof. The splitting field S ramifies only at five and seven. At five, $f(X + 1) = X^3 - 5X - 5$ is Eisenstein, so the cubic ramification is tame. The discriminant has nonsquare unit part -7, yielding the unramified quadratic field and hence S_3. At seven its inertia is quadratic. Its ramified quadratic field is $\mathbf{Q}_7(\sqrt{-7}) = \mathbf{Q}_7(\sqrt{21})$, since -3 is a square in $\mathbf{Q}_7$. Base change to K removes this ramification. At two the irreducible reduction of f gives an unramified cubic; its unramified quadratic field already lies in F. □

We now specify four unit squareclasses in A without using a class-group calculation or a fundamental-unit computation. Choose a root u of f and $z = \sqrt{-3}$ in A and set

$$v = \frac{9 + 26u - 8u^2 + (2u - 7)z}{2}.$$

The choices of roots below are made compatibly with τ. Direct substitution gives

$$v^3 + \frac{15z - 1}{2}v^2 + \frac{15z + 1}{2}v - 1 = 0.$$

In particular u, v are integral units. Their minimal polynomials over $\mathbf{Q}$ are

$$f(X), \qquad X^6 - X^5 + 170X^4 + 335X^3 + 170X^2 - X + 1.$$

Both have norm one for A/H.

Lemma 29.8. *In $W = \mathcal{O}_A^\times/(\mathcal{O}_A^\times)^2$, the four classes*

$$[u],\ [\tau u],\ [v],\ [\tau v]$$

are independent and span $U = (\tau - 1)W$. At either dyadic prime of A their image has dimension two. At both five-adic primes their image is zero.

Proof. Here A has signature $(0, 6)$, so $\dim_{\mathbf{F}_2} W = 6$ by Dirichlet's unit theorem, including the torsion squareclass. Since A/H has odd degree, the τ-invariants in W are the images of the unit squareclasses of H: the norm of an invariant class is its third power and hence that same squareclass. The latter space has dimension two. Thus $(\tau - 1)W$ has dimension four, and every norm-one class belongs to it.

For completeness we give a finite exact certificate of independence and of the dyadic ranks. In the ordered list $(u, \tau u, v, \tau v)$, encode a product by its four-bit mask $m = \sum_{j=0}^{3} m_j 2^j$. The masks that are squares in the two dyadic completions are respectively

first dyadic prime	second dyadic prime
$0, 6, 11, 13$	$0, 7, 9, 14$

Each kernel has order four and their intersection is zero. This proves local rank two and global rank four. The calculation can be performed by reduction modulo eight: over an unramified dyadic field a unit is a square precisely when its residue modulo eight has a square root. Appendix E specifies an exact number-field certificate for this table. The exact calculation is supplied in [52, `d21`].

Finally, inertia at five contains τ. Every class of the form $[w/\tau w]$ for a global unit w has residue one, and is a local square because the residue characteristic is odd. This proves the last assertion for all of U. □

Define $B = A(\sqrt{U})$. Conjugation in $\mathrm{Gal}(A/K)$ exchanges τ and τ^{-1}, so U is stable under this group. Lemma 29.8 gives

$$[B : K] = 96, \qquad B/K \text{ Galois and solvable.} \tag{45}$$

The field B/A is split at five. At two it has ramified group C_2^2 with upper break one over the unramified cubic stage. Indeed its local defining classes are units, and their nontrivial C_3-module has no unramified quadratic constituent.

Lemma 29.9. *If E/F is the kernel field of a finite-flat representation killed by two, then adjoining a dyadic completion of B preserves Fontaine's bound $d < 2$. The same is true after composing two such finite-flat fields.*

Proof. A quadratic extension defined by a unit is the point field of a finite flat extension of $\mathbf{Z}/2\mathbf{Z}$ by μ_2; the usual Kummer construction gives a model over the ring of integers. Direct sums handle a space of unit classes. Restriction of scalars from the unramified cubic extension preserves finite flatness. A finite étale representation accounts for that unramified stage. Taking direct sums with the original finite-flat group schemes realizes the compositum of their point fields, still killed by two. Fontaine's corollary now applies. □

29.1.2 The stable degree-four image at two

The following refinement is the only place where the local rank two in Lemma 29.8 is essential.

Lemma 29.10. *Let $\bar{\rho} : G_F \to \mathrm{SL}_2(\mathbf{F}_{16})$ be finite flat, with kernel field E. Suppose its action extends to a semilinear projective action of $G_{\mathbf{Q}_2}$ with nontrivial field automorphism $x \mapsto x^4$. Then the compositum with a dyadic completion of B satisfies*

$$d(EB/F) \le 15/8.$$

Proof. Write T/F for the unramified cubic extension. The quadratic classes contributed by B over T form a two-dimensional, nontrivial $\mathrm{Gal}(T/F)$-module of units.

If $\bar{\rho}$ is locally irreducible, its tame inertia has order five, as in Lemma 11.9. After tame base change of degree five the three nonzero quadratic characters from B have conductor six. The different tower gives

$$d(EB/F) \le \frac{4}{5} + \frac{3 \cdot 6}{4 \cdot 5} = \frac{17}{10} < \frac{15}{8}.$$

If the tame diagonal order is three, let $P \le 16$ be the order of the original translation group. Its nontrivial quadratic characters have conductor two over the tame cubic stage by Lemma 11.9. The B-characters there have conductor four and trivial tame action; they are independent of the first module. Counting characters gives

$$d(EB/F) \le \frac{2}{3} + \frac{3P \cdot 4 + (P-1) \cdot 2}{3 \cdot 4P} = \frac{11}{6} - \frac{1}{6P} < \frac{15}{8}.$$

It remains to consider unramified diagonal characters. If wild inertia is zero, E/F is unramified and the contribution of B alone is $3/2$. Otherwise its unique fixed line is preserved by the semilinear extension. Let s lift the rational Frobenius. The diagonal ratio of s^2 is $aa^4 \in \mathbf{F}_4^\times$. Thus the unramified diagonal ratio over F has order one or three.

In the order-three case, over T the original Kummer module and the module from B both lie in the nontrivial $\mathrm{Gal}(T/F)$-part of the unit squareclasses. That part has dimension four: the total unit squareclass dimension over T is seven, and the invariant part, from F, has dimension three. All its nonzero ramified characters have conductor two. Hence the combined wild inertia has order at most sixteen and different at most $2(1 - 1/16) = 15/8$.

In the order-one case, the unit squareclasses from F have at most two ramified dimensions, since their dimension is three and they include the unramified quadratic line. The module from B adds at most two ramified dimensions. Again the ramification index is at most sixteen, and every ramified quadratic character has conductor two. The same bound follows. □

29.1.3 The discriminant comparisons and residual modularity

Use the common test function and explicit formula of Section 13, with the finite-tail bound $B_{b,N}$ and the real-place correction $A_{b,4}$.

For every nonsolvable kernel field considered here, the group is $G = \mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^f \geq 4$. Compositing with a solvable Galois extension is disjoint over K, since G is nonabelian simple. At an SC_2 place a degree-n compositum preserving that completion has $n/12$ norm-25 primes. It contributes $c_{25} = 1/6$. At an SP_2 place, adjoining H makes its residue degree two and leaves ramification index at most two, giving $c_{25} \geq 1/4$. The resulting bounds are

pair at five	upper bound for rd	retained c_{25}
$\mathrm{SC}_2, \mathrm{SC}_2$	$4\sqrt{21}\,5^{2/3}$	$1/3$
$\mathrm{SC}_2, \mathrm{SP}_2$	$4\sqrt{21}\,5^{7/12}$	$5/12$
$\mathrm{SP}_2, \mathrm{SP}_2$	$4\sqrt{21}\,5^{1/2}$	$1/2$

The common dyadic factor four comes from Fontaine's strict $d_2 < 2$. In the mixed case it can be replaced by $2^{2-2/q}$ using Lemma 11.9. For the stable $q = 16$ case with two SC_2 places, Lemma 29.10 replaces it by $2^{15/8}$ even after adjoining B.

Table 3 records rigorous evaluations of (11). In every row $N = 100$ and computations use 220-bit real-ball arithmetic. The lower bounds are rounded downwards and the upper bounds upwards at the shown precision. The column m states how many positive norm-25 Euler powers are retained. Every row discards the real-place contribution.

case	$n \geq$	b	m	GRH lower	local upper
two SC_2, distinct conjugates	48 771 072	17.34	8	53.60787465	53.59813048
two SC_2, $q \geq 64$	50 319 360	17.37	8	53.62432870	53.59813048
two SC_2, stable $q = 16$	783 360	13.21	8	50.46443530	49.14970236
mixed, $q = 8$	2 016	7.31	4	39.50635209	39.41346729
mixed, $q = 16$	16 320	9.40	5	46.57108547	42.98069085
mixed, $q \geq 32$	130 944	11.48	7	51.01199785	46.87077572
two SP_2, $q \geq 8$	2 016	7.36	4	41.44651216	40.98780307

Table 3: Certified root-discriminant contradictions.

Proof of Theorem 29.4. The solvable absolutely irreducible images in characteristic two are dihedral and are modular by the usual Hecke theta-series argument, as recalled in [3, proof of Theorem 2.5]. For nonsolvable images Dickson's classification gives, after conjugation, $G = \mathrm{SL}_2(\mathbf{F}_q)$ with $q = 2^f \geq 4$. The case $q = 4$ is modular by [3, Theorem 2.5]. We may assume $q \geq 8$. Let M/K be the kernel field.

Suppose first that at least one place above five satisfies SP_2. Adjoin H. It is disjoint from M, is split at two, and preserves the different bounds at five while supplying the norm-25 coefficients above. Thus $[MH : \mathbf{Q}] = 4q(q^2 - 1)$. For a mixed pair, the three mixed rows of Table 3 exclude $q = 8$, $q = 16$ and all $q \geq 32$. The two-SP_2 row excludes every $q \geq 8$ in the other pair.

We are left with two SC_2 places. Adjoin B. By (45), simplicity and Lemma 29.9,

$$[MB : \mathbf{Q}] = 192q(q^2 - 1), \qquad \mathrm{rd}(MB) < 4\sqrt{21}\,5^{2/3}.$$

Its five-adic completions remain exactly S_3, since both M and A have the unique SC_2 field and B/A is locally split. The second row of Table 3 excludes $q \geq 64$.

Let σ be the nontrivial automorphism of $K/\mathbf{Q}$. If $M \neq M^\sigma$, simplicity gives $M \cap M^\sigma = K$. The group of their compositum over K is $G \times G$, again disjoint from solvable B. Its dyadic field is still the point field of a finite-flat group killed by two, and its five-adic completions are unchanged. Its degree is $[MM^\sigma B : \mathbf{Q}] = 192|G|^2 \geq 48\,771\,072$. The first row of the table excludes this possibility. Hence $M = M^\sigma$ and $M/\mathbf{Q}$ is Galois.

For $q = 8, 16, 32$ the conjugation action defines $C_2 \longrightarrow \mathrm{Out}(\mathrm{SL}_2(\mathbf{F}_{2^f})) = C_f$. If this action is trivial, the centerless extension $\mathrm{Gal}(M/\mathbf{Q})$ is $G \times C_2$: correct a lift of the generator by an inner element; its square then centralizes G and lies in its trivial centre. Projection onto G extends the given residual representation to $G_{\mathbf{Q}}$. Serre's theorem over $\mathbf{Q}$ [43, 44] and quadratic base change [4] prove modularity over K. The base change is cuspidal, since its residual restriction is absolutely irreducible.

For $q = 8$ and 32, f is odd and the outer action is automatically trivial. For $q = 16$ its only nontrivial possibility is the field involution $x \mapsto x^4$. Since two is inert in K, its dyadic action is precisely the semilinear situation of Lemma 29.10. We have

$$[MB : \mathbf{Q}] = 783\,360, \qquad \mathrm{rd}(MB) \leq \sqrt{21}\, 2^{15/8} 5^{2/3}.$$

The third row of the table contradicts this. All cases have therefore been excluded or proved modular. □

Lemma 29.11. *Under the local hypotheses of Theorem 29.4, without GRH, a nonsolvable image satisfies $\bar{r}(c_v) \neq 1$ for at least one real place v of K.*

Proof. Lemma 11.9 bounds the original kernel by $\mathrm{rd}(M) \leq 4\sqrt{21}\, 5^{2/3} 2^{-2/q}$, using the larger of the prescribed five-adic contributions. Since $4\sqrt{21}\, 5^{2/3} < 4\sqrt{220}$, Corollary 13.5(i) excludes a totally real kernel. Thus some complex conjugation acts nontrivially. □

Proof of Corollary 29.5. Choose the two-adic member of $\mathcal{R}$ and an invariant lattice. Its nonsolvable residue is absolutely irreducible, has determinant one, and is finite flat at two because the member is crystalline of weights $\{0, 1\}$. Here we use the integral Barsotti–Tate classification, including residue characteristic two, to pass from a stable lattice to a finite flat torsion model; see [45, Theorems 2.3(3), 4.1 and Corollary 4.3]. All-place compatibility and Lemma 29.6 give the actual local fields required by Theorem 29.4. That theorem provides a regular algebraic cuspidal modular witness for the residue. Lemma 29.11 supplies a nontrivial residual complex conjugation at some real place.

Apply [73, Theorem 6.1]. Its hypotheses are finite ramification, potential crystallinity with labelled weights $\{0, 1\}$ at all dyadic places, an absolutely irreducible nonsolvable residue, a nontrivial residual complex conjugation at some real place, and a regular algebraic essentially self-dual cuspidal modular witness. Each has just been checked; for GL_2 the determinant supplies the essential self-duality. Thorne uses $\mathrm{HT}(\chi_2) = -1$, so formally apply his theorem to the dual of our member and then take the contragredient of the automorphic conclusion. The given two-adic member is modular. Compatibility and Chebotarev identify all other members with the same modular system. □

29.1.4 A separate finite-flat theorem in characteristic five

Let

$$H = K(\sqrt{-3}) = \mathbf{Q}(\sqrt{-3}, \sqrt{-7}), \qquad \eta : G_K \longrightarrow \{1, -1\}$$

be its quadratic character. Section 29.2 identifies a unique S_3 kernel field A_1/K whose cubic character over H has conductor (10), up to replacing that character by its inverse. At both five-adic places its local field is tame S_3 with $e = 3, f = 2$. At two its inertia is tame C_3. Write $\bar{r}_1 : G_K \to \mathrm{SL}_2(\mathbf{F}_2)$ for its standard residual representation.

Theorem 29.12. *Let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be a continuous totally odd lift of $\bar{r}_1$, with determinant χ_2. Suppose that r is unramified outside two and five, is crystalline at two with labelled weights $\{0,1\}$, and has either τ_3 or τ_6 as its specified finite inertia representation independently at each place above five. Then*

$$r \simeq \mathrm{Ind}_{G_H}^{G_K} \psi$$

for a geometric character ψ of G_H. In particular r is modular. This assertion does not require GRH or potential modularity.

The compatible-system assumption in Proposition 29.1 will be used to compare every coefficient conjugate and then to construct an abelian variety. Theorem 29.12 instead uses a quadratic selftwist of the actual two-adic source.

29.2 Solvable residual representations and their fields

Reduction in characteristic two of either cuspidal type has the exact tame S_3 field with $e = 3, f = 2$ at five. Reduction of the special condition is contained in $\mathbf{Q}_5(\sqrt{2}, \sqrt{5})$, with elementary quadratic image. These statements concern the full decomposition image, not only inertia; see Lemma 29.6.

The cuspidal place makes every characteristic-two residue in Proposition 29.1 absolutely irreducible. Its nonsolvable case is already modular as an actual source, by Corollary 29.5. In that result the GRH comparisons exclude simple images $\mathrm{SL}_2(\mathbf{F}_q)$ with $q \geq 8$ in the mixed case, the $q = 4$ residue is modular, and a separate totally real discriminant comparison verifies the real-place hypothesis of Thorne's automorphy lifting theorem [73, Theorem 6.1]. We therefore concentrate on solvable residues.

Such an absolutely irreducible determinant-one image in characteristic two is dihedral D_{2n} with n odd. Let L/K be its quadratic inducing field. We first dispose of the inducing fields nonsplit at two.

Lemma 29.13. *Let $r : G_F \to \mathrm{GL}_2(E)$ be crystalline with labelled weights $\{0,1\}$ and determinant χ_2, where $E/\mathbf{Q}_2$ is finite. If the actual residual local image is a two-group, then r is ordinary.*

Proof. Here $F = K_2$ is unramified over $\mathbf{Q}_2$. A finite two-group has only trivial simple representations in characteristic two, so the residual semisimple constituents are unramified. Lemma 4.16, with $p = 2$ and $e(F/\mathbf{Q}_2) = 1$, therefore proves ordinarity of the actual source. □

Proposition 29.14. *For a residue occurring in the mixed theorem, either its given source is modular by the ordinary theorem of Allen, or its quadratic inducing field is H and splits at two.*

Proof. If L/K ramifies at two, inertia has nontrivial sign in the dihedral image. Its wild image is a normal two-subgroup, and tame inertia is cyclic. Since n is odd, inertia must be a reflection C_2. Its normalizer in D_{2n} is that same C_2, so the local decomposition image is also C_2. The actual finite-flat representation therefore cuts out L_F/F. Fontaine's upper-break bound gives conductor at most two [34, Theorem A]. Its Kummer class has even dyadic valuation.

The actual crystalline source is ordinary by Lemma 29.13. Allen's ordinary modular witness lemma [2, Lemma 5.1.2] supplies a quotient-compatible nearly ordinary cuspidal lift of the globally irreducible dihedral residue. It has no CM nonsplitting hypothesis. The actual-source theorem [2, Theorem 5.2.1] then applies: if L is CM, its dyadic place is nonsplit; for a non-CM L this extra condition is vacuous. Equal residual local characters are allowed in both applications.

It remains to identify the possible inducing fields that are split or unramified at two. Put $\alpha = (1 + \sqrt{21})/2$ and $\epsilon = (5 + \sqrt{21})/2$. The ordinary class number of K is one: Minkowski's bound is $\sqrt{21}/2 < 2.3$, and two is inert. The unit ϵ is fundamental and has norm one. The

squareclasses supported at two and five have basis $[-1]$, $[\epsilon]$, $[2]$, $[\alpha]$, $[\alpha^\sigma]$. Even dyadic valuation removes $[2]$ and gives the alternative basis $[-1], [\epsilon], [5], [\alpha]$. In F we have $[5] = 1$ and $[\epsilon] = [-1]$, the latter because $((3 + \sqrt{21})/2)^2 = 3\epsilon$, and -3 is a square in F. If $w = \alpha \bmod 2 \in \mathbf{F}_4$, then $\alpha^3 \equiv 1 + 2\alpha \pmod 4$; the images of $[-1], [\alpha], [-\alpha]$ in U_1/U_2 are $1, w, 1 + w$. All are nonzero. Thus every nontrivial local class in their span is ramified of conductor two, and none is the unramified quadratic class. Consequently an allowable L is either ramified or split at two, never unramified nonsplit.

The kernel of the local squareclass map is generated by $[5]$ and $[-\epsilon]$. The three nontrivial split possibilities are $5, -\epsilon, -5\epsilon$. At the cuspidal place above five the inducing quadratic field must be unramified quadratic. The first and third candidates ramify there. Hence only $L = K(\sqrt{-\epsilon}) = K(\sqrt{-3}) = H$ remains. □

We next enumerate the odd inducing characters for H by the ray class exact sequence. Put

$$u = \frac{\sqrt{-3} + \sqrt{-7}}{2}, \qquad \zeta = \frac{-1 + \sqrt{-3}}{2},$$

with $\sqrt{-3}\sqrt{-7} = \sqrt{21}$. Then $u^2 = -\epsilon^{-1}$ and $N_{H/K}(u) = \epsilon^{-1}$. Since ϵ is the fundamental totally positive unit of K, every unit of H is a power of u times a root of unity. Indeed, after multiplying by a power of u its norm to K is one, so all its complex conjugates have absolute value one and it is a root of unity. The roots of unity of H are μ_6. Also $h_H = 1$. Its Minkowski bound is $63/(2\pi^2) < 4$, whereas primes above two have norm four and the prime above three has norm nine.

An odd inducing character is tame at two and five, so its conductor divides (10). The two primes of H above two have residue field $\mathbf{F}_4$, and those above five have residue field $\mathbf{F}_{25}$. Thus the odd part of the local residue-unit group modulo (10) is $\mathbf{F}_3^4$. At the five-primes, $u^4 = -1$ modulo five, so u has order eight and trivial image in the odd quotient. The polynomial $X^2 - 5X + 1$ of ϵ is irreducible modulo two. Since $u^2 = -\epsilon^{-1}$, at either dyadic prime u has order three. Choose generators so that the images of u and ζ in the four odd factors are

$$(1, 1, 0, 0), \qquad (1, -1, 1, 1),$$

with the dyadic factors first. The two dyadic residues of u agree under the identification induced by K, whereas those of ζ are inverse. The full unit group was determined above, so the odd ray quotient is exactly $\mathbf{F}_3^2$. The involution of H/K swaps the first two factors and negates the last two. Modulo the first unit row it acts as -1 on this quotient.

A character is represented by a vector $(a, -a, c, d)$ with $c + d = a$. Its four projective lines give exactly four cubic quotients, all dihedral over K, with conductors

$$(5), \quad (10), \quad (2)\mathfrak{q}_1, \quad (2)\mathfrak{q}_2, \tag{46}$$

where $\mathfrak{q}_i$ lies above the corresponding five-prime of K. The first line has $a = 0$ and $c = -d \neq 0$. For $a \neq 0$, normalize $a = 1$ and use $(c, d) = (2, 2), (1, 0), (0, 1)$.

For the mixed orientation, the first two quotients ramify cubically at the special place and are impossible. Exactly one of the last two ramifies at the selected cuspidal place. Denote its field over K by A_{m}. It has image S_3 and is the unique residual representation for that orientation up to $\mathrm{GL}_2(\overline{\mathbf{F}}_2)$-conjugacy: the two nontrivial cubic characters are inverse and give conjugate induced representations. A determinant-one scalar twist in characteristic two is trivial.

For a pair of cuspidal places, the first two fields in (46) are the possibilities. We denote them A_0 and A_1, respectively. The first is unramified at two; the second has tame inertia C_3 there. Both A_{m} and A_1 have local residual characters of order three over $\mathbf{F}_4$ at two, with the two finite-flat signatures $(1, 0)$ and $(0, 1)$. At the special place, A_{m} is unramified quadratic and Frobenius is a nontrivial reflection.

29.2.1 Finite-flat tangent extensions at two

We use the exact Honda-system description at the boundary $e(F) = 1 = p - 1$, reviewed in [13, pp. 4–6]. On biconnected objects the relevant categories are abelian, with the usual schematic kernels and quotients, and the generic-fibre functor is fully faithful. The integral classification and exactness at two can also be expressed using [45, Corollaries 4.3–4.4].

Lemma 29.15. *Let $k = \mathbf{F}_4$, let $R = k[\varepsilon]/(\varepsilon^2)$, and let $\theta : G_F \to k^\times$ have nontrivial tame inertia of order three and a finite-flat model. If $\theta(1+\varepsilon c)$ is finite flat as an underlying $\mathbf{F}_2$-representation, then c is unramified.*

Proof. Every finite-flat model of θ or its selfextensions is biconnected. An étale quotient or a multiplicative subobject would have an unramified generic constituent, whereas every constituent has the nontrivial ramified character θ; modulo two the cyclotomic character is trivial. Full faithfulness therefore extends the actual R-action to the model, and schematic closure and quotient give an exact selfextension of the rank-one model.

By the exact coefficient version of the classification [45, Corollaries 4.3–4.4 and Proposition 5.7.3], its two coefficient factors have exponents $r_0, r_1 \in \{0, 1\}$. The tame exponent is $r_0 + 2r_1$ modulo three. For the two nontrivial order-three characters the unique signatures are $(1, 0)$ and $(0, 1)$. The subobject and quotient therefore have the same rank-one model, including its unramified constants. Lemma 11.8, with $p = 2$, $e = 1$ and $f = 2$, makes the generic extension unramified after dividing by θ, as required. □

Lemma 29.16. *For $A = A_{\mathrm{m}}$ or A_1, every normal finite-flat dual-number deformation at two gives quadratic Kummer characters over the dyadic completions of A of conductor at most four.*

Proof. Over $k = \mathbf{F}_4$ the residual local representation is $\theta \oplus \theta^{-1}$, with both characters nontrivial tame of order three. Its model is a direct sum of the unique biconnected models. A finite-flat $k[\varepsilon]$ deformation is an extension of this model by itself. Pullback along either constituent inclusion and pushout along either constituent projection in the abelian biconnected category extract its two offdiagonal extension classes. These are finite-flat extensions already over the original unramified field F.

Apply Fontaine's theorem there: classical upper breaks are at most one. Killing the two residual characters then makes a tame cubic base change, which multiplies positive upper breaks by three. Each resulting quadratic character has break at most three, hence conductor at most four. □

The order of these operations supplies the bound for every dual-number tangent point. A finite-flat group defined only over the ramified cubic field can instead have conductor six. No division by a dual-number parameter, and no characteristic-zero lift of the tangent point, is used in Lemma 29.16.

29.2.2 The normal Selmer calculations

Write $\bar{r} = \mathrm{Ind}_H^K \chi$ over $k = \mathbf{F}_4$. In an inducing basis we have

$$\mathrm{ad}^0(\bar{r}) = kI \oplus V, \qquad V = \mathrm{Ind}_H^K \chi^2. \tag{47}$$

The two offdiagonal entries give V, and the equal diagonal entries give the scalar summand. This is a direct calculation. The normal module is the scalar extension of the standard $\mathbf{F}_2$-representation of S_3.

Let L_{m} and L_1 be the cubic point fields over K inside A_{m} and A_1. Absolute equations are

$$f_{\mathrm{m}}(X) = X^6 - X^5 - 2X^4 + 3X^3 - 9X + 9,$$
$$f_1(X) = X^6 - 3X^5 + 2X^4 - 19X^3 + 76X^2 - 97X + 37.$$

Both fields have signature $(2, 2)$ and class number one. Their absolute discriminants are respectively 3704400 and 92610000. Each has one dyadic prime, with absolute ramification index three and residue degree two. Over five the residue degrees and ramification indices are $(e, f) = (3, 1), (1, 1), (1, 2)$ for L_{m} and $(3, 1), (3, 1)$ for L_1.

Apply Lemma 11.13 with S the places above two and five together with the infinite places. The space $K(S, 2)$ has dimension five. The corresponding spaces for L_{m} and L_1 have dimensions eight and seven, since their class groups are trivial. Thus the two normal cohomology spaces have dimensions three and two over $\mathbf{F}_2$. Their computation requires only eight and four squareclasses, respectively.

The exact degree-six calculation in Appendix F gives the dyadic conductors

L_{m}	$0, 2, 6, 6, 6, 6, 6, 6$
L_1	$0, 6, 6, 6.$

Here zero denotes the trivial class. The nonzero mixed class of conductor two is unramified and nonsplit at the degree-one prime above the special five-place. In particular its unramified quadratic character takes value one on Frobenius, in additive $\mathbf{F}_2$ notation.

To relate this to the special condition, realize the standard module as the vectors of coordinate sum zero in the permutation module on three letters. A residual special Frobenius is a transposition. Its fixed letter corresponds to the degree-one completion of L_{m}. On the standard module the fixed coordinate is a nonzero functional on the coinvariants of that transposition. The trace derivative gives the same functional. Consequently a normal class unramified at that place has zero first variation of Frobenius trace exactly when its degree-one quadratic character is split. This excludes the nonzero mixed class surviving the dyadic bound.

Proposition 29.17. *For the mixed residue, a finite-flat normal tangent with homogeneous trace zero at the special place is a coboundary. For $\bar{r}_1$, every finite-flat normal tangent is a coboundary.*

Proof. Lemma 29.16 gives conductor at most four after killing the residual cubic characters. The point-field dyadic completion equals either residual-field completion, because H/K splits at two. The displayed conductor lists therefore leave only the mixed line and the zero class in the second case. The special trace condition kills the mixed line. Scalar extension gives the assertion for every finite coefficient field. The class groups are trivial, so no ideal-class contribution is omitted from these squareclass spaces. □

29.2.3 Closed deformation conditions and the full tangent

We now pass from the necessary normal cuts to the actual global deformation functor. Residual S_3 is absolutely irreducible over $\mathbf{F}_2$, so its universal global deformation ring over $\mathbf{Z}_2$ exists. We fix determinant χ_2 and ramification outside two and five. The finite-flat condition at two is a closed deformation condition on Artin coefficient rings; see [13, Definition 1.1.1 and the following discussion]. Here its residual model is biconnected, so models and coefficient actions are unique. Local framed conditions give closed quotients without a local-centralizer restriction.

At a cuspidal five-adic place, require the fixed finite inertia kernel to act trivially and a chosen generator to have the characteristic polynomial for the selected τ_3 or τ_6. The residual

cubic eigenvalues are distinct, so these equations specify the finite inertia representation up to conjugacy. They are closed matrix equations.

For the mixed case we impose a stronger special condition retaining the exact quadratic character. Fix $\alpha : G_{K_{v_{\mathrm{SP}}}} \to \{1,-1\}$ and an arithmetic Frobenius Φ, and write $a = \alpha(\Phi) \in \{1,-1\}$. Let $M = r(\Phi) - aI$. The residual matrix $S - I$ has a unit pivot. Impose

$$\operatorname{tr} r(\Phi) = 6a, \qquad r(g)v = \chi_2(g)\alpha(g)v \quad (g \in G_{K_{v_{\mathrm{SP}}}}), \tag{48}$$

where v is the pivot column of M. Since the determinant is already five, $\det M = 5 - a(6a) + a^2 = 0$. The unit pivot therefore makes $\operatorname{im} M$ a free rank-one direct summand generated by v. Cayley–Hamilton gives $(r(\Phi) - 5aI)M = 0$. The second equation in (48) imposes the exact character $\chi_2\alpha$ on that canonical line; the quotient character is α by the determinant. These are closed equations in the universal ring. Every actual special source with the chosen α satisfies them. This construction avoids any division by the eigenvalue difference four.

Let D_{m} denote this mixed quotient, and D_1 the quotient with the chosen cuspidal condition at both places for $\bar{r}_1$.

Proposition 29.18. *Both $\mathbf{Z}_2 \to D_{\mathrm{m}}$ and $\mathbf{Z}_2 \to D_1$ are surjective. If either problem has a characteristic-zero point, its ring is $\mathbf{Z}_2$.*

Proof. We calculate the true relative tangent, after scalar extension to $k = \mathbf{F}_4$. Fixed determinant makes the cocycles take values in $\mathrm{ad}^0(\bar{r})$, but strict changes of basis use every matrix in $\operatorname{End}(\bar{r})$. The actual quotient is consequently

$$Z^1(\mathrm{ad}^0\,\bar{r})/d\operatorname{End}(\bar{r}), \tag{49}$$

with the stated local conditions. In characteristic two this need not equal $H^1(\mathrm{ad}^0\,\bar{r})$.

For the mixed problem, (48) gives homogeneous trace zero on tangent Frobenius, since $6a$ maps to zero in $k[\varepsilon]$. For D_1 no special condition is needed. Proposition 29.17 kills the normal V-component in (47). After a strict global change of basis the tangent is therefore scalar, cI. Fixed finite cuspidal inertia makes c unramified there. In the mixed case, the marked exact character on the special line makes c trivial on inertia at that place. At two its generic rank-one summands have finite-flat closures and quotients; applying Lemma 29.15 makes c unramified. It is unramified elsewhere by the support condition.

The narrow class number of K is two, whereas its ordinary class number is one. Hence the only everywhere-finite-unramified additive quadratic character, after extension to k, is the character δ_H of H/K. But $\delta_H I$ is a coboundary in the quotient (49). Indeed, for $X = \operatorname{diag}(1,0)$, conjugation by $1 + \varepsilon X = \operatorname{diag}(1+\varepsilon, 1)$ fixes the inducing subgroup and multiplies reflections by $1 + \varepsilon$. Its boundary is exactly $\delta_H I$. The matrix X has trace one; restricting changes of basis to trace-zero matrices would incorrectly retain this tangent. The same strict conjugation transports the marked special line.

The special marking adds no tangent of its own: with representation and eigencharacter fixed, an infinitesimal line motion w satisfies $(S - I)w = 0$ and thus lies in the original line. There is no motion in its quotient. The true relative tangent is zero in both problems.

All the data are defined over $\mathbf{Z}_2$; the $\mathbf{F}_4$ calculation is scalar extension of the $\mathbf{F}_2$ tangent. A complete Noetherian local $\mathbf{Z}_2$-algebra with residue $\mathbf{F}_2$ and zero relative tangent is a quotient of $\mathbf{Z}_2$, by Nakayama's lemma and completeness. Any characteristic-zero point rules out a nonzero ideal in this quotient, so the ring is $\mathbf{Z}_2$ itself. □

29.2.4 The cuspidal pair with tame dyadic inertia

Proof of Theorem 29.12. The actual source r gives a characteristic-zero point of D_1, so Proposition 29.18 gives $D_1 = \mathbf{Z}_2$ and uniqueness of its lift up to conjugacy and coefficient extension.

The quadratic character η is unramified at every finite place and trivial at F, since H/K is the narrow class quadratic extension and splits at two. It reduces to one in characteristic two. Thus $r \otimes \eta$ has the same determinant, support and crystalline dyadic condition as r. At both primes above five it has exactly the same finite inertia representation, including the choice of τ_3 or τ_6. It is another point of the same ring, with the same residual identification. Uniqueness gives $r \simeq r \otimes \eta$. Absolute irreducibility and the nontrivial quadratic selftwist give $r = \operatorname{Ind}_H^K \psi$ by Clifford theory.

The character is geometric as a constituent of $r|_{G_H}$. Global class field theory for geometric rank-one representations identifies it with an algebraic Hecke character, up to the chosen cyclotomic normalization [61, Chapter III]. Since H/K is CM and r is odd and regular, automorphic induction gives the required cuspidal Hilbert modular representation [4, Chapter III]. □

29.2.5 Rational traces and a toric place

We first isolate the rational endomorphism argument needed for the mixed source. It removes a possible Schur-index ambiguity in passing from rational traces to an elliptic curve.

Proposition 29.19. *Let B/K' be an abelian variety of dimension g over a number field, and let ℓ be prime. Suppose that r is an absolutely irreducible two-dimensional $\mathbf{Q}_\ell$-valued representation such that $V_\ell(B) \simeq r^{\oplus g}$. If, at a finite place of residue characteristic different from ℓ, the monodromy of r has rank one after finite local base change, then B is K'-isogenous to E^g for an elliptic curve E/K' with $V_\ell(E) \simeq r$.*

Proof. Make the reduction semistable and its torus split by a finite local extension. Monodromy on $V_\ell(B)$ has rank g, so B has purely toric reduction, with torus T of rank g. Put $C = \operatorname{End}_{K'}(B) \otimes \mathbf{Q}$. Faltings' endomorphism theorem [32] gives

$$C \otimes \mathbf{Q}_\ell = \operatorname{End}_{G_{K'}}(V_\ell B) = M_g(\mathbf{Q}_\ell),$$

so $\dim_{\mathbf{Q}} C = g^2$. Endomorphisms extend to the local Néron model and act on $X_*(T) \otimes \mathbf{Q}$.

Let I be the kernel of this rational action. With a basis of the Tate twist fixed, write N for monodromy on $V = V_\ell(B)$. Purely toric reduction gives $W = \operatorname{im} N = \ker N = V_\ell(T)$. Every $f \in I$ kills W. Since $fN = Nf$, its image lies in $\ker N = W$. Products of any two elements of I therefore act trivially on V, and Faltings' faithfulness gives $I^2 = 0$. The algebra C is semisimple by Poincaré complete reducibility, so $I = 0$.

Thus C embeds in $\operatorname{End}_{\mathbf{Q}}(X_*(T) \otimes \mathbf{Q}) = M_g(\mathbf{Q})$. Equality of dimensions makes this an isomorphism. A primitive rational matrix idempotent cuts out an abelian subvariety with one copy of r as its Tate module, hence an elliptic curve E/K'. The matrix units identify its g factors up to K'-isogeny. □

Proof of Proposition 29.1. Assume that $\mathcal{R}$ is nonmodular. Modularity is preserved by coefficient conjugation. At every coefficient embedding into $\overline{\mathbf{Q}}_2$, the nonsolvable residual case is modular by Corollary 29.5, and all dihedral cases other than H split at two are modular by Proposition 29.14. The ray calculation therefore puts every coefficient conjugate in the unique mixed residual class A_{m}.

The cuspidal type is rational: coefficient conjugation exchanges its two inverse constituents, preserving τ_3 and τ_6 separately. The exact special character α takes values in $\{1, -1\}$. For a fixed arithmetic Frobenius Φ at the special place, $\operatorname{tr} r_2(\Phi) = 6\alpha(\Phi) \in \{6, -6\}$. Every coefficient embedding fixes this number, its sign and the quadratic inertia character. Thus every conjugate belongs to the same closed problem D_{m} of Section 29.2.3. The actual characteristic-zero point

gives $D_{\mathrm{m}} = \mathbf{Z}_2$. All conjugates are consequently the same two-adic representation, defined over $\mathbf{Q}_2$. Every good Frobenius trace has only one $\mathbf{Q}$-conjugate and is rational. This uses all coefficient embeddings; a single $\mathbf{Q}_2$ realization would not give the conclusion.

Potential modularity and the special finite place give a GL_2-type abelian variety B/K realizing the system. More precisely, after potential modularity the square-integrable local component gives an abelian variety by [68, Lemma 9.4.3]; [68, Lemma 9.4.4] descends the realization to K. These lemmas and the realization argument in the proof of Proposition 9.4.1 use the given potential modularity, rather than imposing that proposition's odd-prime residual hypotheses on the present characteristic-two residue.

Let $g = \dim B$ and retain the $\mathbf{Q}_2$ model r_2. The realization may enlarge the coefficient field. For a good place, its trace in that enlarged field maps under one embedding to the original trace; every other embedding gives a $\mathbf{Q}$-conjugate of that same algebraic number. Since it is rational, the enlargement introduces no additional trace values. Chebotarev and Brauer–Nesbitt give

$$V_2(B) \otimes \overline{\mathbf{Q}}_2 \simeq (r_2 \otimes \overline{\mathbf{Q}}_2)^{\oplus g}.$$

The space $\mathrm{Hom}_{G_K}(r_2, V_2(B))$ has $\mathbf{Q}_2$-dimension g by scalar extension. Its evaluation map is an isomorphism after extension to $\overline{\mathbf{Q}}_2$, hence before extension. Therefore $V_2(B) \simeq r_2^{\oplus g}$.

At the special place monodromy on r_2 has rank one, after killing the quadratic twist. Proposition 29.19 supplies an elliptic curve E/K with $V_2(E) \simeq r_2$. Every elliptic curve over a real quadratic field is modular [36, Theorem 1]. Thus r_2 and the compatible system are modular, a contradiction. □

29.3 Switching the pure menus to characteristic three

Proposition 29.20 (Switchable sources)**.** *Assume GRH. Let $\mathcal{R}$ be a weight-two compatible system of determinant cyclotomic and support at five, with either pure bounded menu. If $\bar{r}_3|_{G_H}$ is irreducible, then $\mathcal{R}$ is modular.*

Proof. Apply Lemma 4.15, with $\ell = 5$, at one five-place to change its type and at the other to retain its type. At three retain the original crystalline weight-two component. Prescribe determinant χ_3 and no ramification elsewhere. Lemma 2.6(i) applies because $\bar{r}_3|_{G_H}$ is irreducible; in characteristic three there is no exceptional-five exclusion. It supplies a potentially modular compatible replacement. The special component has Frobenius ratio five, so Section 2 retains its nonzero monodromy. Proposition 29.1 makes the replacement modular. Its residue has a weight-two witness; Lemma 2.2(ii) then applies to the actual old three-adic member and proves modularity of the original system. □

29.4 Reducible residual characters and the ordinary case

Lemma 29.21. *If the three-adic residue of a bounded-menu system of determinant cyclotomic and support at five is reducible, its semisimplification is $\psi \oplus \psi\bar{\chi}_3$, where ψ is quadratic. Up to interchanging the constituents,*

$$\psi = 1, \quad \psi_5, \quad \psi_\gamma, \quad \psi_5\psi_\gamma.$$

Here ψ_x denotes the character of $K(\sqrt{x})/K$. The last two possibilities have common ramified quadratic inertia at three. Reducibility forces the same quadratic inertia parity at the two five-places.

Proof. Write the constituents initially as ψ and $\bar{\chi}_3\psi^{-1}$. At five their semisimplified inertia is scalar quadratic. At three a character with values in characteristic three kills principal units,

and the residue field has multiplicative group of order two. Thus ψ^2 is unramified at every finite place.

The ordinary class number of K is one: its Minkowski bound is $\sqrt{21}/2 < 2.3$ and two is inert. Its units are $\{\pm\epsilon^n\}$ and there is no norm-minus-one unit, since the norm-minus-one equation is impossible modulo three. The narrow class number is two, with class-field character $\eta_0 = \bar{\chi}_3$. Since $\psi(c)^2 = 1$ at complex conjugations and η_0 is totally odd, ψ^2 cannot equal η_0. Hence $\psi^2 = 1$.

The quadratic squareclasses supported at three and five have basis $[-1], [\epsilon], [\gamma], [5], [a]$. Their unramified-at-two subspace is $\langle [-\epsilon], [5], [\gamma] \rangle$. For an explicit check in the unramified quadratic dyadic ring, a unit class is unramified exactly when its cube is one modulo four. The identities $a^3 = 6a + 5, \quad \epsilon^3 = 24a + 43, \quad \gamma^3 = 12a + 21$ give this kernel: the ramified images of -1 and a are independent, while $[\epsilon] = [-1]$. Both 5 and $-\epsilon$ are locally squares; γ is unramified nonsplit, since the residue trace of $(\gamma^3 - 1)/4 = 3a + 5$ is one.

At three both 5 and $-\epsilon$ are nonsquare units, whereas γ has odd valuation. At the five-places $a \equiv 0, 1$, their local squareclasses are

	$a \equiv 0$	$a \equiv 1$
$-\epsilon$	U	U
5	R	R
γ	1	U,

where U is unramified quadratic and R is the character of $\mathbf{Q}_5(\sqrt{5})$. Interchanging the constituents multiplies ψ by η_0. The four stated representatives and the parity assertion follow. □

Proposition 29.22. *A bounded-menu compatible system whose three-adic residue has constituents unramified at three is modular.*

Proof. By Lemmas 4.16 and 29.21, the actual crystalline weight-two three-adic source is ordinary, with constituent quotient $\bar{\chi}_3$. Skinner–Wiles, Theorem A of [66], applies: K is totally real abelian, $K(\bar{\chi}_3) = H$ is abelian over $\mathbf{Q}$, the quotient is totally odd and is nontrivial on G_{K_3}, and the ordinary quotient is unramified. Removing the Teichmüller character of the ordinary quotient gives the required pro-three character with finite inertia. The determinant is χ_3. The theorem imposes no splitting condition at three and proves modularity of the actual source. □

29.5 The cyclotomic-dihedral exception at three

Write $\omega = \bar{\chi}_3$ and let σ be the nontrivial automorphism of $K/\mathbf{Q}$. The two places v_1, v_2 above five have completion $\mathbf{Q}_5$. For determinant χ_3, the Weil representations of τ_3, τ_6 have Frobenius trace zero and square $-5I$.

29.5.1 The residual representation and the quartic fields

Lemma 29.23. *Let $\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ be absolutely irreducible, with determinant ω, unramified outside three and five, and reducible on G_H. Suppose that its inertia at either place above five is scalar quadratic. Then*

$$\bar{r} \simeq \mathrm{Ind}_{G_H}^{G_K} \xi, \qquad \xi^c = \xi^{-1}, \qquad \mathrm{ord}(\xi) = 4,$$

where c is complex conjugation, and ξ^2 cuts out $H(\sqrt{\gamma})/H$. Its linear image is D_8, and it is defined over $\mathbf{F}_3$. If η, η' are the quadratic characters of

$$L = K(\sqrt{\gamma}), \qquad L' = K(\sqrt{-3\gamma}),$$

then

$$\mathrm{ad}^0\,\bar{r} = \omega \oplus \eta \oplus \eta', \qquad \eta' = \omega\eta. \tag{50}$$

The summand normal to induction from H is $\eta \oplus \eta'$. The two scalar quadratic inertia characters at five are different. For a specified ordered pair of these inertia characters there is at most one residual isomorphism class satisfying these hypotheses.

Proof. Clifford theory gives the displayed induction. Restriction of the determinant to G_H gives $\xi^c = \xi^{-1}$. The character ξ^2 is unramified outside three. At three every finite character with values in $\overline{\mathbf{F}}_3^\times$ is tame, so ξ^2 factors through the ray class group of H with modulus its unique prime $\mathfrak{P}_3$ above three.

Here is an elementary calculation of this ray group. The field $H = \mathbf{Q}(\sqrt{-3}, \sqrt{-7})$ has discriminant 441. Its Minkowski bound is $63/(2\pi^2) < 4$. The prime ideals above two have norm four, and the unique prime above three has norm nine. Consequently $h_H = 1$. The algebraic integer $u = \frac{\sqrt{-3}+\sqrt{-7}}{2}$, $\quad u^4 + 5u^2 + 1 = 0$, is a unit. Its image at $\mathfrak{P}_3$ has order four in $\mathbf{F}_9^\times$: its square is -1. The ray class exact sequence therefore bounds $\mathrm{Cl}_H(\mathfrak{P}_3)$ by a group of order two.

The quadratic extension $H(\sqrt{\gamma})/H$ is nontrivial and tame of conductor $\mathfrak{P}_3$. Indeed γ has odd valuation at three and is a unit at every other finite place. At two its squareclass is represented by $\gamma^3 \equiv 1 \pmod 4$, so the quadratic extension is unramified. At all other odd primes it is unramified as well. This proves $\mathrm{Cl}_H(\mathfrak{P}_3) \simeq C_2$. Thus ξ has order at most four. Order at most two would give $\xi^c = \xi$ and make the induction reducible. Hence its order is four, and its square is the unique nontrivial ray character.

The relation $\gamma\gamma^\sigma = -3$ identifies L' with the quadratic conjugate of L. The usual adjoint decomposition of this dihedral representation gives (50). Rotation and reflection may be represented over $\mathbf{F}_3$ by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At each five-prime of H, the unit u has residue of order eight in $\mathbf{F}_{25}^\times$, since $u^4 = -1$ modulo five. An order-four tame character at three takes the value -1 on u. A nontrivial quadratic tame character at either five-prime also takes the value -1 on u. The global unit relation therefore requires an odd number of the two quadratic inertia characters at five to be nontrivial.

Finally the order-four inertia character at three has just two possibilities, inverse to one another. Once it and the two five-adic quadratic inertia characters are fixed, two possible global characters differ by an everywhere unramified character of H. Such a character is trivial because $h_H = 1$. Replacing all characters by their inverses leaves the quadratic characters at five unchanged and gives the same induced representation. This proves uniqueness. □

Remark 29.24. The scalar-inertia hypothesis in Lemma 29.23 is automatic for the cyclotomic-reducible residue of either source considered below. An induction from a character with values in $\overline{\mathbf{F}}_3^\times$ has image of order prime to three. The order-three part of cuspidal inertia, or the unipotent part of special inertia, must therefore disappear in that image. The remaining inertia is scalar quadratic. This argument does not discard a possible nilpotent inertia direction in a deformation.

Lemma 29.25. *Let ι be the nontrivial automorphism of L/K, and put*

$$y = \sqrt{\gamma}, \qquad z = \frac{a+y}{2} = \frac{y^2+y-1}{2}, \qquad g = \frac{1-y^2-5y}{2} = 2a - 5z.$$

Then $h_L = 1$, $\mathrm{disc}(L) = -1323$, and the minus eigenspace of the $S_{3,5}$-unit cubeclasses of L has basis $[g], [z]$ over $\mathbf{F}_3$. There is a unique degree-two prime $\mathfrak{q}$ above five. The classes in that

eigenspace which are cubes in $L_{\mathfrak{q}}$ are precisely

$$1, \quad [g], \quad [g]^2.$$

Each of the two nontrivial classes has cubic conductor six at three after adjoining ζ_3.

Proof. The identities

$$N_{L/K}(z) = 1, \qquad z + z^{-1} = a, \qquad N_{L/K}(g) = -a^3, \qquad g + g^{\iota} = -a \tag{51}$$

follow directly from $y^2 = \gamma$ and $a^2 = a + 5$. The element z is an integral unit, with polynomial $X^4 - X^3 - 3X^2 - X + 1$. The relative discriminant of $\mathcal{O}_K[z]$ is $(a^2 - 4) = (\gamma)$, the prime above three. It is squarefree, so this order is maximal. Since $a = z + z^{-1}$, it follows that $\mathcal{O}_L = \mathbf{Z}[z]$ and $\operatorname{disc}(L) = 21^2 N_{K/\mathbf{Q}}(\gamma) = -1323$. Modulo two the polynomial of z is $\Phi_5(X)$, irreducible over $\mathbf{F}_2$, so two is inert in L. The signature is $(2, 1)$ and Minkowski's bound is less than five. The only possible ideal norm between two and four is therefore three. The unique prime over three is principal, generated by y, whose rational norm is -3. Hence $h_L = 1$.

The unit ranks of L and K are two and one; their torsion groups have no three-primary part. Restriction and norm split the invariant part modulo cubes because two is invertible modulo three. Thus the minus part of the unit cubeclasses has dimension one. The prime above three is fixed by ι. Of the two places of K above five, one splits in L and the other is inert: the values of $\gamma = a + 1$ modulo five are respectively one and two. The minus part of the $S_{3,5}$-divisor space has dimension one. The triviality of the class group consequently makes the minus part of the $S_{3,5}$-unit cubeclasses two-dimensional.

Both g and z belong to this minus part by (51). The element g is integral because $g = 2a - 5z$. Its norm and trace show that its valuations at the two split five-primes are one and two in some order: their sum is three, while their unequal minimum is the valuation one of $-a$. In particular its divisor class modulo three is nonzero.

At $\mathfrak{q}$ one has $a = 1$, $y^2 = 2$ in $\mathbf{F}_{25}$ and $g = 2, \quad z = (1 + y)/2$. The element $2 \in \mathbf{F}_5^{\times}$ is a cube in $\mathbf{F}_{25}^{\times}$. The displayed residue of z has cube -1 and is not ± 1, so it has order six and is not a cube in the group of order twenty-four. Thus $[z]$ is a nonzero unit class, independent of $[g]$, and the local cube condition at $\mathfrak{q}$ kills exactly its coefficient. Units are cubes over this five-adic field exactly when their residues are cubes, since cubing is an automorphism of the principal units.

At three, y is a uniformizer of L_3 and the absolute ramification index is four. The extension $M = L_3(\zeta_3)/L_3$ is unramified quadratic. Since $g \equiv -1 - y \pmod{y^2}$, the polynomial $(X+1)^3 + g$ is Eisenstein over M. At a root its derivative has valuation $v_E(3) = 12$, because $X + 1$ is a unit. An Eisenstein generator gives the full ring of integers. The cubic extension is cyclic because ζ_3 belongs to M, and the conductor–discriminant formula gives conductor $12/(3 - 1) = 6$. For g^2, the polynomial $(X + 1)^3 - g^2$ is Eisenstein and gives the same calculation. □

Corollary 29.26. *A class in $H^1(K, \mathbf{F}_3(\eta'))$ which is unramified outside three and five, is zero at the five-place where $\eta'(\mathrm{Frob}_5) = 1$, and has cubic conductor at most five at three after killing its quadratic character, is zero. The analogous assertion holds for η.*

Proof. Since $\eta' = \omega\eta$, Kummer theory identifies

$$H^1(K, \mathbf{F}_3(\eta')) \simeq (L^{\times}/L^{\times 3})^{-}.$$

The support condition and $h_L = 1$ make such a class an $S_{3,5}$-unit class. At the specified five-place, $\eta(\mathrm{Frob}_5) = -1$, so L/K is inert; local vanishing means being a cube at the degree-two prime $\mathfrak{q}$. Lemma 29.25 leaves only $1, g, g^2$, and its conductor calculation excludes the latter two. Quadratic conjugation exchanges L and L', proving the other assertion. □

29.5.2 The normal tangent space

Lemma 29.27. *Let a class in $H^1(K, \eta \oplus \eta')$ arise from a finite-flat tangent deformation of the residual representation of Lemma 29.23, with determinant fixed and ramification supported at three and five. If the first variation of the Frobenius trace is zero at both places above five, then this normal class is zero.*

Proof. We verify the conditions of Corollary 29.26 for each normal summand.

At a five-place where a normal character ν has Frobenius value $+1$, $H^1(K_v, \mathbf{F}_3(\nu))$ is purely unramified. The tame ramified direction would require $\nu(\mathrm{Frob}_5) = 5 = -1$ in $\mathbf{F}_3$. In the adjoint representation the commuting traceless line is spanned by the residual Frobenius matrix $\bar{\Phi}$, normalized by $\bar{\Phi}^2 = I$. If the unramified class takes the value $b\bar{\Phi}$, its variation of the trace is $b\,\mathrm{tr}(\bar{\Phi}^2) = 2b$. The trace condition therefore kills this unramified class. At the other five-place, where $\nu(\mathrm{Frob}_5) = -1$, we impose no cut on the ramified direction. In particular this proof allows first-order nilpotent inertia of order three.

At three, work first over H_3, an unramified quadratic extension of K_3 with absolute ramification index two. We may compute over $\mathbf{F}_3$, where the residue is defined. To obtain an actual $\mathbf{F}_9$-action on a finite-flat model, take a direct sum of two copies with the regular $\mathbf{F}_9$-action. Over H_3, schematic closures and quotients of the $\mathbf{F}_9$-linear generic constituents extract the upper and lower extensions. These operations preserve the existing scalar action. This construction avoids assuming that an arbitrary generic coefficient action extends to an arbitrary finite-flat model.

Fontaine's ramification bound, applied over this original field of absolute ramification index two, bounds the classical upper breaks of each extracted point field by

$$2\left(1 + \frac{1}{3-1}\right) - 1 = 2. \tag{52}$$

Here the shift by one comes from the convention $G^{(u)} = G^{u-1}$ in [34, Theorem A]. Pass to the projective kernel field, which is a quotient of the point field and retains the bound. The diagonal ratio has order two, although the original diagonal characters have order four. Killing the ratio by its tame quadratic extension multiplies positive upper breaks by two. The resulting cubic characters have upper break at most four, hence conductor at most five.

If extracting a quadratic-adjoint summand requires combining the two extensions, take the compositum of their point fields first. The direct sum of their finite-flat groups satisfies the same bound, and every projected character field is a quotient. The relevant field over L_3 acquires ζ_3 by an unramified extension, which does not alter the conductor. Thus Corollary 29.26 applies to both summands. Scalar extension of its $\mathbf{F}_3$ calculation gives the same conclusion for any finite residual coefficient field. □

Remark 29.28. The projective quotient in this proof is essential. We apply Fontaine's bound to actual finite-flat extensions over H_3 before making any ramified base change. We do not twist a finite-flat representation by a ramified character and assert that it remains finite flat.

29.5.3 Two cuspidal places

Theorem 29.29. *Let $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_3)$ be continuous and totally odd, with determinant χ_3, unramified outside three and five, and crystalline of labelled Hodge–Tate weights $\{0, 1\}$ at three. Suppose its type at each place above five is τ_3 or τ_6. If its residue is absolutely irreducible but reducible over H, then r is induced from a geometric character of G_H. In particular r is modular.*

Proof. By Lemma 29.23, the residue is $\bar{r} = \operatorname{Ind}_H^K \xi$. Enlarge the residue field to contain $\mathbf{F}_9$ and choose an inducing basis. Put $J = \operatorname{diag}(1, -1)$. Then $J\bar{r}(g)J^{-1} = \omega(g)\bar{r}(g)$ and $J^2 = 1$.

Let D be the universal fixed-determinant deformation ring of $\bar{r}$, finite flat at three, unramified outside three and five, and with the prescribed finite inertia at the two five-places and Frobenius trace zero there. These conditions are represented by a closed quotient of the universal deformation ring: finite-flatness is a stable condition, and the local finite-inertia and Frobenius relations are closed algebra relations; see [76, Corollary 3.3.5 and Section 5.2]. The actual source gives a characteristic-zero point of D.

The character ω is unramified at every finite place of K. Consequently the operation

$$r_A \longmapsto J^{-1}(r_A \otimes \omega)J$$

preserves these conditions, the determinant and the identified residue. Unramified twisting preserves finite-flatness by descent from the unramified extension killing the twist. At five, twisting preserves the prescribed inertia and the zero Frobenius trace. This operation defines an involution of D.

On $\operatorname{ad}^0 \bar{r} = \omega \oplus \eta \oplus \eta'$ it acts as $+1$ on the diagonal summand and as -1 on the two normal summands. The anti-invariant projection of an admissible tangent is again admissible, since the involution preserves the deformation conditions and two is invertible. This normal tangent is zero by Lemma 29.27: diagonal variations have zero Frobenius trace at five, since Frobenius is offdiagonal in the inducing basis. Hence the involution is trivial on the entire relative tangent, and therefore on the relative cotangent. Lemma 12.2, with $m = 2$, makes it the identity on D.

It follows that the actual irreducible source satisfies $r \simeq r \otimes \omega$. An intertwiner squares to a scalar by Schur's lemma. After rescaling, its two eigenspaces are preserved by G_H and interchanged by the other coset. Thus $r = \operatorname{Ind}_H^K \psi$. The character ψ is de Rham and finitely ramified, hence algebraic by global class field theory [61, Chapter III]. Since H/K is CM and the two characters are distinct, quadratic automorphic induction is regular algebraic and cuspidal [4, Chapter III]. □

29.5.4 A finite-flat selfextension calculation

Lemma 29.30. *Let $F = H_3$, and let k be a finite coefficient field containing $\mathbf{F}_9$. Let $\theta : G_F \longrightarrow k^\times$ have the order-four tame inertia character occurring in Lemma 29.23. A finite-flat selfextension of θ, equipped with an actual k-action on its finite-flat model, is unramified after dividing by the common character θ on its generic fibre.*

Proof. Take the schematic closure of the generic subline and the finite-flat quotient. The exact finite-flat/Kisin-module equivalence and its coefficient versions apply to this sequence [47, Theorem 1.1.3, Proposition 1.1.11 and Lemmas 1.2.4–1.2.5]. The two coefficient idempotents give rank-one endpoints with Frobenius exponents (r_0, r_1). Since $e(F) = 2$, one has $0 \le r_i \le 2$. The tame exponent is $r_0 + 3r_1$ modulo eight. The two order-four inertia characters have exponents two and six, whose unique possibilities are respectively $(2, 0)$, $(0, 2)$. The rank-one models of the two endpoints are consequently isomorphic, including their unramified constants.

Lemma 11.8 applies with $p = 3$, $e = 2$ and $f = 2$. Thus the only selfextensions are the unramified ones after dividing by θ. This uses the exact rank-two $k[[u]]$ extension furnished above, without assuming freeness over $k[t]/(t^2)[[u]]$. □

29.5.5 Two special places

Theorem 29.31. *Let $\mathcal{R}$ be a totally odd, absolutely irreducible, strongly compatible two-dimensional system over K, with cyclotomic determinant and labelled Hodge–Tate weights*

$\{0,1\}$. *Assume all-place Weil–Deligne compatibility, support at five away from the coefficient characteristic, and a crystalline three-adic member at three. At $v_i \mid 5$ suppose it is special with nonzero monodromy and fixed quadratic twisting character $\alpha_i : G_{\mathbf{Q}_5} \longrightarrow \{\pm 1\}$, including the unramified sign. If the characteristic-three residue is absolutely irreducible but reducible over H, then $\mathcal{R}$ is modular.*

Proof. Again Lemma 29.23 applies. Fix its model over $\mathbf{F}_3$. Form the universal global deformation ring with determinant χ_3, support at three and five, and the finite-flat condition at three, and impose the following closed local equations at v_i. For arithmetic Frobenius Φ_i put $b_i = \alpha_i(\Phi_i) \in \{\pm 1\}$, require

$$\operatorname{tr} r(\Phi_i) = 6b_i, \qquad r(g)|_{\ell_i} = \chi_3(g)\alpha_i(g), \tag{53}$$

where ℓ_i is the $5b_i$-eigenline of $r(\Phi_i)$. The Frobenius eigenvalues $5b_i, b_i$ have difference $4b_i$, a unit at three. Thus the eigenline is functorial and the equations are closed; the determinant supplies the quotient character α_i.

For a precise construction of this quotient, impose the stable condition on finite-cardinality $\mathbf{Z}_3[G_{K,\{3,5\}}]$-modules consisting of finite-flatness at three and the algebra annihilator identities

$$(g - \chi_3(g)\alpha_i(g))(h - \alpha_i(h)) = 0 \qquad (g, h \in G_{K_{v_i}})$$

at each five-place. Finite-flatness is stable under subquotients and finite direct sums, as are these identities. Wake–Wang-Erickson's stable-condition theorem gives the universal pseudodeformation quotient [76, Theorem 2.5.5 and Section 5.2]; impose the fixed determinant χ_3 as well. Since the residual representation is absolutely irreducible, pseudodeformations and ordinary deformations agree, and the universal Cayley–Hamilton algebra is $M_2(D)$ over the resulting ring D. The displayed annihilator identities are equivalent to (53), because the two residual Frobenius eigenvalues are distinct. This constructs the stated deformation condition without requiring uniqueness of finite-flat models at $e = p - 1$. Every actual special source with these characters gives a point of D.

We show that its relative tangent over $\mathbf{Z}_3$ is zero. Its normal projection belongs to $H^1(K, \eta \oplus \eta')$. Equation (53) has zero first trace variation in characteristic three. The tangent construction of Lemma 29.27, including its finite-flat extraction, kills both normal summands. After a strict conjugation, the tangent lies in the remaining diagonal summand ω of (50).

At a five-place, residual inertia is scalar and residual Frobenius is a reflection with distinct eigenlines. In an inducing basis, a traceless diagonal matrix takes either Frobenius eigenline to the other. The prescribed character on ℓ_i in (53) therefore forces the diagonal tangent to vanish on inertia. Movement of the line cannot compensate for this error, since the residual inertia matrix is scalar. The diagonal class is consequently unramified at both places above five.

At three, compute the tangent over $\mathbf{F}_3$ and equip a direct sum of two copies of its finite-flat model with the regular $\mathbf{F}_9$-action. Over H_3, take schematic closures and quotients of its $\mathbf{F}_9$-linear generic constituents. These operations preserve the existing coefficient action and extract the selfextensions of the two order-four characters. Lemma 29.30 shows that the restriction of the diagonal class to H_3 is unramified. The character ω over K_3 is the nontrivial unramified quadratic character. Hence $H^1_{\mathrm{unr}}(K_3, \mathbf{F}_3(\omega)) = \mathbf{F}_3/(\omega(\mathrm{Frob}_3) - 1)\mathbf{F}_3 = 0$. The extension H_3/K_3 is unramified of degree two; restriction is injective and does not change inertia. The diagonal class is therefore locally zero at three.

Globally Kummer theory identifies $H^1(K, \mathbf{F}_3(\omega)) = K^\times/K^{\times 3}$. The class number and unit calculation of Lemma 29.21 applies. The support condition, unramifiedness at five, and the trivial class group leave just $[\epsilon]$ and $[\gamma]$ as possible global classes. A cube at three has valuation divisible by three, so the γ coefficient is zero. Finally $\epsilon = 1 + \gamma$, while a unit cube with residue one is 1 modulo γ^3, since $v_\gamma(3) = 2$. Neither ϵ nor ϵ^2 is such a cube. The diagonal tangent vanishes.

All deformation data are defined over $\mathbf{Z}_3$. The vanishing of the relative tangent and Nakayama's lemma make D a quotient of $\mathbf{Z}_3$. An actual characteristic-zero source point forces

$$D = \mathbf{Z}_3. \tag{54}$$

Thus the source is the unique lift with its ordered pair of exact special characters.

It remains to compare this unique lift with its quadratic conjugate; this also avoids any need to control all other three-adic coefficient places. Let η_0 be the quadratic character of H/K and let ψ_5 be that of $K(\sqrt{5})/K$. At both five-places η_0 is the unramified quadratic character and ψ_5 is the quadratic character of $\mathbf{Q}_5(\sqrt{5})$. They form a basis of the local quadratic characters, with the same restrictions at the two places. Put $\beta = \alpha_1\alpha_2$ and take the unique $\delta \in \{1, \eta_0, \psi_5, \eta_0\psi_5\}$ whose restriction at both places is β. The inertia parity in Lemma 29.23 makes β ramified, so here $\delta = \psi_5 \quad \text{or} \quad \delta = \eta_0\psi_5$. Both characters are unramified at three. The extension H/K is unramified at every finite place; at three the unit five is a nonsquare in $\mathbf{F}_3$ and gives an unramified quadratic extension as well.

The representation $r_3^\sigma \otimes \delta$ has the original ordered pair (α_1, α_2), the same determinant and support, and the same crystalline condition. Its residue has the same restored ordered scalar inertia characters, hence is the same residual isomorphism class by Lemma 29.23. It is another point of D. Equation (54) gives

$$r_3^\sigma \simeq r_3 \otimes \delta. \tag{55}$$

Both possible characters δ are restricted from $G_{\mathbf{Q}}$: ψ_5 is cut out by $\mathbf{Q}(\sqrt{5})$ and $\eta_0\psi_5$ by $\mathbf{Q}(\sqrt{-15})$. Strong compatibility transports (55) to the whole system. Lemma 12.4, with $\eta = \delta$, gives its modularity directly. □

Remark 29.32. The class, unit and conductor calculations above are elementary and use no GRH. The corresponding PARI/GP certificates have, in the basis $[g], [z]$, strict-five masks $0, 1, 2$ and conductor-at-most-five masks $0, 4, 8$, with intersection $\{0\}$.

The exact calculation is supplied in [52, `d21`].

29.6 The nonordinary reducible characteristic-three source

This section proves a source modularity theorem for both pure local menus. Its input is an actual characteristic-zero compatible system. The proof uses finite-flat extensions at the ramified prime above 3, a universal pseudodeformation ring with explicit local conditions, and descent under quadratic conjugation. In particular, it does not replace source modularity by modularity of a semisimplified residue.

Let σ be the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$, and write $\omega = \overline{\chi}_3|_{G_K}$. The character ω cuts out $K(\sqrt{-3})/K$. Let ψ_γ cut out $K(\sqrt{\gamma})/K$. The norm identity gives

$$\psi_\gamma^\sigma = \psi_\gamma\omega. \tag{56}$$

Theorem 29.33 (Nonordinary reducible source modularity)**.** *Let $\mathcal{R}$ be an odd, absolutely irreducible, strongly compatible system over K of parallel weight two and cyclotomic determinant, good outside the two primes above 5. Assume its 5-adic companion is absolutely irreducible, as holds for the potentially modular systems constructed in the controlled reduction chain. Suppose its local types at the two primes over 5 are either both quadratic twists of Steinberg with nonzero monodromy, or both the tame dihedral supercuspidal types with inertia of order 3 or 6.*

Suppose the semisimplification of its characteristic-three residue is globally reducible, and both constituent characters are ramified quadratic at the prime over 3. Then $\mathcal{R}$ is modular.

More precisely, after a quadratic twist unramified outside 5, the system has Frobenius traces invariant under quadratic conjugation. Its 5-adic companion descends to an odd geometric representation of $G_{\mathbf{Q}}$, and modularity follows from Pan's theorem over $\mathbf{Q}$ at the prime 5.

29.6.1 Quadratic normalization

In the reducible characteristic-three case, each constituent has quadratic inertia at a prime over 5. The residues of the fundamental unit ϵ at the two primes over 5 are 2 and 3. The global unit relation therefore forces the two quadratic ramification parities to agree. If both are ramified, twist by the quadratic character ψ_5 of $K(\sqrt{5})/K$. This removes both ramification factors, preserves goodness at 3, and is trivial at the dyadic completion. After this normalization, the two types are either unramified quadratic twists of Steinberg at both places, or the SC types of inertia order 3 at both places.

Lemma 29.21 now gives, up to interchanging labels,

$$\chi_1 = \psi_\gamma, \qquad \chi_2 = \psi_\gamma \omega, \tag{57}$$

with distinct characters defined over $\mathbf{F}_3$. We henceforth work with the normalized system and the labels (57).

29.6.2 The finite-flat extension line at three

Let $F = K_{\mathfrak{p}_3} = \mathbf{Q}_3(\sqrt{21})$. Its ramification index is 2 and its residue field is $\mathbf{F}_3$. The character $\omega|_{G_F}$ is the nontrivial unramified quadratic character: over F, adjoining $\sqrt{-3}$ amounts to adjoining the square root of the nonsquare unit -7. Thus the two characters in (57) have the same ramified quadratic inertia and opposite unramified factors.

Lemma 29.34 (The local finite-flat line)**.** *For either orientation of the pair* (57)*, the finite-flat subspace of $H^1(F, \mathbf{F}_3(\omega))$ has dimension one. Under the Kummer identification it is spanned by $[a\epsilon]$. For any finite $k/\mathbf{F}_3$, the finite-flat subspace with k coefficients is the scalar extension of this line.*

Proof. First work over $\mathbf{F}_3$. A rank-one finite-flat model of either character has unique Breuil–Kisin exponent $r = 1$: the inertia exponent is odd, whereas height one permits only $0 \le r \le 2$. The two unramified factors differ by -1.

Any finite-flat extension gives an exact extension of these rank-one Kisin modules. Indeed, take the schematic closure of the generic rank-one subrepresentation and then the finite-flat quotient. The exact correspondence on objects killed by p is supplied by [47, Theorem 1.1.3 and Proposition 1.1.11]; the coefficient versions are [47, Lemmas 1.2.4–1.2.5]. In suitable bases the Frobenius matrix is, up to a common unramified factor,

$$\Phi = \begin{pmatrix} u & h(u) \\ 0 & -u \end{pmatrix}, \qquad h \in \mathbf{F}_3[[u]]. \tag{58}$$

Every such matrix has height at most one, since $u^2\Phi^{-1}$ is integral. A change of the upper lift by $f(u)$ changes h by a nonzero scalar multiple of $u(f(u) + f(u^3))$. The operator $f \mapsto f + f(u^3)$ is bijective on $\mathbf{F}_3[[u]]$: its constant term is multiplication by 2, and its other coefficients are solved recursively. Thus every extension is represented by a constant h and the Kisin extension space is one-dimensional. This proves that the actual finite-flat Galois extension space has dimension at most one. It uses no injectivity assertion for restriction to G_{F_∞}.

We identify a nonzero class using the curve with number field label `2.2.21.1-25.1-a4` in [48]. In generalized Weierstrass notation its model is

$$E = [a, a, a+1, -a-3, -a-2]. \tag{59}$$

Its conductor is (5) [48], so it has good reduction at 3; counting its points gives $a_{\mathfrak{p}_3}(E) = 0$. Its 3-division polynomial factors as

$$3\left(x - \frac{a+1}{3}\right)\left(x^3 + (2a+2)x^2 + (2a+3)x + \frac{4a+7}{3}\right). \tag{60}$$

The point above the rational root has squared doubled generalized y-coordinate $(61a+109)/9$, and the identity

$$\frac{61a+109}{9\gamma} = \left(\frac{2a+3}{3}\right)^2$$

identifies its squareclass as γ. The stable subgroup therefore has character $\chi_1 = \psi_\gamma$; the quotient has character $\chi_2 = \psi_\gamma\omega$.

Twisting $E[3]$ by χ_2 gives an extension of the trivial character by ω. Its action on the three projective lines other than the invariant line has affine form

$$z \longmapsto \omega(g)z + c(g), \qquad [c] \in H^1(K, \mathbf{F}_3(\omega)).$$

The cubic factor of (60) parametrizes those three lines. Depressing it gives

$$P = -2a-5, \qquad Q = \frac{34a+55}{9}, \qquad \frac{Q^2}{4} + \frac{P^3}{27} = \frac{50a}{9} + \frac{1075}{108} = \left(\frac{5(4a+7)}{18}\right)^2.$$

One Cardano radicand is $-3a-5 = -a\epsilon$, and the other is its inverse times a cube. The splitting field of the cubic is therefore $K(\sqrt[3]{-a\epsilon}, \zeta_3)$. The affine action on these roots identifies the actual extension line with $\langle [a\epsilon] \rangle$; changing an affine coordinate can multiply the cocycle by -1, which does not change the line. This identifies the extension, rather than only its quadratic resolvent. The class $[a\epsilon]$ is locally nonzero at 3, as the unit calculation below proves. Since $E[3]$ is finite-flat at 3, the upper bound is attained. Conjugating the curve supplies the opposite orientation; the same calculation below shows that its local line is unchanged.

Finally let $k/\mathbf{F}_3$ be finite, and let c be a k-valued extension whose underlying $\mathbf{Z}_3$-module is finite-flat. Pull back along the $\mathbf{F}_3$-line in its quotient. For every $\mathbf{F}_3$-linear functional $\lambda : k \to \mathbf{F}_3$, quotient the resulting kernel by $\ker(\lambda)\chi_1$. The resulting two-dimensional $\mathbf{F}_3$-extension is finite-flat, by stability under subquotients, and has class $\lambda \circ c$. Each coordinate class lies in the line just found; therefore c lies in that line tensored with k. Conversely scalar extension of a finite-flat extension is finite-flat by direct sums and subquotients. This argument avoids assuming that arbitrary generic coefficient endomorphisms extend to a chosen finite-flat model. □

29.6.3 The global plane and local evaluations

Write classes in $K^\times/K^{\times 3}$ additively and put

$$b = [a\epsilon], \qquad s = [5], \qquad d = [5a\epsilon] = s + b.$$

Lemma 29.35 (The global finite-flat extension plane)**.** *The global classes unramified outside* $\{3, 5\}$ *and finite-flat at* 3*, for either orientation of* (57)*, form*

$$V = \langle b, s \rangle = \langle s, d \rangle. \tag{61}$$

Quadratic conjugation acts by $\sigma(s) = s$*,* $\sigma(d) = -d$*. If* $\mathfrak{q}$ *is the prime over* 5 *at which* a *has valuation one, the two local restriction maps, in the basis* (s, d)*, are*

$$\ell_{\mathfrak{q}} = (1, -1), \qquad \ell_{\bar{\mathfrak{q}}} = (1, 1). \tag{62}$$

Moreover

$$\chi_1(\mathrm{Frob}_{\mathfrak{q}}) = 1, \qquad \chi_1(\mathrm{Frob}_{\bar{\mathfrak{q}}}) = -1, \tag{63}$$

and the values of χ_2 *are the opposite ones.*

Proof. The class and unit calculation in Lemma 29.21 shows that the global cubic classes with the indicated support have basis

$$[\epsilon], \quad [\gamma], \quad [a], \quad [\bar{a}], \qquad \bar{a} = 1 - a,$$

and dimension four. We compute their local-cube kernel at 3 elementarily. Normalize $v_F(\gamma) = 1$ and put $U_i = 1 + \mathfrak{p}_F^i$. The identity $\epsilon^3 - 5a\epsilon = 9\epsilon$ gives

$$\frac{5a\epsilon}{\epsilon^3} = 1 - \frac{9}{\epsilon^2} \in U_4.$$

The 3-adic logarithm identifies cubing $U_2 \to U_4$ with multiplication by 3, so this map is an isomorphism. Thus d is locally a cube. On the other hand

$$\epsilon = 1 + \gamma \in U_1 \setminus U_2, \qquad -a\epsilon = 1 - 3\epsilon \in U_2 \setminus U_3.$$

Every unit cube lies in $\{\pm 1\}U_3$. Therefore $[\epsilon]$ and $[a\epsilon]$ are independent local cubic classes. The valuation of γ gives a third independent class. Since $\mu_3 \not\subset F$, the dimension of $F^\times/F^{\times 3}$ is $[F : \mathbf{Q}_3] + 1 = 3$. The local-cube kernel on the four-dimensional global space is consequently exactly $\langle d \rangle$. Lemma 29.34 now gives (61).

The norm identity $N(a\epsilon) = -5$ gives $\sigma(b) = s - b$, and hence $\sigma(d) = -d$. In particular $\sigma(b)$ has the same local class as b, since d is locally zero and $-2 = 1$ in $\mathbf{F}_3$. At either 5-adic place, units have no nonzero cubic Kummer classes because $3 \nmid 5 - 1$. The valuations of s are $(1, 1)$ and those of d are $(2, 1)$, giving (62). Finally γ reduces to 1 at $\mathfrak{q}$ and to 2 at $\bar{\mathfrak{q}}$; their quadratic residue symbols give (63). □

29.6.4 A universal pseudoring with explicit local conditions

Fix a finite residue coefficient field $k/\mathbf{F}_3$ containing that of the actual source, and put $W = W(k)$. Let $G = G_{K,\{3,5\}}$. We impose a stable condition $\mathcal{C}$ on finite-cardinality $\mathbf{Z}_3[G]$-modules: finite-flatness at 3, and the local algebra conditions at 5 specified below. Here stability means closure under isomorphisms, subquotients, and finite direct sums. Finite-flatness is stable by [76, §5.2]; an algebra annihilator condition is stable by definition.

The universal pseudodeformation ring with residual pseudorepresentation $\chi_1 \oplus \chi_2$ and condition $\mathcal{C}$ exists by [76, Theorem 2.5.5]. It is a complete noetherian local W-algebra. Impose in addition the closed condition that the determinant equals χ_3, and call the resulting ring R. Every normalized actual source defines a point of R. Indeed, all finite quotients of a stable lattice in its crystalline weight-two representation are finite-flat at 3, and the exact local types at 5 satisfy the algebra identities below.

Because the residual pseudorepresentation is multiplicity-free, its universal Cayley–Hamilton algebra has generalized-matrix-algebra form

$$\mathcal{E} = \begin{pmatrix} R & B \\ C & R \end{pmatrix}.$$

The reducibility ideal J is the image of $B \otimes_R C \to R$; see [76, Proposition 4.2.2]. Writing $\mathfrak{m}$ for the maximal ideal of R, the extension interpretation [76, Theorem 4.3.5] gives

$$(B/\mathfrak{m}B)^\vee \subseteq \operatorname{Ext}^1_{\mathcal{C}}(\chi_2, \chi_1), \qquad (C/\mathfrak{m}C)^\vee \subseteq \operatorname{Ext}^1_{\mathcal{C}}(\chi_1, \chi_2). \tag{64}$$

Before the fixed-determinant quotient these are equalities; the inclusions suffice afterwards. The construction is functorial for quadratic conjugation whenever the local conditions are exchanged.

For a normalized SP type, let α_v be its exact unramified quadratic twisting character. We impose

$$\big(g - \chi_3(g)\alpha_v(g)\big)\big(h - \alpha_v(h)\big) = 0 \qquad (g, h \in G_{K_v}). \tag{65}$$

These are relations in the completed group algebra. They hold on every extension of α_v by $\chi_3\alpha_v$, hence on every stable lattice in the actual Steinberg representation. The two residual Frobenius eigenvalues are distinct. Thus the relations allow ramification only in the orientation whose invariant character is $\chi_3\alpha_v$, and they fix the local characters exactly. The universal condition is permitted to contain points of zero monodromy; the actual source has nonzero monodromy and is included.

For a normalized SC3 type, choose arithmetic Frobenius Φ and a generator T of pro-3 tame inertia, so that $\Phi T\Phi^{-1} = T^5$. Kill wild inertia and the prime-to-3 tame inertia, and impose

$$T^2 + T + 1 = 0, \qquad \Phi^2 + 5 = 0. \tag{66}$$

The actual normalized SC representation has inertial eigenvalues ζ_3, ζ_3^{-1}, exchanged by Frobenius. Its cyclotomic determinant gives $\Phi^2 = -5$, so it satisfies these relations. Replacing Frobenius by an inertial multiple, or changing the tame generator, preserves the conditions on this quotient. Thus quadratic conjugation exchanges the same local condition at the two places. In characteristic 3, its first relation becomes

$$(T - 1)^2 = 0. \tag{67}$$

29.6.5 The diagonal pseudotangent

Lemma 29.36 (Vanishing of the reducible tangent)**.** *For either pure local menu, the ring R satisfies*

$$\mathfrak{m}/(3R + \mathfrak{m}^2 + J) = 0. \tag{68}$$

Consequently multiplication induces a surjection

$$(B/\mathfrak{m}B) \otimes_k (C/\mathfrak{m}C) \twoheadrightarrow \mathfrak{m}/(3R + \mathfrak{m}^2). \tag{69}$$

Proof. A tangent in the reducible quotient R/J has constituent characters

$$\chi_1(1 + \varepsilon f), \qquad \chi_2(1 - \varepsilon f), \qquad f \in H^1(G, k),$$

where the opposite signs follow from the fixed determinant. The constituent character deformations are themselves finite-flat modules at 3, by the stability of $\mathcal{C}$ and [76, Proposition 4.3.4].

Fontaine's bound for a finite-flat module killed by p^n is triviality of $G^{(u)}$ for $u > e(n+1/(p-1))$; the convention in the cited statement is $G^{(u)} = G^{u-1}$, where the latter is classical upper numbering [34, pp. 515–516, Theorem A]. For $p = 3$, $n = 1$, and $e = 2$, the classical upper break is at most 2. The abelian cubic character f therefore has conductor at most $\mathfrak{p}_3^3$. Its tame quadratic factor does not change positive ramification groups. At a prime over 5, a cyclic cubic character is unramified because $3 \nmid 5 - 1$.

We need only the ray group of conductor $\mathfrak{p}_3^3$ and its Frobenius classes at five. These have the following elementary calculation. The group $(\mathcal{O}_K/\mathfrak{p}_3^3)^\times$ has order 18. The unit $\epsilon = 1 + \gamma$ has exact order 3 modulo γ^3, so the image of $\langle -1, \epsilon\rangle$ has order 6. Class number one then gives ray group C_3. The class of $\mathfrak{q} = (a)$ is nontrivial: $-a = 1 - \gamma$ differs from $1, \epsilon, \epsilon^2$ with valuations $1, 1, 2$, respectively, so it does not belong to the unit image modulo γ^3. Its conjugate also has nontrivial class because conjugation acts as an automorphism of the ray group. Infinite places contribute only 2-primary data and do not affect this cubic quotient.

In the SP case, (65) fixes the two Frobenius characters, hence forces $f(\Phi) = 0$. In the SC case, $\Phi^2 = -5$ gives the same conclusion, since the residual Frobenius eigenvalues are ± 1 and 2 is invertible. Therefore $f = 0$, proving (68). The ideal J is generated by products from B and C, and $\mathfrak{m}J \subseteq \mathfrak{m}^2$; this proves (69). □

29.6.6 The SP/SP action of quadratic conjugation

Lemma 29.37 (The two SP signs agree)**.** *An irreducible actual source in the situation of Theorem 29.33 has equal normalized unramified Steinberg signs at the two primes over* 5*. For these signs, quadratic conjugation acts trivially on the relative cotangent space of* R*.*

Proof. Suppose first that both signs are -1. The invariant monodromy character is $\chi_3\alpha_v$ and reduces to the Frobenius eigenvalue $-\alpha_v$. Using (63), the permitted global extension spaces in (64) are contained in

$$\ker \ell_{\bar{\mathfrak{q}}} = \langle b \rangle, \qquad \ker \ell_{\mathfrak{q}} = \langle \sigma(b) \rangle, \tag{70}$$

respectively. If both signs are $+1$, the roles of the lines are exchanged. If the signs are opposite, one orientation must be unramified at both 5-places. Its global extension space is then contained in $\ker \ell_{\mathfrak{q}} \cap \ker \ell_{\bar{\mathfrak{q}}} = 0$. Nakayama gives $B = 0$ or $C = 0$, so $J = 0$ and every point of R is globally reducible. This contradicts the actual irreducible source. Equivalently, Ribet's lattice lemma would require nonsplit residual extensions in both orientations, while one of them has no permitted nonzero extension. The signs therefore agree.

Quadratic conjugation preserves the resulting local conditions, interchanges the two residual characters, and interchanges the two lines (70). Its action on their product is $+1$. One can check the sign directly from $b = d - s$ and $\sigma(b) = s - b = -(d + s)$. For a basis-independent verification, choose a lift $h \in G_{\mathbf{Q}}$ of σ which fixes ζ_3; it exists since $K \cap \mathbf{Q}(\zeta_3) = \mathbf{Q}$. The coefficient action on ω is then trivial, and $\chi_1(h^2) = \chi_2(h^2)$. Squared transport is inner by a residual scalar, so the two exchange factors multiply to $+1$. The surjection (69) now shows that quadratic conjugation fixes the entire relative cotangent space. □

29.6.7 The SC/SC action of quadratic conjugation

Lemma 29.38 (The SC anti-invariant product vanishes)**.** *For the normalized SC/SC menu, quadratic conjugation acts trivially on the relative cotangent space of* R*.*

Proof. Both extension spaces in (64) are subspaces of $V \otimes_{\mathbf{F}_3} k$. Hence the tensor surjecting onto the relative cotangent space is a quotient of $V^\vee \otimes V^\vee$. Quadratic conjugation swaps the factors and acts on each by $s \mapsto s$, $d \mapsto -d$. The scalar in this statement is fixed by the same choice of h as in the preceding proof. The local equation (67) kills the image of $\ell_v \otimes \ell_v$ in the cotangent space. Indeed, in the universal generalized matrix algebra write

$$T - 1 = \begin{pmatrix} x & b_T \\ c_T & y \end{pmatrix}, \qquad x, y \in \mathfrak{m}.$$

Its upper-left squared entry is $x^2 + b_T c_T$. The exact relation over R is $(T-1)^2 = -3T$, so $b_T c_T \in \mathfrak{m}^2 + 3R$; therefore $b_T c_T$ is zero in $\mathfrak{m}/(3R + \mathfrak{m}^2)$. The classes of b_T and c_T are evaluation on local inertia, which are the functionals in (62), up to a harmless nonzero change of local generator.

The evaluations in (62) are $s^\vee - d^\vee$ and $s^\vee + d^\vee$. Thus Lemma 12.3 makes the action on the relative cotangent trivial. The local calculation above concerns products of opposite offdiagonal extensions, not only traces of representation tangents. □

Quadratic conjugation defines a $W(k)$-linear involution of R: its square is induced by an inner automorphism of G_K, which does not change a pseudorepresentation. Lemmas 29.37 and 29.38, followed by Lemma 12.2, show that this involution is the identity. Consequently every normalized actual source satisfies

$$\operatorname{tr} r_3(g) = \operatorname{tr} r_3(hgh^{-1}) \qquad (g \in G_K). \tag{71}$$

No assertion that R is finite over $W(k)$, that its whole cotangent space is zero, or that its traces are rational is needed. The essential fact is that an involution of order prime to the residue characteristic cannot survive when its anti-invariant cotangent is zero.

Completion of the proof of Theorem 29.33. Strong compatibility transports (71), first at every good Frobenius, to all companions of the normalized system. Chebotarev identifies the semisimplifications of the system and its quadratic conjugate. In particular its absolutely irreducible 5-adic member is invariant under σ.

Lemma 12.4 makes the normalized system modular. Undoing the finite quadratic normalization proves the claim. □

Remark 29.39. The seed has $a_v = -1$ at both places over five and $a_{\mathfrak{p}_3} = 0$. The displayed identities determine its local Kummer class and the local-cube kernel explicitly. The source proof uses these finite-flat and pseudodeformation calculations and no dyadic modularity assumption.

The exact calculation is supplied in [52, `d21`].

29.7 Completion of the bounded source theorem

Theorem 29.40. *Assume GRH. Let $\mathcal{R}$ be a potentially modular compatible system over K, of determinant cyclotomic and labelled weights $\{0,1\}$, with conductor supported at five. Suppose each five-place has one of the bounded types* SC *or* SP*. Then $\mathcal{R}$ is modular.*

Proof. The mixed case is Proposition 29.1. A pure special source with reducible five-adic residue is covered directly by Proposition 29.2. For the other pure sources, consider the three-adic member, which is crystalline at three. If its residue remains irreducible on G_H, Proposition 29.20 applies. If the residue is irreducible but its restriction is reducible, the dihedral source theorems of the preceding sections apply: induction from H for the cuspidal pair, and fixed-character rigidity followed by descent under quadratic conjugation for the special pair.

Otherwise Lemma 29.21 gives the complete constituent list. The two unramified-at-three possibilities are modular by Proposition 29.22. In the remaining two, twist by ψ_5 when necessary to remove the common quadratic inertia at five. This twist is unramified at three and preserves the determinant and crystalline condition. The nonordinary reducible source theorem proved above gives invariance under quadratic conjugation and modularity. These alternatives are exhaustive for every finite coefficient field and every residual lattice. □

Theorem 29.41 (Unrestricted residual base at five)**.** *Assume GRH. Every irreducible representation*

$$\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_5), \qquad \det \bar{r} = \bar{\chi}_5,$$

unramified outside five is modular. No local finite-flatness, tameness, coefficient-field or conductor-exponent restriction is imposed.

Proof. Solvable projective image is handled by an odd Artin lift and solvable Artin automorphy, with weight raising to obtain a regular residual witness. In the nonsolvable case the restriction to $G_{K(\zeta_5)}$ is irreducible. The exceptional projective-$\mathrm{PGL}_2(\mathbf{F}_5)$ condition cannot occur: $[K(\zeta_5) : K] = 4$, whereas the abelianization of $\mathrm{PGL}_2(\mathbf{F}_5)$ has order two.

Lemma 4.14 provides actual weight-two local lifts at both five-places with determinant χ_5. Prescribe their components and no ramification elsewhere. Theorem 4.2.1 of [31] gives a global lift. Snowden's potential-modularity and compatible-system construction gives a potentially modular weight-two system with the selected menu. At a selected special component, the purity observation in Section 2 retains nonzero monodromy. The system is crystalline at two and three

by all-place compatibility. Theorem 29.40 makes the actual system modular, and its five-adic reduction is the original $\bar{r}$. □

29.8 The arbitrary-conductor transfer

We verify the terminal character condition and apply Theorem 1.3 with reset prime five. Its terminal source is allowed arbitrary regular weights.

29.8.1 Characters supported at five

Lemma 29.42. *Every finite characteristic-five character of G_K unramified outside five extends to $G_{\mathbf{Q}}$. If a representation of determinant $\bar{\chi}_5$ and support at five is reducible, its constituent quotient is exactly $\bar{\chi}_5$ or $\bar{\chi}_5^{-1}$.*

Proof. Lemma 6.7 gives the quotient and writes a supported character as a cyclotomic power times a narrow class character. The latter is invariant under quadratic conjugation because the narrow class group has order two. It extends across the index-two group by choosing a square root of its value on the square of a lift of the nontrivial automorphism. The extension has finite image. □

Remark 29.43. The generic mixed-case theorem of [20, Theorem A] excludes the local ratio $\bar{\chi}_5^{\pm 1}$. It therefore does not replace the residually reducible endpoint used here. At three its complete-splitting hypothesis also fails. The terminal argument uses the locally irreducible source theorems of Zhang, not the generic mixed theorem.

Completion for $d = 21$ in Theorem 1.2. Two is inert and five splits. Theorem 29.41 supplies the supported irreducible residual base, and Lemma 29.42 gives the extending odd constituent quotient $\bar{\chi}_5^{\pm 1}$. Apply Corollary 6.6. □

30 The characteristic-three base over $\mathbf{Q}(\sqrt{37})$

Put $K = \mathbf{Q}(\sqrt{37})$. We use one wild type and one bounded quadratic or quartic type at the two three-primes. The compatible system thereby has a finite-flat characteristic-two member whose local fields fit inside a fixed solvable auxiliary extension.

Theorem 30.1. *Assume GRH. Every continuous irreducible representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with determinant $\overline{\chi}_3$, unramified outside three, is modular.*

Write

$$w = \frac{1+\sqrt{37}}{2}, \qquad \epsilon = 5 + 2w, \qquad \pi = 3 + w, \qquad \pi' = 4 - w.$$

Thus $w^2 - w - 9 = 0$, $N_{K/\mathbf{Q}}(\epsilon) = -1$ and $\pi\pi' = 3$. Put $\mathfrak{p} = (\pi)$ and $\mathfrak{p}' = (\pi')$. Both completions above three are $\mathbf{Q}_3$, and the unique dyadic completion is the unramified quadratic extension of $\mathbf{Q}_2$.

Lemma 30.2. *The ordinary and narrow class numbers of K are one. Every finite character $G_K \to \overline{\mathbf{F}}_3^{\times}$ unramified outside three has square one.*

Proof. The Minkowski bound is $\sqrt{37}/2 < 3.05$. There is no ideal of norm two, and the two ideals of norm three are principal. Thus the ordinary class number is one. The unit ϵ has norm -1, which also gives narrow class number one. The character assertion follows from Lemma 4.17. □

Proposition 30.3. *Let $\overline{\rho}$ satisfy the hypotheses of Theorem 30.1, and suppose its image is nonsolvable. It lifts to a totally odd, potentially modular, strongly compatible system $\mathcal{R} = (r_\lambda)$ with determinant χ and weights $\{0, 1\}$. The system is unramified away from three, has type τ_+ at $\mathfrak{p}$, and has one of the two local conditions of Lemma 4.20 at $\mathfrak{p}'$.*

Proof. Lemma 4.19 and Lemma 4.20 give the two weight-two local points with determinant χ_3. Choose their components and unramified conditions elsewhere. Nonsolvability excludes induction from $K(\zeta_3)$, so Lemma 2.6(i) supplies the system, crystalline at two. On a selected special component the Frobenius ratio is three; Section 2 retains nonzero monodromy. No global quadratic character is needed. □

Choose a place above two in the coefficient field and write r_2 for this member and $\overline{r}_2$ for its reduction. Lemma 4.19 makes $\overline{r}_2$ irreducible, with determinant one. Let M be its kernel field. Its completion at $\mathfrak{p}$ is E_+, while Lemma 4.20 puts its completion at $\mathfrak{p}'$ inside E'.

30.1 Ramification at two

For a finite Galois extension L/F of dyadic local fields, $d(L/F)$ denotes the valuation of its different, normalized by $v_F(2) = 1$. Thus $d(L/F)$ is the local contribution to the exponent of two in the root discriminant. We use Serre's upper numbering for ramification groups.

Lemma 30.4. *Let $F/\mathbf{Q}_2$ be unramified and let J be a finite flat commutative group scheme over $\mathcal{O}_F$ killed by two. The field generated by $J(\overline{F})$ has no positive upper ramification break greater than one, and its normalized different is strictly less than two.*

Proof. Fontaine's Theorem A and its corollary [34, pp. 515–516] apply to finite flat group schemes of any rank. His notation is $G^{(u)} = G^{u-1}$, where the right side uses Serre's upper numbering. For a group scheme killed by p^n over a field of absolute ramification index e, his bound is

$$G^{(u)} \text{ acts trivially if } u > e\left(n + \frac{1}{p-1}\right).$$

Here $e = n = 1$ and $p = 2$. Thus G^v acts trivially for $v > 1$. The different bound is the cited corollary. □

Corollary 30.5. *A compositum of finite Galois extensions of F having no upper break greater than one has the same property, and its normalized different is strictly less than two.*

Proof. Upper-numbering ramification is compatible with quotients. If all quotient fields have trivial groups above one, so does their compositum. The integral formula for the different then gives $d < 2$; more precisely, for tame inertia order t and wild inertia order Q, one has $d \le 2 - (t+1)/(tQ)$. □

We also need to recognize ordinary lifts in the dihedral case.

Lemma 30.6. *Let $F/\mathbf{Q}_2$ be unramified and let $r : G_F \to \mathrm{GL}_2(E)$ be Barsotti–Tate with determinant χ_2. If its residual image is a two-group, then r is ordinary.*

Proof. A two-group has only trivial irreducible representations in characteristic two, so both residual constituents are unramified. Apply Lemma 4.16 with $p = 2$ and $e(F/\mathbf{Q}_2) = 1$. □

30.2 An explicit solvable auxiliary field

Put

$$Q(X) = X^4 - 2X^3 + (3-2w)X^2 + (4+2w)X - w - 2, \qquad C = K(\sqrt{\pi}, \sqrt{\pi'}, \sqrt{-\epsilon}),$$

and let H be the splitting field of Q over K. We use $N = CH$.

Proposition 30.7. *The extension N/K is solvable Galois of degree 96, is unramified outside two and three, and has no dyadic upper break greater than one. Its completions at $\mathfrak{p}$ and $\mathfrak{p}'$ are E_+ and E', respectively.*

Proof. The cubic resolvent and discriminant of Q are

$$R(T) = T^3 + (2w-3)T^2 + 4 - 4w, \qquad \operatorname{disc}(Q) = \operatorname{disc}(R) = 144\delta, \quad \delta = 99 - 28w = -\epsilon^{-3}\pi^3.$$

The quartic is irreducible modulo the prime above eleven with $w \equiv 7$: its reduction is $X^4 - 2X^3 + 7X + 2$. This polynomial is relatively prime to $X^{121} - X$ and divides $X^{14641} - X$ over $\mathbf{F}_{11}$, so is irreducible. Hence Q is irreducible over K.

At $\mathfrak{p}$ we have $w \equiv 18 \pmod{27}$. The shifted polynomial is

$$Q(Y-1) = Y^4 - 6Y^3 + (15-2w)Y^2 + (6w-12)Y - 5w.$$

Its Newton polygon has segments of slopes -1 and $-1/3$, of lengths one and three. It therefore has one root in $\mathbf{Q}_3$ and an irreducible totally ramified cubic factor. The cubic roots have valuation $1/3$, whereas the linear root has valuation one. The resultant of these factors has valuation one. Since $v_{\mathfrak{p}}(\operatorname{disc}(Q)) = 5$, the cubic factor has discriminant exponent three. Its root is a uniformizer, so this is also the field discriminant exponent. Its discriminant square class is that of $-\epsilon\pi$, namely 3. Thus its splitting field is totally ramified S_3, with normalized different $7/6$ and quadratic subfield $\mathbf{Q}_3(\sqrt{3})$. By Lemma 4.19 it is E_+. The quartic's linear factor shows that its full local splitting field is the same field.

The global quartic group is transitive and contains this point-stabilizer S_3, so it is S_4. Its unique quadratic subfield is $D = K(\sqrt{-\epsilon\pi})$. Valuations at $\mathfrak{p}, \mathfrak{p}'$ and the mixed real signs of $-\epsilon$ give $[C : K] = 8$. The intersection $H \cap C$ is an elementary abelian two-extension and hence is D, the maximal such quotient field of S_4. Therefore $[N : K] = 24 \cdot 8/2 = 96$.

At $\mathfrak{p}'$ the reduction of Q is $X^2(X-1)^2$. Hensel factorization gives two quadratic factors with unit resultant. Their discriminant valuations are each at least one and sum to $v_{\mathfrak{p}'}(\operatorname{disc}(Q)) = 2$, so each is one. Both define ramified quadratic extensions; the local splitting field lies in E'. The local square classes of $\pi, \pi', -\epsilon$ are respectively $3, 1, 1$ at $\mathfrak{p}$, and $1, 3, -1$ at $\mathfrak{p}'$. Thus adjoining C leaves E_+ unchanged and makes the other completion exactly E'.

At two put $h = X^2 + X + 1 - w$. Direct calculation gives

$$Q = h^2 + 2g, \qquad g = -2X^3 + (2w+1)X - 6.$$

Over the residue field $\mathbf{F}_4$, the polynomial $\bar{h}$ is irreducible, since $\operatorname{Tr}_{\mathbf{F}_4/\mathbf{F}_2}(1 - \bar{w}) = 1$, and $\bar{g} = X$ is relatively prime to it. Dedekind's index criterion therefore shows that the quartic order is maximal at two. Its local field has ramification index two and residue degree two, and its relative discriminant exponent is four. These statements also follow by passing to the unramified quadratic extension defined by h and using the Eisenstein factor of Q near a root of h.

The reduction of R is $T^2(T+1)$, so it has a local linear factor and its local splitting group has order at most two. The local splitting group of Q is consequently contained in a subgroup of order eight of S_4, and is a two-group. Thus its lower ramification groups satisfy $G_0 = G_1$. In the permutation action on the four roots, inertia has two orbits, since the quartic field has

residue degree two. The terms for G_0 and G_1 in the Artin conductor are therefore two each. They exhaust its discriminant exponent four, so faithfulness of the permutation action gives $G_2 = 1$. The largest upper break is at most one.

Finally, the radicands defining C are dyadic units. Every corresponding quadratic character has conductor at most two and upper break at most one. Corollary 30.5 gives the same bound for N. The displayed relative discriminant and the radicands also show that there is no ramification away from two and three. □

30.3 The discriminant comparisons

Let L/K be finite Galois and unramified outside two and three. If its dyadic completion has no upper break greater than one, its dyadic different is less than two. If its completion at $\mathfrak{p}$ is E_+ and its completion at $\mathfrak{p}'$ is contained in E', the discriminant tower gives

$$\mathrm{rd}(L) < 4\sqrt{37}\,3^{5/6} < 60.78017. \tag{72}$$

Each three-adic exponent is multiplied by one half, since $[K : \mathbf{Q}] = 2$.

Proposition 30.8. *Assume GRH for the Dedekind zeta function of L. If its completions above $\mathfrak{p}, \mathfrak{p}'$ are E_+, E' and $n = [L : \mathbf{Q}] \geq 96768$, then $\mathrm{rd}(L) > 60.80125$.*

Proof. Use the explicit formula of Section 13 with $B_b \leq B_b^*$ and the finite Euler sums. There are $n/12$ primes of norm three and $n/8$ of norm nine. Thus $c_3 = 1/6$ and $c_9 = 1/4$. Discard the real-place term. At $b = 56/5$ and $n = 96768$, the directed lower bound is 60.8012536506. It increases with n because $C_b > 0$. □

Proposition 30.9. *Assume GRH. The image of $\overline{r}_2$ constructed in Proposition 30.3 is either solvable or conjugate to $\mathrm{SL}_2(\mathbf{F}_4) \simeq A_5$.*

Proof. Dickson's classification makes a nonsolvable finite subgroup of $\mathrm{SL}_2(\overline{\mathbf{F}}_2)$ conjugate to $\mathrm{SL}_2(\mathbf{F}_q)$, for $q = 2^r$, $r \geq 2$. Suppose $q \geq 8$. This group is nonabelian simple, so its kernel field M is linearly disjoint over K from the solvable extension N. The compositum $L = MN$ satisfies $[L : \mathbf{Q}] = 2 \cdot 96 \cdot q(q^2 - 1) \geq 96768$. By Lemma 30.4, Proposition 30.7 and Corollary 30.5, its dyadic different is less than two. Its completions at three are exactly E_+ and E'. Equation (72) contradicts Proposition 30.8. □

30.4 Modularity of the remaining images

The sign of the type τ_+ gives a direct treatment of the dihedral case. We make the required quadratic calculation explicit.

Lemma 30.10. *Let D/K be quadratic, unramified at two and at every finite place away from three. If it ramifies at $\mathfrak{p}$, its completion there is $\mathbf{Q}_3(\sqrt{-3})$.*

Proof. Class number one shows that $D = K(\sqrt{\alpha})$ with α a product of $-1, \epsilon, \pi, \pi'$ modulo squares. At the dyadic completion the residue field is $\mathbf{F}_4$. In the basis $1, \bar{w}$, Lemma 11.12 gives the columns

	-1	ϵ	π	π'
1	1	0	0	1
$\bar{w}$	0	1	1	1

.

For example these are obtained by cubing the four units modulo four and dividing their differences from one by two. The kernel is generated by $[\epsilon\pi]$ and $[-\pi\pi']$. Thus the four global classes that are unramified at two are

$$1, \qquad \epsilon\pi, \qquad -3, \qquad -\epsilon\pi'.$$

At $\mathfrak{p}$ we have $w = 18 \pmod{27}$, hence $\pi/3 = 1 \pmod 3$ and $\epsilon = -1 \pmod 3$. The two ramified classes in this list, $\epsilon\pi$ and -3, therefore both have local square class -3. □

Proposition 30.11. *Assume GRH. The characteristic-two member r_2 of Proposition 30.3 is modular.*

Proof. Suppose first that the residual image is solvable. Since $\overline{r}_2$ is irreducible and has determinant one, its image is an odd dihedral group D_{2m}. Let D/K be its quadratic inducing extension. Its local quadratic field at $\mathfrak{p}$ is $\mathbf{Q}_3(\sqrt{3})$, by Lemma 4.19. Lemma 30.10 shows that D/K ramifies at the unique dyadic place.

The dyadic wild inertia image is a nontrivial normal two-subgroup of the local decomposition image in D_{2m}. It is generated by a reflection. The normalizer of a reflection in an odd dihedral group is its own subgroup, so the entire local residual image is C_2. Lemma 30.6 makes r_2 ordinary. Allen's nearly ordinary residually dihedral theorem [2, Theorem 5.2.1 and Lemma 5.1.2] now gives modularity. The extra condition when the inducing quadratic field is CM is satisfied because the dyadic place ramifies in D; in particular it does not split. Allen's dihedral argument supplies the ordinary modular witness required in that theorem.

Suppose now that the residual image is nonsolvable. Proposition 30.9 makes it $A_5 = \mathrm{SL}_2(\mathbf{F}_4)$. Its residual modularity follows from [3, Theorem 2.5]. We check the real-place hypothesis in Thorne's characteristic-two lifting theorem. If the kernel field M were totally real, then $[M : \mathbf{Q}] = 120$, while its finite ramification gives $\mathrm{rd}(M) \le 4\sqrt{37}\,3^{5/6}2^{-2/4} < 42.97807$, by Lemma 11.9. This contradicts the classical lower bound 43.14954 in Lemma 13.4; no prime term or zero-free hypothesis is needed for this real-place check. Hence $\overline{r}_2(c_v) \neq 1$ for at least one real place v of K. The representation r_2 is Barsotti–Tate at two, its residual image is nonsolvable and modular, and this real-place condition holds. Thorne's Theorem 6.1 [73] applies directly. It permits both ordinary and nonordinary local representations and imposes no condition $K_2 = \mathbf{Q}_2$. □

Proof of Theorem 30.1. Solvable projective images are modular by automorphic induction and Langlands–Tunnell, with the characteristic-three lifting and twisting argument used in [3, proof of Theorem 2.5]. We may therefore suppose that the image is nonsolvable. Proposition 30.3 constructs a compatible system lifting $\overline{\rho}$. Proposition 30.11 makes its characteristic-two member modular. Strong compatibility, Chebotarev and strong multiplicity one then identify the system with the system of that Hilbert modular form. Its characteristic-three residual representation is $\overline{\rho}$, which is consequently modular. □

31 Completion at the odd terminal primes

Completion for $d = 3, 7, 10, 11, 14, 19, 37$ in Theorem 1.2. For each row the first citation gives the irreducible supported base and the second gives the extending quotient in the reducible case.

Two does not split in these fields, and the indicated terminal prime splits. Apply Corollary 6.6.

d	ℓ	Irreducible base	Constituent quotient
3	13	Proposition 22.10	Lemma 6.8
7	3	Theorem 23.1	Lemma 4.17
10	3	Theorem 24.1	Lemma 24.7
11	5	Proposition 25.13	Lemma 6.7
14	5	Theorem 26.14	Lemma 6.7
19	3	Theorem 28.1	Lemma 6.7
37	3	Theorem 30.1	Lemma 4.17

For $d = 10$, the quotient is ω_3 or $\omega_3\nu$ with ν the ordinary Hilbert class character, which is totally even. At the other places in the table, total oddness follows from the cyclotomic determinant. For $d = 15$ use Propositions 2.7 and 6.2 at seven and Theorem 27.1. The full reduction for $d = 21$ is in Section 29. □

32 The four remaining split-dyadic fields

Theorem 32.1. *Assume GRH. For $d \in \{57, 65, 73, 89\}$, every absolutely irreducible characteristic-two representation of G_K unramified outside two is modular.*

Proof. By Lemma 14.5, it suffices to treat $\mathrm{SL}_2(\mathbf{F}_q)$ with $q \geq 8$.

Use the functions F_b, J_b, B_b, C_b of Section 13. Put $a = \pi/(2b)$ and let

$$\log \mathcal{L}(n, b, k) = \gamma + \log(8\pi) - \sum_{j=0}^{3} \left\{ \frac{1}{j + 1/2} - J_b(j + 1/2) \right\} - \frac{2a^2}{49} - \frac{C_b}{n} + \frac{k \log 4}{8} \sum_{m \geq 1} 4^{-m/2} F_b(m \log 4).$$

The tail estimate in that theorem makes this a lower bound for the root discriminant, under GRH, when the indicated norm-four Euler contributions are present. It increases with n at fixed b, k. We shall also write $B(G), C(G)$ for the corresponding integrals with an even test function G.

For M in Lemma 14.4, the following strict comparisons hold with $n = 8q(q^2 - 1)$ and $d \leq 89$:

q	k	b	$\mathcal{L}(n, b, k) >$	upper bound $<$
8	1	7.8	38.07399	36.13605
16	0	9.8	36.64088	36.13605
$q \geq 32$	0	12	38.91320	37.73593

The last row drops Euler terms and uses $n \geq 261888$ and $\mathrm{rd}(M) \leq 4\sqrt{89}$. In the first two rows, raising k by one increases the logarithmic lower bound by $\frac{\log 4}{8} \sum_m 4^{-m/2} F_b(m \log 4)$, exceeding $\log 2/(2q)$ already in the first summand since $F_b(\log 4) > 1/2$. Thus the table excludes every case except $q = 8, k = 0$. The exact calculation is supplied in [52, `split_dyadic`].

Let σ be the nontrivial automorphism of $K/\mathbf{Q}$. If $L^\sigma \neq L$, simplicity and Goursat's lemma give $[LL^\sigma : \mathbf{Q}] = 2 \cdot 504^2 = 508032$. Each local constituent has upper breaks at most one, since $k = 0$. The same is true of the compositum by compatibility of upper numbering with quotients, so $\mathrm{rd}(LL^\sigma) \leq 4\sqrt{89}$. The last table row, by degree monotonicity, contradicts this.

Hence $L/\mathbf{Q}$ is Galois. Its degree is 1008 and $\mathrm{rd}(L) \leq 4\sqrt{89} < 55.46400$, so Lemma 13.4 excludes a totally real kernel. Some real conjugation therefore acts nontrivially, and Lemma 2.5 gives modularity over K. □

Completion for $d = 57, 65, 73, 89$ in Theorem 1.2. The class group and scalar Selmer calculations in Lemmas 14.1 and 14.2 give the scalar source by Theorem 1.4. With Theorem 32.1, apply Theorem 10.1. □

33 Additional supported sources and residual bases

These results treat wider support, different local profiles, or different determinants. Each statement specifies the analytic input it uses.

33.1 Scalar and nonscalar sources over $\mathbf{Q}(\sqrt{2})$

In this section $K = F_2$, $s = \sqrt{2}$, $\ell = 7$ and $\epsilon = 1 + s$. A special type includes its exact quadratic scalar character. We first normalize the ramified scalar factors.

33.1.1 Ordinary scalar sources

Proposition 33.1 (Ordinary scalar sources)**.** *Let $\mathcal{R}$ be a system over $K = \mathbf{Q}(\sqrt{2})$, of cyclotomic determinant and labelled weights $\{0, 1\}$, good outside seven and special at both seven-primes with unramified quadratic scalar characters. If some dyadic member is ordinary with scalar residual semisimplification, the actual system is modular.*

Proof. Write $s = \sqrt{2}$, $\epsilon = 1 + s$, $\pi = 3 + s$, $\pi' = 3 - s$ and $\delta = -7$. The supported quadratic classes with even dyadic valuation form $V = \langle [-1], [\epsilon], [\pi], [\pi'] \rangle$. Localization at K_2 has kernel $\langle [\delta] \rangle$. Indeed unit squares modulo $8\mathcal{O}_K$ have coordinate pairs $(1, 0), (1, 4), (3, 2), (3, 6)$ in the basis $1, s$. Testing the sixteen classes gives precisely $1, \delta$ in the square kernel. The accuracy exceeds twice the valuation of two, so Hensel's lemma makes the test exact. Moreover no nonzero class in V is split at both seven-primes: vanishing valuations removes π, π', and the Legendre-symbol vectors of $-1, \epsilon$ are $(-, -)$ and $(-, +)$.

Let r be the ordinary dyadic member. Lemma 12.1 puts it in the marked basis with real matrix C, ordinary line e_2 and residue $1 + \chi_\delta E_{12}$. Write ϖ for a coefficient uniformizer and k for its residue field.

We compare two actual sources with these normalizations and the same exact unramified ordinary quotient ν at two. Suppose $r_1 \equiv r_2 \pmod{\varpi^n}$, $n \geq 1$, and write

$$r_2(g) = (1 + \varpi^n X(g)) r_1(g) \pmod{\varpi^{n+1}}, \qquad X(g) = \begin{pmatrix} x(g) & y(g) \\ z(g) & x(g) \end{pmatrix}.$$

The determinant gives the trace-zero form. The adjoint action of $1 + \chi_\delta E_{12}$ adds $\chi_\delta(g) z(h)(1 + E_{12})$; hence z is additive. Equality of the exact ordinary characters gives $x|_{G_{K_2}} = y|_{G_{K_2}} = 0$, and the local class of z is a unit Kummer class. Here is the required assertion for the divided difference. The two local ordinary extensions have identical characters $\nu, \chi_2 \nu^{-1}$. The difference of their integral extension cocycles divided by ϖ^n is integral, and its rational class remains in the E-linear crystalline extension space. The corresponding stable lattice is again ordinary Barsotti–Tate. Its reduction is an extension of the etale model by the multiplicative model, hence a unit Kummer class. This uses actual sources and does not assert exactness for arbitrary models.

At a seven-place residual inertia has a unit upper entry, so its special line is uniquely determined. Congruent representations have congruent special lines: take the kernel of a rank-one inertia difference with a unit entry. If the exact special characters agree modulo ϖ^{n+1},

variation of that line by $t_v e_2$ gives

$$z|_{G_v} = 0, \qquad x|_{G_v} = t_v \chi_\delta|_{G_v}.$$

If they differ by an unramified quadratic sign, put $e = v_\varpi(2)$. The same homogeneous equations hold for $n < e$; at $n = e$ the equations are instead

$$z|_{G_v} = 0, \qquad x|_{G_v} = t_v \chi_\delta|_{G_v} + uU_v, \qquad u = \overline{2/\varpi^e} \neq 0,$$

where U_v is the nontrivial unramified quadratic character. Indeed changing the sign of the special eigencharacter adds this scalar diagonal term.

In either case z is a supported quadratic class, has even dyadic valuation, and is split at both seven-places. The calculation of V gives $z = 0$. The remaining coordinates are now additive; their vanishing at two gives $x = A\chi_\delta$, $y = B\chi_\delta$. The fixed real matrix gives $X(c) = 0$, so $A = B = 0$. Thus every homogeneous first difference vanishes. The inhomogeneous equation is impossible because χ_δ is ramified at each seven-place while U_v is nontrivial unramified.

Choose a lift σ of the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$ in the decomposition group at two. The actual unramified ordinary quotient of r^σ is exactly ν: the ramified automorphism of $K_2/\mathbf{Q}_2$ acts trivially on its unramified Galois quotient. Normalize it with the same real matrix and marked ordinary line. The normalized residue is again exactly $1 + \chi_\delta E_{12}$, since δ is rational. Its two special characters are those of r with the places interchanged. If they differ, their difference is the unramified quadratic sign at both places. Homogeneous comparison for $n < e$ gives congruence modulo ϖ^e; the inhomogeneous comparison at $n = e$ is then the contradiction just proved. Hence the special characters agree. Homogeneous comparison for every $n \geq 1$ and completeness give $r^\sigma \simeq r$.

Lemma 12.4, with $\eta = 1$, proves modularity of the actual system. □

33.1.2 Nonordinary scalar sources

Put $K = \mathbf{Q}(\sqrt{2})$, $s = \sqrt{2}$, $\epsilon = 1 + s$, and $\pi = 3 + s$, $\pi' = 3 - s$. A special condition includes its exact quadratic twist, including the unramified character.

Proposition 33.2. *Let $\mathcal{R}$ be a system over K of determinant cyclotomic and labelled weights $\{0, 1\}$. Suppose it is good outside seven and special at both seven-primes. If every dyadic coefficient member is nonordinary with scalar residual semisimplification, its global trace field is $D = \mathbf{Q}(a_2)$ of degree $d \leq 3$, and its dyadic trace order is $R_0 = \mathbf{Z}_2[a_2]$ with $R_0/2R_0 \simeq \mathbf{F}_2[t]/(t^d)$, where $t = \overline{a_2}$.*

Proof. Let D be the global trace field, $d = [D : \mathbf{Q}]$, and apply the common simultaneous integral trace-order construction to all dyadic coefficient embeddings. Write its coefficient order as R_0 and its regular order as E_0, with trace T and norm $\mathcal{D}$. Thus $\mathrm{rank}_{\mathbf{Z}_2} R_0 = d$, and the regular lattice is an actual nonordinary Barsotti–Tate lattice. Write $E = E_0/2E_0$, $R = R_0/2R_0$, and $J = \mathrm{rad}\, E$. Lemma 11.2 bounds the dyadic conductors of the quadratic characters in $(J/J^2)^\vee$ by two.

Corollary 11.4 gives the three-dimensional space of supported quadratic characters of dyadic conductor at most two, with basis $\alpha = [-1]$, $\beta = [5 + 4s] = [\epsilon\pi]$, $\delta = [-7]$. Their relative discriminant norms are $4, 7, 49$, respectively. At two, α has break one, β is the unramified quadratic class, and δ is trivial. The latter follows already from $-7 \equiv 1 \pmod 8$ in $\mathbf{Q}_2$.

Choose the seven-place where $s \equiv 3 \pmod 7$, a real conjugation at the embedding $s > 0$, an arithmetic Frobenius at this seven-place, and a tame inertia generator there. Frobenius can be chosen to be trivial on the ramified quadratic class [7]. Their evaluation vectors on (α, β, δ) are $(1, 0, 1)$, $(1, 1, 1)$, $(0, 0, 1)$. Let A, B, C be their differences from one in E. These

evaluation vectors are independent, so A, B, C span J/J^2 and generate E. At the fixed special place the normalized characteristic-zero local differences $L(g)$ obey $L(g)L(h) = (\chi_2(g) - 1)L(h)$. Their exact quadratic sign disappears modulo two, yielding zero products and trace zero for the corresponding local differences in E. Polarized Cayley–Hamilton therefore gives

$$A^2 = B^2 = C^2 = BC = CB = 0, \quad AB + BA = X = T(AB), \quad AC + CA = Y = T(AC).$$

Every word is in the R-span of $1, A, B, C, AB, AC$.

We calculate in $R' = R/\mathfrak{m}_R^2$, retaining the original trace and norm laws. The traces of the six normal words show that its cotangent space is generated by X, Y. The regular Kisin module is free of rank one on the right over $E_0[[u]]$; its Frobenius generator $\Phi(u)$ has norm divisible by $u^2 - 2$ and trace at $u = 0$ equal to the arithmetic coefficient a_2. The ramified first-layer functional is $(1, 0, 0)$ on (α, β, δ), so

$$\Phi = N + u(1 + M) + O(u^2), \quad N = A + C + Z, \quad M = m_A A + m_B B + m_C C + Z', \quad Z, Z' \in J^2.$$

The unramified functional takes value one on β, hence $m_B = 1$. The norm of Z, its polar term with $A + C$, and all trace terms of length at least three vanish in R' by the six-word relations. The constant coefficient of the norm gives $0 = \mathcal{D}(N) = \mathcal{D}(A + C) = Y$. The linear coefficient of the norm gives

$$a_2 = T(N) = T(NM) = m_B X + (m_A + m_C)Y = X \quad \text{in } R'.$$

Nakayama proves

$$R = \mathbf{F}_2[t]/(t^d), \qquad t = \overline{a_2}, \qquad D = \mathbf{Q}(a_2).$$

If $d \geq 2$, X is another generator of its maximal ideal and $Y = vX$ with $v \in \mathfrak{m}_R$. The degree assertion follows because $1, a_2, \ldots, a_2^{d-1}$ span R_0 by Nakayama.

Suppose $d \geq 4$. Put $Z_0 = C + vB$. It is central and satisfies $Z_0^2 = BZ_0 = XZ_0 = 0$ and $T(Z_0) = T(AZ_0) = 0$. Over $S = \mathbf{F}_2[X]/(X^3)$ use the abstract free rank-four algebra

$$Q = S\langle a, b\rangle/(a^2, b^2, ba + ab - X).$$

There is a surjection $E \to Q$ taking A, B, Z_0 to $a, b, 0$. To verify that this map is well-defined, pair any relation in the normal words $1, A, B, AB, Z_0, AZ_0$ with $1, A, B, AB$ by the original trace. For its first four coefficients f_i the pairings are $Xf_3, Xf_2, Xf_1, Xf_0 + X^2 f_3$. Their vanishing puts every f_i in (X^{d-1}), which vanishes after base change to S. The normal-word multiplication then proves the assertion.

Compose this quotient with $a \mapsto E_{12}$ and $b \mapsto XE_{21}$ to obtain $\theta : G_K \to \mathrm{GL}_2(S)$. Its trace is the original trace reduced to S, and comparison of the two Cayley–Hamilton laws gives determinant one. The two visible linear characters are $\alpha + \beta$ and β: their values on A, B, C are $(1, 0, 0)$ and $(0, 1, 0)$, since v is in the radical. Put $F = K(i, \sqrt{5 + 4s})$. On G_F every matrix has the form $I + XM + X^2 N$, where the lower-left entry of M is zero and its diagonal entries agree. These M's lie in a commutative algebra, so $\theta(G_F)$ is abelian of exponent at most four. If r is the product of the chosen real and seven-Frobenius elements, $\operatorname{tr}\theta(r) = X$, whence $\theta(r)^4 = I + X^2 I \neq I$. Since $r^2 \in G_F$, the abelian image over F has an element of order four.

This contradicts a small ray calculation. The field F has degree eight, class number one, absolute discriminant 3211264, one dyadic prime $\mathfrak{P}$ with absolute $(e, f) = (4, 2)$, and three seven-primes with $(e, f) = (2, 2), (1, 2), (1, 2)$. The only ramified break of F/K at two is one. The original regular point field has upper break at most three over K_2, so Herbrand's subgroup formula bounds the break over F by five. At seven the special identities already show that the whole local image in E has exponent two. Hence $\theta|_{G_F}$ is a quotient of the ray group H of

modulus $\mathfrak{P}^6 \prod_{w|7} w$, with $4H$ and $2I_7$ killed, where I_7 is generated by the three local tame unit images.

A defining polynomial, and the exact ray quotient, are

$$z^8 - 4z^7 + 8z^6 - 8z^5 + 7z^4 - 8z^3 + 8z^2 - 4z + 1,$$
$$H \simeq C_{48} \times C_8 \times C_2^3, \qquad H/(4H + 2I_7) \simeq C_2^5.$$

Equivalently, let H_0 be the ray group for $\mathfrak{P}^6$ alone. The canonical map $H \to H_0$ has kernel I_7, and $H_0 \simeq C_2^2$. Computing the Smith form of $4H + 2\ker(H \to H_0)$ gives the displayed quotient; this is the coordinate-free calculation in `verify_sp7_ray_quotient_independent.gp`. The quotient has exponent two, contradicting the order-four image just constructed. Hence $d \leq 3$. The exact calculation is supplied in [52, `ramified/d2`]. □

Lemma 33.3. *A good/special-seven system as above is modular if its trace field has degree at most three and is generated by a_2, with $\mathbf{Z}_2[a_2]/2 = \mathbf{F}_2[t]/(t^d)$ when $d = 3$.*

Proof. Lemma 11.16 gives a coefficient prime over three with residue field $\mathbf{F}_3$ or $\mathbf{F}_9$, or a coefficient prime over five with residue field $\mathbf{F}_5$. For an irreducible residue, finite-field Brauer descent realizes the representation over this trace residue field.

At three a reducible residue has constituents $1, \omega_3$ and the actual source is ordinary and modular by Lemma 15.2. An irreducible residue remains cyclotomically irreducible by that lemma. Its small-field witness [3, Theorem 2.5], followed by [3, Theorem 2.3], modularizes the actual weight-two source.

At the selected prime over five the residue field is $\mathbf{F}_5$. In the reducible case the square of a constituent is unramified away from five, since every seven-type is good or quadratic-special. Lemma 11.14 modularizes the actual source. In the irreducible case Lemma 11.15, the small-field witness and the actual weight-two AKT theorem apply. □

Theorem 33.4. *Assume GRH. Every pure totally odd absolutely irreducible weight-two cyclotomic compatible system over K, good outside seven, with good or quadratic-special type at both seven-primes, is modular.*

Proof. First remove any ramified quadratic special scalar factors using Lemma 15.1; the corrections are unramified at two and three and at the other seven-prime. Hence all actual special scalar characters may be taken unramified. If some dyadic coefficient residue is nonsolvable, apply Proposition 21.2, which covers both ordinary and nonordinary sources under the present GRH hypothesis. The irreducible solvable alternative is excluded by Lemma 15.5; hence we may assume every dyadic coefficient residue has scalar semisimplification. If some dyadic member is ordinary, the ordinary proposition or its one-good-place corollary proves the assertion. Otherwise all are nonordinary. With exactly one special place, Proposition 15.4 applies. With two special places, Proposition 33.2 and Lemma 33.3 apply.

If both seven-places are good, the first-layer quadratic space of the regular nonordinary trace algebra is just $\langle[-1]\rangle$. Indeed its characters have dyadic conductor at most two by the middle-model calculation, and the narrow ray group of modulus $\mathfrak{p}_2^2\infty_1\infty_2$ has order two. The real difference $A = c - 1$ therefore generates the algebra $E = E_0/2E_0$ and satisfies $A^2 = 0$. The original trace of both 1 and A is zero modulo two. All group traces consequently vanish, so the coefficient algebra $R = R_0/2R_0$ is $\mathbf{F}_2$. This makes E spanned by $1, A$, contradicting $\dim_{\mathbf{F}_2} E = 4[D : \mathbf{Q}] \geq 4$. This treats the last case. Undo the finite quadratic twists. □

An unconditional observation. The separate nonordinary exclusion remains available without GRH. If a nonordinary dyadic member in the preceding theorem had nonsolvable residue, its kernel field M would have degree at least 120. Theorem 11.10 gives $\mathrm{rd}(M) < 4\sqrt{14} < 14.967$, whereas (12) with $b = 7$ gives $\mathrm{rd}(M) > 16.661$. This observation is not needed in the streamlined GRH proof above.

33.1.3 The supported-seven problem over $\mathbf{Q}(\sqrt{2})$

Put $K = \mathbf{Q}(s)$ with $s^2 = 2$, $\epsilon = 1 + s$ and $\mathfrak{l}_\pm = (3 \pm s)$. The prime three is inert and $K_3/\mathbf{Q}_3$ is unramified quadratic.

Proposition 33.5. *Assume GRH. Every irreducible determinant-ω_7 residue supported at seven is modular.*

Proof. A solvable image has a modular Artin witness. For a nonsolvable image the cyclotomic restriction is irreducible. Use Lemma 15.3, prescribed globalization, actual avoiding potential modularity and Brauer descent to construct a weight-two compatible system with the stated bounded menu, good outside seven. Normalize the independent principal-series signs as above.

Reduce at three. A reducible residue already modularizes the actual source by Lemma 15.2. Otherwise that lemma gives cyclotomic irreducibility. There is no exceptional-five exclusion in characteristic three. Prescribe the actual good/special lifts of Lemma 4.9, retaining the old Barsotti–Tate component at three, and globalize by EGPZ. Actual potential modularity and compatibility give the new system. Purity ensures nonzero special monodromy wherever the special component was selected, since its two Frobenius eigenvalues have ratio seven. By Theorem 33.4 this new system is modular. AKT applies backwards to the old weight-two three-adic source. Undoing the quadratic twists and reducing at seven proves the assertion. □

33.2 Exceptional lifts and sources over $\mathbf{Q}(\sqrt{5})$

Use the field notation of Section 16.1. The independent quadratic corrections at eleven are cut out by $\alpha = 1 + 4\phi$ and $\alpha' = 5 - 4\phi$. Their product is -11. The field $H = K(\sqrt{-11})$ is split at two, since K_2 contains $\sqrt{5}$ and $-11/5$ is a square in $\mathbf{Q}_2$. This section treats the larger support at eleven and an alternative terminal prime. The one-place theorem of Section 16 already suffices for the full field result.

33.2.1 A one-prime source obstruction

Put $R = \mathcal{O}_K[1/\pi]$ and $F = K_2$.

Lemma 33.6. *We have* $\mathrm{Ext}^1_R(\mu_2, \mathbf{Z}/2\mathbf{Z}) = 0$.

Proof. Over the completion at two, an extension of μ_2 by $\mathbf{Z}/2\mathbf{Z}$ splits by the connected–etale sequence. Its middle term is therefore killed by two locally, hence globally, by flatness. Its generic extension class lies in $H^1(K, \mathbf{F}_2) = K^\times/K^{\times 2}$, is unramified away from the inverted primes and two, and is a square in F.

For R, the global Kummer space has basis $-1, \phi, 2, \pi$. Its localization in $F^\times/F^{\times 2}$ is injective. An odd power of 2 is excluded by valuation. Taking the norm to $\mathbf{Q}_2$ then excludes nonzero exponents of ϕ and π: the three possible nontrivial norm classes are $-1, 11, -11$, all nonsquare in $\mathbf{Q}_2$. Finally -1 is nonsquare in F. Thus the generic extension is split.

A generically split extension is split over R. Of its two generic splittings, one agrees with the local splitting at two; away from two both group schemes are etale and any generic splitting

extends. These splittings glue. Equivalently this is the finite-flat gluing sequence of [59, Proposition 2.3 and Corollary 2.4].

□

Lemma 33.7. *Let G be a 2-divisible group over R. Suppose that every simple finite flat subquotient of every $G[2^n]$ has trivial generic Galois action. Then G has a multiplicative 2-divisible subgroup with etale quotient.*

Proof. A simple generic factor has order two. Schematic closure realizes a generic composition series as a finite flat composition series. At the unique dyadic prime, the prime-order classification over an unramified ring gives either the etale or the multiplicative model. Away from two the model is etale and its generic action is trivial. Since the generic group of order two has no nontrivial automorphism, gluing gives exactly $\mathbf{Z}/2\mathbf{Z}$ or μ_2 over R.

By Lemma 33.6, adjacent factors in the order $\mathbf{Z}/2\mathbf{Z}$ below μ_2 can be interchanged: the relevant extension splits. Repeating this operation orders the filtration with all multiplicative factors below all etale factors. Extensions of multiplicative groups are multiplicative, by Cartier duality, and extensions of etale groups are etale. Thus each $G[2^n]$ has a multiplicative subgroup with etale quotient.

This subgroup is unique. Its localization at two is the connected part, since its quotient is etale; its global subgroup is the flat closure of that generic subgroup. Consequently it is functorial in $G[2^n]$ and compatible with the transition and multiplication maps. The local connected part is multiplicative at every level, so these subgroups form the multiplicative 2-divisible part, with etale quotient. This is also the filtration argument used in finite-flat approaches to small-conductor abelian varieties [59, 60]. □

Theorem 33.8. *Let $E_0/\mathbf{Q}_2$ be finite and let $r : G_K \to \mathrm{GL}_2(E_0)$ be continuous. Suppose that $\det r = \chi_2$, that r is Barsotti–Tate at two, unramified outside $2\mathfrak{p}$, and semistable at $\mathfrak{p}$ with unipotent inertia. Then r fits into an exact sequence*

$$0 \longrightarrow E_0(1) \longrightarrow r \longrightarrow E_0 \longrightarrow 0.$$

In particular there is no absolutely irreducible such representation. The same assertion holds with $\mathfrak{p}'$ in place of $\mathfrak{p}$. No GRH assumption is needed.

Proof of Theorem 33.8. Choose a global stable $\mathcal{O}_{E_0}$-lattice. The local Barsotti–Tate condition realizes such a lattice after an isogeny: taking the closure of a finite generic subgroup realizes any commensurable stable lattice. The integral coefficient action extends by full faithfulness for Tate modules of 2-divisible groups. Glue this local group to the etale group defined by the lattice away from two. Levelwise finite-flat gluing [59, Proposition 2.3] gives a 2-divisible group over R with $\mathcal{O}_{E_0}$ action.

The reduction modulo the maximal ideal of $\mathcal{O}_{E_0}$ satisfies the one-prime hypotheses of Proposition 16.2. At eleven, unipotent inertia factors through the procyclic tame pro-two quotient. Its reduction is cyclic of exponent two, hence has order at most two. Its semisimplification is trivial. Filtering by the coefficient uniformizer and then over $\mathbf{F}_2$ shows that every simple generic factor of every $G[2^n]$ is trivial. Lemma 33.7 supplies a functorial multiplicative subgroup.

The height of G is $2[E_0 : \mathbf{Q}_2]$, and its dimension is $[E_0 : \mathbf{Q}_2]$, because its labelled Hodge–Tate pairs are $\{0, 1\}$. The multiplicative part has height $[E_0 : \mathbf{Q}_2]$. Functoriality makes its rational Tate module E_0-stable, hence an E_0-line. The quotient character is unramified at two, and also at eleven because inertia in r is unipotent. It is unramified at every other finite prime by hypothesis. Class field theory and narrow class number one make this character trivial. The determinant identifies the subcharacter with χ_2. □

33.2.2 Two cuspidal places above eleven

We use one finite-flat ordinarity observation repeatedly. A nonordinary rank-two Barsotti–Tate source of cyclotomic determinant is biconnected up to isogeny. Its coefficient-uniformizer torsion and finite-flat subquotients are then biconnected. Over an unramified dyadic ring, a prime-order model with trivial generic action is $\mathbf{Z}/2\mathbf{Z}$ or μ_2. For odd p and absolute ramification index $e < p-1$, Raynaud's uniqueness theorem similarly identifies the models of 1 and ω_p with the etale and multiplicative models [57, Theorem 3.3.3 and Corollary 3.3.6]. Neither model is biconnected. Their occurrence as simple factors therefore forces the actual source to be ordinary.

Proposition 33.9. *Let $K = \mathbf{Q}(\sqrt{5})$ and let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, weight-two strongly compatible system of determinant cyclotomic, good outside eleven. Suppose its two eleven-adic types are determinant-one tame cuspidal types with inertial eigencharacters of order three or six. If a dyadic residue has solvable image, the actual system is modular.*

Proof. Modulo two the scalar quadratic factors disappear. At either eleven-adic place tame inertia has eigencharacters of order three, and Frobenius interchanges them because $11 \equiv -1 \pmod 3$. Thus the full local residue is S_3, and the global residue is absolutely irreducible. A solvable irreducible determinant-one subgroup of $\mathrm{GL}_2(\overline{\mathbf{F}}_2)$ is odd dihedral. Let L/K be its inducing quadratic extension.

The quadratic quotient is unramified at eleven: its local inertia is the odd rotation subgroup. It is unramified outside two, and Fontaine's finite-flat upper bound one over the unramified dyadic completion gives dyadic conductor at most two. Since K has class number one and fundamental unit $\phi = (1+\sqrt{5})/2$, its supported quadratic squareclasses are generated by $-1, \phi, 2$. An odd dyadic valuation gives conductor three and is excluded. The possibilities are consequently $-1, \phi, -\phi$.

At the two eleven-primes, the values of ϕ are $8, 4$. The first is nonsquare and the second square in $\mathbf{F}_{11}$; for $-\phi$ they are $3, 7$, respectively square and nonsquare. Both cuspidal places require the quadratic quotient to be nontrivial unramified. Only -1 is nonsquare at both. Hence $L = K(i)$.

At two this quadratic extension is ramified. Its nontrivial wild inertia image is a reflection of the odd dihedral group. The normalizer of a reflection subgroup is that subgroup itself. Since wild inertia is normal in the decomposition group, the whole dyadic decomposition image is C_2. In particular the local residual semisimplification is trivial.

Filtering the coefficient-uniformizer torsion over $\mathbf{F}_2$ gives a simple finite-flat factor of order two with trivial generic action. The preceding observation makes the actual dyadic source ordinary.

The main theorem of [2, Theorem 5.2.1] applies: the source is ordinary and totally odd, the residue is absolutely irreducible and solvable, and its CM inducing quadratic extension does not split at the dyadic place. Equivalently combine his Lemma 5.1.2 with Theorem 5.2.1. This proves modularity of the source and hence of the system. □

Lemma 33.10. *Let $\mathcal{R}$ be a determinant-cyclotomic weight-two system over K, good outside eleven, with local types at eleven special quadratic or tame cuspidal of order three or six. If an associated three-adic residue is globally reducible, the actual three-adic source is modular. If it is globally irreducible, its restriction to $G_{K(\zeta_3)}$ remains absolutely irreducible.*

Proof. The quadratic characters ramified at just one eleven-prime, constructed from $\alpha = 1 + 4\phi$ and its conjugate, are unramified at three. Twist away the scalar quadratic inertial signs at both eleven-primes. Modulo three the remaining inertia is unipotent, for either local type. This twist preserves the determinant, the weights and crystallinity at three.

If the residue is reducible, its characters are unramified at eleven. They factor through the narrow ray modulo three, whose order is two: the local/sign group has order $8 \cdot 4 = 32$, whereas the images of ϕ and -1 generate order sixteen. Indeed ϕ has order eight modulo three, signs $(+,-)$, and -1 is independent because its first sign is negative. The unique nontrivial character is ω_3. The constituents are therefore $1, \omega_3$.

At three the absolute ramification index is $1 < 3 - 1$. The finite-flat observation makes the actual Barsotti–Tate source ordinary. Skinner–Wiles [66, §4.5, Theorem A] now applies. The quotient splitting field $K(\zeta_3)$ is abelian over $\mathbf{Q}$, its quotient character is totally odd and locally nontrivial at three, the ordinary quotient is unramified with residual character one, and the determinant is χ_3.

If the cyclotomic restriction of an irreducible residue were reducible, the residue would be induced from a prime-to-three character θ of $H = K(\zeta_3)$. The unipotent inertia at eleven has order dividing three, while the inducing characters have prime-to-three order; it is therefore trivial on G_H. Thus θ is unramified outside three and tame at three. The absolute discriminant of H is $5(-3)(-15) = 225$. Its degree-four Minkowski bound is $22.5/\pi^2 < 2.28$. It has no norm-two prime, since two has residue degree two. Consequently its class number is one. There is one prime above three, of residue field $\mathbf{F}_9$, whose full unit group is generated by the image of ϕ. The ray group for this prime is therefore trivial. This makes θ trivial, contrary to irreducibility of its induction. □

Lemma 33.11. *Let $r : G_K \to \mathrm{GL}_2(E)$ be totally odd and absolutely irreducible, of determinant χ_5, Barsotti–Tate above five, and unramified outside five and eleven. Suppose its residue is globally reducible with semisimple inertia trivial at both eleven-primes. Then the actual source r is modular.*

Proof. The narrow ray modulo the prime above five is C_2, so the residual constituents are $1, \omega_5$. The source is ordinary by the finite-flat argument at absolute ramification index $2 < 5 - 1$. Apply Skinner–Wiles [66, §4.5, Theorem A]. Its quotient field $K(\omega_5) = \mathbf{Q}(\zeta_5)$ is abelian over $\mathbf{Q}$; the quotient is totally odd and locally nontrivial at five. In the ordinary filtration the quotient has weight zero, is unramified, and reduces to one, so its normalized character factors through a pro-five group and has finite inertial order. The determinant gives $k = 2$. The theorem imposes no requirement that five be unramified in K. □

Theorem 33.12. *Assume GRH for Dedekind zeta functions. Let $K = \mathbf{Q}(\sqrt{5})$, and let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible strongly compatible rank-two system of determinant cyclotomic and labelled weights $\{0, 1\}$. Suppose it is good outside eleven and at both eleven-primes has a tame determinant-one cuspidal type with inertial eigencharacters of order three or six. Then the actual system $\mathcal{R}$ is modular.*

Proof. Choose a coefficient place above two and let M/K be the kernel field of its residual representation. At each eleven-prime, the quadratic scalar sign disappears modulo two and tame inertia has order three. Frobenius interchanges the two eigenlines since $11 \equiv -1 \pmod 3$. Its square is one modulo two, from determinant eleven and characteristic-zero trace zero. The full local residue is consequently S_3, so the global residue is absolutely irreducible.

If its image is solvable, Proposition 33.9 proves modularity. We may therefore suppose the image is $G = \mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^f \geq 4$.

We construct a solvable auxiliary extension. Put $H = K(i)$. Its absolute discriminant is $5(-4)(-20) = 400$, and its Minkowski bound is $30/\pi^2 < 3.04$. It has no prime of norm two or three: their residue degrees are two. Thus its class number is one. Its units are generated by i and $\phi = (1 + \sqrt{5})/2$. Indeed the norm to K of a unit is a totally positive unit ϕ^{2m}; division

by ϕ^m gives a unit all of whose complex absolute values equal one, hence a root of unity. The roots of unity in H are precisely μ_4.

The two eleven-primes of K are inert in H/K, with residue fields $\mathbf{F}_{121}$. The elementary three quotient of the ray group of H for their product is C_3^2. To see this, each residue multiplicative group has order 120 and contributes one C_3. The images of ϕ lie in $\mathbf{F}_{11}^{\times}$, and i has order four; neither unit has a nontrivial three projection. Class number one gives the assertion.

Let B/H correspond to this maximal exponent-three ray quotient. Complex conjugation over K acts by the eleventh-power map on both residue fields, hence by inversion on C_3^2. Thus B/K is Galois with group $C_3^2 \rtimes C_2$ and $[B : \mathbf{Q}] = 36$. At two, B/H is unramified while H/K is ramified quadratic. The inertia group in B/K is therefore a reflection C_2. Its normalizer in $C_3^2 \rtimes C_2$ is itself, so the full dyadic decomposition group is C_2. The local field is $K_2(i)$ and has upper break one.

At either eleven-prime the ray inertia is C_3. Frobenius maps nontrivially to $\mathrm{Gal}(H/K)$, and every element outside C_3^2 has order two. Since the decomposition quotient by inertia is cyclic and generated by Frobenius, its order is two. Both local fields are therefore the tame S_3 extension of $\mathbf{Q}_{11}$ with $(e, f) = (3, 2)$. This extension is unique: the unramified quadratic extension contains μ_3, and the unit classes defining Galois-stable tame cubic extensions give the unique extension because $3 \nmid 10$.

Since G is nonabelian simple, $M \cap B = K$. Write $L = BM$. Its degree is $36q(q^2 - 1)$. Both local fields above eleven remain the same tame S_3 field, giving

$$\frac{2N_{121}(L)}{[L : \mathbf{Q}]} = \frac{1}{3}, \qquad d_{11} = \frac{2}{3}.$$

At two the residual Barsotti–Tate point field has no upper break exceeding one by Fontaine's bound over the unramified base. The same is true of B, so it is true of their compositum. In particular $d_2 < 2$ and $\mathrm{rd}(L) < 4\sqrt{5}\,11^{2/3}$. For $q = 16$ we use a slightly sharper bound. If wild inertia in M is nontrivial, it fixes a unique line. The diagonal characters of the decomposition group are then characters of G_{K_2}; their odd-order inertial part factors through $\mathbf{F}_4^{\times}$. The tame inertia has order at most three and the wild inertia has order at most q. After adjoining B, write w for the wild inertia order and e for the inertia order. We have $w \le 2q$ and $e \le 3w$. The upper-filtration formula, using the break bound one, gives $d_2 \le (1 - 1/e) + (1 - 1/w) \le 2 - \frac{2}{3q}$. If wild inertia in M is trivial, the compositum has wild inertia of order at most two, so the same formula gives $d_2 < 3/2$, which is smaller. This case distinction matters: a tame irreducible residue over K_2 may have inertia of order five, rather than three. Consequently $\mathrm{rd}(BM) \le \sqrt{5}\,2^{2-2/(3q)}11^{2/3}$. Apply (11) with $B_{b,100}$ and the norm-121 prime density $1/3$, retaining all prime powers in the support. Directed 256-bit Arb evaluation gives

case	$n \ge$	b	lower root discriminant	upper root discriminant
$q = 16$	146880	11.6	43.50979406	42.97974824
$q \ge 32$	1178496	13.6	45.77092571	44.23915099
$q = 8,\ M \ne M^\sigma$	9144576	16.4	47.10533308	44.23915099

Here σ denotes the nontrivial automorphism of $K/\mathbf{Q}$. The first two lines exclude every $q \ge 16$. At fixed b and prime density the lower expression increases with the degree, so the second line covers the entire unbounded range. The exact calculation is supplied in [52, `d5`].

For the third line use BMM^σ. If $M \ne M^\sigma$, simplicity makes the two fields disjoint over K, and their product has no solvable quotient. It is also disjoint from B, so its degree is $36 \cdot 504^2$. The compositum retains upper dyadic break at most one and the same local S_3 fields at eleven. This proves the third line's applicability and excludes that case.

For the original kernel field M, Lemma 11.9 gives $\mathrm{rd}(M) \le 4\sqrt{5}\,11^{2/3}2^{-2/q}$. Since $4\sqrt{5}\,11^{2/3} < 4\sqrt{220}$, Corollary 13.5(i) excludes a totally real kernel without GRH. If $q = 8$, the preceding

comparison gives $M = M^\sigma$, and Lemma 2.5 proves residual modularity. Lemma 2.4 now proves modularity of the system for both $q = 4$ and $q = 8$. □

33.2.3 The supported residual base at eleven

Fixed-determinant positivity. We use the following observation for both eleven-adic menus. Let $\bar{r} : G_{\mathbf{Q}_{11}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_{11})$ have determinant ω_{11}. By [9, Propositions 2.6–2.7] it has a regular crystalline lift of determinant χ_{11}. Write its weights as $h_1 < h_2$, with $h_1 + h_2 = 1$. The corresponding algebraic compact representation is $\mathrm{Sym}^{h_2-h_1-1} \otimes \det^{h_1}$. Its scalar central character has exponent $h_1 + h_2 - 1 = 0$. The nonzero crystalline deformation ring and the fixed-determinant Breuil–Mézard formula [74, Theorem 1.2] therefore give a nonzero effective cycle for at least one Serre weight with trivial scalar central character. These cycles are independent of the Hodge type and inertial type for this fixed determinant. Consequently, in any collection of weight-two compact types containing all such Serre weights, at least one type has a nonzero lifting ring. That ring is eleven-torsion-free and has a characteristic-zero point. This argument applies to the actual residual extension.

Proposition 33.13. *Assume* ZF(4809/5000). *Let $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ have determinant $\bar{\chi}_5$, be finite flat at the prime above five, and be unramified outside five and eleven. Suppose that its inertia image at each eleven-adic place has order dividing five. It cannot have projective image S_5 with the quadratic quotient S_5/A_5 cutting out $K(\zeta_5)/K$.*

Proof. At the five-adic place, the absolute ramification index of K is two. Raynaud's finite-flat tame weights lie in $\{0, 1, 2\}$ [57, Theorem 3.4.3 and Corollaries 3.3.6, 3.4.4]. In the reducible local case their sum is two, so the projective tame ratio has order at most two. In the irreducible tame case the niveau-two exponent is $a + 5b$, with $a + b = 2$ and $0 \le a, b \le 2$; its projective inertia order is three.

If projective wild inertia is nontrivial, it is a C_5 and its decomposition group lies in its normalizer $C_5 \rtimes C_4$ in S_5. The tame quotient has order at most two. The subgroup $C_5 \rtimes C_2$ lies in A_5, since inversion of a five-cycle is a double transposition. The quadratic quotient would therefore be unramified at five. This contradicts $K(\zeta_5)/K$, which is ramified quadratic there.

Thus wild inertia is trivial. Since the quadratic quotient is ramified, projective inertia has order two. The absolute ramification index at five in the projective kernel field $N/\mathbf{Q}$ is four, tamely ramified, so its normalized different contribution is $3/4$. At eleven it is at most $4/5$. The degree is $[N : \mathbf{Q}] = 240$, and $\mathrm{rd}(N) \le 5^{3/4}11^{4/5} < 22.76889$. At a ramified eleven-place, inertia is a five-cycle t. Frobenius centralizes it, since $t^{11} = t$, and its centralizer in S_5 is $\langle t \rangle$. Thus the relative decomposition group is C_5 and the residue degree is one. If at most one of the two eleven-places ramifies, the improved upper bound is $5^{3/4}11^{2/5} < 8.72539$; the degree-120 classical bound in Lemma 13.4 already contradicts it.

If both ramify, then $N_{11} = 48$ and $2N_{11}/240 = 2/5$. Use Proposition 13.2 with $a = 2309/5000$, $b = 8$, $n = 240$, and the first two norm-eleven powers. In Lemma 13.7 take $B_0 = 63923/100000$, $M = 100$, $r = 40$ and $s = 100$. The finite bounds are $B(G_{a,8}) < 0.71499188$, $C(G_{a,8}) < 30.19072206$, and the retained prime term is greater than 0.16802233. Directed evaluation before rounding gives $\mathrm{rd}(N) > 22.84250 > 22.76888 > 5^{3/4}11^{4/5}$, a contradiction. The certificate is [83, `analytic/all_q_discriminants.py`]. GRH implies the zero-free hypothesis; Theorem B.4 supplies it under the stated CT analytic inputs as well. □

Proposition 33.14. *Assume GRH for Dedekind zeta functions. Every pure, totally odd, absolutely irreducible strongly compatible weight-two system over K of cyclotomic determinant, good outside eleven and special with unipotent inertia at its two eleven-primes, is modular.*

Proof. Take a coefficient place above three. If its residue is reducible, Lemma 33.10 proves modularity of the actual source. Otherwise the same lemma gives absolute irreducibility after restriction to $K(\zeta_3)$.

At each eleven-prime apply Lemma 4.15 to the actual residual representation of the old special source. It gives a tame determinant-one cuspidal lift of order three, with determinant χ_3, matching the full residual Frobenius extension and either nontrivial inertia orientation.

Prescribe these tame order-three cuspidal types at both eleven-primes, retain the old crystalline weight-two component at three, and retain unramified conditions elsewhere. EGPZ [31, Theorem 4.2.1] gives a global point. Lemma 2.6 gives actual potential modularity and the pure strongly compatible replacement. It is modular by Theorem 33.12. AKT [3, Theorem 2.3], applied to the old weight-two three-adic member, gives the backward implication. □

Theorem 33.15. *Assume GRH for Dedekind zeta functions. Every continuous absolutely irreducible totally odd representation $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_{11})$ of determinant ω_{11}, unramified outside eleven, is modular.*

Proof. A solvable image has an odd solvable Artin witness and hence is modular after regular weight raising. Otherwise the cyclotomic restriction is absolutely irreducible. Use Lemma 16.5 at both split eleven-places, and unramified conditions elsewhere. EGPZ gives a weight-two global lift of exact cyclotomic determinant; there is no exceptional case in characteristic eleven. Lemma 2.6 constructs its pure strongly compatible system.

Make the independent global quadratic corrections described above. The resulting system is good outside eleven and its two local conditions are good, special, or principal series of odd inertial order five. All twists are reversed at the end. Its five-adic member is Barsotti–Tate above five. At eleven its actual residual inertia is trivial or unipotent of order five: the fifth-order eigencharacters coalesce, while a special representation has procyclic tame pro-five unipotent inertia.

If the five-residue is globally reducible, Lemma 33.11 gives modularity of the actual member and hence of the entire system. Otherwise Lemma 16.4 proves cyclotomic irreducibility. The controlled exceptional exclusion Proposition 33.13 applies: the representation is finite flat at five, supported only at five and eleven, and has the stated eleven-inertia. Thus the residual five-representation is nonexceptional in the precise sense required by EGPZ.

At each eleven-prime choose the good or special local lift of Lemma 4.9. Retain a component through the old Barsotti–Tate point at five and unramified conditions elsewhere. EGPZ gives the weight-two replacement, and Lemma 2.6 gives its pure strongly compatible system. By the purity observation in Section 2, the replacement is special or good at both eleven-places, and good away from eleven.

If either eleven-place is good, Theorem 33.8 excludes this pure absolutely irreducible system. If both are special, Proposition 33.14 proves its modularity. In the latter case the common modular residual witness lifts the actual old weight-two five-adic source by [3, Theorem 2.3]. Compatibility, reversal of the quadratic twist and reduction at the original eleven-place prove the theorem. □

33.2.4 A terminal prime at seven

The improved zero-free strip also gives a residual endpoint suited to weight reduction. Here $K = \mathbf{Q}(\sqrt{5})$, and seven is inert.

Proposition 33.16. *Assume* ZF(4809/5000). *Let $\bar{r} : G_K \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ be continuous, finite flat at two, and unramified outside two and seven. If its inertia image at seven has order at most two, its image is solvable. Among nonsolvable images, those with $q \leq 16$, those unramified at*

seven, and those with relative decomposition group C_2 at seven are excluded without a zero-free hypothesis.

Proof. A nonsolvable image is conjugate to $\mathrm{SL}_2(\mathbf{F}_q)$ for a power of two $q \geq 4$, by Dickson's classification. Let L be its kernel field. Lemma 11.9 gives the relative exponent $2 - 2/q$ at two. If inertia at seven is trivial, then $\mathrm{rd}(L) \leq 4\sqrt{5} < 8.945$, whereas $[L : \mathbf{Q}] \geq 120$ and Lemma 13.4 gives $\mathrm{rd}(L) > 16.96717$.

Otherwise inertia is generated by a nontrivial unipotent involution u. Frobenius conjugates u to $u^{49} = u$. Its centralizer in $\mathrm{SL}_2(\mathbf{F}_q)$ is the elementary abelian translation group. Since the decomposition group is generated by inertia and one Frobenius lift, it is C_2 or C_2^2. Tame ramification at seven contributes $7^{1/2}$, and hence

$$[L : \mathbf{Q}] = 2q(q^2 - 1), \qquad \mathrm{rd}(L) \leq \sqrt{5}\, 2^{2-2/q}\sqrt{7} = U(q).$$

Theorem 13.8 gives the contradiction and its unconditional variants. No ordinarity assumption or choice of a new dyadic lattice enters this argument. □

Lemma 33.17. *Let $\mathcal{R}$ be a cyclotomic weight-two compatible system over $K = \mathbf{Q}(\sqrt{5})$, good outside seven. Suppose that its local type at seven is tame and its inertial eigencharacters have two-power order. Then the type is not genuinely supercuspidal, and the reduction of any stable lattice in a dyadic member has inertia of order at most two at seven. Under* ZF(4809/5000) *every such dyadic residual image is solvable.*

Proof. A genuine tame supercuspidal type over the unramified quadratic extension $K_7/\mathbf{Q}_7$ has characters θ, θ^{49} with $\theta^{49} \neq \theta$. Cyclotomic determinant gives $\theta^{50} = 1$ on inertia. A two-power order would therefore divide two, contrary to regularity.

For any stable dyadic lattice, the residual tame inertia is procyclic and both its eigenvalues are one. A generator is unipotent in dimension two and characteristic two, so its order is at most two. The same argument includes a special type with nonzero monodromy. The dyadic member is crystalline of weights $\{0, 1\}$, so its lattice reduction is finite flat; its determinant is one and it is unramified outside two and seven. Proposition 33.16 now applies. □

33.2.5 Reducible dyadic sources

Throughout this subsection put $K = \mathbf{Q}(\sqrt{5})$, $\phi = (1 + \sqrt{5})/2$ and $\delta = \chi_{-7}$. Write $\bar{\delta} : G_K \to \mathbf{F}_2$ for the associated additive quadratic character.

Lemma 33.18. *The quadratic classes supported at two and seven form $\langle [-1], [\phi], [2], [7] \rangle$. Put $V = \langle [-1], [\phi], [7] \rangle$. Then*

$$\ker(V \longrightarrow K_2^\times / K_2^{\times 2}) = \langle [-7] \rangle, \qquad \ker(V \longrightarrow K_7^\times / K_7^{\times 2}) = \langle [-1] \rangle.$$

Every reducible determinant-one characteristic-two representation unramified outside two and seven, with inertia of order at most two at seven, has trivial semisimplification.

Proof. Class number one and the unit group $\langle -1, \phi \rangle$ give the supported classes. Since -7 is a square in $\mathbf{Q}_2$, the first kernel contains $[-7]$. The norm of ϕ to $\mathbf{Q}_2$ is -1, which is nonsquare; hence a square cannot involve an odd power of ϕ. The remaining classes are rational. The kernel of $\mathbf{Q}_2^\times / \mathbf{Q}_2^{\times 2} \to K_2^\times / K_2^{\times 2}$ is $\langle [5] \rangle$, and among $1, -1, 7, -7$ only $1, -7$ are squares in K_2. This proves the first equality. At seven, odd valuation excludes $[7]$, whereas -1 is a square in $\mathbf{F}_{49}$. The relation $\phi^8 = -1$ in $\mathbf{F}_{49}$ gives $\phi^{24} = -1$, so ϕ is a nonsquare there. This proves the second equality.

The constituents of a reducible characteristic-two representation have odd order. They are unramified at seven and tame at two. The image of ϕ generates $\mathbf{F}_4^\times$, so the ray group modulo two is trivial. Thus both characters are trivial. □

Lemma 33.19. *Let $\mathcal{R}$ be a cyclotomic-determinant system good outside seven, tame at seven, whose finite inertial characters there have two-power order. Its type at seven is scalar or principal series, with characters θ, θ^{-1} of order at most sixteen. There is $\eta \in \{1, \delta\}$ such that $\mathcal{R}$ and $\mathcal{R}^\sigma \otimes \eta$ have the same inertial trace at seven, where σ is the nontrivial automorphism of $K/\mathbf{Q}$. One may take $\eta = 1$ unless θ has order sixteen.*

Proof. Wild inertia at seven is pro-seven and therefore acts trivially on the finite type. Its two eigencharacters have product one. If Frobenius interchanges two distinct characters, then $\theta^{49} = \theta^{-1}$, so $\theta^{50} = 1$. A character of two-power order would have order at most two, contradicting regularity. Thus a nonscalar type is principal series and $\theta^{48} = 1$. The local action of σ raises the inertial characters to the seventh power. For order at most eight this permutes θ, θ^{-1}. At order sixteen, $\theta^7 = \theta^8\theta^{-1}$ and $\theta^{-7} = \theta^8\theta$, while δ restricts to the unique nontrivial quadratic character of tame inertia. The assertion follows. Scalar quadratic types, with or without monodromy, have the same inertial trace after conjugation. □

Proposition 33.20. *Let $\mathcal{R}$ be a system over K, of cyclotomic determinant and labelled weights $\{0, 1\}$. Suppose it is good outside seven, tame at seven, and its finite inertial characters there have two-power order. If a crystalline dyadic member has reducible residue, then the actual system $\mathcal{R}$ is modular.*

Proof. The residual inertia at seven has order at most two, since tame inertia is procyclic and its two eigencharacters reduce to one. Lemma 33.18 makes the residual semisimplification trivial. Lemma 4.16, with $p = 2$ and the unramified quadratic field K_2, makes the actual dyadic member ordinary.

Write this member as $r : G_K \to \mathrm{GL}_2(E)$ and its exact unramified ordinary quotient at two as ν. Lemma 12.1, using Lemma 33.18, puts r in the marked basis with real matrix C, ordinary line e_2 and residue $1 + \bar{\delta}E_{12}$.

We compare two actual sources r_1, r_2 with these normalizations, the same exact quotient ν at two and the same inertial trace at seven. Enlarge the coefficient field so that both are defined over its integers $\mathcal{O}$, with uniformizer ϖ and residue field k. Suppose $r_1 \equiv r_2 \pmod{\varpi^n}$, $n \geq 1$, and write

$$r_2(g) = (1 + \varpi^n X(g))r_1(g) \pmod{\varpi^{n+1}}, \qquad X(g) = \begin{pmatrix} x(g) & y(g) \\ z(g) & x(g) \end{pmatrix}.$$

The determinant gives this trace-zero form. Conjugation by $1 + \bar{\delta}E_{12}$ adds $\bar{\delta}z(1 + E_{12})$, so z is additive. At two, the marked ordinary line and the exact quotient imply $x = y = 0$, and the local class of z is a unit Kummer class. Indeed the divided difference of the two integral ordinary extension cocycles remains integral and its rational class lies in the linear crystalline extension space for the identical characters $\nu, \chi_2\nu^{-1}$. It therefore defines another ordinary Barsotti–Tate extension. Its reduction is an extension of the etale model by the multiplicative model, so its Kummer class is a unit. This argument concerns actual sources and their crystalline extension classes, not an arbitrary tangent model.

At seven choose a tame generator T with $\bar{r}(T) = 1 + E_{12}$ and Frobenius Φ. Equality of the exact inertial traces gives $\mathrm{tr}(X(T)\bar{r}(T)) = z(T) = 0$. The relation $\Phi T \Phi^{-1} = T^{49}$ then gives $z(\Phi) = 0$. Explicitly, $\bar{r}(\Phi) = 1 + bE_{12}$ with $b \in \mathbf{F}_2$; $X(T^{49}) = X(T)$ because the sum of its 49 conjugates contains 24 pairs, and $X(T)$ commutes with $\bar{r}(\Phi)$ once $z(T) = 0$. The cocycle relation consequently makes $X(\Phi)$ commute with $1 + E_{12}$. Hence z is trivial on G_{K_7}.

The supported class z has even dyadic valuation, so it belongs to $V \otimes_{\mathbf{F}_2} k$. The second kernel in Lemma 33.18 makes it a multiple of the additive character attached to $[-1]$. Its value at c is one, so $X(c) = 0$ makes z vanish. Now x, y are additive. Their vanishing at two makes each a multiple of $\bar{\delta}$, and $X(c) = 0$ kills both. Thus every first difference vanishes. Induction and completeness give $r_1 = r_2$ in their normalized bases.

Apply this comparison to r and $r^\sigma \otimes \eta$, with η from Lemma 33.19. Choose the lift of σ in a dyadic decomposition group. It fixes the unramified Galois quotient of K_2, so the exact ordinary quotient of r^σ is ν. Moreover η is trivial at two, since -7 is a square in $\mathbf{Q}_2$. Normalize $r^\sigma \otimes \eta$ by its ordinary vector and the same real conjugation as above; its residue is again $1 + \bar{\delta} E_{12}$. The comparison gives $r^\sigma \simeq r \otimes \eta$. Lemma 12.4 proves modularity of the system. □

33.2.6 The remaining solvable image

Proposition 33.21. *Let $\bar{r} : G_K \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ be finite flat at two, unramified outside two and seven, with inertia of order at most two at seven. If its image is solvable, either its semisimplification is trivial or it is the standard representation of a unique S_3 extension M/K. If S is the splitting field of $X^3 + 2X + 2$ over $\mathbf{Q}$, then $M = KS$. Its quadratic subfield over K is $H = K(\sqrt{-7})$, which splits at two. The local image at two is tame C_3, and the local image at seven is C_2.*

Proof. The reducible assertion is Lemma 33.18. An irreducible solvable image is odd dihedral. Write its representation as $\mathrm{Ind}_{G_L}^{G_K} \theta$, with L/K quadratic and θ of odd order. Fontaine's bound gives quadratic conductor at most two at the dyadic place. A radicand of odd dyadic valuation has conductor three, so the possible fields are the seven nontrivial squareclasses in $\langle [-1], [\phi], [7] \rangle$. The rotation character is unramified at seven and tame at two; hence it factors through the odd part of the ray group of L modulo its dyadic primes.

All seven fields have class number one. Up to rational conjugation, the relevant arithmetic is

a ($L = K(\sqrt{a})$)	$\lvert D_L \rvert$	r_2	Minkowski bound	prime norms
-1	400	2	$30/\pi^2 < 4$	none
ϕ	400	1	$15/(2\pi) < 3$	none
7	19600	0	$105/8$	$4, 9$
-7	1225	2	$105/(2\pi^2) < 6$	4
7ϕ	19600	1	$105/(2\pi) < 17$	$4, 5, 11$

The fields for $-\phi$ and -7ϕ are rational conjugates of the displayed mixed-signature fields. For $a \neq -7$ the basis $1, \phi, t, \phi t$, $t^2 = a$, is integral and has the stated discriminant. At two the radicands are nonsquares modulo four, so the relative discriminant has exponent two; the odd ramified places follow by Eisenstein. For $a = -7$ use $u = (1 + \sqrt{-7})/2$ instead of t.

For $a = 7$, the elements $3 + t$ and $2 + t$ have absolute norms four and nine. They generate the dyadic prime and one of the two primes over three; conjugation gives the other. For $a = -7$, u and $1 - u$ generate the two norm-four primes. Finally, for $a = 7\phi$ we have

$$N_{L/\mathbf{Q}}(-1 + 4\phi - \phi t) = 4, \qquad N_{L/\mathbf{Q}}(2\phi + t) = -5, \qquad N_{L/\mathbf{Q}}(2 + \phi + t) = 11.$$

There is one dyadic prime and two primes of each of norms five and eleven. Conjugation over K exchanges each latter pair. Thus every prime needed by Minkowski's theorem is principal.

Except for $a = -7$, the dyadic prime of K ramifies in L. There is a single dyadic residue field $\mathbf{F}_4$, generated multiplicatively by the image of the unit ϕ. Its tame ray quotient is trivial. For $H = K(\sqrt{-7})$, two splits into two primes with residue field $\mathbf{F}_4$. The roots of unity in H are $\{\pm 1\}$ and its units are $\{\pm \phi^n\}$: the norm to K of a unit is ϕ^{2m}, and division by ϕ^m leaves an

algebraic unit of complex absolute value one at every embedding. Consequently

$$\mathrm{Cl}_{\mathfrak{P}_1\mathfrak{P}_2}(H) \simeq (\mathbf{F}_4^\times \times \mathbf{F}_4^\times)/\langle(\phi,\phi)\rangle \simeq C_3.$$

The involution of H/K exchanges the two dyadic primes and acts by inversion on this quotient. Its two nontrivial characters therefore induce the same irreducible representation and give one S_3 extension.

The polynomial X^3+2X+2 is Eisenstein at two and has discriminant -140. Its splitting field S has group S_3 and quadratic subfield $\mathbf{Q}(\sqrt{-35})$, so $S\cap K=\mathbf{Q}$. At two its splitting field is the tame extension $\mathbf{Q}_2(\zeta_3,\sqrt[3]{2})$; the unramified quadratic field K_2 already contains ζ_3. At five the quadratic completion of S is $\mathbf{Q}_5(\sqrt{-35})$, which becomes unramified over $K_5=\mathbf{Q}_5(\sqrt{5})$. At seven its quadratic completion remains ramified after the unramified base change to K_7. Thus KS/K has exactly the required relative ramification, and its quadratic subfield is $K(\sqrt{-35})=H$. It realizes the ray extension just determined.

This residual representation is finite flat at two. The curve $y^2+y=x^3$ has good reduction there, and its two-division field is $\mathbf{Q}_2(\zeta_3,\sqrt[3]{2})$: for a nonzero two-torsion point, $y=-1/2$ and $(-2x)^3=2$. Its restriction to K_2 is the same two-dimensional C_3 representation. □

Corollary 33.22. *Assume* ZF(4809/5000)*. Let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, strongly compatible system over K, of cyclotomic determinant and labelled weights $\{0,1\}$, good outside seven and crystalline at every dyadic coefficient place. Suppose its type at seven is tame with two-power inertial eigencharacters. Then $\mathcal{R}$ is modular unless every dyadic residue is the standard representation of* $\mathrm{Gal}(M/K)$ *from Proposition 33.21. In that remaining case every actual dyadic member is nonordinary.*

Proof. Every dyadic residue has solvable image by Proposition 33.16. A reducible one implies modularity by Proposition 33.20; otherwise the classification applies at every coefficient place. An ordinary cyclotomic Barsotti–Tate source has unramified semisimple residual inertia at two, whereas the stated S_3 representation has tame inertia of order three. □

Remark 33.23. The remaining CM inducing field H splits at two. Allen's ordinary theorem does not apply to these nonordinary sources, and the nonsolvable-residue dyadic lifting theorem does not apply to their S_3 residues. The field M is the compositum with an S_3 field over $\mathbf{Q}$ ramified at five; its residual descent does not give a characteristic-zero descent or a classical level-seven source.

Remark 33.24. Proposition 33.16 excludes nonsolvable residues, and Proposition 33.20 proves modularity of the actual source when its dyadic residue is reducible. A source theorem for the nonordinary S_3 case of Corollary 33.22, and realization of the preceding changes of weight for the actual local extensions, remain necessary to complete a global weight-reduction chain.

33.3 Trace relations over $\mathbf{Q}(\sqrt{6})$

Use the local types, cubic classes and ordinary argument from Section 19, with both five-places now allowed to be ramified. Put $K=\mathbf{Q}(s)$, $s^2=6$.

Theorem 33.25. *For a bounded source with types* SC/SC *or* SP/SP*, a solvable residual image at a coefficient prime above three implies modularity.*

Theorem 33.26. *Assume GRH. A bounded source with types* SP/SP *and nonsolvable residual image at a coefficient prime above two is modular. GRH is unnecessary if that dyadic member is nonordinary.*

Proposition 33.27. *A pure bounded source with common ramified quadratic residual constituents at three is modular.*

Proof. Normalize its residual pair to $\psi_\gamma + \psi_\gamma\omega$. By Lemmas 19.3 and 19.7, the two global offdiagonal extension spaces are contained in $V = \langle q, d\rangle$, with five-adic evaluations $\ell_1 = (1, 1)$ and $\ell_2 = (1, -1)$. Use the finite-flat Cayley–Hamilton construction of Proposition 19.6, now over $\mathbf{Z}_3$ without an ordinary parameter, and impose the exact five-adic conditions by the same closed-ideal construction. After removing a fixed scalar quadratic inertia character, a cuspidal condition kills wild and prime-to-three tame inertia and imposes $T^2 + T + 1 = 0$, $\Phi^2 + 5 = 0$. A special condition of exact character α_v imposes $(g - \chi_3(g)\alpha_v(g))(h - \alpha_v(h)) = 0$ for all local g, h. The diagonal cotangent vanishes by Fontaine and Lemma 19.4. Thus products of the two offdiagonal cotangents surject onto $\mathfrak{m}/(3R + \mathfrak{m}^2)$.

Since $N(\gamma) = -15$, the involution σ followed by twisting by ψ_5 interchanges the residual constituents. For SC/SC it restores the ordered order-three and order-six types. Its action on the product cotangent is factor interchange composed with $\sigma(q) = q$, $\sigma(d) = -d$. At either five-place, after removing its scalar quadratic character, the exact relation $(T - 1)^2 = -3T$ kills $\ell_v \otimes \ell_v$ in the relative cotangent. Lemma 12.3, with $s = q$, therefore makes the combined involution trivial on the cotangent.

For two special conditions, choose the place with $s \equiv 1 \pmod 5$ first. There ψ_γ has character RU, while at the other place it is trivial; here R is the character of $\mathbf{Q}_5(\sqrt{5})$ and U the unramified quadratic. Write the exact special characters as RU^e, U^f. The allowed first extension orientation is ramified at the first place exactly when $e = 0$, and at the second exactly when $f = 1$; the other orientation is complementary. If $e \neq f$, one orientation is forbidden at both places. Independence of ℓ_1, ℓ_2 then makes that entire global extension space zero. Nakayama makes the source reducible, a contradiction. Hence $e = f$. The product of the exact characters is R, and the involution σ followed by ψ_5 preserves the conditions. It interchanges the two permitted extension lines and fixes their tensor product. Its action on the whole cotangent is again trivial.

Lemma 12.2 makes the combined involution trivial on the complete ring. We obtain $r_3^\sigma \simeq r_3 \otimes \psi_5$. Since ψ_5 is restricted from $G_{\mathbf{Q}}$, Lemma 12.4 proves modularity. □

Proof of Theorem 33.25. If the residue is reducible, Propositions 19.6 and 33.27 apply. If it is irreducible, Lemma 19.5 gives irreducibility on G_H. The Fong–Swan and solvable Artin argument supplies a regular cuspidal residual witness; Lemma 2.1 gives one in the weight-zero convention. Then [3, Theorem 2.3] applies to the actual weight-two three-adic source. Strong compatibility gives modularity of the system. □

33.3.1 The mixed reducible case

Proposition 33.28. *A mixed bounded source cannot have a globally reducible three-adic residue.*

Proof. The unramified constituent pair is excluded by Proposition 19.6, since the involution σ exchanges the two monodromy operators. For the other pair normalize the special place to $\mathfrak{q}_- = (1 - s)$ and the cuspidal place to $\mathfrak{q}_+ = (1 + s)$, with residual pair $\psi_\gamma + \psi_\gamma\omega$. The global finite-flat plane is $V = \langle q, d\rangle$, with evaluation functionals $\ell_+ = (1, 1), \ell_- = (1, -1)$.

We need the relation $q^\vee \otimes q^\vee = 0$ in the relative pseudocotangent. We prove it in its finite quotient generalized matrix algebra, retaining the coefficient-valued offdiagonal product. Here are the exactness and coefficient arguments for this relation. Every finite-flat group killed by three whose generic factors are the same ramified quadratic θ has a unique model with its generic identification. The rank-one Kisin exponent is one, and its unramified constant is determined by θ. Closing a generic graph and a common generic θ-line, then inducting on the

finite-flat quotient, proves both full faithfulness and exactness of generic subquotients. This argument is confined to this filtered category; it does not assert full faithfulness on all groups at ramification index $p-1$.

Let R be the common finite-flat pseudoring, with generalized matrix algebra $(R\ B; C\ R)$. Put $R' = R/(3R+\mathfrak{m}^2)$, $B' = B/\mathfrak{m}B$, $C' = C/\mathfrak{m}C$, and let E' be the resulting generalized matrix algebra. The products of B', C' lie in the diagonal $\mathfrak{m}_{R'}$, and its radical I satisfies $I^3 = 0$. The regular left Galois module E' is finite flat by the stable condition of [76, Definition 2.5.2, Theorem 2.5.5]. Every finite left E'-module, being a quotient of a sum of regular modules, has the same finite-flat property and local factors θ.

For such a module V let $\mathfrak{M}(V)$ be the covariant Kisin module, obtained by applying the contravariant classification to the Cartier dual of its finite-flat model. The graph argument above extends every short exact sequence of these generic modules to a short exact sequence of finite-flat groups. Kim's exact classification [45, Corollary 4.3 and Proposition 5.7.3(2)] therefore gives an exact sequence of the underlying Kisin modules. These are finite free $\mathbf{F}_3[[u]]$-modules, so the sequence splits as modules and remains exact modulo u. The presentation argument for an exact linear functor now gives $\mathfrak{M}(V)/u = P_0 \otimes_{E'} V$ for a projective right E'-module P_0. Each of the two simple modules has a rank-one model, so each indecomposable projective occurs in P_0 once. Thus $P_0 \simeq E'$. Lifting a right generator and using Nakayama and equality of ranks gives $\mathfrak{M}(E') \simeq E'[[u]]$ as right modules.

Normalize the simple Frobenius maps to the same cu and write the regular Frobenius as $cu + X(u)$, with $X(u) \in I[[u]]$. Its height-one inverse is

$$u^2(cu + X(u))^{-1} = u/c - X(u)/c^2 + X(u)^2/(c^3 u).$$

Integrality forces $X(0)^2 = 0$. The diagonal radical is square zero and annihilates the offdiagonal modules, so this is exactly the vanishing of the product of the two constant offdiagonal coordinates in R'. On a rank-two extension those coordinates are $h(0)$ in Lemma 19.7. The constant functional vanishes on the unramified line d, represented by $5\epsilon(1-s) \equiv 1+3s \pmod 9$, and is nonzero on q. This identifies the asserted relation with $q^\vee \otimes q^\vee$ in the global product cotangent.

In the mixed problem the special relation kills one extension orientation at $\mathfrak{q}_-$, while the cuspidal relation kills $\ell_+ \otimes \ell_+$. Together with the displayed product relation these kill the product cotangent. The diagonal quotient has the one C_3 ray direction of Lemma 19.4, killed by the exact cuspidal Frobenius trace zero. Thus the relative pseudocotangent is zero, and every characteristic-zero point factors through $\mathbf{Z}_3$.

Compare such a point with the CM curve $E : y^2 = x^3 + \gamma^3(7+3s)^2/4$. It is the γ-twist of $y^2 + (7+3s)y = x^3$, and has exactly the same residual pair. Its minimal conductor is $\mathfrak{q}_+^2$: it is good at two, three and $\mathfrak{q}_-$, and has type II* at $\mathfrak{q}_+$. Its division points with $x = 0$ give the character ψ_γ; its trace at $\mathfrak{q}_-$ is zero, also forced by its CM and the inertness of five in $\mathbf{Q}(\sqrt{-3})$. These assertions are checked by the exact Weierstrass and minimal conductor calculation supplied with the proof.

The unknown source has rational three-adic traces by the vanishing cotangent. Its trace algebra is split: nonzero special monodromy provides a nonzero nilpotent, impossible in a division algebra. Realize it over $\mathbf{Q}_3$. For it and E choose the idempotents of a Frobenius at $\mathfrak{q}_-$; its two eigenvalues are distinct modulo three. Their generalized matrix orders have reducibility ideal (3). Indeed the normalized cuspidal inertia has diagonal entries $1+x, 1+y$ with $x, y \in 3\mathbf{Z}_3$, trace -1, determinant one, and offdiagonal product $xy - 3$, of valuation one. Choose an Eichler lattice whose primitive upper residual extension is the orientation forbidden at the special place. For both sources that class spans $\ker \ell_-$. Scale it and fix the two idempotents so the two residual upper cochains agree exactly, say β. Both $\beta(q^\vee)$ and $\beta(\ell_+)$ are nonzero.

Let f, g be the two pseudopoints modulo nine of the common finite-flat problem with the cuspidal condition at $\mathfrak{q}_+$ and no imposed condition at $\mathfrak{q}_-$. Their difference $\delta = (f-g)/3$ is a derivation on the relative pseudocotangent. Writing c_f, c_g for their divided lower residual functionals gives $\delta(BC) = \beta(B)(c_f - c_g)(C)$. The finite-flat $q^\vee$ product relation and the equality of the two cuspidal inertia traces give $(c_f - c_g)(q^\vee) = (c_f - c_g)(\ell_+) = 0$. These functionals are independent on V, so the derivation kills every offdiagonal product. Its diagonal quotient is zero as above. The two trace functions are equal modulo nine. At $\mathfrak{q}_-$ this asserts $\pm 6 = 0$ modulo nine, a contradiction. □

33.3.2 Nonsolvable characteristic-two sources

We give the complete ramification and explicit-formula argument for Theorem 33.26. The quadratic extension H/K is unramified at all finite places, unramified quadratic at two and at both places over five.

Lemma 11.9 applies to every finite-flat residue over K_2.

At a special five-place the full residual decomposition image is contained in C_2^2. Adjoin $H = K(\zeta_3)$, which is unramified at every finite place and unramified quadratic at two and both five-places. For M/K with nonsolvable image $\mathrm{SL}_2(\mathbf{F}_q)$, simplicity gives $M \cap H = K$, and

$$[MH : \mathbf{Q}] = 4q(q^2-1), \quad \mathrm{rd}(MH) \le \sqrt{24}\, 2^{2-2/q}\sqrt{5}, \quad 2N_{25}/[MH : \mathbf{Q}] \ge \tfrac{1}{2}.$$

If the actual dyadic member is nonordinary, Theorem 11.10 improves the upper bound to $\mathrm{rd}(MH) < 2\sqrt{120} < 21.910$. For $q \ge 8$, (12) with $b = 10$, $n \ge 2016$ and just the first norm-25 term gives $\mathrm{rd}(MH) > 23.29$, a contradiction. A totally real kernel field of degree at least 120 has $\mathrm{rd}(M) > 43.14954$ by Lemma 13.4, contradicting the same upper bound. These two comparisons require no GRH.

For an arbitrary dyadic member, assume GRH and apply the explicit formula of Section 13, with $B_{b,100}$, retaining four nonnegative prime-power terms at norm 25. Outward interval evaluation at 256 bits gives

case	$n \ge$	b	lower rd	upper rd
$q = 8$	2016	7.36	41.44651216	36.84623482
$q \ge 16$	16320	9.4	49.15326236	43.81780461

Each lower entry is strict and each upper entry is a strict outward upper bound. At fixed b, the right side of the explicit formula increases with n, since $C_b > 0$. The common dyadic factor four bounds the unbounded ranges. These comparisons exclude every $q \ge 8$. The exact calculation is supplied in [52, `ramified/d6_d10_d11/shared`].

For $q = 4$, [3, Theorem 2.5] supplies a regular cuspidal residual witness. Lemma 11.9 gives $\mathrm{rd}(M) \le 4\sqrt{120}\, 2^{-2/q}$. Thus Corollary 13.5(i) excludes a totally real kernel for every actual dyadic source, without GRH. Thus some residual real conjugation is nontrivial. All hypotheses of [73, Theorem 6.1] now hold for the actual crystalline two-adic source, including nonsolvability and this real-place condition. Its modularity proves Theorem 33.26.

33.3.3 Two special conditions at two

Proposition 33.29. *For a source of types* SP/SP*, the semisimplified residue $\bar{r}_2$ is either nonsolvable or $1 \oplus 1$. In particular an absolutely irreducible $\bar{r}_2$ is nonsolvable and, under GRH, the source is modular.*

Proof. An absolutely irreducible solvable subgroup of $\mathrm{GL}_2(\overline{\mathbf{F}}_2)$ with determinant one is odd dihedral, by Dickson's classification. Indeed a normal unipotent two-subgroup would give a

nonzero invariant subspace fixed by the whole group, while the irreducible subfield groups $\mathrm{SL}_2(\mathbf{F}_{2^r})$, $r \geq 2$, are nonsolvable. Thus the residue is $\mathrm{Ind}_{G_L}^{G_K} \xi$ for a quadratic L/K and a nontrivial odd-order character ξ.

Special inertia at five reduces to the identity or a group of order two: its scalar quadratic factor becomes trivial in characteristic two. Its intersection with the odd rotation subgroup is trivial. Consequently ξ is unramified at all primes over five, even when L/K is ramified there. It is unramified elsewhere away from two. At two it is tame, so its conductor divides the radical of (2) in L, hence divides (2).

Fontaine's upper-break bound three gives dyadic conductor at most four for L/K. An odd-valuation quadratic Kummer class has conductor $2e + 1 = 5$, since $e(K_2/\mathbf{Q}_2) = 2$. Therefore the possible nontrivial radicands are precisely the fifteen classes in $\langle -1, \epsilon, a, b \rangle$. In mask order for this displayed basis their ordinary ray groups modulo (2) are

$$C_4, 1, C_2, C_4, C_4, C_4, C_4, C_4, C_4, C_4, C_4, C_8 \times C_4, C_2, C_4 \times C_2, C_2.$$

These are exact degree-four ray computations; certification of each class group returns one. All groups are two-primary, so none admits the odd-order character ξ, a contradiction. The exact calculation is supplied in [52, `ramified`].

If the residue is reducible, both constituent characters have odd order. They are unramified at five by the special calculation, and at two since the residue units are $\mathbf{F}_2^\times$ and wild inertia is pro-two. They are unramified at every other finite place and at the real places. The class number one statement makes both trivial. The final modularity assertion follows from Theorem 33.26. □

33.3.4 Ordinary scalar sources

Proposition 33.30. *A bounded source with two special places and scalar residual semisimplification at two is modular if some two-adic member is ordinary.*

Proof. Let r be that member. Put $H_0 = \chi_{-3}|_{G_K}$ and $\psi_5 = \chi_5|_{G_K}$. The exact two special characters θ_1, θ_2 determine a unique $\eta \in \{1, H_0, \psi_5, H_0\psi_5\}$ whose restriction at both five-places is $\theta_1\theta_2$. Both r and $r^\sigma \otimes \eta$ have the same exact special characters, ordinary crystalline condition, determinant and support.

The supported squareclasses in the basis $-1, \epsilon, 1+s, 1-s$ have localization kernel $\langle [-15] \rangle$ at two. Indeed their squares modulo $8\mathcal{O}_{K_2}$ have coordinates $(1,0), (1,4), (7,2), (7,6)$; only $1, [-15]$ lie in the kernel. Lemma 12.1 therefore puts both aligned representations in the same marked problem, with residue $1 + \chi_{-15}E_{12}$, fixed real matrix $C = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and ordinary line e_2.

Kisin's marked ordinary functor, Proposition 2.4.4 and Corollary 2.4.5 of [46], is represented at arbitrary ramification, including two. Pull it back to the global matrix functor with the displayed real matrix and the ordinary section e_2. At five impose the exact character line equations. The line there is unique since a residual unipotent element has unit upper entry, so these too are represented conditions.

We compute the homogeneous tangent in this gauge. Write its cocycle as $X(g) = \left(\begin{smallmatrix} x(g) & y(g) \\ z(g) & x(g) \end{smallmatrix}\right)$ relative to $1 + \chi_{-15}(g)E_{12}$. The adjoint action adds $\chi_{-15}(g)z(g)(1 + E_{12})$, so z is additive. The ordinary finite-flat condition gives $y|_{G_{K_2}} = 0$, $x|_{G_{K_2}}$ unramified, and $z|_{G_{K_2}}$ a unit Kummer class. At a special place allow the line to move to $e_1 + t_v e_2$. The exact characters give $z|_{G_v} = 0$ and $x|_{G_v} = t_v\chi_{-15}|_{G_v}$. The supported squareclass table then gives $z \in k\chi_{-1}$. The real involution relation forces $z(c) = 0$, hence $z = 0$. Now x, y are additive, and the dyadic conditions give $y = B\chi_{-15}$ and $x = A\chi_{-15} + D\psi_5$. At five $\chi_{-15} = R + U$ and $\psi_5 = R$, where R, U are the independent ramified and unramified quadratic characters; thus $D = 0$. The fixed real matrix

forces $A = B = 0$. The represented ring has zero relative tangent and is a quotient of $W(k)$. Its actual characteristic-zero point makes it $W(k)$, so the aligned pair is identical.

We have $r^\sigma = r \otimes \eta$, with η restricted from $G_{\mathbf{Q}}$. Lemma 12.4 proves modularity. □

33.3.5 The scalar two-adic source with two special places

Let $K = \mathbf{Q}(\sqrt{6})$. A special place in this section has nonzero monodromy and its exact scalar twist is quadratic, including its unramified part. Let a_v denote the good-place trace of a pure parallel weight-two system, with arithmetic Frobenius normalization.

Theorem 33.31. *Let $\mathcal{R}$ be a totally odd, absolutely irreducible, pure, strongly compatible rank-two system over K, with compatibility including monodromy. Suppose it has cyclotomic determinant, labelled weights $\{0, 1\}$, is good outside five and is special at both primes above five. Suppose every member at a coefficient place above two is nonordinary and has scalar residual semisimplification. Then the global trace field has degree at most three. The actual system is modular.*

Let D be the global trace field and $d = [D : \mathbf{Q}]$. Apply Section 11 to the tuple at all dyadic coefficient places, and use its notation $R_0, E_0, T, \mathcal{D}$. Then R_0 has $\mathbf{Z}_2$-rank d, while $E_0/2E_0$ is local with residue field $\mathbf{F}_2$.

Put $E = E_0/2E_0$, $R = R_0/2R_0$, and let J be the radical of E. A functional on J/J^2 gives a middle/middle extension by quotienting the regular module. In the calculation of Theorem 11.10, its Kisin matrix is $\left(\begin{smallmatrix} u & h \\ 0 & u \end{smallmatrix}\right)$, and its Artin–Schreier class has the unique form $n/u + m$, with $n, m \in \mathbf{F}_2$. The first coordinate is ramified of break one when nonzero; the second is the unramified character.

At five these quadratic characters are tame. Corollary 11.4 gives their complete three-dimensional space, with basis

$$\alpha = [-1], \qquad \beta = [\epsilon] = [2], \qquad \gamma = [-15], \qquad \epsilon = 5 + 2\sqrt{6}.$$

Their relative discriminant norms are $4, 4, 25$. Thus they span every allowed first-layer character. At a fixed five-place, -1 is square, 2 is an unramified nonsquare and -15 gives a ramified character. Choose a Frobenius lift trivial on γ, and a tame inertia generator. Write A, B, C for the differences from one of a real involution, this Frobenius and this inertia. Their evaluations on (α, β, γ) are respectively $(1, 0, 1), (0, 1, 0), (0, 0, 1)$. Since the group spans E, the first-layer functional is determined by the resulting character; these three elements therefore span J/J^2. Nilpotence of J gives algebra generation of E.

At any one five-place normalize by its fixed rational quadratic special sign. In a basis of the special line, for local g the normalized difference $L(g)$ has top row $(\chi_2(g) - 1, *)$ and bottom row zero. Thus $L(g)L(h) = (\chi_2(g) - 1)L(h)$ in the original joint order. The signs disappear modulo two and all local differences have zero products and trace zero there. This uses equality of the exact quadratic characters across coefficient conjugates. For the real element $(1 + A)^2 = 1$, $T(A) = 0$ modulo two. Polarized Cayley–Hamilton now gives in E

$$\begin{gathered} A^2 = B^2 = C^2 = BC = CB = 0, \\ AB + BA = X = T(AB), \qquad AC + CA = Y = T(AC). \end{gathered} \tag{73}$$

In particular all words reduce to the R-span of $1, A, B, C, AB, AC$.

Lemma 33.32. *The ring R is $\mathbf{F}_2[t]/(t^d)$, where t is the image of a_2. When $d \geq 2$, X is another generator of its maximal ideal and Y/X can be chosen to be a unit. In characteristic zero $D = \mathbf{Q}(a_2)$.*

Proof. First work in the relative cotangent quotient $R' = R/\mathfrak{m}_R^2$, retaining the full algebra over it and its trace and determinant laws. The normal form above shows that the relative cotangent is generated by X, Y: all traces reduce to their R'-span and the trace order is generated by the actual traces. All traces are in $\mathfrak{m}_{R'}$.

The ramified functional n in the preceding middle-extension calculation on (α, β, γ) is $(1, 1, 0)$. Indeed α is dyadically ramified, γ splits there, and $\alpha + \beta + \gamma = [5]$ is the unramified quadratic character. It follows in J/J^2 that the constant term of Φ is $A + B + C$. In the u-coefficient of Φ the corresponding unramified functional takes value one on $[5]$. These statements are unaffected by a first-order basis change, whose correction is $u(\varphi - 1)z$ with zero constant and linear coefficients. Write

$$\Phi = N + u(1 + M) + O(u^2), \qquad N = A + B + C + Z,$$
$$M = m_A A + m_B B + m_C C + Z', \qquad Z, Z' \in J^2.$$

Evaluation of $[5]$ on A, B, C gives $0, 1, 1$, hence $m_B + m_C = 1$. More explicitly, take the functional on the full allowed character space obtained by restricting to the dyadic tower and reading the constant Artin–Schreier coefficient. The local ramification break bound ensures that only the terms of degrees $-1, 0$ can occur. Its restriction to the actual middle extensions gives the coefficients of M, so one may use this lift of the functional even when the actual first layer is smaller.

We give the determinant calculation in R' itself. By the six-word normal form, $J^2 \subset \mathfrak{m}_{R'} E + R'AB + R'AC$. A term from $\mathfrak{m}_{R'} E$ contributes neither a norm nor a polar term, since $\mathfrak{m}_{R'}^2 = 0$ and all traces are in $\mathfrak{m}_{R'}$. The norms of AB, AC are zero by multiplicativity, and their mutual polar term is $XY - T(ABAC) = 0$. Their trace pairings with $A + B + C$ are zero by (73). Thus $\mathcal{D}(N) = \mathcal{D}(A + B + C) = X + Y$. The constant term of the determinant in Lemma 11.6 gives $X = Y$.

The same normal form gives $T(J^3) = 0$: each word of length three reduces to a multiple of a trace-zero generator, and longer products contribute either such terms or quadratic products of X, Y. The coefficient of u in the norm is $T(N)T(1 + M) - T(N(1 + M))$. It vanishes by the determinant divisibility, and the product of the two traces vanishes in R'. Hence $T(N) = T(NM) = (m_B + m_C)X = X \quad$ in R'. Lemma 11.6 identifies $T(N)$ with a_2. Nakayama now makes R a quotient of $\mathbf{F}_2[t]$, with radical generated by t. Its dimension is the rank d of R_0, giving the asserted relation $t^d = 0$. The same calculation gives relative generators X and Y whenever $d \geq 2$, with identical initial coefficient. Finally $1, a_2, \ldots, a_2^{d-1}$ span R_0 over $\mathbf{Z}_2$ by Nakayama. The rational degree of $a_2 \in D$ is therefore at least d, proving $D = \mathbf{Q}(a_2)$. □

Suppose $d \geq 4$ and use X as the parameter of R. Choose $v \in R^\times$ with $Y = vX$, and put $Z = C + vB$. By (73), Z is central, $Z^2 = BZ = XZ = 0$, and $T(Z) = T(AZ) = 0$. Here $XC = (AB + BA)C = BAC = B(Y + CA) = YB$. Consequently E is spanned over R by $1, A, B, AB, Z, AZ$.

Let $S = \mathbf{F}_2[X]/(X^3)$ and $Q = S\langle a, b\rangle/(a^2, b^2, ba + ab - X)$, as in the proof of Proposition 33.2. The same trace-pairing argument gives a surjection $E \to Q$, taking A, B, Z to $a, b, 0$. Indeed, pairing a relation in $1, A, B, AB, Z, AZ$ with $1, A, B, AB$ puts its first four coefficients in (X^{d-1}), which vanishes in S. The free normal basis $1, a, b, ab$ then makes the map multiplicative. This does not require faithfulness of the scalar action on E.

Compose with $a \mapsto E_{12},\ b \mapsto XE_{21}$ to obtain $\theta : G_K \to \mathrm{GL}_2(S)$. On formal coefficients its matrix is

$$\begin{pmatrix} f_0 + Xf_3 & f_1 \\ Xf_2 & f_0 \end{pmatrix}.$$

Its trace is the reduction of T. Comparing the two Cayley–Hamilton identities for a group element gives $\det \theta(g) = 1$. The two linear characters $f_1(0), f_2(0)$ are visible on actual regular

quotients: send a, respectively b, to the generator of $\mathbf{F}_2[z]/(z^2)$, and send the other generator and X to zero. Their values on A, B, C are $(1,0,0), (0,1,1)$. The calculated first-layer space gives $f_1(0) = \alpha,\ f_2(0) = [5]$.

Put $F = K(i, \sqrt{5})$. Each matrix over G_F is $I + XM + X^2N$; the lower left entry of M is zero by $f_2(0) = 0$, and its diagonal entries agree by the determinant. All these M's lie in the commutative algebra generated by I, E_{12}. Multiplication modulo X^3 thus makes $\theta(G_F)$ abelian of exponent at most four. It has a specified nonzero double. Let r be the product of the chosen real and special Frobenius elements. Its image has trace X, so

$$\theta(r)^2 = I + X\theta(r), \qquad \theta(r)^4 = I + X^2 I.$$

The element r^2 lies in G_F, since F/K is elementary abelian, and therefore gives an element of order four there.

Fix a five-prime w of F. Choose a Frobenius h over F_w and a tame inertia element k in the underlying K-decomposition group with $[5](k) = 1$ (additive notation). The special identities give

$$T(h-1) = T(k-1) = 0, \qquad (h-1)(k-1) = (k-1)^2 = 0 \quad \text{in } E,$$

where the trace equalities are in R. In particular the formal ab-coefficient of each difference vanishes in Q, by the original trace pairing $Xf_3 = 0$ in R. Write $h - 1 = P + Ua + Vb,\ k - 1 = P' + U'a + V'b$ in Q. The b- and ab-coefficients of their zero product give

$$PV' + VP' = UV' + VU' = 0.$$

Now V' is a unit and $V \in XS$, by the values of $[5]$. Thus $h - 1 = (V/V')(k-1)$ in the *abstract* algebra Q. The coefficient comparison here requires its free normal basis. The matrix of k is I modulo X, since -1 is square at five. If $\theta(k) - I = XM_0 + O(X^2)$, its square gives $M_0^2 = 0$. We have proved at all four primes that

$$\theta(h) = I + X^2 N_w, \qquad N_w^2 = 0. \tag{74}$$

There is one dyadic prime $\mathfrak{P}$ of F, with absolute $(e, f) = (4, 2)$. Over K, the quadratic $K(i)$ has dyadic conductor two: its absolute discriminant is the product of $24, -4, -24$, and division by 24^2 gives relative discriminant norm four. The factor $K(\sqrt{5})$ is dyadically unramified. Thus the ramified break of F/K is one. By the finite-flat bound and Herbrand's subgroup formula the upper break over F is at most $1 + 2(3-1) = 5$. At five the elementary local image over K has exponent two and its possible quadratic tame inertia is killed by $K(\sqrt{5})$. Hence the abelian representation over F is unramified at five and all its five-Frobenius elements have order dividing two. The field is totally imaginary.

Write H for the ray class group of modulus $\mathfrak{P}^6$ of F. Class field theory makes $\theta|_{G_F}$ a quotient of

$$H/(4H + \langle 2[w] : w \mid 5 \rangle).$$

The following coordinates are from the ordinary ray group, with the ordered primes as in the certificate. They concern abelian extensions of the degree-eight field F.

group or prime	invariants or class
H	$(8, 4, 2, 2)$
$H/(4H + \langle 2[w] \rangle)$	$(4, 2, 2, 2)$
w_1	$(0, 1, 0, 0)$
w_2	$(0, 1, 0, 1)$
w_3	$(2, 1, 0, 1)$
w_4	$(2, 1, 0, 0)$

For reproducibility one defining polynomial for F is

$$y^8 - 4y^7 + 2y^6 + 8y^5 + 13y^4 - 44y^3 + 164y^2 - 140y + 145.$$

The certificate computes a certified class group and units, obtains the ray cyclic generators and the ideal logarithms L of the four primes, and takes the Smith form of the relation lattice generated by $\mathrm{diag}(8,4,2,2)$, $4I$, $2L$. Thus $[w_1] - [w_4]$ is the unique nonzero double of the quotient. It maps to $I + X^2 I$, by the square of $\theta(r^2)$. By (74) this gives $N_{w_1} + N_{w_4} = I$ for square-zero matrices, impossible since $(I + N_{w_1})^2 = I$. This proves $d \leq 3$. The exact calculation is supplied in [52, `ramified/d6_d10_d11/d6`].

The trace-field bound and Lemma 11.16 give a coefficient prime above three with residue field $\mathbf{F}_3$ or $\mathbf{F}_9$, or above five with residue field $\mathbf{F}_5$.

Choose the indicated coefficient place. For an absolutely irreducible residue, the traces in its finite residue field give a realization over that field by finite-field descent (the Brauer group of a finite field is zero). Allen–Khare–Thorne, *Modularity of* $\mathrm{PGL}_2(\mathbf{F}_p)$*-representations over totally real fields*, Theorems 1.1 and 2.3, give residual automorphy and lift the actual weight-two member when its restriction to the cyclotomic subgroup is absolutely irreducible. In the nonsolvable case this restriction is automatic: reducibility on a normal cyclic quotient would force an induced dihedral representation, with solvable image. At three all solvable cases, including globally reducible residues, are covered by the pure special characteristic-three source theorem. At five a solvable irreducible residue either remains irreducible on the cyclotomic subgroup or is induced from $K(\sqrt{5})$. Such induction at a five-place is incompatible with the special ordinary line: a tame generator outside the inducing subgroup has scalar square and projective order two, while the ratio of the ordinary inertial characters has order four. This gives cyclotomic irreducibility in this case as well.

When the five-adic residual semisimplification is reducible, the dual source has determinant χ_5^{-1} and at both five-places has an invariant line of finite character with quotient of character that finite character times χ_5^{-1}. Thus it is ordinary of weight $(0,0)$ in the convention of [28, §1.6]. Corollary 1.2.2 there applies since K is totally real abelian and five is unramified in K. This proves modularity without a calculation of the constituent ratio. Strong compatibility transfers modularity to the entire system, proving Theorem 33.31. The bounded five-adic menu is Lemma 4.14; Lemma 4.15, with $\ell = 5$, gives both replacements of the actual characteristic-three residue, with determinant fixed.

33.3.6 The supported residual base

Proposition 33.33. *Assume GRH. Every bounded system over* $K = \mathbf{Q}(\sqrt{6})$ *is modular. Every absolutely irreducible determinant-*ω_5 *residual representation unramified outside five is modular.*

Proof. Consider first a bounded system of two special types. The quartic-ray calculation excludes an irreducible solvable two-residue. A nonsolvable two-residue gives modularity by the bounded special source theorem under GRH. In the remaining case its semisimplification is $1 \oplus 1$. If some two-adic member is ordinary, the marked ordinary rigidity Proposition 33.30, followed by projective descent, gives modularity. Otherwise Theorem 33.31 applies to all dyadic coefficient places together and proves modularity. Thus every two-special bounded system is modular.

Now consider any bounded system at three. A reducible residue is handled by Theorem 33.25 for a pure menu and is excluded by Proposition 33.28 for a mixed menu. An irreducible solvable residue has a regular cuspidal Artin witness; Lemma 19.5 and [3, Theorem 2.3] then modularize the actual weight-two source.

In the nonsolvable case the residue stays absolutely irreducible over $K(\zeta_3)$. At both five-places use the genuine special lift of Lemma 4.15, which retains the entire actual residue and its quadratic scalar character. Keep the old crystalline weight-two component at the three-place. Emerton–Gee–Pan–Zhu (EGPZ), Theorem 4.2.1, gives a global point on the selected components. Lemma 2.6 gives a pure compatible system. On the special components the Frobenius eigenvalue ratio is five; zero monodromy at a pure point would contradict purity. Thus both local monodromies are nonzero and the new system is modular by the previous paragraph. AKT, Theorem 2.3, applies backwards to the old weight-two three-adic member. This proves modularity of every bounded system.

Let $\bar{r}$ be absolutely irreducible of determinant $\bar{\chi}_5$, unramified outside five. Solvable image has a regular cuspidal Artin witness after weight raising. Otherwise cyclotomic restriction is absolutely irreducible. The EGPZ exception cannot occur: $[K(\zeta_5) : K] = 4$, whereas the abelianization of $\mathrm{PGL}_2(\mathbf{F}_5) \simeq S_5$ has order two. The fixed determinant bounded local lift lemma at each completion $K_w = \mathbf{Q}_5$ supplies a weight-two lift of one of the indicated types for the entire local representation. EGPZ produces the global lift, with unramified conditions away from five. Lemma 2.6 and purity yield a bounded system. Its modularity, followed by reduction, proves the claimed residual base at five.

□

33.4 The characteristic-three base over $\mathbf{Q}(\sqrt{13})$

In this section $K = \mathbf{Q}(\sqrt{13})$. Both completions above three are $\mathbf{Q}_3$, while two is inert. Section 17 gives a classical base with determinant $\bar{\chi}_3$. The following supported theorem allows an arbitrary determinant. Lemma 17.1 and its arithmetic give narrow class number one and quadratic characters for this support.

Lemma 33.34. *Under GRH, a totally imaginary number field of degree n has root discriminant greater than* 39.01 *if $n \geq 531360$, and greater than* 37.63 *if $n \geq 78624$.*

Proof. Use the explicit formula of Section 13, with no Euler terms or real-place term retained, using $B_b \leq B_b^*$. The parameters give

b	n	$\mathrm{rd}(L) >$
45/4	531360	39.01373
21/2	78624	37.63959.

These are outward interval evaluations of the displayed finite expressions. Each bound increases with n, since $C_b > 0$. The exact calculation is supplied in [52, `classical`]. □

Proposition 33.35. *Assume GRH. Every continuous irreducible totally odd representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ unramified outside* 3 *is modular.*

Proof. Let L/K be the linear kernel field, let $G = \mathrm{Gal}(L/K)$, and let Z and $H = G/Z$ be its scalar subgroup and projective image. The field L is totally imaginary. The scalar subgroup has order prime to three. Since three splits in K, the discriminant tower and (4) give $\mathrm{rd}(L) \leq \sqrt{13}\, 3^{13/6} < 38.971$. The solvable case is already modular. Otherwise Dickson's classification gives $H \simeq A_5$, $\mathrm{PSL}_2(\mathbf{F}_{3^f})$ or $\mathrm{PGL}_2(\mathbf{F}_{3^f})$, with $f \geq 2$. If $3^f \geq 81$, then $[L : \mathbf{Q}] \geq 2|\,\mathrm{PSL}_2(\mathbf{F}_{81})| = 531360$, contradicting Lemma 33.34.

Suppose $3^f = 27$. The wild inertia order at either prime above three is at most 27, so $\mathrm{rd}(L) \leq \sqrt{13}\, 3^{13/6-1/27} < 37.417$. If $H = \mathrm{PSL}_2(\mathbf{F}_{27})$, then $|Z| \geq 4$. Indeed, if $|Z| \leq 2$, determinant kills Z and factors through the perfect group H, contradicting oddness; order three is impossible. If $H = \mathrm{PGL}_2(\mathbf{F}_{27})$, then $|Z| \geq 2$: otherwise its perfect subgroup $\mathrm{PSL}_2(\mathbf{F}_{27})$

embeds in $\mathrm{SL}_2(\overline{\mathbf{F}}_3)$, which has only one nonidentity involution, namely $-I$. In either case $[L : \mathbf{Q}] \geq 78624$, again contradicting Lemma 33.34.

Suppose now that $H = \mathrm{PGL}_2(\mathbf{F}_9)$. Its nontrivial determinant-squareclass character is unramified outside three, so narrow class number one forces it to ramify at some $\mathfrak{p}_i$. At a place with nontrivial projective wild inertia, that inertia has a unique fixed point, defined over $\mathbf{F}_9$, and the local projective group lies in the corresponding Borel. Its torus quotient is abelian, so the tame Frobenius relation gives $a^3 = a$ for every inertia torus coordinate. Thus $a = \pm 1$, both squares in $\mathbf{F}_9^\times$. The squareclass character is unramified there. At the place where it ramifies, projective wild inertia, and hence linear wild inertia, is trivial. Consequently $\mathrm{rd}(L) < \sqrt{13}\, 3^{(1+13/6)/2} = \sqrt{13}\, 3^{19/12} < 20.532$. But $[L : \mathbf{Q}] \geq 2|\,\mathrm{PGL}_2(\mathbf{F}_9)| = 1440$, and (12) with $b = 16$ gives $\mathrm{rd}(L) > 21.15$. Thus this image is excluded without GRH.

The remaining nonsolvable groups are A_5 and $\mathrm{PSL}_2(\mathbf{F}_9)$. They embed in $\mathrm{PGL}_2(\mathbf{F}_9)$ with trivial determinant-squareclass character, since they are perfect. The projective representation is modular by [3, Theorem 1.2(5)], and its given linear lift is modular by [3, Lemma 2.1]. $\square$

A The initial exceptional lift over $\mathbf{Q}(\sqrt{5})$

Corollary A.1 (A classical quotient of the prepared KT ring)**.** *In the exact setup of [41, §7], assume the fixed set V_2 of $D^{\mathrm{St(uni)}}$ places is nonempty. The quotient $R_{\mathcal{S}}^{\mathrm{cl}}$ of the global deformation ring at the classical weight-two augmentation of its ordinary Iwasawa algebra Λ is finite over $\mathcal{O}$.*

Proof. In the notation of [41, §7], let $\mathbb{T}$ be the localized ordinary Hecke algebra at infinite ordinary level. The paragraph preceding their Theorem 7.2 makes $\mathbb{T}$ finite over Λ. Theorem 7.2 in the final version of [41] gives a surjection

$$R_{\mathcal{S}} \longrightarrow \mathbb{T}$$

with nilpotent kernel I, since $V_2 \neq \varnothing$. We use the final-version numbering of that reference. The complete deformation ring $R_{\mathcal{S}}$ is Noetherian. Thus each I^j/I^{j+1} is a finite module over $R_{\mathcal{S}}/I = \mathbb{T}$, and hence finite over Λ. The finite filtration by powers of I shows that $R_{\mathcal{S}}$ is finite over Λ. Tensoring along the classical augmentation $\Lambda \to \mathcal{O}$ gives the claimed finiteness. This argument includes the entire deformation ring, including possible characteristic-five components; it uses no assumed characteristic-zero point or flatness of $R_{\mathcal{S}}$. □

Proposition A.2 (An initial exceptional lift)**.** *Let $K = \mathbf{Q}(\sqrt{5})$, and let $\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ be continuous, absolutely irreducible and totally odd. Assume that its projective image is $\mathrm{PGL}_2(\mathbf{F}_5)$ and that its projective kernel field contains $K(\zeta_5)$. Put*

$$\nu = [\det \bar{r}\, \bar{\chi}_5^{-1}],$$

where brackets denote the Teichmuller lift. There is a continuous lift $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_5)$ with determinant $\nu\chi_5$, ramified at finitely many places and de Rham of parallel weight two. It is totally odd. The lift is potentially modular over totally real extensions with avoidance of any prescribed finite extension, and belongs to a strongly compatible system.

Proof. Fixed local rings over K. Let S contain the places above 5 and the ramification of $\bar{r}$. At each $v \in S$, choose a definite-type local lift by [68, Propositions 7.7.1 and 7.8.1], weight two when $v \mid 5$. The determinant correction of [68, §7.9] twists by the square root of a finite character reducing to 1; hence it does not change the residual representation or Hodge–Tate weights and gives determinant $\nu\chi_5$. Choose a corresponding nonzero, reduced, $\mathcal{O}$-flat local fixed-type quotient $R_v^{\dagger}$ from [68, Propositions 7.3.1 and 7.4.1]. Its framed dimension is 4 for $v \nmid 5$, and $4 + [K_v : \mathbf{Q}_5]$ for $v \mid 5$. At 5 the type may be A, B or C; we do not require a potentially Barsotti–Tate lift when only type C was chosen.

Fix finite Galois local extensions killing these fixed inertial types, the local residual representations, and their finite determinant factors. After these extensions, the local characteristic-zero points are unramified or in the Steinberg closure away from 5; at 5 they are ordinary semistable for types A, C, or nonordinary Barsotti–Tate for type B. A single finite extension works for every point of each chosen local ring: the finite inertial type is fixed, while its possible monodromy is retained. Let K_ν be the field cut out by ν. It is totally real since ν is totally even.

Preparation and the modular residual witness. Real approximation and Krasner's lemma, followed by taking a totally real Galois closure, supply a totally real finite extension containing K_ν, for which every completion at a controlled place contains the fixed local extension. This Galois closure is taken before constructing any modular witness. Apply the residual potential-modularity theorem [68, Theorem 5.1.1] over this field. This theorem is for any odd residual representation and has no exceptional-image exclusion. It gives a weight-two modular residual witness after a further totally real extension.

Let R now denote the set of places away from 5 at which the restrictions of the original chosen local rings have Steinberg type. Let σ denote the places above 5 lying over a local type A or C. Apply [41, Lemma 9.2] to the known modular witness, this R, and this σ. After a soluble totally real extension it supplies a weight-two witness π that is ordinary and Iwahori at σ, nonordinary and spherical at the other places above 5, Steinberg at the pullback of R, and unramified at every other finite place. Thus incidental bad primes of the first potential-modularity witness have disappeared. The residual representation is trivial at the places above 5 and R. The already imposed original local conditions remain of the stated type under all these restrictions.

Finally make a totally real quadratic extension split at the current places above 5 and R, and base change π. Denote the resulting field by F. Then $[F : K]$ and $\#R_F$ are even. Its central character is everywhere unramified. In the dual normalization its ratio with the required Teichmuller character $\psi = 1$ is a finite unramified character of 5-power order; twisting by the inverse square root makes the central character exactly 1, without changing residues or local types. The residual representation over F is unramified away from 5 and trivial at 5 and R_F. At each $w \in R_F$, residual triviality and the cyclotomic determinant give $\mathrm{N}\, w \equiv 1 \pmod 5$, as required in [41, §7].

A fresh fixed unipotent prime. Choose the auxiliary prime only after F has been fixed. Let M/K be the projective kernel field and let F^{g}/K be the Galois closure of F/K. Since F^{g} is totally real, $M \cap F^{\mathrm{g}} = K$: the only proper nontrivial normal subextension of the S_5-extension M/K is its imaginary quadratic subfield $K(\zeta_5)$, and M itself is not totally real. Let L/K be the full linear residual kernel field, put $G = \mathrm{Gal}(L/K)$, let Z be its scalar subgroup, and set $N = \mathrm{Gal}(L/(L \cap F^{\mathrm{g}}))$. Projective disjointness gives $ZN = G$, so G/N is a quotient of the prime-to-five group Z. Every element of order five in G therefore lies in N and is an actual nontrivial unipotent matrix. It is trivial on $K(\zeta_5)$.

Chebotarev therefore gives a prime $q \notin S$, split completely in F^{g}, with actual residual Frobenius a nontrivial unipotent of order 5. Then $\mathrm{N}q \equiv 1 \pmod 5$ and $\nu(q) = 1$. Impose $D^{\mathrm{St(uni)}}$ at q on the original K-deformation problem, as well as at every place above q on its F-restriction. This is a nonzero formally smooth local problem of framed dimension 4 by [41, Proposition 6.6]. Write Q_F for the places above q. Its cardinality is $[F : K]$, hence even and positive. Take $V_1 = \varnothing$ and $V_2 = Q_F$ in [41, §7]. No congruence modulo 25 is needed for this fixed local problem.

Choose the good-subgroup auxiliary place a using [41, Lemma 7.1], away from the places above 5, R_F and Q_F. The raising step can be carried out with the prescribed local conditions at five. Use the level-raising argument of [41, Lemma 5.2 and proof of Theorem 9.1], in the form of [72, Lemma 4.13] that keeps the levels and ordinarity at the coefficient places. At a place in σ, keep Iwahori level and the unit U_v eigensummand. At a place above five outside σ, keep spherical level and localize at the zero reduction of T_v. Both conditions are present in the initial witness. In the latter case the spherical Hecke polynomial has constant term of positive five-adic valuation, so the zero reduction of T_v forces both roots to have positive valuation. The degeneracy maps, trace maps and vanishing-cycle sequences at the raising places commute with these operators at five. The support argument therefore applies in the indicated eigensummands, which are obtained by exact localization of the finite Hecke modules. At the new unipotent places the congruence of the two eigenvalues leaves only the required choice of the Steinberg sign.

At the auxiliary place a from [41, Lemma 7.1], the good level is $U_1^1(a)$. Use the trivial-character projector for $U_0(a)/U_1^1(a) \simeq (k_a^\times)^2$ on every level-raising module. Its denominator $(\mathrm{N}a - 1)^2$ is a five-adic unit. The initial witness is fixed by this projector, and the maps at all the raising places commute with it. Hence the resulting representation has Iwahori invariants at a. Its local representation there is spherical or an unramified twist of Steinberg. The Steinberg

possibility would give

$$\frac{\operatorname{tr}(\bar{r}(\mathrm{Frob}_a))^2}{\det(\bar{r}(\mathrm{Frob}_a))} = \frac{(1+\mathrm{N}a)^2}{\mathrm{N}a}$$

in the residue field, excluded by the choice of a. It follows that a contributes no ramification. The new witness has unramified central character, whose reduction is the required Teichmüller character. If a determinant correction is needed, twist by the inverse square root of the unramified five-power ratio. This preserves the local types and all residues.

Enlarge the coefficient field to contain the images of the embeddings at the five-places and all residual eigenvalues. Flatness and the local dimensions are preserved by this coefficient extension. The resulting witness satisfies the conditions of [41, §7], with $V_1 = \varnothing$ and $V_2 = Q_F$.

Restriction at the level of local rings. Let $R^{\dagger}_{K,q}$ be the global fixed-determinant deformation ring over K, with the original local quotients $R^{\dagger}_v$, the new $D^{\mathrm{St(uni)}}$ condition at q transported from the Khare–Thorne dual convention, and unramifiedness outside $S \cup \{q\}$. Let R^{cl}_F be the classical weight-two quotient of the prepared KT ring just constructed. There is a restriction homomorphism

$$R^{\mathrm{cl}}_F \longrightarrow R^{\dagger}_{K,q}.$$

Here it is essential to verify the type conditions before passing to the global ring. On each original reduced, $\mathcal{O}$-flat local ring $R^{\dagger}_v$, restriction kills the defining ideal of the corresponding prepared local condition: this holds on all finite characteristic-zero points by the choice of the fixed local extensions, and these points are dense in the reduced flat local generic fiber. Thus it holds on the whole local ring, as in [68, Lemma 6.4.1].

The ordinary local ring is the reduced, $\mathcal{O}$-torsion-free image quotient of the framed ring obtained from the flag scheme; it forgets the flag [72, §5.4.1 and Proposition 5.13]. Its classical characteristic-zero points require an ordinary line with trivial inertial character, without marking a particular line. After preparation and dualization each type A or C point has such a line. The same density argument therefore gives the ordinary quotient-ring map on the original local ring. Type B maps to the nonordinary Barsotti–Tate problem. Away from 5, finite-inertia type maps to the unramified problem and Steinberg type to its Steinberg closure. The determinant is identical, and the unramified sign at residual-trivial Steinberg places is forced by the residue. At the new q the local fields are unchanged because q splits in F, so the local restriction map preserves $D^{\mathrm{St(uni)}}$ directly. The resulting local ring homomorphisms remain valid after tensoring with any global ring, including one with 5-torsion. No density of global characteristic-zero points has been assumed.

The projective image over F remains S_5, so the residual restriction is absolutely irreducible. Restriction is finite by [68, Lemma 6.3.1]. Passing to the local quotients preserves finiteness, since their defining ideals have already been shown to vanish. Corollary A.1 therefore makes $R^{\dagger}_{K,q}$ finite over $\mathcal{O}$.

A characteristic-zero point. Put $n = \#(S \cup \{q\})$. The completed tensor product of the local rings has dimension $1 + [K : \mathbf{Q}] + 3n$. The framed presentation in the proof of [68, Proposition 6.2.1] uses $r + n - 1$ variables and $r + [K : \mathbf{Q}]$ relations, so its global quotient has dimension at least $4n$. Forgetting the frames has relative dimension $4n - 1$; hence $\dim R^{\dagger}_{K,q} \geq 1$. The invariant vanishing used in that calculation follows from absolute irreducibility over $K(\zeta_5)$ and Schur's lemma; the exceptional-image exclusion is not used. A finite $\mathcal{O}$-algebra of positive dimension has nonzero generic fibre. A closed generic point gives the required weight-two lift, and its fixed determinant makes it totally odd.

Actual potential modularity and the compatible system. For any prescribed finite extension B/K, apply the avoiding version of [68, Theorem 5.1.1] to $\bar{r}$, including the residual and cyclotomic kernel fields in the avoidance data. It gives a totally real Galois field K'/K, disjoint from the prescribed avoidance field, and a weight-two modular residual witness over K'. The

actual lift $r|_{G_{K'}}$ is finitely ramified and weight two, and its residual restriction to $G_{K'(\zeta_5)}$ is absolutely irreducible. Apply [41, Theorem 1.1], after dualizing, to obtain the automorphy of this actual lift.

The proof of [68, Proposition 9.2.1 and Remark 9.2.2], using solvable descent and Brauer induction, now constructs the strongly compatible system. In this use the proposition's old exceptional exclusion is replaced in its proof by the actual avoiding potential modularity just proved; the remaining argument uses only that input and absolute irreducibility, not an adequacy hypothesis. In particular this is an application of that proof with its modularity input supplied by KT, not an invocation of the original proposition outside its stated hypotheses. □

B Two zero-free half-planes from the Cureton–Thorner construction

B.1 Analytic input and normalization

Let $F \supset \mathbf{Q}(\zeta_{12})$ be a number field and let S be as in [15, §5]. We retain the arithmetic construction, the fixed local profiles, the theta transformation and the Poisson identity of [15, §§7–11], together with the holomorphy and growth estimates used there. We call these the *CT analytic inputs*. They are external inputs throughout this appendix; the changed exceptional-character count and numerical parameters are proved below. The correlation argument below uses the general-scale theta bound and distinct primitive characters. The fixed-scale sextic argument uses the inputs of Section B.4 and allows bounded frequency multiplicity. We record the common normalization first.

For $X = Z^\alpha$, $Y = Z^\beta$, write $I_Z = I(Z^\alpha, Z^\beta, Z)$ and put

$$M = \alpha + \beta, \qquad h = 1 - \alpha, \qquad C = \frac{\alpha}{2} + \frac{h}{6} = \frac{\alpha}{3} + \frac{1}{6}.$$

If $0 < \beta < \alpha < 1$ and $M > 1$, the theta evaluation gives

$$|I_Z| \ll_\epsilon Z^{M-1/2-\beta/2+\epsilon}. \tag{75}$$

The general scale is retained in the proofs of [15, §§9.2, 10.3]; the local profiles remain fixed. On a Poisson contour $(x, y, z) = (\Re\xi, \Re w, \Re z)$ the external power of Z is

$$E(x, y, z) = \frac{\alpha}{2} + x - 1 + hz + \beta(y - 1). \tag{76}$$

The Euler correction is holomorphic, with the stated arbitrarily small conductor loss uniformly in height when the real coordinates range over a fixed compact subregion of the domain

$$\begin{gathered} x, y, z > 0, \quad 6z + y > 1, \quad x + 6z > 1, \quad x + y > 1, \\ x + y + 6z > 2, \quad 6x + 6z > 5, \quad x > \tfrac{1}{2}, \quad 3x > 2, \quad 3x + y > \tfrac{5}{2}. \end{gathered} \tag{77}$$

This is the affine form of [15, (11.2)].

In a frequency dyad $U \le q_u < 2U$, the nonprincipal primitive characters ω_u are distinct, their conductors have norm $O_{F,S}(U)$, and their components at S belong to a fixed finite list. There are $O_{F,S}(U)$ such characters [15, Lemma 7.4 and §12.1]. For a nonprincipal primitive character of conductor norm Q, we use conductor-aspect convexity in the form

$$|L(\sigma + it, \omega, F)| \ll_{F,\epsilon} Q^{(1-\sigma)/2+\epsilon}(2 + |t|)^{A_F}, \qquad 0 \le \sigma \le 1. \tag{78}$$

The exponent A_F is enlarged when necessary to cover products of characters from the fixed list. The Mellin kernels of the profiles have arbitrarily rapid vertical decay on the strips used below.

The principal residues at $w = 1$ and $z = 1/6$ give a nonzero constant multiple of

$$f(Z) = \frac{1}{2\pi i} \int_{(2)} Z^{s-(1-C)} e^{(s-5/6)^2} \frac{H(s)}{\zeta_F^S(s)}\, ds. \tag{79}$$

Here H is the principal Euler correction of [15, (11.3)]; it is holomorphic and nowhere zero in $\Re s > 2/3$, with H and $1/H$ bounded on every closed half-plane strictly inside that region. The shift in (79) is fixed by the residue exponent C. For the correlation parameters below, $f(Z) = Z^{-7/2000} f_{\mathrm{CT}}(Z)$.

If the theta saving is S_θ and the Poisson remainders have saving S_P relative to C, then

$$S_\theta = C - M + \frac{1}{2} + \frac{\beta}{2} = \frac{2}{3} - \frac{2}{3}\alpha - \frac{1}{2}\beta.$$

A fixed family of $O(U^\nu)$ frequency rows, on a contour with $0 < y \le 1$ where the reciprocal denominator is bounded by an arbitrarily small conductor power, contributes

$$\ll_\epsilon Z^{E(x,y,z)} U^{\nu+(1-y)/2-z+\epsilon}. \tag{80}$$

Apply the Poisson identity, the Euler-correction bound and (78) at y. Here $\zeta_F^S(6z)$ is bounded. The Mellin profiles absorb the polynomial growth in $\Im w$ and $\Im z$. The Gaussian decays in $\Im\xi + \Im z$, so its integral against the Mellin decay is rapidly decreasing in $\Im\xi$. This also supplies arbitrary power savings in U on horizontal connecting segments at $|\Im\xi| = U^\rho$ and on tails at larger heights. For $U = Z^u$, $u \in [u_1, u_2]$, its saving is

$$C - E(x,y,z) - \max_{u\in[u_1,u_2]} \left(\nu + \frac{1-y}{2} - z\right) u. \tag{81}$$

The maximum is attained at an endpoint.

Lemma B.1. *Use the principal term $f(Z)$ of (79). Suppose $f(Z) \ll Z^{C-\delta_0}$ for $Z \ge 1$, where $0 < \delta_0 < 1/3$. Then $\zeta_F(s)$ is nonzero for $\Re s > 1 - \delta_0$, away from its pole.*

Proof. For small Z, move the defining line to any fixed real part $R > 2$. Absolute convergence and the decay of the Gaussian give $f(Z) = O_R(Z^{R-1+C})$, so the integral over $0 < Z < 1$ in the Mellin transform is entire. The bound at infinity makes the full transform holomorphic in $\Re s > 1 - \delta_0$. Mellin inversion on $\Re s = 2$ identifies it with

$$\int_0^\infty f(Z) Z^{-s+1-C} \frac{dZ}{Z} = e^{(s-5/6)^2} \frac{H(s)}{\zeta_F^S(s)}.$$

The identity extends meromorphically, and H is nowhere zero in this region. Thus ζ_F^S has no zeros there. Omitted Euler factors have no zeros in the positive half-plane. For arbitrary K one may take $F = K(\zeta_{12})$ and descend by the abelian zeta factorization. □

B.2 The exceptional characters

Write $\kappa > 0$ for the exponent of the mollifier length. Fix $1/2 < \sigma_0 < 1$, $0 < c < 1/2$, and $r' > r$. Let ω have conductor norm $O(U^m)$, with $0 < m \le 1$. For the correlation family $m = 1$ suffices. Extend ω by zero at omitted primes when using an incomplete L-function. Define

$$M_U(s,\omega) = \sum_{\mathrm{N}\,\mathfrak{d} \le U^\kappa} \frac{\mu_F(\mathfrak{d})\omega(\mathfrak{d})}{(\mathrm{N}\,\mathfrak{d})^s}, \qquad a_U(\mathfrak{n}) = \sum_{\substack{\mathfrak{d}\mid\mathfrak{n} \\ \mathrm{N}\,\mathfrak{d} \le U^\kappa}} \mu_F(\mathfrak{d}).$$

Thus $a_U(1) = 1$, $a_U(\mathfrak{n}) = 0$ for $1 < \mathrm{N}\,\mathfrak{n} \le U^\kappa$, and $|a_U(\mathfrak{n})| \ll_\eta (\mathrm{N}\,\mathfrak{n})^\eta$.

Suppose $L(\beta_0+i\gamma,\omega,F)=0$, where $\sigma_0\le\beta_0\le1$ and $|\gamma|\le4U^\rho$. The smoothed zero detector in the proof of [15, Lemma 12.1] is obtained by integrating

$$M_U(\beta_0+i\gamma+s,\omega)L(\beta_0+i\gamma+s,\omega,F)\Gamma(s)U^{rs}.$$

Move the line to $\Re s=c-\beta_0$. This lies between -1 and zero, and the residue at zero vanishes. Convexity and the mollifier bound make the new integral a negative power of U whenever

$$\kappa(1-c)+\frac{m(1-c)}{2}+r(c-\sigma_0)+\eta+\rho A_F<0. \tag{82}$$

The unit coefficient in the exponential sum is then cancelled by the nonunit terms. Terms of norm exceeding $U^{r'}$ are negligible, since $r'>r$. Dyadic decomposition and partial summation give some $U^\kappa/2<N\le U^{r'}$ for which

$$\max_{N\le y\le2N}\left|\sum_{N<\mathrm{N}\,\mathfrak{n}\le y}a_U(\mathfrak{n})\omega(\mathfrak{n})(\mathrm{N}\,\mathfrak{n})^{-\sigma_0-i\gamma}\right|\gg(\log U)^{-1}. \tag{83}$$

The estimate also applies to the incomplete L-function: omitting the fixed Euler factors costs only the indicated small loss. A second partial summation can remove $(\mathrm{N}\,\mathfrak{n})^{-i\gamma}$ from the prefix, at a cost $O(1+U^\rho)$. We use that version in the sextic count.

Lemma B.2. *Assume* (82) *and choose $\eta,\epsilon>0$ so that*

$$\kappa(1-2\sigma_0+2\eta+\epsilon)+1+2\epsilon+\rho A_F<0,\qquad 1-2\sigma_0+2\eta+\epsilon<0. \tag{84}$$

The number of characters in the frequency dyad with a zero in $\Re s\ge\sigma_0$, $|\Im s|\le4U^\rho$, is

$$\ll_{F,S,\eta,\epsilon}U^{r'(2-2\sigma_0)+2\eta r'}(\log U)^6.$$

Proof. Choose one zero and one detecting dyad for each exceptional character. Fix a dyad N and order its ideals by norm. Pad the list to length 2^L with zero coefficients, where $L+1\ll\log U$, and let $\mathcal{B}_N$ be its full tree of aligned binary blocks. Each prefix is a union of at most $L+1$ blocks. By (83), every character assigned to N has a block on which the sum has modulus at least $V\asymp(\log U)^{-2}$. Assign it to one such block.

For $B\in\mathcal{B}_N$, let R_B be the number assigned to it, and put

$$G_B=\sum_{\mathfrak{n}\in B}|a_U(\mathfrak{n})|^2(\mathrm{N}\,\mathfrak{n})^{-2\sigma_0},\qquad G_N\ll N^{1-2\sigma_0+2\eta}$$

for the corresponding mass on the whole dyad. Choose a fixed smooth weight $w(\mathfrak{n})=w_0(\mathrm{N}\,\mathfrak{n}/N)$, equal to one on $[1,2]$ and supported in $[1/2,4]$, with $0\le w_0\le1$. Cauchy–Schwarz, after choosing phases to align the detected sums, gives

$$R_B^2V^2\le G_B(R_BD_N+R_B^2C_N),\qquad D_N\ll_F N, \tag{85}$$

where C_N bounds the off-diagonal correlations

$$\sum_{\mathfrak{n}}w(\mathfrak{n})(\omega_j\overline{\omega_k})(\mathfrak{n})(\mathrm{N}\,\mathfrak{n})^{-i(\gamma_j-\gamma_k)}.$$

The majorant is one on every block, so the estimate is uniform in B.

For $j\ne k$, the underlying product character is nonprincipal, with conductor norm $O(U^2)$. Mellin inversion on $\Re z=\epsilon$ and (78) give a factor $U^{1+\epsilon}$. The omitted Euler factors cost a further U^ϵ; the Mellin scale costs N^ϵ. Since $|\gamma_j-\gamma_k|\le8U^\rho$, rapid Mellin decay gives

$$C_N\ll N^\epsilon U^{1+2\epsilon+\rho A_F}.$$

There is no pole in this shift. By (84) and $N > U^{\kappa}/2$, uniformly in B,

$$G_B C_N \le G_N C_N \ll U^{\kappa(1-2\sigma_0+2\eta+\epsilon)+1+2\epsilon+\rho A_F} \le \frac{1}{2}V^2$$

for all sufficiently large U. Absorbing this term in (85) yields $R_B \ll N G_B V^{-2}$.

The blocks at each binary level partition the list. Therefore

$$\sum_{B\in\mathcal{B}_N} G_B = (L+1)G_N, \qquad |E_N| = \sum_B R_B \ll N^{2-2\sigma_0+2\eta}(\log U)^5.$$

Summing over the $O(\log U)$ norm dyads proves the assertion. Distinctness of the primitive characters transfers the same bound to the frequencies. □

B.3 The parameters and the zero-free half-plane

Take

$$\begin{gathered}\alpha = \frac{1319}{2000}, \quad \beta = \frac{377}{1000}, \quad \sigma_0 = \frac{427}{500}, \quad \kappa = \frac{7101}{5000}, \\ r = \frac{22545}{10000}, \quad r' = \frac{22596}{10000}, \quad c = \frac{1}{100}.\end{gathered} \tag{86}$$

Then $M = 2073/2000 > 1$, $h = 681/2000$, and $C = 773/2000$. Fix A_F as above and choose

$$\eta = \epsilon = 10^{-5}, \qquad \rho = \frac{1}{20000(A_F+1)}.$$

Even with the coarser allowance $\rho A_F \le 10^{-3}$, the left sides of (82) and the first inequality in (84) are at most -0.00079 and -0.004438994. Also $r' - r = 51/10000 > 0$. Absorb $(\log U)^6$ into $U^{1/10000}$. Lemma B.2 gives

$$|E(U)| \ll U^{\nu}, \qquad \nu = \frac{206251}{312500} = 0.6600032, \tag{87}$$

since $r'(2-2\sigma_0) + 2\eta r' + 1/10000 = 0.659948392 < \nu$. These choices fix the exceptional sets before any later contour loss is chosen.

Put $\delta = 10^{-7}$, $x_1 = 1+\delta$, $x_g = \sigma_0 + 10^{-4}$, and $u_{\max} = h + 10^{-4} = 0.3406$. Use the contours

contribution	(x, y, z)
small and exceptional frequencies	$(x_1, 1/1000, 2/5)$
good intermediate frequencies	$(x_g, 1-\sigma_0, 2/5)$
principal w remainder	$(x_1, 1/2, 1/6+\delta)$
principal z remainder	$(x_1, 1, 1/50)$.

The small range is $0 \le u \le 1/5$, the intermediate range is $1/5 \le u \le u_{\max}$, and $U = Z^u$. All corners lie in (77); the smallest good-corner margin is 10^{-4}. The affine inequalities also hold along the coordinate moves.

For the good characters, the analytic-log argument of [15, Lemma 12.1] gives $1/L \ll_{\varepsilon} U^{\varepsilon}$ on the finite contour through x_g. To make its geometry explicit, use disks centered at $2+it$, $|t| \le 2U^{\rho}$, with radii $0.999 < 1.1459 < 1.145925 < 1.14595$. The outer left edge is $0.85405 > \sigma_0$, the target reaches $x_g = 0.8541$, and the outer disk stays below height $4U^{\rho}$. The inner disk lies in absolute convergence. Borel–Carathéodory and three circles give the required arbitrarily small conductor loss. The decay of the fixed kernels handles the horizontal and original vertical tails after ρ has been fixed.

The principal shifts pass through $(x_1, x_1, 1/6+\delta)$, cross $w = 1$, and within that residue cross $z = 1/6$. The denominator remains in absolute convergence and the Mellin kernels are

holomorphic in the positive z strips used. The nonprincipal contours stop at $z = 2/5$, so no $z = 1/6$ pole is crossed. For $U > Z^{u_{\max}}$, keep $x = y = 4$ and move z to 50000; the external exponent is

$$E(4, 4, 50000) + u_{\max}(1 - 50000) = -\frac{3973}{20000} = -0.19865.$$

The negative frequency power makes the large dyads summable.

Proposition B.3. *With* (86) *and* (87)*, the exact savings, including the fixed δ shifts, are as follows.*

Contribution	*Exact saving*	*Decimal*
Theta	77/2000	0.0385
Principal w	3769997319/20000000000	0.18849986595
Principal z	499399/10000000	0.0499399
Small	772729/10000000	0.0772729
Exceptional intermediate	60134547/1562500000	0.03848611008
Good intermediate	193059/5000000	0.0386118
Large	11703/20000	0.58515

In particular $S_P = 60134547/1562500000 < S_\theta$.

Proof. Substitute in (76) and (81). The frequency coefficient is positive in each of the small, exceptional and good ranges, so the worse endpoint is respectively $1/5$, $u_{\max}$ and $u_{\max}$. The principal corrections are $-(1 + h)\delta$ in the w row and $-\delta$ in the z row. The other bad rows also retain $-\delta$. These are finite rational calculations. □

Theorem B.4. *Assume the CT analytic inputs of Section B.1 for the number fields used there. For every number field K,*

$$\zeta_K(s) \neq 0 \qquad (\Re s > 0.9618,\ s \neq 1).$$

Every finite-order Hecke L-function over K is nonzero in that half-plane wherever it is holomorphic.

Proof. The count (87) and the contour comparison give

$$I_Z = c_0 f(Z) + O(Z^{C - S_P + 1/4000}), \qquad c_0 \neq 0.$$

All remaining contour losses are chosen with total below $1/4000$; the fixed exceptional sets do not change. Since

$$S_P - \frac{1}{4000} = 0.03823611008 > \frac{191}{5000}, \qquad S_\theta - \frac{1}{10000} = 0.0384 > \frac{191}{5000},$$

equation (75), with its loss fixed below $1/10000$, yields $f(Z) \ll Z^{C - 191/5000} = Z^{3483/10000}$.

Lemma B.1 gives the zeta assertion. For a finite-order Hecke character, apply abelian factorization to the associated abelian extension. At $s = 1$ the ratio of the two simple zeta residues is nonzero, so every nontrivial Hecke factor is nonzero there as well. □

B.4 Cureton–Thorner inputs and normalization

Let $F \supseteq \mathbf{Q}(\zeta_{12})$. We use the arithmetic construction and fixed local profiles of [15, §§7–11]. Write S for its finite set of places, $R \subset F_S$ for its lattice, Γ for its unit group, and $\ell_T \in F_S^\times$ for the chosen element with $|\ell_T|_S = T$. Let $B_m(Z)$ be the weighted cubic theta coefficient of [15, (5.3)], and use the profiles W_0, W_1 and Mellin transform $M(z)$ of [15, (5.5)]. For $X, Y, Z > 0$ the common average is

$$I(X,Y,Z) = \int_{F_S^1/\Gamma} \frac{1}{Y} \sum_{\mathfrak{s}} W_1(q_{\mathfrak{s}}/Y) \frac{1}{\sqrt{q_{\mathfrak{s}} X}} \sum_{m \in R \setminus \{0\}} q_{\mathfrak{s}}^{-1/2} g_{\mathfrak{s}}(-m) W_0\left(\frac{m}{t \ell_{q_{\mathfrak{s}} X}}\right) B_m(Z)\, dt,$$

with the Gauss sums and the indexing of [15, (5.6)]. The following are the analytic inputs.

(CT1) At the fixed scale $M_0 = 419/400$, for a fixed compact $B \subset F_S^\times$, the theta transformation gives

$$\sum_{m \in R \cap \ell_{Z^{M_0}} B} |B_m(Z)|^2 \ll_{\varepsilon,F,S,B} Z^{2M_0 - 1 + \varepsilon}.$$

The bound is uniform after multiplication of B by elements of a fixed compact subset of F_S^1 [15, Proposition 9.1].

(CT2) For distinct reduced fractions $d/\mathfrak{s} \bmod R$ of denominator norm $q_{\mathfrak{s}} \le V$, the additive large-sieve operator on $R \cap \ell_N B$ has squared operator norm $O_{F,S,B}(N + V^2)$, with the same bound for its adjoint [15, (10.2)]. We use the smooth separation of the fixed weight W_0 in the denominator scale in [15, §10.3].

(CT3) The Poisson formula [15, (11.1)], initially on the real lines $(4, 4, 2)$, is

$$\begin{aligned} I(X,Y,Z) &= \frac{1}{(2\pi i)^3} \iiint X^{1/2-z} Z^{\xi+z-1} Y^{w-1} e^{(\xi+z-1)^2} M(z) \widehat{W}_1(w) \\ &\quad \cdot \sum_{u \in \mathcal{U}} q_u^{-z} F_u(\xi, w, z)\, dz\, dw\, d\xi, \\ F_u(\xi, w, z) &= \frac{\zeta_F^S(6z) L^S(w, \chi_u)}{L^S(\xi, \chi_u)} H_u(\xi, w, z). \end{aligned}$$

Here $\mathcal{U}$ is the Kummer frequency family of [15, Lemma 7.4], $q_u = |u|_S$, and χ_u has the Euler factors of that formula. The Euler correction H_u is holomorphic in the domain

$$\begin{gathered} x, y, z_0 > 0, \quad 6z_0 + y > 1, \quad x + 6z_0 > 1, \quad x + y > 1, \\ x + y + 6z_0 > 2, \quad 6x + 6z_0 > 5, \quad x > \tfrac{1}{2}, \quad 3x > 2, \quad 3x + y > \tfrac{5}{2}, \end{gathered} \tag{88}$$

where $(x, y, z_0) = (\Re\xi, \Re w, \Re z)$. On each fixed compact subregion of these real coordinates it is $O_\varepsilon(q_u^\varepsilon)$ uniformly in height [15, (11.2)]. For $u = 1$, the character is principal and $H(s) = H_1(s, 1, 1/6)$ is holomorphic and nowhere zero for $\Re s > 2/3$, with H and $1/H$ bounded on closed half-planes strictly inside that region [15, (11.3) and Lemma 11.1]. For $u \ne 1$ the inducing primitive characters are nonprincipal. In $U \le q_u < 2U$ they have conductor norm $O_{F,S}(U)$ and occur with bounded multiplicity. The exterior ideal of u has norm q_u and valuations in $\{0, \dots, 5\}$; local components at S belong to a fixed finite list [15, Lemmas 7.2 and 7.4 and §12.1]. There are $O_{F,S}(U)$ frequencies in this dyad by ideal counting.

(CT4) The Mellin profiles $\widehat{W}_1(w), M(z)$ have arbitrarily rapid vertical decay on the fixed strips used below, with $\widehat{W}_1$ entire and M holomorphic for $\Re z > 0$. We use $M(1/6)\widehat{W}_1(1) \ne 0$ [15, §§10.2 and 13.1].

(CT5) For a nonprincipal primitive Hecke character of conductor norm Q, conductor convexity gives

$$|L(\sigma + it, \chi)| \ll_{\varepsilon,F} Q^{(1-\sigma)/2+\varepsilon} (2 + |t|)^{A_F}, \qquad 0 \le \sigma \le 1,$$

as in [15, §12.1]. Omitted Euler factors and their reciprocals cost $O_{\varepsilon,\sigma_*}(Q_{\mathrm{om}}^{\varepsilon})$ on each $\Re s \geq \sigma_* > 0$, where Q_{om} is the norm of the product of the omitted primes. This follows by bounding each factor by a constant and using the elementary bound on the number of distinct prime factors of that product. For the Poisson family, Q_{om} is bounded by a fixed power of U.

For $X = Z^{\alpha}$, $Y = Z^{\beta}$ with $\alpha + \beta = M_0$ and $Y \leq X$, put $I_Z = I(X, Y, Z)$. On bounded representatives of the compact quotient, the additive factor is supported in $R \cap \ell_{XY} B'$, for a fixed compact B'. After expanding the Gauss sums its fraction coefficients have size $O(Y^{-1} q_{\mathfrak{s}}^{-1} X^{-1/2})$. There are $O_F(Y)$ denominators of norm comparable to Y and at most $q_{\mathfrak{s}}$ residues for each. The sum of coefficient squares is $O(X^{-1}Y^{-2})$. Separating the smooth weights and using (CT2), the additive factor has squared norm

$$\ll (XY + Y^2) X^{-1} Y^{-2} \ll Y^{-1}.$$

Cauchy–Schwarz and (CT1) therefore give

$$|I_Z| \ll_{\varepsilon} Z^{M_0 - 1/2 - \beta/2 + \varepsilon}. \tag{89}$$

At this fixed scale apply (80) and (81) under (CT3)–(CT5).

B.5 The sextic exceptional-character bound

Lemma B.5. *Put $\sigma_0 = 167/200$. For sufficiently small fixed $\rho > 0$, the number of characters in this frequency dyad with a zero in $\Re s \geq \sigma_0$, $|\operatorname{Im} s| \leq 4U^{\rho}$, is $O_{F,S}(U^{67/100})$.*

Proof. Enlarge the finite set of omitted primes in the detector to meet the hypotheses of the sextic Hecke large sieve [85, Theorem 1.3]. The zeros in $\Re s > 0$ are unchanged by removing these Euler factors. The finitely many extra valuations lie in $\{0, \dots, 5\}$, and may be absorbed into the finite list of twists. All ideal norm comparisons below allow constants depending on the field and these fixed primes. We work with ideals prime to the enlarged set.

Write the exterior ideal of u as $\mathfrak{a}\mathfrak{b}$, where $\mathfrak{a}$ is squarefree and $\mathfrak{b}$ is squarefull, with disjoint support. Group by dyadic ranges $\mathrm{N}\,\mathfrak{b} \asymp B = U^{v}$ and hence $\mathrm{N}\,\mathfrak{a} \ll A = U/B$. There are $O_F(B^{1/2})$ possible $\mathfrak{b}$, since a squarefull ideal is the product of a square and the cube of a squarefree ideal. The conductor norm of the primitive character is $O_{F,S}(U/B^{1/2})$. Outside the fixed set every prime in its conductor occurs to exponent one, and $\mathrm{N}\operatorname{rad}(\mathfrak{b}) \leq (\mathrm{N}\,\mathfrak{b})^{1/2}$. For fixed $\mathfrak{b}$, the primitive character on exterior ideals is the product of the family character $\chi_{\mathfrak{a}}$ of [85], the character $\chi_{\mathfrak{b}}$, and a twist from a fixed finite list. Indeed these characters have the same tame inertia at all exterior primes by the sextic symbol and the construction in [85, Lemma 2.1 and Appendix II]. Their quotient is of order dividing six and unramified outside the fixed set. There are finitely many such characters. On ideals meeting the exterior conductor, the products have the same zero extension. Each exterior ideal has bounded multiplicity by the generator choice of [15, Lemmas 7.2 and 7.4]. For $v > 0.67$, counting the ideals directly gives $O(U^{1-v/2}) = O(U^{0.665})$ in each dyadic range.

For the other ranges use (82) at $c = 1/100$. Use the following constants, taking $r' > r$ for truncation.

v	m	κ	r	r'	exponent before losses
$0 \leq v \leq 1/2$	1	0.51	1.22	1.222	-0.0066
$1/2 < v \leq 0.67$	3/4	0.15	0.65	0.652	-0.0165

Choose the losses small enough to preserve negativity. Use (83) with the oscillatory factor removed, which leaves a maximal prefix of size $\gg U^{-\rho}/\log U$.

For fixed $\mathfrak{b}$ and fixed twist, the squared ℓ^2 norm over the squarefree indices $\mathfrak{a}$ of these maximal prefix sums is

$$\ll_\varepsilon (AN)^\varepsilon\big(A+N+(AN)^{2/3}\big)N^{1-2\sigma_0}.$$

Here the variable twist $\chi_\mathfrak{b}$ multiplies the coefficients, which have the same divisor bounds. To see the estimate with unrestricted detecting ideals, write $\mathfrak{n}=\mathfrak{n}_1\mathfrak{n}_2$ with a squarefree and a squarefull part. For fixed $\mathfrak{n}_2$ the norm cutoff on $\mathfrak{n}_1$ is the original cutoff divided by $\mathrm{N}\,\mathfrak{n}_2$. The factor $\chi_\mathfrak{a}(\mathfrak{n}_2)$ has absolute value at most one. Use the squarefree large sieve on $\mathfrak{n}_1$ with bound $2N/\mathrm{N}\,\mathfrak{n}_2$. For a maximal prefix, write prefixes as unions of disjoint binary blocks. At each level the coefficient square masses sum to their total mass, so Cauchy–Schwarz and summation over levels cost only logarithms. Minkowski then sums the square roots of the bounds over $\mathfrak{n}_2$. Factoring out the square root of the displayed bound leaves at most $\sum_{\mathrm{N}\,\mathfrak{n}_2\le 2N}(\mathrm{N}\,\mathfrak{n}_2)^{-1/2}$ over squarefull ideals, which is $O(\log N)$. This proves the estimate, with logarithms absorbed in ε.

Multiply this estimate by $O(B^{1/2})$ and count detected characters using the prefix lower bound. With $t=\log N/\log U$, the three exponents before the small losses are

$$1-\frac{v}{2}+(1-2\sigma_0)t,\qquad \frac{v}{2}+(2-2\sigma_0)t,\qquad \frac{2}{3}-\frac{v}{6}+(5/3-2\sigma_0)t.$$

For the first range these are at most 0.6583, 0.65326, and 0.664967. For the second they are at most 0.6495, 0.55016, and 0.582834. The dyadic summations, the height loss, and the arbitrary ε losses can be absorbed in the strict margin to 0.67. □

B.6 The contour comparison and zero-free region

Theorem B.6. *Under the Cureton–Thorner analytic inputs of §B.4, every Dedekind zeta function is nonzero in $\Re s>0.9586$, apart from its pole at one.*

Proof. In the notation of [15, §§5,9–12] take

$$\alpha=0.608,\quad \beta=0.4395,\quad h=1-\alpha=0.392,\quad C=\alpha/2+h/6.$$

Here $\alpha+\beta=419/400$ and $\beta<\alpha$. The estimate (89) is

$$|I_Z|\ll_\varepsilon Z^{C-S_\theta+\varepsilon},\qquad S_\theta=\frac{2}{3}-\frac{2}{3}\alpha-\frac{1}{2}\beta=\frac{499}{12000}.$$

For the nonprincipal frequencies we apply (80).

Put $\delta=10^{-7}$, $x_1=1+\delta$, $x_g=0.83502$ and $u_{\max}=0.39202$. Use $(x_1,0.001,0.4)$ for the nonprincipal small and exceptional frequencies, with $0\le u\le 1/5$ and $1/5\le u\le u_{\max}$, respectively. The count for the small frequencies is $O(U)$ and Lemma B.5 gives $O(U^{0.67})$ for the exceptional frequencies. The remaining intermediate frequencies use $(x_g,0.165,0.4)$ and the bound $O(U)$. These corners are in the domain (88); in particular $x_g+0.165=1.00002$. The intermediate corners $(x,4,2)$ and $(x,4,2/5)$, for $x=x_1$ or x_g, are also in the domain. Its convexity then keeps all the coordinate moves inside with fixed positive margins. For the exceptional and small frequencies the denominator stays in absolute convergence. For the good frequencies Lemma B.5 gives the zero-free rectangle used in the analytic-log argument of [15, Lemma 12.1]. One can use disks centered at $2+it$ for $|t|\le 2U^\rho$, with radii $0.999<1.16498<1.16499<1.164995$. The outer disk stays to the right of σ_0 and within the rectangle. The Euler-product logarithm is bounded on the innermost disk, and convexity and Borel–Carathéodory bound the logarithm by $O(\log U)$ on the disk of radius 1.16499. Hadamard's three-circles theorem gives $O((\log U)^\theta)$ on the disk of radius 1.16498, with fixed $\theta<1$. This disk reaches $\Re s=x_g$. Exponentiation bounds the reciprocal by $O_\varepsilon(U^\varepsilon)$ on the target contour

and its horizontal connections. The same bound holds for the imprimitive reciprocal by (CT5). Rapid Mellin decay handles the horizontal and vertical tails as in the cited comparison.

For the principal frequency first move ξ to x_1. Keep $z_0 = 1/6 + \delta$ while moving w to $1/2$, crossing $w = 1$. In the residue then move z_0 to $1/20$, crossing $z = 1/6$. The intermediate corner $(x_1, 4, 1/6 + \delta)$ and the final remainder corners $(x_1, 1/2, 1/6 + \delta)$, $(x_1, 1, 1/20)$ satisfy (88). The reciprocals on $\Re\xi = x_1$ are bounded by absolute convergence. On the nonprincipal contours the entire numerator $L^S(w, \chi_u)$ and $6z_0 > 1$ give no poles in w or z; the reciprocal bound controls the only moves of ξ to the left of one.

For $u > u_{\max}$ use $(4, 4, 500000)$. The exponent at $u_{\max}$ is

$$E(4, 4, 500000) + u_{\max}(1 - 500000) = -4.98548.$$

The dyads sum geometrically, because the coefficient of u is negative. Here we use absolute convergence in w, ξ, the $O(U)$ frequency count, and (CT3)–(CT4). The following lower bounds for the savings relative to C come from the indicated formulas, before the arbitrarily small analytic losses.

Contribution	Saving at least
Theta	0.0415833
Small	0.1276937
Exceptional intermediate	0.0459343
Good intermediate	0.0416154
Principal w remainder	0.2197498
Principal z remainder	0.0457332
Large	5

For the three intermediate and small rows the frequency coefficient is positive, so the upper endpoint gives the lower bound for the saving. Choose the total losses below 10^{-4}. The theta bound and the Poisson identity then give $f(Z) \ll Z^{C-0.0414}$ for the principal term (79). The two principal residues give $c_0 f(Z)$ with

$$c_0 = \tfrac{1}{6}\big(\operatorname*{Res}_{s=1} \zeta_F^S(s)\big)^2 M(1/6)\widehat{W}_1(1) \neq 0.$$

Lemma B.1 gives the region and its descent from $F = K(\zeta_{12})$ to K. □

C Numerical comparisons and finite certificates

C.1 The comparisons for $\mathbf{Q}(\sqrt{6})$

We use (10) with the real-place term retained. For $G = F_b/\cosh(Ax)$, $0 < A \leq 1/2$, its contribution to $\log \mathrm{rd}$ is at least $(r_1/n)(1 - \pi^4/(24b^2))$. Indeed $G \geq F_b/\cosh(x/2)$, and the classical estimate in this appendix applies. The Euler coefficients in Lemma 19.12 have the normalization $c_P = 2N_P/n$. We keep up to twelve powers of each specified norm in the support of the test function.

For the narrower strip put $a = 2293/5000$. Use Lemma 13.7 with $(M, r, s) = (60, 10, 40)$. For B_0 use the finite expansion of (16) with $R = 500$ and final alternating index 80, evaluated at this a. The classical rows use $N = 100$ in (13). The displayed decimals are outward bounds. In a row for L, decreasing e at fixed f lowers the upper bound and increases the retained Euler

coefficients. In a range of q, the lower bound uses the smallest degree.

Field	Range or conditions	b	a_0	Lower	Upper
L	$q = 4,\ f = 2$	5	1/2	20.8677	13.5186
L	$q = 4,\ f = 1$	5	1/2	22.0885	17.8379
L	$q = 8,\ f = 2$	8	1/2	25.6785	20.5922
L	$q = 8,\ f = 1$	8	1/2	26.5040	22.2408
L	$q = 16,\ f = 1$	12	1/2	28.0247	24.3108
L	$q = 16,\ f = 2$	12	1/2	25.9359	24.3108
L	$q = 32,\ f = 2$	20	1/2	25.8819	25.5200
L	$q = 64,\ f = 2$	30	0.4586	27.1375	26.1763
L	$q = 128,\ f = 2$	45	0.4586	26.9654	26.5186
L	$q = 256,\ f = 2$	60	0.4586	26.8955	26.6934
MH	$q \geq 16,\ e \leq 4$	25	1/2	25.9059	24.6406
M	$q \geq 32,\ f = 1$	43	1/2	27.1019	26.8708
M	$q \geq 64,\ f_v = 1$	50	1/2	31.5370	26.8708
M	$q \geq 64,\ e_v = 1$	50	1/2	22.3292	17.9696
MV	$q \geq 512,\ e = 8,\ f = 2$	150	0.4586	26.8754	26.8708

C.2 The comparisons for $\mathbf{Q}(\sqrt{41})$

For Lemma 18.1 all rows use $A = 1/2$ and the classical integrals in (14). The notation is M for the kernel field and $L = M^U$ for the Sylow subfield. The first row retains no Euler or real-place terms. The next two use real fraction at least $r_0 = 1/(q + 1)$, coefficients $2r_0$ and $(1 - r_0)/4$ at norms two and four, and six and five prime powers respectively. The last row retains six norm-four powers with coefficient $1/4$. For ranges of q the lower bound is evaluated at the smallest degree.

Case	Field	b	Lower	Upper
Tame or unramified at one place, $q \geq 4$	M	8	16.82	16.135
Both elementary, $q = 4$	L	9/2	21.785	19.411
Both elementary, $q = 8$	L	7	22.80	21.957
Both elementary, $q \geq 16$	M	30	26.66	25.613

The certificate uses ninety terms of (14) with the stated tail bound and directed intervals.

C.3 Arithmetic and interval certificates

The ray arguments for $d = 2, 5, 13$ use Minkowski's bound and principal ideal relations. The same method checks the two families over $\mathbf{Q}(\sqrt{6})$, giving 51 quartic cases in all. The certificates in [87] give these checks and the arithmetic of the quadratic base fields. For each quartic the program enumerates the primes up to a rational upper bound for the Minkowski cutoff. Factoring principal ideals gives an integer relation lattice. In the $d = 6$ two-special-place calculation we also impose relations killing the prescribed prime classes. The resulting lattice index bounds the order of the class group quotient. Its odd part is trivial except that it can divide three for the field identified in Lemma 19.11. For the local term in the ray sequence, it computes the finite group $(\mathcal{O}_L/(2))^\times$ and the images of explicit units of absolute norm one. The quotient by these images has two-power order. This suffices for the ordinary ray group, and adding real places only introduces factors of order two.

A script verifies the displayed discriminant comparisons for $d = 2, 5, 13, 17, 33$ using exact rationals. It uses the alternating series for arctangent, logarithmic series with geometric tail bounds, and Taylor bounds for exponential, sine and cosine. The Euler constant is bounded below by $H_n - \log n - 1/(2n)$ with $n = 2048$, using the convex trapezoidal estimate for $1/x$. Root-discriminant upper bounds are checked by integer powers. The directed interval programs for $d = 6$ and $d = 41$ evaluate the bounds of Sections C.1 and C.2, respectively. The exponent inequalities in the proof of Theorem B.6 are checked by exact rational arithmetic. The same archive contains the certificate for the CM comparison at five.

D Depth-four normal ray calculation

We give the matrices used in Proposition 24.8. The fields K, A, the standard module W_0 and the binary ray coordinates are those in its proof. For the depth-four calculation write $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on W_0. In the seven binary ray coordinates the class killed by $[\mathfrak{q}_+]$ is $v = (1, 0, 0, 1, 1, 0, 1)^t$. The source matrices for the actions of S, U in those coordinates are

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The solution space to $Xv = 0$, $XC_2 = SX$, $XC_3 = UX$ has basis

$$X_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Chinese remainder lifts of local units, chosen to be one at the other dyadic prime, have binary ray logarithm columns

$$g = (0, 1, 1, 0, 0, 0, 0)^t, \quad h = (0, 0, 0, 1, 0, 0, 0)^t, \quad j = (0, 0, 0, 1, 0, 1, 1)^t.$$

The first prime supplies g, h and the second supplies g, j. For $\alpha^2 + \alpha + 1 = 0$, evaluation of the two equivariant maps followed by the left U-eigenrow $(1, \alpha)$ gives, respectively,

$$\begin{pmatrix} 0 & 1+\alpha \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \alpha \\ 1 & 1+\alpha \end{pmatrix}.$$

Both determinants are nonzero, and the conjugate eigenrow has the same property. Hence vanishing on local units in either one of the two C_3 eigencharacters, at either prime, kills the global depth-four class. These conclusions concern the characteristic-two normal error; the order of tame inertia after reduction in the joint trace order is checked separately. The matrices and local logarithms are reproduced in [52, `ramified/d6_d10_d11/d10`].

E Auxiliary fields over the field of discriminant twenty-one

The unit assertions involve only a number field of degree twelve. One defining polynomial for A is

$$P(X) = X^{12} - 3X^{11} + 8X^{10} - 9X^9 + 9X^8 + 6X^7 + X^6 \\ + 6X^5 + 9X^4 - 9X^3 + 8X^2 - 3X + 1.$$

The field defined by P has discriminant $21^6 5^8$ and signature $(0, 6)$. The certificate reconstructs the embeddings of u and z, checks their equations and the norm-one relations, and tests the sixteen products at the two dyadic primes. Integrality and unit status follow directly from their monic minimal polynomials with constant coefficient ± 1. It uses only `nfinit`, exact field arithmetic, Galois conjugates and local-power tests; no computation of a full unit group or class group is used in the proof. The exact calculation is supplied in [52, `d21`].

The arithmetic and interval certificates are:

- `check_small_unit_formulas.gp`: the two-orbit unit identities and local squareclass kernels in Lemma 29.8;
- `verify_split5_local.py`: the two compact-type Brauer-character identities, calculated in $\mathbf{Z}[\zeta_{24}]$;
- `certify_K21_SC_SC.py`: real-ball evaluations of (11), with each comparison asserted only when its lower and upper balls are disjoint;
- `K21_SC_SC_arb_certificate.json`: the full-precision interval enclosures for Table 3.

The smallest gap, in the distinct-conjugate row, is greater than 0.00974417509. The analytic calculation certifies the displayed inequalities; the group and local hypotheses are proved in the preceding sections. The exact calculation is supplied in [52, `d21/inherited_python`, `d21/split5`].

F Normal Selmer calculations over the field of discriminant twenty-one

The certificate `d21_point_field_selmer.gp` in [52, `audit_simplifications`] computes the two normal spaces of Section 29.2.2 in the cubic point fields. Every class and unit computation has absolute degree six. The script verifies the embeddings into the original residual fields, certifies class number one, and constructs the full S-unit groups for S above two and five. Applying $[u] \mapsto [u/N_{L/K}(u)]$ gives the complete norm kernels. Its rank checks are three and two, respectively.

The script prints its generators and enumerates their eight or four products. Before computing a quadratic relative discriminant it multiplies the radicand by a rational square to make every polynomial coefficient integral. The dyadic discriminant exponent is the quadratic conductor. The mixed conductor-at-most-four subspace has dimension one, and its nonzero element is unramified at the degree-one special prime with Hilbert symbol $(u, 5) = -1$. The second norm kernel has conductor six on every nonzero class. These are basis-independent assertions, checked explicitly by the script. Its successful output contains two `BNF_CERTIFY 1` markers and `STATUS SUCCESS_DEGREE_SIX_NORMAL_SELMER`.

The scripts in [52, `d21/arithmetic`] also verify the elementary quartic ray classification. The nonsolvable source branch uses the directed discriminant comparisons in `certify_K21_SC_SC.py` and `K21_SC_SC_arb_certificate.json` in [52, `d21/split5`]. The normal Selmer computations above have no analytic hypothesis.

G The scalar trace generator over $\mathbf{Q}(\sqrt{14})$

Proposition G.1. *Let $\mathcal{R}$ be a system of weight two and cyclotomic determinant over $K = \mathbf{Q}(\sqrt{14})$, good outside five and special at both five-places. Suppose its dyadic constituents are nonordinary crystalline and have scalar residual semisimplification. In the simultaneous orders*

of Section 11 one has

$$R_0 = \mathbf{Z}_2[a_2] = \mathbf{Z}_2[a_7], \qquad R_0/2R_0 \simeq \mathbf{F}_2[t]/(t^n), \quad n = \operatorname{rank}_{\mathbf{Z}_2} R_0, \quad t = \overline{a_2}.$$

Proof. Put $R = R_0/2R_0$, $E = E_0/2E_0$, and $J = \operatorname{rad}(E)$. Use Lemma 11.6 with $m = -2$, so the Frobenius generator Φ satisfies

$$\mathcal{D}(\Phi(u)) \in (u^2 + 2)R_0[[u]], \qquad T(\Phi(0)) = a_2.$$

Every functional on J/J^2 gives a middle selfextension of the regular group. Lemmas 11.2 and 26.1 put its character in $\langle \alpha, \beta, \gamma \rangle = \langle [-1], [2], [5] \rangle$. Choose a real involution, a Frobenius at one fixed five-place acting trivially on [5], and a tame inertia generator there. Their evaluation vectors on (α, β, γ) are

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1).$$

Their differences from one, denoted A, B, C, span J/J^2 and therefore generate E.

At the selected five-place the common exact special sign can be normalized simultaneously. The normalized local differences satisfy $L(g)L(h) = (\chi_2(g) - 1)L(h)$ in the actual integral order. Modulo two their products and traces are zero. The real involution and polarized Cayley–Hamilton give

$$\begin{gathered} A^2 = B^2 = C^2 = BC = CB = 0, \\ AB + BA = X = T(AB), \qquad AC + CA = Y = T(AC). \end{gathered} \tag{90}$$

All words reduce to the R-span of $1, A, B, C, AB, AC$.

Work in $R' = R/\mathfrak{m}_R^2$, retaining the original trace and norm laws. The trace cotangent is generated by X, Y. At two the ramified functional on (α, β, γ) is $(1, 1, 0)$, while $\gamma = [5]$ is the unramified character. Consequently the regular Kisin generator has

$$\begin{gathered} \Phi = N + u(1 + M) + O(u^2), \qquad N = A + B + Z, \\ M = m_A A + m_B B + m_C C + Z', \qquad Z, Z' \in J^2, \qquad m_C = 1. \end{gathered}$$

These coefficients are obtained by restricting the ramified and unramified functionals on the full allowed character space; this also applies when the actual first layer is a proper quotient.

The six-word relations show $J^2 \subset \mathfrak{m}_R E + RAB + RAC$. Terms in $\mathfrak{m}_R E$ contribute neither norm nor polar trace, since $\mathfrak{m}_R^2 = 0$ and all traces lie in $\mathfrak{m}_R$. The norms of AB, AC vanish, their mutual polar term is $XY - T(ABAC) = 0$, and their trace pairings with $A + B$ vanish. The constant coefficient in the determinant identity gives

$$0 = \mathcal{D}(N) = \mathcal{D}(A + B) = X.$$

Also $T(J^3) = 0$ in R': each word of length three reduces to a trace-zero generator times X or Y, and longer terms have quadratic trace factors. The linear coefficient of the determinant therefore gives

$$a_2 = T(N) = T(NM) = (m_A + m_B)X + m_C Y = Y.$$

Nakayama makes R a quotient of $\mathbf{F}_2[t]$ with radical generated by $\overline{a_2}$. Its dimension is n, whence $R = \mathbf{F}_2[t]/(t^n)$ and $R_0 = \mathbf{Z}_2[a_2]$.

For the second generator, write $g - 1 = aA + bB + cC \pmod{J^2}$, with $a, b, c \in \mathbf{F}_2$. Since $\det(g) = 1$ modulo two, the determinant identity gives $T(g) = \mathcal{D}(g - 1)$. The same polar calculation yields

$$T(g) = abX + acY = acY \quad \text{in } R'.$$

The unique prime above seven is a good place with residue field $\mathbf{F}_7$. At its Frobenius, the classes $(-1, 2, 5)$ evaluate as $(1, 0, 1)$. Hence $a_7 = Y = a_2$ in the relative cotangent, and Nakayama proves $R_0 = \mathbf{Z}_2[a_7]$. □

H Local and scalar comparisons over $\mathbf{Q}(\sqrt{15})$

H.1 Weight-two local replacements at seven

In this section write $\chi = \chi_7$ and $\omega = \overline{\chi}$, and use Hodge–Tate weights with $\mathrm{HT}(\chi) = 1$. Scalar crystalline means crystalline after at most a scalar quadratic twist. Quadratic special means special with nonzero monodromy, after at most such a twist. Write C_4, C_8 for the determinant-one tame cuspidal types whose inertial characters have order four and eight respectively.

Lemma H.1. *Let $\bar{r} : G_{\mathbf{Q}_7} \to \mathrm{GL}_2(\overline{\mathbf{F}}_7)$ have determinant ω. It admits a potentially semistable lift of weights $\{0,1\}$ and determinant χ which is scalar crystalline, quadratic special, of type C_4, or of type C_8. The choices in the split-seven reduction retain the actual residual representation, including its extension class.*

Proof. For Serre weights put $W(a,b) = \mathrm{Sym}^a \mathbf{F}_7^2 \otimes \det^b$, with b taken modulo six. In our determinant normalization the relevant weights have $a + 2b = 0 \bmod 6$, so these weights are self-dual. The tame potentially Barsotti–Tate criterion of Caraiani–Emerton–Gee–Savitt [11, Lemma A.4] gives a potentially crystalline weight-two lift when an appropriate Serre weight occurs in the compact type. Apply it to the dual residual representation, then dualize the resulting lift, since its convention has $\mathrm{HT}(\chi) = -1$ and Hodge type zero $\{0,1\}$. The determinant-one types are self-dual. For the weight calculation in this convention, a crystalline $W(a,b)$-lift of the dual has weights $\{-a-b, 1-b\}$. After dualizing and using our sign convention, shifting both weights by $a + 2b$ gives $\{b, a+b+1\}$ and preserves the residual representation, since $6 \mid a + 2b$. This identifies the weights computed below with those in the criterion. Equivalently one can use the Gee–Kisin Breuil–Mézard criterion [38, Corollary 4.5.6]. The determinant ratio of the lift to χ is unramified and reduces to one, since the type has determinant one. Its inverse square root reducing to one corrects the determinant.

For the tame cuspidal type parametrized by ω_2^{6t}, where ω_2 is a niveau-two fundamental character and $t = 1, 2, 3$, the compact reduction has factors

$$W(2t-2, 1-t), \qquad W(6-2t, t).$$

Indeed, on eigenvalues x, y, with $w = x/y$, their Brauer characters are $\sum_{j=1-t}^{t-1} w^j$ and $y^6 \sum_{j=t}^{6-t} w^j$. On a regular split element $y^6 = 1$, so the sum is zero. On a regular nonsplit element take $y = x^7$; then $y^6 = w$, and the sum is $-w^t - w^{-t}$. These are the cuspidal character values. On a scalar the sum is six, the dimension of the cuspidal representation. Thus $t = 2$ gives C_4 with factors $W(2,2), W(2,5)$. The choices $t = 1, 3$ give the two order-eight types, containing $W(4,1), W(4,4)$ respectively.

Suppose first that $\bar{r}$ is irreducible. Wild inertia has nonzero invariants and is normal, so it acts trivially. The two tame characters are $\omega_2^u, \omega_2^{7u}$, where $u = 1 \bmod 6$. The unordered pairs modulo 48 are

$$(1,7), \quad (25,31), \quad (13,43), \quad (19,37).$$

For $(1,7)$ induce a crystalline character of the unramified quadratic extension with labelled weights zero and one. The given residual unramified factor can be retained, and an unramified scalar corrects the determinant. Twisting by a ramified quadratic character gives $(25,31)$, since the twist adds 24 to both exponents. For the last two pairs, the exponents $a + 1 + 8b$ of $W(2,5)$ and $W(2,2)$ are 43 and 19 respectively. Both pairs therefore have C_4 lifts.

For a reducible $\bar{r}$, the two inertial exponents sum to one modulo six. The unordered possibilities are $\{2,5\}$, $\{0,1\}$, and $\{3,4\}$. For $\{2,5\}$ the ordered exponents $(a+1+b, b)$ of $W(2,2)$ and $W(2,5)$ give the two orderings. The ratio on inertia is ω^3, so there is no exceptional cyclotomic extension class. Thus the explicit Serre weights and the tame criterion give C_4 for the actual extension in either ordering.

For $\{0,1\}$ or $\{3,4\}$ remove a scalar quadratic twist if needed, leaving exponents zero and one. In the split case lift the characters with weights zero and one and with inverse unramified factors. In the nonsplit case with invariant line of exponent one, the residual characters have the form $\omega\bar{\alpha}, \bar{\alpha}^{-1}$ with $\bar{\alpha}$ unramified. If $\bar{\alpha}^2 \neq 1$, use the characters $\chi\alpha, \alpha^{-1}$ for an unramified lift α. Local Tate duality gives $H^2(\mathbf{Q}_7, \overline{\mathbf{F}}_7(\omega\bar{\alpha}^2)) = 0$. The integral H^2 is then zero by reduction and Nakayama, so the given extension class lifts integrally. In characteristic zero the Bloch–Kato dimension formula gives

$$H^1_f(\mathbf{Q}_7, E(\chi\alpha^2)) = H^1(\mathbf{Q}_7, E(\chi\alpha^2)).$$

This yields the scalar crystalline choice. If $\bar{\alpha}^2 = 1$, take the common unramified quadratic lift and use the Kummer description of the resulting cyclotomic extension. A residual class supported on units lifts to the crystalline subspace. If the residual valuation coordinate is nonzero, lift the unit and valuation coordinates, giving a special lift with nonzero monodromy. A special lift of a unit class is also possible by giving its lifted valuation coordinate a nonzero multiple of seven.

Finally, if the invariant line has exponent zero and the quotient has exponent one, the relevant ordered Serre weight is $W(4,1)$. For the original pair $\{3,4\}$ it is $W(4,4)$. The ratio on inertia is inverse-cyclotomic, so again the exceptional cyclotomic extension case does not occur. Both weights lie in the indicated order-eight types. Restore any removed quadratic twist in the scalar crystalline and special cases. It preserves the determinant. □

H.2 Marked ordinary lattices

For the scalar comparisons put $K = \mathbf{Q}(\sqrt{15})$, $s = \sqrt{15}$ and $H = K(i)$. Write χ_d for the quadratic character of $K(\sqrt{d})/K$, written additively in tangent formulas. Put $h = \chi_{-1}$ and write ε_2 for the two-adic cyclotomic character. The class number of K is two and its dyadic prime represents the nontrivial class. Since the fundamental unit $4 + s = (3 + s)^2/6$, the squareclasses with even valuation at every odd prime are generated by $-1, 2, 3$, and all have even dyadic valuation. The finite-unramified quadratic characters are $\langle h, \chi_3 \rangle$, of which precisely $1, h$ split at the dyadic completion $\mathbf{Q}_2(i)$.

The cyclic-lattice construction of Lemma 12.1 will be used with the marked ordinary vector v and a real conjugation c. The actual residual image at two is trivial, since the local group preserves both the global residual line and the transverse mark. The supported squareclasses of even dyadic valuation for the residual extension are generated by $-1, 2, 3, 7$, as in Section 27.2. Whenever one seven-place has scalar inertia, its residual inertia vanishes. The two seven-valuations have equal parity, so 7 is excluded. The dyadic splitting condition then leaves only h. Thus the normalized basis $((c-1)v, v)$ gives residue $1 + hE_{12}$, real matrix $\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and ordinary line e_2.

For a relative tangent to a marked entry order write its matrices as $(1 + \delta X)(1 + hE_{12})$ with $\delta^2 = 0$. The fixed determinant makes $X = \left(\begin{smallmatrix} x & y \\ z & x \end{smallmatrix}\right)$. Multiplication gives additive cochains $z, x + y$ and $dx = dy = h \smile z$. All three coordinates vanish at c. At two the marked line gives $y = 0$, the unramified quotient makes x unramified, and z is a unit Kummer class. Indeed the stable marked lattice has a multiplicative sublattice and an étale quotient. Passing to its torsion tangent quotient by schematic closure, the span of $e_1, \delta e_1, \delta e_2$ modulo δe_1 has étale quotient and multiplicative line, with extension coordinate z. The finite-flat Kummer classification gives the unit condition.

Lemma H.2. *If the tangent cochains are unramified at the odd places, then $z = y = 0$ and $x \in \mathbf{F}_2\chi_3$.*

Proof. The possible nonzero lower coordinates are χ_2, χ_3, χ_6. For $d = 2, 3, 6$, a solution to

$dx = h \smile \chi_d$ restricts to G_H as the Kummer class of $\alpha_d\beta$, for some $\beta \in K^\times/K^{\times 2}$, where

$$\alpha_2 = 1 + i, \qquad \alpha_3 = (1+i)(3+is)/4, \qquad \alpha_6 = \alpha_2\alpha_3.$$

Indeed $\mathrm{N}_{H/K}(\alpha_d) = d$. Tracking the signs of square roots of α_d and its conjugate gives a cochain with coboundary $\chi_d \smile h$, and adding the pointwise product $h\chi_d$ gives the required ordering. All solutions differ by a quadratic character of K.

For $d = 2$, odd unramifiedness forces even odd valuations of β. It has even dyadic valuation, while α_2 has odd valuation, contrary to unramifiedness of x at two. For $d = 3, 6$, the only odd valuation of α_d is over three, where H/K is unramified inert. Then $\beta = (3+s)\beta'$ with β' in the rational squareclass span of $-1, 2, 3$. At a dyadic completion the nontrivial automorphism τ over $\mathbf{Q}_2$ negates s, i. The ratios α_j^τ/α_j for $j = 2, 3$ have class $[i] = [2]$, and $(3+s)^\tau/(3+s)$ has class $[-6]$. Thus $(\alpha_d\beta)^\tau/(\alpha_d\beta)$ has class $[-3]$ or $[-6]$, respectively, and neither is trivial in $\mathbf{Q}_2(i)$. An unramified or split local class would be fixed by τ. This excludes all three nonzero lower coordinates. Now x, y are additive, unramified at finite places, and vanish at c. Since y is split at two, the finite-unramified character calculation gives the assertion. □

H.3 Two scalar crystalline seven-places

Proposition H.3. *Assume GRH for Dedekind zeta functions. Every such system of profile* CR/CR*, good away from seven, is modular, for either choice of the two scalar inertial signs.*

Proof. On every stable dyadic lattice the residual inertia at seven is trivial, since both exact inertia matrices are rational scalars of order dividing two. The unramified-terminal nonsolvable comparison in Section 27.2 applies. The solvable irreducible case is excluded by the dihedral ray calculation, since its unique candidate ramifies at both seven-places. In the reducible case the residual semisimplification is $1 \oplus 1$. For an ordinary member compare it with its h-twist in the marked basis. Their exact scalar inertial signs at each seven-place agree, so the tangent cochains are unramified away from two. By Lemma H.2 their simultaneous entry order is generated by the ordinary quotient value ν on dyadic Frobenius. Indeed, $\nu - 1$ detects the only possible tangent χ_3, and Nakayama gives $\mathbf{Z}_2[\nu]$. The value ν is identical in both factors because h is split at two. Hence the matrices agree, giving the quadratic self-twist and automorphic induction.

Suppose the dyadic source is nonordinary. Fix the two exact scalar characters on inertia, and take a simultaneous packet with this profile. Use the integral trace order R_0 and the actual regular matrix order E_0 of Section 11, restricted to the selected local factors. The left regular lattice is biconnected Barsotti–Tate by Lemma 11.5. Write $E = E_0/2E_0$, of dimension four times the trace rank over $\mathbf{F}_2$. Terminal inertia is trivial on E, since the fixed scalar signs are rational. Let $H = K(i, \sqrt{3})$, the finite-unramified genus field of degree four over K. Let $C \subset E$ be the span of the image of G_H and J its radical. The regular G_H-module C is an actual generic submodule of E, so it has a biconnected finite-flat model at each dyadic place of H by schematic closure.

Functionals on J/J^2 give actual middle/middle finite-flat subquotients of conductor at most two at those places, by the middle-model calculation of Lemma 11.2. They inject into the quadratic ray characters of H of dyadic modulus $\prod_{w|2} w^2$. The class group of H is trivial and the ray group at this modulus is C_2, as certified in [53, `d15_support/scalar_round3/nonordinary_scalar_cr_cr.gp`]. Hence J/J^2 has dimension at most one, and a nonzero character ramifies at some dyadic place. In this case C is monogenic, with radical generated by X. At such a ramified dyadic place its regular Kisin module is free of rank one over $(k \otimes_{\mathbf{F}_2} C)[[u]]$, where k is the local residue field. Indeed right multiplication extends by the middle-model full faithfulness, its radical quotient is the unique middle model, and exactness and Nakayama give freeness by equality of ranks. The Frobenius is $u + V(u)$, with $V(0) \equiv \alpha X \bmod J^2$, $\alpha \neq 0$,

as seen from the ramified Artin–Schreier coordinate of that middle selfextension. Height one makes $u^2(u+V)^{-1}$ integral. Modulo J^3 its coefficient of u^{-1} is $\alpha^2 X^2$, so $J^2 = J^3 = 0$. In either case $\dim_{\mathbf{F}_2} C \le 2$.

The four translates of C span E, giving trace rank at most two for the fixed-profile packet. Apply this bound to the four finite-unramified rational quadratic twists generated by χ_{-1} and χ_3, where here the subscript denotes a quadratic radicand. These twists preserve both exact signs. Without an actual nontrivial self-twist, the stabilizer of each local coefficient orbit acts faithfully on its trace field. The sum of the local trace degrees would then be divisible by four, contrary to the bound of two. A quadratic self-twist gives automorphic induction from a de Rham character. Strong compatibility concludes. □

H.4 Nonordinary scalar sources with one strict place

Proposition H.4. *A comparison system of profile* SP/CR*,* C4/CR *or* C8/CR *is modular if a dyadic member is nonordinary crystalline with trivial residual semisimplification.*

Proof. For these profiles, the first layer of the regular matrix order is detected by χ_{-1}, χ_3, using the exact scalar sign at the crystalline seven-place and the conductor bound of Lemma 11.2. For the fixed special or order-four cuspidal profile, let $A_1 = c - 1$ and $A_2 = F - 1$, with the exact special sign removed where applicable. The trace-order relations are

$$A_1^2 = A_2^2 = 0, \qquad A_1A_2 + A_2A_1 = tI.$$

If the trace rank is s, the four words $1, A_1, A_2, A_1A_2$ are free over $\mathbf{F}_2[t]/(t^s)$. For $s \ge 3$, sending A_1 to E_{12} and A_2 to tE_{21} modulo t^3 gives an abelian image on $G_{K(i,\sqrt{3})}$ containing the order-four element $(cF)^2$. The dyadic conductor is at most four and the strict local inertia is of exponent at most two, with the rational uniformizer killed after the unramified quadratic base change. The corresponding ray quotient has exponent two, a contradiction. Thus $s \le 2$. In the special case, the local products $(r(g) - 1)(r(h) - 1)$ vanish modulo two after sign removal. If $s = 2$, the depth-two ray bound makes the kernel over $K(i, \sqrt{3})$ of order at most two. The group algebra of the resulting quotient of order at most eight must surject onto the regular algebra of dimension eight. Equality would make a product of strict inertia and Frobenius differences nonzero in the group algebra and zero in the regular algebra. Here the kernel has nontrivial strict inertia if it is nontrivial, by the equality of the depth-one and depth-two ray quotients and the class number one of $K(i, \sqrt{3})$. This gives $s \le 1$ for the special profile.

For the order-eight profile the same four-word relations hold. Writing $T = a + bA_1 + cA_2 + dA_1A_2$ for the strict tame generator, the first layer and determinant give $\tau = \operatorname{tr} T = td \in (t^2)$. If $s \ge 5$, then $\tau^2 = 0$ implies $\tau \in (t^3)$. In the preceding quotient modulo t^3, the strict local Frobenius and inertia again generate an elementary abelian group. The same ray contradiction gives $s \le 4$.

For the special profile, twisting by χ_{-3} preserves the full local profile and a packet of two exceeds the rank bound one unless there is an actual self-twist. For the order-four and order-eight profiles the four finite-unramified genus twists preserve the exact scalar inertia and the cuspidal orientation. In the order-four case the rank is at most two. In the order-eight case every local trace field contains $\mathbf{Q}_2(\sqrt{2})$, fixed by the twists, so the rank would be at least eight in the absence of a self-twist. Each nontrivial actual self-twist gives modularity by automorphic induction. The ray computations are given in [53, `d15_support/scalar_work/nonordinary_scalar_sp_cr_ray.gp`].

□

H.5 Ordinary scalar cusp sources

Theorem H.5. *Assume GRH for Dedekind zeta functions. A system with profile* C4/CR *or* C8/CR *is modular if all its dyadic members are ordinary with trivial residual semisimplification.*

Proof. The twist by χ_{-7}, if needed, makes the scalar crystalline place good and is split at two. Fix the cuspidal orientation and take the packet under the four finite-unramified twists $V = \langle \chi_{-1}, \chi_3 \rangle$. The marked ordinary lattice has residual representation $1 + \chi_{-1}E_{12}$ after fixing a real conjugation. Let ν be its ordinary dyadic Frobenius quotient. In the cyclic basis at the cusp, with tame generator T and arithmetic Frobenius F, write

$$T = \begin{pmatrix} a & b \\ 7b & \tau - a \end{pmatrix}, \qquad F^2 = -7I, \qquad 7b^2 = a\tau - a^2 - 1.$$

Here $\tau = 0$ for C4 and $\tau^2 = 2$ for C8. In the marked tangent, the trace of F and the inversion relation $FTF^{-1} = T^{-1}$ give $z(F) = z(T) = 0$. The supported classes and the good seven-place then force $z = 0$ or χ_2, since χ_3 is nonsplit at the cusp. The additive $x + y$ is unramified at two and at the good seven-place. Ideal-class parity makes it unramified at the cusp, so $x(T) = y(T)$. If $z = \chi_2$ and $x(T) = 0$, the cup-product calculation for α_2 in Lemma H.2 contradicts dyadic unramifiedness. Thus b, whose tangent is $y(T)$ in the cyclic basis, detects the nonzero lower direction. When $z = 0$, parity makes x, y unramified at the cusp, so only the χ_3 direction remains. It is detected by ν. Nakayama gives $B = \mathbf{Z}_2[\nu, b]$. The two generators transform by independent characters of V. With $A = \mathbf{Z}_2[\nu^2, b^2]$, the integral grading is $B = A \oplus \nu A \oplus bA \oplus \nu bA$. The trace a_3 at the prime above three has the degree of b, so belongs to bA. Its tangent is $h(\mathrm{Frob}_3)z(\mathrm{Frob}_3)$. Both h and χ_2 are unramified nonsplit at three, so this detects the same direction as b. Thus a_3/b is a unit whenever the b-direction occurs. If it did not occur, the grading and Nakayama would give $b = 0$, which is excluded by the reciprocity calculation below.

Modulo b, the action is monomial over $H = K(i)$. Its diagonal character ϑ satisfies $\vartheta^\sigma \vartheta = \varepsilon_2$ for the nontrivial $\sigma \in \mathrm{Gal}(H/K)$. At the cusp the character takes the value a on a tame generator. Its two dyadic factors at the principal element $1 + i$ have product one, since the marked ordinary quotient is unramified, $(1 + i)/(1 - i) = i$ is a unit, and the cyclotomic character on the local Artin image of $1 - i$ is trivial. The cusp logarithm of $1 + i$ is 2 mod 4 and the other odd factors are unramified. Thus reciprocity gives $a^2 = 1$ in $B/(b)$. The cusp equations give $2 = 0$ there. The anticyclotomic ray calculation, with both dyads unramified, cusp inertia of exponent at most two and the cusp uniformizer killed, gives C_4, by [53, `d15_support/audit15/ordinary_cusp7_ray.gp`]. Therefore $B/(b)$ has length at most four.

The length is the sum of $[D_j : \mathbf{Q}_2]v_2(a_{3,j})$ over the distinct local trace-field factors D_j. These are the marked entry fields because dyadic inertia has distinct rational eigenvalues and the marked ordinary line is defined over the trace field. The length formula is the index of multiplication by b. At exact level $\mathfrak{p}^2$ of norm 49, the computation in [53, `d15_support/audit15/ordinary_cusp7_seeds.m`] and the corresponding residual checks give an ordinary scalar C8 form of degree sixteen with $\mathrm{N}(a_3) = 452^2$, and an ordinary scalar C4 form of degree two with $a_2 = 1$ and $a_3^2 = 2$. The Atkin–Lehner signs distinguish these cuspidal types. The principal-series possibility is excluded by the residual checks in [53, `d15_support/audit15/ORDINARY_SCALAR_CUSP_SEVEN.md`], using the trivial dyadic quotient and the one-sided ray groups. For the degree-sixteen seed, the only nonsolvable possibility is $\mathrm{SL}_2(\mathbf{F}_4)$. Its scalar dyadic quotient bounds the local different by 5/4 in Lemma 11.11. The compositum with $K(i, \sqrt{3})$ has absolute degree 480 and root discriminant less than 35.25, while the Euler terms at two, three, five and the good seven-place give a lower bound greater than 45.33 [53, `d15_support/audit15/ordinary_cusp7_seed_intervals.py`]. A residual inertia of order three at the cusp is excluded by [53, `d15_support/audit15/ordinary_cusp7_ps_ray.gp`], and

the unique solvable irreducible residue with inertia of order at most two ramifies at both seven-places. Conjugation by the nontrivial automorphism of K supplies the other orientation.

The degree-sixteen form contributes four, exhausting the length bound for C8. A nonmodular packet would contribute positive additional length. For C4, a putatively nonmodular four-twist packet has positive even length. If it has length two, the cusp inertia dies in its monomial quotient, as it is the order-two subgroup of the anti ray C_4. Hence $a - 1 \in b^2 A$, and the determinant equation makes b^2 associate to two. The packet then has total trace degree at most four. The orbit calculation gives a coefficient-three realization over $\mathbf{F}_3$ or $\mathbf{F}_9$, and the cusp protects cyclotomic irreducibility. The small-image criterion and weight-two lifting give modularity [3]. The other possible packet length is four, which the degree-two modular form and its twist exceed when adjoined. □

H.6 Ordinary scalar special sources

Proposition H.6. *A comparison system of profile* SP/CR *is modular if all its dyadic members are ordinary crystalline with trivial residual semisimplification.*

Proof. Twist by χ_{-7}, if needed, to make the scalar crystalline place good. Form a packet closed under χ_{-3}, which is trivial at both seven-places. Use h to fix the sign $\lambda_0 = \pm 1$ of the special quotient on arithmetic Frobenius F. Both exact scalar signs on inertia are allowed. Adjoin their tuple μ at a tame generator T to the marked entry order A. Put $N = \mu^{-1} r(T) - I$, so $N^2 = 0$, $NF = \lambda_0 N$, and $FN = 7\lambda_0 N$. In the basis (cv, v) put $n = N_{11}$.

The tangent equations in the real basis give $z(F) = 0$ because F has constant trace. Its eigenline is generated by $(1, \ell)$ with ℓ in the maximal ideal, as $F - \lambda_0 I$ has unit upper-right entry. The tangent of N can therefore have only an upper-right entry at T. Thus $z(T) = 0$, $x(T) = d\mu$, and $y(T)$ is that entry. The same parity and cup-product calculations as for Theorem H.5 give $z = 0$ or χ_2, with $x(T) = y(T) = 1$ in the latter case. Changing to the basis (cv, v) makes n detect this latter direction, and ν detects the remaining χ_3 direction. Hence $A = \mathbf{Z}_2[\nu, n]$. For a fixed inertial sign the tangent of N vanishes and the order is $\mathbf{Z}_2[\nu]$.

Adjoin the h-twists to form B, still with μ adjoined. In the basis (cv, v), an entry at g has h-sign $h(g)$ on the diagonal and $h(g)h(c)$ off the diagonal. The generators of A are even, so $B = A \oplus \lambda I$, with $\lambda = (\lambda_0, -\lambda_0)$ in the rational product and I an ideal of A. Write $N_{12} = \lambda u$ and $F_{22} = \lambda d$. The relation $NF = \lambda N$ gives

$$n = u(1 - d)/F_{12}.$$

Here F_{12} is a unit and d is in the maximal ideal, since F reduces to the swap matrix. The line equations give $N_{21} \in \lambda u A$ and $N_{22} = -n$. Thus the monodromy ideal is $(u, \lambda u)$. The element u is nonzero on every characteristic-zero factor by the nonzero monodromy condition.

Modulo this ideal, the tangent at T is scalar and the additive $x + y$ vanishes on inertia by parity. Hence all tangent cochains are unramified at the odd places. Lemma H.2 and Nakayama make the quotient generated by the even element ν, so $I = (u)$. On G_H the off-diagonal products in the swap basis vanish modulo u^2. Its first diagonal gives a character ϑ over $A/(u^2)$ with $\vartheta^\sigma \vartheta = \varepsilon_2$. Its inertia value is $\mu(1 + n)$. Reciprocity on $1 + i$ as above gives $2n = 0 \bmod u^2$, since the local logarithm is twice an odd number, $\mu^2 = 1$, and $n^2 = 0 \bmod u^2$. As n/u is a unit and u is a nonzerodivisor, $2 \in uA$.

Modulo u the character is anticyclotomic, unramified at both dyadic places, of inertia exponent at most two at the strict place and trivial on its uniformizer over H. For the last assertion F becomes antidiagonal with $F^2 = -7I = I$ in characteristic two. The C_4 ray quotient therefore bounds the length of $A/(u)$ by four. The level-seven forms in [53, `d15_support/certificates/ordinary_special7_seeds.m`] have trace field $\mathbf{Q}(\sqrt{3})$, $a_2^2 = 3$

and $a_3 = \pm 2$. Their opposite Atkin–Lehner signs give the two choices of λ_0. The real nonzero traces at two and three make the narrow central character trivial, and the conductor exponent one gives unramified special type. Their dyadic residue field is $\mathbf{F}_2$, and the solvable classification of Section 27.2 gives trivial residual semisimplification. For the seed with the matching λ_0, the fixed-sign entry order is $\mathbf{Z}_2[a_2] = \mathbf{Z}_2[\sqrt{3}]$. The monodromy entries have zero tangent, hence valuation at least one. The divisibility $2 \in uA$ gives $v_2(u) = 1$ there. Adjoining this seed makes $2/u$ a unit in the joint order. Its rank is therefore at most four. The seed contributes two, leaving at most two for a hypothetical nonmodular packet. Thus its global trace field D is rational or quadratic. In the quadratic case the packet twist χ_{-3} gives the nontrivial coefficient automorphism, negating a_2. Hence $D = \mathbf{Q}(a_2)$ and $a_2^2 \in \{3, 5, 7\}$ by ordinarity and the Weil bound.

If a coefficient-five place has residue field $\mathbf{F}_5$, an irreducible residue is handled by [3, Theorems 1.2 and 2.3], or by [72, Theorem 7.5] when the cyclotomic restriction is reducible. Its cyclotomic quadratic subfield is the real field $K(\sqrt{5})$. For a reducible residue write the constituents as $\psi, \omega_5\psi^{-1}$. The character ψ^2 is quadratic and unramified away from five. The prime above five has the nontrivial ideal class, so valuation parity also makes ψ^2 unramified at five. Its even character is either trivial or the character of $K(\sqrt{5})/K$. The possible finite-flat rank-one tame exponents at five are $0, 1, 2$. Triviality of ψ^2 on inertia excludes the middle exponent. The actual crystalline source is ordinary by the finite-flat uniqueness at absolute ramification index $2 < 5 - 1$. Its residual ratio is odd, extends to $\mathbf{Q}$, and is nontrivial at the unique five-place. Apply [66, Section 4.5, Theorem A], with the residual quotient labelled by the actual ordinary filtration.

If $D = \mathbf{Q}(\sqrt{3})$ or $\mathbf{Q}(\sqrt{7})$, the residue at five is over $\mathbf{F}_{25}$ with inner twist $\eta = \chi_{-3}$. For an irreducible residue, Lemma 12.5 gives projective descent to $\mathbf{F}_5$ with totally odd squareclass, since η is totally odd and -1 is a square in $\mathbf{F}_5$. The preceding irreducible transfer applies. If the residue is reducible, the fifth-power coefficient Frobenius switches the two constituents after twisting by η, as seen at real conjugation. This gives $\psi^6 = \omega_5\eta$. The character ψ has 2-power order by its local inertia and the class group, so it has order eight. The residual ratio is then $\psi^4\eta = \omega_5^2\eta = \chi_{-1}$. At the special seven-place this ratio takes value -1 on Frobenius, whereas the exact special eigenvalues give ratio 7 or 7^{-1}. These are distinct from -1 modulo five. Thus this reducible case is impossible. This treats both scalar inertia signs at the special place. □

I Congruence fields over $\mathbf{Q}(\sqrt{15})$

Retain the ordinary residual cases of Section 27.2 over $K = \mathbf{Q}(\sqrt{15})$. In particular, the residual group is $G = \mathrm{SL}_2(k)$, $k = \mathbf{F}_q$, $q \geq 8$. The dyadic wild inertia is the whole upper-unipotent subgroup of G, and its unramified multiplier is nontrivial. At the ramified seven-place, the residual inertia and a Frobenius generate a Klein four group. We exclude the rows with $q \geq 16$ and obtain a uniqueness statement when $q = 8$.

I.1 Auxiliary fields and the ordinary comparison

Put $J = K(\sqrt{-3})$, $H = K(i)$ and $C = K(i, \sqrt{3})$. These fields are finite-unramified over K. Their dyadic residue degrees are $2, 1, 2$, respectively, and at both seven-places they are $1, 2, 2$. The composita with a nonsolvable point field preserve root discriminants, and their local residue degrees are the least common multiples of the individual degrees.

The class group of H is C_2. Let $\mathfrak{P}$ be its prime over the strict seven-place. The ray group of modulus $\mathfrak{P}$ is C_8, its dyadic Artin classes generate, and both the local uniformizer 7 and the prime over the other seven-place have Artin order two. The cyclic quartic quotient defines A/H. Over K this is a solvable Galois field of degree eight, by stability of the quotient and modulus.

Its strict local indices are $(2,2)$, and its residue degrees at the good seven-place and at two are two and four. The depth-four ray group of A at both dyadic primes is $C_4 \times C_2^7$. Splitting both strict primes cuts the quotient to C_2^7. Its class field A^+ is solvable Galois of degree 1024 over K. Its dyadic upper breaks are at most three because the conductor over the unramified dyadic completion of A is at most four. It retains indices $(2,2)$ at the strict place.

The depth-four ray quotient of A obtained by splitting the strict primes and one dyadic prime is C_2^2. The quadratic subfield B_1/A specified by the ray certificate splits over A at the strict primes and has absolute degree 32. Its three dyadic completions have absolute indices $(2,4),(4,4),(2,4)$. The depth-four ray quotient at these three primes, with the strict primes split, is C_2^7 under GRH. Let B_1^+ be its class field, of degree 2048 over K. Its normal closure is solvable and all strict completions remain those of A. At a dyadic completion over A, both B_1/A and B_1^+/B_1 have upper breaks at most three. The inverse Herbrand function of the first step satisfies $\psi(u) \geq u$; the upper filtration therefore vanishes above three in the compositum of the local conjugates over A. The same upper bound holds over K_2 because A/K is unramified at two. The arithmetic data for both orientations are given by [53, `d15_support/ordinary15/ordinary_one_cr_aux.gp` and `d15_support/ordinary15/ordinary_one_cr_b1.gp`].

A nonsolvable point field is disjoint from each solvable normal closure by simplicity. For $q \geq 512$ apply Proposition 13.3 to MA^+, retaining the strict term $(e,f) = (2,2)$. At $b = 25.9$ the lower bound exceeds 50.70665 at $q = 512$ and increases with degree. For $q = 256$, MB_1^+ and $b = 24.5$ give a lower bound above 50.48934 with the same term. Both exceed the common upper bound 50.39764. For the finite rows use M, MJ, MH, MC, MA with the local bounds of Lemma 11.11 and the available Euler terms at two and both seven-places. The 923 strict comparisons in [53, `d15_support/ordinary15/ordinary_one_cr_intervals.py`] leave precisely the rows in Section 27.2.

I.2 Two order-four cuspidal lifts

The residual tame generator and Frobenius at the ramified seven-place are simultaneously conjugate to $1 + aE_{12}, 1 + bE_{12}$, with $a, b \in k^\times$ distinct. Choose coefficient lifts a_0, b_0, with a_0 a unit, and adjoin i. Set

$$T = \begin{pmatrix} i & a_0 \\ 0 & -i \end{pmatrix}, \qquad F = \begin{pmatrix} x & b_0 \\ -2ix/a_0 & -x \end{pmatrix}, \qquad x^2 - (2ib_0/a_0)x + 7 = 0.$$

A root x is integral and reduces to one. The equations give $T^2 = -I$, $F^2 = -7I$, and $FTF^{-1} = T^{-1} = T^7$. They define a continuous tame local lift of the actual residual representation, with cyclotomic determinant and the order-four cuspidal Weil–Deligne parameter. At the other seven-place choose an unramified lift. This is possible since the residual representation is unramified there. At two retain the ordinary crystalline weight-two component.

The rational prime 11 splits in K. Fix the place v generated by $\pi = 2 + \sqrt{15}$, of norm -11. At v prescribe either the unramified condition, or its twist by a local ramified quadratic character. Such a twist has scalar quadratic inertia, preserves the determinant and reduces to the original local representation. At the other place over 11 and at all remaining finite places away from two and seven impose unramified conditions. Lemma 2.6(iii) applies, with nonsolvable residual image and cyclotomic determinant, and globalizes each choice. Only existence of the two dyadic lifts on the selected components is needed.

For each of the two resulting sources take the packet under the four everywhere-finite-unramified twists $V = \langle \chi_{-1}, \chi_3 \rangle$, with one representative for each distinct dyadic coefficient orbit. No actual self-twist occurs because it would make the characteristic-zero projective image dihedral, contrary to the nonsolvable residue. If D is one local trace field and $W \subset V$ its coefficient-orbit stabilizer, the contribution of the packet to the sum of local trace degrees is

$4[D^W : \mathbf{Q}_2]$. The fixed field contains all squares of actual traces, whose reductions generate k, so $[D^W : \mathbf{Q}_2] \geq f = [k : \mathbf{F}_2]$. The two packets have distinct exact scalar inertia at v, hence disjoint trace orbits.

I.3 The simultaneous lattice and its congruence group

Let A be the closed simultaneous trace order, a subring of the product of local coefficient integer rings. It is finite free over $\mathbf{Z}_2$, local with residue field k. The rational trace product is the product of the local trace fields of the distinct orbits. Indeed, a field isomorphism matching the traces of two factors would identify their coefficient orbits. Consequently

$$\dim_{\mathbf{F}_2} R \geq 8f, \qquad R = A/2A.$$

The simultaneous representations are realized over A. For an elementary verification, choose a common residual element with distinct eigenvalues. Its characteristic polynomial over A has two roots in A by Hensel lifting. Diagonalize it in the product lattice and scale the two basis vectors so that the upper off-diagonal entry of a common element with both residual off-diagonal entries nonzero is one. The two projectors, and then E_{12}, are in the A-algebra generated by the matrices. The opposite off-diagonal entry is in A by taking a trace against E_{12}, and is a unit, so the same algebra contains E_{21}. Taking traces against these matrix units recovers every entry in A.

By Lemmas 11.5 and 11.7, the stable lattice A^2 is Barsotti–Tate at two and its quotient modulo two has upper breaks at most three. At the dyadic place the ordinary line over A is recovered from the simple residual eigenvalues of a Frobenius lift and Hensel projectors, since all ordinary lines reduce to the unique invariant line of the residual wild subgroup. Its diagonal characters are $\chi_2\nu^{-1}, \nu$ with ν unramified. Therefore the image of dyadic inertia in $\mathrm{SL}_2(R)$ is upper unipotent. Conjugation by Frobenius multiplies the upper-right coordinate by a unit d with $d - 1$ a unit. The commutator map on the finite additive inertia image is bijective. Thus every abelian quotient of the simultaneous image is unramified at two.

Write L for the finite linear image in $\mathrm{SL}_2(R)$. Its residual quotient is G. At the cusp the exact T, F identities are common to both packets, so modulo rational two the decomposition group remains a Klein four group. At the CR place the common rational scalar inertia disappears modulo rational two. At v the inertia image is generated by one central scalar of order at most two, corresponding to the two possible exact signs.

I.4 Noncentrality

If the kernel of $L \to G$ were scalar, it would be elementary abelian because its determinant is one in characteristic two. The central extension splits. Over a Sylow two-subgroup of G each lift of a nontrivial transvection has scalar square. The characteristic polynomial identity and a unit off-diagonal entry force its trace to vanish and its square to be one. Thus the Sylow preimage is elementary abelian. Restriction followed by corestriction multiplies the extension class by the odd Sylow index, so the global extension splits. Its scalar factor would then be an elementary abelian global character, unramified at two by the inertia commutator calculation. At the cusp it is unramified by ideal-class parity: that prime is nonprincipal and v is principal, while all other finite odd primes are unramified. It is also unramified at v. In fact any radicand with odd valuation exactly at v and even valuation at the other finite places has rational norm squareclass -11. To see this, divide it by π; the norm of any element of even valuation everywhere is a rational square, since the class group has order two and its even-support squareclasses are generated by $-1, 2$ and the totally positive fundamental unit. But both the split and unramified quadratic squareclasses of $K_2 = \mathbf{Q}_2(i)$ have square norm to $\mathbf{Q}_2$, the latter being represented by

3. The class -11 is not a dyadic square. The scalar factor thus has rank at most two, by the narrow class group C_2^2.

Every trace of the section $G \to L$ lies in the canonical copy of k in R. For an odd-order element this follows by splitting the separable root-of-unity polynomial over an unramified coefficient extension. Every other nonidentity element of G is an involution, whose trace is zero by Cayley–Hamilton and determinant one. Thus R would be generated by k and at most two scalar units squaring to one, with binary dimension at most $4f$. This contradicts the packet bound.

I.5 The perfect core and cancellation at eleven

Let L^{∞} be the last term of the derived series of L, and let $\mathfrak{m}$ be the maximal ideal of R. It maps onto G. At the first noncentral projective congruence layer choose a kernel element which is scalar modulo $\mathfrak{m}^n$ but not modulo $\mathfrak{m}^{n+1}$. The space $M_2(k)/kI$ has no G-fixed vector: scalar conjugacy differences would give an additive character of the perfect group G, and then the centralizer is scalar. Hence its commutator with a suitable element of L^{∞} has a noncentral leading matrix. This commutator belongs to L^{∞} and is identity modulo $\mathfrak{m}^n$. Its leading matrix is trace zero by determinant one.

Project this leading layer along a k-linear functional $\mathfrak{m}^n/\mathfrak{m}^{n+1} \to k$ detecting a nonzero off-diagonal entry. The resulting additive G-stable subgroup of $\mathfrak{sl}_2(k)$ is all of $\mathfrak{sl}_2(k)$. Indeed if a matrix has lower entry $c \neq 0$, upper-unipotent conjugation differences give the matrices with diagonal tc and upper entry t^2c. Torus conjugation differences give all pure upper matrices. Subtraction then gives every scalar matrix, and swapping gives the lower matrices. A matrix initially with only upper entry is handled by swapping first. Therefore

$$|L^{\infty}| \geq q^3|G|.$$

If the scalar inertia at v is nontrivial, adjoin $Q = K(\sqrt{\pi})$. This quadratic field ramifies at v and at two by the norm calculation. Its dyadic radicand is a unit, so its dyadic upper break is at most three. It is disjoint from the linear point field since all quadratic quotients of L are unramified at two. At v the inertia of the compositum is the central diagonal involution formed by the scalar inertia and the nontrivial element of $\mathrm{Gal}(Q/K)$. Quotient by this involution. This kills the ramification at v and leaves $L^{\infty} \times 1$ injected. If the scalar inertia was trivial, take the original point field. The resulting Galois field N/K is unramified away from two and the cusp, has cusp indices $(2, 2)$, has dyadic upper breaks at most three, and its perfect core has order at least $q^3|G|$. Adjoining the unramified-at-seven quadratic Q cannot enlarge the cusp residue degree beyond two.

I.6 The directed comparison

Use the degree-2048 extension B_1^{+}/K just constructed. Its solvable normal closure preserves the same local bounds, and any solvable quotient of N/K kills its perfect core. Thus

$$[NB_1^{+} : \mathbf{Q}] \geq 4096q^4(q^2 - 1), \qquad \mathrm{rd}(NB_1^{+}) \leq 4\sqrt{60}\,7^{1/4}.$$

At $q = 16$, retaining only the cusp Euler term with indices $(2, 2)$, the directed comparison at parameter 24.5 gives lower bound greater than 50.48869 and upper bound less than 50.39764. The degree lower bound is increasing in q, which excludes all $q \geq 16$.

I.7 Uniqueness of a remaining $q = 8$ point field in each orientation

Fix which seven-place ramifies. There is at most one residual point field M/K in the remaining $q = 8$ rows, even allowing both binary Frobenius types of the ordinary multiplier. Indeed,

for two distinct point fields choose the corrected extensions N_i/K above. The kernel of each $\mathrm{Gal}(N_i/K) \to \mathrm{Gal}(M_i/K)$ is a 2-group. A common nonsolvable quotient would therefore have $\mathrm{SL}_2(\mathbf{F}_8)$ as its nonabelian simple quotient, identifying M_1 and M_2. Thus the intersection is solvable and the degree of the compositum is at least twice the product of the two perfect-core orders, namely

$$2(8^3\,|\mathrm{SL}_2(\mathbf{F}_8)|)^2 = 133177540608.$$

The two strict local completions agree, since there is a unique tame elementary quadratic extension of $\mathbf{Q}_7$ with indices $(2,2)$. The dyadic upper breaks are still at most three. The same root-discriminant upper bound applies. At parameter 25.2 the strict-seven Euler term gives a directed lower bound greater than 50.59732, a contradiction.

J The ordinary $\mathrm{SL}_2(\mathbf{F}_8)$ residue over $\mathbf{Q}(\sqrt{15})$

Put $k = \mathbf{F}_8$. We call an auxiliary profile unsigned if it is unramified at $\mathfrak{l}$, and signed if it has ramified scalar inertia there. In this section A denotes the simultaneous trace order of any finite family of the prescribed C4 lifts with aligned residue, closed under the four genus twists, and $R = A/2A$. The local identities and the ordinary filtration hold for every such family. We first prove $\dim_k R \le 8$, then apply the bound to all the dyadic factors with residue field k.

J.1 The marked basis and two trace generators

The ordinary line is defined over A by the distinct dyadic Frobenius eigenspaces. Choose a generator v of this free rank-one module and a conjugate of the strict decomposition group such that the residual line fixed by its tame inertia element T differs from the marked dyadic line. This is possible by residual surjectivity onto $\mathrm{SL}_2(k)$. Then (Tv, v) is a basis. Let F be the strict arithmetic Frobenius and ν the ordinary unit-root unramified character evaluated at a dyadic arithmetic Frobenius F_2. In the marked basis the cusp relations give

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad a^2 + b^2 = -7.$$

Put $u = a + b$. It is a unit with residual value one and $u^2 = 1$ in R.

The literal tuple twists act on the trace order, multiplying each trace by its twist sign. This action respects the matrix entries in the marked basis. Indeed the corresponding two representations with equal traces are conjugate over A by the absolutely irreducible trace realization. The conjugating map commutes with the common matrix of T and preserves the marked line. In the cyclic basis it is therefore scalar. Consequently the entries of $r(g)$ have the degree given by evaluating both genus characters at g. The integral ring is graded by these four degrees, since its trace generators are homogeneous and the rational character eigenspaces are disjoint. Its grading survives reduction modulo two and any quotient by a homogeneous ideal. The canonical coefficient subring and subfield are of degree zero. The units u, ν have independent degrees, namely $(1,1)$ at the strict Frobenius and $(0,1)$ at the dyadic Frobenius in additive binary notation. Multiplication by them gives a strong grading.

In R, let $P = \nu^2 - \bar{\nu}^2$, with $\bar{\nu}$ represented in the canonical copy of k. The ordinary multiplier in $R/(P)$ is constant, namely $\bar{\nu}^{\pm 2}$ of order seven. Suppose its projective image grows. Pass to the first noncentral radical layer. Lemma 11.11 gives the absolute normalized dyadic different at most $2 - 1/q - 1/q^2$. To remove the inertia at eleven, quotient by its central subgroup, of order at most two. The dyadic bound is preserved and the perfect core loses a factor of at most two. The first-layer perfect core before this quotient has order at least $q^3|\mathrm{SL}_2(k)|$. The degree-eight ray auxiliary preserves the dyadic bound and the strict indices $(2,2)$. At absolute

degree 2,064,384, the strict-seven Euler term gives a directed lower bound greater than 46.57449, against the upper bound 45.71701.

It follows that over $R/(P)$ the image is scalar times the constant residual image, up to conjugacy. The scalar character λ has exponent two and is unramified at two by the ordinary local commutator calculation. The ideal class and norm argument of Appendix I makes it a quotient of the narrow class group C_2^2. In a constant-residual realization the marked dyadic line is its residual line lifted constantly. The marking vector in that line differs from the constant vector by a unit, and T is constant because $\lambda(T) = 1$. Thus in the marked basis $u = \lambda(F)$ and $\nu/\bar{\nu} = \lambda(F_2)$ in this quotient. The indicated two Frobenii generate the genus group. Since traces generate the ring, $R/(P)$ is generated by u, ν. Lifting this generation to R uses P nilpotent and already in the algebra they generate: for the subalgebra $B = k[u, \nu]$, the equality $R = B + PR$ implies $R = B$ by iteration.

The invariant degree-zero subring is therefore $k[P]$, because $u^2 = 1$, $\nu^2 = P + \bar{\nu}^2$, and the two degrees are independent. If P has nilpotence index n, the four grading pieces are free rank one over $k[P]/(P^n)$, by their homogeneous units. Equivalently

$$R \simeq k[X, Y]/(X^{2n}, Y^2), \qquad X = \nu - \bar{\nu}, \quad Y = u - 1, \quad P = X^2.$$

J.2 Excluding a second noncentral layer

If $n \geq 3$, the dyadic Frobenius F_2^7 is scalar modulo $\mathfrak{m}^2$ and nonscalar modulo $\mathfrak{m}^3$, where $\mathfrak{m} = (X, Y)$. Indeed its ratio in the ordinary eigenbasis is $(\bar{\nu}^2 + P)^{\pm 7}$, with nonzero first P-term. Its square F_2^{14} is scalar modulo $\mathfrak{m}^4$ and nonscalar modulo $\mathfrak{m}^5$, since the first term is a nonzero multiple of $P^2 = X^4$. At each step, commuting with a suitable residual element produces a nonscalar element in the linear square-zero kernel. The preceding conjugation calculation puts a block of order q^3 in the perfect core at each step. The perfect core of the deeper image surjects onto that of the earlier image. The two contributions therefore multiply, giving perfect core order at least $q^6|\mathrm{SL}_2(k)|$.

Transfer the auxiliary scalar ramification as before. The resulting field has dyadic upper break at most three and strict indices $(2, 2)$. Compositum with A^+/K, of relative degree 1024, keeps those bounds. The absolute degree is at least $2048q^6|\mathrm{SL}_2(k)| = 270582939648$. The second directed row gives lower root discriminant above 50.70436 and upper below 50.39764. This excludes $n \geq 3$, and proves the stated dimension bound.

With the two opposite auxiliary signs of Appendix I, the integral rank is at least $8[k : \mathbf{F}_2]$. It is therefore exactly that rank, and $n = 2$. As a further consequence, in this saturated characteristic-two ring the entries of the dyadic inertia matrices lie in the invariant ring $k[P]/(P^2)$, since both genus characters are unramified at two. The ordinary unipotent inertia consequently has order at most q^2 even when the multiplier is not constant.

Suppose the residue is nonmodular. Its image is $\mathrm{SL}_2(\mathbf{F}_8)$, its dyadic wild inertia is the full translation group, and its multiplier generates $\mathbf{F}_8^\times$. Its strict seven-place has decomposition group V_4 with nontrivial inertia, and the other seven-place is unramified in the residue. Prescribe order-four cuspidal type at the strict place, good reduction at the other, and the actual ordinary crystalline parallel-weight-two component at two. The preceding local matrices lift the actual residue at the strict place. Make two choices at the principal prime $\mathfrak{l} = (2 + \sqrt{15})$ of norm eleven: the unramified type and the ramified quadratic scalar twist of that type. Lemma 2.6(iii) gives the two pure strongly compatible systems. Their dyadic residues at the chosen factors are the given nonmodular residue. Consequently neither system can be modular at any coefficient.

J.3 The two fixed trace fields

At every dyadic factor the residue is unramified at eleven, because the exact auxiliary inertia is a rational scalar sign. The existing one-sided dyadic comparisons, and the congruence comparison leave only ordinary nonsolvable $q = 8$ factors or scalar semisimplifications. The solvable irreducible dyadic residue would require ramification at both seven-primes. A reducible residue has trivial diagonal characters, by the odd-order character and class-group calculation. For a nonordinary scalar factor use the fixed-profile rank comparison. The common auxiliary scalar sign vanishes on the simultaneous order modulo rational two, so it contributes no ramification to the middle-model ray calculation. The bound of two for the trace rank with one fixed cuspidal orientation follows as in the nonordinary C4/CR comparison. Its packet of four finite-unramified genus twists preserves that exact profile. Without a self-twist its trace rank is divisible by four, a contradiction. A self-twist gives modularity by automorphic induction. Thus no nonordinary scalar factor occurs in either nonmodular system.

We next bound the unsigned ordinary scalar packet and exclude signed ordinary scalar factors. Use the fixed-orientation marked entry order and the notation ν, b, A of the ordinary cuspidal argument. For an unsigned scalar packet from a nonmodular system, adjoin the unsigned modular seed of degree two and its twist with opposite a_2. Their disjoint factors contribute at least two to the index of b. Hence any unsigned scalar packet from the nonmodular system has index at most two. Its index is positive and even by the four-twist argument. If present, it has index two, rank four, and b^2 is associated to two on this packet.

For a signed ordinary scalar packet, adjoin the unsigned degree-two seed and take the four-twist packets together. In the marked tangent calculation an auxiliary inertia matrix is scalar. At an auxiliary inertia generator its tangent is xI, with $y = z = 0$ in the notation of the ordinary scalar calculation. The additive class $x + y$ is unramified at two and the good seven-place. Among odd places it can ramify only at the cusp and the principal auxiliary $\mathfrak{l} = (2 + \sqrt{15})$. The cusp prime is nonprincipal, so the even-valuation ideal-class condition excludes ramification there. The rational norm obstruction of the preceding congruence argument then excludes ramification at $\mathfrak{l}$. Thus the auxiliary inertia has zero tangent and the joint marked entry order still has the form

$$B = A \oplus \nu A \oplus bA \oplus \nu bA, \qquad A = \mathbf{Z}_2[\nu^2, b^2].$$

Let $s_{\mathfrak{l}}$ be the exact scalar tuple on a generator of auxiliary inertia. It equals 1 on the unsigned seed and -1 on the signed factors. It belongs to the degree-zero suborder. Modulo b, apply reciprocity on $1 + i$ to the first diagonal character over $K(i)$ in the swap basis. The two dyadic factors have product one, as in the unsigned calculation. The cusp factor is -1, since the diagonal entry there has square -1. The auxiliary factor is $s_{\mathfrak{l}}$: the local norm of $1 + i$ at the inert eleven-place is 2, a nonsquare modulo eleven. Hence $1 + s_{\mathfrak{l}} \in bB \cap A = b^2 A$. Writing $1 + s_{\mathfrak{l}} = b^2 u_0$, the unsigned seed has $v_2(b^2) = 1$, so u_0 is a unit on that factor and in the local order A. Thus $b = 0$ on every signed factor. In characteristic zero the swap-basis representation on such a factor is monomial over $K(i)$, so its quadratic self-twist gives automorphic induction. This contradicts nonmodularity.

Include all the $q = 8$ dyadic factors of the two systems, with the four genus twists of each. Their residual point fields agree by uniqueness for the fixed strict orientation. Automorphisms of $\mathrm{SL}_2(\mathbf{F}_8)$ are inner up to coefficient Frobenius, so align the embeddings to have the same actual residual representation. Put $k = \mathbf{F}_8$ and $R = A/2A$, where A is the resulting simultaneous trace order. For each of the two systems its $q = 8$ packet contributes at least $4[k : \mathbf{F}_2]$, by the orbit argument with squared traces. The two profiles are disjoint by the exact scalar sign at eleven. The rank bound above gives equality, with each of the two $q = 8$ contributions exactly twelve.

For either system write D for the global trace field, $W \subset V_0 = \langle \chi_{-1}, \chi_3 \rangle$ for the global twist stabilizer, and $F = D^W$. There is no genuine self-twist, since the dyadic nonsolvable residue

excludes quadratic induction. The sum of all local trace degrees of the global genus packet is $4[F : \mathbf{Q}]$. The signed system has no scalar dyadic factors, so its invariant field is cubic. The unsigned system has at most four additional trace degrees, so its invariant field is cubic or quartic. For both systems the $q = 8$ dyadic place of F is unramified of degree three, since the squared traces reduce onto k. If the unsigned field is quartic, the other dyadic place of F has degree one. The scalar packet bound and a_3/b being a unit then give $v_2(U_3) = 1$ at that place.

Write $U_j = a_j^2$, where a_j is the good-place trace at the unique prime of norm $j = 2, 3$, with the crystalline Frobenius trace for $j = 2$. These squares lie in F. The dyadic crystalline trace is $\nu + 2/\nu$, where ν is the ordinary unramified quotient value. The residue of ν has order seven. Therefore U_2 generates F, since its residue already has degree three and $[F : \mathbf{Q}]$ is three or four. It is totally positive, integral, with all conjugates at most eight. The fields D and F are totally real by purity.

J.4 Odd coefficient constraints

The extension D/F is unramified at odd coefficient places. Otherwise a nontrivial coefficient inertia element gives a genuine residual quadratic self-twist by its inner twist in W. At every odd coefficient prime other than seven the order-four cusp remains locally absolutely irreducible, with projective dihedral decomposition. Thus the global residue is irreducible and the self-twist makes it dihedral. The inducing quadratic character from the inner twist is finite-unramified. At odd primes other than three, five and seven, the good or scalar-twisted-good finite-flat condition at a coefficient place over p gives projective tame inertia of order greater than two, so the global projective group cannot be V_4. Its inducing quadratic character is then unique. The cyclotomic quadratic subfield is ramified at the coefficient prime, hence different. At three the cyclotomic restriction is already irreducible at the cusp, since seven splits in $K(\zeta_3)/K$. At five the cyclotomically reducible case uses Thorne's theorem for the real inducing field $K(\sqrt{5})$. The other cases use the odd dihedral witness and the weight-two lifting theorem.

At coefficient seven the global residue is also irreducible. Indeed a globally reducible residue has diagonal exponents zero and one at the good seven-place by finite flatness. At the strict seven-place an order-four cuspidal potentially Barsotti–Tate lift of a locally reducible residue requires exponents two and five, by the compact constituents $W(2, 2), W(2, 5)$. Reciprocity on the fundamental unit $4 + \sqrt{15}$, whose values at the two seven-places are inverse generators, makes the two exponents of each global constituent equal when there is no auxiliary inertia. If the scalar sign at eleven is ramified, the exponents instead differ by three, since the unit is nonsquare at $\mathfrak{l}$. Neither relation permits the strict exponents two and five. The restriction to $K(\zeta_7)$ is irreducible by the finite-flat good-seven argument: an inducing character from $K(\sqrt{-7})$ has projective tame inertia of order at most two at the good place, whereas the good-place finite-flat inertia has tame order six or eight. Therefore a residual self-twist at coefficient seven again gives modularity by the odd dihedral witness and weight-two lifting. This proves the unramified assertion in all cases.

It follows that U_2 has even valuation at every odd place of F. It is a unit at both possible dyadic places because every actual dyadic factor is ordinary. In particular its rational norm is a square. The same even-valuation condition away from two holds for nonzero U_3.

At a degree-one coefficient place of F over three the projective residue is defined over $\mathbf{F}_3$, and at a degree-one place over five it is defined over $\mathbf{F}_5$. The former case is modular by the small-image criterion. In the latter case the projective determinant-squareclass has uniform real parity because it is the coefficient Frobenius inner twist times the determinant squareclass, and the inner twists are rational genus characters. Both parity cases of Allen–Khare–Thorne at five apply, since $[K(\zeta_5) : K] = 4$. The actual weight-two transfer, including the real residually dihedral theorem when needed, gives modularity. Thus F has no degree-one prime at three

or five. Likewise at a degree-one seven-place, an even coefficient Frobenius inner twist gives a totally odd projective squareclass, so Allen–Khare–Thorne at seven and weight-two transfer apply. For these projective descents use Lemma 12.5. The coefficient extension is unramified and the residual Frobenius twist comes from W, so its parity gives the descended determinant squareclass as in that lemma.

J.5 Cubic coefficient fields

We first impose a local constraint at the strict seven-place. Fix an ordinary dyadic line vector and move it by a global group element to a vector v outside the unique residual cusp line. This is possible by global residual irreducibility. With the order-four cusp inertia matrix T, the pair (Tv, v) is a basis. Divide the cusp Frobenius by the rational square root of -7 in $\mathbf{Q}_2$, obtaining the involution

$$\begin{pmatrix} x & y \\ y & -x \end{pmatrix}, \qquad x^2 + y^2 = 1.$$

The residual cusp V_4 condition makes x and y units: if either vanished the involution would reduce to the identity or to T. Hence the unit $w = x + y$ satisfies $w \equiv 1$ in the residue and $v_2(w-1) = 1/2$, using $(w-1)(w+1) = 2xy$.

For the local coefficient twist stabilizer this marked basis is compatible with its semilinear action. The ordinary line is unique, v was obtained from it by a fixed global group element, and T is fixed by every finite-unramified quadratic twist. Thus w transforms by the sign of the twist on the cusp Frobenius. Since F is unramified at its cubic dyadic place, that sign is nontrivial on the local stabilizer. The global quadratic eigenfield for the sign therefore has a totally positive squareclass, with even odd valuations and even valuation at this dyadic place. In degree three this is the only dyadic place. Its local quadratic extension is ramified, by the valuation $v_2(w-1) = 1/2$. The other relevant sign is evaluation at dyadic Frobenius, under which the ordinary quotient ν, and hence a_2, transform. With $h = \chi_{-1}$ and $\eta = \chi_3$, the cusp sign is $h\eta$ and the dyadic sign is η, where these symbols in sign-degree statements denote the two evaluation coordinates on V_0.

The cubic computation in [53, `d15_support/round4/ordinary/q8_weil_certificate.gp`] leaves ten candidates for U_2, in six reduced cubic fields of discriminants 49, 361, 1489, 3889, 5521, and 7249. Their ordinary class groups are trivial, so the supported positive squareclasses are generated by the positive unit classes and 2. The cusp constraint eliminates fields with no suitable positive unit class. At a degree-one seven-place where every supported positive quadratic class splits, the coefficient Frobenius inner twist is trivial, also eliminating the field. The surviving possibilities are discriminants 5521 and 7249. In each, the exact positive supported squareclasses are generated by 2 and U_2. Their local classes at the cubic dyadic place are independent, the first of odd valuation and the second a ramified unit class. Thus the full stabilizer $W = V_0$ is impossible: its local stabilizer would also be full, and the two independent sign eigenfields for cusp and dyad would both be generated by units. Indeed the corresponding generators can be taken to be w and ν locally. The nontrivial dyadic sign leaves $W = \langle \chi_3 \rangle$ or $W = \langle \chi_{-3} \rangle$, and the nontrivial cusp sign excludes the latter. The degree-one seven-place of discriminant 5521 now eliminates that field because χ_3 is even.

The remaining cubic case has

$$F = \mathbf{Q}(x), \quad x^3 - x^2 - 12x + 1 = 0, \qquad U_2 = 4 - x, \qquad P_0(X) = X^3 - 11X^2 + 28X - 1.$$

At the prime of norm three the stabilizer acts by its nontrivial sign. Every nonzero U_3 in a cubic candidate therefore belongs to the squareclass of U_2 in F. Exact enumeration of the integral elements of this class with all conjugates in $(0, 12]$ gives only $U_3 = U_2$.

In fact the common residual Frobenius at the prime of norm three has nonzero trace. If the trace were zero, its order would be at most two. The finite-unramified quadratic extension $J = K(\sqrt{-3})$ splits at three and at the cusp. Its compositum with the residual point field has absolute degree 2016 and root discriminant at most $2^{7/4}\sqrt{60}\,7^{1/4}$. Retain the cusp term with $(e, f) = (2, 2)$ and the term at three with $e \le 2$ and $f \le 2$. The term with $f = 2$ is a lower bound because F_b is decreasing on its support. At $b = 7.4$ the directed lower bound is greater than 42.91435, whereas the upper bound is less than 42.37919. Consequently $U_3 \neq 0$ in both systems.

J.6 The final comparison

If the unsigned field is quartic, its exact coefficient sieve in Section J.7 gives $\overline{U}_3 \neq \overline{U}_2$ at the cubic dyadic place. This contradicts the signed cubic identity $U_3 = U_2$, since the two systems have the same aligned dyadic residue. Both fields are therefore cubic, so $U_3 = U_2$ holds throughout the simultaneous trace order.

The saturated order modulo two has

$$R = k[X, Y]/(X^4, Y^2), \qquad P = X^2, \qquad R_{\text{degree}\,0} = k[P]/(P^2), \qquad \nu^2 = \bar{\nu}^2 + P.$$

Here degree refers to the four genus twists and $P \neq 0$. Both a_3 and the marked cusp unit w have the same degree, since the cusp Frobenius and the norm-three Frobenius have the same values on the genus characters. The unit w satisfies $w^2 = 1$ in R. Hence a_3/w has degree zero and its square lies in k. Thus U_3 is constant in R. On the other hand, $a_2 = \nu + 2/\nu$, so $U_2 = \bar{\nu}^2 + P$ in R. This contradicts $U_3 = U_2$ and excludes the ordinary order-eight residue.

J.7 The quartic alternative in the order-eight comparison

We use the notation of the order-eight comparison. Let D be the trace field of the source unramified at eleven, let W be its genus-twist stabilizer, and put $F = D^W$. Write $U_j = a_j^2$ for $j = 2, 3$. Assume $[F : \mathbf{Q}] = 4$ and denote its degree-three and degree-one dyadic primes by $\mathfrak{p}_c$ and $\mathfrak{p}_s$, respectively. Both are unramified. The $\mathfrak{p}_s$ packet is ordinary and scalar residually. In its marked entry order the determinant index comparison with the modular order-four witness has length two. The cusp equation and the integral grading give

$$a - 1 \in b^2 A, \qquad a^2 + 7b^2 = -1, \qquad b^2 \sim 2.$$

Here A is the degree-zero order and a, b are the marked cusp coordinates used in the ordinary scalar comparison. The ratio a_3/b is a unit of A, since a_3 and b have the same sign degree. On projecting to $F_{\mathfrak{p}_s} = \mathbf{Q}_2$, write $a = 1 + 2c$ with $c \in \mathbf{Z}_2$. It follows that

$$\frac{b^2}{2} = -\frac{1 + 2c(c+1)}{7} \equiv 1 \pmod 4, \qquad U_3 \equiv 2 \pmod 8 \quad \text{in } F_{\mathfrak{p}_s}.$$

In particular $a_3 \neq 0$.

At $\mathfrak{p}_c$, use the order-four comparison basis and its unit w from the cusp, with $v_2(w-1) = 1/2$. The local coefficient stabilizer acts on w by the same sign as on a_3, since the two Frobenius elements have the same evaluations on the two genus characters. Thus $a_3/w \in F_{\mathfrak{p}_c}^{\times}$. We have $w^2 \in F_{\mathfrak{p}_c}^{\times}$ and $v_2(1 - w^2) = 1$, since $w + 1 = (w - 1) + 2$ also has valuation $1/2$. Over the unramified local ring, the polynomial of $w - 1$ is Eisenstein with discriminant $4w^2$. Hence U_3 has even valuation at $\mathfrak{p}_c$, and the quadratic extension defined by its squareclass has discriminant exponent two.

The odd coefficient restrictions in the order-eight comparison give even valuations of U_2, U_3 at all odd primes of F. For a degree-one prime of F over seven or a degree-two prime over three,

let $\eta \in W$ be the Frobenius in D/F. Since this extension is unramified there, the two squareclass signs of U_2, U_3 are the eigenvalues of η on a_2, a_3. If a valuation is positive, compute its sign after dividing by the corresponding even power of a local uniformizer. The signs agree exactly when η is totally even. At a degree-one seven-prime this yields the projective $\mathbf{F}_7$ comparison in the text. At a degree-two three-prime the projective representation descends to $\mathbf{F}_9$ by the same cocycle argument. Since -1 is a square in $\mathbf{F}_9$, an even inner twist makes its determinant squareclass totally even. The cusp gives absolute irreducibility on the cyclotomic restriction at three. Thus [3, Theorems 1.2(5) and 2.3 and Lemma 2.1] give residual modularity and its transfer to the source. Consequently the squareclass signs must differ at each prime of either kind.

The sieve in [53, `certificates/d15_quartic/q8_weil_certificate.sh`] first enumerates the twenty-six possibilities for U_2, using its Weil bound, square norm, and the absence of degree-one primes over three and five. It then enumerates the integral U_3 with all conjugates in $(0, 12]$ satisfying the conditions just proved. The interval proof of completeness for the integral basis boxes is the same as for the cubic sieve. The unique surviving row is

$$F = \mathbf{Q}(x), \quad x^4 - x^3 - 8x^2 + x + 8 = 0, \qquad U_2 = -x^2 + x + 7, \quad U_3 = x^2 - 2x + 2.$$

At $\mathfrak{p}_c$ the reduction of x has minimal polynomial $x^3 + x^2 + 1$, so U_3 is nonzero modulo two and its residue differs from that of U_2.

The source with the ramified sign at eleven has cubic fixed field. Its reductions at the cubic dyadic prime agree with those of the source unramified at eleven, since the residual representations were aligned. Hence its U_3 is nonzero. The cubic calculation gives $U_3 = U_2$ for that source, contradicting the distinct residues in the quartic row. This excludes the quartic alternative.

The cubic and quartic sieves use the programs and certificates in [53, `d15_support/round4/ordinary/` and `certificates/d15_quartic/`]. The polynomial bounds come from the Weil house and square norm. Exact Sturm counts certify the real roots and the sign conditions, and rational brackets with interval interpolation bound the integral basis coordinates. The remaining class group, decomposition and valuation tests are exact.

K Dyadic trace orders over $\mathbf{Q}(\sqrt{19})$

We use the notation of Section 28. The scalar order bounds come from the middle model and the ordinary filtration, respectively. For the residual S_3 representation we compare the simple models in a finite-flat filtration.

K.1 The nonordinary scalar order

Proof of Proposition 28.7. Let R_0, E_0 be the actual orders. Put $R = R_0/2R_0$, $E = E_0/2E_0$ and $J = \mathrm{rad}(E)$. The algebra E is local with residue $\mathbf{F}_2$ by Section 11. The biconnected regular lattice and Lemma 11.2 place the characters dual to J/J^2 in V_2.

Suppose first that $\mathfrak{v}_1$ is special. After removing its fixed quadratic character, a local matrix difference $L(g) = r(g) - 1$ has top row $(\chi_2(g) - 1, *)$ and bottom row zero. Thus

$$L(g)L(h) = (\chi_2(g) - 1)L(h). \tag{91}$$

Choose Frobenius and inertia differences A, B whose evaluations on V_2 are $(1, 0, 0)$ and $(0, 1, 0)$. Modulo rational two,

$$A^2 = B^2 = AB = BA = 0, \qquad T(A) = T(B) = D(A) = D(B) = 0. \tag{92}$$

If $\mathfrak{v}_2$ is crystalline, exact scalar inertia cuts the first layer to $\langle -1, 6a\rangle$. The two evaluations above span its dual, so A, B generate E. Equation (92) gives $\dim_{\mathbf{F}_2} E \le 3$, contrary to $\operatorname{rank}_{\mathbf{Z}_2} E_0 = 4 \operatorname{rank}_{\mathbf{Z}_2} R_0 \ge 4$. For CR/SP the surviving space is $\langle -1, 2a\rangle$, and the two local special evaluations at $\mathfrak{v}_2$ give the same contradiction. For CR/CR the first layer is contained in $\langle -1\rangle$. A real involution generates E and gives dimension at most two. This excludes every menu containing CR.

For SP/SP, choose in addition the difference $C = c - 1$ of a negative-real conjugation. Its evaluation is $(1, 0, 1)$, so A, B, C generate E. This remains true when the actual first layer is a proper quotient of the one described by V_2. We have $C^2 = T(C) = D(C) = 0$. Polarized Cayley–Hamilton gives

$$AC + CA = X1, \qquad BC + CB = Y1, \qquad X = T(AC), \quad Y = T(BC).$$

Every word reduces to the R-span of

$$1, A, B, C, AC, BC. \tag{93}$$

The maximal ideal $\mathfrak{m}$ of R is generated by X, Y. The trace and determinant laws here are the original R-valued laws. All scalar identities below are identities of the determinant law, which takes values in R.

Work over $R' = R/\mathfrak{m}^2$. The normal form (93) gives

$$J^2 \subset \mathfrak{m}E + R'AC + R'BC, \qquad T(J^3) = 0, \qquad D(J^2) = 0,$$

and, for every $Z \in J^2$,

$$D(A + C + Z) = X. \tag{94}$$

Indeed $D(AC) = D(BC) = 0$. Their mutual polar term is a product of X, Y, and their polar pairings with $A + C$ have trace zero. Pairings with $\mathfrak{m}E$ lie in $\mathfrak{m}^2$. These computations establish (94) in the determinant law over R'. At $\mathfrak{v}_2$, the Frobenius and inertia differences have first layers

$$A' = C \pmod{J^2}, \qquad B' = A + B + C \pmod{J^2}.$$

The special identity $T(A'B') = 0$ gives $X + Y = 0$ in R'.

Let $\Phi(u)$ be the regular Frobenius generator of Lemma 11.6. The Eisenstein polynomial $u^2 - 26u - 2$ reduces to u^2 modulo two. The ramified coordinate of the first-layer extensions in Lemma 28.2 gives

$$\Phi(0) = A + C + Z, \qquad Z \in J^2.$$

Its determinant has zero constant term modulo two. By (94), $X = 0$ in R', and then $Y = 0$. Nakayama yields $\mathfrak{m} = 0$, hence $R = \mathbf{F}_2$ and $\operatorname{rank}_{\mathbf{Z}_2} R_0 = 1$. The argument applies without change to a simultaneous order of members with the same exact special characters.

Let σ denote the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$. If the two special characters are θ_1, θ_2, choose a rational quadratic $\eta \in \{1, \kappa_{-1}, \kappa_3, \kappa_{-3}\}$ whose restriction at three is $\theta_1\theta_2$. It is unramified at $K_{\mathfrak{p}_2}$, and r and $r^\sigma \otimes \eta$ have the same exact local types. Both are nonordinary scalar-residual. Their simultaneous trace order has rank one, so their traces agree. Thus

$$r^\sigma \simeq r \otimes \eta. \tag{95}$$

Apply Lemma 12.4. □

K.2 The ordinary scalar order

Lemma K.1. *There is no supported continuous cochain $x : G_K \to \mathbf{F}_2$ satisfying*

$$dx = \kappa_3 \smile \kappa_{-2}, \qquad x|_{G_{K_{\mathfrak{p}_2}}} = 0, \qquad x|_{G_{K_{\mathfrak{v}}}} \in \mathbf{F}_2\kappa_3 \quad (\mathfrak{v} \mid 3).$$

Proof. Put $H = K(\sqrt{3})$ and $\alpha_0 = 1+\sqrt{3}$. Since $\mathrm{N}_{H/K}(\alpha_0) = -2$, the usual dihedral embedding construction gives a cochain x_0 with $dx_0 = \kappa_3 \smile \kappa_{-2}$ whose restriction to G_H is the Kummer character of α_0. All other supported solutions differ by a quadratic character χ_β of K. The asserted local conditions would make $\alpha_0\beta$ square at every H-prime over three and at a chosen H-prime over two.

At three, α_0 is square in $\mathbf{Q}_3(\sqrt{3})$, since it is one modulo the uniformizer. At two it has odd valuation. Write

$$\beta = \pi(-1)^i\epsilon^j a^k (a')^l, \qquad i,j,k,l \in \mathbf{F}_2.$$

At $\mathfrak{v}_1$, a is a nonsquare unit and $a'/3$ is a square unit. At $\mathfrak{v}_2$ their roles are reversed. Both $-1, \epsilon$ are nonsquare units, and π is square at both places. The two conditions over three are $i+j+k = i+j+l = 0$. The four possibilities are

$$\beta \in \pi\{1, 3, -6, -2\} \quad \text{in } K^\times/K^{\times 2}.$$

Choose $\sqrt{3} = cs$ in $K_{\mathfrak{p}_2}$ with $c^2 = 3/19$ and $c \equiv 9 \pmod{16}$. In $\mathcal{O}_{K_{\mathfrak{p}_2}}/8$,

$$\alpha_0/\pi = (57c - 13)/2 + (3 - 13c)s/2 = 2 + 7s.$$

Since 3 is square and $[\epsilon] = [2]$ locally, the four tests reduce to $2+7s$ and $(2+7s)(-\epsilon) = 1+4s$ modulo eight. The complete list of unit squares in $(\mathbf{Z}/8)[s]/(s^2-19)$ is

$$1, \quad 3, \quad 5+4s, \quad 7+4s.$$

Neither candidate occurs. □

Proof of Proposition 28.8. Choose a vector v on the marked ordinary line and take its global orbit lattice. Its reduction is cyclic and nontrivial, with trivial semisimplification by Lemma 28.4. The invariant line of its nonzero unipotent extension is transverse to the marked ordinary line. Otherwise the orbit of v would span only that line. At two both lines are preserved and both characters are trivial, so the local residue is trivial. The localization kernel in (37) makes the global residue, after scaling,

$$\bar{r} = I + \kappa_3 E_{12}. \tag{96}$$

This is ramified at both three-primes. It cannot occur at a CR place, whose actual inertia is the fixed scalar quadratic character and reduces identically to one.

Fix an inertia element g at $\mathfrak{v}_1$ with $\kappa_3(g) = 1$, and write $\theta = \theta_1(g) \in \{\pm 1\}$. Special monodromy gives $(r(g) - \theta)^2 = 0$. The basis $((r(g)-\theta)v, v)$ is integral, because its residual vectors are independent, and in this basis

$$r(g) = \begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix}, \qquad \text{marked ordinary line} = \langle e_2 \rangle. \tag{97}$$

The basis is unchanged by rescaling v, up to a common scalar. The centralizer of this matrix intersected with the stabilizer of e_2 consists only of scalars. Hence this normalization removes the framing directions for every member of the collection.

Let B be the closed $\mathbf{Z}_2$-algebra generated by all simultaneous matrix entries in these normalized bases. It is finite free and local, with residue exactly $\mathbf{F}_2$: every residual entry in (96)

belongs to $\mathbf{F}_2$, including after passing to any factor of the normalization. The representation on B^2 is an actual stable lattice in ordinary crystalline constituents, hence is Barsotti–Tate. Its marked line is the multiplicative sublattice Be_2, and its quotient is etale. Coefficient actions extend to the Barsotti–Tate group by full faithfulness. For a quotient of B, close the generic coefficient kernel and take the finite-flat quotient in this ordinary group. The image in the etale quotient is finite etale, and the remaining subgroup is multiplicative. Thus the coefficient quotients retain the actual ordinary finite-flat condition.

Take a tangent $B \to \mathbf{F}_2[\delta]/(\delta^2)$ and write

$$r_\delta(h) = (I + \delta X(h))\bar{r}(h), \qquad X = \begin{pmatrix} x & y \\ z & x \end{pmatrix}.$$

Fixed determinant gives the trace-zero shape. The cocycle equations for (96) are

$$z \text{ additive}, \qquad dx = dy = \kappa_3 \smile z, \qquad q := x + y \text{ additive}. \tag{98}$$

The exact normalization gives $X(g) = 0$. At two, the marked line and the etale quotient imply

$$y = 0, \qquad x = \kappa U_2, \qquad \kappa \in \mathbf{F}_2,$$

where U_2 is the unramified quadratic character. The class z is a unit Kummer class. At each three-prime the unit upper entry of special inertia recovers its line as a direct summand over B. Varying that line to $e_1 + \delta t_{\mathfrak{v}} e_2$, with its two exact characters fixed, gives

$$z|_{G_{K_{\mathfrak{v}}}} = 0, \qquad x|_{G_{K_{\mathfrak{v}}}} = t_{\mathfrak{v}}\kappa_3 \quad (\mathfrak{v} \mid 3).$$

The arithmetic preceding Lemma 28.2 now forces $z \in \mathbf{F}_2\kappa_{-2}$.

If $z = 0$, the additive x is unramified at two and lies in $\langle \kappa_{-1}, \kappa_3 \rangle$. The absence of an unramified component at three kills the first coefficient, and $X(g) = 0$ kills the second. Thus $x = \kappa = 0$. Then $y = q$ is split at two, so it is a multiple of κ_3, again killed by $y(g) = 0$. Since the entries generate B, this tangent is zero. The cotangent dimension is therefore at most one.

For a nonzero tangent, $z = \kappa_{-2}$. Lemma K.1 shows $\kappa \neq 0$, hence $\kappa = 1$. The additive q is unramified at two with value U_2. Since $q(g) = 0$, we have $q = \kappa_{-1}$. At a real conjugation c we have $\kappa_3(c) = 0$ and $z(c) = q(c) = 1$, so

$$X(c) = \begin{pmatrix} x(c) & x(c) + 1 \\ 1 & x(c) \end{pmatrix}, \qquad \det X(c) = x(c)^2 + x(c) + 1 = 1. \tag{99}$$

The last equality uses the residue field $\mathbf{F}_2$ of the actual normalized order.

Put $T = B/2B$. If its maximal ideal is nonzero, the cotangent bound gives $T = \mathbf{F}_2[t]/(t^n)$ with $n \geq 2$. At a real conjugation, in $M_2(T)$ we have $r(c) - I = tX(c) + t^2Y$. Total oddness gives the exact identity $\det(r(c) - I) = 0$ in B. Reducing it to T and using (99), we obtain

$$0 = t^2 \det(X(c) + tY),$$

and the second factor is a unit. Hence $t^2 = 0$ and $\operatorname{rank}_{\mathbf{Z}_2} B = \dim_{\mathbf{F}_2} T \leq 2$. The trace order is a submodule of B and has the same upper bound.

Finally let ν be the unramified ordinary character, in the crystalline normalization. We have $a_2 = \nu(\mathrm{Frob}_2) + 2/\nu(\mathrm{Frob}_2)$. In the unique nonzero tangent, $\nu(\mathrm{Frob}_2) = 1 + \delta$, so $a_2 - 1$ generates the cotangent. Nakayama and completeness show $B = \mathbf{Z}_2[a_2]$. This also holds when the cotangent is zero. □

K.3 The global norm kernel for S_3

Let $L = K(\alpha)$, where $\alpha^3 - 2\alpha - 2 = 0$. This field has signature $(2, 2)$, discriminant $2^8 19^3 = 1755904$, and class number one. An absolute polynomial is

$$y^6 - 61y^4 + 4y^3 + 1087y^2 + 220y - 5487.$$

There is one dyadic prime, of absolute ramification index six and residue degree one, and four primes over three: a degree-one and a degree-two prime over each $\mathfrak{v}_i$. The degree-six arithmetic, including certification of the class group, is given in `s3_extension_layers.gp` of [52].

The three-letter permutation module is $\mathbf{F}_2 \oplus V$. Shapiro's lemma therefore gives

$$H^1(G_{K,S}, V) = \ker\bigl(L(S_L, 2) \xrightarrow{\mathrm{N}_{L/K}} K(S, 2)\bigr). \tag{100}$$

The two squareclass spaces have dimensions nine and five. The norm is surjective since $[L : K] = 3$, so the kernel has dimension four. Define

$$\beta = \pi/\alpha, \quad \tau = \frac{2s}{3\alpha^2 - 2}, \quad z = \alpha + \tau, \quad z_{19} = \alpha - 3 - s, \quad q = \alpha^2 - \alpha + 1,$$

and the squareclasses

$$e = \alpha + 1, \quad b = 3(\alpha - 1), \quad c = (5 + s)z, \quad d = -\frac{z_{19}\beta^8 q}{sz^2}. \tag{101}$$

They are supported at two, three and infinity. The square roots of their norms to K can be chosen as

$$1, \quad 9, \quad 2(5 + s)^2, \quad \frac{(\pi^3/2)^4}{4},$$

respectively. In particular they lie in (100). The identities

$$(3 - \alpha)q^2 = \frac{19}{\alpha + 1}, \qquad (\alpha - 3)^2 - 19 = (\alpha - 3)(\alpha + 2)^2,$$

$$(\alpha^2 - 2)^2 = 2(2 - \alpha)(\alpha + 1), \qquad \alpha^4(\alpha - 1) = 2(\alpha + 2)$$

give the support and norm assertions directly. For the last class, denote the primes in the corresponding factorizations by P_2, A_3, P_{19}, Q_{19}. We have

$$(z_{19}) = P_2^2 A_3^2 P_{19}, \quad (q) = Q_{19}, \quad (s) = P_{19}Q_{19}, \quad (z) = P_2^5 A_3, \quad (\beta) = P_2.$$

thus d is a unit.

At the degree-one primes over $(\mathfrak{v}_1, \mathfrak{v}_2)$, the unramified quadratic coordinates, and at the degree-two primes the ramification coordinates, are as follows:

	degree-one Frobenius		degree-two inertia	
	$\mathfrak{v}_1$	$\mathfrak{v}_2$	$\mathfrak{v}_1$	$\mathfrak{v}_2$
e	1	1	0	0
b	0	0	1	1
c	0	1	1	0
d	0	1	0	0

(102)

These coordinates include the nonsquare-unit tests after removing even valuations. For example, at the degree-one prime over $\mathfrak{v}_1$ one may use $\alpha \equiv 4$ and $s \equiv 10 \pmod{27}$ to test d.

The dyadic conductors of all fifteen nonzero combinations are

class	e	b	eb	c	ec	bc	ebc	every class containing d
conductor	8	8	2	6	8	8	6	12

(103)

Every combination is locally nontrivial, so the four classes in (101) are independent and form the full global kernel. One way to compute the table is to normalize $v(\beta) = 1$, so $v(\alpha) = 2$ and $v(2) = 6$, and use

$$\alpha\beta = \pi, \qquad \alpha^2\beta^2 - 26\alpha\beta - 2 = 0, \qquad \alpha^3 = 2(\alpha + 1).$$

For instance, $\alpha = \beta^2 + \beta^4 - \beta^5 \pmod{\beta^6}$. The quadratic defect of e and b is five, that of c and ebc is seven, and that of eb is eleven. Every class containing d has defect one. The conductor formula is $13 - \text{defect}$. Equivalently these are the exact dyadic valuations of the relative discriminants of $X^2 - u$ for the fifteen explicit u, as computed by the accompanying script.

For a normal tangent in the V summand of $\mathrm{ad}^0(V)$, the necessary special Frobenius-trace condition at $\mathfrak{v}_i$ is the vanishing of the degree-one coordinate in (102). A residual Frobenius is a transposition. In the three-letter permutation module its fixed letter is the degree-one completion, and the trace derivative is that coordinate. A crystalline scalar condition instead kills the degree-two inertia coordinate. If a class is written additively as $Ee + Bb + Cc + Dd$, the four conditions are

$$\mathrm{SP}_1 : E = 0, \quad \mathrm{SP}_2 : E + C + D = 0, \qquad \mathrm{CR}_1 : B + C = 0, \quad \mathrm{CR}_2 : B = 0.$$

Their kernels and the directions of smaller conductor are

$(\mathfrak{v}_1, \mathfrak{v}_2)$	candidate space	smaller-conductor line	upper break over L_2
SP, SP	$\langle b, c + d\rangle$	b	7
SP, CR	$\langle c, d\rangle$	c	5
CR, SP	$\langle e + b + c, e + d\rangle$	$e + b + c$	5
CR, CR	$\langle e, d\rangle$	e	7

(104)

In each row the other two nonzero elements have upper break eleven.

K.4 The three local models

Over $\mathbf{F}_2((u))$, the local generic module for V has Frobenius matrix A_1 below. Its height-one integral models, with their generic identifications, are exactly

$$A_1 = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ u^2 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & u^2 \\ u & 0 \end{pmatrix}. \tag{105}$$

To see exhaustiveness, take the apartment of the two coordinate lines in the Bruhat–Tits tree, with vertex $M_k = \mathbf{F}_2[[u]]u^k e_1 + \mathbf{F}_2[[u]]e_2$. Frobenius sends k to $1 - 2k$ and doubles distances from the apartment. A vertex at distance r from it, projecting to k, has distance $3r + |1 - 3k|$ from its Frobenius image. Height one at absolute ramification two requires this to be at most two. Thus $r = 0$ and $k = 0$ or 1. The permitted homotheties give exactly (105).

Write an extension matrix as $\left(\begin{smallmatrix} A_i & B \\ 0 & A_j \end{smallmatrix}\right)$, and write a, d for the diagonal entries of B. If $A_i = \left(\begin{smallmatrix} 0 & u^{r_i} \\ u^{s_i} & 0 \end{smallmatrix}\right)$, then $(r_i, s_i) = (1, 0), (0, 2), (2, 1)$. Integrality of $u^2 A_i^{-1} B A_j^{-1}$ gives

$$v_u(a) \geq \max(0, r_i + s_j - 2), \qquad v_u(d) \geq \max(0, s_i + r_j - 2). \tag{106}$$

Put $u = t^3$. The diagonal trivializations have exponent pairs

$$P_1 = (-1, -2), \qquad P_2 = (-4, -2), \qquad P_3 = (-4, -5).$$

The raw pole orders in the V summand are bounded by

$$P_{i1} - 2P_{j1} - 3v_u(a), \qquad P_{i2} - 2P_{j2} - 3v_u(d).$$

Further coefficient terms subtract three, and Artin–Schreier reduction divides every even pole by two. Substituting (106) gives the following maxima

submodule/quotient	A_1	A_2	A_3
A_1	1	1	7
A_2	none	1	1
A_3	none	none	1

(107)

The A_1/A_3 entry also permits pole five: its second raw sequence begins $8, 5, 2$, whose reduced orders are $1, 5, 1$. Thus both breaks five and seven force submodule A_1 and quotient A_3, while break eleven is impossible. The offdiagonal entries of B give the scalar summand. The same calculation, with bounds $r_i + r_j - 2$ and $s_i + s_j - 2$, gives maximum pole three after the tame cubic extension, hence scalar upper break at most one over $K_{\mathfrak{p}_2}$.

The field-of-norms comparison preserves the breaks just used. After the tame cubic extension, the first ramification break of the dyadic radical tower is twelve. The breaks five, seven and eleven in (103) are smaller. The usual ramification comparison for this tower therefore identifies them with the Artin–Schreier poles of the generic Kisin modules by [79, Proposition 3.3.2 and Corollary 3.3.6]. This is the same comparison as in Lemma 11.2, with the absolute ramification index multiplied by three.

K.5 The cotangent and the length

Proof of Proposition 28.9. In characteristic two we have $\mathrm{ad}^0(V) = \mathbf{F}_2 I \oplus V$. The V component of a permissible tangent belongs to the corresponding row of (104). Table (107) excludes the two break-eleven directions, so it lies in the single displayed line.

Suppose its V component is zero. At a special place let F be Frobenius, with eigenvalues $3\theta, \theta$, and let N denote the scalar-normalized inertia difference. The exact source identities include

$$(F - 3\theta)N = 0, \qquad N(F - \theta) = 0.$$

The residue is unramified there, and $\bar{F}$ is a nonidentity transposition. Scalar infinitesimal inertia $\delta c I$ would give $c(\bar{F} - I) = 0$, hence $c = 0$. At a crystalline place the inertia is exactly the fixed scalar and gives the same unramifiedness directly. The scalar local pole bound and Lemma 28.2 now leave only κ_{-1} globally. This is a strict change of basis. If S is an order-three matrix in the residual image, then a transposition conjugates S to $S^2 = S + I$, whereas rotations centralize it. Thus

$$gSg^{-1} - S = \kappa_{-1}(g) I.$$

Strict equivalence permits every infinitesimal GL_2 change of basis. The fixed determinant does not restrict conjugating matrices to trace zero. The scalar direction therefore vanishes in the trace deformation problem.

Let R_0 be the simultaneous trace order. Absolute residual irreducibility reconstructs the image algebra as $M_2(R_0)$ and supplies the actual lattice R_0^2. Its dyadic Barsotti–Tate group has finite-flat quotient modulo rational two. The cotangent calculation gives

$$R_0/2R_0 = \mathbf{F}_2[t]/(t^n)$$

for some $n \geq 1$. If $n \geq 2$, its first-order V class is the line of break five or seven in (104).

Suppose $n \geq 3$. Close the generic t-adic filtration inside this single finite-flat group and take its successive finite-flat quotients. Apply the exact Kisin anti-equivalence of [45, Corollary 4.3 and Proposition 5.7.3], reversing the filtration. Every consecutive two-factor quotient has the same nonzero generic first-order class, up to a nonzero scalar. By (107), its lower simple model is A_1 and its upper simple model is A_3. For three successive factors the middle object would therefore have to be A_3 as one quotient and A_1 as the next subobject. These models are not isomorphic. This contradiction proves $n \leq 2$ and $\operatorname{rank}_{\mathbf{Z}_2} R_0 \leq 2$. □

Remark K.2. The length bound uses the common middle object of two adjacent extensions. The three simple models in (105) are essential for the comparison of breaks five and seven.

K.6 The Hilbert comparison form

The newspace of parallel-weight-two Hilbert cusp forms of level (3) and trivial central character has dimension 14, with constituent dimensions $2, 2, 5, 5$. On each two-dimensional constituent the operator T_2 has polynomial $X^2 - 2$, the operators at the two three-places are opposite scalar signs, and the operator at the tested prime above five is $-I \pm 2T_2$. The complete Magma program `hilbert_seed_certificate.m` in [52] constructs the space and checks these identities. The proof uses the two exact local signs and the odd trace at the tested prime.

AI disclosure

During the preparation of this article, the authors used ChatGPT (OpenAI) as an assistant in mathematical research and manuscript preparation. Its use included exploring proof strategies, checking mathematical arguments, investigating possible simplifications, assisting with computations and verification scripts, checking references, and drafting and revising portions of the exposition. AI-generated suggestions and calculations were critically examined and revised where necessary. The authors take full responsibility for the mathematical content, proofs, computations, and conclusions of the article.

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