Serre modularity over real quadratic fields

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Abstract

Assuming GRH for Dedekind zeta functions, we prove Serre's conjecture over the real quadratic fields $\mathbf{Q}(\sqrt{d})$ for

$$d \in \{2, 3, 5, 6, 7, 10, 11, 13, 17, 21, 33, 37, 41, 57, 65, 73, 89\}.$$

More precisely, every continuous, absolutely irreducible, totally odd two-dimensional Galois representation over these fields in odd residual characteristic is modular. The result for $d = 17$ is unconditional; for $d = 33$, GRH can be replaced by the assertion that Dedekind zeta functions have no zeros with real part greater than $31/32$. The proof combines conductor reduction with local ramification estimates and global discriminant bounds. Its principal new ingredients are a modularity criterion for 2-adic representations with scalar reduction and integral trace arguments that force quadratic descent or bound coefficient fields. Uniform local lifting constructions allow these methods to treat arbitrary conductor.

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1 Introduction

Let K be a real quadratic field. We study Serre's conjecture for continuous absolutely irreducible totally odd representations $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, with p odd and with arbitrary conductor. Here modularity means occurrence in the reduction of the Galois representation of a regular algebraic cuspidal Hilbert modular form. We do not prescribe its weight or level.

Over $\mathbf{Q}$, the conjecture originated in Serre's questions about modular eigenvalues modulo a prime and was subsequently refined by his predictions for weight, level and character [58, Section 3, Problem 1], [60]. The lifting theorems of Wiles, Taylor–Wiles and Skinner–Wiles, together with potential modularity, laid the foundations for its proof [74, 67, 62, 66].

An essential step in this development was the construction of compatible systems containing a given representation. Dieulefait proved such an existence theorem over $\mathbf{Q}$, in its original crystalline range, using potential modularity and descent [18, Theorem 1.1]. This permits a change of residual characteristic while retaining a single characteristic-zero compatible system. The method was used to establish cases of the Fontaine–Mazur conjecture and small-level cases of Serre's conjecture [19, 39]. Khare and Wintenberger subsequently proved Serre's conjecture in full generality [40, 41], with Kisin's two-adic modularity lifting theorem providing the final dyadic input [43]. Later lifting theorems permit shorter versions of this proof, as in Dieulefait–Pacetti [24]. Over totally real fields, the corresponding strategy of connecting residual representations through compatible systems is formulated in the Rayuela conjecture [23].

Over a real quadratic field, discriminant bounds need not exclude all small-characteristic representations of small conductor. For example, the elliptic curve

$$y^2 + \sqrt{6}xy + (1+\sqrt{6})y = x^3 + (1-\sqrt{6})x^2 - (1+3\sqrt{6})x - 1 - 2\sqrt{6}$$

has everywhere good reduction over $\mathbf{Q}(\sqrt{6})$; its label is `2.2.24.1-1.1-a1` in [45]. Thus the base cases must establish modularity of surviving representations, rather than only nonexistence. In addition, residual reducibility and ordinarity of a characteristic-zero source are different conditions. A congruence is useful only when modularity can be transferred back to that actual source.

Our starting point is the characteristic-three method of Dieulefait–Pacetti over $\mathbf{Q}(\sqrt{13})$ [25]. Successive discriminant comparisons restrict the residual images, and a conductor reduction relates arbitrary representations to the remaining base case. We retain this distinction between the residual base and the conductor argument, remove the restriction at two, and supply new source theorems where residual modularity alone is insufficient. Fontaine's local bounds, refined local ramification calculations, and global discriminant estimates provide the arithmetic input [31, 46, 47]; finite-flat groups and integral trace orders control the scalar and dihedral cases.

The results below use the lifting and automorphic inputs specified in Sections 2 and 5. In particular, statements involving the split-dyadic scalar argument are relative to the dyadic structural and propagation results listed there, including the cited preprints. These inputs are distinct from the analytic hypothesis GRH. Write d for a squarefree radicand.

Theorem 1.1. *Let*

$$d \in \{2,3,5,6,7,10,11,13,17,21,33,37,41,57,65,73,89\}, \qquad K = \mathbf{Q}(\sqrt{d}).$$

Assume GRH for Dedekind zeta functions. For every odd prime p, every continuous absolutely irreducible totally odd representation

$$\bar{\rho} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_p)$$

is modular.

For $d = 17$ the theorem holds without GRH. For $d = 33$ the proof also works with the zero-free hypothesis stated in Corollary 29.2. The analytic arguments for both variants are included in Section 31. Further nonordinary source arguments and individual residual-image exclusions hold without GRH; these refinements are recorded in the corresponding proofs.

1.1 Structural theorems

The field list is an application of three mechanisms: conductor reduction to a specified terminal characteristic, scalar modularity from a quadratic Selmer condition, and coefficient-field rigidity from simultaneous integral trace orders. We state these mechanisms separately, since their hypotheses have different roles.

Conductor reduction. A strongly compatible system, and the conventions for weight two, are specified in Section 2. The following theorem concerns a system already constructed in characteristic zero; it does not supply an initial lift of an arbitrary residual representation.

Theorem 1.2 (Conductor reduction at a split terminal prime)**.** *Let K be real quadratic, suppose two does not split in K, and let ℓ be an odd rational prime that splits in K. Assume:*

(i) every absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ of determinant $\bar{\chi}_\ell$ and unramified outside ℓ is modular;
(ii) for every reducible representation with that determinant and ramification support, its constituent quotient is totally odd and extends to $G_{\mathbf{Q}}$.

Then every nondihedral potentially modular weight-two strongly compatible system over K, whose determinant is cyclotomic times a finite totally even character, is modular.

The proof is given in Section 6. The quotient in hypothesis (ii) is automatically totally odd, since its two constituents have product $\bar{\chi}_\ell$.

A scalar Selmer condition. Suppose two splits in K, with dyadic places w_0, w_1. Put

$$V^+_{w_0}(K) = \left\{ [a] \in K^\times/K^{\times 2} : \begin{array}{l} a \text{ is totally positive,} \\ v_{\mathfrak{q}}(a) \equiv 0 \pmod 2 \quad (\mathfrak{q} \nmid 2), \\ a \in K^{\times 2}_{w_1} \end{array} \right\}. \tag{1}$$

This finite squareclass space classifies totally real quadratic extensions unramified outside w_0 and split at w_1. Quadratic conjugation exchanges the two choices of dyadic place. The condition we use is

$$V^+_{w_0}(K) = 0. \tag{SV}$$

It is not an odd-narrow-class-number condition.

Theorem 1.3 (Scalar dyadic modularity)**.** *Let K be a real quadratic field in which two splits and* (SV) *holds. Under the dyadic structural, propagation and classicality inputs of Section 5, let $E/\mathbf{Q}_2$ be finite and let $r : G_K \to \mathrm{GL}_2(E)$ be continuous and absolutely irreducible, unramified outside finitely many places, with $\det r = \varepsilon_2$ and $\bar{r}^{\mathrm{ss}} = \mathbf{1} \oplus \mathbf{1}$. Suppose that at both dyadic places r is absolutely irreducible, de Rham and has distinct Hodge–Tate weights. Then r is automorphic.*

Sections 7–8 construct the auxiliary fields and prove the theorem. Together with a residual base at two and a two-power ordinary class number, it gives the split-dyadic Serre criterion of Theorem 10.1. The seven split-dyadic fields in the main theorem satisfy these arithmetic conditions. The Selmer vanishing is also valid when the narrow class number is even.

Integral trace orders and symmetry. Let $\mathcal{R}$ be a strongly compatible system of cyclotomic determinant over a real quadratic field K, with trace field D. For finitely many coefficient orbits, form their simultaneous integral matrix order E_0, including the trace scalars and every dyadic coefficient place, as in Section 11. Its rank is four times the sum of the trace-field degrees. The integral order, rather than a single residual two-dimensional realization, is essential here.

Theorem 1.4 (A trace-order bound and quadratic symmetry)**.** *Let $K/\mathbf{Q}$ be real quadratic with conjugation σ, and let $\mathcal{R}$ be an absolutely irreducible strongly compatible rank-two system with cyclotomic determinant and trace field D. Let h, η be restrictions of quadratic characters of $G_{\mathbf{Q}}$. Form the coefficient orbits generated by*

$$H(\mathcal{R}) = \mathcal{R} \otimes h, \qquad G(\mathcal{R}) = \mathcal{R}^{\sigma} \otimes \eta.$$

Suppose $\mathcal{R} \not\simeq \mathcal{R} \otimes h$ and their simultaneous integral order satisfies $\dim_{\mathbf{F}_2}(E_0/2E_0) < 32$. Then either

$$\mathcal{R}^{\sigma} \simeq \mathcal{R} \otimes \eta \quad \text{or} \quad \mathcal{R}^{\sigma} \simeq \mathcal{R} \otimes \eta h,$$

or $[D : \mathbf{Q}] \leq 2$ or $D/\mathbf{Q}$ is biquadratic. In the latter alternatives every coefficient prime has residue degree at most two. In the former alternatives, a totally odd regular system is modular by quadratic descent.

The arithmetic work in applying Theorem 1.4 lies in bounding the actual order. The finite-flat ramification theorem 11.7 and the regular-order construction in Section 11 supply different parts of this argument. The bound of twenty-six over $\mathbf{Q}(\sqrt{11})$ then gives the coefficient-field alternative directly; the bound of ten over $\mathbf{Q}(\sqrt{3})$ uses Corollary 12.4.

Two further source theorems have a separate role. Proposition 20.3 constructs the initial lift in the exceptional characteristic-five case over $\mathbf{Q}(\sqrt{5})$, which the general prescribed-lift theorem excludes. Theorem 21.36 proves modularity of the nonordinary reducible characteristic-three source over $\mathbf{Q}(\sqrt{21})$. Its proof kills the anti-invariant part of the full pseudodeformation cotangent space and applies Lemma 12.1; no finiteness theorem for the whole pseudodeformation ring is needed. These are source theorems, not consequences of residual-image bounds alone.

1.2 Field applications

The terminal characteristics of the outer conductor reductions are listed below. A source proof may use other coefficient characteristics internally. At every backward step the lifting theorem is applied to the actual old source, with its weight and local ordering retained.

Radicands d	Terminal prime	Source arguments
$7, 13, 37$	3	Supported residual bases and auxiliary composita.
$2, 3, 6, 10, 11$	$7, 13, 5, 3, 5$	Finite-flat extensions, trace orders and descent.
5	11	Exceptional initial lift and split-eleven sources.
21	5	Mixed local types and scalar or dihedral rigidity.
$17, 33, 41, 57, 65, 73, 89$	2	The scalar Selmer criterion and a residual base at two.

Part I develops the lifting constructions, conductor criteria, scalar theorem and integral-order methods. Part II proves the required source theorems. Part III verifies the remaining residual bases and assembles the field results. The appendices specify the longer finite calculations; their programs and outputs are provided in [49].

Buzzard–Diamond–Jarvis formulated the weight conjecture at unramified residual primes [10]; see also [34]. Our modularity results yield the weight part of Serre's conjecture under the image hypotheses of Gee–Liu–Savitt [36, Theorem 6.2.1]. Namely, suppose the residual representation remains absolutely irreducible after adjoining the roots of unity of order the residual characteristic and, in characteristic five, that the projective image of this restriction is not the alternating group on five letters. Then the occurring Serre weights are exactly those prescribed by the local crystalline-lift recipes, including at ramified residual primes and for all predicted companion weights. These consequences inherit the assumptions of the corresponding modularity theorem; the weight theorem introduces no further analytic hypothesis.

Part I
General criteria and local methods

2 Compatible systems and congruences

We use d for the squarefree radicand, D for a fundamental discriminant, and q for the residual coefficient-field cardinality. The finite arithmetic and numerical checks are supplied in [49]; the three longer certificate descriptions appear in the appendices.

Throughout, $G_F = \mathrm{Gal}(\overline{F}/F)$ and the cyclotomic character χ_p has labelled Hodge–Tate weight one. Irreducibility means absolute irreducibility. A representation is totally odd if every real complex conjugation has determinant -1.

A system over a totally real field is a pure, regular, totally odd, absolutely irreducible, strongly compatible rank-two system. We require compatibility of the Frobenius-semisimplified Weil–Deligne representations at every finite place, including a place above the coefficient characteristic, and compatibility of labelled weights. Its conductor support consists of the places where the common Weil–Deligne representation is ramified or has nonzero monodromy. Outside this support its p-adic member is crystalline above p and unramified at the other places. Weight two means labelled weights $\{0, 1\}$. Regular weights may include negative integers.

On the closure of a special local component the Frobenius eigenvalues have ratio $\mathrm{N}(v)$. For a pure system, monodromy cannot vanish at such a place: otherwise both eigenvalues would have the same complex absolute value, contradicting that ratio.

We write $\mathcal{R} \rightsquigarrow \mathcal{R}'$ when a congruence, possibly followed by a finite character twist, has the property that modularity of $\mathcal{R}'$ implies modularity of $\mathcal{R}$. Every such implication below is an application of a lifting theorem to the actual old characteristic-zero member. Finite twists preserve modularity. Equality of good Frobenius polynomials and Chebotarev then propagate modularity from one member to the whole system.

2.1 Residual witnesses and lifting

Lemma 2.1. *Let F be totally real of even degree. An irreducible residual representation with a regular algebraic cuspidal Hilbert modular witness has a parallel-weight-two cuspidal witness, after allowing arbitrary level.*

Proof. Transfer the witness to the quaternion algebra over F ramified at every real place and split at every finite place. At sufficiently small level $U = U^p U_p$, reduction of a primitive integral eigenvector is nonzero, with coefficient module V over a finite field k. Its U_p-action factors through a finite quotient G; write U_p' for the kernel. The map $V \hookrightarrow \mathrm{Map}(G, V)$ given by $v \mapsto (g \mapsto gv)$ is equivariant for $(hf)(g) = f(gh)$. The target is a sum of regular representations.

Pointwise application and induction in stages give the same away-from-p Hecke eigencharacter in scalar forms of level $U^pU'_p$. Their integral space is $\mathcal{O}^X$ for a finite double-coset set X, so reduction is surjective. Deligne–Serre's lifting lemma [17, Lemme 6.11] lifts the eigencharacter to characteristic zero. A norm character would have reducible residue; inverse Jacquet–Langlands therefore gives a parallel-weight-two cuspidal form.

Retaining both operators T_v, S_v, with polynomial $X^2 - T_vX + \mathrm{N}(v)S_v$, preserves the residual trace and determinant. Chebotarev and Brauer–Nesbitt identify the residue. No exact characteristic-zero central character or ordinary ordering is prescribed by this argument. □

Totally odd solvable residual representations have solvable Artin witnesses, followed by regular weight raising, in the scopes of automorphic induction and Langlands–Tunnell; see [3, Theorem 1.2(1), Lemma 2.1 and proof of Theorem 2.5]. Lemma 2.1 gives weight two whenever it is needed. We use Dickson's classification and the rank-two adequacy criterion [6, Proposition A.2.1] throughout.

Lemma 2.2. *Let K be real quadratic, $p > 2$, and let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be finitely ramified and totally odd, with modular residue whose restriction to $G_{K(\zeta_p)}$ is irreducible. Then r is modular in either case:*

(i) p splits in K, and r is potentially semistable with distinct labelled weights at the places above p;
(ii) r is de Rham of labelled weights $\{0, 1\}$ above p, and its residue has a parallel-weight-two modular witness.

Proof. Apply respectively [70, Theorem 4.7] and [3, Theorem 2.3], dualizing to their inverse-cyclotomic normalization when required. In (ii), the parallel-weight-two witness gives the required weight-zero automorphic convention. □

Lemma 2.3. *Let F be a totally real extension of a real quadratic field, and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be finitely ramified and totally odd, with determinant $\chi_2\psi$, where ψ has finite order. Suppose its residue is nonsolvable. Then r is modular in either case:*

(i) r is nearly ordinary of regular weights at every dyadic place and its residue has a nearly ordinary regular modular witness with the required residual ordinary ordering;
(ii) r is potentially Barsotti–Tate and potentially nonordinary at every dyadic place and its residue has a regular modular witness.

Proof. The first case is [2, Theorem 5.2.1]. If the line and quotient have weights $b_\tau > a_\tau$ with sum one, its parameters are $w_\tau = a_\tau$ and $k_\tau = b_\tau - a_\tau + 1$, so $k_\tau + 2w_\tau = 2$. Nonsolvability excludes the CM-dihedral exception; residual distinctness is not imposed.

In the second case choose finite Galois extensions of the dyadic completions over which the source is Barsotti–Tate. The solvable local realization theorem [5, Chapter X, Section 2], in the form [14, Theorem 3.1 and Example 3.3], realizes them in a solvable totally real Galois extension. No degree is prescribed. The nonsolvable residual projective image is $\mathrm{PSL}_2(\mathbf{F}_{2^a})$, $a \geq 2$; its image after solvable Galois restriction is normal with solvable quotient and is unchanged. Solvable base change retains a cuspidal modular witness, put in weight two by Lemma 2.1. Apply [43, Theorem 3.3.5]. Its condition that a dyadic completion equal $\mathbf{Q}_2$ applies only at potentially ordinary places, and there are none.

For descent take a cyclic series. Strong multiplicity one makes the automorphic representation invariant at each step because the Galois representation extends downstairs. The descended Galois representation differs from the given one by a character of the cyclic quotient. Twisting and iterating proves modularity over F; see [7, Lemma 2.2.2]. □

2.2 Globalization and compatibility

Lemma 2.4. *Let K be real quadratic, fix a finite set S of places containing the coefficient places and all residual ramification, and let $\bar{r}$ be totally odd with irreducible cyclotomic restriction. At coefficient characteristic two assume nonsolvable image. Fix a totally odd global de Rham determinant lifting its determinant and unramified outside S, and nonempty compatible local components with regular weights at the coefficient places. Outside the exceptional characteristic-five case of [28, Definition 4.1.1], these components admit a global lift unramified outside S. Once such a lift has been constructed, it is potentially modular and belongs to a strongly compatible system in the following cases:*

(i) odd coefficient characteristic and parallel weight two, with irreducible cyclotomic residual restriction;

(ii) odd coefficient characteristic, adequate cyclotomic residue, and regular potentially diagonalizable crystalline local components;

(iii) coefficient characteristic two, determinant χ_2, and either ordinary crystalline components of regular weights at all dyadic places, or potentially nonordinary crystalline components of weight two at all dyadic places.

Proof. Existence is [28, Theorem 4.2.1 and Remark 1.1.2]. Away from the coefficient prime, inertial conditions and conductor bounds specify unions of components. At that prime, take components through the supplied points; potential diagonalizability persists after a fixed sufficiently large local extension.

For weight two first use the avoiding weight-two residual witness of [64, Theorem 5.1.1], followed by the actual-source theorem [3, Theorem 2.3]. The potential-modularity field is chosen Galois and disjoint from the residual kernel and cyclotomic fields. Thus the actual source becomes modular after this extension. This argument starts with the weight-two point already constructed; it does not assume the existence of an exceptional initial characteristic-five point.

For regular potentially diagonalizable sources with $p \geq 7$, use [7, Theorem C, Theorem 4.5.1 and Corollary 4.5.2], with avoidance imposed before the CM quadratic base change. For the smaller odd characteristics, Snowden's avoiding residual theorem gives an ordinary weight-two witness over a totally real field, disjoint from the residual and cyclotomic fields. After a further disjoint CM quadratic extension, its local representations are potentially diagonalizable by [7, Lemma 1.4.3(1)]. Apply the potentially diagonalizably automorphic alternative of [22, Theorem 1] to the actual source, with polarization multiplier its determinant; adequacy and exclusion of ζ_p are retained. Solvable descent returns potential Hilbert modularity.

At two, first make the finite local preparation which trivializes the residual dyadic restrictions, inside a totally real extension avoiding the global residual kernel. The ordinary weight-two residual witness of [69, Proposition 6.7] can then be chosen over a further avoiding field, with trivial finite central character. Its residual ordinary ordering agrees with that of the source because the prepared residual characters are trivial. Apply Lemma 2.3 to the actual source. This proves potential modularity before a compatible system is invoked.

The construction of compatible systems by potential modularity and descent originates in Dieulefait's work over $\mathbf{Q}$ [18, Theorem 1.1]. For the totally real fields and local conditions here, use the Brauer-induction construction of [64, Proposition 9.2.1 and Remark 9.2.2], retaining the finite characters in the Brauer identity. Mackey's formula and Frobenius reciprocity express the self-pairing of each virtual member in terms of pairings of restricted modular representations with finite character twists. Compatibility and Chebotarev make these pairings independent of the coefficient prime. At the original prime the self-pairing is one, so each virtual member is an irreducible or its negative. Its dimension two excludes the negative. Local–global compatibility for the modular systems [61, Theorem 1 and equation (1)] includes the coefficient prime

and arbitrary cohomological weights. Restriction–induction identities give the common Weil representations; purity determines monodromy [7, Lemma 1.3.6(2)–(3)]. Labelled weights are compatible by the same restriction argument. □

For the quadratic fields here other than $\mathbf{Q}(\sqrt{5})$, the characteristic-five exception is excluded uniformly: an exceptional projective S_5 extension would contain $K(\zeta_5)$ and therefore give an abelian quotient of order four, whereas $S_5^{\mathrm{ab}} = C_2$. The exceptional initial lift over $\mathbf{Q}(\sqrt{5})$ is treated separately below.

3 An auxiliary prime for congruences

In this section K is real quadratic. The finite set S may include any prescribed terminal reset primes.

Proposition 3.1. *Let $\mathcal{R}$ be a nondihedral weight-two system of determinant $\chi\psi$, where ψ has finite two-power order. Fix a finite set S of rational primes containing $2, 3$, all primes dividing the discriminant of K, and every conductor prime. There are a prime $N > \max(S \cup \{5\})$, a degree-one prime λ of K of norm $l > N$, and a congruent weight-two system $\mathcal{R}'$ such that $\mathcal{R} \rightsquigarrow \mathcal{R}'$ in characteristic N, its determinant remains $\chi\psi$, and its only new conductor prime is λ. At λ its type is induced from the unramified quadratic extension with inverse inertia characters of order N.*

While this type is retained and conductor is supported at places over $S \cup \{l\}$, the residual representations at every rational prime in S have nonsolvable image. At odd such characteristics their cyclotomic restrictions are adequate.

Proof. Let K_S/K be the maximal elementary quadratic extension unramified outside the places over S and infinity. It is finite. Write 2^a for the order of ψ. By [23, Proposition 4.6], choose $N > \max S$, $N > 5$, split in K and the coefficient field, with $N \equiv 1 \pmod{2^{\max\{2,a+1\}}}$, such that a residual member contains $\mathrm{SL}_2(\mathbf{F}_N)$. Its determinant is surjective on inertia above N, so its image is $\mathrm{GL}_2(\mathbf{F}_N)$.

Let R/K be the projective kernel field. Prescribe Frobenius to act trivially on K_S, by inversion on $K(\zeta_N)$, and as projective complex conjugation on R. The first two prescriptions agree on the intersection, since $K(\zeta_N)/K$ is totally ramified above N, whereas K_S/K is unramified there. The only nontrivial solvable quotient of $\mathrm{PGL}_2(\mathbf{F}_N)$ is its determinant-squareclass quotient. Our congruence makes $\bar{\psi}$ square-valued, so that quotient is cut out by $\overline{\chi}_N^{(N-1)/2}$ and lies in $K(\zeta_N)$. Inversion fixes it, and projective complex conjugation belongs to $\mathrm{PSL}_2(\mathbf{F}_N)$, since -1 is square modulo N. The prescriptions therefore agree on $R \cap K_S K(\zeta_N)$.

Chebotarev gives a positive-density set of such prime ideals. Those of residue degree greater than one have zero prime-ideal density. We may consequently choose λ of degree one and arbitrarily large norm l, outside the rational primes already used. Thus l splits in K, $l \equiv -1 \pmod{N}$, and the residual Frobenius eigenvalues are $a, -a$.

Put $\delta = \chi_N\psi$. Choose α lifting a with $\alpha^2 = -\delta(\mathrm{Frob}_\lambda)$, and define local inertia and Frobenius by

$$T = \begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_N^{-1} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}.$$

They satisfy $\Phi T \Phi^{-1} = T^l$ and have determinant δ; reduction gives the original unramified local representation. Retain the old crystalline weight-two lifts above N and compatible conditions elsewhere. Lemma 2.4 gives a global weight-two lift and its system. Lemma 2.2 at the split prime N gives backward modularity.

For a later coefficient characteristic $t \in S$, the two order-N inertia characters at λ remain distinct and are exchanged by Frobenius. The residual representation is therefore irreducible and its projective image contains an element of order $N > 5$, excluding A_4, S_4, A_5. If it were dihedral, the inducing quadratic extension would be unramified outside S: at λ its odd-order inertia lies in the rotations. But Frobenius at λ exchanges the rotation eigenspaces, making λ inert in that quadratic extension, contrary to its complete splitting in K_S. Dickson's classification forces nonsolvable projective image. Cyclotomic restriction retains its perfect derived subgroup, hence irreducibility. The order-N element excludes the inadequate characteristic-three and characteristic-five cases; for $t > 5$, the usual rank-two criterion applies. In characteristic two, nonsolvability is the required conclusion. □

4 Local lifts and persistent inertial types

Lemma 4.1. *Let $F/\mathbf{Q}_p$ be finite, with $p > 2$, and let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ have determinant $\overline{\chi}_p$. Then $\overline{r}$ has a regular crystalline, potentially diagonalizable lift with determinant exactly χ_p.*

Proof. Apply [9, Proposition 2.7] with prescribed determinant χ_p. In rank two, its construction also gives potential diagonalizability. An irreducible residual representation is induced from a character of the unramified quadratic extension; the constructed lift is induced from a crystalline character, so becomes a sum of crystalline characters after restriction. A reducible representation is lifted as a crystalline extension of crystalline characters, to which [7, Lemma 1.4.3(1)] applies. The unramified scalar correction in the fixed-determinant construction preserves both properties. □

Lemma 4.2. *Let $F/\mathbf{Q}_p$ be finite, and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be potentially semistable with labelled Hodge–Tate weights $\{0, 1\}$ and determinant χ_p times a finite-order character. If its monodromy is nonzero, then r is potentially ordinary. Thus a potentially nonordinary representation with these weights is potentially Barsotti–Tate.*

If $F = \mathbf{Q}_p$ and r is attached to a weight-two newform which is Steinberg at p, with determinant χ_p and $a_p = s \in \{1, -1\}$, its ordinary quotient is unramified with Frobenius value s.

Proof. Pass to a finite extension over which r is semistable and the finite determinant factor is trivial. Apply the filtered-module functor to $r^\vee$, with filtration jumps zero and one. If the residue degree is f and $N \neq 0$, the relation $N\varphi = p\varphi N$ gives eigenvalues $\alpha, p^f\alpha$ for φ^f, with the first eigenline equal to $\ker N$. Equality of the total Hodge and Newton numbers gives $v_p(\alpha) = 0$.

Every labelled Hodge number on this line is zero or one. Weak admissibility bounds their average by the Newton slope zero, so all are zero. The line is weakly admissible and corresponds to a crystalline weight-zero, hence unramified, subrepresentation of $r^\vee$. Its dual is an ordinary quotient of r. Consequently potential nonordinarity forces $N = 0$, and the resulting potentially crystalline representation with these weights is potentially Barsotti–Tate.

In the last assertion, local–global compatibility identifies the Frobenius eigenvalues on the dual module as s, ps [61, equation (1) and Theorem 1]. The same argument gives its unramified line with unit eigenvalue s; dualizing replaces the Frobenius value by $s^{-1} = s$. □

Lemma 4.3. *Let $F/\mathbf{Q}_2$ be quadratic, and let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ have determinant one.*

(i) If $\overline{r}$ is reducible, it has a crystalline ordinary lift of determinant χ_2 with regular labelled weights. It may retain any specified invariant residual line. Its invariant line has the larger weight at every embedding, and the lift lies on an ordinary component of the fixed-determinant crystalline space. If $F/\mathbf{Q}_2$ is ramified, the weights may be chosen to be $\{1 - n, n\}$ at both embeddings for some $n \geq 2$.

(ii) If $\overline{r}$ is irreducible, it has a crystalline lift of determinant χ_2 and weights $\{0,1\}$. Every potentially Barsotti–Tate lift with these weights is potentially nonordinary.

Proof. In the reducible case use the proof of [9, Proposition 2.7], starting with the given residual line and quotient. Choose crystalline character lifts with product χ_2. Multiplying the first by a sufficiently large positive power of χ_2 and the second by the inverse power preserves their reductions and product, since $\overline{\chi}_2 = 1$. Their weights are respectively positive and negative at every embedding. The crystalline extension constructed there lifts the given extension and has those labelled weights. Its unramified scalar determinant correction reduces to one and preserves the filtration. We obtain weights $a_\tau < b_\tau$, with $a_\tau + b_\tau = 1$, and the invariant line has weight b_τ.

For fixed regular weights, the ordinary crystalline deformation ring is a reduced quotient of the crystalline ring, with the same pure dimension, and both generic fibres are smooth [7, Section 1.4]. The ordinary locus is therefore a union of irreducible components of the crystalline generic fibre. Smoothness makes this union open and closed; its intersection with the fixed-determinant fibre is again a union of components. We choose a component through the constructed point.

For a ramified quadratic $F/\mathbf{Q}_2$, every odd-order character of $\mathcal{O}_F^\times$ is trivial, so the two residual diagonal characters are unramified. Starting with their Teichmuller lifts times χ_2^n and χ_2^{1-n} in the same construction gives the asserted constant labelled weights for large n.

Suppose now that $\overline{r}$ is irreducible. Wild inertia acts trivially, since its fixed space is nonzero and stable under the whole local group. Tame inertia has two distinct characters exchanged by Frobenius. If $F/\mathbf{Q}_2$ is ramified, Frobenius acts by squaring. If one is θ, the other is θ^2, and determinant one gives $\theta^3 = 1$. Thus θ has order three. A Frobenius matrix exchanges the two eigenlines; determinant one makes its square the identity, and diagonal conjugation makes both off-diagonal entries one. Hence there is exactly one such representation up to isomorphism, with image $\mathrm{GL}_2(\mathbf{F}_2) \simeq S_3$.

The elliptic curve

$$A/\mathbf{Q}: \quad y^2 + y = x^3,$$

with Cremona label `27a3` (LMFDB label `27.a4`) [45], realizes it over $\mathbf{Q}_2$. Its discriminant is -27, so it has good reduction at two. Its nonzero two-torsion points have $y = -1/2$ and $4x^3 + 1 = 0$, giving

$$\mathbf{Q}_2(A[2]) = \mathbf{Q}_2(\zeta_3, \sqrt[3]{2}).$$

This is an S_3 extension, since $X^3 - 2$ is Eisenstein and $\mathbf{Q}_2(\zeta_3)/\mathbf{Q}_2$ is unramified quadratic. Its unique quadratic subfield is therefore unramified, so it is disjoint from the ramified quadratic extension F. Thus $A[2]|_{G_F}$ is the required residual representation, and $V_2(A)|_{G_F}$ is a crystalline Barsotti–Tate lift by good reduction, with determinant χ_2 by the Weil pairing [30, Theorem 6.2].

Suppose instead that $F/\mathbf{Q}_2$ is unramified. For the unramified quadratic extension E/F we may write $\overline{r} = \mathrm{Ind}_{G_E}^{G_F} \bar{\theta}$, with $\bar{\theta}|_I = \omega_4^e$, where ω_4 takes values in $\mathbf{F}_{16}^\times$. Frobenius over F acts by the fourth power. Determinant one gives $5e \equiv 0 \pmod{15}$, and irreducibility excludes $e = 0$, so $e \in \{3, 6, 9, 12\}$.

Index the four embeddings of E by $0, 1, 2, 3$, with i and $i+2$ restricting to the same embedding of F, and normalize ω_4 so a crystalline character of weights k_i reduces on inertia to exponent $\sum_i 2^i k_i$. The profiles

$$(k_0, k_1, k_2, k_3) = (a, b, 1-a, 1-b), \qquad a, b \in \{0, 1\},$$

have paired sums one and exponents $12 - 3a - 6b \in \{3, 6, 9, 12\}$. Lift $\bar{\theta}$ crystallinely with the appropriate profile and correct its unramified value. Induction along the unramified extension gives a crystalline lift with weights $\{0, 1\}$ at both embeddings of F. Its determinant differs from

χ_2 by an unramified character δ reducing to one. An unramified character ν with $\nu^2 = \delta^{-1}$ exists after enlarging coefficients and reduces to one, because characteristic two has no other square root of one. Twisting by ν fixes the determinant.

Finally, a potentially ordinary lift of these weights has, over a finite Galois extension, a line of constant labelled weight one. This line is unique, since two such lines would give weight one twice at every embedding. Its Galois conjugates are the same line, so it descends to G_F. Its intersection with a stable lattice is saturated; reduction gives an invariant line in $\overline{r}$, a contradiction. □

Lemma 4.4. *For a finite extension $F/\mathbf{Q}_l$ with $l \neq 2$, every continuous $\overline{r} : G_F \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ has a lift of determinant χ_2, with finite inertia or Steinberg type. An unramified $\overline{r}$ has an unramified such lift.*

Proof. If $\overline{r}$ is reducible with diagonal characters α, α^{-1} and $\alpha \neq 1$, then $H^1(G_F, \alpha^2) = 0$: local Euler characteristic and duality away from two give this from the vanishing of $H^0(G_F, \alpha^2)$ and $H^0(G_F, \alpha^{-2}\overline{\chi}_2)$, since $\overline{\chi}_2 = 1$. Thus $\overline{r}$ is split, and $\widetilde{\alpha} \oplus \chi_2\widetilde{\alpha}^{-1}$ is the required lift.

If $\alpha = 1$, write $\overline{r} = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$. The additive character c kills wild inertia and odd-order tame inertia. If τ generates pro-two tame inertia and Φ is arithmetic Frobenius, with residue cardinality Q, their relation is $\Phi\tau\Phi^{-1} = \tau^Q$. Choose lifts n, x of $c(\tau), c(\Phi)$ and set

$$\tau \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad \Phi \longmapsto \begin{pmatrix} Q & x \\ 0 & 1 \end{pmatrix}.$$

This satisfies the relation and has determinant χ_2. Take $n = 0$ when $\overline{r}$ is unramified; otherwise the lift has nonzero rank-one monodromy.

If $\overline{r}$ is irreducible, its finite local image is solvable. Dickson's classification in characteristic two makes it odd dihedral; its scalar kernel is trivial because its determinant is one. Write $\overline{r} = \mathrm{Ind}_{G_E}^{G_F} \bar{\theta}$ for a quadratic extension E/F and an odd-order character $\bar{\theta}$. The induced-determinant formula gives $\bar{\theta}|_{F^\times} = 1$. Its Teichmüller lift θ_0 has the same restriction. Let $\epsilon_{E/F}$ be the quadratic character of $F^\times$. Choose a character ϕ of $E^\times$ reducing to one and restricting to $\chi_2\epsilon_{E/F}$ on $F^\times$ as follows.

For unramified E/F, take ϕ unramified with the required value on a common uniformizer. For ramified E/F, which is tamely quadratic, take the quadratic residue character on $\mathcal{O}_E^\times$. It restricts to $\epsilon_{E/F}$ on $\mathcal{O}_F^\times$. Write $\pi_F = u\pi_E^2$ and choose $\phi(\pi_E)$ to be a square root of $(\chi_2\epsilon_{E/F})(\pi_F)/\phi(u)$. All these values reduce to one. Then $\mathrm{Ind}_{G_E}^{G_F}(\theta_0\phi)$ has determinant $\epsilon_{E/F}(\theta_0\phi)|_{F^\times} = \chi_2$, the required reduction, and finite inertia. An irreducible representation over an algebraically closed field cannot be unramified, since an unramified group is procyclic. □

4.1 A type containing every relevant Serre weight

Let $F/\mathbf{Q}_p$ be finite, where $p > 2$, with residue field k of size q. We give the cyclotomic character χ_p Hodge–Tate weight one. Put $K_0 = \mathrm{GL}_2(\mathcal{O}_F)$, choose a uniformizer π, put $K_1 = 1 + \pi M_2(\mathcal{O}_F)$, and write

$$I^+ = \left\{ \begin{pmatrix} 1 + \pi a & b \\ \pi c & 1 + \pi d \end{pmatrix} : a, b, c, d \in \mathcal{O}_F \right\}, \qquad J = \mathcal{O}_F^\times I^+.$$

Fix a nontrivial additive character $\psi : k \to \overline{\mathbf{Q}}_p^\times$. Define an affine generic character of J by

$$\theta\left(z \begin{pmatrix} 1 + \pi a & b \\ \pi c & 1 + \pi d \end{pmatrix} \right) = \psi(\bar{b} + \bar{c}), \qquad z \in \mathcal{O}_F^\times.$$

The ambiguity in z does not affect the two displayed residue coordinates. Let Z and U denote the scalar subgroup and the upper unipotent subgroup of $\mathrm{GL}_2(k)$.

Lemma 4.5. *Every irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(k)$ with trivial central character is a quotient of $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1$.*

Proof. Let V be such a representation. A finite p-group acting on a nonzero vector space in characteristic p has a nonzero fixed vector. Thus $V^U \neq 0$. Since Z acts trivially, $V^{ZU} \neq 0$. Frobenius reciprocity gives a nonzero map $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1 \to V$, which is surjective by irreducibility. □

Lemma 4.6. *The representation $\sigma = \mathrm{Ind}_J^{K_0} \theta$ is irreducible of dimension $q^2 - 1$. It is the compact type of a simple-supercuspidal inertial type τ with determinant one. Its natural integral lattice has reduction equal to the inflation of $\mathrm{Ind}_{ZU}^{\mathrm{GL}_2(k)} 1$.*

Proof. On $K_1 = 1 + \pi M_2(\mathcal{O}_F)$, the character is $1 + \pi A \mapsto \psi(\bar{A}_{21})$. Under conjugation by $\mathrm{GL}_2(k)$, its stabilizer is ZU. Indeed the trace pairing on $M_2(k)$ is nondegenerate, including after composing with the nontrivial additive character of k, and the centralizer of a nonzero nilpotent matrix is ZU. The restriction of σ to K_1 is therefore a sum of $q^2 - 1$ distinct characters permuted transitively by K_0. This proves irreducibility and the dimension assertion.

The simple-supercuspidal construction identifies σ as the compact type of the representations obtained by extending θ to $\langle g \rangle F^\times I^+$, where $g = \left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$; see [1, Section 3.1]. To check the occurrence condition, let an irreducible smooth representation contain θ on J, and use its central character to extend θ to $F^\times I^+$. By Frobenius reciprocity the compact induction from this latter group surjects onto the representation. That compact induction is the direct sum of the two irreducible simple-supercuspidal representations in [1, Section 3.1, p. 6]. Changing the extension on g, or its central character on a uniformizer, gives an unramified twist. Conversely each such twist contains σ. The occurrence characterization of inertial types therefore identifies σ with $\sigma(\tau)$, up to duality; compare [35, Theorem 1.1.3]. The Langlands parameter is the induced parameter of [1, Section 3.4 and Theorem 3.18]. The central character on $\mathcal{O}_F^\times$ is trivial, so $\det \tau = 1$.

All values of θ are p-power roots of unity and reduce to one. The coset basis of the induced lattice therefore reduces to the permutation representation on K_0/J. Since $K_1 \subset J$ and $J/K_1 = ZU$, this is the asserted inflation. Taking the dual does not change its Jordan–Hölder support. □

Theorem 4.7 (A universal wild inertial type)**.** *Let $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be continuous with determinant $\bar{\chi}_p$. Then $\bar{r}$ admits a potentially crystalline lift of Hodge–Tate weights $\{0, 1\}$ at every embedding, determinant exactly χ_p, and inertial type τ from Lemma 4.6. For $F = \mathbf{Q}_p$, write this type as $\tau_{p,w}$. Its inertia remains absolutely irreducible after reduction in every coefficient characteristic greater than p, for every stable lattice.*

Proof. We use the potentially Barsotti–Tate Breuil–Mézard theorem of Gee–Kisin, [35, Corollaries 3.5.3 and 3.5.6]. To match their conventions, apply their theorem to the dual residual representation, and dualize the resulting lift at the end. Their cyclotomic character has Hodge–Tate weight -1. The central-character restriction in Corollary 3.5.3 is therefore trivial in the present determinant situation.

At least one of the nonnegative multiplicities μ_V is positive. To see this, use the existence of some potentially Barsotti–Tate lift of any rank-two residual representation. This existence is explicitly justified in the proof of [35, Theorem A.2]: the irreducible case is contained in [64, Proposition 7.8.1], and the reducible case follows from the crystalline-extension construction cited there. The nonzero potentially Barsotti–Tate ring for that lift has positive Hilbert–Samuel multiplicity; the Breuil–Mézard formula makes at least one μ_V positive.

Every V with positive multiplicity has trivial central character. By Lemmas 4.5 and 4.6, it occurs in the reduction of the simple-supercuspidal type. The Breuil–Mézard formula for Hodge type zero and type τ consequently has positive right side. Its potentially crystalline lifting ring is nonzero and p-torsion free, hence has nonempty generic fibre. Dualizing supplies a lift r with the desired Hodge–Tate weights and type: a two-dimensional determinant-one type is self-dual.

Finally $\det r/\chi_p$ is unramified and reduces to one. It is unramified because it is potentially crystalline of weight zero with trivial inertial type. As p is odd, its inverse has an unramified square root reducing to one. Twisting r by this square root fixes its determinant without changing its reduction, weights or inertial type.

For $F = \mathbf{Q}_p$, the parameter is induced from a ramified quadratic extension. Its restriction to wild inertia has two distinct characters of order p, exchanged by a quadratic inertia element [1, Section 3.4 and Theorem 3.18]. In coefficient characteristic greater than p the two characters remain distinct, and the exchanging element prevents either eigenline from being stable. The finite inertial image has order divisible only by 2 and p, so semisimplicity makes this conclusion independent of the stable lattice. □

Remark 4.8. For $p = 3$ this theorem applies to both unramified and ramified quadratic completions. It gives a local lift of weight two. It makes no assertion that an arbitrary-weight global source is modular whenever such a replacement is modular. A backward modularity implication must be verified separately for the actual global source.

Lemma 4.9. *Let $\ell \neq 3$ be an odd prime with $\ell \equiv -1 \pmod 3$. Let $\bar{s}$ be the reduction, with any stable lattice, of a determinant-χ_3 representation of $G_{\mathbf{Q}_\ell}$ which is either a quadratic twist of special type or tame dihedral with inertia of order three or six. After a finite coefficient extension, $\bar{s}$ has lifts of both kinds with determinant exactly χ_3. The scalar quadratic inertial character determines whether the cuspidal lift has inertia of order three or six, and the special lift may be chosen with nonzero monodromy.*

Proof. Remove the scalar quadratic character. Wild inertia is trivial; tame inertia is trivial or unipotent of order three. Frobenius has characteristic polynomial $X^2 - 1$ modulo three: in the special case its eigenvalues are $\ell\alpha, \alpha$, $\alpha = \pm 1$, and in the cuspidal case its trace is zero and determinant is ℓ. The two eigenvalues are distinct. A nonzero inertial extension has one Frobenius eigenline as its invariant line and is normalized by diagonal conjugation. There are two nonsplit orientations and the split case.

For a cuspidal lift take

$$T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Phi = bS, \quad b^2 = -\ell, \quad b \equiv 1 \pmod 3.$$

Hensel's lemma supplies b. We have $T^3 = 1$, $STS = T^{-1}$ and $\det \Phi = \ell$. The standard lattice gives nontrivial unipotent inertia with eigenvalue -1 on its invariant line; replacing Φ by $-\Phi$ gives the other orientation. For trivial residual inertia use $T = \operatorname{diag}(\zeta_3, \zeta_3^{-1})$ over $\mathbf{Q}_3(\zeta_3)$ and the same exchanging Frobenius.

For a special lift use

$$\Phi = \operatorname{diag}(\ell, 1), \qquad U(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \Phi U(t) \Phi^{-1} = U(\ell t).$$

Choose its three-primary tame parameter to be a unit multiple when residual inertia is nontrivial, and three times a unit multiple otherwise. Both choices have nonzero monodromy. An unramified quadratic sign selects either orientation. Restoring the removed quadratic character preserves the determinant and gives the required inertial order. □

Lemma 4.10. *Let $p \in \{2,3\}$, let $F/\mathbf{Q}_p$ have absolute ramification index $p-1$, and let $r : G_F \to \mathrm{GL}_2(E)$ be crystalline of labelled weights $\{0,1\}$ and determinant χ_p. If both residual constituents are unramified, then r is ordinary.*

Proof. After a finite unramified extension the residual image is a p-group, so its underlying $\mathbf{F}_p$-representation has a fixed nonzero vector. Let G be a Barsotti–Tate group with coefficient action realizing a lattice. A nonzero etale quotient has coefficient rank one, by the weights, and proves ordinarity. Otherwise G is connected. Its Cartier dual is also connected, since $r^\vee(1) \simeq r$ and connectedness is invariant under isogeny. Coefficient-uniformizer torsion is therefore biconnected.

The schematic closure of the fixed $\mathbf{F}_p$-line has order p and unramified generic character. Its Oort–Tate parameter valuation lies between zero and $p-1$. The endpoints give etale and multiplicative models. For $p = 3$ the only other valuation is one, which gives a ramified quadratic generic character. Thus the closure is etale or multiplicative, both impossible in a biconnected group: the latter would give an etale quotient of its Cartier dual. The canonical maximal etale quotient descends through the unramified extension, proving ordinarity over F. □

5 Dyadic automorphy and pro-modularity inputs

The local weight-two lifts and their support-preserving global construction are proved in Section 9, using the fixed-determinant Breuil–Mézard theorem [71, Corollary 8.0.2] and the prescribed-lift theorem [28, Theorem 4.2.1]. Potential modularity is established separately there. For backward modularity we use Kisin's and Tung's theorems with their local-source and residual-image hypotheses [43, Theorem 3.3.5], [71, Theorem 8.0.3].

The scalar argument uses the representation spaces, relative presentations and local geometry of [48, Section 3], and the completed-forms and local block comparisons of [48, Propositions 4.1, 4.2 and 4.4, Lemma 4.6]. The localized propagation theorem [48, Theorem 5.1] applies over totally real solvable fields of even degree completely split at two, to a potentially pro-modular characteristic-two curve whose generic representation is not virtually solvable and whose allowed odd local images are finite. In these conventions the completed-form central character is $(\det r)\varepsilon_2$ and the local block parameter is $r \otimes \varepsilon_2$. The regular de Rham locally irreducible parameters are treated by the local correspondence [12, Theorems 1.1 and 1.3].

For irreducible solvable residue we use [21, Proposition 10.3 and Theorem 10.6]. The application is over an abelian totally real field completely split at two, with every dyadic restriction of the actual source absolutely irreducible, so there are no locally reducible places. The auxiliary fields and scalar-source deduction needed here are proved in Sections 7 and 8.

6 Conductor reduction at a split odd prime

We prove Theorem 1.2. The local constructions above preserve the actual residual extension; the lifting implications in Section 2 concern the old characteristic-zero source at each congruence.

Proof of Theorem 1.2. Twist by the inverse square root of the odd-order determinant factor, leaving determinant $\chi\psi$ with ψ of two-power order. Let S contain $2, 3, 5, \ell$, all primes dividing the discriminant of K, and all rational conductor primes. Apply Proposition 3.1; write $\mathfrak{q}$ for the auxiliary place, of split rational norm $q > N > \max S$. The order-N type gives nonsolvable residual images and adequate cyclotomic restrictions at all later coefficient characteristics in S.

In characteristic two, use Lemma 4.3 at the unique quadratic dyadic completion to choose a crystalline target of exact determinant χ_2. For a reducible local residue take an ordinary

regular target, retaining the old ordinary filtration when present; for an irreducible local residue take a weight-two target, which is potentially nonordinary. Lemma 4.4 supplies determinant-χ_2 lifts at odd conductor places and keeps unramified places unramified. Retain the protective type at $\mathfrak{q}$, correcting its unramified scalar. Lemma 2.4 gives the new system, good at two. For the old weight-two source, Lemma 4.2 and descent of the unique potentially ordinary line give the two cases of Lemma 2.3; in the ordinary case the target has matching residual ordering. Thus modularity returns to the old system. The new weights may be higher than two.

Reset in characteristic ℓ, imposing the type of Theorem 4.7 at both ℓ-places and retaining $\mathfrak{q}$ and all cleared conditions. Lemma 2.4 gives a weight-two system, with backward modularity by Lemma 2.2(i).

For each remaining odd conductor characteristic $t \neq \ell$, use the crystalline potentially diagonalizable components of Lemma 4.1, retaining the two wild ℓ-types, the protective place and all cleared conditions. If t is nonsplit, first reset at ℓ whenever the source is not of weight two; then apply Lemma 2.2(ii) backwards, using Lemma 2.1. If t splits, use Lemma 2.2(i) without a reset. In either case Lemma 2.4 supplies the replacement system. The protective order-N element ensures adequacy and excludes the exceptional characteristic-five image. Each step clears t, may change the weights, and introduces no conductor prime outside $S \cup \{q\}$.

Finally work in coefficient characteristic $q > \max\{5, \ell\}$. The retained wild inertia at an ℓ-place remains irreducible modulo q, by Theorem 4.7. It lies in the cyclotomic subgroup, since $K(\zeta_q)/K$ is unramified there. Thus the residual cyclotomic restriction is absolutely irreducible and adequate. If the residual image is solvable, an odd solvable Artin witness and split-q lifting already prove modularity of the old system. Otherwise the crystalline replacements above q, Lemma 2.4 and split-q lifting remove $\mathfrak{q}$. The resulting system has exact cyclotomic determinant, arbitrary regular weights and conductor supported at ℓ.

Its actual ℓ-adic member is absolutely irreducible at both ℓ-places: a Galois line would give a Weil–Deligne line, contradicting the retained irreducible inertia type. If its residue is irreducible, (i) and [76, Theorem 6.1.1] give modularity. If it is reducible, apply (ii) and [76, Theorem 5.1.1]. Here K is totally real abelian, ℓ splits completely, and the constituent quotient is also nontrivial at both ℓ-places: a residual constituent has tame inertia ω_ℓ^a, so the quotient has inertia ω_ℓ^{1-2a}, an odd power modulo the even integer $\ell - 1$. Propagate modularity backwards and undo the finite twist. □

7 Auxiliary fields with scalar Selmer vanishing

Let $K = \mathbf{Q}(\sqrt{d})$ be real quadratic, with $d > 1$ squarefree and $d \equiv 1 \pmod 8$. The space $V_{w_0}^+(K)$ and condition (SV) are defined in the introduction. No class-number hypothesis is imposed in this section.

Lemma 7.1 (Cyclic towers with bounded odd splitting)**.** *Let $K = \mathbf{Q}(\sqrt{d})$ be real quadratic with two split, let Σ be a finite set of odd rational primes, and let $n \geq 2$. There is an odd prime $r \notin \Sigma$ and a totally real cyclic field $L/\mathbf{Q}$ of degree 2^n such that:*

1. *L is ramified only at r, where it is totally ramified;*
2. *r is inert in K;*
3. *2 splits completely in L;*
4. *every $\ell \in \Sigma$ has at most two primes above it in KL.*

The prime r can avoid any additional prescribed finite set.

Proof. Let P be the union of Σ and the prime divisors of d, and put

$$B_n = \mathbf{Q}(\zeta_{2^{n+1}}, 2^{1/2^n}), \qquad C_n = B_n\big(t^{1/2^n} : t \in P\big).$$

The field B_n is Galois over $\mathbf{Q}$ and ramified only at two. At a prime of B_n above an odd rational prime t, the normalized valuation of t is one. These valuations show that the classes of the $t \in P$ generate a direct sum of copies of $\mathbf{Z}/2^n\mathbf{Z}$ in $B_n^\times/(B_n^\times)^{2^n}$: a product $\prod_{t\in P} t^{b_t}$ that is a 2^n-th power has $2^n \mid b_t$ for each t. Kummer theory consequently gives

$$\mathrm{Gal}(C_n/B_n) \simeq (\mathbf{Z}/2^n\mathbf{Z})^P,$$

where a vector (a_t) acts by $t^{1/2^n} \mapsto \zeta_{2^n}^{a_t} t^{1/2^n}$.

Let j be the number of prime factors of d. If j is odd, choose $a_t = 1$ for every $t \in P$. If j is even, choose $a_t = 2$ for exactly one prime dividing d, and $a_t = 1$ for every other $t \in P$. Then $\sum_{t\mid d} a_t$ is odd and each a_t has two-adic valuation at most one. The field C_n is Galois over $\mathbf{Q}$. Apply Chebotarev to the conjugacy class of the prescribed automorphism $h \in \mathrm{Gal}(C_n/B_n)$. We obtain a prime r split in B_n for which the Kummer exponents have the stated 2-adic valuations. Conjugating h can multiply all exponents by an odd integer, which does not change those valuations or their parity. Since $h(\sqrt{d}) = -\sqrt{d}$, the prime r is inert in K. Since r splits in B_n, it satisfies

$$r \equiv 1 \pmod{2^{n+1}}, \qquad 2 \in (\mathbf{F}_r^\times)^{2^n}.$$

Take L to be the unique subfield of $\mathbf{Q}(\zeta_r)$ of degree 2^n. The first congruence makes L totally real; the second makes two split completely. For $t \in P$, its Frobenius in $\mathrm{Gal}(L/\mathbf{Q})$ has order 2^n when a_t is odd, and order 2^{n-1} when $a_t = 2$. This follows from the 2^n-th power residue symbol at r. If $\ell \nmid d$, we chose $a_\ell = 1$, so L has one prime above ℓ and the quadratic extension KL/L yields at most two. If $\ell \mid d$, it ramifies in K and is unramified in L. The unramified local extension supplied by L and the ramified quadratic extension supplied by K are disjoint. Thus the number of primes over ℓ in KL equals their number in L, which is at most two. This proves all assertions. □

Lemma 7.2 (Propagation of one-dyadic scalar Selmer vanishing)**.** *Let K be a totally real field with two places w_1, w_2 above two, and suppose that the only totally real quadratic extension of K unramified outside w_1 and split at w_2 is the trivial extension. Let F/K be cyclic of degree 2^n, split at w_1, w_2, unramified outside a single finite prime $\mathfrak{r} \nmid 2$, and totally ramified there. Then the only totally real quadratic extension of F unramified outside the full fibre above w_1 and split at every place above w_2 is trivial.*

Proof. Let V be the corresponding subspace of $H^1(G_F, \mathbf{F}_2)$. It is stable under the 2-group $\Gamma = \mathrm{Gal}(F/K)$. If $V \neq 0$, the elementary invariant-vector theorem for representations of a finite 2-group in characteristic two gives a nonzero $c \in V^\Gamma$. Let H/F be the quadratic extension cut out by c. Then H/K is Galois, and its group G is a central extension of the cyclic group Γ by C_2. Every such central extension is abelian: a lift of a generator of Γ, together with the central kernel, generates G.

Because H/F is unramified at $\mathfrak{r}$ and F/K is totally ramified there, the inertia subgroup $I \subset G$ maps isomorphically onto Γ and intersects $\mathrm{Gal}(H/F)$ trivially. Consequently $G = I \times \mathrm{Gal}(H/F)$. The fixed field $E = H^I$ is quadratic over K, totally real, and unramified at $\mathfrak{r}$. At every finite place away from w_1 and $\mathfrak{r}$, both steps H/F and F/K are unramified, so E/K is unramified there. At w_2, both steps split completely, so E/K splits. This contradicts the assumed vanishing over K. □

Theorem 7.3 (Uniform scalar field preparation)**.** *Let K be real quadratic with two split and satisfying* (SV)*. Let Σ be a fixed finite set of odd rational primes, and fix integers $n > a \geq 1$. There are totally real abelian two-extensions $K \subset A \subset F$, split completely at two, and a dyadic place v_0 of A, with the following properties. Put*

$$D = [F : \mathbf{Q}], \qquad M = \{v \mid v_0 \text{ in } F\}, \qquad m = |M|,$$

and let S_A, S_F consist of the real places, the dyadic places, and the places above Σ.

1. *$D = 2^{n+1}$, $[F : A] = 2^{n-a}$, and $m/D = 2^{-a-1}$. The numbers u_A, u_F of allowed odd places are at most $2|\Sigma|$.*
2. *For every finite field k of characteristic two,*

$$\{c \in H^1(G_{F,S_F}, k) : \operatorname{loc}_v c = 0 \text{ for all } v \in S_F \setminus M\} = 0.$$

   Here and below k has trivial Galois action.
3. *Restriction $H^1(G_{A,S_A}, k) \longrightarrow H^1(G_{F,S_F}, k)$ is injective.*
4. *Let $H^1_{\mathrm{str}}(F, k)$ have zero local condition at M, full local condition at $S_F \setminus M$, including the real places, and unramified condition outside S_F. Then*

$$\dim_k H^1_{\mathrm{str}}(F, k) \le 2D - 3m + u_F.$$

The degrees $[A : \mathbf{Q}]$ and $[F : A]$ can both be made arbitrarily large, while the allowed odd-place bound remains fixed.

Proof. Apply Lemma 7.1, take the fields $A = KL_a$ and $F = KL$ in Lemma 7.2, using (SV) as its base vanishing hypothesis. The first three assertions follow from the two preceding lemmas and complete splitting at two. For injectivity, a class killed under restriction factors through the cyclic group $\mathrm{Gal}(F/A)$. Its only nonzero quadratic character is ramified at the fresh conductor prime r, which is outside S_A. Such a class is excluded from $H^1(G_{A,S_A}, k)$, proving injectivity.

For the last assertion use the Selmer dimension formula from local and global Tate duality, with the real local conditions included [51, Theorem 4.2]. The scalar module k is self-dual with Tate twist, since the mod-two cyclotomic character is trivial. The dual local conditions are full at M, zero at $S_F \setminus M$, and unramified elsewhere. Their global Selmer group vanishes by the second assertion. The global invariant terms therefore cancel. Each unmarked dyadic place contributes $\dim H^1(G_{\mathbf{Q}_2}, k) - 1 = 2$, and each marked place contributes -1. A real place with full local condition contributes $1 - 1 = 0$; an allowed odd finite place contributes at most one. Unramified local conditions contribute zero outside S_F. Consequently the strict Selmer dimension is at most

$$2(D - m) - m + u_F = 2D - 3m + u_F,$$

as claimed. Taking a and $n - a$ large proves the final assertion. □

When $\Sigma = \varnothing$, taking $n = a + 2$ in Theorem 7.3 gives $[F : A] = 4$ with $[A : \mathbf{Q}]$ arbitrarily large. There are no odd local Euler terms, so the same Selmer dimension formula gives $\dim_k H^1_{\mathrm{str}}(F, k) = 2[F : \mathbf{Q}] - 12$.

Remark 7.4. Condition (SV) is a genuine restriction. For $K = \mathbf{Q}(\sqrt{113})$, the totally positive element $\alpha = (11 + \sqrt{113})/2$ has norm two and valuation one at one dyadic prime. At the other its unit value u satisfies $u + 2/u = 11$, so $u \equiv 1 \pmod 8$ and is a square in $\mathbf{Q}_2$. Thus $[\alpha]$ is a nonzero element of the corresponding $V^+_{w_0}(K)$. This is a limitation of the criterion, not an obstruction to modularity over that field.

8 The scalar dyadic endpoint

We prove Theorem 1.3 using the field preparation of Section 7 and the dyadic inputs listed in Section 5. The argument treats arbitrary finite ramification and regular weights.

Proof of Theorem 1.3. First suppose r is virtually solvable, in the sense that the identity component of its algebraic monodromy group is solvable. An irreducible regular two-dimensional

geometric representation with this property is induced from a character of a quadratic extension of K. Indeed, a nontrivial unipotent radical would have a unique invariant line preserved by the whole monodromy group, while scalar identity component would give finite projective image and, after finite extension, equal Hodge–Tate weights. The remaining torus case has two lines permuted by a quadratic quotient. The inducing quadratic field must be CM. The inducing character is algebraic. If its field had a real place, the absolute-value relation on units and Dirichlet's unit theorem would force all real algebraic exponents to agree. At the two real embeddings above any split real place of K, the induced weights would therefore coincide. This contradicts regularity and also excludes mixed signature. Its inducing character is de Rham and finitely ramified, hence is the 2-adic avatar of an algebraic Hecke character. Each dyadic place is nonsplit in the inducing extension: splitting would make the local induction a sum of characters, contrary to local absolute irreducibility. Quadratic automorphic induction gives a cuspidal automorphic representation with Galois representation r; its regular algebraic infinity type follows from the given regular Hodge–Tate weights. We henceforth assume that r is not virtually solvable. Its restriction to every finite extension is then absolutely irreducible.

The seed and the auxiliary fields. A cuspidal modular seed with scalar semisimple reduction is available over $\mathbb{Q}$: take the compatible system of Ramanujan's form Δ, twisted by ε_2^{-5}. Its determinant is ε_2, it is non-CM, and its residual semisimplification at 2 is $\mathbf{1} \oplus \mathbf{1}$. Indeed Jacobi's identity gives $\Delta \equiv \sum_{j\geq 0} q^{(2j+1)^2} \pmod 2$, so the trace at every odd prime is zero and the residual determinant is one. Base change supplies a cuspidal seed over K and every abelian auxiliary field used below. Its determinant agrees exactly with the target; hence its quaternionic transfer has the required central character. It has tame level one, and the small uniform subgroups for completed forms are chosen at two.

Choose a finite set Σ of odd rational primes supporting the ramification of r and the seed, and enlarge it when necessary to include the prime factors of d. Theorem 7.3 supplies

$$K \subset A \subset F, \qquad D_A = [A : \mathbb{Q}], \quad D = [F : \mathbb{Q}], \qquad m = [F : A],$$

with $A/\mathbb{Q}$ and $F/\mathbb{Q}$ abelian, 2 completely split, and uniformly bounded numbers u_A, u_F of allowed odd places. The marked set M consists of the m places of F over one selected dyadic place of A. Both D_A and m may be made arbitrarily large, with $m/D = 1/D_A$. The same theorem gives injectivity of scalar residual extension classes under $A \subset F$, and gives

$$\{c \in H^1(G_{F,S_F}, k) : \mathrm{loc}_v\, c = 0 \text{ for all } v \in S_F \setminus M\} = 0. \tag{S1}$$

Here S_F consists of the real and dyadic places and the allowed odd places. The coefficient field k can be enlarged without changing the vanishing. The real local conditions in (S1) are zero conditions, so the assertion uses the totally positive squareclass condition (SV), not only an ordinary class-number condition.

For the strict Selmer group, Theorem 7.3(iv) gives

$$\dim_k H^1_{\mathrm{str}}(F, k) \leq 2D - 3m + u_F. \tag{S2}$$

Let c be a fixed generator bound for the augmentation ideal of the characteristic-two universal local pseudodeformation ring of $G_{\mathbf{Q}_2}$ with residual pseudorepresentation $\mathbf{1} \oplus \mathbf{1}$ and fixed determinant one. This bound is independent of A: every selected dyadic completion is $\mathbf{Q}_2$, and extending the residue field preserves a fixed bound. Choose

$$D_A > \max\{4, u_A + c + 3\}, \qquad m > u_F + 4. \tag{S3}$$

The uniform bound on u_A, u_F and independent growth of D_A, m make these choices possible. Since $D = mD_A$, they imply $D > u_F + 4$ and

$$(2D - 2m - 4 - u_F) - D = m(D_A - 2) - (u_F + 4) > 0.$$

They also give the dimension bound needed for entry from A below.

Connected transverse charts. Over the characteristic-two fixed-determinant pseudodeformation base for G_F, take the finite-type framed representation scheme and record an invariant projective line at each mark. In its residual fibre retain the open locus where no marked line is globally invariant. At such a point the residual representation at a mark is scalar: a nonzero unipotent extension has only one invariant line, whereas the global invariant line and the chosen transverse line are different. An upper extension with diagonal characters frozen must therefore be zero locally at every mark.

At a transverse point, the product of the real and marked dyadic local line problems has dimension $2D+5m$, by [48, Lemma 3.12]. The relative global presentation of [48, Proposition 3.6] has number of generators minus relations at least $-6m$. Every component of the completed chart consequently has dimension at least $2D-m$, and its components can be joined with intersections of dimension at least $2D-m-1$, by complete-local connectedness [65, Lemma 52.14.1, Tag 0ECR].

The globally reducible locus has dimension at most

$$2D-2m+u_F+3.$$

Indeed, fixed-product diagonal characters contribute at most one parameter by Leopoldt for the abelian field F; moving the global invariant line and an upper coboundary contribute at most two; the strict extension class contributes at most $2D-3m+u_F$ by (S2); and moving the m marked lines contributes at most m. This is a relative generator bound after freezing the diagonal characters. Adding the global invariant line by proper incidence and then forgetting it bounds the entire reducible image. By (S3), this bound is strictly less than $2D-m-1$.

Therefore a maximal-dimensional intersection in a component chain is generically globally irreducible. Forgetting the global frame and marked lines loses at most $3+m$ dimensions. The remaining trace image has dimension at least $2D-2m-4$. Prescribing one Frobenius trace at each allowed odd place costs at most u_F equations. The consequence of (S3) displayed above leaves dimension greater than D. The dihedral-locus bound in [48, Lemma 3.8] and its curve-avoidance argument supply a non-virtually-solvable characteristic-two curve. The finite local-image conclusion is exactly [48, Lemma 3.9], applied after its fixed preliminary local extensions; the determinant on the characteristic-two curve is one. The prescribed residual Frobenius traces thus make all allowed odd local images finite. Theorem 5.1 of [48] then propagates potential pro-modularity across this intersection. Thus all components through a transverse residual point have the same potential pro-modularity status.

The transverse residual fibre is geometrically irreducible. Fixing the global line, its extensions form the nonzero vectors of the strict scalar cohomology space. This space contains the strict space with no odd ramification, of dimension $2D-3m>0$ by the same duality calculation. The complementary marked lines form $\mathbb{A}^m$. Conjugation gives a surjection from the product of these two spaces with GL_2, an irreducible parameter space. Since the representation scheme is noetherian, one irreducible component Y_0 contains this whole transverse residual fibre. Local chart propagation at the seed and target entry points, through Y_0, therefore connects their components. Completion is faithfully flat, so the completed-chart conclusion applies to components of the representation scheme.

Entering the transverse charts from the seed and target. We include this step because connecting deformation components for one fixed residual extension would not suffice. Work first over A. The seed gives a pro-modular component in the fixed-determinant pseudo-base of dimension at least $1+2D_A$, by [48, Proposition 4.4]. An unrestricted fixed-determinant framed component through $r|_{G_A}$ has trace image of dimension at least $2D_A-2$, by the relative presentation over the real local rings and the loss of three frame parameters. Its generic representation is absolutely irreducible because it contains the given target.

In each component pass to the special fibre, prescribe the residual Frobenius traces at the allowed odd places, and prescribe the constant scalar local pseudorepresentation at the selected dyadic place of A. The last condition costs at most the c local generators fixed above. The total cost is at most $1 + u_A + c$ equations. Even for the smaller target component, the remaining dimension is at least

$$2D_A - 2 - (1 + u_A + c) = 2D_A - u_A - c - 3 > D_A$$

by (S3). The bad-locus bound and curve avoidance therefore supply a non-virtually-solvable characteristic-two curve in each component. Lemma 3.9 of [48] makes its odd local images finite, as above.

At the selected dyadic place the curve representation has an invariant line after a finite coefficient extension. Choose a nonzero vector on that line and take its full G_A-orbit lattice over the curve's valuation ring. Compactness bounds it, and global irreducibility makes it full rank. The vector is primitive: otherwise all its translates lie in the maximal ideal times the orbit lattice, contradicting Nakayama's lemma. Its residual orbit spans the whole rank-two residual space. Consequently its residual line is not globally invariant. The residual representation is a nonzero additive extension. Injectivity of scalar extension classes under $A \subset F$ ensures that it remains nonzero over F. Transporting the local line to all marks gives transverse lines, because F/A is Galois. Thus both curves enter the transverse residual fibre over F.

The seed curve is potentially pro-modular by solvable base change. Propagation through the transverse charts, through the common component Y_0, makes the target curve potentially pro-modular. Applying [48, Theorem 5.1] back over A to the original target curve makes its entire containing component potentially pro-modular. In particular, r becomes pro-modular over one finite solvable totally real extension of K, completely split at 2. Only finitely many component chains and propagation steps occur; their field extensions and tame levels can be replaced by one common Galois compositum and one common deeper tame level. This proves the scalar version of the pro-modularity input directly.

Classicality and descent. At every dyadic place of the resulting field the local representation is still the original absolutely irreducible regular de Rham representation over $\mathbb{Q}_2$, since the auxiliary extensions split completely above 2. Proposition 4.2 of [48], together with the all-prime local correspondence [12, Theorems 1.1 and 1.3], gives a nonzero locally algebraic vector in its Hecke eigenmodule, including when the residual local block is scalar. Proposition 4.1 identifies this vector with a regular algebraic definite-quaternionic eigensystem. Global irreducibility excludes reduced-norm characters, so Jacquet–Langlands produces a Hilbert cusp form. Local-global compatibility and Chebotarev identify its Galois representation with the restriction of r.

Finally descend through a tower of prime-degree cyclic extensions to K. At each step, Galois invariance of r and strong multiplicity one make the upstairs cuspidal representation invariant under the cyclic Galois group. Cyclic automorphic descent [4, Chapter III, Theorems 4.2(d) and 5.1] yields a cuspidal representation downstairs. Its Galois representation differs from the given one by a character of the cyclic quotient, because their restrictions are absolutely irreducible and isomorphic upstairs; twisting corrects that character. The archimedean types remain regular algebraic because every real place splits in a totally real extension. Iterating gives modularity of r over K. This is the locally irreducible part of the classicality and descent argument in [48, Section 8.8]; none of its ordinary dense-point arguments over $\mathbb{Q}$ is used here. □

9 Dyadic lifts and compatible systems

We construct two local types which suffice at every dyadic place and then globalize them without enlarging the prescribed ramification support. Steinberg lifts enter only in the local multiplicity calculation; the global lifts will all be potentially Barsotti–Tate and potentially nonordinary.

Let k be finite of characteristic two. For a continuous $\bar{s} : G_{\mathbf{Q}_2} \to \mathrm{GL}_2(k)$ put $\psi_{\bar{s}} = \varepsilon_2[\det \bar{s}]$, with brackets denoting Teichmüller lift. Choose $E/\mathbf{Q}_2$ with residue field k, enlarging both when necessary, and write $\mathcal{O}$ for its ring of integers and ϖ for a uniformizer. Let $\tau_0 = \widetilde{\omega}_2 \oplus \widetilde{\omega}_2^2$, where ω_2 has niveau two, and let τ_* be the inertial type of the elliptic curve

$$y^2 = x^3 - 2x,$$

with Cremona label `256b1` (LMFDB label `256.b1`) [45].

Proposition 9.1. *Every $\bar{s}$ has a potentially Barsotti–Tate lift of determinant $\psi_{\bar{s}}$ and labelled weights $\{0,1\}$, with type τ_0 if it admits a crystalline lift of those weights, and τ_* otherwise. Both types force actual local irreducibility and potential nonordinarity. The inertia image of τ_* is Q_8.*

Proof. Every character $G_{\mathbf{Q}_2} \to k^\times$ is unramified: by local class field theory its inertial image is a quotient of the pro-two group $\mathbf{Z}_2^\times$, whereas $k^\times$ has odd order. If a crystalline weight-two lift exists, its determinant differs from $\psi_{\bar{s}}$ by an unramified character δ reducing to one. After enlarging coefficients, an unramified character ν with $\nu^2 = \delta^{-1}$ also reduces to one. Twisting by ν therefore fixes the determinant without changing the residue, weights or inertial type.

We first identify the residual representations for which no such crystalline lift exists. An irreducible $\bar{s}$ has trivial wild inertia, since the fixed space of this normal two-subgroup is nonzero and $G_{\mathbf{Q}_2}$-stable. After an unramified scalar twist its determinant is one; its two tame inertia characters are exchanged by squaring and have order three. It is the unique such representation, realized by $A[2]$ for the good-reduction curve $A : y^2 + y = x^3$ used in the proof of Lemma 4.3. Thus it has a crystalline weight-two lift. A split residue $\bar{\alpha} \oplus \bar{\beta}$ lifts to $\varepsilon_2[\bar{\alpha}] \oplus [\bar{\beta}]$. For a nonsplit reducible residue, untwist its quotient and write its character ratio as $\bar{\eta}$. If $\bar{\eta} \neq 1$, local Euler characteristic and the Bloch–Kato formula [8, Corollary 3.8.4] give

$$\dim_E H^1(G_{\mathbf{Q}_2}, E(\varepsilon_2[\bar{\eta}])) = \dim_E H^1_f(G_{\mathbf{Q}_2}, E(\varepsilon_2[\bar{\eta}])) = 1.$$

Moreover $H^2(G_{\mathbf{Q}_2}, \mathcal{O}(\varepsilon_2[\bar{\eta}])) = 0$ by local Tate duality, since $[\bar{\eta}]^{-1} - 1$ takes a unit value. Reduction of H^1 is consequently surjective, and the actual residual extension lifts crystallinely. Hence failure of crystalline lifting is possible only for

$$\bar{s} = \bar{\alpha} \otimes \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \qquad 0 \neq c \in H^1(G_{\mathbf{Q}_2}, k),$$

with $\bar{\alpha}$ unramified.

For this last representation, Kummer theory identifies $H^1(G_{\mathbf{Q}_2}, \mathcal{O}(\varepsilon_2))$ with $\widehat{\mathbf{Q}_2^\times} \otimes_{\mathbf{Z}_2} \mathcal{O}$, where the hat denotes pro-two completion. Its reduction onto $H^1(G_{\mathbf{Q}_2}, k)$ is surjective. Choose a lift κ of c with nonzero valuation; adding a suitable $\varpi\mathcal{O}$-multiple of the Kummer class of two achieves this without changing its reduction. The extension of 1 by ε_2 defined by κ is semistable and noncrystalline: Kummer classes are semistable, and the crystalline subspace is the unit subspace. Twisting by $[\bar{\alpha}]$ gives determinant $\varepsilon_2[\bar{\alpha}]^2 = \psi_{\bar{s}}$ and preserves the actual residual extension.

We now apply the full fixed-determinant Breuil–Mézard formula [71, Conjectures 5.1.2–5.1.3 and Corollary 8.0.2]. The two Serre weights are $W_0 = 1$ and $W_1 = \mathrm{Sym}^1$. Write $\mu_0(\bar{s}), \mu_1(\bar{s})$ for their nonnegative multiplicities. For a compact type σ with semisimplified reduction containing

W_i with multiplicity a_i, the Hilbert–Samuel multiplicity of the corresponding fixed-determinant weight-two ring modulo ϖ is

$$a_0\mu_0(\bar{s}) + a_1\mu_1(\bar{s}).$$

These deformation rings are taken with their reduced, $\mathcal{O}$-torsion-free structure; positivity is equivalent to a nonempty characteristic-zero fibre. The crystalline trivial type has reduction W_0, so $\mu_0 > 0$ exactly when the crystalline lift exists, with determinant fixed as above. The compact type of τ_0 is the inflation of the sign character of $\mathrm{GL}_2(\mathbf{F}_2) \simeq S_3$, whose reduction is also W_0. This proves the τ_0 assertion. Otherwise $\mu_0 = 0$, whereas the semistable lift just constructed gives $\mu_1 > 0$: the compact type for unrestricted monodromy and trivial inertia is the finite Steinberg representation, whose reduction is W_1. The determinant compatibility in both comparisons follows from $\det \tau_0 = 1$ and the unramifiedness of $[\det \bar{s}]$.

For τ_* we use one test representation. The two-torsion of the displayed curve has Kummer class two, which has odd valuation and is not finite flat [11, Definition A.3 and Lemma A.4]; in particular it has no crystalline weight-two lift. Its three-division polynomial is $3X^4 - 12X^2 - 4$. Choose $u^2 = -3$, $i^2 = -1$, $s = iu$. Its roots are $\pm\alpha, \pm\beta$, with

$$\alpha^2 = 2 + 4/s, \qquad \beta^2 = 2 - 4/s, \qquad \alpha\beta = 2u/3.$$

They recover s, u, so their splitting field is $L = \mathbf{Q}_2(u, i, \alpha)$. The field $U = \mathbf{Q}_2(u)$ is unramified quadratic, and $T = U(i)$ is ramified quadratic. For $\pi = 1 - i$, one has $\alpha^2/\pi^2 = i + 2/u$ and $v_\pi(i + 2/u - 1) = 1$. This unit is nonsquare even after unramified extension of T: a putative square root has residue one, while its square is one modulo π^2. Hence L/T is ramified quadratic, with $e(L/\mathbf{Q}_2) = 4$, $f(L/\mathbf{Q}_2) = 2$. Projective inertia on the four roots is $V_4 \subset A_4$. Its inverse image in $\mathrm{SL}_2(\mathbf{F}_3)$ is Q_8, with no proper subgroup surjecting onto V_4. The curve has potentially good reduction since $j = 1728$; its finite inertia on T_3 reduces injectively modulo three. Compatibility identifies its faithful irreducible Q_8 inertia with τ_*.

The Tate module of this curve gives a nonempty type-τ_* ring, while its residue has $\mu_0 = 0$. The multiplicity formula therefore forces $a_1(\tau_*) > 0$. For every $\bar{s}$ without a crystalline weight-two lift we already have $\mu_1(\bar{s}) > 0$, so the same formula gives a nonempty type-τ_* fibre. An unramified scalar twist removes and restores $\bar{\alpha}$ without changing the compact type, giving the required determinant in general.

The two characters of τ_0 are exchanged by Frobenius, and τ_* is irreducible on inertia. Their Weil parameters are therefore irreducible, which forces Galois irreducibility. An ordinary line after finite Galois base change would be the unique line of labelled weight one and would descend, a contradiction. Thus neither type is potentially ordinary. □

Lemma 9.2. *Let F be totally real and let $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be finitely ramified, with nonsolvable residue,* $\det r = \varepsilon_2\eta$ *for a finite totally even character η, and potentially Barsotti–Tate and potentially nonordinary at every dyadic place. Then r is potentially modular.*

Proof. A finite character twist makes the residual determinant one. Apply [69, Proposition 6.7], avoiding its kernel field, to make the residue modular without shrinking its image. A further solvable totally real extension makes every dyadic source Barsotti–Tate. Nonsolvability and residual modularity persist. Kisin's theorem [43, Theorem 3.3.5] applies: its condition that a completion equal $\mathbf{Q}_2$ concerns only potentially ordinary places, and there are none. Restore the finite twist. □

Corollary 9.3 (Prescribed dyadic lifts in compatible systems)**.** *Let K be real quadratic with two split, and let $\bar{r} : G_K \to \mathrm{SL}_2(k)$ be continuous with nonsolvable image. Fix a finite set S containing the dyadic places and all residual ramification. At each odd place in S, choose a nonempty union of components of the generic fibre of the local lifting ring with determinant ε_2. Then $\bar{r}$ lifts to the two-adic member of a potentially modular weight-two strongly compatible system of cyclotomic determinant, unramified outside S, on the chosen odd local components, and with dyadic types prescribed by Proposition 9.1.*

Proof. Lemma 4.4 makes the odd local lifting rings nonempty and permits an unramified choice wherever the residue is unramified. At the dyadic places use Proposition 9.1. Choose a finite extension of each completion killing its type. In the corresponding potentially semistable generic fibre of weights $\{0, 1\}$, the type is locally constant, so the prescribed type selects a nonempty union of components. The associated Weil parameters are irreducible, so monodromy vanishes, and Proposition 9.1 gives potential nonordinarity.

The global determinant ε_2 lifts the residual determinant and is totally odd. Since $K(\zeta_2) = K$, the nonsolvable image gives the required absolute irreducibility. The prescribed-lift theorem [28, Theorem 4.2.1] therefore supplies a characteristic-zero point unramified outside S on these components. Lemma 9.2 makes this actual lift potentially modular. The compatible-system construction in the proof of Lemma 2.4 then gives the assertion, including the local conditions and ramification support. □

10 A Serre criterion when two splits

Theorem 10.1 (The split-dyadic criterion)**.** *Let K be real quadratic with two split. Use the lifting and propagation inputs of Sections 2 and 5, and suppose:*

(i) $V_{w_0}^{+}(K) = 0$*;*
(ii) the ordinary ideal class group $\mathrm{Cl}(K)$ *has two-power order;*
(iii) every continuous absolutely irreducible representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ *of determinant one, unramified outside two, is modular.*

Then every continuous absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, for every odd prime p and arbitrary conductor, is modular.

Condition (iii) is a residual base case, separate from the scalar source theorem. The proof below has no analytic assumption beyond whatever is needed to establish that condition. Strong compatibility includes the coefficient prime, so a crystalline replacement clears that conductor from every companion.

Proof. **10.1 Initial lifts and the protective place**

Use the global construction and compatible systems of Lemma 2.4, the modular witnesses of Lemma 2.1, and the backward lifting statements in Lemma 2.2. For arbitrary regular dyadic sources at the split completions $\mathbf{Q}_2$, we use [71, Theorem 8.0.3]. The initial characteristic-five exception is absent for this K. Indeed $K \neq \mathbf{Q}(\sqrt{5})$, so $[K(\zeta_5) : K] = 4$, whereas $\mathrm{PGL}_2(\mathbf{F}_5)$ has abelianization of order two. Its projective kernel field therefore cannot contain $K(\zeta_5)$; this verifies the nonexceptional hypothesis of [28, Definition 4.1.1 and Proposition 4.1.2]. The subsequent protective images exclude all small exceptional images as well.

The protective place in Proposition 3.1 can be chosen with the additional property that the order of two modulo its rational norm is odd. We give the simultaneous field conditions, since this property is needed when its own inertia ceases to protect the last congruence.

Write $K = \mathbf{Q}(\sqrt{d})$ with $d > 1$ squarefree. Start with the initial weight-two compatible system. A reversible finite twist removes the odd-order part of its finite determinant, leaving $\det r_t = \varepsilon_t \psi_2$, with ψ_2 of two-power order and with no new rational conductor prime. Choose a fresh odd reset prime $s > 5$, above the needed prime bounds and split in K, by Chebotarev. Include it, two, three, five, every conductor prime and every prime dividing d in a finite set S before choosing the protective place.

Choose N and the projective residual field as in the proof of Proposition 3.1, with N large enough for the fixed weight bounds. Thus N splits in K and the coefficient field, the projective image is $\mathrm{PGL}_2(\mathbf{F}_N)$, and ψ_2 is square-valued modulo N.

Let K_S/K be the maximal elementary quadratic extension unramified outside S and infinity. Choose a degree-one place λ of rational norm ℓ with the standard good-dihedral Frobenius prescription and

$$\ell \equiv -1 \pmod{N}, \qquad \lambda \text{ split in } K_S, \qquad \ell \equiv 9 \pmod{16}, \qquad 2 \in (\mathbf{F}_\ell^\times)^8. \tag{2}$$

The last two conditions make $\mathrm{ord}_\ell(2)$ odd. We verify compatibility of the simultaneous conditions. The Galois group of $B = \mathbf{Q}(\zeta_8, 2^{1/8})$ has order sixteen. Writing its elements as (a, b) for the actions on $2^{1/8}$ and ζ_8, multiplication is $(a, b)(c, e) = (a + bc, be)$. The commutator of $(1, 3), (0, 7)$ is $(2, 1)$, generating $\mathrm{Gal}(B/\mathbf{Q}(\zeta_8))$. Thus its maximal abelian subfield is $\mathbf{Q}(\zeta_8)$. Since B is ramified only at two and K is real quadratic with two split, $K \cap B = \mathbf{Q}$. The same statement holds with $\mathbf{Q}(\zeta_{16})$ in place of B. The compositum with the elementary extension K_S/K has abelianization of exponent two: the same commutators survive with trivial K_S coordinate. Hence $K(\zeta_{16}) \not\subset BK_S$ and its intersection with BK_S is $K(\zeta_8)$. Splitting in BK_S is compatible with Frobenius nine on ζ_{16}. This two-extension is unramified at N; the only nontrivial solvable quotient of the projective residual field is its quadratic cyclotomic quotient, ramified at N. Its intersection with the compositum of the preceding two-extension and $K(\zeta_N)$ is exactly that quotient. Projective complex conjugation and inversion on ζ_N agree on it, since $N \equiv 1 \pmod 4$. Chebotarev therefore gives (2), with ℓ split in K and arbitrarily large.

Install the order-N induced type at λ and apply the forward construction and split-N backward lifting of Proposition 3.1. Its image argument makes every residue in S nonsolvable, with adequate cyclotomic restriction in odd characteristic. The additional conditions (2) do not alter that argument.

10.2 Dyadic preparation and odd conductor

Reduce the protected system modulo two. Its determinant is one, since ψ_2 has two-power order. At each odd ramified place use Lemma 4.4 to choose a component of determinant ε_2; its type need not equal the old type. At λ retain the order-N induction: take inertia $\mathrm{diag}(\zeta_N, \zeta_N^{-1})$ and antidiagonal Frobenius with off-diagonal product $-\ell$, giving determinant ℓ and the same residual local representation. Corollary 9.3 gives a potentially modular weight-two system of exact cyclotomic determinant, with dyadic types τ_0 or τ_* and no new conductor prime. For the backward arrow apply Tung over K itself, where the dyadic completions are $\mathbf{Q}_2$: the old regular source may have arbitrary dyadic conductor. Once the replacement is modular, the common nonsolvable residue is modular and the theorem applies.

Retain the dyadic types τ_0, τ_* in every subsequent odd-characteristic congruence. Remove all odd rational conductor primes except s, ℓ. Before removing a nonsplit prime t, change coefficient characteristic to s and choose a weight-two replacement, retaining the dyadic and protective types and every previously cleared place away from s. Weight-two local existence is [64, Proposition 7.8.1], followed by prescribed lifting. At s, the finite determinant correction reduces to one and has s-power order; its inverse square root fixes the determinant without changing the residue or weights [64, Section 7.9]. Unramified local components preserve cleared

places, and the old components preserve retained types. Since s splits, Tung gives the backward implication for the actual old regular source.

In characteristic t, replace the source by regular crystalline potentially diagonalizable lifts at all places above t. This clears its conductor in the new compatible system. For nonsplit t, the old source has weight two by the reset, and [3, Theorem 2.3], with the witness from Lemma 2.1, gives the backward implication. For split t, use Tung without a weight restriction on the old source. The forward construction applies in characteristic three regardless of its splitting in K. All conditions away from t retain the dyadic and protective types. Finally clear s by a regular crystalline replacement in characteristic s and split-prime lifting. The weights are allowed to grow; no crystalline weight-two replacement has been asserted at a removal prime.

10.3 The last prime and the terminal source

Remove the protective place last, in characteristic ℓ. Its own order-N type can no longer protect that residue, so use the dyadic types. If one is τ_*, its Q_8 inertia stays absolutely irreducible modulo odd ℓ: the center acts as -1, while every one-dimensional character kills it. The local cyclotomic extension at two is unramified and preserves this inertia. If both are τ_0, the two order-three characters remain distinct modulo $\ell > 3$ and Frobenius interchanges them. By (2), the cyclotomic residue degree is odd, so that interchange survives restriction to $G_{K(\zeta_\ell)}$. The global residual cyclotomic restriction is therefore absolutely irreducible in both cases.

If the residue is solvable, its Artin lift and regular weight raising give a modular witness; Tung at the split prime ℓ already proves modularity of the old source, completing the backward chain. Otherwise ℓ is large enough for adequacy. Choose regular crystalline potentially diagonalizable lifts above ℓ, preserving all dyadic types and cleared places. Prescribed lifting and potential modularity give the system with λ removed. Tung gives the backward implication.

The terminal system has cyclotomic determinant, conductor supported above two, and both dyadic supercuspidal types. Its actual 2-adic member r_2 is locally absolutely irreducible, de Rham and regular, with no weight-two restriction. If its residue is irreducible and nonsolvable, hypothesis (iii) and Tung give modularity. If the residue is irreducible and solvable, apply [21, Theorem 10.6]: $K/\mathbf{Q}$ is abelian, two splits, all places belong to Σ_n, and $\Sigma_o = \varnothing$, so its nonsplitness requirement is vacuous. If the residual semisimplification is reducible, both characters have odd order. At the dyadic completions $\mathbf{Q}_2$ they kill the pro-two unit group, and they also kill the real sign groups. They factor through the ordinary class group and are trivial by (ii). Hypothesis (i) and Theorem 1.3 apply. Virtually solvable actual sources are alternatively covered by the CM-induction argument in its proof.

In every case the actual terminal source is modular. Compatibility identifies its entire system with the automorphic system, and all backward implications return to the initial lift. Reducing proves Theorem 10.1. □

11 Integral orders and finite-flat groups

For the quadratic applications write $F_d = \mathbf{Q}(\sqrt{d})$. We first retain the full coefficient information in an integral order, then establish the finite-flat bounds used on its lattices.

11.1 Trace orders

We use simultaneous realizations at every dyadic coefficient place. This is essential when a local calculation bounds a global trace-field degree. Let $\mathcal{R}_1, \dots, \mathcal{R}_m$ be distinct coefficient orbits of absolutely irreducible strongly compatible rank-two systems, with trace fields D_i and

cyclotomic determinants. Write $d_i = [D_i : \mathbf{Q}]$. Let R_0 be the closed $\mathbf{Z}_2$-algebra generated by their simultaneous traces and let E_0 be the actual order generated by their simultaneous group images and the trace scalars, in integral matrix realizations at all the coefficient places above two.

Lemma 11.1. *The rings R_0, E_0 are finite free $\mathbf{Z}_2$-modules, and*

$$R_0[1/2] = \prod_i (D_i \otimes_{\mathbf{Q}} \mathbf{Q}_2), \qquad \operatorname{rank}_{\mathbf{Z}_2} R_0 = \sum_i d_i, \qquad \operatorname{rank}_{\mathbf{Z}_2} E_0 = 4 \sum_i d_i.$$

The order E_0 is the closed $\mathbf{Z}_2$-span of the group image. Its trace and determinant laws take values in R_0. If every residual semisimplification is scalar, $E_0/2E_0$ is a local algebra with residue field $\mathbf{F}_2$.

Proof. All the chosen matrices preserve integral lattices, so their span is contained in a finite free $\mathbf{Z}_2$-module. Cayley–Hamilton gives

$$T(g)1 = g + \det(g) g^{-1}.$$

Since $\det(g) \in \mathbf{Z}_2^{\times}$, the closed group span already contains every trace scalar and is an order. The determinant of a sum lies in R_0 by the polarization identity

$$\det(x + y) = \det(x) + \det(y) + T(x)T(y) - T(xy),$$

first for linear combinations of group elements and then by continuity.

Good Frobenius traces generate each D_i. A rational subalgebra of $\prod_i D_i$ surjecting onto every factor identifies two simple factors only if an embedding matches their good traces. Brauer–Nesbitt would then identify their coefficient orbits. The rational simultaneous trace algebra is therefore the displayed product. On each simple trace factor, absolute irreducibility makes the image algebra central simple of dimension four; it becomes M_2 after a coefficient extension. This proves both ranks.

In characteristic two a scalar residual semisimplification with cyclotomic determinant is trivial. Each actual residual image is unipotent, and the principal congruence kernels are pro-two. The simultaneous image is therefore pro-two. Its finite image in $E_0/2E_0$ spans that algebra, which is consequently a quotient of a finite two-group algebra over $\mathbf{F}_2$. Such a group algebra is local with residue field $\mathbf{F}_2$. □

The next finite-flat assertion is restricted to a category in which the order-two model is unique. Let $F/\mathbf{Q}_2$ have absolute ramification index two, and allow a finite unramified extension of F. We call a finite flat group biconnected if its special fibre and the special fibre of its Cartier dual are connected.

Lemma 11.2. *Among biconnected finite flat groups killed by two whose generic Jordan–Hölder factors over $\mathbf{F}_2$ are trivial, every morphism of generic fibres extends uniquely. Generic subobjects and quotients have biconnected finite-flat models, and their exact sequences extend to those models. The simple model has Breuil–Kisin Frobenius u. Every self-extension in this category has generic quadratic characters of conductor at most two.*

Proof. Schematic closure and finite-flat quotient preserve biconnectedness. Indeed a closed subgroup of a connected finite group over the perfect residue field has local coordinate algebra and is connected; connectedness of quotients follows from surjectivity. Cartier duality gives the other two assertions.

For a group of order two, the height-one exponent is j with $0 \le j \le 2$. Biconnectedness excludes $j = 0, 2$, leaving $j = 1$. The unit factor is removed by change of generator when the

generic character is trivial. This is the unique middle model, including its generic identification; see the connected classification in [42, Theorem 4.1, Proposition 4.2 and Corollaries 4.3–4.4].

A generic isomorphism between two objects extends as an isomorphism by induction on their common order. Close its graph in the product. For each projection, choose a generic subgroup of order two and close it in source and target. The restriction of the projection is an isomorphism by uniqueness of the middle model. Its induced map on the finite-flat quotients is an isomorphism by induction, hence the projection is an isomorphism. For an arbitrary generic morphism, apply this to the first projection of its graph closure. The second projection then extends the morphism. Flatness gives uniqueness. Taking schematic closures and finite-flat quotients gives the asserted exact sequences.

Let k_0 be the residue field. Kim's exact classification [42, Corollary 4.3 and Proposition 5.7.3] represents a self-extension of the simple model by

$$\Phi = \begin{pmatrix} u & h(u) \\ 0 & u \end{pmatrix}, \qquad h \in k_0[[u]].$$

Changing the upper lift replaces h by $h + u(\varphi(f) - f)$. The map $\varphi - 1$ is bijective on $uk_0[[u]]$, so we may write $h = h_0 + h_1 u$. Rescaling the endpoint bases by u^{-1} gives the étale φ-module class $h_0/u + h_1$, whose positive upper break is at most one over the field of norms.

To compare ramification, choose a uniformizer π of F and put $F_n = F(\pi^{1/2^n})$. The transition F_n/F_{n-1} has different exponent $2e(F_{n-1}) + 1$ and upper break $4 \cdot 2^{n-1}$. Fontaine's bound gives upper break at most three for the generic quadratic character of our finite flat group [31, Theorem A]; his numbering is shifted by one from classical upper numbering. Herbrand's theorem therefore preserves this character's break at each transition. Wintenberger's comparison [75, Proposition 3.3.2 and Corollary 3.3.6] identifies it with the field-of-norms break. Its conductor is consequently at most two. The argument is unchanged after finite unramified extension. □

Lemma 11.3. *Let $F/\mathbf{Q}_2$ be finite, and let V be a direct sum of crystalline rank-two representations, with finite coefficient fields, labelled weights $\{0, 1\}$ and determinant χ_2. Every stable $\mathbf{Z}_2$-lattice in V is Barsotti–Tate. If all the constituents are nonordinary, its Barsotti–Tate group and its finite-flat subquotients are biconnected. This applies, in particular, to the left regular lattice of E_0.*

Proof. The crystalline lattice classification gives a Barsotti–Tate group for each constituent. Every stable lattice in their sum is realized by an isogenous group, by schematic closure of the finite generic isogeny kernel. Connectedness and dual connectedness are invariant under isogeny.

The connected–étale decomposition is stable under coefficient actions. A nonzero rational étale factor in a rank-two constituent has coefficient rank one and gives an unramified quotient; the determinant and weights then give the ordinary filtration. Nonordinarity excludes this. Applying the same argument to $r^\vee(1) \simeq r$ excludes an étale factor of the dual. Subgroups and quotients preserve biconnectedness by the first paragraph of the proof of Lemma 11.2. After extending coefficients, the rational regular representation of E_0 is a sum of columns of the original constituents, so the same argument applies to it. □

Here the trace and determinant are those of the original integral order. A subsequent nonfaithful matrix quotient need not preserve the information they carry. In particular a vanishing matrix in such a quotient does not imply vanishing of its original trace.

Lemma 11.4. *Suppose every dyadic coefficient constituent is crystalline of labelled weights $\{0, 1\}$ and determinant χ_2, is nonordinary, and has scalar residual semisimplification. Let $m = 2$ or 6 and work at the dyadic completion of F_m. There the regular lattice E_0 is Barsotti–Tate.*

Its Breuil–Kisin module is free of rank one as a right $E_0[[u]]$-module. If its Frobenius generator is $\Phi(u)$, then

$$\mathcal{D}(\Phi(u)) \in (u^2 - m)R_0[[u]], \qquad T(\Phi(0)) = a_2 \in R_0.$$

The point field of $E_0/2E_0$ has upper breaks at most three over $(F_m)_2$.

Proof. The regular lattice is Barsotti–Tate and biconnected by Lemma 11.3. Lemma 11.2 gives exactness and full faithfulness for its torsion subquotients with trivial simple generic factors. We use the covariant Breuil–Kisin functor $\mathfrak{M}_{\mathrm{cov}}(G) = \mathfrak{M}_{\mathrm{contra}}(G^D)$. Cartier duality followed by Kim's anti-equivalence is covariant and exact [42, Corollary 4.3 and Proposition 5.7.3], so it preserves the right action and the quotient used below.

The commuting right action of E_0 extends by Tate full faithfulness. Write $J_0 = \mathrm{rad}(E_0)$ and let $\mathfrak{M}$ be the Breuil–Kisin module of the regular group. In the scheme killed by two take the sum of the right multiplication maps for a set of generators of $J_0/2E_0$. Its finite-flat quotient has generic fibre $E_0/J_0 = \mathbf{F}_2$ and is the unique middle model. Exactness of the covariant functor gives $\mathfrak{M}/\mathfrak{M}J_0 = \mathbf{F}_2[[u]]$ with Frobenius u. A generator lifts by Nakayama to a surjection $E_0[[u]] \to \mathfrak{M}$ of right modules. These have the same rank over $\mathbf{Z}_2[[u]]$, so this is an isomorphism.

Height one supplies Ψ with $\Phi\Psi = \Psi\Phi = u^2 - m$. In each integral matrix realization these two identities give a rank-two height-one matrix. Its rational étale Frobenius module is the corresponding column constituent of the regular representation, by its commuting right matrix action after extending scalars. The connected finite-flat comparison thus identifies its rank-two generic realization with the given crystalline member. Kisin's determinant comparison [43, Lemmas 2.3.2 and 2.3.4] gives divisibility of its determinant by $u^2 - m$. Division by this distinguished monic polynomial commutes with each actual coefficient embedding, hence its remainder already vanishes in $R_0[[u]]$. At $u = 0$ the covariant comparison gives crystalline Frobenius on $V(-1) = V^\vee$ for the rank-two member V. Its determinant is 2 and its trace is the arithmetic coefficient a_2, since $V^\vee \simeq V(-1)$ has the arithmetic Frobenius normalization. This fixes the convention in both identities. Finally Fontaine's finite-flat ramification bound for the group killed by two gives classical upper break at most $2e((F_m)_2) - 1 = 3$. □

Lemma 11.5. *Let $F/\mathbf{Q}_2$ be ramified quadratic and let M/F be the point field of a finite flat group scheme killed by two. Every positive upper break is at most three, and the normalized relative different contribution is strictly less than two, with $v_2(2) = 1$. In particular this applies to the reduction of any stable coefficient lattice of a crystalline rank-two source of weights $\{0, 1\}$.*

Proof. Fontaine's bound [31, Theorem A], converted to classical upper numbering, is $2e(F) - 1 = 3$. Integration of the upper filtration bounds the relative different exponent divided by the relative ramification index strictly below four. Dividing by $e(F/\mathbf{Q}_2) = 2$ gives the normalized bound. For the last assertion, realize the stable crystalline lattice by a Barsotti–Tate group. Its coefficient-uniformizer torsion is a finite-flat subgroup of rational two-torsion. □

11.2 Rank-one selfextensions at the endpoint exponents

Lemma 11.6. *Let $F/\mathbf{Q}_p$ have absolute ramification index e and residue degree f, and let k be a finite coefficient field containing its residue field. Consider a rank-one Breuil–Kisin module with coefficient factors indexed by $i \in \mathbf{Z}/f\mathbf{Z}$, whose Frobenius maps are $b_i u^{r_i}$, where $b_i \in k^\times$ and $r_i \in \{0, e\}$. On these factors Frobenius is k-linear and sends u to u^p. Every height-one selfextension, exact on the underlying free $k[[u]]$-modules, has Frobenius matrices*

$$A_i = b_i u^{r_i} \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}, \qquad a_i \in k,$$

after changing the lifts of the quotient generators. The extension classes form a one-dimensional k-space. Whenever the exact finite-flat comparison identifies the given models with these modules, their generic selfextensions become unramified after dividing by the common character.

Proof. Before changing lifts, write $A_i = \left(\begin{smallmatrix} b_i u^{r_i} & h_i \\ 0 & b_i u^{r_i} \end{smallmatrix}\right)$. The height-one condition makes $u^e A_i^{-1}$ integral. Its upper-right entry gives $u^e \mid h_i$ when $r_i = e$; when $r_i = 0$ no divisibility is needed. Thus in either case $h_i = b_i u^{r_i} f_i(u)$ with $f_i \in k[[u]]$. Changing the quotient lift by μ_i changes f_i by $\varphi(\mu_{i-1}) - \mu_i$.

On $uk[[u]]$ this cyclic coboundary map is bijective: after one cycle its solution is obtained by inverting $1 - \varphi^f$, using the convergent series $\sum_{n \geq 0} \varphi^{fn}$. Hence all positive powers of u can be removed. On constants the coboundaries are $(\mu_{i-1} - \mu_i)_i$, whose cokernel is k, measured by $\sum_i a_i$. This proves the dimension assertion. Unramified selfextensions of the common character form a one-dimensional space and have finite-flat models: tensor the rank-one model with an unramified two-dimensional unipotent representation and descend from a finite unramified extension. The exact comparison identifies this nonzero line with the whole extension space, proving the generic assertion. □

The restriction $r_i \in \{0, e\}$ is used in the divisibility of h_i. The middle-exponent calculations below are separate.

11.3 Ramification of nonordinary torsion

We first prove a local statement independent of K.

Theorem 11.7. *Let $F/\mathbf{Q}_2$ be ramified quadratic, and let $r_F : G_F \to \mathrm{GL}_2(E)$ be nonordinary, crystalline of labelled weights $\{0, 1\}$ and determinant χ_2. For any stable lattice, the point field of its reduction has positive upper breaks at most one and normalized relative different contribution less than one, with $v_2(2) = 1$. If the reduction is locally absolutely irreducible, its full image is S_3 and its wild inertia is trivial.*

Proof. As in Lemma 11.3, the lattice is realized by a biconnected Barsotti–Tate group G. The same holds after isogeny, and $G[\varpi_E]$ and all its finite-flat subquotients are biconnected.

After enlarging the residue coefficient field, suppose first that the reduction is reducible. Its diagonal characters are unramified, since $\mathcal{O}_F^\times$ is pro-two. After an unramified extension it is $1 + c(g)E_{12}$, where c has values in a finite field k of characteristic two. For any $\mathbf{F}_2$-linear functional $\lambda : k \to \mathbf{F}_2$, pull the quotient line back to $\mathbf{F}_2$ and quotient the invariant line by $\ker \lambda$. This gives a biconnected self-extension of the simple model with generic character $\lambda \circ c$. Lemma 11.2 bounds its conductor by two. These characters separate the elementary abelian point-field group, so every positive upper break is at most one. If its ramification index is e, its different exponent is at most $2(e - 1)$; division by $2e$ gives the asserted normalized bound. Unramified base change preserves these ramification bounds.

If the reduction is locally absolutely irreducible, the local classification in Lemma 4.3(ii) gives full image S_3 and trivial wild inertia. Tame index three contributes $1/3$ to the normalized relative different. □

11.4 Inert odd coefficient primes

Lemma 11.8. *Let (K, p) be one of $(F_2, 5), (F_3, 5), (F_3, 7)$. Let r be a member of a compatible system as above, crystalline of weight two at p and of cyclotomic determinant. Suppose its residue is globally reducible with constituents $\psi, \omega_p \psi^{-1}$ and ψ^2 is unramified at every finite place away from p. Then r is ordinary and modular.*

Proof. The completion K_p is unramified quadratic. A residual character is prime to p and therefore kills wild inertia. If a is the tame exponent of ψ modulo p^2-1, the rank-one finite-flat exponents and the unit relation for the totally even character ψ^2 give

K	p	$\mathrm{ord}(\epsilon \bmod p)$	a allowed by units	a finite flat
F_2	5	12	$6 \mid a$	$0, 1, 5, 6$
F_3	5	3	$3 \mid a$	$0, 1, 5, 6$
F_3	7	8	$4 \mid a$	$0, 1, 7, 8$

For the first order, $(1+\sqrt{2})^3 = 2$ and $(1+\sqrt{2})^6 = -1$ modulo five. For the second, $\epsilon_3^2+\epsilon_3+1 = 0$ modulo five. Modulo seven, $\epsilon_3^4 = -1$ and $\epsilon_3^8 = 1$. In each row only $a = 0, p+1$ survive: these are the etale and multiplicative endpoints. The tame exponent list is the rank-one finite-flat classification over the unramified quadratic base [53].

This implies actual ordinarity. Otherwise the Barsotti–Tate group of a stable coefficient lattice and its dual would be connected: a coefficient-stable etale factor would give the ordinary coefficient-rank-one filtration. Schematic closure and finite-flat quotient, after killing unramified character factors, produce a rank-one finite-flat subquotient with one of the two endpoint generic characters. At absolute ramification index one, finite-flat uniqueness [53, Theorem 3.3.3 and Corollary 3.3.6] identifies its model with the etale or multiplicative endpoint, contradicting biconnectedness. The unique ordinary filtration descends through the unramified extension. Apply [26, Corollary 1.2.2] to $r^\vee$: K is totally real abelian, p is unramified, the actual source is ordinary, its determinant is inverse cyclotomic, and its residue is reducible. This proves modularity of r. □

Lemma 11.9. *Let $p \geq 5$ be inert and unramified in a real quadratic field K, with $[K(\zeta_p) : K] = p-1$. An absolutely irreducible residue of an actual weight-two crystalline cyclotomic p-adic source remains absolutely irreducible over $K(\zeta_p)$.*

Proof. Otherwise Clifford theory induces it from the unique quadratic subfield of $K(\zeta_p)/K$. The image is dihedral of order prime to p, and that quadratic field is ramified at p. Cyclic tame projective inertia therefore contains a reflection and has order two. This contradicts the finite-flat inertia possibilities. Over the unramified quadratic completion the rank-one exponents are $0, 1, p, p+1$ modulo p^2-1. Cyclotomic determinant gives pairs $(0, p+1)$ or $(1, p)$, with projective orders $p-1$ or $p+1$. In the locally irreducible case the niveau-four exponents are

$$p+1, \quad p^2+p, \quad p^3+1, \quad p^3+p^2 \pmod{p^4-1}.$$

They are paired by multiplication by p^2. Their projective inertia order is $(p^2+1)/2$: subtracting the paired exponent reduces the computation to $\gcd(p^2+1, p \pm 1) = 2$. None of the displayed orders is two for $p \geq 5$. These are precisely the labelled finite-flat tame profiles [53], so the induced case is impossible. □

For either field at five, the cyclotomic degree is four; for F_3 at seven it is six. A small-field modular witness [3, Theorem 2.5], or the projective witness specified below, can therefore be lifted to the actual source by [3, Theorem 2.3] using Lemma 11.9. The prime three is treated separately, since its order $p-1 = 2$ would not give this exclusion.

11.5 A common cubic coefficient-field screen

Lemma 11.10. *Let D be a totally real number field of degree $d \leq 3$. If $d = 3$, suppose $D = \mathbf{Q}(\alpha)$, where α is an algebraic integer, every conjugate of α lies in $[-\sqrt{8}, \sqrt{8}]$, and its minimal polynomial is congruent to X^3 modulo two. Then D has a prime above three with residue field $\mathbf{F}_3$ or $\mathbf{F}_9$, or a prime above five with residue field $\mathbf{F}_5$.*

Proof. For $d \le 2$, any prime above three has the required residue field. In degree three write the minimal polynomial as $f(z) = z^3 + Az^2 + Bz + C$. Its nonleading coefficients are even, and purity gives $|A| \le 8$, $-8 \le B \le 24$, $|C| \le 22$. The bound on B follows by evaluating the pairwise-product sum at the vertices of the cube $[-\sqrt{8}, \sqrt{8}]^3$. Exact Sturm counting among these even coefficient triples leaves four irreducible polynomials and their conjugates $-f(-z)$:

$f(z)$	p	local certificate
$z^3 - 2z^2 - 4z + 6$	3	$f(0) \equiv 0, \quad f'(0) \not\equiv 0 \pmod 3$,
$z^3 - 6z - 2$	3	$f(z-1) = z^3 - 3z^2 - 3z + 3$ is Eisenstein,
$z^3 - 2z^2 - 2z + 2$	5	$f(3) \equiv 0, \quad f'(3) \not\equiv 0 \pmod 5$,
$z^3 - 4z - 2$	5	$f(1) \equiv 0, \quad f'(1) \not\equiv 0 \pmod 5$.

The enumeration discards $C = 0$ and reducible polynomials and counts roots on the indicated interval by Sturm's theorem; endpoint signs are evaluated exactly in $\mathbf{Q}(\sqrt{8})$. Hensel's lemma supplies a degree-one completion in the first, third and fourth rows. The second has a totally ramified cubic completion at three and hence residue field $\mathbf{F}_3$. The same conclusions hold for $-f(-z)$. The exact calculation is supplied in [49, `common`]. □

12 Symmetry and quadratic descent

The integral-order bounds can force either a small coefficient field or an isomorphism with a quadratic conjugate. We record the two rigidity mechanisms used below. The prime-to-characteristic cotangent argument is distinct from the dyadic order argument.

12.1 Finite-order actions on complete local rings

Lemma 12.1 (Cotangent rigidity)**.** *Let $\mathcal{O}$ be a complete noetherian local ring, and let A be a complete noetherian local $\mathcal{O}$-algebra with the same residue field. Let a finite group Γ, with $|\Gamma| \in \mathcal{O}^\times$, act continuously by local $\mathcal{O}$-algebra automorphisms on A. If its action on $\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}}A)$ is trivial, its action on A is trivial.*

Proof. Average lifts $x_1, \dots, x_n$ of a relative cotangent basis over Γ. The resulting invariant elements have the same cotangent classes. Nakayama and completeness give a surjection from $\mathcal{O}[[X_1, \dots, X_n]]$ to A sending its variables to these elements. The entire image is fixed. □

The invertibility hypothesis is essential: $T \mapsto -T$ is a nontrivial involution of $\mathbf{Z}_2[[T]]$ acting trivially on its relative cotangent space. In applications to pseudodeformations, the cotangent calculation must include products of opposite extensions, not only tangent vectors of representations.

Lemma 12.2. *Let k be a field of characteristic different from two, and let $V = ks \oplus kd$ carry the involution $\sigma(s) = s$, $\sigma(d) = -d$. On $V^\vee \otimes_k V^\vee$ put*

$$\iota(a \otimes b) = \sigma^\vee(b) \otimes \sigma^\vee(a), \qquad \ell_\pm = s^\vee \pm d^\vee.$$

If an ι-equivariant quotient kills both $\ell_+ \otimes \ell_+$ and $\ell_- \otimes \ell_-$, then ι acts trivially on that quotient.

Proof. The (-1)-eigenspace of ι is the line generated by $s^\vee \otimes d^\vee + d^\vee \otimes s^\vee$. It is killed because

$$\ell_+ \otimes \ell_+ - \ell_- \otimes \ell_- = 2(s^\vee \otimes d^\vee + d^\vee \otimes s^\vee).$$

Since two is invertible, the two eigenspaces split and the quotient has only the eigenvalue one. □

12.2 Descent from a twisted conjugacy

Lemma 12.3 (Quadratic descent)**.** *Let $K/\mathbf{Q}$ be real quadratic, with nontrivial automorphism σ. Let $\mathcal{R} = (r_\lambda)$ be a totally odd, absolutely irreducible strongly compatible rank-two system over K, with cyclotomic determinant and regular labelled Hodge–Tate weights. Suppose that, for one coefficient place,*

$$r_\lambda^\sigma \simeq r_\lambda \otimes \eta,$$

where η is the restriction of a quadratic character of $G_{\mathbf{Q}}$ (the trivial character is allowed). Then $\mathcal{R}$ is modular.

Proof. The identity of good Frobenius traces transfers the displayed isomorphism to a companion r in characteristic $p \geq 5$, by compatibility and Chebotarev. Write $H = G_K$ and choose $t \in G_{\mathbf{Q}} \setminus H$. There is an intertwiner A with

$$Ar(h)A^{-1} = \eta(h)r(tht^{-1}) \qquad (h \in H).$$

Since $\eta^\sigma\eta = 1$, Schur's lemma gives $A^2 r(t^2)^{-1}$ scalar. Thus $[r]$ and $[A]$ define a continuous projective representation of $G_{\mathbf{Q}}$.

After a finite coefficient extension, scale A so that $\det A = \chi_p(t)$. Define a continuous choice of matrices by $L(h) = r(h)$ and $L(ht) = r(h)A$ for $h \in H$. Then $\det L(g) = \chi_p(g)$ for every g, so its scalar cocycle

$$c(g_1, g_2) = L(g_1)L(g_2)L(g_1g_2)^{-1}$$

takes values in $\{\pm 1\}$. It is locally constant and is trivial on $H \times H$. Tate's theorem in [54, Theorem 6.3], for the *discrete trivial* $G_{\mathbf{Q}}$-module $\overline{\mathbf{Q}}^\times$, gives a continuous cochain $b : G_{\mathbf{Q}} \to \overline{\mathbf{Q}}^\times$ such that $c = \delta b$. Compactness makes the image of b finite. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ and enlarge the coefficient field by these finitely many algebraic values. Then $\widetilde{r} = b^{-1}L$ is a continuous representation of $G_{\mathbf{Q}}$. Because $c|_{H\times H} = 1$, the restriction $b|_H$ is a finite-order character. In fact $c^2 = 1$ implies that b^2 is a finite-order character of $G_{\mathbf{Q}}$.

The representation $\widetilde{r}$ is absolutely irreducible, since its restriction to H is a finite twist of r. The locally constant function b factors through a finite Galois quotient. Outside the ramification of that quotient, $K/\mathbf{Q}$ and r, inertia lies in H and acts trivially, so $\widetilde{r}$ is finitely ramified. Finite twists preserve de Rham representations and their weights; de Rhamness descends through finite local extensions. Thus $\widetilde{r}$ is regular de Rham at p. Every complex conjugation belongs to H, and the square of its value under $b|_H$ is one, so $\widetilde{r}$ is odd. Apply [50, Theorem 1.0.4]. Cyclic base change to K is cuspidal because its Galois restriction is irreducible; untwisting by $b|_H$ proves modularity of r, and hence of the compatible system. □

12.3 Coefficient orbits and the integral-order bound

Proof of Theorem 1.4. The two operations commute and are involutions on isomorphism classes. Let $V = C_2^2$ be the resulting abstract group, let $m \in \{1, 2, 4\}$ be the number of coefficient orbits, and put $e = [D : \mathbf{Q}]$. Each operation preserves the trace-field degree, since it permutes the good primes and changes their traces only by rational signs. Lemma 11.1 gives $4me = \dim_{\mathbf{F}_2}(E_0/2E_0) < 32$.

The stabilizer V_0 of the coefficient orbit of $\mathcal{R}$ has order $4/m$. Its elements induce unique automorphisms of D, because the good traces generate D. These automorphisms form a homomorphism $V_0 \to \mathrm{Aut}(D)$ whose kernel consists of actual isomorphisms of systems. The element H is not in the kernel. Any nontrivial kernel therefore contains G or GH and gives one of the asserted twisted-conjugacy identities.

Otherwise V_0 embeds in $\mathrm{Aut}(D)$. The fixed-field theorem gives $4/m \mid e$, hence $4 \mid me$. Since $0 < me < 8$, we have $me = 4$. Thus

$$(m, e) = (4, 1), (2, 2), (1, 4).$$

In the last case $D/\mathbf{Q}$ has a Klein four group of automorphisms and degree four, so is biquadratic. A residue-field Galois group is then a cyclic quotient of a subgroup of C_2^2, and has order at most two. The other cases have degree at most two. The descent assertion is Lemma 12.3. □

Corollary 12.4 (The two-orbit alternative)**.** *Let $\mathcal{R}$ be an absolutely irreducible strongly compatible rank-two system over a real quadratic field, with cyclotomic determinant and trace field D. Let h be a rational quadratic character restricted to its base field. Suppose there is no actual h-selftwist. If the simultaneous order of the coefficient orbits generated by $\mathcal{R}$ and $\mathcal{R} \otimes h$ has reduction of dimension less than sixteen, then $D = \mathbf{Q}$, or D is quadratic and its nontrivial automorphism realizes the twist by h.*

Proof. Write $m \in \{1, 2\}$ for the number of orbits and $e = [D : \mathbf{Q}]$. The rank formula gives $me < 4$. If $m = 2$, then $e = 1$. If $m = 1$, the stabilizer C_2 acts faithfully on D, since there is no actual selftwist. Thus $2 \mid e$ and $e < 4$, so $e = 2$ and the nontrivial automorphism gives the stated identity. □

The bounds concern the actual simultaneous order, not one residual realization. The small-residue-degree alternative must still be combined with residual modularity and a lifting theorem for the actual source.

Part II

Terminal source theorems

13 The common reduction for five ramified dyadic fields

Write $F_d = \mathbf{Q}(\sqrt{d})$. The five pairs considered here are

d	2	3	6	10	11
$\mathrm{disc}(F_d)$	8	12	24	40	44
ℓ	7	13	5	3	5

In every case ℓ splits in F_d, and the unique dyadic completion has absolute ramification index two and residue field $\mathbf{F}_2$. The quadratic Minkowski bounds $\sqrt{8}/2 < 2$ and $\sqrt{12}/2 < 2$ give class number one for F_2, F_3; the arithmetic for the other three fields is established below.

Theorem 13.1. *Assume GRH for Dedekind zeta functions. For $d \in \{2, 3, 6, 10, 11\}$, every continuous absolutely irreducible and totally odd representation $G_{F_d} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, for every odd prime p and arbitrary conductor, is modular.*

The field-dependent arguments below prove the supported residual bases. We first give the reduction which is common to all five fields.

The required local choices are Lemma 4.1 at odd removal characteristics, Lemma 4.3 at the ramified dyadic completion, and Theorem 4.7 at the split terminal prime. Each retains the actual residual representation and has exact cyclotomic determinant.

13.1 Reduction to the supported residual base

Proposition 13.2. *Let (K,ℓ) be any pair in the displayed table. Suppose the following two field-dependent assertions hold:*

1. *every absolutely irreducible totally odd residual representation of determinant $\bar{\chi}_\ell$, unramified outside ℓ, is modular;*
2. *for every totally odd reducible representation of determinant $\bar{\chi}_\ell$, unramified outside ℓ, the quotient of its two constituents is totally odd and extends to $G_{\mathbf{Q}}$.*

Then every continuous absolutely irreducible totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, for every odd p and arbitrary conductor, is modular.

Proof. The solvable-image case has an odd solvable Artin witness and is modular after weight raising. Otherwise the cyclotomic restriction is irreducible. The initial weight-two lift of [3, Theorem 2.2] and Lemma 2.4 supply a potentially modular compatible system, with cyclotomic determinant times a finite totally even character. Its nonsolvable residue makes it nondihedral. Apply Theorem 1.2 and reduce at the original coefficient place. □

Lemma 13.3. *Let $(K,\ell) = (F_2,7)$ or $(F_3,13)$. Let $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ have determinant ω_ℓ and reducible semisimplification, and be unramified outside ℓ. The quotient of its constituents is ω_ℓ^e for an odd e, up to inversion. In particular it is totally odd, extends to $G_{\mathbf{Q}}$, and is nontrivial at both ℓ-adic places.*

Proof. Write the constituents as $\psi, \omega_\ell\psi^{-1}$. The finite character ψ has prime-to-ℓ order, hence kills wild inertia and factors through the narrow ray modulo ℓ and the two real places. Write a, b for its two tame exponents modulo $\ell - 1$, and $u, v \in \mathbf{Z}/2$ for its signs.

For $K = \mathbf{Q}(\sqrt{2})$ use the primitive root 3 modulo seven. At the two seven-primes the unit $1+\sqrt{2}$, of signs $(+,-)$, has residues $4 = 3^4, 5 = 3^5$. The unit relations are

$$a + b + u + v \equiv 0 \pmod 2, \qquad 4a + 5b + 3v \equiv 0 \pmod 6.$$

The second relation modulo three gives $a \equiv b \pmod 3$, so $2a \equiv 2b \pmod 6$. For $K = F_3$ the positive unit $2+\sqrt{3}$ has residues $6 = 2^5$ and $11 = 2^7$ modulo thirteen, with primitive root 2. Its relation $5a + 7b \equiv 0 \pmod{12}$ gives $a \equiv b \pmod{12}$. Thus in every case $\psi^2\omega_\ell^{-2a}$ is unramified at all finite places and totally even. Ordinary class number one makes it trivial. The quotient is consequently ω_ℓ^{1-2a}, up to inversion. Its exponent is odd, so it is nontrivial at both split coefficient places and odd at every real place. □

14 Nonsolvable dyadic sources

The following ray auxiliary is used only over F_3; over F_2 a quadratic auxiliary suffices. Its construction and local data are separated from the common source argument.

Proposition 14.1. *Let $K = \mathbf{Q}(\sqrt{3})$. Assume GRH for Dedekind zeta functions. There is a finite Galois solvable extension A/K with $[A : \mathbf{Q}] = 6144$ having the following properties:*

1. *A/K is unramified away from two and thirteen;*
2. *its dyadic classical upper ramification groups are trivial above three;*
3. *at both primes over thirteen its completion is the full tame biquadratic extension of $\mathbf{Q}_{13}$, with $(e,f) = (2,2)$.*

In particular, if M/K is a Galois extension unramified outside two and thirteen, with dyadic upper breaks at most three and with each thirteen-adic completion contained in that biquadratic field, then $L = AM$ has

$$\mathrm{rd}(L) < 8\sqrt{39}, \qquad \frac{2N_{169}(L)}{[L:\mathbf{Q}]} = \frac{1}{2}.$$

Here $N_{169}(L)$ counts the primes of absolute norm 169.

Proof. Put

$$\epsilon = 2 + \sqrt{3}, \qquad \pi = 4 + \sqrt{3}, \qquad \pi' = 4 - \sqrt{3},$$

and let

$$B = K(\sqrt{-1}, \sqrt{\epsilon}, \sqrt{\pi}, \sqrt{\pi'}).$$

The four squareclasses are independent: the valuations at the two thirteen-primes separate π, π', and the real signs separate -1 from the nonsquare positive unit ϵ. Alternatively the explicit successive composita below verify the degree. Thus $[B : K] = 16$, and B/K is elementary abelian and unramified away from two and thirteen. We have $\epsilon = (1 + \sqrt{3})^2/2$ and $\pi\pi' = 13$.

There are two primes of B above two, each with absolute $(e, f) = (8, 2)$; there are eight primes above thirteen, each with $(e, f) = (2, 2)$. The latter completions are the full biquadratic extensions of $\mathbf{Q}_{13}$: ϵ has the squareclass of the nonsquare unit two, and one of π, π' is a uniformizer at each of the two split places. Thus the unramified and ramified quadratic directions both occur.

We record the dyadic filtration explicitly. In the sixteen-element squareclass group generated above, the counts of relative quadratic conductor exponents $0, 2, 4$ at K_2 are respectively $4, 4, 8$; exponent zero includes the locally trivial classes. The local kernel has order two, so the eight local characters have conductor multiplicities $2, 2, 4$. Thus the local extension has an unramified quadratic part, inertia of order four, and positive upper groups of orders four for $0 < u \leq 1$ and two for $1 < u \leq 3$. Its inverse Herbrand function satisfies

$$\psi_{B_w/K_2}(3) = 1 + 2(3 - 1) = 5.$$

These conductor counts follow directly by taking the dyadic valuation of the relative discriminant of $X^2 - a$ for the fifteen nontrivial products a. They are checked in the accompanying GP certificate; no finite-flat refinement is needed here. The exact calculation is supplied in [49, `ramified/d2`].

Let $\mathfrak{P}_1, \mathfrak{P}_2$ be the two dyadic primes of B and put $\mathfrak{m} = \mathfrak{P}_1^6\mathfrak{P}_2^6$. Take the full ray class field of B for $\mathfrak{m}$ and then its largest subfield A in which every prime over thirteen splits completely over B. There is no infinite part of the modulus, since B is totally imaginary. The modulus and the subgroup generated by all thirteen-prime classes are stable under $\mathrm{Gal}(B/K)$. Consequently A/K is Galois and solvable. It is unramified outside two and thirteen, and its thirteen-adic completions equal those of B.

The ray exponent six implies that the classical upper breaks of A/B at two are at most five. Herbrand's subgroup formula and the displayed inverse Herbrand value imply that the upper breaks of A/K at two are at most three. This step uses the general Fontaine-compatible bound three, not a stronger assertion for nonordinary joint trace orders.

Here are the exact arithmetic data defining the extension. An absolute equation for B is obtained by adjoining i, $\sqrt{2}$ and $\sqrt{13}$ successively to a root of $X^4 - 8X^2 + 13$, reducing the polynomial after each step. The result has degree 32, discriminant

$$2^{72}3^{16}13^{16},$$

and class group C_{12}. The full ray group at $\mathfrak{m}$ is

$$C_{24} \times C_8 \times C_2^4.$$

Quotienting by the classes of all eight thirteen-primes gives

$$\mathrm{Gal}(A/B) \simeq C_{12} \times C_2^4.$$

Its order is 192, so $[A : \mathbf{Q}] = 32 \cdot 192 = 6144$. The program `verify_d3_aux6144.gp` constructs the degree-32 polynomial and computes these ray-class data under GRH. The exact calculation is supplied in [49, `ramified/d3/aux32`].

For the last assertion, upper-break bounds are preserved under compositum. At two, the relative different contribution over K_2, normalized by $v_2(2) = 1$, is strictly below $(1+3)/2 = 2$. Adding the base different contribution one gives an absolute contribution strictly below three. At three the extension is unramified over K, contributing the base exponent $1/2$. At thirteen the tame ramification index two contributes $1/2$. Thus $\mathrm{rd}(L) < 2^3\sqrt{3}\sqrt{13} = 8\sqrt{39}$. Every thirteen-adic completion has degree four over $\mathbf{Q}_{13}$ and residue degree two, so $N_{169}(L) = [L : \mathbf{Q}]/4$, as asserted. □

Proposition 14.2. *Assume GRH. Let (K, ℓ) be either $(F_2, 7)$ or $(F_3, 13)$, and let $\mathcal{R}$ be a cyclotomic weight-two compatible system, good outside ℓ. Over F_2 allow good or quadratic-special type at each seven-prime; over F_3 require quadratic-special type at both thirteen-primes. If some dyadic residue is nonsolvable, the actual system is modular. No dyadic ordinarity assumption is required.*

Proof. Let M/K be the point field of the actual dyadic residue. Its determinant is one, so Dickson's classification gives image $G = \mathrm{SL}_2(\mathbf{F}_q)$ with $q = 2^f \geq 4$. Its coefficient-uniformizer torsion is a finite-flat subgroup of rational two-torsion of the Barsotti–Tate group realizing the actual source. Fontaine's bound gives upper break at most three over K_2. Integration gives normalized relative different strictly below two, since $e(K_2) = 2$. The base dyadic contributions are $3/2$ over F_2 and 1 over F_3; thus the absolute contributions are strictly below $7/2$ and 3. At a special odd place the quadratic scalar disappears and inertia has order at most two. At a good place inertia is trivial. Hence

K	auxiliary A	$\mathrm{rd}(AM)$ upper bound	$2N_{169}(AM)/[AM : \mathbf{Q}]$
F_2	$K(i)$	$8\sqrt{14}$	not used
F_3	Proposition 14.1	$8\sqrt{39}$	$1/2$

For F_2, the biquadratic discriminant formula gives $\mathrm{disc}(K(i)) = 8(-4)(-8) = 256$. Its relative dyadic conductor is two, because $256/8^2 = 4$, and its positive upper break is one. The compositum preserves upper bound three and remains tame of index at most two at seven; no decomposition-group assertion at a good seven-place is needed. This proves the displayed upper bound.

For F_3, special Frobenius has trace ± 14 and determinant thirteen. It squares to one modulo two and commutes with the unipotent inertia. Thus the whole thirteen-decomposition image is a quotient of C_2^2, and its local field lies in the maximal biquadratic extension of $\mathbf{Q}_{13}$. The auxiliary in Proposition 14.1 contains precisely that local field, has dyadic upper bound three, and has absolute degree 6144. That proposition proves the displayed discriminant bound and the norm-169 density. Over F_2, only the inertia bound is used.

The group G is nonabelian simple, so M is disjoint from the solvable auxiliary over K. Apply the common GRH formula with $B_{b,100}$ and the positive compact test F_b. The directed 256-bit Arb comparisons are

K	n lower bound	b	rd lower bound	prime terms
$F_2,\ q \geq 8$	$4 \cdot 504 = 2016$	7	31.19016847	0
$F_3,\ q \geq 16$	$6144 \cdot 4080 = 25067520$	16.8	50.07905369	N_{169}

These exceed respectively $8\sqrt{14} < 29.93325910$ and $8\sqrt{39} < 49.95998399$. At fixed b and prime density the lower expression increases with degree. They therefore exclude the full displayed ranges, without a nonordinary ramification refinement. The exact calculation is supplied in [49, `ramified/d2`].

For $K = F_3$ and $q = 8$, let σ be the nontrivial automorphism of $K/\mathbf{Q}$. If $M \neq M^\sigma$, their simple groups make the fields linearly disjoint over K, and their product, with group $G \times G$, is disjoint from the solvable auxiliary. The field AMM^σ has degree $6144 \cdot 504^2 = 1560674304$, the same root-discriminant upper bound and norm-169 density. The common formula at $b = 20$ gives the directed lower bound 51.79576050, a contradiction. If $M = M^\sigma$, the outer automorphism group of $\mathrm{SL}_2(\mathbf{F}_8)$ has odd order three. The action of $\mathrm{Gal}(K/\mathbf{Q})$ is inner. Since G is centerless, conjugation gives a homomorphism $\mathrm{Gal}(M/\mathbf{Q}) \to G$ restricting to the identity on $\mathrm{Gal}(M/K)$. Compose it with the actual residual matrix realization. Rational Serre modularity and quadratic base change give a cuspidal residual witness over K; the restriction is irreducible, so cuspidality is retained. For $q = 4$ over either field, [3, Theorem 2.5] supplies the witness.

In every remaining case some residual real conjugation is nontrivial. Otherwise M is totally real of degree at least 120 and root discriminant below one of the displayed upper bounds. The common formula at $n = 120, b = 4.8$, with real contribution $J_b(1/2) - J_b(3/2) + J_b(5/2) - J_b(7/2)$ and no prime terms, gives $\mathrm{rd}(M) > 56.06306031$, a contradiction in both fields. Apply [69, Theorem 6.1] to the dual actual crystalline two-adic source and dualize back. The nonsolvable modular residue and checked real-place condition meet its hypotheses. It imposes no $\mathbf{Q}_2$-completion requirement and allows ordinary and nonordinary sources. Strong compatibility proves modularity of $\mathcal{R}$. □

15 Scalar and nonscalar sources over $\mathbf{Q}(\sqrt{2})$

In this section $K = F_2$, $s = \sqrt{2}$, $\ell = 7$ and $\epsilon = 1 + s$. A special type includes its exact quadratic scalar character. We first normalize the ramified scalar factors.

Lemma 15.1. *Put $\alpha_\pm = 5 \pm 4s$. The quadratic characters attached to $K(\sqrt{\alpha_\pm})/K$ ramify at exactly one of the two seven-primes, are unramified at two and three, and their product is the character of $K(\sqrt{-7})/K$, split at two.*

Proof. We have $\alpha_+ = \epsilon(3 + s)$ and $\alpha_- = \alpha_+^\sigma$, with $\alpha_+\alpha_- = -7$. Both are units away from seven. The integral element $(1 + \sqrt{\alpha_\pm})/2$ has polynomial $X^2 - X - (1 \pm s)$. Its reduction at the dyadic prime is $X^2 + X + 1$, so the quadratic extensions there are unramified. At three the defining radicands are units, hence the quadratic extensions are unramified. Their odd seven-adic valuations give exactly the asserted seven-ramification. Finally -7 is a square in $\mathbf{Q}_2$ because it is one modulo eight. □

15.1 Nonscalar dyadic residues

Lemma 15.2. *Let $K = \mathbf{Q}(\sqrt{2})$ and let $\bar{r} : G_K \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$ be finite flat at two, unramified away from two and seven, with inertia of order at most two at both seven-primes. If its image is solvable, then $\bar{r}^{\mathrm{ss}} = 1 \oplus 1$.*

Proof. If the representation is reducible, its diagonal characters have odd order and are unramified at seven. Their inertia at two is trivial because the dyadic residue field is $\mathbf{F}_2$ and the wild inertia is pro-two. They are therefore everywhere unramified; narrow class number one makes them trivial.

Otherwise its image is an odd dihedral group, and it is induced from an odd-order character θ of a quadratic extension L/K. The quadratic quotient is unramified outside two and seven. Since K has class number one and unit group $\langle -1, \epsilon \rangle$, its squareclass is generated by $-1, \epsilon, s, 3+s, 3-s$, where $s^2 = 2$. Fontaine's finite-flat bound makes the quadratic conductor at two at most four. A quadratic extension of K_2 defined by an odd valuation radicand has conductor $2e(K_2) + 1 = 5$. Thus the exponent of s is zero, and there are precisely the fifteen nontrivial classes generated by

$$-1, \quad \epsilon, \quad 3 + s, \quad 3 - s.$$

The character θ is unramified at seven. If L/K ramifies there, its residual inertia is a reflection of order two and has trivial intersection with the odd rotation subgroup. If L/K is unramified there, inertia lies in that odd subgroup and has order dividing two, so is trivial. Its wild inertia at two is also trivial. Consequently θ factors through the odd quotient of the ray group of L modulo the primes above two.

That quotient is trivial in all fifteen cases. Here is a complete finite certificate: use the bits of an integer $1 \le m \le 15$ to select the four displayed generators, form their product $A(y)$, and take

$$P_m(X) = \mathrm{Res}_y \left((X - y)^2 - A(y), y^2 - 2 \right).$$

The accompanying program `verify_d2_solvable_two.gp` computes the maximal order and certified class group of each quartic field, and its ray group modulo two. It applies `bnfcertify` in every row and asserts that every ray invariant is a power of two. The possible absolute discriminants and ray invariants are

$\mathrm{disc}(L)$	$\mathrm{Cl}_{(2)}(L)$
256	1
-1024	1
7168	1 or $C_4 \times C_2$
-448	1
-1792	1
12544	1
3136	C_2^2
-50176	C_2^2.

The modulus (2) may contain higher dyadic powers; these add only two-primary ray factors, so its odd quotient equals that of the radical modulus required here. Infinite-place factors are also two-primary. Thus θ is trivial, contradicting the irreducibility of its induction. The exact calculation is supplied in [49, `ramified/d2`]. □

15.2 Ordinary scalar sources

Proposition 15.3 (Ordinary scalar sources)**.** *Let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, strongly compatible rank-two system over $K = \mathbf{Q}(\sqrt{2})$, of cyclotomic determinant and labelled weights $\{0, 1\}$, good outside seven and special at both seven-primes with unramified quadratic scalar characters. If some dyadic member is ordinary with scalar residual semisimplification, the actual system is modular.*

Proof. Write $s = \sqrt{2}$, $\epsilon = 1 + s$, $\pi = 3 + s$, $\pi' = 3 - s$ and $\delta = -7$. The supported quadratic classes with even dyadic valuation form

$$V = \langle [-1], [\epsilon], [\pi], [\pi'] \rangle.$$

Localization at K_2 has kernel $\langle [\delta] \rangle$. Indeed unit squares modulo $8\mathcal{O}_K$ have coordinate pairs $(1, 0), (1, 4), (3, 2), (3, 6)$ in the basis $1, s$. Testing the sixteen classes gives precisely $1, \delta$ in the

square kernel. The accuracy exceeds twice the valuation of two, so Hensel's lemma makes the test exact. Moreover no nonzero class in V is split at both seven-primes: vanishing valuations removes π, π', and the Legendre-symbol vectors of $-1, \epsilon$ are $(-,-)$ and $(-,+)$.

Let r be an ordinary dyadic member over a finite extension $E/\mathbf{Q}_2$, with uniformizer ϖ and residue field k. Choose a vector v on its ordinary line and take the lattice generated by its global orbit. Its residue cannot be trivial, because a cyclic trivial module has dimension one. It is therefore a nonzero unipotent extension, whose invariant line is transverse to the reduction of the ordinary line. Both lines are locally preserved at two, forcing the actual local residue to be trivial. The localization kernel above makes the global residue $1 + \chi_\delta E_{12}$. Since δ is negative at each real place, choose a real conjugation c and the integral basis $((r(c) - 1)v, v)$. Its reductions are independent, and it gives

$$r(c) = C := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{ordinary line } e_2, \qquad \bar{r} = 1 + \chi_\delta E_{12}.$$

The normalized residue is defined over $\mathbf{F}_2$.

We compare two actual sources with these normalizations and the same exact unramified ordinary quotient ν at two. Suppose $r_1 \equiv r_2 \pmod{\varpi^n}$, $n \geq 1$, and write

$$r_2(g) = (1 + \varpi^n X(g)) r_1(g) \pmod{\varpi^{n+1}}, \qquad X(g) = \begin{pmatrix} x(g) & y(g) \\ z(g) & x(g) \end{pmatrix}.$$

The determinant gives the trace-zero form. The adjoint action of $1 + \chi_\delta E_{12}$ adds $\chi_\delta(g) z(h)(1 + E_{12})$; hence z is additive. Equality of the exact ordinary characters gives $x|_{G_{K_2}} = y|_{G_{K_2}} = 0$, and the local class of z is a unit Kummer class. Here is the required assertion for the divided difference. The two local ordinary extensions have identical characters $\nu, \chi_2 \nu^{-1}$. The difference of their integral extension cocycles divided by ϖ^n is integral, and its rational class remains in the E-linear crystalline extension space. The corresponding stable lattice is again ordinary Barsotti–Tate. Its reduction is an extension of the etale model by the multiplicative model, hence a unit Kummer class. This uses actual sources and does not assert exactness for arbitrary models.

At a seven-place residual inertia has a unit upper entry, so its special line is uniquely determined. Congruent representations have congruent special lines: take the kernel of a rank-one inertia difference with a unit entry. If the exact special characters agree modulo ϖ^{n+1}, variation of that line by $t_v e_2$ gives

$$z|_{G_v} = 0, \qquad x|_{G_v} = t_v \chi_\delta|_{G_v}.$$

If they differ by an unramified quadratic sign, put $e = v_\varpi(2)$. The same homogeneous equations hold for $n < e$; at $n = e$ the equations are instead

$$z|_{G_v} = 0, \qquad x|_{G_v} = t_v \chi_\delta|_{G_v} + u U_v, \qquad u = \overline{2/\varpi^e} \neq 0,$$

where U_v is the nontrivial unramified quadratic character. Indeed changing the sign of the special eigencharacter adds this scalar diagonal term.

In either case z is a supported quadratic class, has even dyadic valuation, and is split at both seven-places. The calculation of V gives $z = 0$. The remaining coordinates are now additive; their vanishing at two gives $x = A\chi_\delta$, $y = B\chi_\delta$. The fixed real matrix gives $X(c) = 0$, so $A = B = 0$. Thus every homogeneous first difference vanishes. The inhomogeneous equation is impossible because χ_δ is ramified at each seven-place while U_v is nontrivial unramified.

Choose a lift σ of the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$ in the decomposition group at two. The actual unramified ordinary quotient of r^σ is exactly ν: the ramified automorphism of

$K_2/\mathbf{Q}_2$ acts trivially on its unramified Galois quotient. Normalize it with the same real matrix and marked ordinary line. The normalized residue is again exactly $1+\chi_\delta E_{12}$, since δ is rational. Its two special characters are those of r with the places interchanged. If they differ, their difference is the unramified quadratic sign at both places. Homogeneous comparison for $n<e$ gives congruence modulo ϖ^e; the inhomogeneous comparison at $n=e$ is then the contradiction just proved. Hence the special characters agree. Homogeneous comparison for every $n\geq 1$ and completeness give $r^\sigma \simeq r$.

Lemma 12.3, with $\eta=1$, proves modularity of the actual system. □

Corollary 15.4. *There is no ordinary scalar-residual source as above which is good at one seven-prime and good or special at the other.*

Proof. The cyclic ordinary lattice supplies a nonzero actual unipotent residue trivial at two. Its supported quadratic class must be χ_δ. But $\delta=-7$ ramifies at both seven-primes, contrary to good reduction at one of them. □

15.3 Nonordinary scalar sources

Proposition 15.5. *Let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, strongly compatible rank-two system over $K=\mathbf{Q}(\sqrt{2})$, of determinant cyclotomic and labelled weights $\{0,1\}$. Suppose it is good outside one seven-prime $\mathfrak{q}$ and special there. If every dyadic coefficient residue has scalar semisimplification, every dyadic member is nonordinary and its global trace field is $\mathbf{Q}$. In particular the actual system is modular.*

Proof. After conjugating $K/\mathbf{Q}$, we may assume $\beta=5+4\sqrt{2}$ ramifies at $\mathfrak{q}$. Twist by the character of $K(\sqrt{\beta})/K$ if necessary to remove the ramified quadratic scalar factor at $\mathfrak{q}$. This character is unramified at two and three and at the other seven-prime. It preserves all hypotheses and the trace field. We may therefore assume the special scalar twist is unramified at $\mathfrak{q}$. Corollary 15.4 excludes an ordinary dyadic member.

Apply the common simultaneous regular trace-order construction to all dyadic coefficient factors; they are all nonordinary. Write its orders as R_0, E_0 and put $R=R_0/2R_0$, $E=E_0/2E_0$, $J=\operatorname{rad} E$. Applying Lemma 11.2 to the finite-flat quotients realizing $(J/J^2)^\vee$ bounds their dyadic conductors by two. The elementary quadratic ray quotient supported there and at $\mathfrak{q}$ has basis

$$\alpha=[-1], \qquad \beta=[5+4\sqrt{2}].$$

Choose the real conjugation at $\sqrt{2}>0$ and a tame inertia element at $\mathfrak{q}$, and call their differences A, C. Their evaluations on (α,β) are $(1,0),(0,1)$, so A, C generate E. The real matrix and the exact special identities give

$$A^2=C^2=0, \qquad AC+CA=X=T(AC).$$

All words lie in the R-span of $1, A, C, AC$. At $\mathfrak{q}$, -1 is unramified nonsquare; change an arithmetic Frobenius lift by inertia so that its value on β is zero. Its difference B has the same first-layer values as A, hence $B-A\in J^2$. The special identity $BC=0$ gives $AC\in J^3$.

Work over $R'=R/\mathfrak{m}_R^2$. The relative coefficient cotangent is generated by X, since the coefficient order is generated by the actual traces. All traces of $1, A, C$ vanish modulo two. The normal relations show that $T(J^3)=0$ over R': the length-three words are $ACA=XA$ and $CAC=XC$, with trace zero, and the length-four trace $T(ACAC)=X^2$ vanishes in R'. Higher words reduce to the same relations; factors from $\mathfrak{m}_{R'}E$ contribute trace zero because all traces lie in $\mathfrak{m}_{R'}$. It follows that $X=T(AC)=0$ in R'. Nakayama gives $R=\mathbf{F}_2$, so R_0 has rank one and the trace field is $\mathbf{Q}$.

Lemma 15.7 now proves modularity. □

Put $K = \mathbf{Q}(\sqrt{2})$, $s = \sqrt{2}$, $\epsilon = 1 + s$, and $\pi = 3 + s$, $\pi' = 3 - s$. A special condition includes its exact quadratic twist, including the unramified character.

Proposition 15.6. *Let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, strongly compatible rank-two system over K of determinant cyclotomic and labelled weights $\{0, 1\}$. Suppose it is good outside seven and special at both seven-primes. If every dyadic coefficient member is nonordinary with scalar residual semisimplification, its global trace field is $D = \mathbf{Q}(a_2)$ of degree $d \le 3$, and its dyadic trace order is $R_0 = \mathbf{Z}_2[a_2]$ with $R_0/2R_0 \simeq \mathbf{F}_2[t]/(t^d)$, where $t = \overline{a_2}$.*

Proof. Let D be the global trace field, $d = [D : \mathbf{Q}]$, and apply the common simultaneous integral trace-order construction to all dyadic coefficient embeddings. Write its coefficient order as R_0 and its regular order as E_0, with trace T and norm $\mathcal{D}$. Thus $\operatorname{rank}_{\mathbf{Z}_2} R_0 = d$, and the regular lattice is an actual nonordinary Barsotti–Tate lattice. Write $E = E_0/2E_0$, $R = R_0/2R_0$, and $J = \operatorname{rad} E$. Lemma 11.2 bounds the dyadic conductors of the quadratic characters in $(J/J^2)^\vee$ by two.

The elementary quadratic quotient of the narrow ray group of modulus $\mathfrak{p}_2^2\pi\pi'\infty_1\infty_2$ has rank three. A basis is

$$\alpha = [-1], \qquad \beta = [5 + 4s] = [\epsilon\pi], \qquad \delta = [-7].$$

Indeed the full narrow ray group has invariants $(6, 2, 2)$; these classes are independent by their seven-valuations and real signs. Their relative discriminant norms are $4, 7, 49$, respectively. At two, α has break one, β is the unramified quadratic class, and δ is trivial. The latter follows already from $-7 \equiv 1 \pmod 8$ in $\mathbf{Q}_2$.

Choose the seven-place where $s \equiv 3 \pmod 7$, a real conjugation at the embedding $s > 0$, an arithmetic Frobenius at this seven-place, and a tame inertia generator there. Frobenius can be chosen to be trivial on the ramified quadratic class [7]. Their evaluation vectors on (α, β, δ) are

$$(1, 0, 1), \qquad (1, 1, 1), \qquad (0, 0, 1).$$

Let A, B, C be their differences from one in E. These evaluation vectors are independent, so A, B, C span J/J^2 and generate E. At the fixed special place the normalized characteristic-zero local differences $L(g)$ obey $L(g)L(h) = (\chi_2(g) - 1)L(h)$. Their exact quadratic sign disappears modulo two, yielding zero products and trace zero for the corresponding local differences in E. Polarized Cayley–Hamilton therefore gives

$$A^2 = B^2 = C^2 = BC = CB = 0, \quad AB + BA = X = T(AB), \quad AC + CA = Y = T(AC).$$

Every word is in the R-span of $1, A, B, C, AB, AC$.

We calculate in $R' = R/\mathfrak{m}_R^2$, retaining the original trace and norm laws. The traces of the six normal words show that its cotangent space is generated by X, Y. The regular Kisin module is free of rank one on the right over $E_0[[u]]$; its Frobenius generator $\Phi(u)$ has norm divisible by $u^2 - 2$ and trace at $u = 0$ equal to the arithmetic coefficient a_2. The ramified first-layer functional is $(1, 0, 0)$ on (α, β, δ), so

$$\Phi = N + u(1 + M) + O(u^2), \quad N = A + C + Z, \quad M = m_A A + m_B B + m_C C + Z', \quad Z, Z' \in J^2.$$

The unramified functional takes value one on β, hence $m_B = 1$. The norm of Z, its polar term with $A + C$, and all trace terms of length at least three vanish in R' by the six-word relations. The constant coefficient of the norm gives $0 = \mathcal{D}(N) = \mathcal{D}(A + C) = Y$. The linear coefficient of the norm gives

$$a_2 = T(N) = T(NM) = m_B X + (m_A + m_C)Y = X \quad \text{in } R'.$$

Nakayama proves

$$R = \mathbf{F}_2[t]/(t^d), \qquad t = \overline{a_2}, \qquad D = \mathbf{Q}(a_2).$$

If $d \geq 2$, X is another generator of its maximal ideal and $Y = vX$ with $v \in \mathfrak{m}_R$. The degree assertion follows because $1, a_2, \ldots, a_2^{d-1}$ span R_0 by Nakayama.

Suppose $d \geq 4$. Put $Z_0 = C + vB$. It is central and satisfies $Z_0^2 = BZ_0 = XZ_0 = 0$ and $T(Z_0) = T(AZ_0) = 0$. Over $S = \mathbf{F}_2[X]/(X^3)$ use the abstract free rank-four algebra

$$Q = S\langle a, b\rangle/(a^2, b^2, ba + ab - X).$$

There is a surjection $E \to Q$ taking A, B, Z_0 to $a, b, 0$. To verify that this map is well-defined, pair any relation in the normal words $1, A, B, AB, Z_0, AZ_0$ with $1, A, B, AB$ by the original trace. For its first four coefficients f_i the pairings are $Xf_3, Xf_2, Xf_1, Xf_0 + X^2 f_3$. Their vanishing puts every f_i in (X^{d-1}), which vanishes after base change to S. The normal-word multiplication then proves the assertion.

Compose this quotient with $a \mapsto E_{12}$ and $b \mapsto XE_{21}$ to obtain $\theta : G_K \to \mathrm{GL}_2(S)$. Its trace is the original trace reduced to S, and comparison of the two Cayley–Hamilton laws gives determinant one. The two visible linear characters are $\alpha + \beta$ and β: their values on A, B, C are $(1, 0, 0)$ and $(0, 1, 0)$, since v is in the radical. Put

$$F = K(i, \sqrt{5 + 4s}).$$

On G_F every matrix has the form $I + XM + X^2N$, where the lower-left entry of M is zero and its diagonal entries agree. These M's lie in a commutative algebra, so $\theta(G_F)$ is abelian of exponent at most four. If r is the product of the chosen real and seven-Frobenius elements, $\operatorname{tr}\theta(r) = X$, whence

$$\theta(r)^4 = I + X^2 I \neq I.$$

Since $r^2 \in G_F$, the abelian image over F has an element of order four.

This contradicts a small ray calculation. The field F has degree eight, class number one, absolute discriminant 3211264, one dyadic prime $\mathfrak{P}$ with absolute $(e, f) = (4, 2)$, and three seven-primes with $(e, f) = (2, 2), (1, 2), (1, 2)$. The only ramified break of F/K at two is one. The original regular point field has upper break at most three over K_2, so Herbrand's subgroup formula bounds the break over F by five. At seven the special identities already show that the whole local image in E has exponent two. Hence $\theta|_{G_F}$ is a quotient of the ray group H of modulus $\mathfrak{P}^6 \prod_{w|7} w$, with $4H$ and $2I_7$ killed, where I_7 is generated by the three local tame unit images.

A defining polynomial, and the exact ray quotient, are

$$z^8 - 4z^7 + 8z^6 - 8z^5 + 7z^4 - 8z^3 + 8z^2 - 4z + 1,$$
$$H \simeq C_{48} \times C_8 \times C_2^3, \qquad H/(4H + 2I_7) \simeq C_2^5.$$

Equivalently, let H_0 be the ray group for $\mathfrak{P}^6$ alone. The canonical map $H \to H_0$ has kernel I_7, and $H_0 \simeq C_2^2$. Computing the Smith form of $4H + 2\ker(H \to H_0)$ gives the displayed quotient; this is the coordinate-free calculation in `verify_sp7_ray_quotient_independent.gp`. The quotient has exponent two, contradicting the order-four image just constructed. Hence $d \leq 3$. The exact calculation is supplied in [49, `ramified/d2`]. □

Lemma 15.7. *A good/special-seven system as above is modular if its trace field has degree at most three and is generated by a_2, with $\mathbf{Z}_2[a_2]/2 = \mathbf{F}_2[t]/(t^d)$ when $d = 3$.*

Proof. Lemma 11.10 gives a coefficient prime over three with residue field $\mathbf{F}_3$ or $\mathbf{F}_9$, or a coefficient prime over five with residue field $\mathbf{F}_5$. For an irreducible residue, finite-field Brauer descent realizes the representation over this trace residue field.

At three a reducible residue has constituents $1, \omega_3$ and the actual source is ordinary and modular by Lemma 15.9. An irreducible residue remains cyclotomically irreducible by that lemma. Its small-field witness [3, Theorem 2.5], followed by [3, Theorem 2.3], modularizes the actual weight-two source.

At the selected prime over five the residue field is $\mathbf{F}_5$. In the reducible case the square of a constituent is unramified away from five, since every seven-type is good or quadratic-special. Lemma 11.8 modularizes the actual source. In the irreducible case Lemma 11.9, the small-field witness and the actual weight-two AKT theorem apply. □

Theorem 15.8. *Assume GRH. Every pure totally odd absolutely irreducible weight-two cyclotomic compatible system over K, good outside seven, with good or quadratic-special type at both seven-primes, is modular.*

Proof. First remove any ramified quadratic special scalar factors using Lemma 15.1; the corrections are unramified at two and three and at the other seven-prime. Hence all actual special scalar characters may be taken unramified. If some dyadic coefficient residue is nonsolvable, apply Proposition 14.2, which covers both ordinary and nonordinary sources under the present GRH hypothesis. The irreducible solvable alternative is excluded by Lemma 15.2; hence we may assume every dyadic coefficient residue has scalar semisimplification. If some dyadic member is ordinary, the ordinary proposition or its one-good-place corollary proves the assertion. Otherwise all are nonordinary. With exactly one special place, Proposition 15.5 applies. With two special places, Proposition 15.6 and Lemma 15.7 apply.

If both seven-places are good, the first-layer quadratic space of the regular nonordinary trace algebra is just $\langle [-1] \rangle$. Indeed its characters have dyadic conductor at most two by the middle-model calculation, and the narrow ray group of modulus $\mathfrak{p}_2^2 \infty_1 \infty_2$ has order two. The real difference $A = c - 1$ therefore generates the algebra $E = E_0/2E_0$ and satisfies $A^2 = 0$. The original trace of both 1 and A is zero modulo two. All group traces consequently vanish, so the coefficient algebra $R = R_0/2R_0$ is $\mathbf{F}_2$. This makes E spanned by $1, A$, contradicting $\dim_{\mathbf{F}_2} E = 4[D : \mathbf{Q}] \geq 4$. This treats the last case. Undo the finite quadratic twists. □

An unconditional observation. The separate nonordinary exclusion remains available without GRH. If a nonordinary dyadic member in the preceding theorem had nonsolvable residue, its kernel field M would have degree at least 120. Theorem 11.7 gives $\mathrm{rd}(M) < 4\sqrt{14} < 14.967$, whereas (40) with $b = 7$ gives $\mathrm{rd}(M) > 16.661$. This observation is not needed in the streamlined GRH proof above.

15.4 The supported-seven problem over $\mathbf{Q}(\sqrt{2})$

Put $K = \mathbf{Q}(s)$ with $s^2 = 2$, $\epsilon = 1 + s$ and $\mathfrak{l}_\pm = (3 \pm s)$. The prime three is inert and $K_3/\mathbf{Q}_3$ is unramified quadratic.

Lemma 15.9. *Let $\mathcal{R}$ be a pure, irreducible, totally odd weight-two strongly compatible system with cyclotomic determinant, good outside seven, whose seven-types are good, quadratic-special, or principal series of order three after the corrections above. If its three-adic residue is reducible, the actual system is modular. If that residue is irreducible, it remains irreducible on $G_{K(\zeta_3)}$.*

Proof. Twist away the scalar quadratic inertia using Lemma 15.1. Modulo three the inertia at both seven-primes is unipotent. A reducible global residue therefore has characters unramified outside three. The prime-to-three narrow ray group modulo three is C_2: ϵ has order eight in $\mathbf{F}_9^\times$ and signs $(+, -)$, while -1 supplies the independent sign relation. Its characters are $1, \omega_3$. Thus the residual constituents are exactly $1, \omega_3$.

Since $e(K_3) = 1 < 3 - 1$, finite-flat uniqueness identifies the rank-one models with the etale and multiplicative models. A nonordinary Barsotti–Tate source would be biconnected, as would its finite-flat subquotients; these rank-one models contradict biconnectedness. The actual source is ordinary, with unramified quotient reducing to one. Skinner–Wiles [62, §4.5, Theorem A] applies: the ratio is locally nontrivial, totally odd, and its splitting field $K(\zeta_3)$ is abelian over $\mathbf{Q}$. The determinant is χ_3 and the local quotient has weight zero. Undo the twists.

For the second assertion, a cyclotomic-reducible irreducible residue would be induced from a prime-to-three character θ of $H = K(\zeta_3)$. Its restriction to seven-inertia is trivial since that inertia has order dividing three. The character is therefore unramified outside three and tame there. The absolute discriminant of H is $8 \cdot (-3) \cdot (-24) = 576$, and its degree-four Minkowski bound is $36/\pi^2 < 4$. There is no prime of norm two or three: their norms are four and nine. Hence H has class number one. Its unique prime above three has residue field $\mathbf{F}_9$, and ϵ generates the residue units. The tame ray group is trivial. This makes θ trivial, contradicting irreducibility. □

Lemma 15.10. *Every local determinant-ω_7 residue over $\mathbf{Q}_7$ has a determinant-χ_7 weight-two lift that is good, special, or principal series of order three after a scalar quadratic twist.*

Proof. The central-trivial Serre weights are $\operatorname{Sym}^r \det^b$, with $r = 0, 2, 4, 6$ and $r + 2b = 0 \pmod 6$. The two principal-series types with characters $\widetilde{\omega}_7^t, \widetilde{\omega}_7^{-t}$ have reductions

t	first constituent	second constituent
1	$(2, 5)$	$(4, 1)$
2	$(4, 4)$	$(2, 2)$.

Together with the two scalar crystalline and two scalar semistable weights, these cover all eight weights. The fixed-determinant formula [70, Theorem 1.2] therefore gives a nonempty weight-two local ring in this menu, for the actual residual extension. A scalar quadratic twist interchanges the principal-series rows and changes order six to order three. The independent global characters in Lemma 15.1 implement these choices at the two split seven-primes without disturbing good reduction at two or three. □

Lemma 15.11. *Let $\bar{r}$ be a local characteristic-three representation at a seven-prime, of determinant one and with tame inertia unipotent. It admits an unramified or genuine special determinant-cyclotomic lift preserving the full residual representation.*

Proof. If inertia is trivial, lift the entire unramified Frobenius matrix, with fixed determinant seven; unequal Frobenius eigenvalues are allowed. Otherwise write tame inertia as $1 + tE_{12}$, $t \neq 0$. Because $7 = 1 \pmod 3$, Frobenius centralizes this matrix and has shape $aI + bE_{12}$, $a = \pm 1$. Its lift

$$\Phi = \begin{pmatrix} 7a & \widetilde{b} \\ 0 & a \end{pmatrix}, \qquad U(x) = 1 + \widetilde{t}xE_{12}$$

has determinant seven, satisfies $\Phi U(x)\Phi^{-1} = U(7x)$, and has nonzero monodromy. It retains both the residual inertia parameter and the Frobenius extension parameter. □

Proposition 15.12. *Assume GRH. Every irreducible determinant-ω_7 residue supported at seven is modular.*

Proof. A solvable image has a modular Artin witness. For a nonsolvable image the cyclotomic restriction is irreducible. Use Lemma 15.10, prescribed globalization, actual avoiding potential modularity and Brauer descent to construct a weight-two compatible system with the stated bounded menu, good outside seven. Normalize the independent principal-series signs as above.

Reduce at three. A reducible residue already modularizes the actual source by Lemma 15.9. Otherwise that lemma gives cyclotomic irreducibility. There is no exceptional-five exclusion in characteristic three. Prescribe the actual good/special lifts of Lemma 15.11, retaining the old Barsotti–Tate component at three, and globalize by EGPZ. Actual potential modularity and compatibility give the new system. Purity ensures nonzero special monodromy wherever the special component was selected, since its two Frobenius eigenvalues have ratio seven. By Theorem 15.8 this new system is modular. AKT applies backwards to the old weight-two three-adic source. Undoing the quadratic twists and reducing at seven proves the assertion. □

16 Integral-order rigidity over $\mathbf{Q}(\sqrt{3})$

In this section $K = F_3$, $\ell = 13$ and $\epsilon = 2 + \sqrt{3}$. Every special type retains its exact quadratic scalar character. No global correction of the two signs is required.

16.1 Nonscalar dyadic sources

Proposition 16.1. *Let a pure cyclotomic weight-two system over K be good outside thirteen and quadratic-special at both thirteen-primes. If some dyadic residue is irreducible and solvable, its actual member is ordinary and modular.*

Proof. An irreducible solvable residue is necessarily odd dihedral. Write it as the induction of an odd-order character θ from a quadratic field L/K. Its quadratic quotient is unramified outside two and thirteen. The local finite-flat bound excludes odd dyadic valuation, whose quadratic conductor would be five. Since K has class number one, its possible quadratic squareclasses are in

$$\langle -1, \epsilon, a, b\rangle, \qquad \epsilon = 2 + \sqrt{3}, \quad a = 4 + \sqrt{3}, \quad b = 4 - \sqrt{3}, \quad ab = 13.$$

If the source is nonordinary, Theorem 11.7 makes the inducing dyadic conductor at most two. The seven nontrivial possibilities are the masks $1, 6, 7, 10, 11, 12, 13$ in this displayed basis. Their ordinary ray groups modulo (2) are, respectively,

$$1,\ 1,\ C_8,\ 1,\ C_8,\ C_2,\ C_4 \times C_2.$$

The odd rotation character θ is unramified at thirteen because its inertia image there has order at most two, and at two it is tame. Its conductor divides the radical of (2), hence divides (2). Every displayed ray group is two-primary, so θ is trivial. This contradicts irreducibility. Therefore no solvable irreducible residue occurs in the nonordinary case.

For an ordinary source use the conductor-at-most-four list, which includes all fifteen nontrivial classes above. Their ray groups modulo (2) are two-primary except for mask fifteen,

$$L = K(\sqrt{-13\epsilon}) = K(\sqrt{-26}), \qquad \mathrm{Cl}_{(2)}(L) = C_{12} \times C_2 \times C_2.$$

Here the equality of fields follows from ϵ having squareclass 2 in K. This field is CM and is ramified at the dyadic place: its relative quadratic conductor there is four. In particular it does not split dyadically. Allen's ordinary residually dihedral theorem, including its ordinary residual-witness construction, applies to the actual source. This proves the second assertion.

The program `d3_quads13_all.gp` computes these fifteen ray groups, certifying each class group with `bnfcertify`. The exact calculation is supplied in [49, `ramified/d3`]. □

16.2 The scalar envelope

Lemma 16.2. *Put $K = \mathbf{Q}(\sqrt{3})$, $\epsilon = 2 + \sqrt{3}$, $\delta_1 = \epsilon(4 + \sqrt{3})$ and $\delta_2 = \epsilon(4 - \sqrt{3})$, and let*

$$F_0 = K(i, \sqrt{\delta_1}, \sqrt{\delta_2}).$$

This is a Galois extension of K of degree eight. Its absolute discriminant is $2^{24}3^8 13^8$, and its class group is C_2. Let F_1 be its Hilbert class field. Then F_1/K is Galois of degree sixteen. Write $\Gamma = \mathrm{Gal}(F_1/K)$. Its group has generators a, b, c, t with

$$a^2 = b^2 = c^2 = t^2 = 1, \qquad t \text{ central},$$
$$[a, b] = [a, c] = t, \qquad [b, c] = 1.$$

Here b and c can be inertia generators at the two thirteen-primes, and the corresponding decomposition groups are $\langle b, t\rangle$ and $\langle c, t\rangle$. Thus the group is $D_8 \times C_2$.

Let $\mathfrak{P}_1, \mathfrak{P}_2$ be the dyadic primes of F_0. The ray group for the dyadic modulus

$$\mathfrak{P}_1^6 \mathfrak{P}_2^6$$

is C_4. The class of every prime above thirteen is 3 in a common cyclic generator. Its quotient imposing Frobenius order at most two at all these primes is precisely the Hilbert quotient C_2.

Proof. We have $\delta_1\delta_2 = 13\epsilon^2$, and both δ_j are totally positive. Thus $F_0 = K(i, \sqrt{\delta_1}, \sqrt{13})$. The independent real sign and thirteen-adic valuation classes give $[F_0 : K] = 8$. Its dyadic completions have absolute $(e, f) = (4, 2)$; its eight thirteen-primes have $(e, f) = (2, 1)$. The discriminant and these local degrees give absolute dyadic different exponent 6. Subtracting twice the different exponent 2 of $K_2/\mathbf{Q}_2$ gives relative different exponent 2; the local ramification index is two, so its unique positive upper break is 1. Hence the local inverse Herbrand function has $\psi(3) = 1 + 2(3 - 1) = 5$.

The exact arithmetic computation starts from the displayed radical description of F_0. It verifies degree sixteen and absolute discriminant $2^{24}3^8 13^8$, and certifies its class group C_2 by `bnfcertify`=1. Computing the ray group at the stated dyadic modulus gives C_4, with all eight thirteen-prime classes equal to 3. Adding twice those classes to the relation lattice gives C_2, necessarily the Hilbert quotient.

The program `verify_d3_classfield_source.gp` constructs the Hilbert quadratic extension from the certified class field of F_0. Its absolute degree is 32 and its discriminant is $2^{48}3^{16}13^{16}$. Exact Galois computations, restricting through an embedding of $X^2 - 3$, give an order-sixteen group with derived subgroup $\langle t\rangle$. The thirteen-adic inertia generators b, c commute, and an involution a outside $\langle b, c, t\rangle$ has both commutators equal to t. The decomposition groups are $\langle b, t\rangle$ and $\langle c, t\rangle$. These computations use no class group of F_1. The exact calculation is supplied in [49, `ramified/d3`].

The group relation follows; putting $z = bc$ gives a central involution independent of t, with $\langle a, b\rangle \simeq D_8$ and $\Gamma = \langle a, b\rangle \times \langle z\rangle$. □

Lemma 16.3. *Let Γ be the order-sixteen group of Lemma 16.2. In its binary group algebra the quotient by the two-sided ideal generated by*

$$(1 + t)(1 + b), \qquad (1 + t)(1 + c)$$

has dimension ten over $\mathbf{F}_2$.

Proof. Put $z = bc$. The center is $\langle t, z\rangle \simeq C_2^2$, and $\mathbf{F}_2[\Gamma]$ is free of rank four over its central group algebra, with basis $1, a, b, ab$. Write $D = 1 + t$ and $Z = 1 + z$. Both are central and

square to zero. The ideal in the statement is generated equivalently by $D(1+b)$ and DZ, since $c = bz$. The central ideal generated by D has dimension eight. Quotienting it by DZ first leaves dimension four, with basis D, Da, Db, Dab. The remaining relation and its product with a identify $Db = D$ and $Dab = Da$, leaving dimension two. The quotient modulo D has dimension eight. The required total dimension is therefore $8 + 2 = 10$.

The multiplication-table computation in `verify_d3_group_algebra_independent.py` gives the same quotient directly. The exact calculation is supplied in [49, `ramified`]. □

16.3 The characteristic-five inner-twist endpoint

Lemma 16.4 (Finite-field projective descent)**.** *Let $k = \mathbf{F}_{25}$, let $\phi : x \mapsto x^5$, and let $\bar{r} : G \to \mathrm{GL}_2(k)$ be absolutely irreducible. Suppose $\eta : G \to \{\pm 1\}$ is a quadratic character and*

$$\bar{r}^{\phi} \simeq \bar{r} \otimes \eta.$$

Then the projectivization of $\bar{r}$ is conjugate to a representation with values in $\mathrm{PGL}_2(\mathbf{F}_5)$. If $G = G_F$ for a totally real field, $\bar{r}$ is totally odd and η is totally odd, the determinant squareclass of this descended projective representation is totally odd.

Proof. Choose $A \in \mathrm{GL}_2(k)$ such that $\bar{r}^{\phi} = A\bar{r}A^{-1}\eta$. Applying ϕ twice shows that $A^{\phi}A$ commutes with the linear representation $\bar{r}$. Absolute irreducibility gives $A^{\phi}A = cI$. Applying ϕ and conjugating by A gives $c^{\phi} = c$, so $c \in \mathbf{F}_5^{\times}$. The norm $k^{\times} \to \mathbf{F}_5^{\times}$ is surjective. Rescale A so that $A^{\phi}A = I$. Matrix Hilbert 90 supplies $B \in \mathrm{GL}_2(k)$ with $A = B^{\phi}B^{-1}$. Hence

$$\bar{r}' = B^{-1}\bar{r}B, \qquad (\bar{r}')^{\phi} = \eta\bar{r}'.$$

Every matrix class $[\bar{r}'(g)]$ is therefore fixed by ϕ and belongs to $\mathrm{PGL}_2(\mathbf{F}_5)$. More explicitly, when $\eta(g) = -1$, multiply $\bar{r}'(g)$ by a nonzero $u \in k$ satisfying $u^{\phi} = -u$; the resulting matrix has entries in $\mathbf{F}_5$. This proves descent without any hypothesis on the projective centralizer.

For a real conjugation c_v, put $C = \bar{r}'(c_v)$. Total oddness and $c_v^2 = 1$ give $C^2 = I$ and $\det C = -1$. If $\eta(c_v) = -1$, then $C^{\phi} = -C$. The preceding $\mathbf{F}_5$-representative uC has determinant $-u^2$. Here $u^2 \in \mathbf{F}_5^{\times}$ is a nonsquare: otherwise $u^2 = a^2$ with $a \in \mathbf{F}_5$, forcing $u = \pm a \in \mathbf{F}_5$. Since -1 is a square in $\mathbf{F}_5$, $\det(uC)$ is a nonsquare. This holds at every real place. □

Proposition 16.5 (The quadratic inner-twist source at five)**.** *Put $K = \mathbf{Q}(\sqrt{3})$ and $h = \chi_{-1}$, the character of $K(i)/K$. Let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, parallel-weight-two strongly compatible system over K, with cyclotomic determinant. Assume it is good outside thirteen and has quadratic-special type, with nonzero monodromy, at the two thirteen-primes. Let D be its trace field. Suppose either $D = \mathbf{Q}$, or D is quadratic and its nontrivial automorphism τ satisfies*

$$\mathcal{R}^{\tau} \simeq \mathcal{R} \otimes h. \tag{$*$}$$

Then $\mathcal{R}$ is modular.

Proof. Choose a coefficient place above five and write r for the actual five-adic source and $\bar{r}$ for its semisimple residue. The source is crystalline of labelled weights $\{0, 1\}$ at five. Its residual trace field is contained in $\mathbf{F}_5$ or $\mathbf{F}_{25}$. If $\bar{r}$ is absolutely irreducible, finite-field Brauer descent realizes it over that field. If it is reducible, its two characters are individually defined over that field: coefficient Frobenius cannot interchange them because their values at a real conjugation are the distinct elements $1, -1$.

If $\bar{r}$ is globally reducible, the square of a constituent is unramified away from five: good places give no inertia and each special thirteen-place contributes at most a common quadratic scalar and unipotent inertia. Lemma 11.8 therefore proves actual-source modularity.

Now suppose $\bar{r}$ is absolutely irreducible. If $D = \mathbf{Q}$, or five splits or ramifies in the quadratic field D, its residual field is $\mathbf{F}_5$. The small-field modularity theorem [3, Theorem 2.5] supplies a modular witness. In the remaining case five is inert in D and τ induces the fifth-power automorphism of $\mathbf{F}_{25}$. Reducing $(*)$ gives

$$\bar{r}^{\phi} \simeq \bar{r} \otimes h.$$

Lemma 16.4 descends its projectivization to $\mathrm{PGL}_2(\mathbf{F}_5)$, with totally odd determinant squareclass. Moreover $[K(\zeta_5) : K] = 4$, since the unique quadratic subfield of $\mathbf{Q}(\zeta_5)$ is $\mathbf{Q}(\sqrt{5})$. The projective modularity theorem [3, Theorem 1.2(3)] applies. Its Lemma 2.1(2) returns a modular witness for the original linear representation $\bar{r}$, including its scalar character.

Lemma 11.9 proves the required absolute irreducibility over $K(\zeta_5)$. The actual weight-two automorphy lifting theorem [3, Theorem 2.3] now applies to r (or to its dual to match the cohomological normalization): it is finitely ramified, de Rham of weights $\{0, 1\}$ and has a modular, cyclotomically irreducible residue. This proves modularity of $\mathcal{R}$. □

16.4 All scalar dyadic residues: a ten-dimensional group algebra

Put $K = \mathbf{Q}(\sqrt{3})$ and let h be the character of $K(i)/K$. It is unramified at every finite place, nontrivial unramified at two, and trivial at both thirteen-places. Indeed the absolute discriminants of $K(i)$ and K are 144 and 12, and -1 is a square in $\mathbf{Q}_{13}$. A special condition includes its exact quadratic scalar character.

Proposition 16.6. *Let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, strongly compatible rank-two system over K, with cyclotomic determinant and labelled weights $\{0, 1\}$. Suppose it is good outside thirteen and special at both thirteen-places. If every dyadic coefficient member has scalar residual semisimplification, then the actual system is modular. No dyadic ordinarity assumption is needed.*

Proof. We first give a bound for a finite packet of such systems whose exact quadratic special characters agree at each thirteen-place. Apply Lemmas 11.1 and 11.3 to the whole simultaneous coefficient tuple. Write R_0 for its trace order, E_0 for its actual matrix order, and s for the sum of the degrees of the distinct coefficient orbits. Then

$$\operatorname{rank}_{\mathbf{Z}_2} R_0 = s, \qquad \operatorname{rank}_{\mathbf{Z}_2} E_0 = 4s.$$

Put $E = E_0/2E_0$. Its joint Galois image G is a finite two-group and spans E, since every coefficient residue has scalar semisimplification. Let L/K be its point field. The left regular lattice is actual Barsotti–Tate. The general Fontaine bound on its rational-two torsion gives upper break at most three over K_2 for the whole field L; no coefficient-uniformizer improvement is used.

At a thirteen-place the integral special matrices have the form

$$\xi(\tau) \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \qquad \xi(F) \begin{pmatrix} 13 & w \\ 0 & 1 \end{pmatrix}, \quad \xi(\tau), \xi(F) \in \{\pm 1\}.$$

Normalize by the common rational signs. The inertia difference is in the actual order and squares to zero. The normalized Frobenius has polynomial $X^2 - 14X + 13$ and hence squares to

one modulo two. It commutes with inertia there. Its whole local image is therefore elementary abelian of exponent at most two. Moreover

$$(F-1)(\tau-1)=(\tau-1)(F-1)=0 \quad \text{in } E. \tag{3}$$

The integral normalized products are zero in one order and twelve times the inertia difference in the other. Exact common signs justify these normalizations in the actual joint order.

Put $\epsilon=2+\sqrt{3}$, $\delta_1=\epsilon(4+\sqrt{3})$, $\delta_2=\epsilon(4-\sqrt{3})$, and $F_0=K(i,\sqrt{\delta_1},\sqrt{\delta_2})$. Lemma 16.2 gives $[F_0:K]=8$, dyadic ramified break one, and ray group C_4 for all dyadic primes to exponent six. Its quotient killing twice every thirteen-prime class is C_2. The group $H=\mathrm{Gal}(LF_0/F_0)$ is a finite two-group. The compositum has upper dyadic break at most three over K_2, since F_0/K has break one. Herbrand's formula $\psi_{F_0/K}(3)=1+2(3-1)=5$ bounds the conductors of characters of H^{ab} by six.

At thirteen the local field of L lies in the unique maximal elementary biquadratic extension of $\mathbf{Q}_{13}$. The field F_0 already has ramification index two, so LF_0/F_0 is unramified there and its Frobenii square to one. There is no other finite ramification. Thus H^{ab} is a quotient of the displayed C_2. By the finite-two-group Frattini theorem H itself is cyclic of order at most two. The unique possible quadratic ray extension is the Hilbert class field F_1/F_0, since $\mathrm{Cl}(F_0)=C_2$. Consequently $L\subset F_1$. This does not assume $F_0\subset L$ or that F_0 contains all quadratic characters of G.

Lemma 16.2 identifies $\Gamma=\mathrm{Gal}(F_1/K)$, of order sixteen. Its thirteen-inertias are generated by b,c, and the corresponding decomposition groups are $\langle b,t\rangle,\langle c,t\rangle$ for one central involution t. Its Frobenius classes are t modulo inertia. Equation (3) makes E a quotient of

$$\mathbf{F}_2[\Gamma]/\big((1+t)(1+b),(1+t)(1+c)\big).$$

By Lemma 16.3 this algebra has dimension ten. We have proved the packet bound

$$4s\le 10. \tag{4}$$

It bounds the actual matrix algebra and assumes no faithfulness of the scalar action $R_0/2\longrightarrow E_0/2$.

Let D be the global trace field. Apply the bound also to $\{\mathcal{R},\mathcal{R}\otimes h\}$. The twist preserves every exact thirteen-character and is unramified at two. An actual h-selftwist would give induction from $K(i)$, contrary to nonzero monodromy at the split thirteen-places. Corollary 12.4 therefore gives a rational trace field, or a quadratic trace field D with nontrivial automorphism ι satisfying

$$\mathcal{R}^{\iota}\simeq\mathcal{R}\otimes h. \tag{5}$$

The hypotheses of Proposition 16.5 are now satisfied: either $D=\mathbf{Q}$, or D is quadratic with the displayed inner twist. That proposition proves actual-source modularity in characteristic five and hence modularity of the system. □

Theorem 16.7. *Assume GRH. Every pure, totally odd, absolutely irreducible, cyclotomic-determinant weight-two strongly compatible system over K, good outside thirteen and quadratic-special at both thirteen-primes, is modular. The exact quadratic scalar signs are arbitrary.*

Proof. If a dyadic residue is nonsolvable, Proposition 14.2 applies. If it is irreducible and solvable, Proposition 16.1 applies. Otherwise every dyadic coefficient residue is reducible. Its characters have odd order and are unramified at thirteen because inertia has order at most two. At two wild inertia is pro-two and the residue unit group is $\mathbf{F}_2^{\times}=1$, so those characters are unramified there too. They are unramified everywhere else and have trivial real-place action. Class number one makes them trivial. Thus every residue has scalar semisimplification, and Proposition 16.6 proves modularity. □

16.5 The supported residual base at thirteen

Lemma 16.8. *Every local residual representation of $G_{\mathbf{Q}_{13}}$ with cyclotomic determinant admits a weight-two lift of exact cyclotomic determinant whose type is quadratic special or determinant-one tame cuspidal with inertial order seven or fourteen.*

Proof. Write $W(r,b) = \mathrm{Sym}^r \mathbf{F}_{13}^2 \otimes \det^b$. There are fourteen central-trivial weights: $r + 2b = 0$ modulo twelve, with $r \in \{0, 2, 4, 6, 8, 10, 12\}$ and b modulo twelve. For the cuspidal type indexed by $t = 1, \ldots, 6$, with niveau-two character exponent $12t$, the compact-type reduction has constituents

$$W(2t-2, 1-t) \quad \text{and} \quad W(12-2t, t).$$

These twelve weights are distinct and exhaust those with $r < 12$. The two remaining Steinberg weights occur in the scalar semistable compact types. The corresponding crystalline scalar compact types also occur in the cuspidal menu, so if a semistable component has only crystalline points one may choose a cuspidal lift instead. The full fixed-determinant formula [70, Theorem 1.2], with its nonnegative multiplicities therefore gives the required nonempty local lifting ring for the actual residual extension. The inertial orders are fourteen for odd t and seven for even t. No global quadratic correction of these signs is required. □

Lemma 16.9. *Let $\mathcal{R}$ be a determinant-cyclotomic weight-two strongly compatible system over $K = \mathbf{Q}(\sqrt{3})$, good outside thirteen, with the preceding cuspidal or special menu at thirteen. If a seven-adic residue is globally reducible, the actual seven-adic source is modular. If it is irreducible, its cyclotomic restriction remains absolutely irreducible.*

Proof. Suppose first that the constituents are ψ and $\omega_7\psi^{-1}$. At a thirteen-place both finite inertial eigencharacters reduce to the same quadratic sign; any surviving seventh-order inertia is unipotent. Consequently ψ^2 is unramified at thirteen. It is unramified at every other finite place away from seven as well. Lemma 11.8 with $(K, p) = (F_3, 7)$ proves actual ordinarity and modularity, without removing the scalar signs by a global twist. If the residue is irreducible, Lemma 11.9 gives cyclotomic irreducibility. □

Lemma 16.10. *At either thirteen-place, the actual reduction in characteristic seven of any member of the cuspidal/special menu admits a genuine quadratic-special lift of cyclotomic determinant. This includes the case of trivial residual inertia.*

Proof. Retain the exact quadratic scalar inertial character and first remove it locally. Cuspidal Frobenius has trace zero and determinant thirteen, so its residual eigenvalues are $1, -1$, which are distinct modulo seven. Special Frobenius has the same pair, up to an unramified sign. The relation with inertia, since $13 = -1$ modulo seven, places any nontrivial unipotent inertia in one offdiagonal orientation in this Frobenius eigenbasis. Choose $u = \pm 1$ for that orientation and take

$$T(x) = 1 + txE_{12}, \qquad \Phi = \mathrm{diag}(13u, u),$$

interchanging the basis vectors if needed. Choose $t \neq 0$ with the required residual value; if inertia was trivial choose t divisible by the coefficient uniformizer. These matrices have determinant one on inertia and thirteen on Frobenius and satisfy $\Phi T(x)\Phi^{-1} = T(13x)$. Restoring the quadratic character gives the asserted actual special lift, with nonzero monodromy. □

Proposition 16.11. *Assume GRH. Every irreducible cyclotomic-determinant residual representation unramified outside thirteen is modular.*

Proof. Solvable residual images have modular Artin witnesses. Otherwise the cyclotomic residual restriction in characteristic thirteen is irreducible. Use Lemma 16.8 and prescribed global lifting to construct a weight-two source, then establish its actual avoiding potential modularity and compatible system. In characteristic seven, a reducible residue gives modularity by Lemma 16.9. For an irreducible residue that lemma verifies the cyclotomic hypothesis. Prescribe the genuine special lifts of Lemma 16.10 at both thirteen-primes and retain the old Barsotti–Tate component above seven. Global lift existence in characteristic seven and actual potential modularity give the replacement system. Purity makes its special monodromy nonzero. Theorem 16.7 modularizes this actual target. Allen–Khare–Thorne then applies backwards to the old actual weight-two seven-adic source. Compatibility and reduction at thirteen return the required original residual witness. □

16.6 The bounded source conditions for $d = 6, 11$

For $K = F_6$ or F_{11}, a bounded system is pure, totally odd, absolutely irreducible, of cyclotomic determinant and labelled weights $\{0, 1\}$, good outside five. Each five-adic type is quadratic-special, denoted SP, or tame determinant-one cuspidal, denoted SC, with inertial eigencharacters of order three or six. A special condition retains its exact quadratic character, including the unramified sign, and has nonzero monodromy. Its characters are $\alpha\chi_p, \alpha$ with α quadratic. At a coefficient-five place its monodromy-stable line gives the special filtration by weak admissibility.

17 Trace relations over $\mathbf{Q}(\sqrt{6})$

Put $K = \mathbf{Q}(s)$, $s^2 = 6$.

Theorem 17.1. *For a bounded source with types* SC/SC *or* SP/SP*, a solvable residual image at a coefficient prime above three implies modularity.*

Theorem 17.2. *Assume GRH. A bounded source with types* SP/SP *and nonsolvable residual image at a coefficient prime above two is modular. GRH is unnecessary if that dyadic member is nonordinary.*

17.1 Characters and cubic classes

Write

$$\epsilon = 5 + 2s, \qquad a = 1 + s, \qquad b = 1 - s, \qquad \pi = 3 + s, \qquad \gamma = 3 + 2s, \qquad H = K(\zeta_3).$$

Then $N(\epsilon) = 1$, $ab = -5$, $N(\pi) = 3$ and $N(\gamma) = -15$. Let σ be the nontrivial automorphism of $K/\mathbf{Q}$ and put $\omega = \bar{\chi}_3$. We identify quadratic characters with squareclasses when convenient.

Lemma 17.3. *The class number of K is one. The quadratic classes supported at three and five and unramified at two form the group generated by*

$$[-3] = [-\epsilon], \quad [5], \quad [\gamma].$$

For a reducible $\bar{r}_3$ with determinant ω arising from a bounded source, its constituents are quadratic. Up to interchange and twisting by the character of $K(\sqrt{5})/K$, their pair is either $\{1, \omega\}$ or $\{\psi_\gamma, \psi_\gamma\omega\}$.

Proof. The Minkowski bound is $\sqrt{24}/2 < 2.45$, and the prime of norm two is generated by $2 + s$, whose norm is -2. The unit ϵ is fundamental, by the least positive solution $(5, 2)$ of

$x^2 - 6y^2 = 1$; norm -1 is impossible modulo three. The squareclasses supported at three and five have basis $[-1], [\epsilon], [\pi], [a], [b]$.

All five generators are dyadic units. Such a unit u gives an unramified quadratic extension, possibly split, precisely when u is a square modulo $4\mathcal{O}_{K_2}$. A square of a unit $A + Bs$ has coordinates $(1, 0)$ or $(3, 2)$ modulo four, according as B is even or odd. Enumerating the thirty-two products gives exactly eight allowable classes, generated as stated. The identity $3/\epsilon = (3 - s)^2$ identifies $[-\epsilon] = [-3]$.

Let a residual constituent be ψ. At five, its square is unramified because semisimplified residual inertia of the bounded types is scalar quadratic. At three, a finite characteristic-three character kills principal units and the residue field is $\mathbf{F}_3$, so its square is unramified there too. It is unramified elsewhere. The narrow class group has order two; its nontrivial character is ω, which is odd at both real places. A square character is even, so $\psi^2 = 1$. The preceding eight classes give the two pairs. Finally, $K_3 = \mathbf{Q}_3(\zeta_3)$ because -2 is a square in $\mathbf{Q}_3$. The first pair is locally unramified, while both characters of the second pair are ramified quadratic. □

Let $F = K_3$, normalize $v_F(s) = 1$, and put $U_i = 1 + \mathfrak{p}_F^i$.

Lemma 17.4. *The global S-unit cubeclass space for $S = \{v : v \mid 15\}$ has basis $[\epsilon], [\pi], [a], [b]$ and maps isomorphically to $F^\times/F^{\times 3}$. The preimage of U_2/U_4 is*

$$V = \langle q, d \rangle, \qquad q = [5], \qquad d = [5] + [\epsilon b].$$

We have $\sigma(q) = q$, $\sigma(d) = -d$. In this basis the two valuations at five are $(1, 1)$ and $(1, -1)$, in some order.

Proof. Since $\mu_3 \subset F$, the class space has dimension four. Moreover $U_1 = \mu_3 U_2$, and the logarithm identifies cubing $U_2 \to U_4$ with multiplication by three. Hence a unit is a cube exactly when it is congruent to 1 or -1 modulo $9\mathcal{O}_F$.

The π exponent is detected by valuation. Of the twenty-seven products $\epsilon^i a^j b^k$, $0 \le i, j, k < 3$, only the trivial one is ± 1 modulo nine. This finite calculation takes place in $(\mathbf{Z}/9\mathbf{Z})[s]/(s^2 - 6)$ and proves injectivity, hence the asserted isomorphism. Modulo three, $\epsilon = -(1 + s)$, $a = 1 + s$, $b = 1 - s$ and $s^2 = 0$. Membership in $\{\pm 1\}U_2$ is therefore the equation $i + j - k = 0$. This plane is generated by $[5] = [ab]$ and $w = [\epsilon b]$. Since $N(\epsilon b) = -5$, we have $\sigma(w) = q - w$. The assertions for $d = q + w$ and the valuations follow. □

Lemma 17.5. *The three-part of the ray class group modulo $\mathfrak{p}_3^3$, with or without the real places, is C_3. The two Frobenius classes at five coincide and generate it. The involution σ acts trivially on its three-part.*

Proof. The unit group of $\mathcal{O}_K/\mathfrak{p}_3^3$ has order eighteen. The images of -1 and ϵ generate a subgroup of order six, giving C_3. Here $\mathfrak{p}_3^3 = (9, 3s)$. The residues of a and b belong to neither the image of the units nor its trivial class: the unit residues are $\pm 1, \pm(5 + 2s), \pm(4 + 2s)$ in this quotient. Thus both ray classes are nontrivial. The identity $a - \epsilon^2 b = 72 + 30s \in (9, 3s)$ makes their classes equal. The involution σ exchanges the two primes and fixes their common generating class, hence acts trivially on the three-part. Infinite places add only two-primary data. □

Lemma 17.6. *No absolutely irreducible characteristic-three residue of a bounded source becomes reducible over H.*

Proof. The field $H = \mathbf{Q}(\sqrt{-2}, \sqrt{-3})$ has discriminant 576 and class number one. Indeed its Minkowski bound is $36/\pi^2 < 4$. There is no prime of norm two. If $A^2 = -2$ and $u^2 - u + 1 = 0$,

the element $1 + Au$ has relative norm $-1 + A$ over $\mathbf{Q}(A)$, hence absolute norm three. Its conjugate gives the other prime of norm three.

A cyclotomic-reducible irreducible residue would be $\mathrm{Ind}_H^K \xi$ with $\xi^c = \xi^{-1}$. Its image has order prime to three. At five, its inertia is scalar quadratic, so ξ^2 is unramified. The two primes of H over three have residue field $\mathbf{F}_3$, so ξ^2 is unramified there as well. It is unramified elsewhere, hence trivial by the class number. Then $\xi = \xi^c$, contradicting irreducibility of the induction. □

17.2 Ordinary sources and an unramified quotient

We prove the ordinary part of Theorem 17.1 without a local-distinctness assumption at three. Such an assumption would fail here because $\omega|_{G_F} = 1$. The argument gives a stronger result in which the two five-adic types need not agree.

Proposition 17.7. *Suppose that the residual constituents are unramified at three. Then the actual system is modular. After the quadratic normalization of Lemma 17.3, it is invariant under σ. Consequently a genuinely mixed* SC/SP *source cannot have this residual branch, and two special conditions in this branch have the same exact character after normalization.*

Proof. By Lemma 17.3, a twist by ψ_5 makes the residual pseudorepresentation $1 + \omega$. This quadratic character is invariant under σ and unramified at three.

Lemma 4.10 makes the crystalline source at F ordinary. Write its unramified ordinary quotient as ν; the subcharacter is $\chi_3\nu^{-1}$.

Put $A = \mathbf{Z}_3[[U]]$ and let ψ_U be the unramified character of G_F sending arithmetic Frobenius to $1 + U$. Begin with the universal finite-flat-at-three fixed-determinant Cayley–Hamilton pair for $1 + \omega$, unramified outside three and five, provided by [72, Theorems 2.5.3 and 2.5.5], and extend its coefficients to A. Impose, for every $g, h \in G_F$, the algebra relations

$$(g - \chi_3(g)\psi_U(g)^{-1})(h - \psi_U(h)) = 0. \tag{6}$$

No local type is imposed at five. To construct the Cayley–Hamilton quotient, take the closed two-sided ideal I generated by these relations, then the scalar ideal generated by the nonconstant coefficients of $D(1 - xt)$, $x \in I$, and use [72, Lemma 2.4.5]. Thus no coefficient-A version of the printed representability theorem is assumed. Denote the scalar ring by R, its maximal ideal by $\mathfrak{m}$, and its GMA by $\left(\begin{smallmatrix} R & B \\ C & R \end{smallmatrix}\right)$. The actual source factors through it with $U = \nu(\Phi_3) - 1$, since upper triangular multiplication proves (6).

The offdiagonal cotangent duals embed into the finite-flat global extension spaces by [72, Theorem 4.3.5]; scalar extension and further quotients retain these inclusions at $3 = U = 0$. Locally such an extension is unipotent. Fontaine's bound gives classical upper break at most two, hence cubic conductor at most three. A nonunit Kummer class instead has conductor four: after replacing the radicand by an inverse and a cube, $X^3 - u$ is Eisenstein with $v_F(u) = 1$, and its different exponent is eight. Thus both global extension spaces lie in

$$V_3 = \langle [\epsilon], [a], [b] \rangle.$$

Their restriction to G_F is injective by Lemma 17.4. For larger coefficient fields the same conclusion follows by extracting $\mathbf{F}_3$ coordinate extensions by pullback and quotient.

Let J be the reducibility ideal and put

$$Q = \mathfrak{m}/(3R + UR + \mathfrak{m}^2), \qquad Q_{\mathrm{red}} = \mathfrak{m}/(3R + UR + \mathfrak{m}^2 + J).$$

For $g, h \in G_F$ both residual diagonal characters equal one. The upper-left entry of (6), reduced in Q, is $b_g c_h = 0$; its other term is a product of elements of $\mathfrak{m}$. Evaluations on all of G_F span

the dual of each allowed global extension space, by local injectivity. Hence every product in $(B/\mathfrak{m}B) \otimes (C/\mathfrak{m}C)$ has zero image in Q. These products generate the image of J, so $Q \simeq Q_{\mathrm{red}}$.

A diagonal tangent relative to A has characters $1 + \eta f$ and $\omega(1 - \eta f)$, with $\eta^2 = 0$. Finite-flatness and Fontaine give conductor at most $\mathfrak{p}_3^3$ for f. It cannot ramify at five: wild inertia is pro-five and the tame unit quotient has order four. It is unramified elsewhere by construction. Therefore the dual of Q_{red} is a subspace of $\mathrm{Hom}(\mathrm{Cl}_{\mathfrak{p}_3^3}(K), \mathbf{F}_3)$. The involution σ acts trivially on this space by Lemma 17.5. It consequently acts trivially on Q; the diagonal tangent need not itself vanish.

The involution σ preserves the support, finite-flat condition, determinant and ordinary relations. It fixes U, because the unramified Frobenius quotient of G_F is unchanged by the automorphism of $F/\mathbf{Q}_3$. Thus it acts on R over A. Its action on Q is the identity. Lemma 12.1, applied over A, makes the involution σ trivial on R. Thus $r_3^{\sigma} \simeq r_3$.

Lemma 12.3, with $\eta = 1$, now proves modularity of the original system.

Finally, σ exchanges the two places above five, and invariance identifies their local Weil–Deligne representations. Their monodromy cannot be zero at one place and nonzero at the other. For two special types it also preserves the exact quadratic character. This proves the asserted exclusions. □

17.3 Nonordinary reducible sources

The other pair in Lemma 17.3 requires a different finite-flat extension space.

Lemma 17.8. *For either ramified quadratic character θ of G_F, the finite-flat selfextensions of θ form exactly the Kummer plane U_2/U_4.*

Proof. The rank-one model has Breuil–Kisin exponent one. An extension has Frobenius matrix

$$\begin{pmatrix} cu & h(u) \\ 0 & cu \end{pmatrix}.$$

The height-one condition imposes no further restriction. A change of upper lift changes h by $cu(f(u^3) - f(u))$. Positive powers are removed recursively, while constants in f disappear. The extension space has dimension at most two, represented by $1, u$. Schematic closure and flat quotient give the rank-one endpoints; the exact finite-flat description used here is [44, Theorem 1.1.3, Proposition 1.1.11 and Lemmas 1.2.4–1.2.5]. Coordinate pullback and quotient give the same bound for arbitrary coefficients.

The unramified selfextension is finite flat and gives U_3/U_4. A generator is $1+6s$: substituting $\alpha = 1 + sz$ in $\alpha^3 = 1 + 6s$ gives

$$z^3 + (s/2)z^2 + (1/2)z - 1 = 0,$$

whose reduction $z^3 - z - 1$ is irreducible and separable over $\mathbf{F}_3$.

For an independent class take

$$E : y^2 = x^3 - \frac{s}{8}x^2 + x.$$

Its discriminant is $-125/2$, and its reduction is supersingular. Its division polynomial factors as

$$\psi_3(x) = (x - 1/s)(3x^3 + 6x + s).$$

At $x = 1/s$, the ordinate square is $s(5/12)^2$, so the stable subgroup has character θ_s. Cardano's radicand for the other three lines is $2/(3s)$, whose product with s^3 is four. Hence its actual

extension line is [2]. It lies in $U_2 \setminus U_3$ after multiplying by the cube -1. An unramified quadratic twist gives the other ramified character; specifically $\gamma/(2s) = 1 + s/4$ is a local square. These two lines exhaust U_2/U_4. □

Proposition 17.9. *A pure bounded source with common ramified quadratic residual constituents at three is modular.*

Proof. Normalize its residual pair to $\psi_\gamma + \psi_\gamma\omega$. By Lemmas 17.4 and 17.8, the two global offdiagonal extension spaces are contained in $V = \langle q, d\rangle$, with five-adic evaluations $\ell_1 = (1, 1)$ and $\ell_2 = (1, -1)$. Use the finite-flat Cayley–Hamilton construction of Proposition 17.7, now over $\mathbf{Z}_3$ without an ordinary parameter, and impose the exact five-adic conditions by the same closed-ideal construction. After removing a fixed scalar quadratic inertia character, a cuspidal condition kills wild and prime-to-three tame inertia and imposes $T^2 + T + 1 = 0$, $\Phi^2 + 5 = 0$. A special condition of exact character α_v imposes $(g - \chi_3(g)\alpha_v(g))(h - \alpha_v(h)) = 0$ for all local g, h. The diagonal cotangent vanishes by Fontaine and Lemma 17.5. Thus products of the two offdiagonal cotangents surject onto $\mathfrak{m}/(3R + \mathfrak{m}^2)$.

Since $N(\gamma) = -15$, the involution σ followed by twisting by ψ_5 interchanges the residual constituents. For SC/SC it restores the ordered order-three and order-six types. Its action on the product cotangent is factor interchange composed with $\sigma(q) = q$, $\sigma(d) = -d$. At either five-place, after removing its scalar quadratic character, the exact relation $(T - 1)^2 = -3T$ kills $\ell_v \otimes \ell_v$ in the relative cotangent. Lemma 12.2, with $s = q$, therefore makes the combined involution trivial on the cotangent.

For two special conditions, choose the place with $s \equiv 1 \pmod 5$ first. There ψ_γ has character RU, while at the other place it is trivial; here R is the character of $\mathbf{Q}_5(\sqrt{5})$ and U the unramified quadratic. Write the exact special characters as RU^e, U^f. The allowed first extension orientation is ramified at the first place exactly when $e = 0$, and at the second exactly when $f = 1$; the other orientation is complementary. If $e \neq f$, one orientation is forbidden at both places. Independence of ℓ_1, ℓ_2 then makes that entire global extension space zero. Nakayama makes the source reducible, a contradiction. Hence $e = f$. The product of the exact characters is R, and the involution σ followed by ψ_5 preserves the conditions. It interchanges the two permitted extension lines and fixes their tensor product. Its action on the whole cotangent is again trivial.

Lemma 12.1 makes the combined involution trivial on the complete ring. We obtain $r_3^\sigma \simeq r_3 \otimes \psi_5$. Since ψ_5 is restricted from $G_{\mathbf{Q}}$, Lemma 12.3 proves modularity. □

Proof of Theorem 17.1. If the residue is reducible, Propositions 17.7 and 17.9 apply. If it is irreducible, Lemma 17.6 gives irreducibility on G_H. The Fong–Swan and solvable Artin argument supplies a regular cuspidal residual witness; Lemma 2.1 gives one in the weight-zero convention. Then [3, Theorem 2.3] applies to the actual weight-two three-adic source. Strong compatibility gives modularity of the system. □

17.3.1 The mixed reducible case

Proposition 17.10. *A mixed bounded source cannot have a globally reducible three-adic residue.*

Proof. The unramified constituent pair is excluded by Proposition 17.7, since the involution σ exchanges the two monodromy operators. For the other pair normalize the special place to $\mathfrak{q}_- = (1 - s)$ and the cuspidal place to $\mathfrak{q}_+ = (1 + s)$, with residual pair $\psi_\gamma + \psi_\gamma\omega$. The global finite-flat plane is $V = \langle q, d\rangle$, with evaluation functionals $\ell_+ = (1, 1), \ell_- = (1, -1)$.

We need the relation $q^\vee \otimes q^\vee = 0$ in the relative pseudocotangent. We prove it in its finite quotient generalized matrix algebra, retaining the coefficient-valued offdiagonal product. Here are the exactness and coefficient arguments for this relation. Every finite-flat group killed

by three whose generic factors are the same ramified quadratic θ has a unique model with its generic identification. The rank-one Kisin exponent is one, and its unramified constant is determined by θ. Closing a generic graph and a common generic θ-line, then inducting on the finite-flat quotient, proves both full faithfulness and exactness of generic subquotients. This argument is confined to this filtered category; it does not assert full faithfulness on all groups at ramification index $p-1$.

Let R be the common finite-flat pseudoring, with generalized matrix algebra $(R\ B; C\ R)$. Put $R' = R/(3R+\mathfrak{m}^2)$, $B' = B/\mathfrak{m}B$, $C' = C/\mathfrak{m}C$, and let E' be the resulting generalized matrix algebra. The products of B', C' lie in the diagonal $\mathfrak{m}_{R'}$, and its radical I satisfies $I^3 = 0$. The regular left Galois module E' is finite flat by the stable condition of [72, Definition 2.5.2, Theorem 2.5.5]. Every finite left E'-module, being a quotient of a sum of regular modules, has the same finite-flat property and local factors θ.

For such a module V let $\mathfrak{M}(V)$ be the covariant Kisin module, obtained by applying the contravariant classification to the Cartier dual of its finite-flat model. The graph argument above extends every short exact sequence of these generic modules to a short exact sequence of finite-flat groups. Kim's exact classification [42, Corollary 4.3 and Proposition 5.7.3(2)] therefore gives an exact sequence of the underlying Kisin modules. These are finite free $\mathbf{F}_3[[u]]$-modules, so the sequence splits as modules and remains exact modulo u. The presentation argument for an exact linear functor now gives $\mathfrak{M}(V)/u = P_0 \otimes_{E'} V$ for a projective right E'-module P_0. Each of the two simple modules has a rank-one model, so each indecomposable projective occurs in P_0 once. Thus $P_0 \simeq E'$. Lifting a right generator and using Nakayama and equality of ranks gives $\mathfrak{M}(E') \simeq E'[[u]]$ as right modules.

Normalize the simple Frobenius maps to the same cu and write the regular Frobenius as $cu + X(u)$, with $X(u) \in I[[u]]$. Its height-one inverse is

$$u^2(cu + X(u))^{-1} = u/c - X(u)/c^2 + X(u)^2/(c^3 u).$$

Integrality forces $X(0)^2 = 0$. The diagonal radical is square zero and annihilates the offdiagonal modules, so this is exactly the vanishing of the product of the two constant offdiagonal coordinates in R'. On a rank-two extension those coordinates are $h(0)$ in Lemma 17.8. The constant functional vanishes on the unramified line d, represented by $5\epsilon(1-s) \equiv 1+3s \pmod 9$, and is nonzero on q. This identifies the asserted relation with $q^\vee \otimes q^\vee$ in the global product cotangent.

In the mixed problem the special relation kills one extension orientation at $\mathfrak{q}_-$, while the cuspidal relation kills $\ell_+ \otimes \ell_+$. Together with the displayed product relation these kill the product cotangent. The diagonal quotient has the one C_3 ray direction of Lemma 17.5, killed by the exact cuspidal Frobenius trace zero. Thus the relative pseudocotangent is zero, and every characteristic-zero point factors through $\mathbf{Z}_3$.

Compare such a point with the CM curve

$$E : y^2 = x^3 + \gamma^3(7+3s)^2/4.$$

It is the γ-twist of $y^2 + (7+3s)y = x^3$, and has exactly the same residual pair. Its minimal conductor is $\mathfrak{q}_+^2$: it is good at two, three and $\mathfrak{q}_-$, and has type II* at $\mathfrak{q}_+$. Its division points with $x = 0$ give the character ψ_γ; its trace at $\mathfrak{q}_-$ is zero, also forced by its CM and the inertness of five in $\mathbf{Q}(\sqrt{-3})$. These assertions are checked by the exact Weierstrass and minimal conductor calculation supplied with the proof.

The unknown source has rational three-adic traces by the vanishing cotangent. Its trace algebra is split: nonzero special monodromy provides a nonzero nilpotent, impossible in a division algebra. Realize it over $\mathbf{Q}_3$. For it and E choose the idempotents of a Frobenius at $\mathfrak{q}_-$; its two eigenvalues are distinct modulo three. Their generalized matrix orders have reducibility

ideal (3). Indeed the normalized cuspidal inertia has diagonal entries $1+x, 1+y$ with $x, y \in 3\mathbf{Z}_3$, trace -1, determinant one, and offdiagonal product $xy-3$, of valuation one. Choose an Eichler lattice whose primitive upper residual extension is the orientation forbidden at the special place. For both sources that class spans $\ker \ell_-$. Scale it and fix the two idempotents so the two residual upper cochains agree exactly, say β. Both $\beta(q^\vee)$ and $\beta(\ell_+)$ are nonzero.

Let f, g be the two pseudopoints modulo nine of the common finite-flat problem with the cuspidal condition at $\mathfrak{q}_+$ and no imposed condition at $\mathfrak{q}_-$. Their difference $\delta = (f-g)/3$ is a derivation on the relative pseudocotangent. Writing c_f, c_g for their divided lower residual functionals gives $\delta(BC) = \beta(B)(c_f - c_g)(C)$. The finite-flat $q^\vee$ product relation and the equality of the two cuspidal inertia traces give

$$(c_f - c_g)(q^\vee) = (c_f - c_g)(\ell_+) = 0.$$

These functionals are independent on V, so the derivation kills every offdiagonal product. Its diagonal quotient is zero as above. The two trace functions are equal modulo nine. At $\mathfrak{q}_-$ this asserts $\pm 6 = 0$ modulo nine, a contradiction. □

17.4 Nonsolvable characteristic-two sources

We give the complete ramification and explicit-formula argument for Theorem 17.2. The quadratic extension H/K is unramified at all finite places, unramified quadratic at two and at both places over five.

Lemma 17.11. *Let a finite-flat representation $G_{K_2} \to \mathrm{SL}_2(\mathbf{F}_q)$ have point field L. Its normalized dyadic different contribution is at most $2 - 2/q$.*

Proof. Fontaine's classical upper-break bound is $2(1 + 1/(2-1)) - 1 = 3$. If wild inertia is nontrivial, its unique invariant line is preserved by the decomposition group. The diagonal characters are characters of G_{K_2}; their odd-order restrictions to units factor through $\mathbf{F}_2^\times$ and are trivial. Thus inertia is elementary abelian of order at most q. The upper-filtration formula, divided by the absolute ramification index two of the base, gives

$$d_2 \le \tfrac{1}{2}\{1 - 1/q + 3(1 - 1/q)\} = 2 - 2/q.$$

If wild inertia is trivial, a nontrivial irreducible tame inertia has order three, since Frobenius squares and exchanges its determinant-one eigencharacters. Its contribution is $1/3$, which is smaller. □

At a special five-place the full residual decomposition image is contained in C_2^2. Adjoin $H = K(\zeta_3)$, which is unramified at every finite place and unramified quadratic at two and both five-places. For M/K with nonsolvable image $\mathrm{SL}_2(\mathbf{F}_q)$, simplicity gives $M \cap H = K$, and

$$[MH : \mathbf{Q}] = 4q(q^2-1), \quad \mathrm{rd}(MH) \le \sqrt{24}\, 2^{2-2/q}\sqrt{5}, \quad 2N_{25}/[MH : \mathbf{Q}] \ge \tfrac{1}{2}.$$

If the actual dyadic member is nonordinary, Theorem 11.7 improves the upper bound to $\mathrm{rd}(MH) < 2\sqrt{120} < 21.910$. For $q \ge 8$, (40) with $b = 10$, $n \ge 2016$ and just the first norm-25 term gives $\mathrm{rd}(MH) > 23.29$, a contradiction. A totally real kernel field of degree at least 120 has $\mathrm{rd}(M) > 41.69$ by the same formula with $b = 7$, contradicting the same upper bound. These two comparisons require no GRH.

For an arbitrary dyadic member, assume GRH and apply the explicit formula of Section 22, with $B_{b,100}$, retaining four nonnegative prime-power terms at norm 25. For the totally real

comparison retain $A_{b,4}$ and no prime terms. Outward interval evaluation at 256 bits gives

case	$n \geq$	b	lower rd	upper rd
$q = 8$	2016	7.36	41.44651216	36.84623482
$q \geq 16$	16320	9.4	49.15326236	43.81780461
totally real kernel	120	4.8	56.06306031	43.81780461

Each lower entry is strict and each upper entry is a strict outward upper bound. At fixed b, the right side of the explicit formula increases with n, since $C_b > 0$. The common dyadic factor four bounds the unbounded ranges. These comparisons exclude every $q \geq 8$. The exact calculation is supplied in [49, `ramified/d6_d10_d11/shared`].

For $q = 4$, [3, Theorem 2.5] supplies a regular cuspidal residual witness. If all real conjugations acted trivially, the kernel field would be totally real of degree at least 120, contradicting the preceding unconditional comparison for a nonordinary member, or the last table row under GRH. Thus some residual real conjugation is nontrivial. All hypotheses of [69, Theorem 6.1] now hold for the actual crystalline two-adic source, including nonsolvability and this real-place condition. Its modularity proves Theorem 17.2.

17.5 Two special conditions at two

Proposition 17.12. *For a source of types* SP/SP*, the semisimplified residue $\bar{r}_2$ is either nonsolvable or $1 \oplus 1$. In particular an absolutely irreducible $\bar{r}_2$ is nonsolvable and, under GRH, the source is modular.*

Proof. An absolutely irreducible solvable subgroup of $\mathrm{GL}_2(\overline{\mathbf{F}}_2)$ with determinant one is odd dihedral, by Dickson's classification. Indeed a normal unipotent two-subgroup would give a nonzero invariant subspace fixed by the whole group, while the irreducible subfield groups $\mathrm{SL}_2(\mathbf{F}_{2^r})$, $r \geq 2$, are nonsolvable. Thus the residue is $\mathrm{Ind}_{G_L}^{G_K} \xi$ for a quadratic L/K and a nontrivial odd-order character ξ.

Special inertia at five reduces to the identity or a group of order two: its scalar quadratic factor becomes trivial in characteristic two. Its intersection with the odd rotation subgroup is trivial. Consequently ξ is unramified at all primes over five, even when L/K is ramified there. It is unramified elsewhere away from two. At two it is tame, so its conductor divides the radical of (2) in L, hence divides (2).

Fontaine's upper-break bound three gives dyadic conductor at most four for L/K. An odd-valuation quadratic Kummer class has conductor $2e + 1 = 5$, since $e(K_2/\mathbf{Q}_2) = 2$. Therefore the possible nontrivial radicands are precisely the fifteen classes in $\langle -1, \epsilon, a, b \rangle$. In mask order for this displayed basis their ordinary ray groups modulo (2) are

$$C_4, 1, C_2, C_4, C_4, C_4, C_4, C_4, C_4, C_4, C_4, C_8 \times C_4, C_2, C_4 \times C_2, C_2.$$

These are exact degree-four ray computations; certification of each class group returns one. All groups are two-primary, so none admits the odd-order character ξ, a contradiction. The exact calculation is supplied in [49, `ramified`].

If the residue is reducible, both constituent characters have odd order. They are unramified at five by the special calculation, and at two since the residue units are $\mathbf{F}_2^\times$ and wild inertia is pro-two. They are unramified at every other finite place and at the real places. The class number one statement makes both trivial. The final modularity assertion follows from Theorem 17.2. □

17.5.1 Ordinary scalar sources

Proposition 17.13. *A bounded source with two special places and scalar residual semisimplification at two is modular if some two-adic member is ordinary.*

Proof. Let r be that member. Put $H_0 = \chi_{-3}|_{G_K}$ and $\psi_5 = \chi_5|_{G_K}$. The exact two special characters θ_1, θ_2 determine a unique $\eta \in \{1, H_0, \psi_5, H_0\psi_5\}$ whose restriction at both five-places is $\theta_1\theta_2$. Both r and $r^\sigma \otimes \eta$ have the same exact special characters, ordinary crystalline condition, determinant and support.

Choose a vector v on the ordinary line and the lattice generated by its global orbit. The vector is primitive and cyclic. Hence the actual residue is a nonzero unipotent extension, and the reduction of its ordinary line is transverse to its unique global invariant line. Its restriction at two is trivial. The kernel of localization of the supported quadratic classes at two is generated by $[-15]$: in the unit basis $-1, \epsilon, 1+s, 1-s$ this follows by taking squares modulo $8\mathcal{O}_{K_2}$. The squares have coordinates $(1,0), (1,4), (7,2), (7,6)$, and only the identity and $[-15]$ occur. Thus the actual residue is $1 + \chi_{-15}E_{12}$. For a real involution c, the basis $((r(c)-1)v, v)$ is integral and gives

$$r(c) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{ordinary mark } e_2.$$

It puts both aligned representations in the same marked problem.

Kisin's marked ordinary functor, Proposition 2.4.4 and Corollary 2.4.5 of [43], is represented at arbitrary ramification, including two. Pull it back to the global matrix functor with the displayed real matrix and the ordinary section e_2. At five impose the exact character line equations. The line there is unique since a residual unipotent element has unit upper entry, so these too are represented conditions.

We compute the homogeneous tangent in this gauge. Write its cocycle as $X(g) = \left(\begin{smallmatrix} x(g) & y(g) \\ z(g) & x(g) \end{smallmatrix}\right)$ relative to $1 + \chi_{-15}(g)E_{12}$. The adjoint action adds $\chi_{-15}(g)z(g)(1+E_{12})$, so z is additive. The ordinary finite-flat condition gives $y|_{G_{K_2}} = 0$, $x|_{G_{K_2}}$ unramified, and $z|_{G_{K_2}}$ a unit Kummer class. At a special place allow the line to move to $e_1 + t_v e_2$. The exact characters give $z|_{G_v} = 0$ and $x|_{G_v} = t_v\chi_{-15}|_{G_v}$. The supported squareclass table then gives $z \in k\chi_{-1}$. The real involution relation forces $z(c) = 0$, hence $z = 0$. Now x, y are additive, and the dyadic conditions give $y = B\chi_{-15}$ and $x = A\chi_{-15} + D\psi_5$. At five $\chi_{-15} = R + U$ and $\psi_5 = R$, where R, U are the independent ramified and unramified quadratic characters; thus $D = 0$. The fixed real matrix forces $A = B = 0$. The represented ring has zero relative tangent and is a quotient of $W(k)$. Its actual characteristic-zero point makes it $W(k)$, so the aligned pair is identical.

We have $r^\sigma = r \otimes \eta$, with η restricted from $G_{\mathbf{Q}}$. Lemma 12.3 proves modularity. □

17.6 The scalar two-adic source with two special places

Let $K = \mathbf{Q}(\sqrt{6})$. A special place in this section has nonzero monodromy and its exact scalar twist is quadratic, including its unramified part. Let a_v denote the good-place trace of a pure parallel weight-two system, with arithmetic Frobenius normalization.

Theorem 17.14. *Let $\mathcal{R}$ be a totally odd, absolutely irreducible, pure, strongly compatible rank-two system over K, with compatibility including monodromy. Suppose it has cyclotomic determinant, labelled weights $\{0,1\}$, is good outside five and is special at both primes above five. Suppose every member at a coefficient place above two is nonordinary and has scalar residual semisimplification. Then the global trace field has degree at most three. The actual system is modular.*

17.6.1 The integral trace order and its regular group

Let D be the global trace field and $d = [D : \mathbf{Q}]$. Apply Lemma 11.1 to the tuple at all dyadic coefficient places, and use its notation $R_0, E_0, T, \mathcal{D}$. Then R_0 has $\mathbf{Z}_2$-rank d, while $E_0/2E_0$ is local with residue field $\mathbf{F}_2$.

17.6.2 The two special generators and the dyadic trace

Put $E = E_0/2E_0,\ R = R_0/2R_0$, and let J be the radical of E. A functional on J/J^2 gives a middle/middle extension by quotienting the regular module. In the calculation of Theorem 11.7, its Kisin matrix is $\left(\begin{smallmatrix} u & h \\ 0 & u \end{smallmatrix}\right)$, and its Artin–Schreier class has the unique form $n/u + m$, with $n, m \in \mathbf{F}_2$. The first coordinate is ramified of break one when nonzero; the second is the unramified character.

At five these quadratic characters are tame. They belong to the quadratic dual of the narrow ray group modulo the square of the dyadic prime and both five-primes. Its invariants are $4, 2, 2$. Three independent characters, in squareclass notation, are

$$\alpha = [-1], \qquad \beta = [\epsilon] = [2], \qquad \gamma = [-15], \qquad \epsilon = 5 + 2\sqrt{6}.$$

Their relative discriminant norms are $4, 4, 25$; independence also follows from the independent quadratic generators $i, \sqrt{2}, \sqrt{5}$ over K. Thus they span every allowed first-layer character. At a fixed five-place, -1 is square, 2 is an unramified nonsquare and -15 gives a ramified character. Choose a Frobenius lift trivial on γ, and a tame inertia generator. Write A, B, C for the differences from one of a real involution, this Frobenius and this inertia. Their evaluations on (α, β, γ) are respectively $(1, 0, 1), (0, 1, 0), (0, 0, 1)$. Since the group spans E, the first-layer functional is determined by the resulting character; these three elements therefore span J/J^2. Nilpotence of J gives algebra generation of E.

At any one five-place normalize by its fixed rational quadratic special sign. In a basis of the special line, for local g the normalized difference $L(g)$ has top row $(\chi_2(g) - 1, *)$ and bottom row zero. Thus $L(g)L(h) = (\chi_2(g) - 1)L(h)$ in the original joint order. The signs disappear modulo two and all local differences have zero products and trace zero there. This uses equality of the exact quadratic characters across coefficient conjugates. For the real element $(1 + A)^2 = 1,\ T(A) = 0$ modulo two. Polarized Cayley–Hamilton now gives in E

$$\begin{gathered} A^2 = B^2 = C^2 = BC = CB = 0, \\ AB + BA = X = T(AB), \qquad AC + CA = Y = T(AC). \end{gathered} \tag{7}$$

In particular all words reduce to the R-span of $1, A, B, C, AB, AC$.

Lemma 17.15. *The ring R is $\mathbf{F}_2[t]/(t^d)$, where t is the image of a_2. When $d \geq 2$, X is another generator of its maximal ideal and Y/X can be chosen to be a unit. In characteristic zero $D = \mathbf{Q}(a_2)$.*

Proof. First work in the relative cotangent quotient $R' = R/\mathfrak{m}_R^2$, retaining the full algebra over it and its trace and determinant laws. The normal form above shows that the relative cotangent is generated by X, Y: all traces reduce to their R'-span and the trace order is generated by the actual traces. All traces are in $\mathfrak{m}_{R'}$.

The ramified functional n in the preceding middle-extension calculation on (α, β, γ) is $(1, 1, 0)$. Indeed α is dyadically ramified, γ splits there, and $\alpha + \beta + \gamma = [5]$ is the unramified quadratic character. It follows in J/J^2 that the constant term of Φ is $A + B + C$. In the u-coefficient of Φ the corresponding unramified functional takes value one on $[5]$. These statements are unaffected by a first-order basis change, whose correction is $u(\varphi - 1)z$ with zero constant and linear coefficients. Write

$$\begin{gathered} \Phi = N + u(1 + M) + O(u^2), \qquad N = A + B + C + Z, \\ M = m_A A + m_B B + m_C C + Z', \qquad Z, Z' \in J^2. \end{gathered}$$

Evaluation of $[5]$ on A, B, C gives $0, 1, 1$, hence $m_B + m_C = 1$. More explicitly, take the functional on the full allowed character space obtained by restricting to the dyadic tower and

reading the constant Artin–Schreier coefficient. The local ramification break bound ensures that only the terms of degrees $-1, 0$ can occur. Its restriction to the actual middle extensions gives the coefficients of M, so one may use this lift of the functional even when the actual first layer is smaller.

We give the determinant calculation in R' itself. By the six-word normal form, $J^2 \subset \mathfrak{m}_{R'}E + R'AB + R'AC$. A term from $\mathfrak{m}_{R'}E$ contributes neither a norm nor a polar term, since $\mathfrak{m}_{R'}^2 = 0$ and all traces are in $\mathfrak{m}_{R'}$. The norms of AB, AC are zero by multiplicativity, and their mutual polar term is $XY - T(ABAC) = 0$. Their trace pairings with $A+B+C$ are zero by (7). Thus $\mathcal{D}(N) = \mathcal{D}(A + B + C) = X + Y$. The constant term of the determinant in Lemma 11.4 gives $X = Y$.

The same normal form gives $T(J^3) = 0$: each word of length three reduces to a multiple of a trace-zero generator, and longer products contribute either such terms or quadratic products of X, Y. The coefficient of u in the norm is $T(N)T(1 + M) - T(N(1 + M))$. It vanishes by the determinant divisibility, and the product of the two traces vanishes in R'. Hence

$$T(N) = T(NM) = (m_B + m_C)X = X \qquad \text{in } R'.$$

Lemma 11.4 identifies $T(N)$ with a_2. Nakayama now makes R a quotient of $\mathbf{F}_2[t]$, with radical generated by t. Its dimension is the rank d of R_0, giving the asserted relation $t^d = 0$. The same calculation gives relative generators X and Y whenever $d \geq 2$, with identical initial coefficient. Finally $1, a_2, \ldots, a_2^{d-1}$ span R_0 over $\mathbf{Z}_2$ by Nakayama. The rational degree of $a_2 \in D$ is therefore at least d, proving $D = \mathbf{Q}(a_2)$. □

17.6.3 An abelian restriction of the regular image

Suppose $d \geq 4$ and use X as the parameter of R. Choose $v \in R^\times$ with $Y = vX$, and put $Z = C + vB$. By (7), Z is central, $Z^2 = BZ = XZ = 0$, and $T(Z) = T(AZ) = 0$. Here $XC = (AB + BA)C = BAC = B(Y + CA) = YB$. Consequently E is spanned over R by $1, A, B, AB, Z, AZ$.

Let $S = \mathbf{F}_2[X]/(X^3)$ and $Q = S\langle a, b\rangle/(a^2, b^2, ba+ab-X)$, as in the proof of Proposition 15.6. The same trace-pairing argument gives a surjection $E \to Q$, taking A, B, Z to $a, b, 0$. Indeed, pairing a relation in $1, A, B, AB, Z, AZ$ with $1, A, B, AB$ puts its first four coefficients in (X^{d-1}), which vanishes in S. The free normal basis $1, a, b, ab$ then makes the map multiplicative. This does not require faithfulness of the scalar action on E.

Compose with $a \mapsto E_{12},\ b \mapsto XE_{21}$ to obtain $\theta : G_K \to \mathrm{GL}_2(S)$. On formal coefficients its matrix is

$$\begin{pmatrix} f_0 + Xf_3 & f_1 \\ Xf_2 & f_0 \end{pmatrix}.$$

Its trace is the reduction of T. Comparing the two Cayley–Hamilton identities for a group element gives $\det \theta(g) = 1$. The two linear characters $f_1(0), f_2(0)$ are visible on actual regular quotients: send a, respectively b, to the generator of $\mathbf{F}_2[z]/(z^2)$, and send the other generator and X to zero. Their values on A, B, C are $(1, 0, 0), (0, 1, 1)$. The calculated first-layer space gives $f_1(0) = \alpha,\ f_2(0) = [5]$.

Put $F = K(i, \sqrt{5})$. Each matrix over G_F is $I + XM + X^2N$; the lower left entry of M is zero by $f_2(0) = 0$, and its diagonal entries agree by the determinant. All these M's lie in the commutative algebra generated by I, E_{12}. Multiplication modulo X^3 thus makes $\theta(G_F)$ abelian of exponent at most four. It has a specified nonzero double. Let r be the product of the chosen real and special Frobenius elements. Its image has trace X, so

$$\theta(r)^2 = I + X\theta(r), \qquad \theta(r)^4 = I + X^2 I.$$

The element r^2 lies in G_F, since F/K is elementary abelian, and therefore gives an element of order four there.

17.6.4 The four Frobenius classes above five

Fix a five-prime w of F. Choose a Frobenius h over F_w and a tame inertia element k in the underlying K-decomposition group with $[5](k) = 1$ (additive notation). The special identities give

$$T(h-1) = T(k-1) = 0, \qquad (h-1)(k-1) = (k-1)^2 = 0 \quad \text{in } E,$$

where the trace equalities are in R. In particular the formal ab-coefficient of each difference vanishes in Q, by the original trace pairing $Xf_3 = 0$ in R. Write $h - 1 = P + Ua + Vb$, $k - 1 = P' + U'a + V'b$ in Q. The b- and ab-coefficients of their zero product give

$$PV' + VP' = UV' + VU' = 0.$$

Now V' is a unit and $V \in XS$, by the values of [5]. Thus $h - 1 = (V/V')(k - 1)$ in the *abstract* algebra Q. The coefficient comparison here requires its free normal basis. The matrix of k is I modulo X, since -1 is square at five. If $\theta(k) - I = XM_0 + O(X^2)$, its square gives $M_0^2 = 0$. We have proved at all four primes that

$$\theta(h) = I + X^2 N_w, \qquad N_w^2 = 0. \tag{8}$$

There is one dyadic prime $\mathfrak{P}$ of F, with absolute $(e, f) = (4, 2)$. Over K, the quadratic $K(i)$ has dyadic conductor two: its absolute discriminant is the product of $24, -4, -24$, and division by 24^2 gives relative discriminant norm four. The factor $K(\sqrt{5})$ is dyadically unramified. Thus the ramified break of F/K is one. By the finite-flat bound and Herbrand's subgroup formula the upper break over F is at most $1 + 2(3 - 1) = 5$. At five the elementary local image over K has exponent two and its possible quadratic tame inertia is killed by $K(\sqrt{5})$. Hence the abelian representation over F is unramified at five and all its five-Frobenius elements have order dividing two. The field is totally imaginary.

Write H for the ray class group of modulus $\mathfrak{P}^6$ of F. Class field theory makes $\theta|_{G_F}$ a quotient of

$$H/(4H + \langle 2[w] : w \mid 5\rangle).$$

The following coordinates are from the ordinary ray group, with the ordered primes as in the certificate. They concern abelian extensions of the degree-eight field F.

group or prime	invariants or class
H	$(8, 4, 2, 2)$
$H/(4H + \langle 2[w]\rangle)$	$(4, 2, 2, 2)$
w_1	$(0, 1, 0, 0)$
w_2	$(0, 1, 0, 1)$
w_3	$(2, 1, 0, 1)$
w_4	$(2, 1, 0, 0)$

For reproducibility one defining polynomial for F is

$$y^8 - 4y^7 + 2y^6 + 8y^5 + 13y^4 - 44y^3 + 164y^2 - 140y + 145.$$

The certificate computes a certified class group and units, obtains the ray cyclic generators and the ideal logarithms L of the four primes, and takes the Smith form of the relation lattice generated by $\mathrm{diag}(8, 4, 2, 2)$, $4I$, $2L$. Thus $[w_1] - [w_4]$ is the unique nonzero double of the quotient. It maps to $I + X^2 I$, by the square of $\theta(r^2)$. By (8) this gives $N_{w_1} + N_{w_4} = I$ for square-zero matrices, impossible since $(I + N_{w_1})^2 = I$. This proves $d \le 3$. The exact calculation is supplied in [49, `ramified/d6_d10_d11/d6`].

17.6.5 Passage to an odd coefficient prime

The trace-field bound and Lemma 11.10 give a coefficient prime above three with residue field $\mathbf{F}_3$ or $\mathbf{F}_9$, or above five with residue field $\mathbf{F}_5$.

Choose the indicated coefficient place. For an absolutely irreducible residue, the traces in its finite residue field give a realization over that field by finite-field descent (the Brauer group of a finite field is zero). Allen–Khare–Thorne, *Modularity of* $\mathrm{PGL}_2(\mathbf{F}_p)$*-representations over totally real fields*, Theorems 1.1 and 2.3, give residual automorphy and lift the actual weight-two member when its restriction to the cyclotomic subgroup is absolutely irreducible. In the nonsolvable case this restriction is automatic: reducibility on a normal cyclic quotient would force an induced dihedral representation, with solvable image. At three all solvable cases, including globally reducible residues, are covered by the pure special characteristic-three source theorem. At five a solvable irreducible residue either remains irreducible on the cyclotomic subgroup or is induced from $K(\sqrt{5})$. Such induction at a five-place is incompatible with the special ordinary line: a tame generator outside the inducing subgroup has scalar square and projective order two, while the ratio of the ordinary inertial characters has order four. This gives cyclotomic irreducibility in this case as well.

When the five-adic residual semisimplification is reducible, the dual source has determinant χ_5^{-1} and at both five-places has an invariant line of finite character with quotient of character that finite character times χ_5^{-1}. Thus it is ordinary of weight $(0,0)$ in the convention of [26, §1.6]. Corollary 1.2.2 there applies since K is totally real abelian and five is unramified in K. This proves modularity without a calculation of the constituent ratio. Strong compatibility transfers modularity to the entire system, proving Theorem 17.14.

17.7 Local lifts and a switch in characteristic three

Lemma 17.16 (Bounded five-adic lifts)**.** *Every continuous representation* $\bar{r} : G_{\mathbf{Q}_5} \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ *of determinant* $\bar{\chi}_5$ *has a lift of determinant* χ_5*, labelled weights* $\{0,1\}$ *and type* SC *or* SP*.*

Proof. The weights of $\mathrm{GL}_2(\mathbf{F}_5)$ with trivial scalar central character are

$$1, \quad \overset{2}{\det}, \quad \mathrm{Sym}^2 \det, \quad \mathrm{Sym}^2 \overset{3}{\det}, \quad \mathrm{Sym}^4, \quad \mathrm{Sym}^4 \overset{2}{\det}.$$

The two cuspidal compact types reduce with constituents

$$\overline{\sigma}(\tau_6)^{\mathrm{ss}} = 1 \oplus \mathrm{Sym}^2 \det, \qquad \overline{\sigma}(\tau_3)^{\mathrm{ss}} = \overset{2}{\det} \oplus \mathrm{Sym}^2 \overset{3}{\det}.$$

Here $\tau_3 = \widetilde{\omega}_2^8 \oplus \widetilde{\omega}_2^{16}$ and $\tau_6 = \widetilde{\omega}_2^4 \oplus \widetilde{\omega}_2^{20}$, for a niveau-two character ω_2 of order 24. The identities follow by evaluating the cuspidal Brauer characters on split and nonsplit regular elements. These finite character identities determine the semisimplified reductions. The full fixed-determinant Breuil–Mézard formula [70, Theorem 1.2] has nonnegative multiplicities; some multiplicity is positive because a potentially Barsotti–Tate lift exists. The determinant confines it to the six displayed weights. The first four therefore give the cuspidal alternatives.

The last two give the semistable compact type for a scalar quadratic inertial type. If the resulting point has nonzero monodromy, it is SP. If all such points were crystalline, the corresponding crystalline compact type, 1 or $\det^2$, would have positive multiplicity, and the cuspidal alternative would again exist. This is a statement about the actual residual extension, with any finite coefficient field. The determinant is fixed in the deformation rings; equivalently an unramified determinant error reducing to one is corrected by its square root. □

Lemma 17.17 (The actual-residual switch)**.** *Let* $\bar{s}$ *be the reduction of a determinant-*χ_3 *representation of* $G_{\mathbf{Q}_5}$ *of type* SC *or* SP*, with any stable lattice. After finite coefficient extension,*

$\bar{s}$ has lifts of both types, with determinant exactly χ_3. The scalar quadratic inertia character determines whether the cuspidal lift has type τ_3 or τ_6.

Proof. Apply Lemma 4.9 with $\ell = 5$. Its hypotheses include the actual residual lattice, both nonsplit orientations and the split case. Restoring the scalar quadratic character gives τ_3 or τ_6 and preserves the fixed determinant. □

17.8 The supported residual base

Proposition 17.18. *Assume GRH. Every bounded system over $K = \mathbf{Q}(\sqrt{6})$ is modular. Every absolutely irreducible determinant-ω_5 residual representation unramified outside five is modular.*

Proof. Consider first a bounded system of two special types. The quartic-ray calculation excludes an irreducible solvable two-residue. A nonsolvable two-residue gives modularity by the bounded special source theorem under GRH. In the remaining case its semisimplification is $1 \oplus 1$. If some two-adic member is ordinary, the marked ordinary rigidity Proposition 17.13, followed by projective descent, gives modularity. Otherwise Theorem 17.14 applies to all dyadic coefficient places together and proves modularity. Thus every two-special bounded system is modular.

Now consider any bounded system at three. A reducible residue is handled by Theorem 17.1 for a pure menu and is excluded by Proposition 17.10 for a mixed menu. An irreducible solvable residue has a regular cuspidal Artin witness; Lemma 17.6 and [3, Theorem 2.3] then modularize the actual weight-two source.

In the nonsolvable case the residue stays absolutely irreducible over $K(\zeta_3)$. At both five-places use the genuine special lift of Lemma 17.17, which retains the entire actual residue and its quadratic scalar character. Keep the old crystalline weight-two component at the three-place. Emerton–Gee–Pan–Zhu (EGPZ), Theorem 4.2.1, gives a global point on the selected components. Lemma 2.4 gives a pure compatible system. On the special components the Frobenius eigenvalue ratio is five; zero monodromy at a pure point would contradict purity. Thus both local monodromies are nonzero and the new system is modular by the previous paragraph. AKT, Theorem 2.3, applies backwards to the old weight-two three-adic member. This proves modularity of every bounded system.

Let $\bar{r}$ be absolutely irreducible of determinant $\bar{\chi}_5$, unramified outside five. Solvable image has a regular cuspidal Artin witness after weight raising. Otherwise cyclotomic restriction is absolutely irreducible. The EGPZ exception cannot occur: $[K(\zeta_5) : K] = 4$, whereas the abelianization of $\mathrm{PGL}_2(\mathbf{F}_5) \simeq S_5$ has order two. The fixed determinant bounded local lift lemma at each completion $K_w = \mathbf{Q}_5$ supplies a weight-two lift of one of the indicated types for the entire local representation. EGPZ produces the global lift, with unramified conditions away from five. Lemma 2.4 and purity yield a bounded system. Its modularity, followed by reduction, proves the claimed residual base at five.

□

18 Dihedral sources over $\mathbf{Q}(\sqrt{10})$

Put $K = \mathbf{Q}(s)$, $s^2 = 10$.

Theorem 18.1. *Assume GRH. Every continuous irreducible representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ of determinant $\bar{\chi}_3$ unramified outside three is modular.*

The two places v_+, v_- above three are distinguished by $s \equiv 1, -1$, respectively. Both completions are $\mathbf{Q}_3$, and the unique dyadic completion $F = K_2$ has ramification index two

and residue field $\mathbf{F}_2$. In proving Theorem 18.1, we prescribe wild binary dihedral inertia at v_+ and either a special or a quartic supercuspidal type at v_-. A discriminant bound reduces the nonsolvable characteristic-two case to an A_5 lifting theorem. The soluble case requires the modularity of an actual lift, since the relevant CM inducing field splits at the dyadic prime. We prove the special case by bounding the global trace field across all of its dyadic embeddings. Theorem 13.1 then follows by a conductor-removal argument with protected residual images.

18.1 The local conditions at three

Lemma 18.2. *Let $\overline{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ have determinant $\overline{\chi}_3$. It has a lift of weights $\{0,1\}$ and determinant χ_3 which is either a quadratic twist of a semistable representation with nonzero monodromy, or potentially crystalline of tame supercuspidal type*

$$\tau_4 = \widetilde{\omega}_2^2 \oplus \widetilde{\omega}_2^6,$$

where ω_2 is a fundamental character of niveau two. For either choice in a determinant-cyclotomic compatible system, the characteristic-two residual local kernel field is contained in

$$E' = \mathbf{Q}_3(\sqrt{3}, \sqrt{-1}), \qquad e(E'/\mathbf{Q}_3) = 2, \quad f(E'/\mathbf{Q}_3) = 2, \quad d(E'/\mathbf{Q}_3) = \tfrac{1}{2}. \tag{9}$$

Proof. Apply Lemma 23.5; its two local choices have the same determinant, weights and characteristic-two kernel envelope. □

At v_+ use the affine-generic construction of Lemma 23.3, with its notation $K_0, J, \vartheta, \sigma_+$ and τ_+. The sign determining the characteristic-two quadratic subfield is the same as in that lemma.

Proposition 18.3. *The representation $\sigma_+ = \mathrm{Ind}_J^{K_0} \vartheta$ is the eight-dimensional type of a supercuspidal inertial type τ_+ with determinant one. Every $\overline{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with determinant $\overline{\chi}_3$ has a potentially crystalline lift of weights $\{0,1\}$, determinant χ_3, and inertial type τ_+.*

Its inertia remains irreducible after reduction in every characteristic greater than three. For a determinant-cyclotomic compatible system of this type, its residual characteristic-two kernel field at this place is the unique totally ramified S_3 extension $E_+/\mathbf{Q}_3$ with quadratic subfield $\mathbf{Q}_3(\sqrt{3})$ and normalized different $7/6$.

Proof. The construction is that of Lemma 23.3, and Proposition 23.4 gives its lifts with the specified determinant and weights. Lemma 23.3 also gives the irreducibility of inertia in characteristic greater than three and the unique characteristic-two kernel field E_+, with its quadratic subfield and normalized different. □

Proposition 18.4. *Let $\overline{\rho}$ satisfy the hypotheses of Theorem 18.1, and suppose its image is nonsolvable. It lifts to a totally odd, potentially modular, strongly compatible system $\mathcal{R} = (r_\lambda)$ with determinant χ and weights $\{0,1\}$. The system is unramified away from three, has type τ_+ at v_+, and has one of the two local conditions of Lemma 18.2 at v_-.*

Proof. Proposition 18.3 and Lemma 18.2 supply the local lifts at v_+ and v_-, respectively. At the second place prescribe the tame supercuspidal type or a component through the special lift supplied by the lemma; no global quadratic character is required. In the latter case, the closure of the special component fixes Weil Frobenius eigenvalue ratio three, so the purity observation in Section 2 retains nonzero monodromy. Nonsolvability implies that the restriction to $G_{K(\zeta_3)}$ is irreducible, since an irreducible representation becoming reducible over a quadratic extension has dihedral projective image.

Apply Lemma 2.4 to these nonempty weight-two components, with determinant χ_3 and unramified conditions away from three. It supplies an actual potentially modular lift and its

strongly compatible system, including local compatibility at the coefficient places. In particular its two-adic member is crystalline of weights $\{0, 1\}$ at two.

□

Choose a place above two in the coefficient field and write r_2 for this member and $\overline{r}_2$ for its reduction. Proposition 18.3 makes $\overline{r}_2$ irreducible, with determinant one. Let M be its kernel field. Its completion at v_+ is E_+, while Lemma 18.2 puts its completion at v_- inside E'.

18.2 The controlled characteristic-two image

Lemma 18.5. *The splitting field A/K of $X^3 - 3X + 1 + s$ has Galois group S_3. It is unramified away from three, and its dyadic completion is the unramified cubic extension of K_2. Its completions at v_+ and v_- are E_+ and $\mathbb{Q}_3(\sqrt{3})$, respectively.*

Proof. The cubic reduces to $X^3 + X + 1$ at the unique dyadic prime. This polynomial is irreducible and separable over $\mathbb{F}_2$, proving both global irreducibility and the dyadic assertion. Its discriminant is

$$-27(7 + 2s) = -6\left(3 + \frac{3}{2}s\right)^2, \qquad N_{K/\mathbb{Q}}(-27(7 + 2s)) = 3^8.$$

It follows that $\mathrm{Gal}(A/K) = S_3$, with quadratic subfield $K(\sqrt{-6})$, and that no finite prime away from three ramifies.

At v_+ put $d = s - 1$. Since $d(s + 1) = 9$, we have $v_3(d) = 2$. Writing $X = 1 + y$ gives $y^3 + 3y^2 + d = 0$, whose roots have valuation $2/3$. The element $z = y^2/3$ satisfies

$$z^3 - 3z^2 - \frac{2d}{3}z - \frac{d^2}{27} = 0.$$

This polynomial is Eisenstein. In the resulting cubic extension the three derivative terms have valuations $5, 4, 3$, respectively, so its different exponent is three. The lower ramification groups of the Galois closure consequently have orders $6, 3, 1$, giving normalized different $(5+2)/6 = 7/6$. Its quadratic subfield is $\mathbb{Q}_3(\sqrt{-6}) = \mathbb{Q}_3(\sqrt{3})$, so the closure is E_+.

At v_-, the constant term $1 + s$ has valuation two. The Newton polygon has slopes -1 and $-1/2$, of lengths one and two. There is one linear factor and one ramified quadratic factor. Its splitting field is the quadratic field defined by the discriminant, namely $\mathbb{Q}_3(\sqrt{3})$. □

Proposition 18.6. *There is a solvable Galois extension N/K of degree 768, unramified outside two and three, whose dyadic completions have no upper break greater than three. Its completion at v_+ is E_+, and its completion at v_- is contained in E'.*

Proof. We use an abelian ray extension of the field A in Lemma 18.5. An absolute equation for A is

$$\begin{aligned} P_A(X) = X^{12} - 2X^{11} + 5X^{10} + 19X^8 + 2X^7 + 31X^6 \\ + 2X^5 + 19X^4 + 5X^2 - 2X + 1. \end{aligned}$$

It is obtained by taking the compositum of $X^6 - 6X^4 + 2X^3 + 9X^2 - 6X - 9$ with $X^2 + 6$; if u is the root of the degree-six polynomial, the embedding of K is given explicitly by $s = -u^3 + 3u - 1$. The class number is four. There are two primes $\mathfrak{P}_1, \mathfrak{P}_2$ above two, each of absolute ramification index two and residue degree three. Above three there is one prime of absolute ramification index six and residue degree one, lying over v_+, and three primes of ramification index two and

residue degree one, lying over v_-. The field has discriminant 241864704000000 and signature $(0, 6)$.

Let R be the ray class group of modulus $\mathfrak{P}_1^4\mathfrak{P}_2^4$. Quotient R by the class of the prime over v_+ and by twice the class of each prime over v_-. Exact ideal and unit arithmetic gives

$$R \simeq C_4^2 \times C_2^5, \qquad R/\langle [\mathfrak{q}_+], 2[\mathfrak{q}_-] \rangle \simeq C_4 \times C_2^5.$$

Here the bracket in the denominator includes all three primes over v_-. These are calculations in degree twelve. The accompanying PARI/GP certificate computes maximal orders and ray class groups, checks the quotient order 128 by Hermite and Smith normal forms, and returns `bnfcertify` $= 1$ for the class-group calculation. The exact conductor of the quotient has exponent four at both dyadic primes. The exact calculation is supplied in [49, `ramified/d6_d10_d11/d10`].

Let N/A be the associated ray extension. The modulus and the relations are preserved by $\mathrm{Gal}(A/K)$, so N/K is Galois. Its group is an extension of an abelian group by S_3, hence is solvable, and $[N : K] = 128 \cdot 6 = 768$. At v_+ the ray relation makes N/A split, so its completion remains E_+. At each prime over v_- it is unramified of degree at most two, giving a subfield of E'. At two, conductor exponent at most four gives upper break at most three over A. Since A/K is unramified there, the same upper bound holds over K_2. □

Theorem 18.7. *Assume GRH for Dedekind zeta functions. There is no finite Galois extension M/K, unramified outside two and three, satisfying all the following conditions:*

1. $\mathrm{Gal}(M/K) \simeq \mathrm{SL}_2(\mathbb{F}_q)$ *for* $q = 2^r \geq 8$;
2. *its dyadic completions have no upper break greater than three;*
3. *its completion at v_+ is E_+ and its completion at v_- is contained in E'.*

Proof. The group $\mathrm{SL}_2(\mathbb{F}_q)$ is nonabelian simple in characteristic two. Thus $M \cap N = K$ for the solvable Galois extension in Proposition 18.6. For $L = MN$ we obtain

$$[L : \mathbb{Q}] = 2 \cdot 768\, q(q^2 - 1) \geq 774144.$$

Upper numbering is compatible with quotients, so the compositum still has no dyadic upper break greater than three. Since K_2 has absolute ramification index two, its normalized relative different is strictly less than two. The completions at three remain in the stated extensions. The global discriminant formula therefore gives

$$\mathrm{rd}(L) < 4\sqrt{40}\, 3^{5/6} = 63.1962098567\ldots.$$

Apply the GRH explicit formula of Section 22 with the retained coefficients $c_3 = 1/6$ and $c_9 = 1/4$. The v_+ primes have norm three. If the other completion has residue degree two, its norm-nine primes give the second coefficient. If it has residue degree one, retaining only the even powers of its norm-three Euler terms gives at least the same contribution.

At $b = 13.3$ and degree 774144, interval arithmetic gives $\mathrm{rd}(L) > 64.5683408714$. The lower expression increases with the degree. This contradicts the preceding upper bound and proves the theorem. The exact calculation is supplied in [49, `ramified/d6_d10_d11/shared`]. □

Corollary 18.8. *Assume GRH. Let $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ be a continuous representation of determinant χ_2, unramified outside finitely many primes and Barsotti–Tate at two. Suppose that its residual representation is irreducible and nonsolvable, is unramified outside two and three, and has the local extensions specified in Theorem 18.7. Then r is modular.*

Proof. Dickson's classification and Theorem 18.7 leave only residual image $\mathrm{SL}_2(\mathbb{F}_4) \simeq A_5$. The characteristic-two A_5 modularity theorem [3, Theorem 2.5] supplies residual modularity. If its kernel field M were totally real, it would have degree 120 and root discriminant less than 63.19621. The same explicit formula, now retaining the real-place term and only the selected norm-three primes, gives $\mathrm{rd}(M) > 66.10839$ at $b = 5$. Thus some real complex conjugation has nontrivial residual image. All the hypotheses of [69, Theorem 6.1] are satisfied, so r is modular. □

Lemma 18.9. *Every finite characteristic-three character ψ of G_K, unramified outside three, has square equal to 1 or to the quadratic character ν of $K(\sqrt{2})/K$. In particular $\bar{\chi}_3\psi^{-2}$ is totally odd and extends to $G_{\mathbb{Q}}$.*

Proof. The Minkowski bound is $\sqrt{40}/2 = \sqrt{10} < 3.17$. The dyadic prime $\mathfrak{p}_2$ is nonprincipal: an element of norm ± 2 would give $x^2 \equiv \pm 2 \pmod 5$. Its square is (2). The element $2 + s$ has norm -6, so each prime above three represents the same ideal class as $\mathfrak{p}_2$; the two triadic classes are inverse and have order two. Thus $h_K = 2$. The unit $3 + s$ has norm -1, so the narrow class number is also two.

The principal units at three are pro-three, and each residue field is $\mathbb{F}_3$. Consequently ψ^2 is unramified at every finite place. The narrow class-group calculation makes it 1 or ν. The biquadratic field $K(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ has discriminant $8 \cdot 5 \cdot 40 = 40^2$; hence it is the quadratic Hilbert class field of K. Its character is the restriction of the quadratic character of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, which is totally even. Multiplication by $\bar{\chi}_3$ proves the last assertion. □

18.3 The residual field at two

Let $K = \mathbf{Q}(s)$, $s^2 = 10$, and label the places v_+ and v_- above three by $s \equiv 1$ and $s \equiv -1$, respectively. Let the two-adic member of the chosen weight-two system have the binary dihedral type at v_+ and the special or quartic type at v_-. Its determinant is cyclotomic and it is crystalline of labelled weights $\{0, 1\}$ at the dyadic place. Throughout this section the residual point field is the field cut out by the actual representation on a stable lattice. The local group at v_+ has standard image S_3 in characteristic two. In particular the residual representation is absolutely irreducible.

18.3.1 The possible inducing quadratics

For a nonsolvable residual image, Proposition 18.6 and Theorem 18.7 leave only $\mathrm{SL}_2(\mathbf{F}_4)$; Corollary 18.8 proves modularity of the actual source in that case. These statements apply to either choice of the mild type, including special monodromy.

In the solvable case Dickson's classification gives induction from an odd-order character θ over a quadratic field H/K. The character defining H is unramified outside two and three, has conductor exponent at most four at two by the finite-flat bound, and cuts out $\mathbf{Q}_3(\sqrt{3})$ at v_+. The squareclasses unramified away from this support have basis

$$-1, \qquad \epsilon = 3 + s, \qquad 2, \qquad 3, \qquad 2 + s.$$

Lemma 18.9 gives the class group of order two, generated by the dyadic class, and identifies both triadic classes with it. The fundamental unit is $\epsilon = 3 + s$. The valuations of $2, 3, 2 + s$ generate the admissible index-two lattice on these three primes, proving the displayed squareclass basis.

For a representative a of each of the 31 nonzero classes, test $3a$ for squareness at v_+ and evaluate the dyadic valuation of the relative discriminant of $X^2 - a$. The complete list at

conductor at most four is

a	3	3ϵ	-6	-6ϵ
dyadic conductor	2	4	0	4.

Only $L = K(\sqrt{-6})$ is CM. If the source is ordinary, Allen's residually dihedral theorem [2, Lemma 5.1.2 and Theorem 5.2.1] applies for each of the three non-CM choices, with its ordinary witness construction and without a dyadic nonsplitting hypothesis for these choices. If the source is nonordinary, its residual dyadic point field has upper breaks at most one, by Theorem 11.7. Both conductor-four quadratics are therefore impossible.

18.3.2 The odd character and the cubic

It remains to consider $H = K(\sqrt{d})$ for $d = 3, -6$. Every dyadic completion of either field has residue field $\mathbf{F}_2$. Its local units are pro-two, and hence θ is unramified there. At v_+ the relative quadratic different exponent is one and the residue degree is one. The conductor formula for induction gives $3 = 1 + a_H(\theta)$, so its conductor exponent is two. At the prime above v_- the full local image is a two-group for either mild type; thus θ kills the entire decomposition group there. At real places an odd-order character is trivial.

Write P_+, P_- for the two primes of H above v_+, v_-. The table records the ordinary ray group for P_+^2, the class of P_- in the indicated generator, and the quotient by $[P_-]$ and $1 + \sigma$, where σ generates $\mathrm{Gal}(H/K)$:

d	(e, f) above 2	$\mathrm{Cl}_{P_+^2}(H)$	$[P_-]$	quotient
3	$(4, 1)$	C_2	1	0
-6	$(2, 1), (2, 1)$	C_{12}	9	C_3.

Here $1 + \sigma$ imposes inversion on the rotation character, since the determinant is trivial modulo two. These computations can be performed in the quartic model $z = s + \sqrt{d}$ with polynomial $z^4 - 2(10 + d)z^2 + (10 - d)^2$. The expression $s = (z + (10 - d)/z)/2$ identifies the two triadic primes; $z \mapsto 2s - z$ identifies σ. Compute the ray group and its automorphism by `bnrinit` and `bnrgaloismatrix`, adjoin the column `bnrisprincipal`(P_-) and the columns of $1 + \sigma$ to the diagonal relation matrix, and take its integer Smith form. The class groups of H are C_2 and C_4, respectively, and `bnfcertify` returns one in both fields. For L either triadic prime has class 1 in the chosen generator of $\mathrm{Cl}(L) = C_4$. The exact calculation is supplied in [49, `ramified/d6_d10_d11/d10`].

There is consequently no such residual representation induced from $K(\sqrt{3})$. The two nontrivial characters of the cubic quotient over L cut out a single field A. Indeed the splitting field of $X^3 - 3X + 1 + s$ has the requisite local data by Lemma 18.5, so it realizes this cubic quotient. In particular, once a conjugate with one of the modular images has been disposed of, every remaining two-adic residual representation has this same kernel and is the standard representation over $\mathbf{F}_2$.

18.3.3 The two normal ray modules

Proposition 18.10. *Let V_n be the binary quotient of the ray group of A of dyadic depth n in which the positive triadic prime splits. For the standard S_3-module W_0 we have $\mathrm{Hom}_{S_3}(V_2, W_0) = 0$. At depth four this space has dimension two, and projection onto either nontrivial C_3-eigencharacter after scalar extension to $\mathbf{F}_4$ is injective on local units at either dyadic prime.*

Proof. Write W_0 for the two-dimensional standard module over $\mathbf{F}_2$ for $\mathrm{Gal}(A/K) = S_3$. Let $\mathfrak{P}_1, \mathfrak{P}_2$ lie above two and $\mathfrak{q}_+$ above v_+. Set

$$D_n = \mathrm{Cl}_{\mathfrak{P}_1^n \mathfrak{P}_2^n}(A), \qquad V_n = D_n/(2D_n + \langle [\mathfrak{q}_+] \rangle).$$

There is no infinite modulus. The normal first-error module in the basis of the wild order-three element is W_0 after extension of residue scalars. A class unramified at the mild prime, split at $\mathfrak{q}_+$, and of dyadic conductor at most n therefore belongs to $\mathrm{Hom}_{S_3}(V_n, W_0)$ after extending the coefficients. To see the identification on the global cohomology class, restrict to G_A. The kernel and cokernel terms in inflation–restriction vanish since W_0 restricts to the regular module for a Sylow two-subgroup of S_3 and is projective. The restricted class is an equivariant additive character.

Use the maximal order and certified class group of A from Proposition 18.6. Computing the binary ray quotient and its nontrivial C_3 part gives

n	D_n	$\dim V_n$	$\dim(V_n)_{C_3\neq 1}$
1	C_4	0	0
2	$C_4 \times C_2$	1	0
3	C_4^2	1	0
4	$C_4^2 \times C_2^5$	6	4.

In particular $\mathrm{Hom}_{S_3}(V_2, W_0) = 0$.

The exact matrices and local unit columns in Appendix A give a two-dimensional space of equivariant maps. For $\alpha^2 + \alpha + 1 = 0$, the determinants of the two local evaluation matrices for either C_3-eigencharacter are $1 + \alpha$ and α, up to Galois conjugation. Both are nonzero. Vanishing of either eigenprojection on local units at either dyadic prime therefore kills the global depth-four class. These statements concern the normal error in characteristic two; the tame inertia in the actual joint trace order is checked below. □

Lemma 18.11. *Let R be a complete local order in a finite product of finite extensions of $\mathbf{Q}_2$, finite free over the integers of an unramified extension containing ζ_3. Let $r : G_K \to \mathrm{GL}_2(R)$ be an integral realization of a tuple of the controlled weight-two systems of Proposition 18.4. Suppose its residue is the standard representation with kernel field A. At the dyadic place, suppose either all constituents are nonordinary, or all are ordinary with the same residual quotient character. Let J be a proper closed ideal of R, and assume that the image of inertia above v_- in G_A is scalar modulo J. Then the image of r modulo J is monomial, with component character $\eta = \chi_{-6}$, in the fixed eigenbasis of the order-three element at v_+.*

Proof. The order-three eigenprojectors are integral. In their basis the normal space to the diagonal torus is the standard S_3-module W_0, extended to the residue field; it has no invariants under the fixed order-three element u, and $H^i(S_3, W_0) = 0$ for $i > 0$. Write $\mathfrak{m}$ for the maximal ideal of R/J. Assume monomiality modulo $\mathfrak{m}^n$ and work modulo $\mathfrak{m}^{n+1}$, with square-zero ideal $I = \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Erasing the entries forbidden by $\eta(g)$ in each matrix gives a monomial representation, since products of two erased entries vanish. Its first error is a cocycle κ with values in $W_0 \otimes I$, and $\kappa(u) = 0$.

At v_+ the actual source normalizes the u-eigenspaces, so the error vanishes on the full decomposition group. At v_- the hypothesis on inertia makes $\kappa|_{G_A}$ unramified. Thus, once its dyadic conductor is bounded, inflation–restriction places the class in the normal ray module of Proposition 18.10.

Here the conductor bounds apply to actual finite-flat subquotients. At the dyadic place L/K is split, so the local component character is trivial. Put $S = (R/J)/\mathfrak{m}^{n+1}$. The module S^2 is a finite 2-primary quotient of the stable $\mathbf{Z}_2$-lattice underlying R^2; schematic closure and quotient in its Barsotti–Tate group realize it as a finite-flat group. The submodule $Se_2 + Ie_1$ and its submodule $\mathfrak{m}Se_2$ are stable. Their quotient, followed by a residue-linear functional on I, extracts one offdiagonal extension; interchanging the indices extracts the other. Over the unramified cubic completion of A, its prime-field coordinates are quadratic extensions. In a

nonordinary tuple they have conductor at most two by the middle-model calculation in the proof of Theorem 11.7. The depth-two vanishing in Proposition 18.10 therefore kills the global class.

For an ordinary tuple, the two dyadic residual Frobenius eigenvalues are distinct. The common ordinary line is consequently the same Frobenius projector in every constituent, and extends over R. Straightening it at first order makes one normal orientation a local coboundary, which vanishes on G_A. Fontaine's bound puts both quadratic coordinates at conductor at most four. The injectivity of the corresponding local eigenprojection in Proposition 18.10 again kills the global class. The unramified local class is zero because Frobenius minus one is invertible on the nontrivial unramified cubic character.

Thus κ is a coboundary. Its value at u forces the matrix defining that coboundary into $W_0^u \otimes I = 0$, so the error itself vanishes in the fixed basis. Induction and completeness prove the claim. □

18.4 The quartic mild type

Proposition 18.12. *Suppose that the controlled two-adic source has the residual field A and the tame quartic supercuspidal type at v_-. Then it is induced from a geometric character of G_L, where $L = K(\sqrt{-6})$, and is modular.*

Proof. Choose a coefficient ring containing ζ_3 and the integral eigenbasis of the actual order-three wild inertia at v_+. At v_- the quartic tame generator has scalar square. Its projective quadratic character is $\eta = \chi_{-6}$, so inertia inside G_A has scalar image. Apply Lemma 18.11 with $J = 0$ to this single source, using its actual ordinary or nonordinary condition at two. The image is monomial in characteristic zero.

The component character is η, so $r = \operatorname{Ind}_{G_L}^{G_K} \theta$ for a character θ. It is de Rham and finitely ramified, as a summand of the restriction of r. By global class field theory it is attached to an algebraic Hecke character [57, Chapter III]. Its conjugate is distinct, by the absolute irreducibility of r. Automorphic induction from the CM field L is cuspidal and gives r, with its regular weights [4, Chapter 3]. □

18.5 The coefficient field of the special source

Let $\mathcal{R}$ be a weight-two strongly compatible system with determinant cyclotomic, good outside the two places above three. Suppose that its type at v_+ is the binary supercuspidal type of Proposition 18.3 and its type at v_- is special, with nonzero monodromy and a fixed quadratic scalar character. Let B be the number field generated by its traces at the good places. We consider the case in which all its residual representations at coefficient places above two are the standard representation of $\mathrm{Gal}(A/K)$. Put $k = \mathbf{F}_4$, let $\mathcal{W} = W(k)$, and fix a primitive cube root of unity $\zeta \in \mathcal{W}$.

Proposition 18.13. *Let d_0 be the number of nonordinary embeddings of B in $\overline{\mathbf{Q}}_2$. Then $d_0 \le 2$. For each of the two possible residual unramified quotient characters at the dyadic place, there is at most one ordinary embedding. Every coefficient completion associated to a nonordinary embedding is ramified over $\mathbf{Q}_2$. In particular $[B : \mathbf{Q}] \le 4$.*

We first describe the integral representation to which we shall apply the finite-flat conditions. Partition the embeddings of B into a nonordinary set and two ordinary sets, according to the eigenvalue $\bar{\zeta}$ or $\bar{\zeta}^{-1}$ of residual Frobenius on the unramified quotient. Each set is stable under the absolute Galois group of $\mathcal{W}[1/2]$. For any nonempty set of this partition, project $B \otimes_{\mathbf{Q}} \mathcal{W}[1/2]$ to the product of field factors corresponding to the set. Let R be the closed

$\mathcal{W}$-subalgebra of this product generated by the traces of the associated tuple of representations. Its fraction algebra is the entire product, since already the traces at the good places generate B over $\mathbf{Q}$. Thus R is finite free over $\mathcal{W}$ of rank the cardinality of the set. It is local with residue field k. Indeed all constituent reductions give the same residual trace, and every maximal ideal of this integral order lifts to a prime of its normalization.

The joint representation has a model on R^2. To see this, choose four group elements whose residual matrices form a k-basis of $M_2(k)$, using absolute irreducibility. Their trace-pairing Gram determinant is a unit. Pairing any group matrix with these four elements expresses it uniquely in their joint span, with coefficients in R; the same argument applies to their products. Their span is thus a free rank-four algebra with reduction $M_2(k)$. Lift a primitive matrix idempotent to this complete algebra and take the corresponding left ideal. This ideal is free of rank two over R, and the action on it identifies the algebra with $M_2(R)$ by reduction and Nakayama. Denote the resulting representation by r. Its rational constituents are those of the original tuple. In particular the stable $\mathbf{Z}_2$-lattice underlying R^2 in their sum is realized by an isogenous Barsotti–Tate group over K_2, by the lattice classification [42, Theorem 4.1]. All its generic subquotients killed by two acquire finite-flat models by schematic closure and quotient. For the nonordinary set their special fibres and their Cartier duals are connected, by the isogeny argument in the proof of Theorem 11.7.

18.5.1 Induction modulo rational two

Write $T = R/2R$ with maximal ideal $\mathfrak{n}$. Choose a wild inertia element u at v_+ whose image generates the actual order-three subgroup and use its two integral eigenprojectors to put $r(u) = \operatorname{diag}(\zeta, \zeta^{-1})$. The standard residual representation in this basis is monomial with component character $\eta = \chi_{-6}$. Here χ_a denotes the quadratic character of $K(\sqrt{a})/K$. The offdiagonal matrices with residual conjugation action form the standard module $W \otimes_{\mathbf{F}_2} k$ of Proposition 18.10. In particular $W^u = 0$ and $H^i(S_3, W) = 0$ for $i > 0$, since W is projective.

Lemma 18.14. *In the fixed u-eigenbasis the image of r over T is monomial with component character η.*

Proof. A tame generator U at v_- has the two characteristic-zero eigenvalues ϵ, ϵ, with one fixed quadratic sign $\epsilon = \pm 1$ across the tuple. Hence $U^2 = 2\epsilon U - 1$ in the joint order and $U^2 = 1$ modulo rational two. The inertia in G_A is cut out by the ramified quadratic subfield, so it acts trivially modulo two; higher inertia already acts trivially in the special source. Lemma 18.11, with $J = 2R$, proves the assertion for the nonordinary block and for each ordinary orientation. □

18.5.2 The residual scalar character

Lemma 18.15. *For the nonordinary set $\dim_k \mathfrak{n}/\mathfrak{n}^2 \le 1$. If this space is nonzero, its visible first-order scalar character is a nonzero multiple of χ_{-2} modulo $k\eta$. For either ordinary set $\mathfrak{n} = 0$.*

Proof. A k-linear functional on $\mathfrak{n}/\mathfrak{n}^2$ gives a specialization to $k[\delta]/(\delta^2)$. Lemma 18.14 makes the specialized representation monomial. Compare it with the constant residual matrices. Their determinant agrees, since the cyclotomic determinant is one in characteristic two. The diagonal infinitesimal error therefore has equal entries, even on an antidiagonal coset after multiplying by the inverse residual matrix. The error is $\delta c(g)$ times the identity for an additive character $c : G_K \to k$. Changing basis by an infinitesimal diagonal matrix can add a multiple of η; conjugation of such a matrix by an antidiagonal element interchanges its diagonal entries. Accordingly the class of c modulo $k\eta$ depends linearly on the functional. This map is injective:

if the character is a multiple of η, the representation is conjugate to its constant reduction, so every trace has zero infinitesimal part and the trace generators force the functional to vanish.

At v_+ choose an antidiagonal inertia element t with $t^2 = -1$. Multiply an arithmetic Frobenius lift by t if necessary so that it centralizes u. It takes t to $-u^i t$ for some i, since it acts by cubing on the tame quotient of order four. Multiplying the lift by a power of u makes its diagonal ratio exactly -1. Its eigenvalues are then $\lambda, -\lambda$ with $\lambda^2 = -3$. The trace after multiplication by u is $\lambda(\zeta - \zeta^{-1}) = \pm 3$. It is rational and common to the tuple by compatibility of the local Weil representations at v_+. Thus λ is one fixed value $\pm(\zeta - \zeta^{-1})$ in $\mathcal{W}$. The scalar error vanishes on this Frobenius and on u, so $c|_{D_{v_+}}$ is a multiple of η. At the dyadic place the nonordinary scalar errors have conductor at most two, by extracting the middle self-extension on the unramified cubic base as in the proof of Lemma 18.11. Unramified base change preserves positive breaks. In an ordinary set the exact unramified quotient shows instead that c is unramified at two.

Use the supported squareclass basis $\langle -1, 3+s, 2, 3, 2+s \rangle$ of Section 18.3. The condition at v_+ and dyadic conductor at most two leave

a	1	-2	3	-6
dyadic conductor exponent	0	2	2	$0.$

For an explicit verification form the 32 products a of the displayed basis, keep those with a or $-6a$ a square at v_+, and compute the dyadic exponent of the relative discriminant of $x^2 - a$ (zero for the trivial class). Equivalently this is the calculation in `scalar_characters.gp`. Scalar extension from $\mathbf{F}_2$ gives the same kernel conditions over k. Modulo η only χ_{-2} survives, and it is ramified at two. Hence the nonordinary cotangent dimension is at most one and an ordinary set has zero cotangent dimension. Nakayama's lemma gives $\mathfrak{n} = 0$ in the latter case. The exact calculation is supplied in [49, `ramified`]. □

18.5.3 The local height condition

Proof of Proposition 18.13. Lemma 18.15 shows that each nonempty ordinary set has cardinality $\dim_k T = 1$. For the nonordinary set we may assume the cotangent dimension is one. Then $T = k[z]/(z^h)$, where $h = \dim_k T$, and its nonzero tangent is represented by χ_{-2} modulo $k\eta$ after scaling z. Restrict to the unramified cubic extension F/K_2 killing the residual dyadic characters. Since L/K splits at 2, Lemma 18.14 splits the local representation as two rank-one T-characters. The restriction of η is trivial, and the first-order error of each character is ramified over F, since χ_{-2} has conductor two there.

Take one of the rank-one summands as a finite-flat subquotient of the regular lattice modulo two. It and its dual have connected special fibres. By Lemma 11.2, the generic coefficient action, the images of powers of z, and the resulting exact filtrations extend to the finite-flat model. Its Kisin module is free of rank one over $T[[u]]$ on every residue embedding factor after enlarging the finite constant field to accommodate the residue field of F. Indeed the z-filtration is exact, with rank-one successive quotients on each factor. The quotient by z has rank one over the residue power-series ring, so lifting its generator gives a surjection by Nakayama. The ranks over that power-series ring are equal, giving the assertion. Contravariance interchanges the subobject and quotient filtrations and gives the same lengths. Frobenius permutes the residue embedding factors.

The Frobenius multiplier on each factor reduces modulo z to u times a unit. Weierstrass preparation writes it as a unit times $u - \lambda_i$ with $\lambda_i \in (z)$. Height one implies $u^2 \in (u - \lambda_i)$, since the Eisenstein polynomial of a uniformizer reduces modulo two to u^2. Evaluating at λ_i gives $\lambda_i^2 = 0$. If all λ_i vanished modulo z^2, the first-order multipliers would each be u times a power-series unit. After the middle exponent is removed, the corresponding Artin–Schreier

class has no pole on any residue factor and is unramified. This is the middle self-extension calculation in the proof of Theorem 11.7, including its comparison of breaks with the field of norms. It contradicts the ramification just proved. Thus one λ_i is a unit times z, and $z^2 = 0$. This gives $h \leq 2$.

Finally suppose that the completion of B belonging to a nonordinary embedding is unramified over $\mathbf{Q}_2$. The trace-order construction above, applied to a single completion before and after unramified coefficient extension, realizes its representation on an integral rank-two lattice there. The residual local representation splits into two distinct unramified lines after unramified coefficient extension. Over the unramified cubic base, Lemma 11.2 splits its finite-flat model as the direct sum of middle models. Therefore Frobenius at $u = 0$ on the integral Kisin module vanishes modulo two. This can be checked after unramified base extension by faithful flatness. Over the original base with residue field $\mathbf{F}_2$, the integral module has coefficient rank two: the underlying finite-free module with unramified coefficient action is free over the coefficient power-series ring, by reduction and Nakayama. Since two is a uniformizer, every entry of its Frobenius-at-zero matrix belongs to $2\mathcal{O}_E$, where E is the enlarged unramified coefficient field. Its determinant would have valuation at least two. On the other hand the determinant of crystalline Frobenius has valuation one because the determinant character is cyclotomic [42, Proposition 4.2]. This contradiction proves the ramification assertion. Summing the three bounds proves the proposition. □

18.6 The small coefficient prime

Let B be the minimal global trace field of the system with a special condition at v_-. We are in the case in which the residual representation at every embedding $B \hookrightarrow \overline{\mathbf{Q}}_2$ is the standard representation of the same S_3 extension A/K. Modularity at any one coefficient prime suffices for modularity of the entire system.

Proposition 18.16. *If all the dyadic embeddings are ordinary, or all are nonordinary, then $[B : \mathbf{Q}] \leq 2$. Otherwise B is quartic, ramified at two, and is generated by the Frobenius trace a_2 at the dyadic prime. Its minimal polynomial is congruent to $X^2(X^2 + X + 1)$ modulo two.*

Proof. The trace order in each set bounds the number of embeddings in that set. Residue Frobenius interchanges the ordinary sets, so if they occur each has exactly one embedding. A nonordinary completion must ramify and its block has at most two embeddings. In the mixed case it has exactly two. At an ordinary embedding one crystalline Frobenius root is the unramified quotient and the other is divisible by the maximal ideal; a_2 thus reduces to the two distinct primitive cube roots in the two orientations. At each nonordinary embedding it is in the maximal ideal by the Newton slope criterion. Strong compatibility supplies $a_2 \in B$ by taking a coefficient prime away from two. Each ordinary root occurs once, whereas restricting the embeddings to a proper subfield would repeat every root $[B : \mathbf{Q}(a_2)]$ times. This proves the proposition. □

Lemma 18.17. *Let B be a mixed quartic field as in Proposition 18.16. Then at least one of the following holds:*

(i) Every prime of B above three has residue degree at most two.
(ii) There is a degree-one prime of B above five or seven.
(iii) Up to replacing a_2 by $-a_2$, its minimal polynomial is

$$X^4 - X^3 - 7X^2 + 8X + 2 \quad \text{or} \quad X^4 - X^3 - 5X^2 + 2X + 2.$$

Proof. Every conjugate of the algebraic integer a_2 lies in $[-\sqrt{8}, \sqrt{8}]$. Write its minimal polynomial as $P = X^4 + aX^3 + bX^2 + cX + d$. By Proposition 18.16, a, b are odd and c, d are even.

The elementary symmetric functions and Cauchy's inequality give

$$|a| \le 11, \quad |b| \le 48, \quad |c| \le 90, \quad 0 < |d| \le 64, \quad 0 \le a^2 - 2b \le 32, \quad 8b \le 3a^2.$$

Replacing a_2 by $-a_2$, we take $a > 0$. Exact rational Sturm sequences on $[-283/100, 283/100]$, with the integral coefficient bounds above, leave the twelve irreducible polynomials in the following table. For a row marked (p, r, e) we have $P(r) \equiv 0$ and $P'(r) \equiv e \not\equiv 0 \pmod{p}$.

a	b	c	d	Local check
1	-9	-6	14	$(5, 2, 2)$
1	-7	-10	-2	one prime above 3, of residue degree 2
1	-7	-8	2	$(3, 2, 2)$; first exceptional polynomial
1	-7	-6	2	$(7, 2, 3)$
1	-7	-6	4	$(7, 1, 1)$
1	-7	-4	4	$(5, 1, 4)$
1	-7	-4	8	$(5, 4, 4)$
1	-7	-4	10	$(5, 0, 1)$
1	-5	-4	2	$(5, 1, 3)$
1	-5	-2	2	$(3, 1, 1)$; second exceptional polynomial
1	-5	-2	4	odd field discriminant
3	-3	-12	-6	Eisenstein at 3

The accompanying certificate uses only integer and rational arithmetic: it tests irreducibility by Gauss's lemma and counts real roots by Sturm's theorem. Thus this step requires no maximal-order computation.

A simple root modulo five or seven gives an embedding $B \hookrightarrow \mathbf{Q}_p$ by Hensel's lemma, proving (ii). For the second row, set $Q = X^2 - X - 1$. Then

$$P = Q^2 + 3(X^3 - 2X^2 - 4X - 1).$$

The polynomial Q is irreducible modulo three, and the remainder of the second factor modulo Q is $-4X - 2$. Dedekind's index criterion shows that three does not divide $[\mathcal{O}_B : \mathbf{Z}[a_2]]$; hence three has one prime in B, of ramification index two and residue degree two. The final row is Eisenstein at three and has one prime of residue degree one. Both rows give (i).

For the penultimate row, put $\alpha = a_2$ and $\beta = (\alpha^3 + \alpha^2 + \alpha)/2$. Direct reduction modulo P gives

$$\beta^4 - 29\beta^2 + 10\beta + 19 = 0.$$

Thus β is integral. Since $\operatorname{disc} P = 8900$, the integral lattice spanned by $1, \alpha, \alpha^2, \beta$ has discriminant 2225. The field discriminant is therefore odd, contrary to Proposition 18.16. The two remaining rows are exactly the polynomials in (iii), after replacing X by $-X$. □

Lemma 18.18. *If an irreducible residual representation of determinant $\bar{\chi}_3$, unramified outside three, is restricted to $G_{K(\zeta_3)}$, it remains irreducible.*

Proof. Otherwise the representation is induced from a character θ of $H = K(\zeta_3)$ with $\theta^\sigma = \theta^{-1}$. Such a character has order prime to three. Its ray conductor divides the product of the two primes above three, both of residue degree one and ramification index two. An absolute equation of H is $X^4 + 10X^2 + 100$. Its class group is cyclic of order four, certified by `bnfcertify`; its ray group for this modulus is also C_4. If S is the action of the nonidentity automorphism over K, the Smith invariant of the lattice generated by 4 and $1 + S$ is two. Equivalently this ray group modulo $1 + \sigma$ is C_2. Thus $\theta^2 = 1$ and $\theta = \theta^\sigma$, contradicting irreducibility. These data are reproduced by `cyclotomic_three_ray.gp`. □

Proof of Theorem 18.1. Soluble residues have solvable odd Artin lifts, which give modularity by Hecke induction or Langlands–Tunnell and regular weight raising. Assume therefore that the prescribed three-residue is nonsolvable. The bounded lift, the discriminant estimate and the soluble-image classification reduce the proof to the special source with the fixed S_3 residue, the quartic source having already been treated. Apply Proposition 18.16. The exact calculation is supplied in [49, `ramified`].

If the degree of B is at most two, or every coefficient prime above three has degree at most two, choose the originally prescribed prime. Its residue is realizable over $\mathbf{F}_3$ or $\mathbf{F}_9$ by its traces, absolute irreducibility and the vanishing of the Brauer group of a finite field. The small-image theorem of Allen–Khare–Thorne [3, Theorem 2.5] gives the required cuspidal residual witness.

At a degree-one coefficient prime of characteristic $p = 5$ or 7, the binary inertia at v_+ remains irreducible modulo p; the order-three element has distinct eigenvalues and a tame element interchanges its eigenlines. The local cyclotomic extension at v_+ is unramified, so the residual representation is absolutely irreducible even on $G_{K(\zeta_p)}$. Theorems 2.5 and 2.3 of [3] give respectively modularity of its residue and modularity of the actual de Rham member with weights $\{0, 1\}$. This includes the ramified prime five. Strong compatibility gives modularity at the original prime.

The last two polynomials in Lemma 18.17 have a coefficient place above three with residue field $\mathbf{F}_3$. If its residue were reducible, write the characters as ψ and $\bar{\chi}_3\psi^{-1}$. Lemma 18.9 gives $\psi^2 = 1$ or ν, the Hilbert class character, which is nontrivial unramified at both triadic places. On $G_{K_{v_-}}$ a quadratic Steinberg twist has semisimplification $\alpha \oplus \alpha\chi_3$ with α quadratic, so ψ^2 is locally trivial. It follows that $\psi^2 = 1$. At the good dyadic place the residual trace would be $\psi(\mathrm{Frob}_2)(1 + 2) = 0$. For the selected coefficient prime $\lambda \mid 3$, this gives $a_2 \in \lambda$. Hence $3 \mid N_{B/\mathbf{Q}}(a_2)$. Both polynomials have constant term 2, so $N_{B/\mathbf{Q}}(a_2) = 2$, a contradiction. The residue is therefore irreducible. Lemma 18.18 and [3, Theorems 2.5 and 2.3] make the actual member modular. This proves the residual theorem. □

19 Quadratic symmetry over $\mathbf{Q}(\sqrt{11})$

Put $K = \mathbf{Q}(s)$, $s^2 = 11$.

19.1 Arithmetic and modular witnesses

We will use the elements

$$\epsilon = 10 + 3s, \qquad \pi = 3 + s, \qquad \alpha = 4 + s, \qquad \alpha' = 4 - s.$$

They satisfy $N(\epsilon) = 1$, $N(\pi) = -2$ and $N(\alpha) = N(\alpha') = 5$.

Lemma 19.1. *The class number of K is one, and its units are generated by -1 and ϵ. Its finite ray class groups of moduli* (3) *and* (5) *are both cyclic of order four. With both real places added to the modulus, the groups are $C_8 \times C_2$ at three and $C_4 \times C_2$ at five.*

Proof. Minkowski's bound is $\sqrt{44}/2 < 3.32$. Three is inert, and the dyadic prime is generated by π, so the class number is one. The least nontrivial positive solution of $x^2 - 11y^2 = 1$ is $(10, 3)$; checking $y = 1, 2$ proves that ϵ is fundamental. The negative Pell equation has no solution because -1 is not a square modulo eleven.

Modulo three the residue group is $\mathbf{F}_9^\times$ and $\epsilon = 1$. The image of the units is $\{\pm 1\}$, giving a cyclic quotient of order four. Adding the real signs gives

$$(C_8 \times C_2^2)/\langle(-1, -1, -1)\rangle \simeq C_8 \times C_2.$$

At the two five-places, ordered by $s = 1, -1 \bmod 5$, ϵ has residues $(3, 2)$, of order four, and its square is the image $(-1, -1)$ of -1. The product map $(u, v) \mapsto uv$ identifies the finite ray quotient with $\mathbf{F}_5^\times$. Adding the signs leaves one independent sign factor and gives $C_4 \times C_2$. □

19.2 Characteristic-three sources

Lemma 19.2. *Let $F/\mathbf{Q}_3$ be unramified quadratic, and let r be crystalline of labelled weights $\{0, 1\}$ with determinant χ_3. If its residual constituents have inertia characters $1, \omega_3$, then r is ordinary.*

Proof. Since $e(F) = 1 < 3 - 1$, the rank-one finite-flat models of these characters are uniquely etale and multiplicative. Choose a Barsotti–Tate group with coefficient action realizing a stable lattice. If it has a nonzero etale rational quotient, the coefficient rank is one, and the determinant gives the ordinary filtration. Otherwise the group and its Cartier dual are connected, using $r^\vee(1) \simeq r$ and invariance of connectedness under isogeny. The closure of a residual invariant line is etale or multiplicative. The former contradicts connectedness, and the latter gives an etale quotient of the connected dual. Thus the source is ordinary. □

Proposition 19.3. *If $\bar{r}_3^{\mathrm{ss}} = \psi \oplus \omega_3\psi^{-1}$ for a bounded system $\mathcal{R}$, then $\psi^2 = 1$ and the actual source r_3 is ordinary and modular.*

Proof. At a five-place the semisimplified residual inertia of our two types is scalar quadratic: the cubic part of the cuspidal inertia disappears modulo three, as does the unipotent part of the special condition. Hence ψ^2 is unramified there. A characteristic-three finite character kills principal units at three, so ψ^2 factors through the finite ray group of modulus (3) from Lemma 19.1.

Fontaine–Laffaille theory [32] over the unramified quadratic K_3 gives the possible inertia exponents $0, 1, 3, 4 \bmod 8$ of a rank-one constituent in weight two. The determinant has exponent four. Thus the reducible pairs are $\{0, 4\}$ and $\{1, 3\}$. In the second case ψ^2 is a faithful quartic ray character. Both five-primes generate that ray group: their positive generators α, α' reduce to $1 + s, 1 - s$ in $\mathbf{F}_9$, and

$$(1 + s)^2 = -s, \qquad (1 + s)^4 = -1, \qquad 1 - s = (1 + s)^3.$$

It follows that $\psi^2(\mathrm{Frob}_v)$ is a primitive fourth root at either five-place.

The quotient of the two residual local characters would therefore have Frobenius value of order four. For both of our five-adic types that quotient is instead -1 in characteristic three: the special eigenvalues have ratio five, while the cuspidal parameter has trace zero and determinant five. Scalar quadratic factors do not change the quotient. This excludes $\{1, 3\}$.

For the pair $\{0, 4\}$, ψ^2 is unramified also at three and is even at both real places. The ordinary class number is one, so $\psi^2 = 1$. Lemma 19.2 makes the actual source ordinary, with unramified quotient. Its residual constituent ratio is ω_3, which is totally odd, extends to $G_{\mathbf{Q}}$, and is nontrivial at K_3. The field $K(\zeta_3)$ is abelian over $\mathbf{Q}$. All the hypotheses of Skinner–Wiles [62, Theorem A] are satisfied, so r_3 is modular. □

Proposition 19.4. *Let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_3)$ be continuous, irreducible, totally odd and finitely ramified, with determinant χ_3, and crystalline of weight two at three. If $\bar{r}$ is absolutely irreducible but becomes reducible on $G_{K(\zeta_3)}$, then r is ordinary at three and modular.*

Proof. Put $H = K(\zeta_3)$. Clifford theory gives $\bar{r} = \mathrm{Ind}_{G_H}^{G_K} \xi$, where ξ has finite order prime to three. Wild inertia at three is therefore trivial. The extension H_3/K_3 is ramified quadratic. A

tame generator t outside G_{H_3} acts antidiagonally in an inducing basis, and

$$\det \bar{r}(t) = -\xi(t^2) = \omega_3(t) = -1.$$

Thus $\bar{r}(t)^2 = 1$. A lift of Frobenius can be chosen in G_{H_3}, and its diagonal matrix commutes with $\bar{r}(t)$ because the residue cardinality is nine. Its two eigenvalues are consequently equal, and the determinant makes that common value ± 1. The local pair is exactly $\beta, \beta\omega_3$ with β unramified quadratic. Lemma 19.2 applies to the actual source.

Induce the Teichmuller lift of ξ to obtain a finite-image, irreducible, totally odd Artin representation. Automorphic induction supplies its weight-one Hilbert theta series. At three its Euler factor has degree one and unramified root ± 1, so the form is ordinary and has the correct residual ordinary quotient. Wiles’s Hilbert ordinary family theorem [73, Theorem 1.4.1] gives a regular ordinary specialization with the same residual quotient. This application is also explicit in [27, pp. 1133–1134].

We have obtained the quotient-compatible modular lift required by [63, Theorem 5.1]. The actual source is ordinary and distinguished at three, its determinant is cyclotomic, and its remaining geometric and parity hypotheses are the stated ones. That theorem permits reducibility on the cyclotomic subgroup and proves modularity of r. □

19.3 Nonsolvable and dihedral dyadic residues

In this section the system is special at both five-primes. Let M be the field of a semisimple dyadic residue. At five its full decomposition image is elementary abelian of exponent at most two: removing its quadratic sign leaves an upper unipotent group, generated by tame inertia and Frobenius. In particular every local field is contained in the maximal elementary quadratic extension of $\mathbf{Q}_5$.

Lemma 19.5. *Put $F = K(i, \sqrt{5})$ and let H be its Hilbert class field. Then $[H : \mathbf{Q}] = 16$, its dyadic primes have absolute $(e, f) = (2, 4)$ and its four five-primes have $(e, f) = (2, 2)$. Its fourth-power dyadic ray, with all five-prime classes killed, has group $C_4^2 \times C_2^4$.*

Proof. The defining polynomials for F, H are respectively

$$y^8 - 4y^7 + 24y^6 - 58y^5 + 141y^4 - 190y^3 + 186y^2 - 100y + 20,$$
$$y^{16} - 9y^{14} + 39y^{12} - 111y^{10} + 214y^8 - 189y^6 + 69y^4 - 6y^2 + 1.$$

The class groups are $C_2, 1$; the second field is the class field of the first and has its squared discriminant. Over H the full ray invariants are $12, 4, 4, 2, 2, 2, 2, 2$. Adding the four reciprocity columns of the five-primes to the diagonal relation matrix gives the displayed quotient by Smith reduction. The certificate `sp_auxiliary.gp` checks the polynomial identification, both class groups with `bnfcertify`, the prime decompositions and the Smith factors. The exact calculation is supplied in [49, `ramified/d6_d10_d11/d11`]. □

Proposition 19.6. *Assume GRH. A two-special bounded system with a nonsolvable dyadic residue is modular.*

Proof. Its image is $\mathrm{SL}_2(q)$, $q = 2^f \geq 4$, by Dickson and determinant one. Let B be the ray field of Lemma 19.5, of absolute degree 4096 and Galois over K by invariance of the modulus and quotient. The dyadic breaks of BM/K are at most three: this follows from Fontaine’s bound for the actual Barsotti–Tate lattice, and from the fourth-power ray over the unramified dyadic extension H/K. The two-adic contribution to the logarithmic relative different, in valuation $v_2(2) = 1$, is at most $4/2 = 2$. At five the field H contains the ramified field $\mathbf{Q}_5(\sqrt{5})$ and

its unramified quadratic extension. This is already the full elementary quadratic envelope. Consequently for $L = BM$ and also $L = BMM^\sigma$,

$$\mathrm{rd}(L) \le 4\sqrt{44 \cdot 5} < 59.329588, \qquad 2N_{25}(L)/[L : \mathbf{Q}] = 1/2.$$

Here N_Q counts primes of norm Q.

Apply Section 22 with $B_{b,100}$ and the retained prime powers specified below. For the totally real comparison retain $A_{b,4}$. Directed 256-bit evaluation gives

	$n \ge$	b	powers of 25	$\mathrm{rd}(L) >$
$q \ge 16$	$4096 \cdot 4080$	16.4	9	59.981834
$q = 8,\ M \ne M^\sigma$	$4096 \cdot 504^2$	20.5	10	62.360509
totally real field	120	4.8	0	56.063060

Simplicity gives disjointness from the solvable auxiliary. Distinct simple conjugate fields are disjoint over K, and their product still has no nontrivial solvable quotient. The first two lines exclude their respective cases. The exact calculation is supplied in [49, `ramified/d6_d10_d11/shared`].

When $q = 8$ and $M = M^\sigma$, conjugation by $\mathrm{Gal}(M/\mathbf{Q})$ induces an inner automorphism of $\mathrm{SL}_2(8)$ since its outer group has order three. The group is centerless, so conjugation realizes an extension of the actual residue to $G_{\mathbf{Q}}$. Rational Serre modularity [40, 41] and quadratic base change give a regular cuspidal witness over K; cuspidality follows by irreducibility of the restriction. For $q = 4$ use [3, Theorem 2.5]. A real conjugation is nontrivial in either case: otherwise M would be totally real of degree at least 120. If wild inertia at two is nontrivial, it fixes a unique line, the diagonal characters are unramified since the base residue field is $\mathbf{F}_2$, and inertia has order at most q. Integrating the upper bound three gives exponent at most $2 - 2/q$; the tame case is smaller. Thus $\mathrm{rd}(M) \le \sqrt{44 \cdot 5}\, 2^{7/4} < 49.890038$, contrary to the last table entry. Lemma 2.1 and [69, Theorem 6.1], applied to the dual in the opposite weight convention, give modularity of the actual dyadic source. □

Proposition 19.7. *A two-special system with irreducible solvable two-residue is modular. Every reducible two-residue has trivial semisimplification.*

Proof. An irreducible solvable subgroup in determinant one is odd dihedral. Let L/K be its inducing quadratic field. The odd rotation character is unramified away from two, trivial at every five-decomposition group over L, and at two is tame. Its conductor divides the product of the dyadic primes of L. The quadratic squareclasses are supported on $-1, \epsilon, \pi, \alpha, \alpha'$ by class number one. Odd dyadic valuation gives conductor five by an Eisenstein square-root polynomial and its derivative, contradicting Fontaine's upper break three. Among the fifteen nontrivial classes in $\langle -1, \epsilon, \alpha, \alpha' \rangle$, the odd part of the ray modulo all dyadic primes with all five-primes split is trivial except at -5ϵ, where it is C_3. The certificate `sp_dihedral_rays.gp` verifies these quartic ray quotients. Since $\pi^2 = 2\epsilon$ the survivor is $K(\sqrt{-10})$. Its dyadic relative conductor is four. Theorem 11.7 excludes a nonordinary actual source. The ordinary source has a nonsplit dyadic prime in its imaginary inducing field and the specified near-ordinary line; Allen's ordinary dihedral theorem [2, Theorem 5.2.1] applies. For a reducible residue the odd-order constituents are unramified at two and five and everywhere else, so class number one makes both trivial. The exact calculation is supplied in [49, `ramified/d6_d10_d11/d11`]. □

19.4 Ordinary scalar systems

Proposition 19.8. *A two-special system with a scalar semisimple two-residue and an ordinary member above two is modular.*

Proof. In that member choose a vector v of the ordinary line and the lattice generated by the global orbit of v. Its reduction is cyclic, with unique global invariant line transverse to v. The local decomposition image at two preserves both lines, so it is trivial. The supported quadratic characters splitting locally at two form the single line $\chi = \chi_{-5}$. Indeed the sixteen classes $\langle -1, \epsilon, \alpha, \alpha' \rangle$ contain precisely $1, 5, -1, -5$ unramified at two and precisely $1, -5$ split there; a class of odd dyadic valuation cannot split. These local computations are recorded in `ordinary_squareclasses.gp`. Every coordinate of the additive residual extension is a multiple of χ. For a real involution c, the basis $((r(c) - 1)v, v)$ is therefore integral and normalizes the residue, the real matrix and the ordinary mark to

$$1 + \chi E_{12}, \qquad \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad e_2.$$

Use the marked ordinary crystalline functor of [43, Proposition 2.4.4, Corollary 2.4.5], pulled back to the matrix problem with this fixed real matrix, support and determinant. At five fix the quadratic characters on inertia, and allow the unramified special signs to vary. The special line is represented without an extra free parameter: for a tame generator normalized by the given sign, its difference from one has unit upper-right entry, so the coordinate of a fixed line in the e_1 chart is determined by its first row. Stability and the character equations on that line and its quotient, including square one of the latter, are then closed equations. The exact calculation is supplied in [49, `ramified`].

Write a homogeneous tangent as $(1 + tX(g))\bar{r}(g)$, where $X(g) = \left(\begin{smallmatrix} x(g) & y(g) \\ z(g) & x(g) \end{smallmatrix}\right)$. Conjugation adds $\chi z(1 + E_{12})$, so z is additive. At two, $y = 0$, x is unramified and z is a unit Kummer class by the finite-flat ordinary extension condition. At both five-primes the special line has the form $e_1 + tt_w e_2$, whence $z = 0$ on the whole decomposition group. The two valuations exclude α, α'; ϵ is a nonsquare at both places. Thus z is a multiple of χ_{-1} and the fixed real matrix kills it. The remaining x, y are additive. The split-at-two calculation gives $y \in k\chi_{-5}$, and $y(c) = 0$ kills it. The unramified-at-two calculation and $x(c) = 0$ leave $x \in k\chi_5$. This last direction is detected by the marked unramified quotient ν on dyadic Frobenius, since χ_5 is locally nontrivial unramified. Cotangent Nakayama makes the complete ring a quotient of $W(k)[[\nu(\mathrm{Frob}_2) - 1]]$.

Choose $\eta = 1$ or χ_{-5} aligning the inertial characters of r and $r^\sigma \otimes \eta$ at both five-primes. The twist is dyadically trivial. The two normalized points agree on the sole generator, since the local involution fixes unramified Frobenius. They agree as marked representations. Lemma 12.3 applies. □

19.5 The integral order of nonordinary scalar systems

Take the tuple at all dyadic coefficient places of the bounded systems under consideration. Every member is nonordinary with trivial residual semisimplification, and the systems have common quadratic inertial signs at each special place. Use the orders R_0, E_0 of Lemma 11.1. For m distinct coefficient orbits of common trace-field degree d,

$$\mathrm{rank}_{\mathbf{Z}_2} R_0 = md, \qquad \mathrm{rank}_{\mathbf{Z}_2} E_0 = 4md.$$

Write $E = E_0/2E_0$; its radical has residue field $\mathbf{F}_2$.

Lemma 19.9. *The left regular lattice of E_0 is Barsotti–Tate at two and biconnected. For the simple factors of its torsion with trivial generic character, the middle order-two model is unique. If J is the radical of E, the differences A, B, C from one of a real involution and of the two normalized tame five-inertias generate E as an $\mathbf{F}_2$-algebra, and all three have square zero.*

Proof. Apply Lemmas 11.3 and 11.2 to the regular lattice and its trivial-simple torsion subquotients. These lemmas also hold after the unramified base changes used below.

A radical functional gives a middle/middle extension by quotienting the regular model. At the dyadic place its quadratic conductor is at most two by the height-one Artin–Schreier calculation in the proof of Theorem 11.7. These characters belong to the narrow ray modulo $\pi^2(5)$, with invariants $4, 2, 2$. Its quadratic dual is spanned by

$$[-1], \quad [\epsilon\alpha], \quad [\epsilon\alpha'].$$

Their dyadic discriminant exponents are respectively $0, 2, 2$. The latter two ramify at distinct five-primes and split at the opposite prime. Real conjugation and the two tame inertias evaluate in the three coordinate directions. Hence A, B, C span J/J^2. Nilpotence gives algebra generation. The real involution squares to one and each normalized tame generator is unipotent with square-zero difference, proving the lemma. The arithmetic is certified in `scalar_regular_rays.gp`. The exact calculation is supplied in [49, `ramified`]. □

19.6 A quaternion obstruction over the Hilbert class field

The next result is used both for the original reduction and a central quotient of it. Suppose E' is such an actual regular finite-flat quotient, with elementary abelian dyadic-characteristic decomposition image at both five-primes of exponent two. Let M' be its point field, and $P = \mathrm{Gal}(M'H/H)$. All five-primes split in $M'H/H$ because H contains the maximal elementary quadratic completion at five. Its dyadic primes have absolute ramification index two; Fontaine's bound gives upper breaks at most three.

Lemma 19.10. *Assume GRH. The group P is elementary abelian of rank at most two. Its span in E' has dimension at most four and at most two radical generators.*

Proof. Denote the dyadic primes of H by v_1, v_2. The ray groups with all five-prime classes killed at dyadic conductor exponents $(4, 0)$ and $(0, 4)$ are trivial; the corresponding $(2, 2)$ group is C_2^2. If either inertia image were proper in the finite two-group P, it would lie in an index-two subgroup. The associated quadratic extension would contradict one of the first two ray calculations. Thus inertia at either dyadic place is all of P.

Let C_0 be the span of P in E' and J_0 its radical. Its regular representation is the actual submodule $C_0 \subset E'$. At either dyadic completion it therefore has a biconnected finite-flat model. A functional on J_0/J_0^2 gives a middle/middle extension, whose conductor is at most two by the calculation in the proof of Theorem 11.7. The $(2, 2)$ ray calculation gives $\dim J_0/J_0^2 \le 2$. At each dyadic place the ramified Artin–Schreier coordinates of independent such functionals are independent over $\mathbf{F}_2$: a nonzero linear combination with zero ramified coordinate would be unramified, whereas inertia is all of P.

The right action of C_0 extends to its finite-flat model by Lemma 11.2. The quotient by the right radical is the unique middle model. Exactness, Nakayama's lemma and equality of ranks give a Kisin module free of rank one over $(k \otimes_{\mathbf{F}_2} C_0)[[u]]$, where $k = \mathbf{F}_{16}$. Normalize its Frobenius to $u + V(u)$, with $V(u) \in (k \otimes J_0)[[u]]$. Height one makes $u^2(u + V)^{-1}$ integral. Modulo J_0^3 this expression is $u - V + V^2/u$, so $V(0)^2 = 0$.

With one radical generator its nonzero ramified coefficient gives $J_0^2 = J_0^3$, hence $J_0^2 = 0$. With two generators write $V(0) = \alpha X + \beta Y \pmod{J_0^2}$. Here $\alpha, \beta \in \mathbf{F}_{16}$ are $\mathbf{F}_2$-independent. If $\beta/\alpha \notin \mathbf{F}_4$, then $1, \beta/\alpha, (\beta/\alpha)^2$ are independent; otherwise their unique relation is $1 + t + t^2 = 0$. Consequently the classes of $X^2, XY + YX, Y^2$ in J_0^2/J_0^3 are equal to a common element z, which is zero in the first case. If $z = 0$, the associated graded algebra is a quotient of $\mathbf{F}_2[X, Y]/(X^2, Y^2)$, so $J_0^3 = 0$ and C_0 is itself a quotient of this commutative algebra.

If $z \neq 0$, choose two group elements whose augmentation differences form a cotangent basis. In the quotient modulo J_0^3 their squares and commutator are $1 + z$. The kernel over the

two-dimensional additive quotient is central elementary abelian. Quotienting it by a complement to $\langle 1+z\rangle$ gives a quaternion quotient of P. Its three quadratic subfields are the subfields of the unique split-five conductor-$(2,2)$ biquadratic extension of H. For the selected quadratic subfield H'/H, the certificate `scalar_quaternion.gp` gives dyadic break one at both places and the split-five ray group $C_4\times C_2^5$ at conductor $(6,6)$. Its quotient on which the involution of H'/H acts by inversion is C_2^6; the latter assertion is the Smith reduction obtained by adding the columns of $S+I$ to the ray relation matrix. A quaternion quotient would supply an order-four character of this quotient: Herbrand's formula bounds its break by $1+2(3-1)=5$, and conjugation inverts the rotation character. This is impossible. The class and unit calculation in degree thirty-two uses GRH. The exact calculation is supplied in [49, `ramified/d6_d10_d11/d11`].

We have proved that C_0 is a quotient of $\mathbf{F}_2[X,Y]/(X^2,Y^2)$, so P is elementary abelian and $|P|\le 8$. To exclude order eight, suppose it occurs. Then C_0 is exactly this four-dimensional algebra and P is its group of augmentation-one units. At either dyadic place write the regular Frobenius as

$$u(1+fX+gY+hXY),\qquad f,g,h\in u^{-1}k[[u]].$$

After removing the simple factor u, a fundamental solution $1+xX+yY+wXY$ satisfies

$$x^2-x=f,\qquad y^2-y=g,\qquad (w+xy)^2-(w+xy)=h+fg.$$

Multiplication by $1+aX+bY+cXY\in P$ translates $(x,y,w+xy)$ by $(a,b,c+ab)$. These three additive coordinates therefore form a basis of the quadratic characters of P. The first two right sides have poles of order at most one, and the third has order at most two. A double pole is equivalent to a simple pole modulo Frobenius minus identity because k is perfect. Every character of P, being an $\mathbf{F}_2$-linear combination of these three, thus has field-of-norms upper break at most one. Independently, the actual finite-flat point field gives classical upper break at most three for every such character by Fontaine's bound. The first break of the uniformizer tower is four and all later transition breaks are larger. Herbrand base change preserves these original breaks, and the ramification comparison in the proof of Theorem 11.7 identifies them with the field-of-norms breaks. Every character consequently has conductor at most two over each H_{v_i}. Thus P is a quotient of the split-five ray group of conductor $(2,2)$, which is C_2^2, a contradiction. □

19.7 Quadratic conjugation and the scalar theorem

Lemma 19.11. *In the setting of Lemma 19.10, if E' is generated by the three real and tame elements above, then* $\dim_{\mathbf{F}_2} E'\le 24$.

Proof. The group $\mathrm{Gal}(H/K)$ is dihedral of order eight: it has the biquadratic quotient $\mathrm{Gal}(F/K)$, a dyadic Frobenius of order four, and involutions given by real conjugation and five-inertia with independent images in that quotient. The two five-inertias belong to the same reflection class.

Set $\Gamma=\mathrm{Gal}(M'H/K)$. Choose real and tame involutions a,b,c so that b,c have the same image in $\mathrm{Gal}(H/K)$, and put $r=ab$, $c=jb$ with $j\in P$. Write R,B for conjugation by r,b on P. Thus $brb=r^{-1}$ and $Bj=j$. The image of the two-group Γ in $\mathrm{Aut}(P)$ is either trivial or generated by one transvection on a two-dimensional space. In particular $R^2=1$.

At the five-place belonging to b, a Frobenius lift, after multiplication by inertia if necessary, has the form $f=pr^2$, $p\in P$. The five-decomposition image in the compositum $M'H$ is elementary abelian, so $f^2=1$. Since $R^2=1$ and P has exponent two, $f^2=r^4$, and therefore $r^4=1$. Thus $\langle r,b\rangle$ maps isomorphically to the dihedral quotient of order eight. The subgroup $\Gamma_0=\langle a,b,c\rangle$ is $P_0\rtimes\langle r,b\rangle$, where $P_0=\langle j,Rj\rangle$; indeed B fixes j and commutes with R. Its group span maps onto E', even when Γ_0 is proper in Γ.

If $\dim P_0 \le 1$, this gives $\dim E' \le |\Gamma_0| \le 16$. Otherwise $P_0 = P$ has rank two, $R = U$ is a transvection, and j does not belong to its fixed line. Since $B \in \{1, U\}$ fixes j, we have $B = 1$.

Now $\Gamma = \Gamma_0 = P \rtimes D_8$ has order thirty-two. Let $q = R(j)j$ in multiplicative notation. The elements q and r^2 are independent central involutions. The Cayley–Hamilton identity with determinant one makes $r + r^{-1}$ central in E'. Commuting it with j gives

$$(1 + q)(1 + r^2) = 0 \qquad \text{in } E'.$$

Indeed $rj = qjr$ and $r^{-1}j = qjr^{-1}$, and the other factors in this commutator are units. The group algebra $\mathbf{F}_2[\Gamma]$ is free of rank eight over the central subalgebra $\mathbf{F}_2[\langle q, r^2\rangle]$. In the latter, the ideal generated by $(1+q)(1+r^2)$ has dimension one. It therefore generates an eight-dimensional ideal of $\mathbf{F}_2[\Gamma]$, and $\dim E' \le 32 - 8 = 24$. □

Theorem 19.12. *Assume GRH. A two-special bounded system all of whose dyadic members are nonordinary and residually scalar is modular.*

Proof. Let $h = \chi_{-1}$ and choose $\eta = 1$ or χ_{-5} aligning the two exact quadratic *inertia* signs of r and $r^\sigma \otimes \eta$. Both characters are invariant under σ; h is unramified at two, trivial at five, and η is trivial at two. The commuting operations $r \mapsto r \otimes h$ and $g : r \mapsto r^\sigma \otimes \eta$ give a packet of at most four coefficient orbits. Denote their number by m and the common degree of their trace fields by d. Each member retains the biconnected dyadic hypothesis. Use their joint reduction E. If their full special signs agree at both places, the five-group already has exponent two. Otherwise at each place, with Frobenius lifts corresponding under σ, the signs on the r, rh packet differ from those on the $g(r), g(r)h$ packet. The discrepancy is the same at both places, since the involution σ exchanges their characters, h splits at five, and η is invariant under σ. A difference of rational signs separates the two coefficient packets.

Write S for the tuple of $+1$ on the first packet and -1 on the second. At each five-place $T(F) = \pm 6S$, so $t = 2S \in R_0$. For normalized tame differences B, C at the two places, the exact special matrices give $BF = \pm SB$ or $CF = \pm SC$. In E this yields $tB = tC = 0$ and $t^2 = 0$. The equation $F^2 - T(F)F + 5 = 0$ makes the Frobenius an involution in $E' = E/tE$. Tame inertia there is an involution commuting with Frobenius by the fifth-power relation. If all signs agree, take $E' = E$ instead. In both cases the five decomposition groups have exponent two. The stable central quotient inherits the finite-flat regular model at two, by closure and quotient of the actual right radical maps; its regular group span, or its restricted group span over H, is precisely the torsion representation used in Lemma 19.10. Lemma 19.9 gives generation of E by A, B, C independently of the Frobenius signs, and therefore generation of E'. Put $R = R_0/2R_0$ and let $\mathfrak{m}$ be its maximal ideal. The polarized identities identify the anticommutators of A, B, C with the scalars $X = T(AB)$, $Y = T(AC)$, $Z = T(BC)$. Ordering words with these relations and the three zero squares leaves eight square-free monomials. Their traces show that X, Y, Z and $W = T(ABC)$ generate $\mathfrak{m}$: the actual trace order is generated by traces of group elements, and each such element belongs to the algebra generated by A, B, C over $\mathbf{F}_2$. Applying the original R-valued trace to $tAB, tAC, tBC, tABC$ gives $tX = tY = tZ = tW = 0$ in R, independently of faithfulness of its reduced scalar action. Thus $t\mathfrak{m} = 0$. Since t is central and kills B, C, the ideal tE is spanned over $\mathbf{F}_2$ by t, tA. Hence Lemma 19.11 gives

$$4md = \dim E \le 24 + 2 = 26.$$

An actual selftwist by h is impossible: it would give induction from $K(i)$, while this quadratic extension splits at five and the source has nonzero monodromy there. The bound $\dim E \le 26 < 32$ and Theorem 1.4 now give either an actual twisted-conjugacy identity involving σ, in which case Lemma 12.3 proves modularity, or a trace field of degree at most two or biquadratic over $\mathbf{Q}$. Every prime above three then has residue degree at most two.

In the small-residue-field cases use the triadic source results above for a solvable residue. A nonsolvable residue stays absolutely irreducible over $K(\zeta_3)$ by Clifford theory. Its trace field realizes it over $\mathbf{F}_3$ or $\mathbf{F}_9$ by Brauer–Nesbitt and the vanishing of the Brauer group of a finite field. Theorems 2.5 and 2.3 of [3], with Lemma 2.1, give modularity of the actual weight-two member and hence of the system. □

19.8 The supported residual base at five

Proposition 19.13. *Assume GRH. Every bounded system is modular. Every absolutely irreducible $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ of determinant ω_5, unramified outside five, is modular.*

Proof. For the two-special menu, Propositions 19.6, 19.7, 19.8 and Theorem 19.12 exhaust the dyadic reductions. Consider now any bounded system in characteristic three. The preceding triadic results handle reducible and cyclotomic-dihedral residues. For an irreducible solvable residue not of the latter form, solvable Artin modularity and weight raising supply a weight-zero witness, and [3, Theorem 2.3] applies. Otherwise the residue remains absolutely irreducible on $K(\zeta_3)$. At both five-places prescribe the genuine special lifts of Lemma 17.17, and retain the given crystalline weight-two component at the unique triadic place. The prescribed-component theorem [28, Theorem 4.2.1] and the weight-two construction in Lemma 2.4 give a pure compatible replacement. A point on the special component has Frobenius eigenvalue ratio five; with zero monodromy this would contradict purity. Thus the replacement is special at both places and modular. A common modular residual witness lifts the actual old weight-two triadic member by [3, Theorem 2.3].

Finally, an irreducible solvable supported five-residue has a regular cuspidal witness by solvable Artin modularity. In the nonsolvable case its restriction to $K(\zeta_5)$ is irreducible. The EGPZ exceptional case cannot occur: $[K(\zeta_5) : K] = 4$, whereas the abelianization of $\mathrm{PGL}_2(\mathbf{F}_5)$ has order two. Use Lemma 17.16 at the split five-places to fix bounded weight-two lifts of the actual local representations. EGPZ gives a global lift, unramified off five, and Lemma 2.4 gives a bounded strongly compatible system. Its modularity proves the supported residual base by reduction. □

20 Exceptional lifts and sources over $\mathbf{Q}(\sqrt{5})$

Throughout this section put $K = \mathbf{Q}(\sqrt{5})$. The prime eleven supplies both a bounded residual base and the wild reset used in the common conductor reduction. The initial characteristic-five exception is constructed before any actual-source modularity theorem is applied.

20.1 Arithmetic and terminal characters

Write

$$\phi = (1 + \sqrt{5})/2, \quad \pi = 3 + \phi, \quad \pi' = 4 - \phi, \qquad \mathfrak{p} = (\pi), \quad \mathfrak{p}' = (\pi').$$

Then $\pi\pi' = 11$, $N(\phi) = -1$, and $N(\pi) = 11$. Minkowski's bound $\sqrt{5}/2 < 2$ gives class number one; the norm-minus-one unit gives narrow class number one. The units are $\{\pm\phi^n : n \in \mathbf{Z}\}$. Two and three are inert, five ramifies, and eleven splits. The completion K_2 is unramified quadratic over $\mathbf{Q}_2$.

Proposition 20.1. *Let $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_{11})$ be unramified outside eleven, with determinant $\bar{\chi}_{11}$ and reducible semisimplification. The ratio of its two constituent characters is the restriction of*

$$\bar{\chi}_{11}^{e}, \qquad e \in \{1, 3, 5, 7, 9\},$$

up to inversion. In particular it extends to $G_{\mathbf{Q}}$ and is nontrivial at both eleven-adic places.

Proof. Write $\bar{r}^{\mathrm{ss}} = \psi \oplus \bar{\chi}_{11}\psi^{-1}$. A finite characteristic-eleven character has order prime to eleven. Wild inertia at eleven is therefore killed. At the two primes above eleven, ϕ has residues $8 = 2^3$ and $4 = 2^2$ in $\mathbf{F}_{11}^{\times}$, where 2 is a generator. The signs of ϕ at the two real places are $(+,-)$.

Write $a, b \in \mathbf{Z}/10\mathbf{Z}$ for the two tame character exponents and $u, v \in \mathbf{Z}/2\mathbf{Z}$ for the signs. The relations imposed by the units -1 and ϕ are

$$a + b + u + v \equiv 0 \pmod 2, \qquad 3a + 2b + 5v \equiv 0 \pmod{10}.$$

They give $a \equiv b \pmod 5$, $v \equiv a \pmod 2$ and $u \equiv b \pmod 2$. There are exactly twenty characters. Squaring makes the two tame exponents equal and kills the signs. Comparison with a power of $\bar{\chi}_{11}^2$ leaves a character unramified at all finite and infinite places, hence trivial. Thus ψ^2 is an even power of $\bar{\chi}_{11}$, and $\bar{\chi}_{11}\psi^{-2}$ is an odd power. □

Put

$$\alpha = \phi\pi = 1 + 4\phi, \qquad \alpha' = 5 - 4\phi, \qquad \alpha\alpha' = -11.$$

The quadratic extensions $K(\sqrt{\alpha})$ and $K(\sqrt{\alpha'})$ ramify at exactly one of the two primes above eleven, respectively, and are unramified at two. Indeed $(1 + \sqrt{\alpha})/2$ satisfies $X^2 - X - \phi = 0$; its reduction at two is irreducible over $\mathbf{F}_4$ because $\mathrm{Tr}_{\mathbf{F}_4/\mathbf{F}_2}(\phi) = 1$. The same holds for α'. These two characters permit independent quadratic corrections at eleven without destroying good reduction at two. Their product cuts out

$$H = K(\sqrt{-11}),$$

which is split at two: K_2 contains $\sqrt{5}$, and $-11/5$ is a square in $\mathbf{Q}_2$ since it is 1 modulo eight.

We use Lemma 2.4 for prescribed lifts and compatible systems; the exceptional initial characteristic-five point is constructed separately in Section 20.2.

The notation F_b, J_b, B_b, C_b and the real-place correction are as in Section 22. We use the explicit formula (39), including its finite-tail bound and retained prime densities. The unconditional comparison below uses the positive kernel $F_b(x)/\cosh(x/2)$.

20.2 Starting an exceptional characteristic-five representation

The ring-theoretic theorem of Khare–Thorne gives the finiteness needed below before a characteristic-zero global point has been found. We use its published nilpotent-kernel statement, rather than its finite-length approximation corollary.

Corollary 20.2 (A classical quotient of the prepared KT ring)**.** *In the exact setup of [38, §7], assume the fixed set V_2 of $D^{\mathrm{St(uni)}}$ places is nonempty. The quotient $R_{\mathcal{S}}^{\mathrm{cl}}$ of the global deformation ring at the classical weight-two augmentation of its ordinary Iwasawa algebra Λ is finite over $\mathcal{O}$.*

Proof. In the notation of [38, §7], let $\mathbb{T}$ be the localized ordinary Hecke algebra at infinite ordinary level. The paragraph preceding their Theorem 7.2 makes $\mathbb{T}$ finite over Λ. Theorem 7.2 in the final version of [38] gives a surjection

$$R_{\mathcal{S}} \longrightarrow \mathbb{T}$$

with nilpotent kernel I, since $V_2 \neq \varnothing$. We use the final-version numbering of that reference. The complete deformation ring $R_{\mathcal{S}}$ is Noetherian. Thus each I^j/I^{j+1} is a finite module over

$R_{\mathcal{S}}/I = \mathbb{T}$, and hence finite over Λ. The finite filtration by powers of I shows that $R_{\mathcal{S}}$ is finite over Λ. Tensoring along the classical augmentation $\Lambda \to \mathcal{O}$ gives the claimed finiteness. This argument includes the entire deformation ring, including possible characteristic-five components; it uses no assumed characteristic-zero point or flatness of $R_{\mathcal{S}}$. □

Proposition 20.3 (An initial exceptional lift)**.** *Let $K = \mathbf{Q}(\sqrt{5})$, and let $\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ be continuous, absolutely irreducible and totally odd. Assume that its projective image is $\mathrm{PGL}_2(\mathbf{F}_5)$ and that its projective kernel field contains $K(\zeta_5)$. Put*

$$\nu = [\det \bar{r}\, \bar{\chi}_5^{-1}],$$

where brackets denote the Teichmuller lift. There is a continuous lift $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_5)$ with determinant $\nu\chi_5$, ramified at finitely many places and de Rham of parallel weight two. It is totally odd. The lift is potentially modular over totally real extensions with avoidance of any prescribed finite extension, and belongs to a strongly compatible system.

Proof. Fixed local rings over K. Let S contain the places above 5 and the ramification of $\bar{r}$. At each $v \in S$, choose a definite-type local lift by [64, Propositions 7.7.1 and 7.8.1], weight two when $v \mid 5$. The determinant correction of [64, §7.9] twists by the square root of a finite character reducing to 1; hence it does not change the residual representation or Hodge–Tate weights and gives determinant $\nu\chi_5$. Choose a corresponding nonzero, reduced, $\mathcal{O}$-flat local fixed-type quotient $R_v^{\dagger}$ from [64, Propositions 7.3.1 and 7.4.1]. Its framed dimension is 4 for $v \nmid 5$, and $4 + [K_v : \mathbf{Q}_5]$ for $v \mid 5$. At 5 the type may be A, B or C; we do not require a potentially Barsotti–Tate lift when only type C was chosen.

Fix finite Galois local extensions killing these fixed inertial types, the local residual representations, and their finite determinant factors. After these extensions, the local characteristic-zero points are unramified or in the Steinberg closure away from 5; at 5 they are ordinary semistable for types A, C, or nonordinary Barsotti–Tate for type B. A single finite extension works for every point of each chosen local ring: the finite inertial type is fixed, while its possible monodromy is retained. Let K_ν be the field cut out by ν. It is totally real since ν is totally even.

Preparation and the modular residual witness. Real approximation and Krasner's lemma, followed by taking a totally real Galois closure, supply a totally real finite extension containing K_ν, for which every completion at a controlled place contains the fixed local extension. This Galois closure is taken before constructing any modular witness. Apply the residual potential-modularity theorem [64, Theorem 5.1.1] over this field. This theorem is for any odd residual representation and has no exceptional-image exclusion. It gives a weight-two modular residual witness after a further totally real extension.

Let R now denote the set of places away from 5 at which the restrictions of the original chosen local rings have Steinberg type. Let σ denote the places above 5 lying over a local type A or C. Apply [38, Lemma 9.2] to the known modular witness, this R, and this σ. After a soluble totally real extension it supplies a weight-two witness π that is ordinary and Iwahori at σ, nonordinary and spherical at the other places above 5, Steinberg at the pullback of R, and unramified at every other finite place. Thus incidental bad primes of the first potential-modularity witness have disappeared. The residual representation is trivial at the places above 5 and R. The already imposed original local conditions remain of the stated type under all these restrictions.

Finally make a totally real quadratic extension split at the current places above 5 and R, and base change π. Denote the resulting field by F. Then $[F : K]$ and $\#R_F$ are even. Its central character is everywhere unramified. In the dual normalization its ratio with the required Teichmuller character $\psi = 1$ is a finite unramified character of 5-power order; twisting by the inverse square root makes the central character exactly 1, without changing residues or local types. The residual representation over F is unramified away from 5 and trivial at 5 and R_F.

A fresh fixed unipotent prime. Choose the auxiliary prime only after F has been fixed. Let M/K be the projective kernel field and let F^{g}/K be the Galois closure of F/K. Since F^{g} is totally real, $M \cap F^{\mathrm{g}} = K$: the only proper nontrivial normal subextension of the S_5-extension M/K is its imaginary quadratic subfield $K(\zeta_5)$, and M itself is not totally real. Let L/K be the full linear residual kernel field, put $G = \mathrm{Gal}(L/K)$, let Z be its scalar subgroup, and set $N = \mathrm{Gal}(L/(L \cap F^{\mathrm{g}}))$. Projective disjointness gives $ZN = G$, so G/N is a quotient of the prime-to-five group Z. Every element of order five in G therefore lies in N and is an actual nontrivial unipotent matrix. It is trivial on $K(\zeta_5)$.

Chebotarev therefore gives a prime $q \notin S$, split completely in F^{g}, with actual residual Frobenius a nontrivial unipotent of order 5. Then $\mathrm{N}q \equiv 1 \pmod 5$ and $\nu(q) = 1$. Impose $D^{\mathrm{St(uni)}}$ at q on the original K-deformation problem, as well as at every place above q on its F-restriction. This is a nonzero formally smooth local problem of framed dimension 4 by [38, Proposition 6.6]. Write Q_F for the places above q. Its cardinality is $[F : K]$, hence even and positive. Take $V_1 = \varnothing$ and $V_2 = Q_F$ in [38, §7]. No congruence modulo 25 is needed for this fixed local problem.

Choose the good-subgroup auxiliary place a using [38, Lemma 7.1], away from the places above 5, R_F and Q_F. Apply [38, Lemma 5.2 and the proof of Theorem 9.1] to π at the even set Q_F. The ordinary/nonordinary variant of [68, Lemma 4.13] preserves the local conditions at 5 and R_F, makes every place of Q_F Steinberg, and leaves the other finite places unramified. Its degeneracy maps and pairings commute with the Hecke operators at 5. The place a is used in this argument and adds no ramification to the deformation problem. An unramified five-power square-root twist, if necessary, restores central character one. This is the witness required in [38, §7], with $V_1 = \varnothing$ and $V_2 = Q_F$.

Restriction at the level of local rings. Let $R^{\dagger}_{K,q}$ be the global fixed-determinant deformation ring over K, with the original local quotients $R^{\dagger}_v$, the new $D^{\mathrm{St(uni)}}$ condition at q, and unramifiedness outside $S \cup \{q\}$. Let R^{cl}_F be the classical weight-two quotient of the prepared KT ring just constructed. There is a restriction homomorphism

$$R^{\mathrm{cl}}_F \longrightarrow R^{\dagger}_{K,q}.$$

Here it is essential to verify the type conditions before passing to the global ring. On each original reduced, $\mathcal{O}$-flat local ring $R^{\dagger}_v$, restriction kills the defining ideal of the corresponding prepared local condition: this holds on all finite characteristic-zero points by the choice of the fixed local extensions, and these points are dense in the reduced flat local generic fiber. Thus it holds on the whole local ring, as in [64, Lemma 6.4.1].

The ordinary local ring is the reduced, $\mathcal{O}$-torsion-free image quotient of the framed ring obtained from the flag scheme; it forgets the flag [68, §5.4.1 and Proposition 5.13]. Its classical characteristic-zero points require an ordinary line with trivial inertial character, without marking a particular line. After preparation and dualization each type A or C point has such a line. The same density argument therefore gives the ordinary quotient-ring map on the original local ring. Type B maps to the nonordinary Barsotti–Tate problem. Away from 5, finite-inertia type maps to the unramified problem and Steinberg type to its Steinberg closure. The determinant is identical, and the unramified sign at residual-trivial Steinberg places is forced by the residue. At the new q the local fields are unchanged because q splits in F, so the local restriction map preserves $D^{\mathrm{St(uni)}}$ directly. The resulting local ring homomorphisms remain valid after tensoring with any global ring, including one with 5-torsion. No density of global characteristic-zero points has been assumed.

The projective image over F remains S_5, so the residual restriction is absolutely irreducible. Restriction is finite by [64, Lemma 6.3.1]. Passing to the local quotients preserves finiteness, since their defining ideals have already been shown to vanish. Corollary 20.2 therefore makes $R^{\dagger}_{K,q}$ finite over $\mathcal{O}$.

A characteristic-zero point. Put $n = \#(S \cup \{q\})$. The completed tensor product of the local rings has dimension $1 + [K : \mathbf{Q}] + 3n$. The framed presentation in the proof of [64, Proposition 6.2.1] uses $r + n - 1$ variables and $r + [K : \mathbf{Q}]$ relations, so its global quotient has dimension at least $4n$. Forgetting the frames has relative dimension $4n - 1$; hence $\dim R^{\dagger}_{K,q} \geq 1$. The invariant vanishing used in that calculation follows from absolute irreducibility over $K(\zeta_5)$ and Schur's lemma; the exceptional-image exclusion is not used. A finite $\mathcal{O}$-algebra of positive dimension has nonzero generic fibre. A closed generic point gives the required weight-two lift, and its fixed determinant makes it totally odd.

Actual potential modularity and the compatible system. For any prescribed finite extension B/K, apply the avoiding version of [64, Theorem 5.1.1] to $\bar{r}$, including the residual and cyclotomic kernel fields in the avoidance data. It gives a totally real Galois field K'/K, disjoint from the prescribed avoidance field, and a weight-two modular residual witness over K'. The actual lift $r|_{G_{K'}}$ is finitely ramified and weight two, and its residual restriction to $G_{K'(\zeta_5)}$ is absolutely irreducible. Apply [38, Theorem 1.1], after dualizing, to obtain the automorphy of this actual lift.

The proof of [64, Proposition 9.2.1 and Remark 9.2.2], using solvable descent and Brauer induction, now constructs the strongly compatible system. In this use the proposition's old exceptional exclusion is replaced in its proof by the actual avoiding potential modularity just proved; the remaining argument uses only that input and absolute irreducibility, not an adequacy hypothesis. In particular this is an application of that proof with its modularity input supplied by KT, not an invocation of the original proposition outside its stated hypotheses. □

Remark 20.4 (Scope). This construction permits the extra prime q in the initial lift. It asserts no minimal conductor or prescribed minimal weight theorem, and does not assume residual modularity over K. The auxiliary prime is imposed on the original deformation problem as well as the prepared one: restricting an arbitrary unramified deformation cannot create a new Steinberg eigenvalue relation. Its use of Theorem 7.2 is a finiteness argument with a fixed auxiliary prime; the varying-prime approximate level lowering in KT §9 is not needed.

20.3 A one-prime source obstruction

Proposition 20.5. *Let $\bar{r} : G_K \to \mathrm{SL}_2(k)$ be finite flat at two, unramified outside $2\mathfrak{p}$, with inertia of order at most two at $\mathfrak{p}$, where k is finite of characteristic two. Then $\bar{r}^{\mathrm{ss}} = 1 \oplus 1$. This assertion is unconditional.*

Proof. Fontaine's finite-flat bound over the unramified dyadic base gives classical upper breaks at most one and normalized different $d_2 < 2$ [31, §2.1, Theorem 1, and §1.2(i)]. The latter reference converts his upper numbering to the classical one. At eleven the ramification is tame of index at most two at just one of the two places. A nonsolvable image is $\mathrm{SL}_2(\mathbf{F}_q)$, $q \geq 4$, so its kernel field has degree at least 120 and

$$\mathrm{rd}(M) < 4\sqrt{5}\,11^{1/4} < 16.28896.$$

With $b = 7$ and $[M : \mathbf{Q}] \geq 120$, (40) gives $\mathrm{rd}(M) > 16.661$, excluding the nonsolvable case. The exact calculation is supplied in [49, `d5/new`].

An irreducible solvable image is odd dihedral, induced from an odd-order character θ of a quadratic field L/K. Its rotation character is unramified at eleven and tame at two. The quadratic quotient has dyadic conductor at most two; an odd dyadic valuation would have conductor three. Thus L belongs to the seven nontrivial classes in $\langle -1, \phi, \pi \rangle$. The following

table gives the fields up to rational conjugation.

a	$\lvert D_{K(\sqrt{a})}\rvert$	r_2	Minkowski bound
-1	400	2	3.1
ϕ	400	1	2.4
π	4400	0	6.3
$-\pi$	4400	2	10.2
$\phi\pi$	275	1	2
$-\phi\pi$	4400	1	8

The bound in the fifth row is strict. Integral bases are $1, \phi, t, \phi t$ with $t^2 = a$, except in that row, where $t^2 - t = \phi$. The relative discriminants at odd primes follow from Eisenstein; at two maximality follows by testing unit squares modulo four. The first, second and fifth rows have no prime ideal below their Minkowski bounds, hence class number one. In the π row the possible generators have norms four or five; the dyadic class has square (2) and $N(t-1) = 5$ makes the norm-five classes principal. In the $-\pi$ row the identity $N(t - \phi) = 20$ makes the norm-five classes equal to the dyadic class, and three has residue degree four. In the last row only the norm-four prime occurs; its square is principal. Thus all these class groups have two-power order.

If L/K is ramified at two, its involution acts trivially on the residue field. Anti-invariance makes the odd tame inertia character trivial. It is then everywhere unramified, hence trivial by the preceding class groups. The only nontrivial class in this list unramified at two is $\alpha = \phi\pi$. For $L = K(\sqrt{\alpha})$ the integral unit $t = (1 + \sqrt{\alpha})/2$ reduces to a root of $X^4 + X + 1$ and has order fifteen in $\mathbf{F}_{16}^{\times}$. It kills the entire odd tame ray group at two. This also makes θ trivial, a contradiction.

Finally, in the reducible case the odd-order constituent characters are unramified at eleven. At two their inertia factors through $\mathbf{F}_4^{\times}$, generated by the image of ϕ. Class number one therefore makes both constituents trivial. □

Put $R = \mathcal{O}_K[1/\pi]$ and $F = K_2$.

Lemma 20.6. *We have* $\mathrm{Ext}^1_R(\mu_2, \mathbf{Z}/2\mathbf{Z}) = 0$.

Proof. Over the completion at two, an extension of μ_2 by $\mathbf{Z}/2\mathbf{Z}$ splits by the connected–etale sequence. Its middle term is therefore killed by two locally, hence globally, by flatness. Its generic extension class lies in $H^1(K, \mathbf{F}_2) = K^{\times}/K^{\times 2}$, is unramified away from the inverted primes and two, and is a square in F.

For R, the global Kummer space has basis $-1, \phi, 2, \pi$. Its localization in $F^{\times}/F^{\times 2}$ is injective. An odd power of 2 is excluded by valuation. Taking the norm to $\mathbf{Q}_2$ then excludes nonzero exponents of ϕ and π: the three possible nontrivial norm classes are $-1, 11, -11$, all nonsquare in $\mathbf{Q}_2$. Finally -1 is nonsquare in F. Thus the generic extension is split.

A generically split extension is split over R. Of its two generic splittings, one agrees with the local splitting at two; away from two both group schemes are etale and any generic splitting extends. These splittings glue. Equivalently this is the finite-flat gluing sequence of [55, Proposition 2.3 and Corollary 2.4].

□

Lemma 20.7. *Let G be a 2-divisible group over R. Suppose that every simple finite flat subquotient of every $G[2^n]$ has trivial generic Galois action. Then G has a multiplicative 2-divisible subgroup with etale quotient.*

Proof. A simple generic factor has order two. Schematic closure realizes a generic composition series as a finite flat composition series. At the unique dyadic prime, the prime-order classification over an unramified ring gives either the etale or the multiplicative model. Away from two the model is etale and its generic action is trivial. Since the generic group of order two has no nontrivial automorphism, gluing gives exactly $\mathbf{Z}/2\mathbf{Z}$ or μ_2 over R.

By Lemma 20.6, adjacent factors in the order $\mathbf{Z}/2\mathbf{Z}$ below μ_2 can be interchanged: the relevant extension splits. Repeating this operation orders the filtration with all multiplicative factors below all etale factors. Extensions of multiplicative groups are multiplicative, by Cartier duality, and extensions of etale groups are etale. Thus each $G[2^n]$ has a multiplicative subgroup with etale quotient.

This subgroup is unique. Its localization at two is the connected part, since its quotient is etale; its global subgroup is the flat closure of that generic subgroup. Consequently it is functorial in $G[2^n]$ and compatible with the transition and multiplication maps. The local connected part is multiplicative at every level, so these subgroups form the multiplicative 2-divisible part, with etale quotient. This is also the filtration argument used in finite-flat approaches to small-conductor abelian varieties [55, 56]. □

Theorem 20.8. *Let $E_0/\mathbf{Q}_2$ be finite and let $r : G_K \to \mathrm{GL}_2(E_0)$ be continuous. Suppose that $\det r = \chi_2$, that r is Barsotti–Tate at two, unramified outside $2\mathfrak{p}$, and semistable at $\mathfrak{p}$ with unipotent inertia. Then r fits into an exact sequence*

$$0 \longrightarrow E_0(1) \longrightarrow r \longrightarrow E_0 \longrightarrow 0.$$

In particular there is no absolutely irreducible such representation. The same assertion holds with $\mathfrak{p}'$ in place of $\mathfrak{p}$. No GRH assumption is needed.

Proof of Theorem 20.8. Choose a global stable $\mathcal{O}_{E_0}$-lattice. The local Barsotti–Tate condition realizes such a lattice after an isogeny: taking the closure of a finite generic subgroup realizes any commensurable stable lattice. The integral coefficient action extends by full faithfulness for Tate modules of 2-divisible groups. Glue this local group to the etale group defined by the lattice away from two. Levelwise finite-flat gluing [55, Proposition 2.3] gives a 2-divisible group over R with $\mathcal{O}_{E_0}$ action.

The reduction modulo the maximal ideal of $\mathcal{O}_{E_0}$ satisfies the one-prime hypotheses of Proposition 20.5. At eleven, unipotent inertia factors through the procyclic tame pro-two quotient. Its reduction is cyclic of exponent two, hence has order at most two. Its semisimplification is trivial. Filtering by the coefficient uniformizer and then over $\mathbf{F}_2$ shows that every simple generic factor of every $G[2^n]$ is trivial. Lemma 20.7 supplies a functorial multiplicative subgroup.

The height of G is $2[E_0 : \mathbf{Q}_2]$, and its dimension is $[E_0 : \mathbf{Q}_2]$, because its labelled Hodge–Tate pairs are $\{0, 1\}$. The multiplicative part has height $[E_0 : \mathbf{Q}_2]$. Functoriality makes its rational Tate module E_0-stable, hence an E_0-line. The quotient character is unramified at two, and also at eleven because inertia in r is unipotent. It is unramified at every other finite prime by hypothesis. Class field theory and narrow class number one make this character trivial. The determinant identifies the subcharacter with χ_2. □

20.4 Two cuspidal places above eleven

We use one finite-flat ordinarity observation repeatedly. A nonordinary rank-two Barsotti–Tate source of cyclotomic determinant is biconnected up to isogeny. Its coefficient-uniformizer torsion and finite-flat subquotients are then biconnected. Over an unramified dyadic ring, a prime-order model with trivial generic action is $\mathbf{Z}/2\mathbf{Z}$ or μ_2. For odd p and absolute ramification index $e < p - 1$, Raynaud's uniqueness theorem similarly identifies the models of 1 and ω_p with

the etale and multiplicative models [53, Theorem 3.3.3 and Corollary 3.3.6]. Neither model is biconnected. Their occurrence as simple factors therefore forces the actual source to be ordinary.

Proposition 20.9. *Let $K = \mathbf{Q}(\sqrt{5})$ and let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible, weight-two strongly compatible system of determinant cyclotomic, good outside eleven. Suppose its two eleven-adic types are determinant-one tame cuspidal types with inertial eigencharacters of order three or six. If a dyadic residue has solvable image, the actual system is modular.*

Proof. Modulo two the scalar quadratic factors disappear. At either eleven-adic place tame inertia has eigencharacters of order three, and Frobenius interchanges them because $11 \equiv -1 \pmod 3$. Thus the full local residue is S_3, and the global residue is absolutely irreducible. A solvable irreducible determinant-one subgroup of $\mathrm{GL}_2(\overline{\mathbf{F}}_2)$ is odd dihedral. Let L/K be its inducing quadratic extension.

The quadratic quotient is unramified at eleven: its local inertia is the odd rotation subgroup. It is unramified outside two, and Fontaine's finite-flat upper bound one over the unramified dyadic completion gives dyadic conductor at most two. Since K has class number one and fundamental unit $\phi = (1 + \sqrt{5})/2$, its supported quadratic squareclasses are generated by $-1, \phi, 2$. An odd dyadic valuation gives conductor three and is excluded. The possibilities are consequently $-1, \phi, -\phi$.

At the two eleven-primes, the values of ϕ are $8, 4$. The first is nonsquare and the second square in $\mathbf{F}_{11}$; for $-\phi$ they are $3, 7$, respectively square and nonsquare. Both cuspidal places require the quadratic quotient to be nontrivial unramified. Only -1 is nonsquare at both. Hence $L = K(i)$.

At two this quadratic extension is ramified. Its nontrivial wild inertia image is a reflection of the odd dihedral group. The normalizer of a reflection subgroup is that subgroup itself. Since wild inertia is normal in the decomposition group, the whole dyadic decomposition image is C_2. In particular the local residual semisimplification is trivial.

Filtering the coefficient-uniformizer torsion over $\mathbf{F}_2$ gives a simple finite-flat factor of order two with trivial generic action. The preceding observation makes the actual dyadic source ordinary.

The main theorem of [2, Theorem 5.2.1] applies: the source is ordinary and totally odd, the residue is absolutely irreducible and solvable, and its CM inducing quadratic extension does not split at the dyadic place. Equivalently combine his Lemma 5.1.2 with Theorem 5.2.1. This proves modularity of the source and hence of the system. □

Lemma 20.10. *Let $\mathcal{R}$ be a determinant-cyclotomic weight-two system over K, good outside eleven, with local types at eleven special quadratic or tame cuspidal of order three or six. If an associated three-adic residue is globally reducible, the actual three-adic source is modular. If it is globally irreducible, its restriction to $G_{K(\zeta_3)}$ remains absolutely irreducible.*

Proof. The quadratic characters ramified at just one eleven-prime, constructed from $\alpha = 1 + 4\phi$ and its conjugate, are unramified at three. Twist away the scalar quadratic inertial signs at both eleven-primes. Modulo three the remaining inertia is unipotent, for either local type. This twist preserves the determinant, the weights and crystallinity at three.

If the residue is reducible, its characters are unramified at eleven. They factor through the narrow ray modulo three, whose order is two: the local/sign group has order $8 \cdot 4 = 32$, whereas the images of ϕ and -1 generate order sixteen. Indeed ϕ has order eight modulo three, signs $(+, -)$, and -1 is independent because its first sign is negative. The unique nontrivial character is ω_3. The constituents are therefore $1, \omega_3$.

At three the absolute ramification index is $1 < 3 - 1$. The finite-flat observation makes the actual Barsotti–Tate source ordinary. Skinner–Wiles [62, §4.5, Theorem A] now applies. The

quotient splitting field $K(\zeta_3)$ is abelian over $\mathbf{Q}$, its quotient character is totally odd and locally nontrivial at three, the ordinary quotient is unramified with residual character one, and the determinant is χ_3.

If the cyclotomic restriction of an irreducible residue were reducible, the residue would be induced from a prime-to-three character θ of $H = K(\zeta_3)$. The unipotent inertia at eleven has order dividing three, while the inducing characters have prime-to-three order; it is therefore trivial on G_H. Thus θ is unramified outside three and tame at three. The absolute discriminant of H is $5(-3)(-15) = 225$. Its degree-four Minkowski bound is $22.5/\pi^2 < 2.28$. It has no norm-two prime, since two has residue degree two. Consequently its class number is one. There is one prime above three, of residue field $\mathbf{F}_9$, whose full unit group is generated by the image of ϕ. The ray group for this prime is therefore trivial. This makes θ trivial, contrary to irreducibility of its induction. □

Lemma 20.11. *Let $r : G_K \to \mathrm{GL}_2(E)$ be totally odd and absolutely irreducible, of determinant χ_5, Barsotti–Tate above five, and unramified outside five and eleven. Suppose its residue is globally reducible with semisimple inertia trivial at both eleven-primes. Then the actual source r is modular.*

Proof. The narrow ray modulo the prime above five is C_2, so the residual constituents are $1, \omega_5$. The source is ordinary by the finite-flat argument at absolute ramification index $2 < 5 - 1$. Apply Skinner–Wiles [62, §4.5, Theorem A]. Its quotient field $K(\omega_5) = \mathbf{Q}(\zeta_5)$ is abelian over $\mathbf{Q}$; the quotient is totally odd and locally nontrivial at five. In the ordinary filtration the quotient has weight zero, is unramified, and reduces to one, so its normalized character factors through a pro-five group and has finite inertial order. The determinant gives $k = 2$. The theorem imposes no requirement that five be unramified in K. □

Theorem 20.12. *Assume GRH for Dedekind zeta functions. Let $K = \mathbf{Q}(\sqrt{5})$, and let $\mathcal{R}$ be a pure, totally odd, absolutely irreducible strongly compatible rank-two system of determinant cyclotomic and labelled weights $\{0, 1\}$. Suppose it is good outside eleven and at both eleven-primes has a tame determinant-one cuspidal type with inertial eigencharacters of order three or six. Then the actual system $\mathcal{R}$ is modular.*

Proof. Choose a coefficient place above two and let M/K be the kernel field of its residual representation. At each eleven-prime, the quadratic scalar sign disappears modulo two and tame inertia has order three. Frobenius interchanges the two eigenlines since $11 \equiv -1 \pmod 3$. Its square is one modulo two, from determinant eleven and characteristic-zero trace zero. The full local residue is consequently S_3, so the global residue is absolutely irreducible.

If its image is solvable, Proposition 20.9 proves modularity. We may therefore suppose the image is $G = \mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^f \geq 4$.

We construct a solvable auxiliary extension. Put $H = K(i)$. Its absolute discriminant is $5(-4)(-20) = 400$, and its Minkowski bound is $30/\pi^2 < 3.04$. It has no prime of norm two or three: their residue degrees are two. Thus its class number is one. Its units are generated by i and $\phi = (1 + \sqrt{5})/2$. Indeed the norm to K of a unit is a totally positive unit ϕ^{2m}; division by ϕ^m gives a unit all of whose complex absolute values equal one, hence a root of unity. The roots of unity in H are precisely μ_4.

The two eleven-primes of K are inert in H/K, with residue fields $\mathbf{F}_{121}$. The elementary three quotient of the ray group of H for their product is C_3^2. To see this, each residue multiplicative group has order 120 and contributes one C_3. The images of ϕ lie in $\mathbf{F}_{11}^\times$, and i has order four; neither unit has a nontrivial three projection. Class number one gives the assertion.

Let B/H correspond to this maximal exponent-three ray quotient. Complex conjugation over K acts by the eleventh-power map on both residue fields, hence by inversion on C_3^2. Thus

B/K is Galois with group $C_3^2 \rtimes C_2$ and $[B : \mathbf{Q}] = 36$. At two, B/H is unramified while H/K is ramified quadratic. The inertia group in B/K is therefore a reflection C_2. Its normalizer in $C_3^2 \rtimes C_2$ is itself, so the full dyadic decomposition group is C_2. The local field is $K_2(i)$ and has upper break one.

At either eleven-prime the ray inertia is C_3. Frobenius maps nontrivially to $\mathrm{Gal}(H/K)$, and every element outside C_3^2 has order two. Since the decomposition quotient by inertia is cyclic and generated by Frobenius, its order is two. Both local fields are therefore the tame S_3 extension of $\mathbf{Q}_{11}$ with $(e, f) = (3, 2)$. This extension is unique: the unramified quadratic extension contains μ_3, and the unit classes defining Galois-stable tame cubic extensions give the unique extension because $3 \nmid 10$.

Since G is nonabelian simple, $M \cap B = K$. Write $L = BM$. Its degree is $36q(q^2 - 1)$. Both local fields above eleven remain the same tame S_3 field, giving

$$\frac{2N_{121}(L)}{[L : \mathbf{Q}]} = \frac{1}{3}, \qquad d_{11} = \frac{2}{3}.$$

At two the residual Barsotti–Tate point field has no upper break exceeding one by Fontaine's bound over the unramified base. The same is true of B, so it is true of their compositum. In particular $d_2 < 2$ and

$$\mathrm{rd}(L) < 4\sqrt{5}\, 11^{2/3}.$$

For $q = 16$ we use a slightly sharper bound. If wild inertia in M is nontrivial, it fixes a unique line. The diagonal characters of the decomposition group are then characters of G_{K_2}; their odd-order inertial part factors through $\mathbf{F}_4^\times$. The tame inertia has order at most three and the wild inertia has order at most q. After adjoining B, write w for the wild inertia order and e for the inertia order. We have $w \le 2q$ and $e \le 3w$. The upper-filtration formula, using the break bound one, gives

$$d_2 \le (1 - 1/e) + (1 - 1/w) \le 2 - \frac{2}{3q}.$$

If wild inertia in M is trivial, the compositum has wild inertia of order at most two, so the same formula gives $d_2 < 3/2$, which is smaller. This case distinction matters: a tame irreducible residue over K_2 may have inertia of order five, rather than three. Consequently

$$\mathrm{rd}(BM) \le \sqrt{5}\, 2^{2-2/(3q)} 11^{2/3}.$$

Apply (39) with $B_{b,100}$ and the norm-121 prime density $1/3$, retaining all prime powers in the support. Directed 256-bit Arb evaluation gives

case	$n \ge$	b	lower root discriminant	upper root discriminant
$q = 16$	146880	11.6	43.50979406	42.97974824
$q \ge 32$	1178496	13.6	45.77092571	44.23915099
$q = 8,\ M \ne M^\sigma$	9144576	16.4	47.10533308	44.23915099

Here σ denotes the nontrivial automorphism of $K/\mathbf{Q}$. The first two lines exclude every $q \ge 16$. At fixed b and prime density the lower expression increases with the degree, so the second line covers the entire unbounded range. The exact calculation is supplied in [49, `d5`].

For the third line use BMM^σ. If $M \ne M^\sigma$, simplicity makes the two fields disjoint over K, and their product has no solvable quotient. It is also disjoint from B, so its degree is $36 \cdot 504^2$. The compositum retains upper dyadic break at most one and the same local S_3 fields at eleven. This proves the third line's applicability and excludes that case.

If $q = 8$ and $M = M^\sigma$, then $M/\mathbf{Q}$ is Galois. Put $\Gamma = \mathrm{Gal}(M/\mathbf{Q})$ and $G = \mathrm{Gal}(M/K) = \mathrm{SL}_2(8)$. Conjugation maps Γ to $\mathrm{Aut}(G)$. The induced map $\Gamma/G \to \mathrm{Out}(G)$ is trivial, since

its source has order two and its target order three. As G is centerless, this gives a retraction $\Gamma \to \mathrm{Inn}(G) = G$. Composing with the given $G \hookrightarrow \mathrm{SL}_2(\mathbf{F}_8)$ extends the actual residual representation to $G_{\mathbf{Q}}$. Rational Serre modularity [40, 41] and quadratic base change give a cuspidal modular witness over K; its restriction is irreducible, so the base change is cuspidal. For $q = 4$ use [3, Theorem 2.5].

Finally some real complex conjugation acts nontrivially in the residue. Otherwise M would be totally real of degree at least 120 and root discriminant below $4\sqrt{5}\,11^{2/3}$. The same explicit formula at $n = 120$, $b = 4.8$, retaining the real term $J_b(1/2) - J_b(3/2) + J_b(5/2) - J_b(7/2)$ and no prime terms, gives $\mathrm{rd}(M) > 56.06306031$, a contradiction. Apply [69, Theorem 6.1] to the dual of the actual crystalline two-adic source, in his Hodge–Tate convention: its residual image is nonsolvable, its residue is modular, and the required real-place condition holds. This proves modularity of the source and hence of $\mathcal{R}$. □

20.5 The supported residual base at eleven

Fixed-determinant positivity. We use the following observation for both eleven-adic menus. Let $\bar{r} : G_{\mathbf{Q}_{11}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_{11})$ have determinant ω_{11}. By [9, Propositions 2.6–2.7] it has a regular crystalline lift of determinant χ_{11}. Write its weights as $h_1 < h_2$, with $h_1 + h_2 = 1$. The corresponding algebraic compact representation is

$$\mathrm{Sym}^{h_2-h_1-1} \otimes \overset{h_1}{\det}.$$

Its scalar central character has exponent $h_1 + h_2 - 1 = 0$. The nonzero crystalline deformation ring and the fixed-determinant Breuil–Mézard formula [70, Theorem 1.2] therefore give a nonzero effective cycle for at least one Serre weight with trivial scalar central character. These cycles are independent of the Hodge type and inertial type for this fixed determinant. Consequently, in any collection of weight-two compact types containing all such Serre weights, at least one type has a nonzero lifting ring. That ring is eleven-torsion-free and has a characteristic-zero point. This argument applies to the actual residual extension.

Lemma 20.13 (The bounded eleven-adic menu)**.** *Every continuous $\bar{r} : G_{\mathbf{Q}_{11}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_{11})$ of determinant ω_{11} has a determinant-χ_{11} weight-two lift which is either potentially crystalline of principal-series type, potentially crystalline of scalar quadratic type, or a quadratic twist of a semistable representation with nonzero monodromy. Up to a quadratic twist, the principal-series eigencharacters have order five.*

Proof. Write $W(r, b) = \mathrm{Sym}^r(\mathbf{F}_{11}^2) \otimes \det^b$. The twelve weights of trivial scalar central character satisfy $0 \le r \le 10$, $0 \le b < 10$ and $r + 2b \equiv 0 \pmod{10}$. For the determinant-one principal-series type with tame characters ω^t, ω^{-t}, its compact type has constituents

t	constituents
1	$W(2,9), W(8,1)$
2	$W(4,8), W(6,2)$
3	$W(4,3), W(6,7)$
4	$W(2,4), W(8,6)$.

The trivial crystalline type and its quadratic twist contribute $W(0,0), W(0,5)$, and the semistable compact types contribute $W(10,0), W(10,5)$. These identities follow from the Brauer characters on split and nonsplit semisimple elements. They cover all twelve weights. Fixed-determinant positivity therefore gives a nonempty lifting ring for at least one condition in this list. For a scalar semistable condition we retain a crystalline point if there is one, and otherwise a component through a point with nonzero monodromy. The statement concerns the actual residual extension. The determinant is fixed in the lifting rings.

A common quadratic twist adds five to b. It interchanges the first and fourth rows, and the second and third rows. Thus the principal-series rows may be taken to be $t = 2, 4$, both of order five. The scalar alternatives similarly become untwisted after a quadratic twist. □

The global quadratic characters cut out by $K(\sqrt{\phi(3+\phi)})$ and its conjugate under $\mathrm{Gal}(K/\mathbf{Q})$ ramify at the two eleven-primes independently and are unramified at two. They therefore make the quadratic corrections in Lemma 20.13 simultaneously without destroying good reduction at two. We use them only as finite twists, which preserve the determinant and can be reversed after proving modularity.

Lemma 20.14 (The actual residual switch at eleven)**.** *Let $\bar{s} : G_{\mathbf{Q}_{11}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ have determinant ω_5. Suppose its wild inertia is trivial and its tame inertia is trivial or unipotent of order five. It has a determinant-χ_5 lift which is unramified or special with unramified quadratic scalar character. In the nontrivial-inertia case the special lift can be chosen with nonzero monodromy.*

Proof. If inertia is trivial, lift the full residual Frobenius matrix unramifiedly and correct its determinant to eleven by a scalar square root reducing to one. This includes unequal unramified eigencharacters; no special lift is asserted in that case.

Otherwise choose a tame generator with matrix $T = 1 + tE_{12}$, $t \neq 0$. Arithmetic Frobenius Φ commutes with T, since $\Phi T \Phi^{-1} = T^{11} = T$ in characteristic five. Thus $\Phi = aI + bE_{12}$, and determinant one gives $a = \pm 1$. Choose lifts $\widetilde{t}, \widetilde{b}$ and put

$$\widetilde{\Phi} = \begin{pmatrix} 11a & \widetilde{b} \\ 0 & a \end{pmatrix}, \qquad U(x) = 1 + \widetilde{t}xE_{12}.$$

On the tame $\mathbf{Z}_5$ quotient these satisfy $\widetilde{\Phi}U(x)\widetilde{\Phi}^{-1} = U(11x)$. They have determinant cyclotomic, match the actual residue, including its Frobenius extension, and have nonzero monodromy. □

Lemma 20.15. *Let $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ be absolutely irreducible, unramified outside five and eleven. If its inertia image at both eleven-primes has order dividing five, then $\bar{r}|_{G_{K(\zeta_5)}}$ is absolutely irreducible.*

Proof. Otherwise Clifford theory gives induction from a prime-to-five character θ of $H = K(\zeta_5) = \mathbf{Q}(\zeta_5)$. The character is unramified at eleven, since its inertia there has both prime-to-five order and order dividing five. It is unramified away from five, and tame at the unique prime $\mathfrak{P}$ above five. The discriminant of H is 125, and its degree-four Minkowski bound is $3\sqrt{125}/(2\pi^2) < 2$, so its class number is one. The image of ϕ modulo $\mathfrak{P}$ is 3, which generates $\mathbf{F}_5^\times$. The ray group modulo $\mathfrak{P}$ is therefore trivial. This makes θ trivial, contrary to the irreducibility of its induction. □

Proposition 20.16. *Assume GRH. Let $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ have determinant $\bar{\chi}_5$, be finite flat at the prime above five, and be unramified outside five and eleven. Suppose that its inertia image at each eleven-adic place has order dividing five. It cannot have projective image S_5 with the quadratic quotient S_5/A_5 cutting out $K(\zeta_5)/K$.*

Proof. At the five-adic place, the absolute ramification index of K is two. Raynaud's finite-flat tame weights lie in $\{0, 1, 2\}$ [53, Theorem 3.4.3 and Corollaries 3.3.6, 3.4.4]. In the reducible local case their sum is two, so the projective tame ratio has order at most two. In the irreducible tame case the niveau-two exponent is $a + 5b$, with $a + b = 2$ and $0 \leq a, b \leq 2$; its projective inertia order is three.

If projective wild inertia is nontrivial, it is a C_5 and its decomposition group lies in its normalizer $C_5 \rtimes C_4$ in S_5. The tame quotient has order at most two. The subgroup $C_5 \rtimes C_2$ lies

in A_5, since inversion of a five-cycle is a double transposition. The quadratic quotient would therefore be unramified at five. This contradicts $K(\zeta_5)/K$, which is ramified quadratic there.

Thus wild inertia is trivial. Since the quadratic quotient is ramified, projective inertia has order two. The absolute ramification index at five in the projective kernel field $N/\mathbf{Q}$ is four, tamely ramified, so its normalized different contribution is $3/4$. At eleven it is at most $4/5$. The degree is $[N : \mathbf{Q}] = 240$, and

$$\mathrm{rd}(N) \leq 5^{3/4}11^{4/5} < 22.76889.$$

The GRH lower bound (39) at $n = 240, b = 5$ is greater than 23.24040, a contradiction. □

Proposition 20.17. *Assume GRH for Dedekind zeta functions. Every pure, totally odd, absolutely irreducible strongly compatible weight-two system over K of cyclotomic determinant, good outside eleven and special with unipotent inertia at its two eleven-primes, is modular.*

Proof. Take a coefficient place above three. If its residue is reducible, Lemma 20.10 proves modularity of the actual source. Otherwise the same lemma gives absolute irreducibility after restriction to $K(\zeta_3)$.

At each eleven-prime apply Lemma 4.9 to the actual residual representation of the old special source. It gives a tame determinant-one cuspidal lift of order three, with determinant χ_3, matching the full residual Frobenius extension and either nontrivial inertia orientation.

Prescribe these tame order-three cuspidal types at both eleven-primes, retain the old crystalline weight-two component at three, and retain unramified conditions elsewhere. EGPZ [28, Theorem 4.2.1] gives a global point. Lemma 2.4 gives actual potential modularity and the pure strongly compatible replacement. It is modular by Theorem 20.12. AKT [3, Theorem 2.3], applied to the old weight-two three-adic member, gives the backward implication. □

Theorem 20.18. *Assume GRH for Dedekind zeta functions. Every continuous absolutely irreducible totally odd representation $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_{11})$ of determinant ω_{11}, unramified outside eleven, is modular.*

Proof. A solvable image has an odd solvable Artin witness and hence is modular after regular weight raising. Otherwise the cyclotomic restriction is absolutely irreducible. Use Lemma 20.13 at both split eleven-places, and unramified conditions elsewhere. EGPZ gives a weight-two global lift of exact cyclotomic determinant; there is no exceptional case in characteristic eleven. Lemma 2.4 constructs its pure strongly compatible system.

Make the independent global quadratic corrections described above. The resulting system is good outside eleven and its two local conditions are good, special, or principal series of odd inertial order five. All twists are reversed at the end. Its five-adic member is Barsotti–Tate above five. At eleven its actual residual inertia is trivial or unipotent of order five: the fifth-order eigencharacters coalesce, while a special representation has procyclic tame pro-five unipotent inertia.

If the five-residue is globally reducible, Lemma 20.11 gives modularity of the actual member and hence of the entire system. Otherwise Lemma 20.15 proves cyclotomic irreducibility. The controlled exceptional exclusion Proposition 20.16 applies: the representation is finite flat at five, supported only at five and eleven, and has the stated eleven-inertia. Thus the residual five-representation is nonexceptional in the precise sense required by EGPZ.

At each eleven-prime choose the good or special local lift of Lemma 20.14. Retain a component through the old Barsotti–Tate point at five and unramified conditions elsewhere. EGPZ gives the weight-two replacement, and Lemma 2.4 gives its pure strongly compatible system. By the purity observation in Section 2, the replacement is special or good at both eleven-places, and good away from eleven.

If either eleven-place is good, Theorem 20.8 excludes this pure absolutely irreducible system. If both are special, Proposition 20.17 proves its modularity. In the latter case the common modular residual witness lifts the actual old weight-two five-adic source by [3, Theorem 2.3]. Compatibility, reversal of the quadratic twist and reduction at the original eleven-place prove the theorem. □

20.6 Removal of arbitrary conductor

Proof for $K = \mathbf{Q}(\sqrt{5})$. A solvable residual image has an odd solvable Artin witness and is modular after regular weight raising. Otherwise its cyclotomic restriction is absolutely irreducible. For an initial weight-two lift use [3, Theorem 2.2], except in exceptional characteristic five, where Proposition 20.3 supplies the point with its allowed auxiliary ramification. Lemma 2.4, or the potential-modularity assertion of that proposition, gives a potentially modular compatible system. Its nonsolvable residue makes it nondihedral, and its determinant is cyclotomic times a finite totally even character.

Apply Theorem 1.2 with $\ell = 11$. Two is inert and eleven splits. The supported irreducible residual assertion is Theorem 20.18; Proposition 20.1 gives the constituent quotient $\bar{\chi}_{11}^{e}$, with $e \in \{1, 3, 5, 7, 9\}$ up to inversion. It is totally odd and extends to $G_{\mathbf{Q}}$. Thus the common criterion proves modularity of the system and hence of the original residue. □

21 Mixed sources and pseudodeformation rigidity over $\mathbf{Q}(\sqrt{21})$

Put

$$K = \mathbf{Q}(\sqrt{21}), \qquad a = \frac{1+\sqrt{21}}{2}, \qquad \epsilon = a + 2, \qquad \gamma = a + 1, \qquad H = K(\zeta_3).$$

Thus $a^2 - a - 5 = 0$, $\gamma^2 = 3\epsilon$, and $N(\gamma) = -3$.

The proof first handles the bounded local menu at the two places above five. Nonsolvable dyadic residues are controlled by auxiliary fields and the common discriminant inequality. The remaining mixed source is treated by finite-flat rigidity and a toric realization. The pure menus are then reduced to three, where actual ordinary, dihedral and residually reducible sources require different arguments.

We use the compatible-system conventions of Section 2. Potential modularity supplies a cuspidal realization after base change, so every companion used below is absolutely irreducible.

Let ω_2 be a fundamental character of niveau two over $\mathbf{Q}_5$, of order 24. The bounded cuspidal types are

$$\tau_3 = \widetilde{\omega}_2^{8} \oplus \widetilde{\omega}_2^{16}, \qquad \tau_6 = \widetilde{\omega}_2^{4} \oplus \widetilde{\omega}_2^{20}.$$

The notation SC means one of these types, and SP means a fixed quadratic twist of Steinberg with nonzero monodromy. The exact quadratic character, including its unramified sign, is retained whenever a special deformation condition is imposed.

Proposition 21.1 (Mixed source)**.** *Assume GRH. A potentially modular compatible system over K, of labelled weights $\{0, 1\}$ and determinant cyclotomic, unramified away from five and crystalline at two, is modular if its types at the two places above five are* SC/SP *in either order. The same conclusion holds if potential modularity is replaced by the existence over K of a* GL_2*-type abelian variety realizing the compatible system.*

The proof of this proposition is given below, including both the nonsolvable and solvable dyadic branches. The special condition retains its actual nonzero monodromy.

For a pure special menu with reducible five-adic residue, Proposition 21.2 applies directly. For either pure menu, the three-adic member r_3 has four exhaustive alternatives:

residual condition	source argument
$\bar{r}_3\vert_{G_H}$ irreducible	switch one five-type to the mixed menu
$\bar{r}_3$ irreducible, $\bar{r}_3\vert_{G_H}$ reducible	dihedral order-eight rigidity
$\bar{r}_3^{\mathrm{ss}}$ reducible, constituents unramified at three	ordinarity and Skinner–Wiles
$\bar{r}_3^{\mathrm{ss}}$ reducible, common ramified quadratic at three	quadratic descent.

The last line is essential: good supersingular elliptic curves in the present conductor class realize it.

Proposition 21.2. *Let $\mathcal{R}$ be an absolutely irreducible, totally odd weight-two compatible system over K with cyclotomic determinant. Suppose its parameters at both places above five are quadratic-special with nonzero monodromy. If a five-adic residual semisimplification is reducible, then $\mathcal{R}$ is modular.*

Proof. After removing the scalar quadratic character, the filtered-module argument of Lemma 4.2 gives an ordinary quotient at each five-place. Restoring that character preserves ordinarity. Apply [26, Corollary 1.2.2] to the dual of the actual five-adic source: $K/\mathbf{Q}$ is totally real abelian, five is unramified in K, the determinant is χ_5^{-1}, and the source is ordinary at both five-places with reducible residual semisimplification. The corollary has no residual constituent-ratio restriction. Dualizing back proves modularity of $\mathcal{R}$. □

21.1 Local lifts and a switch in characteristic three

Lemma 21.3 (Bounded five-adic lifts)**.** *Every continuous representation $\bar{r} : G_{\mathbf{Q}_5} \to \mathrm{GL}_2(\overline{\mathbf{F}}_5)$ of determinant $\bar{\chi}_5$ has a lift of determinant χ_5, labelled weights $\{0, 1\}$ and type* SC *or* SP*.*

Proof. The weights of $\mathrm{GL}_2(\mathbf{F}_5)$ with trivial scalar central character are

$$1, \quad \overset{2}{\det}, \quad \mathrm{Sym}^2 \det, \quad \mathrm{Sym}^2 \overset{3}{\det}, \quad \mathrm{Sym}^4, \quad \mathrm{Sym}^4 \overset{2}{\det}.$$

The two cuspidal compact types reduce with constituents

$$\overline{\sigma}(\tau_6)^{\mathrm{ss}} = 1 \oplus \mathrm{Sym}^2 \det, \qquad \overline{\sigma}(\tau_3)^{\mathrm{ss}} = \overset{2}{\det} \oplus \mathrm{Sym}^2 \overset{3}{\det}.$$

These identities follow by evaluating the cuspidal Brauer characters on split and nonsplit regular elements. The cuspidal character is zero on split regular elements, equals $-\theta(z) - \theta(z^5)$ on nonsplit regular elements and $4\theta(a)$ on a scalar a. Substituting the exponents four and eight gives the displayed identities. The fixed-determinant positivity argument in Section 20.5, applied at five, gives a nonzero cycle for at least one of these six weights for the actual residual extension. The first four weights give the cuspidal alternatives.

The last two give the semistable compact type for a scalar quadratic inertial type. If the resulting point has nonzero monodromy, it is SP. If all such points were crystalline, the corresponding crystalline compact type, 1 or $\det^2$, would have positive multiplicity, and the cuspidal alternative would again exist. This is a statement about the actual residual extension, with any finite coefficient field. The determinant is fixed in the deformation rings; equivalently an unramified determinant error reducing to one is corrected by its square root. □

Lemma 21.4 (The actual-residual switch)**.** *Let $\bar{s}$ be the reduction of a determinant-χ_3 representation of $G_{\mathbf{Q}_5}$ of type* SC *or* SP*, with any stable lattice. After finite coefficient extension, $\bar{s}$ has lifts of both types, with determinant exactly χ_3. The scalar quadratic inertia character determines whether the cuspidal lift has type τ_3 or τ_6.*

Proof. Apply Lemma 4.9 with $\ell = 5$. □

21.2 Controlled characteristic-two residues

Put $F = K_2$, the unramified quadratic extension of $\mathbf{Q}_2$. We distinguish the residual conditions $\mathrm{SC}_2, \mathrm{SP}_2$ from the characteristic-zero conditions SC, SP.

Definition 21.5. The two permitted conditions at a place $v \mid 5$ of K are:

(i) SC_2: the kernel field is the tame S_3 extension of $\mathbf{Q}_5$ with ramification index three and residue degree two;
(ii) SP_2: the kernel field is contained in $E_5' = \mathbf{Q}_5(\sqrt{2}, \sqrt{5})$.

The SC_2 field is unique in a fixed algebraic closure and is the splitting field of $X^3 - 5$. Over the unramified quadratic field, Frobenius has eigenvalues $+1$ and -1 on the uniformizer and unramified lines of cube classes. Only the first is a ramified stable line, and it gives the inversion action on cubic inertia.

Theorem 21.6. *Assume GRH for Dedekind zeta functions. Let*

$$\bar{r} : G_K \longrightarrow \mathrm{SL}_2(\overline{\mathbf{F}}_2)$$

be continuous and absolutely irreducible. Suppose that $\bar{r}$ is unramified outside two and five, is the generic fibre of a finite flat group scheme killed by two at the dyadic place, and satisfies either condition of Definition 21.5 independently at each place above five. Then $\bar{r}$ is modular.

The theorem covers the pairs $(\mathrm{SC}_2, \mathrm{SC}_2)$, $(\mathrm{SC}_2, \mathrm{SP}_2)$ and $(\mathrm{SP}_2, \mathrm{SP}_2)$. The first pair requires the unit extension constructed below. A quadratic extension suffices for the other two pairs.

Corollary 21.7. *Assume GRH. Let $\mathcal{R} = (r_\lambda)_\lambda$ be a totally odd, strongly compatible two-dimensional system over K, of determinant the cyclotomic character and labelled Hodge–Tate weights $\{0, 1\}$. Suppose it has all-place Weil–Deligne compatibility, is unramified at $v \nmid 5$ for members of residue characteristic different from that of v, has crystalline two-adic member at two, and at both places above five has either a quadratic twist of special type with nonzero monodromy or one of the two potentially crystalline types in Lemma 21.3. If its characteristic-two residual representation has nonsolvable image, then the given system $\mathcal{R}$ is modular.*

The distinction between these two assertions matters. Theorem 21.6 provides a modular witness for a residue. Corollary 21.7 uses an automorphy lifting theorem to identify the given characteristic-zero representation with a modular one.

Lemma 21.8. *In a determinant-cyclotomic compatible system with all-place Weil–Deligne compatibility, reduction at a place above two of a local member of quadratic-special type satisfies* SP_2. *Reduction of either cuspidal type satisfies* SC_2.

Proof. In the special case the diagonal characters are, up to a quadratic twist, $\chi_2\nu, \nu$ with ν unramified quadratic. They both reduce to one. The full residual image, including Frobenius, is therefore elementary abelian of exponent two. Local class field theory puts its field inside E_5'. Consequently $e \leq 2$, $f \leq 2$ and $d \leq 1/2$.

For either cuspidal type the reduced inertia has exact order three. A Frobenius lift swaps its two eigenlines. In characteristic zero its square is $-5I$, by its determinant and zero trace. After reduction its square is one. Thus the full residual decomposition group is S_3, with $e = 3$, $f = 2$ and $d = 2/3$. □

21.2.1 Finite-flat ramification at two

We use Fontaine's ramification theorem [31, Theorem A and its corollary, pp. 515–516]. His numbering is $G^{(u)} = G^{u-1}$ in Serre's upper numbering. For a group scheme killed by two over an unramified dyadic field, the theorem therefore gives upper breaks at most one, and its corollary gives $d < 2$. More generally, for a group scheme killed by p it gives $d < 1 + 1/(p-1)$.

Lemma 21.9. *Let $q = 2^f \geq 4$ and let $\bar{\rho} : G_F \to \mathrm{SL}_2(\mathbf{F}_q)$ be finite flat. Its kernel field E satisfies*

$$d(E/F) \leq 2 - \frac{2}{q}.$$

If its tame diagonal characters are nontrivial and wild inertia is nontrivial, the stronger bound $d(E/F) < 4/3$ holds.

Proof. First suppose wild inertia is trivial. If the representation is reducible over $\overline{\mathbf{F}}_2$, local class field theory makes the tame characters have orders dividing $\#\mathbf{F}_4^\times = 3$. The different is at most $2/3$. If it is locally irreducible, its tame characters are θ, θ^4, so determinant one gives $\theta^5 = 1$. Its different is at most $4/5$. Both bounds suffice.

If wild inertia is nontrivial, its unique fixed line is rational over $\mathbf{F}_q$ and is stable under the local decomposition group. Write the diagonal characters as η, η^{-1}. Their tame order is one or three. Killing the diagonal characters gives a field T over which the image is an elementary quadratic translation group of order $P \leq q$. Unramified enlargements do not alter normalized differents.

If η is unramified, so is T/F. Fontaine's upper-break bound shows that all quadratic characters of E/T have conductor at most two. The conductor–discriminant formula gives $d(E/F) \leq 2(1 - 1/P) \leq 2 - 2/q$.

Suppose the tame diagonal order is three. Now $e(T/F) = 3$. Tame conjugation acts nontrivially on the translation module and its Kummer dual. The quadratic unit filtration over T, whose absolute ramification index is three, has its nontrivial tame grades at levels one and five. They give quadratic conductors six and two, respectively; the level-three and unramified directions are tame-invariant and cannot occur. These are the usual Kummer filtration and orthogonality formulas; see [16, §1]. A conductor-six character has upper break five over T, hence $5/3$ over F, contradicting Fontaine's bound. Only conductor two remains, and

$$d(E/F) \leq \frac{2}{3} + \frac{2}{3}(1 - 1/P) < \frac{4}{3}.$$

□

21.2.2 The solvable auxiliary fields

Let

$$H = K(\sqrt{-3}) = \mathbf{Q}(\sqrt{-3}, \sqrt{-7}).$$

It is quadratic over K, unramified at every finite place, split at two, and unramified quadratic at both places above five. The discriminant product $(-3)(-7)(21) = 21^2$ verifies the first assertion; the other two follow from the quadratic local square classes. This field alone will handle the pairs involving SP_2.

Let S be the splitting field over $\mathbf{Q}$ of

$$f(X) = X^3 - 3X^2 - 2X - 1,$$

and put $A = KS$. The polynomial is irreducible modulo two and has discriminant -175. Thus $S/\mathbf{Q}$ has group S_3, with quadratic subfield $\mathbf{Q}(\sqrt{-7})$, and A/K has group S_3 and fixed quadratic field H. Write τ for a generator of its normal subgroup of order three.

Lemma 21.10. *The extension A/K is unramified outside five. Its dyadic completions are unramified cubic over F; at both places above five its completion is the* SC_2 *field.*

Proof. The splitting field S ramifies only at five and seven. At five, $f(X+1) = X^3 - 5X - 5$ is Eisenstein, so the cubic ramification is tame. The discriminant has nonsquare unit part -7, yielding the unramified quadratic field and hence S_3. At seven its inertia is quadratic. Its ramified quadratic field is $\mathbf{Q}_7(\sqrt{-7}) = \mathbf{Q}_7(\sqrt{21})$, since -3 is a square in $\mathbf{Q}_7$. Base change to K removes this ramification. At two the irreducible reduction of f gives an unramified cubic; its unramified quadratic field already lies in F. □

We now specify four unit squareclasses in A without using a class-group calculation or a fundamental-unit computation. Choose a root u of f and $z = \sqrt{-3}$ in A and set

$$v = \frac{9 + 26u - 8u^2 + (2u - 7)z}{2}.$$

The choices of roots below are made compatibly with τ. Direct substitution gives

$$v^3 + \frac{15z - 1}{2}v^2 + \frac{15z + 1}{2}v - 1 = 0.$$

In particular u, v are integral units. Their minimal polynomials over $\mathbf{Q}$ are

$$f(X), \qquad X^6 - X^5 + 170X^4 + 335X^3 + 170X^2 - X + 1.$$

Both have norm one for A/H.

Lemma 21.11. *In $W = \mathcal{O}_A^\times/(\mathcal{O}_A^\times)^2$, the four classes*

$$[u],\ [\tau u],\ [v],\ [\tau v]$$

are independent and span $U = (\tau - 1)W$. At either dyadic prime of A their image has dimension two. At both five-adic primes their image is zero.

Proof. Here A has signature $(0, 6)$, so $\dim_{\mathbf{F}_2} W = 6$ by Dirichlet's unit theorem, including the torsion squareclass. Since A/H has odd degree, the τ-invariants in W are the images of the unit squareclasses of H: the norm of an invariant class is its third power and hence that same squareclass. The latter space has dimension two. Thus $(\tau - 1)W$ has dimension four, and every norm-one class belongs to it.

For completeness we give a finite exact certificate of independence and of the dyadic ranks. In the ordered list $(u, \tau u, v, \tau v)$, encode a product by its four-bit mask $m = \sum_{j=0}^{3} m_j 2^j$. The masks that are squares in the two dyadic completions are respectively

first dyadic prime	second dyadic prime
$0, 6, 11, 13$	$0, 7, 9, 14$

Each kernel has order four and their intersection is zero. This proves local rank two and global rank four. The calculation can be performed by reduction modulo eight: over an unramified dyadic field a unit is a square precisely when its residue modulo eight has a square root. Appendix B specifies an exact number-field certificate for this table. The exact calculation is supplied in [49, `d21`].

Finally, inertia at five contains τ. Every class of the form $[w/\tau w]$ for a global unit w has residue one, and is a local square because the residue characteristic is odd. This proves the last assertion for all of U. □

Define

$$B = A(\sqrt{U}).$$

Conjugation in $\mathrm{Gal}(A/K)$ exchanges τ and τ^{-1}, so U is stable under this group. Lemma 21.11 gives

$$[B : K] = 96, \qquad B/K \text{ Galois and solvable.} \tag{10}$$

The field B/A is split at five. At two it has ramified group C_2^2 with upper break one over the unramified cubic stage. Indeed its local defining classes are units, and their nontrivial C_3-module has no unramified quadratic constituent.

Lemma 21.12. *If E/F is the kernel field of a finite-flat representation killed by two, then adjoining a dyadic completion of B preserves Fontaine's bound $d < 2$. The same is true after composing two such finite-flat fields.*

Proof. A quadratic extension defined by a unit is the point field of a finite flat extension of $\mathbf{Z}/2\mathbf{Z}$ by μ_2; the usual Kummer construction gives a model over the ring of integers. Direct sums handle a space of unit classes. Restriction of scalars from the unramified cubic extension preserves finite flatness. A finite étale representation accounts for that unramified stage. Taking direct sums with the original finite-flat group schemes realizes the compositum of their point fields, still killed by two. Fontaine's corollary now applies. □

21.2.3 The stable degree-four image at two

The following refinement is the only place where the local rank two in Lemma 21.11 is essential.

Lemma 21.13. *Let $\bar{\rho} : G_F \to \mathrm{SL}_2(\mathbf{F}_{16})$ be finite flat, with kernel field E. Suppose its action extends to a semilinear projective action of $G_{\mathbf{Q}_2}$ with nontrivial field automorphism $x \mapsto x^4$. Then the compositum with a dyadic completion of B satisfies*

$$d(EB/F) \le 15/8.$$

Proof. Write T/F for the unramified cubic extension. The quadratic classes contributed by B over T form a two-dimensional, nontrivial $\mathrm{Gal}(T/F)$-module of units.

If $\bar{\rho}$ is locally irreducible, its tame inertia has order five, as in Lemma 21.9. After tame base change of degree five the three nonzero quadratic characters from B have conductor six. The different tower gives

$$d(EB/F) \le \frac{4}{5} + \frac{3 \cdot 6}{4 \cdot 5} = \frac{17}{10} < \frac{15}{8}.$$

If the tame diagonal order is three, let $P \le 16$ be the order of the original translation group. Its nontrivial quadratic characters have conductor two over the tame cubic stage by Lemma 21.9. The B-characters there have conductor four and trivial tame action; they are independent of the first module. Counting characters gives

$$d(EB/F) \le \frac{2}{3} + \frac{3P \cdot 4 + (P-1) \cdot 2}{3 \cdot 4P} = \frac{11}{6} - \frac{1}{6P} < \frac{15}{8}.$$

It remains to consider unramified diagonal characters. If wild inertia is zero, E/F is unramified and the contribution of B alone is $3/2$. Otherwise its unique fixed line is preserved by the semilinear extension. Let s lift the rational Frobenius. The diagonal ratio of s^2 is $aa^4 \in \mathbf{F}_4^\times$. Thus the unramified diagonal ratio over F has order one or three.

In the order-three case, over T the original Kummer module and the module from B both lie in the nontrivial $\mathrm{Gal}(T/F)$-part of the unit squareclasses. That part has dimension four: the

total unit squareclass dimension over T is seven, and the invariant part, from F, has dimension three. All its nonzero ramified characters have conductor two. Hence the combined wild inertia has order at most sixteen and different at most $2(1 - 1/16) = 15/8$.

In the order-one case, the unit squareclasses from F have at most two ramified dimensions, since their dimension is three and they include the unramified quadratic line. The module from B adds at most two ramified dimensions. Again the ramification index is at most sixteen, and every ramified quadratic character has conductor two. The same bound follows. □

21.2.4 The discriminant comparisons and residual modularity

Use the common test function and explicit formula of Section 22, with the finite-tail bound $B_{b,N}$ and the real-place correction $A_{b,4}$.

For every nonsolvable kernel field considered here, the group is $G = \mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^f \geq 4$. Compositing with a solvable Galois extension is disjoint over K, since G is nonabelian simple. At an SC_2 place a degree-n compositum preserving that completion has $n/12$ norm-25 primes. It contributes $c_{25} = 1/6$. At an SP_2 place, adjoining H makes its residue degree two and leaves ramification index at most two, giving $c_{25} \geq 1/4$. The resulting bounds are

pair at five	upper bound for rd	retained c_{25}
$\mathrm{SC}_2, \mathrm{SC}_2$	$4\sqrt{21}\,5^{2/3}$	$1/3$
$\mathrm{SC}_2, \mathrm{SP}_2$	$4\sqrt{21}\,5^{7/12}$	$5/12$
$\mathrm{SP}_2, \mathrm{SP}_2$	$4\sqrt{21}\,5^{1/2}$	$1/2$

The common dyadic factor four comes from Fontaine's strict $d_2 < 2$. In the mixed case it can be replaced by $2^{2-2/q}$ using Lemma 21.9. For the stable $q = 16$ case with two SC_2 places, Lemma 21.13 replaces it by $2^{15/8}$ even after adjoining B.

Table 1 records rigorous evaluations of (39). In every row $N = 100$ and computations use 220-bit real-ball arithmetic. The lower bounds are rounded downwards and the upper bounds upwards at the shown precision. The column m states how many positive norm-25 Euler powers are retained. The totally real row uses $r_1/n = 1$ and no prime terms; the other rows discard the real-place contribution.

case	$n \geq$	b	m	GRH lower	local upper
two SC_2, distinct conjugates	48 771 072	17.34	8	53.60787465	53.59813048
two SC_2, $q \geq 64$	50 319 360	17.37	8	53.62432870	53.59813048
two SC_2, stable $q = 16$	783 360	13.21	8	50.46443530	49.14970236
mixed, $q = 8$	2 016	7.31	4	39.50635209	39.41346729
mixed, $q = 16$	16 320	9.40	5	46.57108547	42.98069085
mixed, $q \geq 32$	130 944	11.48	7	51.01199785	46.87077572
two SP_2, $q \geq 8$	2 016	7.36	4	41.44651216	40.98780307
totally real kernel	120	4.80	0	56.06306031	53.59813048

Table 1: Certified root-discriminant contradictions.

21.2.5 Proof of the residual theorem

Proof of Theorem 21.6. The solvable absolutely irreducible images in characteristic two are dihedral and are modular by the usual Hecke theta-series argument, as recalled in [3, proof of Theorem 2.5]. For nonsolvable images Dickson's classification gives, after conjugation, $G = \mathrm{SL}_2(\mathbf{F}_q)$ with $q = 2^f \geq 4$. The case $q = 4$ is modular by [3, Theorem 2.5]. We may assume $q \geq 8$. Let M/K be the kernel field.

Suppose first that at least one place above five satisfies SP_2. Adjoin H. It is disjoint from M, is split at two, and preserves the different bounds at five while supplying the norm-25 coefficients above. Thus

$$[MH : \mathbf{Q}] = 4q(q^2 - 1).$$

For a mixed pair, the three mixed rows of Table 1 exclude $q = 8$, $q = 16$ and all $q \geq 32$. The two-SP_2 row excludes every $q \geq 8$ in the other pair.

We are left with two SC_2 places. Adjoin B. By (10), simplicity and Lemma 21.12,

$$[MB : \mathbf{Q}] = 192q(q^2 - 1), \qquad \mathrm{rd}(MB) < 4\sqrt{21}\,5^{2/3}.$$

Its five-adic completions remain exactly S_3, since both M and A have the unique SC_2 field and B/A is locally split. The second row of Table 1 excludes $q \geq 64$.

Let σ be the nontrivial automorphism of $K/\mathbf{Q}$. If $M \neq M^\sigma$, simplicity gives $M \cap M^\sigma = K$. The group of their compositum over K is $G \times G$, again disjoint from solvable B. Its dyadic field is still the point field of a finite-flat group killed by two, and its five-adic completions are unchanged. Its degree is

$$[MM^\sigma B : \mathbf{Q}] = 192|G|^2 \geq 48\,771\,072.$$

The first row of the table excludes this possibility. Hence $M = M^\sigma$ and $M/\mathbf{Q}$ is Galois.

For $q = 8, 16, 32$ the conjugation action defines

$$C_2 \longrightarrow \mathrm{Out}(\mathrm{SL}_2(\mathbf{F}_{2^f})) = C_f.$$

If this action is trivial, the centerless extension $\mathrm{Gal}(M/\mathbf{Q})$ is $G \times C_2$: correct a lift of the generator by an inner element; its square then centralizes G and lies in its trivial centre. Projection onto G extends the given residual representation to $G_{\mathbf{Q}}$. Serre's theorem over $\mathbf{Q}$ [40, 41] and quadratic base change [4] prove modularity over K. The base change is cuspidal, since its residual restriction is absolutely irreducible.

For $q = 8$ and 32, f is odd and the outer action is automatically trivial. For $q = 16$ its only nontrivial possibility is the field involution $x \mapsto x^4$. Since two is inert in K, its dyadic action is precisely the semilinear situation of Lemma 21.13. We have

$$[MB : \mathbf{Q}] = 783\,360, \qquad \mathrm{rd}(MB) \leq \sqrt{21}\,2^{15/8}5^{2/3}.$$

The third row of the table contradicts this. All cases have therefore been excluded or proved modular. □

21.2.6 Lifting the given nonsolvable source

Lemma 21.14. *Under the hypotheses of Theorem 21.6, if the image is nonsolvable, then $\bar{r}(c_v) \neq 1$ for at least one real place v of K.*

Proof. If all complex conjugations were trivial, the kernel field M would be totally real. Its degree would be at least $2|\,\mathrm{SL}_2(\mathbf{F}_4)| = 120$, and its root discriminant would be less than the largest bound $4\sqrt{21}\,5^{2/3}$. The last row of Table 1 contradicts this, even without retaining any Euler term. □

Proof of Corollary 21.7. Choose the two-adic member of $\mathcal{R}$ and an invariant lattice. Its nonsolvable residue is absolutely irreducible, has determinant one, and is finite flat at two because the member is crystalline of weights $\{0, 1\}$. Here we use the integral Barsotti–Tate classification, including residue characteristic two, to pass from a stable lattice to a finite flat torsion model;

see [42, Theorems 2.3(3), 4.1 and Corollary 4.3]. All-place compatibility and Lemma 21.8 give the actual local fields required by Theorem 21.6. That theorem provides a regular algebraic cuspidal modular witness for the residue. Lemma 21.14 supplies a nontrivial residual complex conjugation at some real place.

Apply [69, Theorem 6.1]. Its hypotheses are finite ramification, potential crystallinity with labelled weights $\{0,1\}$ at all dyadic places, an absolutely irreducible nonsolvable residue, a nontrivial residual complex conjugation at some real place, and a regular algebraic essentially self-dual cuspidal modular witness. Each has just been checked; for GL_2 the determinant supplies the essential self-duality. Thorne uses $\mathrm{HT}(\chi_2) = -1$, so formally apply his theorem to the dual of our member and then take the contragredient of the automorphic conclusion. The given two-adic member is modular. Compatibility and Chebotarev identify all other members with the same modular system. □

21.2.7 A separate finite-flat theorem in characteristic five

Let

$$H = K(\sqrt{-3}) = \mathbf{Q}(\sqrt{-3}, \sqrt{-7}), \qquad \eta : G_K \longrightarrow \{1, -1\}$$

be its quadratic character. Section 21.3 identifies a unique S_3 kernel field A_1/K whose cubic character over H has conductor (10), up to replacing that character by its inverse. At both five-adic places its local field is tame S_3 with $e = 3, f = 2$. At two its inertia is tame C_3. Write $\bar{r}_1 : G_K \to \mathrm{SL}_2(\mathbf{F}_2)$ for its standard residual representation.

Theorem 21.15. *Let $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be a continuous totally odd lift of $\bar{r}_1$, with determinant χ_2. Suppose that r is unramified outside two and five, is crystalline at two with labelled weights $\{0,1\}$, and has either τ_3 or τ_6 as its specified finite inertia representation independently at each place above five. Then*

$$r \simeq \mathrm{Ind}_{G_H}^{G_K} \psi$$

for a geometric character ψ of G_H. In particular r is modular. This assertion does not require GRH or potential modularity.

The compatible-system assumption in Proposition 21.1 will be used to compare every coefficient conjugate and then to construct an abelian variety. Theorem 21.15 instead uses a quadratic selftwist of the actual two-adic source.

21.3 Solvable residual representations and their fields

Reduction in characteristic two of either cuspidal type has the exact tame S_3 field with $e = 3, f = 2$ at five. Reduction of the special condition is contained in $\mathbf{Q}_5(\sqrt{2}, \sqrt{5})$, with elementary quadratic image. These statements concern the full decomposition image, not only inertia; see Lemma 21.8.

The cuspidal place makes every characteristic-two residue in Proposition 21.1 absolutely irreducible. Its nonsolvable case is already modular as an actual source, by Corollary 21.7. In that result the GRH comparisons exclude simple images $\mathrm{SL}_2(\mathbf{F}_q)$ with $q \geq 8$ in the mixed case, the $q = 4$ residue is modular, and a separate totally real discriminant comparison verifies the real-place hypothesis of Thorne's automorphy lifting theorem [69, Theorem 6.1]. We therefore concentrate on solvable residues.

Such an absolutely irreducible determinant-one image in characteristic two is dihedral D_{2n} with n odd. Let L/K be its quadratic inducing field. We first dispose of the inducing fields nonsplit at two.

Lemma 21.16. *Let $r : G_F \to \mathrm{GL}_2(E)$ be crystalline with labelled weights $\{0,1\}$ and determinant χ_2, where $E/\mathbf{Q}_2$ is finite. If the actual residual local image is a two-group, then r is ordinary.*

Proof. Here $F = K_2$ is unramified over $\mathbf{Q}_2$. A finite two-group has only trivial simple representations in characteristic two, so the residual semisimple constituents are unramified. Lemma 4.10, with $p = 2$ and $e(F/\mathbf{Q}_2) = 1$, therefore proves ordinarity of the actual source. □

Proposition 21.17. *For a residue occurring in the mixed theorem, either its given source is modular by the ordinary theorem of Allen, or its quadratic inducing field is H and splits at two.*

Proof. If L/K ramifies at two, inertia has nontrivial sign in the dihedral image. Its wild image is a normal two-subgroup, and tame inertia is cyclic. Since n is odd, inertia must be a reflection C_2. Its normalizer in D_{2n} is that same C_2, so the local decomposition image is also C_2. The actual finite-flat representation therefore cuts out L_F/F. Fontaine's upper-break bound gives conductor at most two [31, Theorem A]. Its Kummer class has even dyadic valuation.

The actual crystalline source is ordinary by Lemma 21.16. Allen's ordinary modular witness lemma [2, Lemma 5.1.2] supplies a quotient-compatible nearly ordinary cuspidal lift of the globally irreducible dihedral residue. It has no CM nonsplitting hypothesis. The actual-source theorem [2, Theorem 5.2.1] then applies: if L is CM, its dyadic place is nonsplit; for a non-CM L this extra condition is vacuous. Equal residual local characters are allowed in both applications.

It remains to identify the possible inducing fields that are split or unramified at two. Put $\alpha = (1+\sqrt{21})/2$ and $\epsilon = (5+\sqrt{21})/2$. The ordinary class number of K is one: Minkowski's bound is $\sqrt{21}/2 < 2.3$, and two is inert. The unit ϵ is fundamental and has norm one. The squareclasses supported at two and five have basis

$$[-1],\ [\epsilon],\ [2],\ [\alpha],\ [\alpha^\sigma].$$

Even dyadic valuation removes [2] and gives the alternative basis $[-1], [\epsilon], [5], [\alpha]$. In F we have $[5] = 1$ and $[\epsilon] = [-1]$, the latter because

$$((3+\sqrt{21})/2)^2 = 3\epsilon,$$

and -3 is a square in F. If $w = \alpha \bmod 2 \in \mathbf{F}_4$, then $\alpha^3 \equiv 1 + 2\alpha \pmod 4$; the images of $[-1], [\alpha], [-\alpha]$ in U_1/U_2 are $1, w, 1+w$. All are nonzero. Thus every nontrivial local class in their span is ramified of conductor two, and none is the unramified quadratic class. Consequently an allowable L is either ramified or split at two, never unramified nonsplit.

The kernel of the local squareclass map is generated by [5] and $[-\epsilon]$. The three nontrivial split possibilities are $5, -\epsilon, -5\epsilon$. At the cuspidal place above five the inducing quadratic field must be unramified quadratic. The first and third candidates ramify there. Hence only $L = K(\sqrt{-\epsilon}) = K(\sqrt{-3}) = H$ remains. □

We next enumerate all odd inducing characters for this last quadratic field. The quartic H has defining polynomial $Y^4 + 20Y^2 + 16$, discriminant 441, and class number one. The class-number assertion and the following ray groups are certified by `bnfcertify`:

$$\mathrm{Cl}_H(5) = C_{24}, \qquad \mathrm{Cl}_H(10) = C_{24} \times C_3.$$

An odd inducing character is tame at two and five; its conductor divides (10). The odd ray quotient has exponent three and dimension two over $\mathbf{F}_3$. The involution of H/K acts on ray generators as $\mathrm{diag}(5,2)$, hence as $-I$ on that quotient. There are exactly four cubic quotients, all dihedral over K, with conductors

$$(5), \quad (10), \quad (2)\mathfrak{q}_1, \quad (2)\mathfrak{q}_2, \tag{11}$$

where $\mathfrak{q}_i$ is the prime of H above the corresponding prime of K over five.

For the mixed orientation, the first two quotients ramify cubically at the special place and are impossible. Exactly one of the last two ramifies at the selected cuspidal place. Denote its field over K by A_{m}. It has image S_3 and is the unique residual representation for that orientation up to $\mathrm{GL}_2(\overline{\mathbf{F}}_2)$-conjugacy: the two nontrivial cubic characters are inverse and give conjugate induced representations. A determinant-one scalar twist in characteristic two is trivial.

For a pair of cuspidal places, the first two fields in (11) are the possibilities. We denote them A_0 and A_1, respectively. The first is unramified at two; the second has tame inertia C_3 there. Both A_{m} and A_1 have local residual characters of order three over $\mathbf{F}_4$ at two, with the two finite-flat signatures $(1,0)$ and $(0,1)$. At the special place, A_{m} is unramified quadratic and Frobenius is a nontrivial reflection.

21.3.1 Finite-flat tangent extensions at two

We use the exact Honda-system description at the boundary $e(F) = 1 = p - 1$, reviewed in [13, pp. 4–6]. On biconnected objects the relevant categories are abelian, with the usual schematic kernels and quotients, and the generic-fibre functor is fully faithful. The integral classification and exactness at two can also be expressed using [42, Corollaries 4.3–4.4].

Lemma 21.18. *Let $k = \mathbf{F}_4$, let $R = k[\varepsilon]/(\varepsilon^2)$, and let $\theta : G_F \to k^\times$ have nontrivial tame inertia of order three and a finite-flat model. If $\theta(1+\varepsilon c)$ is finite flat as an underlying $\mathbf{F}_2$-representation, then c is unramified.*

Proof. Every finite-flat model of θ or its selfextensions is biconnected. An étale quotient or a multiplicative subobject would have an unramified generic constituent, whereas every constituent has the nontrivial ramified character θ; modulo two the cyclotomic character is trivial. Full faithfulness therefore extends the actual R-action to the model, and schematic closure and quotient give an exact selfextension of the rank-one model.

By the exact coefficient version of the classification [42, Corollaries 4.3–4.4 and Proposition 5.7.3], its two coefficient factors have exponents $r_0, r_1 \in \{0,1\}$. The tame exponent is $r_0 + 2r_1$ modulo three. For the two nontrivial order-three characters the unique signatures are $(1,0)$ and $(0,1)$. The subobject and quotient therefore have the same rank-one model, including its unramified constants. Lemma 11.6, with $p = 2$, $e = 1$ and $f = 2$, makes the generic extension unramified after dividing by θ, as required. □

Lemma 21.19. *For $A = A_{\mathrm{m}}$ or A_1, every normal finite-flat dual-number deformation at two gives quadratic Kummer characters over the dyadic completions of A of conductor at most four.*

Proof. Over $k = \mathbf{F}_4$ the residual local representation is $\theta \oplus \theta^{-1}$, with both characters nontrivial tame of order three. Its model is a direct sum of the unique biconnected models. A finite-flat $k[\varepsilon]$ deformation is an extension of this model by itself. Pullback along either constituent inclusion and pushout along either constituent projection in the abelian biconnected category extract its two offdiagonal extension classes. These are finite-flat extensions already over the original unramified field F.

Apply Fontaine's theorem there: classical upper breaks are at most one. Killing the two residual characters then makes a tame cubic base change, which multiplies positive upper breaks by three. Each resulting quadratic character has break at most three, hence conductor at most four. □

The order of these operations supplies the bound for every dual-number tangent point. A finite-flat group defined only over the ramified cubic field can instead have conductor six. No

division by a dual-number parameter, and no characteristic-zero lift of the tangent point, is used in Lemma 21.19.

21.3.2 The complete normal Selmer calculations

Write $\bar{r} = \mathrm{Ind}_H^K \chi$ over $k = \mathbf{F}_4$. In an inducing basis one has the explicit decomposition

$$\mathrm{ad}^0(\bar{r}) = kI \oplus V, \qquad V = \mathrm{Ind}_H^K \chi^2. \tag{12}$$

The scalar summand consists of the equal diagonal entries; the two offdiagonal entries form V. This is a direct calculation, without averaging over an order-two group.

Shapiro's lemma and Kummer theory identify the global normal classes with the appropriate nontrivial C_3-part of squareclasses in A, where $C_3 = \mathrm{Gal}(A/H)$. The finite computations below are exhaustive because the unit, divisor and class contributions are all included.

For A_{m} an absolute defining polynomial is

$$\begin{aligned} P_{\mathrm{m}}(X) = X^{12} - 3X^{11} + 3X^{10} - X^8 + 3X^7 - 6X^6 \\ + 3X^5 + X^4 + 6X^3 - 3X^2 - 9X + 9. \end{aligned}$$

Its class group is trivial, with certified full unit group. The unit-squareclass space has dimension six; there are four primes over five, giving an S_5-unit space of dimension ten. Its C_3-invariants descend by odd norm to the units and two five-adic divisors of H, a space of dimension four. Hence its full nontrivial C_3-part has dimension six over $\mathbf{F}_2$. The two dyadic primes are fixed by C_3, so adding their divisors adds no normal contribution. The trivial class group leaves no missing two-class term.

Enumerate the 64 products in the ordered basis of the attached certificate, encoded by their six-bit masks. The conductor bound from Lemma 21.19 retains

$$0, 17, 33, 48.$$

At the special place the fixed exact line character will force homogeneous Frobenius trace zero in the tangent. Its necessary cut retains

$$0, 5, 8, 13, 16, 21, 24, 29, 34, 39, 42, 47, 50, 55, 58, 63.$$

The intersection consists only of zero. To identify the second cut, take residual Frobenius $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For $z = \left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right)$, the first trace variation is $\mathrm{tr}(zS) = b + c$. Under Shapiro this is the unramified local Kummer character evaluated at Frobenius. In the residue field $\mathbf{F}_{25}$ it is computed by the Hilbert symbol $(u, 5)$; the uniformizer direction contributes trivially since $25 \equiv 1 \pmod 4$. The exact calculation is supplied in [49, `d21`].

For A_1, use

$$\begin{aligned} P_1(X) = X^{12} - 9X^{11} + 32X^{10} - 57X^9 + 66X^8 - 171X^7 \\ + 559X^6 - 1020X^5 + 1092X^4 - 396X^3 - 40X^2 - 180X + 148. \end{aligned}$$

Its field discriminant is 8576612100000000, its certified class group is C_3, and its nontrivial C_3-part of unit squareclasses has dimension four. The primes over two and five are fixed by C_3, so their divisors add no normal part. The odd class number excludes an additional two-class contribution. All 16 classes therefore suffice. The exact quadratic conductor pairs at its two dyadic primes are

four-bit masks	conductor pair
0	$(0,0)$
$5, 9, 12$	$(0,6)$
$6, 11, 13$	$(6,0)$
other nonzero masks	$(6,6)$

Only zero satisfies conductor at most four at both primes. At either five-adic place the normal character χ^2 has nontrivial cubic inertia. Local H^0 and H^2 vanish, and the local Euler characteristic formula gives $H^1 = 0$. Thus fixed cuspidal inertia introduces no additional normal direction there.

Proposition 21.20. *For the mixed residue, a finite-flat normal tangent with homogeneous trace zero at the special place is a coboundary. For $\bar{r}_1$, every finite-flat normal tangent is a coboundary.*

Proof. The exhaustive squareclass computations above and Lemma 21.19 leave only zero in each of the corresponding global Kummer spaces. □

These computations use certified class and unit data only in degrees four and twelve. The script names and exact outputs are listed in Appendix C. The exact calculation is supplied in [49, `d21`].

21.3.3 Closed deformation conditions and the full tangent

We now pass from the necessary normal cuts to the actual global deformation functor. Residual S_3 is absolutely irreducible over $\mathbf{F}_2$, so its universal global deformation ring over $\mathbf{Z}_2$ exists. We fix determinant χ_2 and ramification outside two and five. The finite-flat condition at two is a closed deformation condition on Artin coefficient rings; see [13, Definition 1.1.1 and the following discussion]. Here its residual model is biconnected, so models and coefficient actions are unique. Local framed conditions give closed quotients without a local-centralizer restriction.

At a cuspidal five-adic place, require the fixed finite inertia kernel to act trivially and a chosen generator to have the characteristic polynomial for the selected τ_3 or τ_6. The residual cubic eigenvalues are distinct, so these equations specify the finite inertia representation up to conjugacy. They are closed matrix equations.

For the mixed case we impose a stronger special condition retaining the exact quadratic character. Fix $\alpha : G_{K_{v_{\mathrm{SP}}}} \to \{1, -1\}$ and an arithmetic Frobenius Φ, and write $a = \alpha(\Phi) \in \{1, -1\}$. Let $M = r(\Phi) - aI$. The residual matrix $S - I$ has a unit pivot. Impose

$$\operatorname{tr} r(\Phi) = 6a, \qquad r(g)v = \chi_2(g)\alpha(g)v \quad (g \in G_{K_{v_{\mathrm{SP}}}}), \tag{13}$$

where v is the pivot column of M. Since the determinant is already five,

$$\det M = 5 - a(6a) + a^2 = 0.$$

The unit pivot therefore makes $\operatorname{im} M$ a free rank-one direct summand generated by v. Cayley–Hamilton gives $(r(\Phi) - 5aI)M = 0$. The second equation in (13) imposes the exact character $\chi_2\alpha$ on that canonical line; the quotient character is α by the determinant. These are closed equations in the universal ring. Every actual special source with the chosen α satisfies them. This construction avoids any division by the eigenvalue difference four.

Let D_{m} denote this mixed quotient, and D_1 the quotient with the chosen cuspidal condition at both places for $\bar{r}_1$.

Proposition 21.21. *Both $\mathbf{Z}_2 \to D_{\mathrm{m}}$ and $\mathbf{Z}_2 \to D_1$ are surjective. If either problem has a characteristic-zero point, its ring is $\mathbf{Z}_2$.*

Proof. We calculate the true relative tangent, after scalar extension to $k = \mathbf{F}_4$. Fixed determinant makes the cocycles take values in $\mathrm{ad}^0(\bar{r})$, but strict changes of basis use every matrix in $\mathrm{End}(\bar{r})$. The actual quotient is consequently

$$Z^1(\mathrm{ad}^0\,\bar{r})/d\,\mathrm{End}(\bar{r}), \tag{14}$$

with the stated local conditions. In characteristic two this need not equal $H^1(\mathrm{ad}^0\,\bar{r})$.

For the mixed problem, (13) gives homogeneous trace zero on tangent Frobenius, since $6a$ maps to zero in $k[\varepsilon]$. For D_1 no special condition is needed. Proposition 21.20 kills the normal V-component in (12). After a strict global change of basis the tangent is therefore scalar, cI. Fixed finite cuspidal inertia makes c unramified there. In the mixed case, the marked exact character on the special line makes c trivial on inertia at that place. At two its generic rank-one summands have finite-flat closures and quotients; applying Lemma 21.18 makes c unramified. It is unramified elsewhere by the support condition.

The narrow class number of K is two, whereas its ordinary class number is one. Hence the only everywhere-finite-unramified additive quadratic character, after extension to k, is the character δ_H of H/K. But $\delta_H I$ is a coboundary in the quotient (14). Indeed, for $X = \mathrm{diag}(1, 0)$, conjugation by $1 + \varepsilon X = \mathrm{diag}(1 + \varepsilon, 1)$ fixes the inducing subgroup and multiplies reflections by $1 + \varepsilon$. Its boundary is exactly $\delta_H I$. The matrix X has trace one; restricting changes of basis to trace-zero matrices would incorrectly retain this tangent. The same strict conjugation transports the marked special line.

The special marking adds no tangent of its own: with representation and eigencharacter fixed, an infinitesimal line motion w satisfies $(S - I)w = 0$ and thus lies in the original line. There is no motion in its quotient. The true relative tangent is zero in both problems.

All the data are defined over $\mathbf{Z}_2$; the $\mathbf{F}_4$ calculation is scalar extension of the $\mathbf{F}_2$ tangent. A complete Noetherian local $\mathbf{Z}_2$-algebra with residue $\mathbf{F}_2$ and zero relative tangent is a quotient of $\mathbf{Z}_2$, by Nakayama's lemma and completeness. Any characteristic-zero point rules out a nonzero ideal in this quotient, so the ring is $\mathbf{Z}_2$ itself. □

21.3.4 The cuspidal pair with tame dyadic inertia

Proof of Theorem 21.15. The actual source r gives a characteristic-zero point of D_1, so Proposition 21.21 gives $D_1 = \mathbf{Z}_2$ and uniqueness of its lift up to conjugacy and coefficient extension.

The quadratic character η is unramified at every finite place and trivial at F, since H/K is the narrow class quadratic extension and splits at two. It reduces to one in characteristic two. Thus $r \otimes \eta$ has the same determinant, support and crystalline dyadic condition as r. At both primes above five it has exactly the same finite inertia representation, including the choice of τ_3 or τ_6. It is another point of the same ring, with the same residual identification. Uniqueness gives

$$r \simeq r \otimes \eta.$$

Absolute irreducibility and the nontrivial quadratic selftwist give $r = \mathrm{Ind}_H^K \psi$ by Clifford theory.

The character is geometric as a constituent of $r|_{G_H}$. Global class field theory for geometric rank-one representations identifies it with an algebraic Hecke character, up to the chosen cyclotomic normalization [57, Chapter III]. Since H/K is CM and r is odd and regular, automorphic induction gives the required cuspidal Hilbert modular representation [4, Chapter III]. □

21.3.5 Rational traces and a toric place

We first isolate the rational endomorphism argument needed for the mixed source. It removes a possible Schur-index ambiguity in passing from rational traces to an elliptic curve.

Proposition 21.22. *Let B/K' be an abelian variety of dimension g over a number field, and let ℓ be prime. Suppose that r is an absolutely irreducible two-dimensional $\mathbf{Q}_\ell$-valued representation such that $V_\ell(B) \simeq r^{\oplus g}$. If, at a finite place of residue characteristic different from ℓ, the monodromy of r has rank one after finite local base change, then B is K'-isogenous to E^g for an elliptic curve E/K' with $V_\ell(E) \simeq r$.*

Proof. Make the reduction semistable and its torus split by a finite local extension. Monodromy on $V_\ell(B)$ has rank g, so B has purely toric reduction, with torus T of rank g. Put $C = \mathrm{End}_{K'}(B) \otimes \mathbf{Q}$. Faltings' endomorphism theorem [29] gives

$$C \otimes \mathbf{Q}_\ell = \mathrm{End}_{G_{K'}}(V_\ell B) = M_g(\mathbf{Q}_\ell),$$

so $\dim_{\mathbf{Q}} C = g^2$. Endomorphisms extend to the local Néron model and act on $X_*(T) \otimes \mathbf{Q}$.

Let I be the kernel of this rational action. With a basis of the Tate twist fixed, write N for monodromy on $V = V_\ell(B)$. Purely toric reduction gives

$$W = \mathrm{im}\, N = \ker N = V_\ell(T).$$

Every $f \in I$ kills W. Since $fN = Nf$, its image lies in $\ker N = W$. Products of any two elements of I therefore act trivially on V, and Faltings' faithfulness gives $I^2 = 0$. The algebra C is semisimple by Poincaré complete reducibility, so $I = 0$.

Thus C embeds in $\mathrm{End}_{\mathbf{Q}}(X_*(T) \otimes \mathbf{Q}) = M_g(\mathbf{Q})$. Equality of dimensions makes this an isomorphism. A primitive rational matrix idempotent cuts out an abelian subvariety with one copy of r as its Tate module, hence an elliptic curve E/K'. The matrix units identify its g factors up to K'-isogeny. □

Proof of Proposition 21.1. Assume that $\mathcal{R}$ is nonmodular. Modularity is preserved by coefficient conjugation. At every coefficient embedding into $\overline{\mathbf{Q}}_2$, the nonsolvable residual case is modular by Corollary 21.7, and all dihedral cases other than H split at two are modular by Proposition 21.17. The ray calculation therefore puts every coefficient conjugate in the unique mixed residual class A_{m}.

The cuspidal type is rational: coefficient conjugation exchanges its two inverse constituents, preserving τ_3 and τ_6 separately. The exact special character α takes values in $\{1, -1\}$. For a fixed arithmetic Frobenius Φ at the special place,

$$\mathrm{tr}\, r_2(\Phi) = 6\alpha(\Phi) \in \{6, -6\}.$$

Every coefficient embedding fixes this number, its sign and the quadratic inertia character. Thus every conjugate belongs to the same closed problem D_{m} of Section 21.3.3. The actual characteristic-zero point gives $D_{\mathrm{m}} = \mathbf{Z}_2$. All conjugates are consequently the same two-adic representation, defined over $\mathbf{Q}_2$. Every good Frobenius trace has only one $\mathbf{Q}$-conjugate and is rational. This uses all coefficient embeddings; a single $\mathbf{Q}_2$ realization would not give the conclusion.

Potential modularity and the special finite place give a GL_2-type abelian variety B/K realizing the system. More precisely, after potential modularity the square-integrable local component gives an abelian variety by [64, Lemma 9.4.3]; [64, Lemma 9.4.4] descends the realization to K. These lemmas and the realization argument in the proof of Proposition 9.4.1

use the given potential modularity, rather than imposing that proposition's odd-prime residual hypotheses on the present characteristic-two residue.

Let $g = \dim B$ and retain the $\mathbf{Q}_2$ model r_2. The realization may enlarge the coefficient field. For a good place, its trace in that enlarged field maps under one embedding to the original trace; every other embedding gives a $\mathbf{Q}$-conjugate of that same algebraic number. Since it is rational, the enlargement introduces no additional trace values. Chebotarev and Brauer–Nesbitt give

$$V_2(B) \otimes \overline{\mathbf{Q}}_2 \simeq (r_2 \otimes \overline{\mathbf{Q}}_2)^{\oplus g}.$$

The space $\mathrm{Hom}_{G_K}(r_2, V_2(B))$ has $\mathbf{Q}_2$-dimension g by scalar extension. Its evaluation map is an isomorphism after extension to $\overline{\mathbf{Q}}_2$, hence before extension. Therefore $V_2(B) \simeq r_2^{\oplus g}$.

At the special place monodromy on r_2 has rank one, after killing the quadratic twist. Proposition 21.22 supplies an elliptic curve E/K with $V_2(E) \simeq r_2$. Every elliptic curve over a real quadratic field is modular [33, Theorem 1]. Thus r_2 and the compatible system are modular, a contradiction. □

21.4 Switching the pure menus to characteristic three

Proposition 21.23 (Switchable sources)**.** *Assume GRH. Let $\mathcal{R}$ be a weight-two compatible system of determinant cyclotomic and support at five, with either pure bounded menu. If $\bar{r}_3|_{G_H}$ is irreducible, then $\mathcal{R}$ is modular.*

Proof. At one five-place apply Lemma 21.4 to obtain the opposite kind of lift; at the other retain its kind. At three retain a crystalline weight-two component containing the original member. Prescribe no ramification elsewhere and determinant χ_3. Theorem 4.2.1 of [28] gives a global lift with precisely these component choices. Its hypotheses in characteristic three are total oddness and irreducibility on G_H; there is no extra adequacy assumption. Snowden's avoiding potential-modularity theorem and compatible-system construction [64, Theorem 5.1.2 and Section 9.2] give a potentially modular compatible system with support at five.

Choose the special component through the constructed point with nonzero monodromy. The purity observation in Section 2 retains this condition in the global system. Proposition 21.1 makes the replacement modular. Apply [3, Theorem 2.3] to the old weight-two three-adic member, whose cyclotomic residual restriction is irreducible, to return modularity to the original system. □

21.5 Reducible residual characters and the ordinary case

Lemma 21.24. *If the three-adic residue of a bounded-menu system of determinant cyclotomic and support at five is reducible, its semisimplification is $\psi \oplus \psi\bar{\chi}_3$, where ψ is quadratic. Up to interchanging the constituents,*

$$\psi = 1, \quad \psi_5, \quad \psi_\gamma, \quad \psi_5\psi_\gamma.$$

Here ψ_x denotes the character of $K(\sqrt{x})/K$. The last two possibilities have common ramified quadratic inertia at three. Reducibility forces the same quadratic inertia parity at the two five-places.

Proof. Write the constituents initially as ψ and $\bar{\chi}_3\psi^{-1}$. At five their semisimplified inertia is scalar quadratic. At three a character with values in characteristic three kills principal units, and the residue field has multiplicative group of order two. Thus ψ^2 is unramified at every finite place.

The ordinary class number of K is one: its Minkowski bound is $\sqrt{21}/2 < 2.3$ and two is inert. Its units are $\{\pm\epsilon^n\}$ and there is no norm-minus-one unit, since the norm-minus-one equation is impossible modulo three. The narrow class number is two, with class-field character $\eta_0 = \bar{\chi}_3$. Since $\psi(c)^2 = 1$ at complex conjugations and η_0 is totally odd, ψ^2 cannot equal η_0. Hence $\psi^2 = 1$.

The quadratic squareclasses supported at three and five have basis $[-1], [\epsilon], [\gamma], [5], [a]$. Their unramified-at-two subspace is

$$\langle [-\epsilon], [5], [\gamma] \rangle.$$

For an explicit check in the unramified quadratic dyadic ring, a unit class is unramified exactly when its cube is one modulo four. The identities

$$a^3 = 6a + 5, \quad \epsilon^3 = 24a + 43, \quad \gamma^3 = 12a + 21$$

give this kernel: the ramified images of -1 and a are independent, while $[\epsilon] = [-1]$. Both 5 and $-\epsilon$ are locally squares; γ is unramified nonsplit, since the residue trace of $(\gamma^3 - 1)/4 = 3a + 5$ is one.

At three both 5 and $-\epsilon$ are nonsquare units, whereas γ has odd valuation. At the five-places $a \equiv 0, 1$, their local squareclasses are

	$a \equiv 0$	$a \equiv 1$
$-\epsilon$	U	U
5	R	R
γ	1	U,

where U is unramified quadratic and R is the character of $\mathbf{Q}_5(\sqrt{5})$. Interchanging the constituents multiplies ψ by η_0. The four stated representatives and the parity assertion follow. □

Proposition 21.25. *A bounded-menu compatible system whose three-adic residue has constituents unramified at three is modular.*

Proof. By Lemmas 4.10 and 21.24, the actual crystalline weight-two three-adic source is ordinary, with constituent quotient $\bar{\chi}_3$. Skinner–Wiles, Theorem A of [62], applies: K is totally real abelian, $K(\bar{\chi}_3) = H$ is abelian over $\mathbf{Q}$, the quotient is totally odd and is nontrivial on G_{K_3}, and the ordinary quotient is unramified. Removing the Teichmüller character of the ordinary quotient gives the required pro-three character with finite inertia. The determinant is χ_3. The theorem imposes no splitting condition at three and proves modularity of the actual source. □

21.6 The cyclotomic-dihedral exception at three

Write $\omega = \bar{\chi}_3$ and let σ be the nontrivial automorphism of $K/\mathbf{Q}$. The two places v_1, v_2 above five have completion $\mathbf{Q}_5$. For determinant χ_3, the Weil representations of τ_3, τ_6 have Frobenius trace zero and square $-5I$.

21.6.1 The residual representation and the quartic fields

Lemma 21.26. *Let $\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ be absolutely irreducible, with determinant ω, unramified outside three and five, and reducible on G_H. Suppose that its inertia at either place above five is scalar quadratic. Then*

$$\bar{r} \simeq \mathrm{Ind}_{G_H}^{G_K} \xi, \qquad \xi^c = \xi^{-1}, \qquad \mathrm{ord}(\xi) = 4,$$

where c is complex conjugation, and ξ^2 cuts out $H(\sqrt{\gamma})/H$. Its linear image is D_8, and it is defined over $\mathbf{F}_3$. If η, η' are the quadratic characters of

$$L = K(\sqrt{\gamma}), \qquad L' = K(\sqrt{-3\gamma}),$$

then

$$\mathrm{ad}^0\, \bar{r} = \omega \oplus \eta \oplus \eta', \qquad \eta' = \omega\eta. \tag{15}$$

The summand normal to induction from H is $\eta \oplus \eta'$. The two scalar quadratic inertia characters at five are different. For a specified ordered pair of these inertia characters there is at most one residual isomorphism class satisfying these hypotheses.

Proof. Clifford theory gives the displayed induction. Restriction of the determinant to G_H gives $\xi^c = \xi^{-1}$. The character ξ^2 is unramified outside three. At three every finite character with values in $\overline{\mathbf{F}}_3^\times$ is tame, so ξ^2 factors through the ray class group of H with modulus its unique prime $\mathfrak{P}_3$ above three.

Here is an elementary calculation of this ray group. The field $H = \mathbf{Q}(\sqrt{-3}, \sqrt{-7})$ has discriminant 441. Its Minkowski bound is $63/(2\pi^2) < 4$. The prime ideals above two have norm four, and the unique prime above three has norm nine. Consequently $h_H = 1$. The algebraic integer

$$u = \frac{\sqrt{-3} + \sqrt{-7}}{2}, \qquad u^4 + 5u^2 + 1 = 0,$$

is a unit. Its image at $\mathfrak{P}_3$ has order four in $\mathbf{F}_9^\times$: its square is -1. The ray class exact sequence therefore bounds $\mathrm{Cl}_H(\mathfrak{P}_3)$ by a group of order two.

The quadratic extension $H(\sqrt{\gamma})/H$ is nontrivial and tame of conductor $\mathfrak{P}_3$. Indeed γ has odd valuation at three and is a unit at every other finite place. At two its squareclass is represented by $\gamma^3 \equiv 1 \pmod 4$, so the quadratic extension is unramified. At all other odd primes it is unramified as well. This proves

$$\mathrm{Cl}_H(\mathfrak{P}_3) \simeq C_2.$$

Thus ξ has order at most four. Order at most two would give $\xi^c = \xi$ and make the induction reducible. Hence its order is four, and its square is the unique nontrivial ray character.

The relation $\gamma\gamma^\sigma = -3$ identifies L' with the quadratic conjugate of L. The usual adjoint decomposition of this dihedral representation gives (15). Rotation and reflection may be represented over $\mathbf{F}_3$ by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At each five-prime of H, the unit u has residue of order eight in $\mathbf{F}_{25}^\times$, since $u^4 = -1$ modulo five. An order-four tame character at three takes the value -1 on u. A nontrivial quadratic tame character at either five-prime also takes the value -1 on u. The global unit relation therefore requires an odd number of the two quadratic inertia characters at five to be nontrivial.

Finally the order-four inertia character at three has just two possibilities, inverse to one another. Once it and the two five-adic quadratic inertia characters are fixed, two possible global characters differ by an everywhere unramified character of H. Such a character is trivial because $h_H = 1$. Replacing all characters by their inverses leaves the quadratic characters at five unchanged and gives the same induced representation. This proves uniqueness. □

Remark 21.27. The scalar-inertia hypothesis in Lemma 21.26 is automatic for the cyclotomic-reducible residue of either source considered below. An induction from a character with values in $\overline{\mathbf{F}}_3^\times$ has image of order prime to three. The order-three part of cuspidal inertia, or the unipotent part of special inertia, must therefore disappear in that image. The remaining inertia

is scalar quadratic. This argument does not discard a possible nilpotent inertia direction in a deformation.

Lemma 21.28. *Let ι be the nontrivial automorphism of L/K, and put*

$$y = \sqrt{\gamma}, \qquad z = \frac{a+y}{2} = \frac{y^2+y-1}{2}, \qquad g = \frac{1-y^2-5y}{2} = 2a - 5z.$$

Then $h_L = 1$, $\operatorname{disc}(L) = -1323$, and the minus eigenspace of the $S_{3,5}$-unit cubeclasses of L has basis $[g], [z]$ over $\mathbf{F}_3$. There is a unique degree-two prime $\mathfrak{q}$ above five. The classes in that eigenspace which are cubes in $L_{\mathfrak{q}}$ are precisely

$$1, \quad [g], \quad [g]^2.$$

Each of the two nontrivial classes has cubic conductor six at three after adjoining ζ_3.

Proof. The identities

$$N_{L/K}(z) = 1, \qquad z + z^{-1} = a, \qquad N_{L/K}(g) = -a^3, \qquad g + g^{\iota} = -a \tag{16}$$

follow directly from $y^2 = \gamma$ and $a^2 = a + 5$. The element z is an integral unit, with polynomial

$$X^4 - X^3 - 3X^2 - X + 1.$$

The relative discriminant of $\mathcal{O}_K[z]$ is $(a^2 - 4) = (\gamma)$, the prime above three. It is squarefree, so this order is maximal. Since $a = z + z^{-1}$, it follows that $\mathcal{O}_L = \mathbf{Z}[z]$ and $\operatorname{disc}(L) = 21^2 N_{K/\mathbf{Q}}(\gamma) = -1323$. Modulo two the polynomial of z is $\Phi_5(X)$, irreducible over $\mathbf{F}_2$, so two is inert in L. The signature is $(2, 1)$ and Minkowski's bound is less than five. The only possible ideal norm between two and four is therefore three. The unique prime over three is principal, generated by y, whose rational norm is -3. Hence $h_L = 1$.

The unit ranks of L and K are two and one; their torsion groups have no three-primary part. Restriction and norm split the invariant part modulo cubes because two is invertible modulo three. Thus the minus part of the unit cubeclasses has dimension one. The prime above three is fixed by ι. Of the two places of K above five, one splits in L and the other is inert: the values of $\gamma = a + 1$ modulo five are respectively one and two. The minus part of the $S_{3,5}$-divisor space has dimension one. The triviality of the class group consequently makes the minus part of the $S_{3,5}$-unit cubeclasses two-dimensional.

Both g and z belong to this minus part by (16). The element g is integral because $g = 2a - 5z$. Its norm and trace show that its valuations at the two split five-primes are one and two in some order: their sum is three, while their unequal minimum is the valuation one of $-a$. In particular its divisor class modulo three is nonzero.

At $\mathfrak{q}$ one has $a = 1$, $y^2 = 2$ in $\mathbf{F}_{25}$ and

$$g = 2, \qquad z = (1 + y)/2.$$

The element $2 \in \mathbf{F}_5^{\times}$ is a cube in $\mathbf{F}_{25}^{\times}$. The displayed residue of z has cube -1 and is not ± 1, so it has order six and is not a cube in the group of order twenty-four. Thus $[z]$ is a nonzero unit class, independent of $[g]$, and the local cube condition at $\mathfrak{q}$ kills exactly its coefficient. Units are cubes over this five-adic field exactly when their residues are cubes, since cubing is an automorphism of the principal units.

At three, y is a uniformizer of L_3 and the absolute ramification index is four. The extension $M = L_3(\zeta_3)/L_3$ is unramified quadratic. Since

$$g \equiv -1 - y \pmod{y^2},$$

the polynomial $(X+1)^3+g$ is Eisenstein over M. At a root its derivative has valuation $v_E(3)=12$, because $X+1$ is a unit. An Eisenstein generator gives the full ring of integers. The cubic extension is cyclic because ζ_3 belongs to M, and the conductor–discriminant formula gives conductor $12/(3-1)=6$. For g^2, the polynomial $(X+1)^3-g^2$ is Eisenstein and gives the same calculation. □

Corollary 21.29. *A class in $H^1(K,\mathbf{F}_3(\eta'))$ which is unramified outside three and five, is zero at the five-place where $\eta'(\mathrm{Frob}_5)=1$, and has cubic conductor at most five at three after killing its quadratic character, is zero. The analogous assertion holds for η.*

Proof. Since $\eta'=\omega\eta$, Kummer theory identifies

$$H^1(K,\mathbf{F}_3(\eta')) \simeq (L^\times/L^{\times 3})^-.$$

The support condition and $h_L=1$ make such a class an $S_{3,5}$-unit class. At the specified five-place, $\eta(\mathrm{Frob}_5)=-1$, so L/K is inert; local vanishing means being a cube at the degree-two prime $\mathfrak{q}$. Lemma 21.28 leaves only $1, g, g^2$, and its conductor calculation excludes the latter two. Quadratic conjugation exchanges L and L', proving the other assertion. □

21.6.2 The normal tangent space

Lemma 21.30. *Let a class in $H^1(K,\eta\oplus\eta')$ arise from a finite-flat tangent deformation of the residual representation of Lemma 21.26, with determinant fixed and ramification supported at three and five. If the first variation of the Frobenius trace is zero at both places above five, then this normal class is zero.*

Proof. We verify the conditions of Corollary 21.29 for each normal summand.

At a five-place where a normal character ν has Frobenius value $+1$, $H^1(K_v,\mathbf{F}_3(\nu))$ is purely unramified. The tame ramified direction would require $\nu(\mathrm{Frob}_5)=5=-1$ in $\mathbf{F}_3$. In the adjoint representation the commuting traceless line is spanned by the residual Frobenius matrix $\bar\Phi$, normalized by $\bar\Phi^2=I$. If the unramified class takes the value $b\bar\Phi$, its variation of the trace is

$$b\,\mathrm{tr}(\bar\Phi^2)=2b.$$

The trace condition therefore kills this unramified class. At the other five-place, where $\nu(\mathrm{Frob}_5)=-1$, we impose no cut on the ramified direction. In particular this proof allows first-order nilpotent inertia of order three.

At three, work first over H_3, an unramified quadratic extension of K_3 with absolute ramification index two. We may compute over $\mathbf{F}_3$, where the residue is defined. To obtain an actual $\mathbf{F}_9$-action on a finite-flat model, take a direct sum of two copies with the regular $\mathbf{F}_9$-action. Over H_3, schematic closures and quotients of the $\mathbf{F}_9$-linear generic constituents extract the upper and lower extensions. These operations preserve the existing scalar action. This construction avoids assuming that an arbitrary generic coefficient action extends to an arbitrary finite-flat model.

Fontaine's ramification bound, applied over this original field of absolute ramification index two, bounds the classical upper breaks of each extracted point field by

$$2\left(1+\frac{1}{3-1}\right)-1=2. \tag{17}$$

Here the shift by one comes from the convention $G^{(u)}=G^{u-1}$ in [31, Theorem A]. Pass to the projective kernel field, which is a quotient of the point field and retains the bound. The diagonal ratio has order two, although the original diagonal characters have order four. Killing

the ratio by its tame quadratic extension multiplies positive upper breaks by two. The resulting cubic characters have upper break at most four, hence conductor at most five.

If extracting a quadratic-adjoint summand requires combining the two extensions, take the compositum of their point fields first. The direct sum of their finite-flat groups satisfies the same bound, and every projected character field is a quotient. The relevant field over L_3 acquires ζ_3 by an unramified extension, which does not alter the conductor. Thus Corollary 21.29 applies to both summands. Scalar extension of its $\mathbf{F}_3$ calculation gives the same conclusion for any finite residual coefficient field. □

Remark 21.31. The projective quotient in this proof is essential. We apply Fontaine's bound to actual finite-flat extensions over H_3 before making any ramified base change. We do not twist a finite-flat representation by a ramified character and assert that it remains finite flat.

21.6.3 Two cuspidal places

Theorem 21.32. *Let $r : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_3)$ be continuous and totally odd, with determinant χ_3, unramified outside three and five, and crystalline of labelled Hodge–Tate weights $\{0,1\}$ at three. Suppose its type at each place above five is τ_3 or τ_6. If its residue is absolutely irreducible but reducible over H, then r is induced from a geometric character of G_H. In particular r is modular.*

Proof. By Lemma 21.26, the residue is $\bar{r} = \mathrm{Ind}_H^K \xi$. Enlarge the residue field to contain $\mathbf{F}_9$ and choose an inducing basis. Put $J = \mathrm{diag}(1,-1)$. Then $J\bar{r}(g)J^{-1} = \omega(g)\bar{r}(g)$ and $J^2 = 1$.

Let D be the universal fixed-determinant deformation ring of $\bar{r}$, finite flat at three, unramified outside three and five, and with the prescribed finite inertia at the two five-places and Frobenius trace zero there. These conditions are represented by a closed quotient of the universal deformation ring: finite-flatness is a stable condition, and the local finite-inertia and Frobenius relations are closed algebra relations; see [72, Corollary 3.3.5 and Section 5.2]. The actual source gives a characteristic-zero point of D.

The character ω is unramified at every finite place of K. Consequently the operation

$$r_A \longmapsto J^{-1}(r_A \otimes \omega)J$$

preserves these conditions, the determinant and the identified residue. Unramified twisting preserves finite-flatness by descent from the unramified extension killing the twist. At five, twisting preserves the prescribed inertia and the zero Frobenius trace. This operation defines an involution of D.

On $\mathrm{ad}^0 \bar{r} = \omega \oplus \eta \oplus \eta'$ it acts as $+1$ on the diagonal summand and as -1 on the two normal summands. The anti-invariant projection of an admissible tangent is again admissible, since the involution preserves the deformation conditions and two is invertible. This normal tangent is zero by Lemma 21.30: diagonal variations have zero Frobenius trace at five, since Frobenius is offdiagonal in the inducing basis. Hence the involution is trivial on the entire relative tangent, and therefore on the relative cotangent. Lemma 12.1, with $m = 2$, makes it the identity on D.

It follows that the actual irreducible source satisfies $r \simeq r \otimes \omega$. An intertwiner squares to a scalar by Schur's lemma. After rescaling, its two eigenspaces are preserved by G_H and interchanged by the other coset. Thus $r = \mathrm{Ind}_H^K \psi$. The character ψ is de Rham and finitely ramified, hence algebraic by global class field theory [57, Chapter III]. Since H/K is CM and the two characters are distinct, quadratic automorphic induction is regular algebraic and cuspidal [4, Chapter III]. □

21.6.4 A finite-flat selfextension calculation

Lemma 21.33. *Let $F = H_3$, and let k be a finite coefficient field containing $\mathbf{F}_9$. Let $\theta : G_F \longrightarrow k^\times$ have the order-four tame inertia character occurring in Lemma 21.26. A finite-flat selfextension of θ, equipped with an actual k-action on its finite-flat model, is unramified after dividing by the common character θ on its generic fibre.*

Proof. Take the schematic closure of the generic subline and the finite-flat quotient. The exact finite-flat/Kisin-module equivalence and its coefficient versions apply to this sequence [44, Theorem 1.1.3, Proposition 1.1.11 and Lemmas 1.2.4–1.2.5]. The two coefficient idempotents give rank-one endpoints with Frobenius exponents (r_0, r_1). Since $e(F) = 2$, one has $0 \leq r_i \leq 2$. The tame exponent is $r_0 + 3r_1$ modulo eight. The two order-four inertia characters have exponents two and six, whose unique possibilities are respectively

$$(2, 0), \qquad (0, 2).$$

The rank-one models of the two endpoints are consequently isomorphic, including their unramified constants.

Lemma 11.6 applies with $p = 3$, $e = 2$ and $f = 2$. Thus the only selfextensions are the unramified ones after dividing by θ. This uses the exact rank-two $k[[u]]$ extension furnished above, without assuming freeness over $k[t]/(t^2)[[u]]$. □

21.6.5 Two special places

Theorem 21.34. *Let $\mathcal{R}$ be a totally odd, absolutely irreducible, strongly compatible two-dimensional system over K, with cyclotomic determinant and labelled Hodge–Tate weights $\{0, 1\}$. Assume all-place Weil–Deligne compatibility, support at five away from the coefficient characteristic, and a crystalline three-adic member at three. At $v_i \mid 5$ suppose it is special with nonzero monodromy and fixed quadratic twisting character $\alpha_i : G_{\mathbf{Q}_5} \longrightarrow \{\pm 1\}$, including the unramified sign. If the characteristic-three residue is absolutely irreducible but reducible over H, then $\mathcal{R}$ is modular.*

Proof. Again Lemma 21.26 applies. Fix its model over $\mathbf{F}_3$. Form the universal global deformation ring with determinant χ_3, support at three and five, and the finite-flat condition at three, and impose the following closed local equations at v_i. For arithmetic Frobenius Φ_i put $b_i = \alpha_i(\Phi_i) \in \{\pm 1\}$, require

$$\operatorname{tr} r(\Phi_i) = 6b_i, \qquad r(g)|_{\ell_i} = \chi_3(g)\alpha_i(g), \tag{18}$$

where ℓ_i is the $5b_i$-eigenline of $r(\Phi_i)$. The Frobenius eigenvalues $5b_i, b_i$ have difference $4b_i$, a unit at three. Thus the eigenline is functorial and the equations are closed; the determinant supplies the quotient character α_i.

For a precise construction of this quotient, impose the stable condition on finite-cardinality $\mathbf{Z}_3[G_{K,\{3,5\}}]$-modules consisting of finite-flatness at three and the algebra annihilator identities

$$(g - \chi_3(g)\alpha_i(g))(h - \alpha_i(h)) = 0 \qquad (g, h \in G_{K_{v_i}})$$

at each five-place. Finite-flatness is stable under subquotients and finite direct sums, as are these identities. Wake–Wang-Erickson's stable-condition theorem gives the universal pseudodeformation quotient [72, Theorem 2.5.5 and Section 5.2]; impose the fixed determinant χ_3 as well. Since the residual representation is absolutely irreducible, pseudodeformations and ordinary deformations agree, and the universal Cayley–Hamilton algebra is $M_2(D)$ over the resulting ring D. The displayed annihilator identities are equivalent to (18), because the two residual Frobenius eigenvalues are distinct. This constructs the stated deformation condition without

requiring uniqueness of finite-flat models at $e = p - 1$. Every actual special source with these characters gives a point of D.

We show that its relative tangent over $\mathbf{Z}_3$ is zero. Its normal projection belongs to $H^1(K, \eta \oplus \eta')$. Equation (18) has zero first trace variation in characteristic three. The tangent construction of Lemma 21.30, including its finite-flat extraction, kills both normal summands. After a strict conjugation, the tangent lies in the remaining diagonal summand ω of (15).

At a five-place, residual inertia is scalar and residual Frobenius is a reflection with distinct eigenlines. In an inducing basis, a traceless diagonal matrix takes either Frobenius eigenline to the other. The prescribed character on ℓ_i in (18) therefore forces the diagonal tangent to vanish on inertia. Movement of the line cannot compensate for this error, since the residual inertia matrix is scalar. The diagonal class is consequently unramified at both places above five.

At three, compute the tangent over $\mathbf{F}_3$ and equip a direct sum of two copies of its finite-flat model with the regular $\mathbf{F}_9$-action. Over H_3, take schematic closures and quotients of its $\mathbf{F}_9$-linear generic constituents. These operations preserve the existing coefficient action and extract the selfextensions of the two order-four characters. Lemma 21.33 shows that the restriction of the diagonal class to H_3 is unramified. The character ω over K_3 is the nontrivial unramified quadratic character. Hence

$$H^1_{\mathrm{unr}}(K_3, \mathbf{F}_3(\omega)) = \mathbf{F}_3/(\omega(\mathrm{Frob}_3) - 1)\mathbf{F}_3 = 0.$$

The extension H_3/K_3 is unramified of degree two; restriction is injective and does not change inertia. The diagonal class is therefore locally zero at three.

Globally Kummer theory identifies $H^1(K, \mathbf{F}_3(\omega)) = K^\times/K^{\times 3}$. The class number and unit calculation of Lemma 21.24 applies. The support condition, unramifiedness at five, and the trivial class group leave just $[\epsilon]$ and $[\gamma]$ as possible global classes. A cube at three has valuation divisible by three, so the γ coefficient is zero. Finally $\epsilon = 1 + \gamma$, while a unit cube with residue one is 1 modulo γ^3, since $v_\gamma(3) = 2$. Neither ϵ nor ϵ^2 is such a cube. The diagonal tangent vanishes.

All deformation data are defined over $\mathbf{Z}_3$. The vanishing of the relative tangent and Nakayama's lemma make D a quotient of $\mathbf{Z}_3$. An actual characteristic-zero source point forces

$$D = \mathbf{Z}_3. \tag{19}$$

Thus the source is the unique lift with its ordered pair of exact special characters.

It remains to compare this unique lift with its quadratic conjugate; this also avoids any need to control all other three-adic coefficient places. Let η_0 be the quadratic character of H/K and let ψ_5 be that of $K(\sqrt{5})/K$. At both five-places η_0 is the unramified quadratic character and ψ_5 is the quadratic character of $\mathbf{Q}_5(\sqrt{5})$. They form a basis of the local quadratic characters, with the same restrictions at the two places. Put $\beta = \alpha_1\alpha_2$ and take the unique

$$\delta \in \{1, \eta_0, \psi_5, \eta_0\psi_5\}$$

whose restriction at both places is β. The inertia parity in Lemma 21.26 makes β ramified, so here

$$\delta = \psi_5 \quad \text{or} \quad \delta = \eta_0\psi_5.$$

Both characters are unramified at three. The extension H/K is unramified at every finite place; at three the unit five is a nonsquare in $\mathbf{F}_3$ and gives an unramified quadratic extension as well.

The representation $r_3^\sigma \otimes \delta$ has the original ordered pair (α_1, α_2), the same determinant and support, and the same crystalline condition. Its residue has the same restored ordered scalar inertia characters, hence is the same residual isomorphism class by Lemma 21.26. It is another point of D. Equation (19) gives

$$r_3^\sigma \simeq r_3 \otimes \delta. \tag{20}$$

Both possible characters δ are restricted from $G_{\mathbf{Q}}$: ψ_5 is cut out by $\mathbf{Q}(\sqrt{5})$ and $\eta_0\psi_5$ by $\mathbf{Q}(\sqrt{-15})$. Strong compatibility transports (20) to the whole system. Lemma 12.3, with $\eta = \delta$, gives its modularity directly. □

Remark 21.35. The class, unit and conductor calculations above are elementary and use no GRH. The corresponding PARI/GP certificates have, in the basis $[g], [z]$, strict-five masks $0, 1, 2$ and conductor-at-most-five masks $0, 4, 8$, with intersection $\{0\}$.

The exact calculation is supplied in [49, `d21`].

21.7 The nonordinary reducible characteristic-three source

This section proves a source modularity theorem for both pure local menus. Its input is an actual characteristic-zero compatible system. The proof uses finite-flat extensions at the ramified prime above 3, a universal pseudodeformation ring with explicit local conditions, and descent under quadratic conjugation. In particular, it does not replace source modularity by modularity of a semisimplified residue.

Let σ be the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$, and write $\omega = \overline{\chi}_3|_{G_K}$. The character ω cuts out $K(\sqrt{-3})/K$. Let ψ_γ cut out $K(\sqrt{\gamma})/K$. The norm identity gives

$$\psi_\gamma^\sigma = \psi_\gamma\omega. \tag{21}$$

Theorem 21.36 (Nonordinary reducible source modularity)**.** *Let $\mathcal{R}$ be an odd, absolutely irreducible, strongly compatible system over K of parallel weight two and cyclotomic determinant, good outside the two primes above 5. Assume its 5-adic companion is absolutely irreducible, as holds for the potentially modular systems constructed in the controlled reduction chain. Suppose its local types at the two primes over 5 are either both quadratic twists of Steinberg with nonzero monodromy, or both the tame dihedral supercuspidal types with inertia of order 3 or 6.*

Suppose the semisimplification of its characteristic-three residue is globally reducible, and both constituent characters are ramified quadratic at the prime over 3. Then $\mathcal{R}$ is modular.

More precisely, after a quadratic twist unramified outside 5, the system has Frobenius traces invariant under quadratic conjugation. Its 5-adic companion descends to an odd geometric representation of $G_{\mathbf{Q}}$, and modularity follows from Pan's theorem over $\mathbf{Q}$ at the prime 5.

21.7.1 Quadratic normalization

In the reducible characteristic-three case, each constituent has quadratic inertia at a prime over 5. The residues of the fundamental unit ϵ at the two primes over 5 are 2 and 3. The global unit relation therefore forces the two quadratic ramification parities to agree. If both are ramified, twist by the quadratic character ψ_5 of $K(\sqrt{5})/K$. This removes both ramification factors, preserves goodness at 3, and is trivial at the dyadic completion. After this normalization, the two types are either unramified quadratic twists of Steinberg at both places, or the SC types of inertia order 3 at both places.

Lemma 21.24 now gives, up to interchanging labels,

$$\chi_1 = \psi_\gamma, \qquad \chi_2 = \psi_\gamma\omega, \tag{22}$$

with distinct characters defined over $\mathbf{F}_3$. We henceforth work with the normalized system and the labels (22).

21.7.2 The finite-flat extension line at three

Let $F = K_{\mathfrak{p}_3} = \mathbf{Q}_3(\sqrt{21})$. Its ramification index is 2 and its residue field is $\mathbf{F}_3$. The character $\omega|_{G_F}$ is the nontrivial unramified quadratic character: over F, adjoining $\sqrt{-3}$ amounts to adjoining the square root of the nonsquare unit -7. Thus the two characters in (22) have the same ramified quadratic inertia and opposite unramified factors.

Lemma 21.37 (The local finite-flat line)**.** *For either orientation of the pair* (22)*, the finite-flat subspace of $H^1(F, \mathbf{F}_3(\omega))$ has dimension one. Under the Kummer identification it is spanned by $[a\epsilon]$. For any finite $k/\mathbf{F}_3$, the finite-flat subspace with k coefficients is the scalar extension of this line.*

Proof. First work over $\mathbf{F}_3$. A rank-one finite-flat model of either character has unique Breuil–Kisin exponent $r = 1$: the inertia exponent is odd, whereas height one permits only $0 \le r \le 2$. The two unramified factors differ by -1.

Any finite-flat extension gives an exact extension of these rank-one Kisin modules. Indeed, take the schematic closure of the generic rank-one subrepresentation and then the finite-flat quotient. The exact correspondence on objects killed by p is supplied by [44, Theorem 1.1.3 and Proposition 1.1.11]; the coefficient versions are [44, Lemmas 1.2.4–1.2.5]. In suitable bases the Frobenius matrix is, up to a common unramified factor,

$$\Phi = \begin{pmatrix} u & h(u) \\ 0 & -u \end{pmatrix}, \qquad h \in \mathbf{F}_3[[u]]. \tag{23}$$

Every such matrix has height at most one, since $u^2\Phi^{-1}$ is integral. A change of the upper lift by $f(u)$ changes h by a nonzero scalar multiple of

$$u\big(f(u) + f(u^3)\big).$$

The operator $f \mapsto f + f(u^3)$ is bijective on $\mathbf{F}_3[[u]]$: its constant term is multiplication by 2, and its other coefficients are solved recursively. Thus every extension is represented by a constant h and the Kisin extension space is one-dimensional. This proves that the actual finite-flat Galois extension space has dimension at most one. It uses no injectivity assertion for restriction to G_{F_∞}.

We identify a nonzero class using the curve with number field label `2.2.21.1-25.1-a4` in [45]. In generalized Weierstrass notation its model is

$$E = [a, a, a+1, -a-3, -a-2]. \tag{24}$$

Its conductor is (5) [45], so it has good reduction at 3; counting its points gives $a_{\mathfrak{p}_3}(E) = 0$. Its 3-division polynomial factors as

$$3\left(x - \frac{a+1}{3}\right)\left(x^3 + (2a+2)x^2 + (2a+3)x + \frac{4a+7}{3}\right). \tag{25}$$

The point above the rational root has squared doubled generalized y-coordinate $(61a + 109)/9$, and the identity

$$\frac{61a + 109}{9\gamma} = \left(\frac{2a+3}{3}\right)^2$$

identifies its squareclass as γ. The stable subgroup therefore has character $\chi_1 = \psi_\gamma$; the quotient has character $\chi_2 = \psi_\gamma\omega$.

Twisting $E[3]$ by χ_2 gives an extension of the trivial character by ω. Its action on the three projective lines other than the invariant line has affine form

$$z \longmapsto \omega(g)z + c(g), \qquad [c] \in H^1(K, \mathbf{F}_3(\omega)).$$

The cubic factor of (25) parametrizes those three lines. Depressing it gives

$$P = -2a - 5, \qquad Q = \frac{34a + 55}{9}, \qquad \frac{Q^2}{4} + \frac{P^3}{27} = \frac{50a}{9} + \frac{1075}{108} = \left(\frac{5(4a + 7)}{18}\right)^2.$$

One Cardano radicand is $-3a - 5 = -a\epsilon$, and the other is its inverse times a cube. The splitting field of the cubic is therefore $K(\sqrt[3]{-a\epsilon}, \zeta_3)$. The affine action on these roots identifies the actual extension line with $\langle [a\epsilon] \rangle$; changing an affine coordinate can multiply the cocycle by -1, which does not change the line. This identifies the extension, rather than only its quadratic resolvent. The class $[a\epsilon]$ is locally nonzero at 3, as the unit calculation below proves. Since $E[3]$ is finite-flat at 3, the upper bound is attained. Conjugating the curve supplies the opposite orientation; the same calculation below shows that its local line is unchanged.

Finally let $k/\mathbf{F}_3$ be finite, and let c be a k-valued extension whose underlying $\mathbf{Z}_3$-module is finite-flat. Pull back along the $\mathbf{F}_3$-line in its quotient. For every $\mathbf{F}_3$-linear functional $\lambda : k \to \mathbf{F}_3$, quotient the resulting kernel by $\ker(\lambda)\chi_1$. The resulting two-dimensional $\mathbf{F}_3$-extension is finite-flat, by stability under subquotients, and has class $\lambda \circ c$. Each coordinate class lies in the line just found; therefore c lies in that line tensored with k. Conversely scalar extension of a finite-flat extension is finite-flat by direct sums and subquotients. This argument avoids assuming that arbitrary generic coefficient endomorphisms extend to a chosen finite-flat model. □

21.7.3 The global plane and local evaluations

Write classes in $K^\times/K^{\times 3}$ additively and put

$$b = [a\epsilon], \qquad s = [5], \qquad d = [5a\epsilon] = s + b.$$

Lemma 21.38 (The global finite-flat extension plane)**.** *The global classes unramified outside* $\{3, 5\}$ *and finite-flat at* 3*, for either orientation of* (22)*, form*

$$V = \langle b, s \rangle = \langle s, d \rangle. \tag{26}$$

Quadratic conjugation acts by $\sigma(s) = s$*,* $\sigma(d) = -d$*. If* $\mathfrak{q}$ *is the prime over* 5 *at which* a *has valuation one, the two local restriction maps, in the basis* (s, d)*, are*

$$\ell_{\mathfrak{q}} = (1, -1), \qquad \ell_{\bar{\mathfrak{q}}} = (1, 1). \tag{27}$$

Moreover

$$\chi_1(\mathrm{Frob}_{\mathfrak{q}}) = 1, \qquad \chi_1(\mathrm{Frob}_{\bar{\mathfrak{q}}}) = -1, \tag{28}$$

and the values of χ_2 *are the opposite ones.*

Proof. The class and unit calculation in Lemma 21.24 shows that the global cubic classes with the indicated support have basis

$$[\epsilon], \quad [\gamma], \quad [a], \quad [\bar{a}], \qquad \bar{a} = 1 - a,$$

and dimension four. We compute their local-cube kernel at 3 elementarily. Normalize $v_F(\gamma) = 1$ and put $U_i = 1 + \mathfrak{p}_F^i$. The identity

$$\epsilon^3 - 5a\epsilon = 9\epsilon$$

gives

$$\frac{5a\epsilon}{\epsilon^3} = 1 - \frac{9}{\epsilon^2} \in U_4.$$

The 3-adic logarithm identifies cubing $U_2 \to U_4$ with multiplication by 3, so this map is an isomorphism. Thus d is locally a cube. On the other hand

$$\epsilon = 1 + \gamma \in U_1 \setminus U_2, \qquad -a\epsilon = 1 - 3\epsilon \in U_2 \setminus U_3.$$

Every unit cube lies in $\{\pm 1\}U_3$. Therefore $[\epsilon]$ and $[a\epsilon]$ are independent local cubic classes. The valuation of γ gives a third independent class. Since $\mu_3 \not\subset F$, the dimension of $F^\times/F^{\times 3}$ is $[F : \mathbf{Q}_3] + 1 = 3$. The local-cube kernel on the four-dimensional global space is consequently exactly $\langle d \rangle$. Lemma 21.37 now gives (26).

The norm identity $N(a\epsilon) = -5$ gives $\sigma(b) = s - b$, and hence $\sigma(d) = -d$. In particular $\sigma(b)$ has the same local class as b, since d is locally zero and $-2 = 1$ in $\mathbf{F}_3$. At either 5-adic place, units have no nonzero cubic Kummer classes because $3 \nmid 5 - 1$. The valuations of s are $(1, 1)$ and those of d are $(2, 1)$, giving (27). Finally γ reduces to 1 at $\mathfrak{q}$ and to 2 at $\bar{\mathfrak{q}}$; their quadratic residue symbols give (28). □

21.7.4 A universal pseudoring with explicit local conditions

Fix a finite residue coefficient field $k/\mathbf{F}_3$ containing that of the actual source, and put $W = W(k)$. Let $G = G_{K,\{3,5\}}$. We impose a stable condition $\mathcal{C}$ on finite-cardinality $\mathbf{Z}_3[G]$-modules: finite-flatness at 3, and the local algebra conditions at 5 specified below. Here stability means closure under isomorphisms, subquotients, and finite direct sums. Finite-flatness is stable by [72, §5.2]; an algebra annihilator condition is stable by definition.

The universal pseudodeformation ring with residual pseudorepresentation $\chi_1 \oplus \chi_2$ and condition $\mathcal{C}$ exists by [72, Theorem 2.5.5]. It is a complete noetherian local W-algebra. Impose in addition the closed condition that the determinant equals χ_3, and call the resulting ring R. Every normalized actual source defines a point of R. Indeed, all finite quotients of a stable lattice in its crystalline weight-two representation are finite-flat at 3, and the exact local types at 5 satisfy the algebra identities below.

Because the residual pseudorepresentation is multiplicity-free, its universal Cayley–Hamilton algebra has generalized-matrix-algebra form

$$\mathcal{E} = \begin{pmatrix} R & B \\ C & R \end{pmatrix}.$$

The reducibility ideal J is the image of $B \otimes_R C \to R$; see [72, Proposition 4.2.2]. Writing $\mathfrak{m}$ for the maximal ideal of R, the extension interpretation [72, Theorem 4.3.5] gives

$$(B/\mathfrak{m}B)^\vee \subseteq \mathrm{Ext}^1_{\mathcal{C}}(\chi_2, \chi_1), \qquad (C/\mathfrak{m}C)^\vee \subseteq \mathrm{Ext}^1_{\mathcal{C}}(\chi_1, \chi_2). \tag{29}$$

Before the fixed-determinant quotient these are equalities; the inclusions suffice afterwards. The construction is functorial for quadratic conjugation whenever the local conditions are exchanged.

For a normalized SP type, let α_v be its exact unramified quadratic twisting character. We impose

$$\big(g - \chi_3(g)\alpha_v(g)\big)\big(h - \alpha_v(h)\big) = 0 \qquad (g, h \in G_{K_v}). \tag{30}$$

These are relations in the completed group algebra. They hold on every extension of α_v by $\chi_3\alpha_v$, hence on every stable lattice in the actual Steinberg representation. The two residual Frobenius eigenvalues are distinct. Thus the relations allow ramification only in the orientation whose invariant character is $\chi_3\alpha_v$, and they fix the local characters exactly. The universal condition is permitted to contain points of zero monodromy; the actual source has nonzero monodromy and is included.

For a normalized SC3 type, choose arithmetic Frobenius Φ and a generator T of pro-3 tame inertia, so that $\Phi T \Phi^{-1} = T^5$. Kill wild inertia and the prime-to-3 tame inertia, and impose

$$T^2 + T + 1 = 0, \qquad \Phi^2 + 5 = 0. \tag{31}$$

The actual normalized SC representation has inertial eigenvalues ζ_3, ζ_3^{-1}, exchanged by Frobenius. Its cyclotomic determinant gives $\Phi^2 = -5$, so it satisfies these relations. Replacing Frobenius

by an inertial multiple, or changing the tame generator, preserves the conditions on this quotient. Thus quadratic conjugation exchanges the same local condition at the two places. In characteristic 3, its first relation becomes

$$(T-1)^2 = 0. \tag{32}$$

21.7.5 The diagonal pseudotangent

Lemma 21.39 (Vanishing of the reducible tangent)**.** *For either pure local menu, the ring R satisfies*

$$\mathfrak{m}/(3R + \mathfrak{m}^2 + J) = 0. \tag{33}$$

Consequently multiplication induces a surjection

$$(B/\mathfrak{m}B) \otimes_k (C/\mathfrak{m}C) \twoheadrightarrow \mathfrak{m}/(3R + \mathfrak{m}^2). \tag{34}$$

Proof. A tangent in the reducible quotient R/J has constituent characters

$$\chi_1(1+\varepsilon f), \qquad \chi_2(1-\varepsilon f), \qquad f \in H^1(G,k),$$

where the opposite signs follow from the fixed determinant. The constituent character deformations are themselves finite-flat modules at 3, by the stability of $\mathcal{C}$ and [72, Proposition 4.3.4].

Fontaine's bound for a finite-flat module killed by p^n is triviality of $G^{(u)}$ for $u > e(n+1/(p-1))$; the convention in the cited statement is $G^{(u)} = G^{u-1}$, where the latter is classical upper numbering [31, pp. 515–516, Theorem A]. For $p = 3$, $n = 1$, and $e = 2$, the classical upper break is at most 2. The abelian cubic character f therefore has conductor at most $\mathfrak{p}_3^3$. Its tame quadratic factor does not change positive ramification groups. At a prime over 5, a cyclic cubic character is unramified because $3 \nmid 5 - 1$.

We need only the ray group of conductor $\mathfrak{p}_3^3$ and its Frobenius classes at five. These have the following elementary calculation. The group $(\mathcal{O}_K/\mathfrak{p}_3^3)^\times$ has order 18. The unit $\epsilon = 1 + \gamma$ has exact order 3 modulo γ^3, so the image of $\langle -1, \epsilon \rangle$ has order 6. Class number one then gives ray group C_3. The class of $\mathfrak{q} = (a)$ is nontrivial: $-a = 1 - \gamma$ differs from $1, \epsilon, \epsilon^2$ with valuations $1, 1, 2$, respectively, so it does not belong to the unit image modulo γ^3. Its conjugate also has nontrivial class because conjugation acts as an automorphism of the ray group. Infinite places contribute only 2-primary data and do not affect this cubic quotient.

In the SP case, (30) fixes the two Frobenius characters, hence forces $f(\Phi) = 0$. In the SC case, $\Phi^2 = -5$ gives the same conclusion, since the residual Frobenius eigenvalues are ± 1 and 2 is invertible. Therefore $f = 0$, proving (33). The ideal J is generated by products from B and C, and $\mathfrak{m}J \subseteq \mathfrak{m}^2$; this proves (34). □

21.7.6 The SP/SP action of quadratic conjugation

Lemma 21.40 (The two SP signs agree)**.** *An irreducible actual source in the situation of Theorem 21.36 has equal normalized unramified Steinberg signs at the two primes over 5. For these signs, quadratic conjugation acts trivially on the relative cotangent space of R.*

Proof. Suppose first that both signs are -1. The invariant monodromy character is $\chi_3\alpha_v$ and reduces to the Frobenius eigenvalue $-\alpha_v$. Using (28), the permitted global extension spaces in (29) are contained in

$$\ker \ell_{\bar{\mathfrak{q}}} = \langle b \rangle, \qquad \ker \ell_{\mathfrak{q}} = \langle \sigma(b) \rangle, \tag{35}$$

respectively. If both signs are $+1$, the roles of the lines are exchanged. If the signs are opposite, one orientation must be unramified at both 5-places. Its global extension space is then contained in

$$\ker \ell_{\mathfrak{q}} \cap \ker \ell_{\bar{\mathfrak{q}}} = 0.$$

Nakayama gives $B = 0$ or $C = 0$, so $J = 0$ and every point of R is globally reducible. This contradicts the actual irreducible source. Equivalently, Ribet's lattice lemma would require nonsplit residual extensions in both orientations, while one of them has no permitted nonzero extension. The signs therefore agree.

Quadratic conjugation preserves the resulting local conditions, interchanges the two residual characters, and interchanges the two lines (35). Its action on their product is $+1$. One can check the sign directly from $b = d - s$ and $\sigma(b) = s - b = -(d + s)$. For a basis-independent verification, choose a lift $h \in G_{\mathbf{Q}}$ of σ which fixes ζ_3; it exists since $K \cap \mathbf{Q}(\zeta_3) = \mathbf{Q}$. The coefficient action on ω is then trivial, and $\chi_1(h^2) = \chi_2(h^2)$. Squared transport is inner by a residual scalar, so the two exchange factors multiply to $+1$. The surjection (34) now shows that quadratic conjugation fixes the entire relative cotangent space. □

21.7.7 The SC/SC action of quadratic conjugation

Lemma 21.41 (The SC anti-invariant product vanishes)**.** *For the normalized SC/SC menu, quadratic conjugation acts trivially on the relative cotangent space of R.*

Proof. Both extension spaces in (29) are subspaces of $V \otimes_{\mathbf{F}_3} k$. Hence the tensor surjecting onto the relative cotangent space is a quotient of $V^\vee \otimes V^\vee$. Quadratic conjugation swaps the factors and acts on each by $s \mapsto s$, $d \mapsto -d$. The scalar in this statement is fixed by the same choice of h as in the preceding proof. The local equation (32) kills the image of $\ell_v \otimes \ell_v$ in the cotangent space. Indeed, in the universal generalized matrix algebra write

$$T - 1 = \begin{pmatrix} x & b_T \\ c_T & y \end{pmatrix}, \qquad x, y \in \mathfrak{m}.$$

Its upper-left squared entry is $x^2 + b_T c_T$. The exact relation over R is $(T - 1)^2 = -3T$, so $b_T c_T \in \mathfrak{m}^2 + 3R$; therefore $b_T c_T$ is zero in $\mathfrak{m}/(3R + \mathfrak{m}^2)$. The classes of b_T and c_T are evaluation on local inertia, which are the functionals in (27), up to a harmless nonzero change of local generator.

The evaluations in (27) are $s^\vee - d^\vee$ and $s^\vee + d^\vee$. Thus Lemma 12.2 makes the action on the relative cotangent trivial. The local calculation above concerns products of opposite offdiagonal extensions, not only traces of representation tangents. □

21.7.8 From the cotangent to the entire pseudoring

Quadratic conjugation defines a $W(k)$-linear involution of R: its square is induced by an inner automorphism of G_K, which does not change a pseudorepresentation. Lemmas 21.40 and 21.41, followed by Lemma 12.1, show that this involution is the identity. Consequently every normalized actual source satisfies

$$\operatorname{tr} r_3(g) = \operatorname{tr} r_3(hgh^{-1}) \qquad (g \in G_K). \tag{36}$$

No assertion that R is finite over $W(k)$, that its whole cotangent space is zero, or that its traces are rational is needed. The essential fact is that an involution of order prime to the residue characteristic cannot survive when its anti-invariant cotangent is zero.

21.7.9 Whole-system descent and the rational theorem

Completion of the proof of Theorem 21.36. Strong compatibility transports (36), first at every good Frobenius, to all companions of the normalized system. Chebotarev identifies the semisimplifications of the system and its quadratic conjugate. In particular its absolutely irreducible 5-adic member is invariant under σ.

Lemma 12.3 makes the normalized system modular. Undoing the finite quadratic normalization proves the claim. □

Remark 21.42. The seed has $a_v = -1$ at both places over five and $a_{\mathfrak{p}_3} = 0$. The displayed identities determine its local Kummer class and the local-cube kernel explicitly. The source proof uses these finite-flat and pseudodeformation calculations and no dyadic modularity assumption.

The exact calculation is supplied in [49, `d21`].

21.8 Completion of the bounded source theorem

Theorem 21.43. *Assume GRH. Let $\mathcal{R}$ be a potentially modular compatible system over K, of determinant cyclotomic and labelled weights $\{0,1\}$, with conductor supported at five. Suppose each five-place has one of the bounded types* SC *or* SP*. Then $\mathcal{R}$ is modular.*

Proof. The mixed case is Proposition 21.1. A pure special source with reducible five-adic residue is covered directly by Proposition 21.2. For the other pure sources, consider the three-adic member, which is crystalline at three. If its residue remains irreducible on G_H, Proposition 21.23 applies. If the residue is irreducible but its restriction is reducible, the dihedral source theorems of the preceding sections apply: induction from H for the cuspidal pair, and fixed-character rigidity followed by descent under quadratic conjugation for the special pair.

Otherwise Lemma 21.24 gives the complete constituent list. The two unramified-at-three possibilities are modular by Proposition 21.25. In the remaining two, twist by ψ_5 when necessary to remove the common quadratic inertia at five. This twist is unramified at three and preserves the determinant and crystalline condition. The nonordinary reducible source theorem proved above gives invariance under quadratic conjugation and modularity. These alternatives are exhaustive for every finite coefficient field and every residual lattice. □

Theorem 21.44 (Unrestricted residual base at five)**.** *Assume GRH. Every irreducible representation*

$$\bar{r} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_5), \qquad \det \bar{r} = \bar{\chi}_5,$$

unramified outside five is modular. No local finite-flatness, tameness, coefficient-field or conductor-exponent restriction is imposed.

Proof. Solvable projective image is handled by an odd Artin lift and solvable Artin automorphy, with weight raising to obtain a regular residual witness. In the nonsolvable case the restriction to $G_{K(\zeta_5)}$ is irreducible. The exceptional projective-$\mathrm{PGL}_2(\mathbf{F}_5)$ condition cannot occur: $[K(\zeta_5) : K] = 4$, whereas the abelianization of $\mathrm{PGL}_2(\mathbf{F}_5)$ has order two.

Lemma 21.3 provides actual weight-two local lifts at both five-places with determinant χ_5. Prescribe their components and no ramification elsewhere. Theorem 4.2.1 of [28] gives a global lift. Snowden's potential-modularity and compatible-system construction gives a potentially modular weight-two system with the selected menu. At a selected special component, the purity observation in Section 2 retains nonzero monodromy. The system is crystalline at two and three by all-place compatibility. Theorem 21.43 makes the actual system modular, and its five-adic reduction is the original $\bar{r}$. □

21.9 The arbitrary-conductor transfer

We verify the terminal character condition and apply Theorem 1.2 with reset prime five. Its terminal source is allowed arbitrary regular weights.

21.9.1 Characters supported at five

Lemma 21.45. *Every finite characteristic-five character of G_K unramified outside five extends to $G_{\mathbf{Q}}$. If a representation of determinant $\bar{\chi}_5$ and support at five is reducible, its constituent quotient is exactly $\bar{\chi}_5$ or $\bar{\chi}_5^{-1}$.*

Proof. Such a character kills principal units at five. Since $h_K = 1$, it is determined by two exponents m, n modulo four at the two residue groups and two real signs. The positive unit ϵ has residues $2, 3 = 2^{-1}$ at the two five-places, so the unit relation gives $m = n$. The relation from -1 then says that the real signs agree. The ray-character data are invariant under quadratic conjugation, hence so is the character. An invariant character extends across the index-two group after adjoining a square root of its value on the square of a lift of the nontrivial automorphism. The target $\overline{\mathbf{F}}_5^{\times}$ is divisible, and the extension still has finite image.

The equality of the tame exponents shows that ψ^2 has at both five-places either trivial inertia or the inertia of $\bar{\chi}_5^2$. It is trivial at the real places. Dividing by 1 or $\bar{\chi}_5^2$ therefore gives an everywhere unramified character trivial at infinity, which is trivial since $h_K = 1$. Thus $\psi^2 \in \{1, \bar{\chi}_5^2\}$, and the quotient $\bar{\chi}_5\psi^{-2}$ is $\bar{\chi}_5^{\pm 1}$. □

Remark 21.46. The generic mixed-case theorem of [20, Theorem A] excludes the local ratio $\bar{\chi}_5^{\pm 1}$. It therefore does not replace the residually reducible endpoint used here. At three its complete-splitting hypothesis also fails. The terminal argument uses the locally irreducible source theorems of Zhang, not the generic mixed theorem.

Removal of arbitrary conductor over $\mathbf{Q}(\sqrt{21})$. A solvable residual image is modular by an odd solvable Artin lift and regular weight raising. Otherwise its cyclotomic restriction is absolutely irreducible. The initial weight-two lift and Lemma 2.4 give a potentially modular compatible system with cyclotomic determinant times a finite totally even character. In characteristic five the exception is excluded by $[K(\zeta_5) : K] = 4$, as in Theorem 21.44. The nonsolvable residue makes the system nondihedral.

Apply Theorem 1.2 with $\ell = 5$. Two is inert and five splits. Theorem 21.44 is the supported irreducible residual assertion, and Lemma 21.45 gives constituent quotient $\bar{\chi}_5^{\pm 1}$, which is totally odd and extends to $G_{\mathbf{Q}}$. The common criterion therefore proves modularity of the original residual representation. □

Part III

Residual bases and field applications

22 A common discriminant comparison

For $b > 0$ put $a = \pi/(2b)$ and define the even function

$$F_b(x) = \begin{cases} (1 - |x|/(2b))\cos(a|x|) + \pi^{-1}\sin(a|x|), & |x| \le 2b, \\ 0, & |x| > 2b. \end{cases}$$

This is the normalized autocorrelation of $\cos(at)$ on $[-b,b]$, extended by zero. Thus $F_b(0)=1$, $0\le F_b\le 1$, and its Fourier transform is nonnegative. Its Laplace transform is

$$J_b(s)=\int_0^{2b} F_b(x)e^{-sx}\,dx=\frac{s}{s^2+a^2}+\frac{a^2(1+e^{-2bs})}{b(s^2+a^2)^2}. \tag{37}$$

Put

$$B_b=\int_0^\infty \frac{1-F_b(x)}{2\sinh(x/2)}\,dx, \qquad C_b=2\{J_b(-1/2)+J_b(1/2)\}.$$

Expansion of $(2\sinh(x/2))^{-1}$ gives $B_b=\sum_{j\ge 0}((j+1/2)^{-1}-J_b(j+1/2))$. Every summand lies between zero and $a^2(j+1/2)^{-3}$. For $N\ge 1$ this gives

$$B_b\le B_{b,N}:=\sum_{j=0}^{N-1}\Big((j+1/2)^{-1}-J_b(j+1/2)\Big)+\frac{a^2}{2(N-1/2)^2}. \tag{38}$$

The midpoint inequality for x^{-3} also gives the convenient one-term bound $B_b\le 2-J_b(1/2)+a^2/2$. The alternating four-term sum

$$A_{b,4}=J_b(1/2)-J_b(3/2)+J_b(5/2)-J_b(7/2)$$

is a lower bound for $\int_0^\infty F_b(x)/(2\cosh(x/2))\,dx$.

Proposition 22.1. *Assume GRH for the Dedekind zeta function of L. Let $n=[L:\mathbf{Q}]$, let r_1 be its number of real embeddings, and let N_Q count its prime ideals of norm Q. If B_b^* is either upper bound for B_b just given, then*

$$\log \mathrm{rd}(L)\ge \gamma+\log(8\pi)-B_b^*-\frac{C_b}{n}+\frac{r_1}{n}A_{b,4}+\sum_Q\frac{2N_Q}{n}\log Q\sum_{m\ge 1}Q^{-m/2}F_b(m\log Q). \tag{39}$$

Any subset of the prime terms may be retained. At fixed b, signature ratio and retained splitting densities, this lower bound increases with n.

Proof. Apply Poitou's explicit formula [52, equation (7) and Proposition 4] with F_b. Under GRH its zero terms have the required sign. Replace B_b by its upper bound and the real-place integral by its displayed lower bound. All omitted prime terms are nonnegative, and $C_b>0$. The inner sums are finite because F_b has compact support. □

For comparisons without GRH, use $G_b(x)=F_b(x)/\cosh(x/2)$, and write $B(G_b), C(G_b)$ for the preceding integrals with F_b replaced by G_b. Its zero terms in the explicit formula are nonnegative throughout the critical strip; see [47, equation (2.3)] and the strip positivity calculation in Section 31. The bound $1-F_b(x)\le \pi^2x^2/(8b^2)$ gives

$$B(G_b)\le \log 2+\frac{7\pi^2\zeta(3)}{16b^2}, \qquad C(G_b)=\frac{32b}{\pi^2}, \qquad \int_0^\infty \frac{F_b(x)}{2\cosh^2(x/2)}\,dx\ge 1-\frac{\pi^4}{24b^2}.$$

Consequently, without any hypothesis on the zeros,

$$\log \mathrm{rd}(L)\ge \log(4\pi)+\gamma-\frac{7\pi^2\zeta(3)}{16b^2}-\frac{32b}{\pi^2 n}+\frac{r_1}{n}\left(1-\frac{\pi^4}{24b^2}\right) +\sum_Q\frac{2N_Q}{n}\log Q\sum_{m\ge 1}\frac{2F_b(m\log Q)}{Q^m+1}. \tag{40}$$

As before, one may discard prime terms or the real-place term. The numerical uses below take $\zeta(3)<1.203$ and, when convenient, retain just the first prime-power term, using the same

quadratic lower bound for F_b; their certificates are in [49, `common/verify_unconditional_refinements.py`].

The comparisons below specify the degree, signature, retained Euler densities and local discriminant upper bound. Their strict numerical inequalities include the tail in (38); no truncated infinite sum is used without a remainder bound.

23 Local constructions for a split terminal prime

We record the local characteristic-three calculations used in the three supported residual bases below. The universal existence statement is Theorem 4.7; here we also need the exact characteristic-two reduction of its type over $\mathbf{Q}_3$.

Lemma 23.1. *Let K be a real quadratic field of ordinary class number one in which three splits. Every finite character $G_K \to \overline{\mathbf{F}}_3^{\times}$ unramified outside three has square one.*

Proof. Such a character has order prime to three and kills the principal units at the two places above three. The ray-class sequence, with both real places included, makes its group a quotient of $(\mathbf{F}_3^{\times})^2 \times \{\pm 1\}^2$, which has exponent two. □

For a finite Galois extension $L/\mathbf{Q}_3$, write $d(L) = v_3(\Delta(L/\mathbf{Q}_3))/[L : \mathbf{Q}_3]$, with $v_3(3) = 1$. Unramified base extension does not change this contribution.

Lemma 23.2. *Let $\overline{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ have determinant $\overline{\chi}_3$, and let L be its linear kernel field. If wild inertia is nontrivial, its fixed line makes $\overline{r}$ triangular, with diagonal quotient $\overline{\chi}_3\alpha$ for an unramified character α. Write Q for the wild inertia order.*

(i) If $\alpha \neq 1$, then $d(L) \le 3/2 - 1/Q < 3/2$.
(ii) If $\alpha = 1$, then $Q \in \{3, 9\}$ and $d(L) \le 13/6 - 1/Q$. Its projective kernel field is totally ramified of degree $2Q$.
(iii) In the second case with $Q = 9$, the projective kernel field is

$$E_0 = \mathbf{Q}_3(\zeta_3, \sqrt[3]{3}, \sqrt[3]{4}), \qquad [E_0 : \mathbf{Q}_3] = 18, \qquad d(E_0) = 37/18. \tag{41}$$

In the tame case $d(L) < 1$. In all cases $d(L) \le 37/18$; the same upper bounds hold for projective subfields.

Proof. A nontrivial normal 3-subgroup has a unique fixed line, which the decomposition group preserves. Rank-one characters over $\mathbf{Q}_3$ have inertia of order dividing two. The diagonal inertia characters are therefore $1, \overline{\chi}_3$ in some order, and their quotient is $\overline{\chi}_3\alpha$ with α unramified.

The kernel P of the two diagonal characters in the finite image consists of translations. It equals wild inertia. Indeed, diagonalize a tame inertia involution, whose conjugation on translations is inversion. The commutator with any translation belongs to inertia and is twice that translation. Since 2 is invertible in characteristic three, every translation lies in wild inertia. In particular P is elementary abelian of order Q.

Put $T = L^P$. It contains μ_3 by the determinant condition and has ramification index two. Thus $T = U(\zeta_3)$ for an unramified extension $U/\mathbf{Q}_3$ of degree prime to three. Let k be its residue field and $\varpi = \zeta_3 - 1$. Local reciprocity gives an equivariant surjection $T^{\times}/(T^{\times})^3 \to P$. Write $U_j = 1 + \varpi^j \mathcal{O}_T$. The logarithm gives $U_2^3 = U_4$, and

$$(1 + a\varpi)^3 \equiv 1 + (\bar{a}^3 - \bar{a})\varpi^3 \pmod{\varpi^4},$$

since $\varpi^2 = -3(1 + \varpi)$. Thus the possible unit grades modulo cubes are $1, 2, 3$, and the last is $k/\{a^3 - a : a \in k\}$, of dimension one over $\mathbf{F}_3$. The tame involution acts on grade j by $(-1)^j$,

and on the valuation quotient trivially. Only grades one and three can map to P. A last nonzero grade j gives conductor $j+1$; these are the usual local reciprocity and conductor conventions of [59, 46].

A Frobenius fixing ζ_3 acts trivially on the top quotient. If $\alpha \neq 1$, its action on P is multiplication by a nontrivial scalar; none of the associated $\mathbf{F}_3$-eigenvalues is one. The top grade cannot contribute, so every nontrivial character of P has conductor at most two. The tower and conductor–discriminant formulas give

$$d(L) = \frac{1}{2} + \frac{1}{2Q}\sum_{\xi\neq 1} a_T(\xi) \leq \frac{3}{2} - \frac{1}{Q}. \tag{42}$$

If $\alpha = 1$, divide by the lower diagonal character. The extension class belongs to

$$H^1(\mathbf{Q}_3, \overline{\mathbf{F}}_3(\overline{\chi}_3)) = (\mathbf{Q}_3^\times/(\mathbf{Q}_3^\times)^3) \otimes_{\mathbf{F}_3} \overline{\mathbf{F}}_3.$$

The rational Kummer space has basis the classes of 3 and 4, coming from valuation and principal units. The additive image therefore has rank at most two. At least $Q/3$ characters kill the image of the one-dimensional top unit grade. Their nontrivial members have conductor at most two; the others have conductor at most four. Hence

$$\sum_{\xi\neq 1} a_T(\xi) \leq 2(Q/3-1) + 4(2Q/3) = 10Q/3 - 2.$$

Equation (42) gives $d(L) \leq 13/6 - 1/Q \leq 37/18$.

In the projective representation, a Frobenius fixing ζ_3 is a translation. Its commutator with the tame inertia involution again shows that it belongs to inertia. Projective decomposition therefore equals inertia, of order $2Q$. When $Q = 9$, the two Kummer coordinates are independent, so their common kernel cuts out precisely E_0. Over $\mathbf{Q}_3(\zeta_3)$ its eight nontrivial characters have conductors two, twice, and four, six times. Thus

$$d(E_0) = \frac{1}{2} + \frac{2\cdot 2 + 6\cdot 4}{18} = \frac{37}{18}.$$

The tame formula is $1 - 1/e < 1$, and normalized differents cannot increase on passage to a subfield. □

The bound $Q \leq 9$ here is confined to an exactly cyclotomic diagonal quotient. For an arbitrary determinant, the bound we use is Moon–Taguchi's inequality

$$d(L) \leq \frac{13}{6} - \frac{1}{Q} \tag{43}$$

when the wild ramification index is $Q > 1$; tame extensions have $d(L) < 1$ [46, Theorem 3 and the following Remark (1)].

23.1 A wild type over $\mathbf{Q}_3$

Specialize the construction of Lemma 4.6 to $F = \mathbf{Q}_3$, with uniformizer $+3$ and additive character $\psi(x) = \zeta_3^x$. Retain its notation K_0, K_1, J and write ϑ for the affine generic character. The sign of this uniformizer fixes the quadratic subfield in the following lemma.

Lemma 23.3. *The representation $\sigma_+ := \operatorname{Ind}_J^{K_0}\vartheta$ is the irreducible eight-dimensional type of a supercuspidal inertial type τ_+ with determinant one. Its reduction satisfies*

$$\overline{\sigma}_+^{\mathrm{ss}} = 1 \oplus \det \oplus \operatorname{Sym}^2 \mathbf{F}_3^2 \oplus (\det \otimes \operatorname{Sym}^2 \mathbf{F}_3^2). \tag{44}$$

The inertia representation τ_+ is irreducible after reduction in every characteristic greater than three. For a determinant-cyclotomic compatible system of this type, its characteristic-two residual local kernel field is the unique totally ramified S_3 extension $E_+/\mathbf{Q}_3$ with quadratic subfield $\mathbf{Q}_3(\sqrt{3})$ and normalized different $7/6$.

Proof. The construction of Theorem 4.7 at $\ell = 3$ identifies σ_+ as the irreducible compact type of dimension eight, with determinant-one inertial parameter τ_+.

In the induced integral lattice, ϑ reduces to one, so $\overline{\sigma}_+ = \operatorname{Ind}_{ZU}^{\mathrm{GL}_2(\mathbf{F}_3)} 1$. If B is the upper triangular Borel, then B/ZU has order two, with characters $1, \det$. Induction in stages gives

$$\overline{\sigma}_+ = \operatorname{Ind}_B^{\mathrm{GL}_2(\mathbf{F}_3)} 1 \oplus \det \otimes \operatorname{Ind}_B^{\mathrm{GL}_2(\mathbf{F}_3)} 1.$$

The first summand is the permutation module on $\mathbf{P}^1(\mathbf{F}_3)$. Its constant line splits, since four is invertible in characteristic three, and its augmentation module is the Steinberg representation $\operatorname{Sym}^2 \mathbf{F}_3^2$. For example, evaluating a quadratic at the four projective points identifies it with that augmentation module. This proves (44); its four constituents are exactly the Serre weights with trivial scalar central character.

By [1, Section 3.4 and Theorem 3.18], the parameter is induced from a character of a ramified quadratic extension of $\mathbf{Q}_3$, of order three on the first principal-unit quotient, with conjugate the inverse there. Its inertia contains, in a suitable basis,

$$u = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The off-diagonal matrix has this form because inertia has determinant one. These matrices generate a group of order twelve. In characteristic $l > 3$, the two u-eigenlines remain distinct and v exchanges them. Thus reduction is irreducible; semisimplicity for this group of order prime to l makes the conclusion independent of the stable lattice. We check the quadratic field in the local parameter. In the parametrization of [1, Sections 3.1, 3.4 and Theorem 3.18], the affine-generic character $b + c$ uses uniformizer $+3$. The inducing quadratic extension is consequently $\mathbf{Q}_3(\sqrt{3})$. Reduction modulo two gives the irreducible group S_3. Its normalizer in $\mathrm{SL}_2(\overline{\mathbf{F}}_2)$ is itself: all automorphisms of S_3 are inner, and its centralizer consists of determinant-one scalars, hence is trivial. Thus the entire local image equals its inertia image and the residual field has $e = 6$, $f = 1$.

The wild character has conductor two over the ramified quadratic field. The two-dimensional Artin conductor is three and remains three after reduction modulo two: the wild inertia has odd order and the inertia invariants remain zero. The lower groups of the residual field have orders $6, 3, 1$, giving normalized different $(5 + 2)/6 = 7/6$. Finally, over $\mathbf{Q}_3(\sqrt{3})$ the conductor-two anti-invariant character space has dimension one over $\mathbf{F}_3$. Its unique nonzero line gives a unique S_3 extension. This proves the stated uniqueness; the splitting field of $x^3 - 3x - 3$ is one realization. □

Proposition 23.4. *Every continuous $\overline{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with determinant $\overline{\chi}_3$ has a potentially crystalline lift of determinant χ_3, Hodge–Tate weights $\{0, 1\}$, and inertial type τ_+.*

Proof. Apply Theorem 4.7 with $F = \mathbf{Q}_3$ and the affine-generic character fixed above. □

Lemma 23.5. *Let $\overline{r} : G_{\mathbf{Q}_3} \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ have determinant $\overline{\chi}_3$. It has a lift of weights $\{0, 1\}$ and determinant χ_3 which is either a quadratic twist of a semistable representation with nonzero monodromy, or potentially crystalline of tame supercuspidal type*

$$\tau_4 = \widetilde{\omega}_2^2 \oplus \widetilde{\omega}_2^6,$$

where ω_2 is a fundamental character of niveau two. For either choice in a determinant-cyclotomic compatible system, the characteristic-two residual local kernel field is contained in

$$E' = \mathbf{Q}_3(\sqrt{3}, \sqrt{-1}), \qquad e(E'/\mathbf{Q}_3) = 2, \quad f(E'/\mathbf{Q}_3) = 2, \quad d(E'/\mathbf{Q}_3) = \tfrac{1}{2}. \tag{45}$$

Proof. We first obtain a quadratic twist of a semistable lift. If $\overline{r}$ is reducible, a ramified quadratic twist, when necessary, puts its invariant character in the form $\overline{\chi}_3\lambda$ and its quotient in the form λ^{-1}, with λ unramified. Let η be the unramified Teichmüller lift of λ. If $\lambda^2 \neq 1$, local duality gives $H^2(G_{\mathbf{Q}_3}, \mathcal{O}(\chi_3\eta^2)) = 0$. Thus the actual residual extension lifts to an extension of η^{-1} by $\chi_3\eta$; the one-dimensional characteristic-zero extension space is crystalline. If $\lambda^2 = 1$, take $\eta^2 = 1$. Kummer theory lifts both the unit and valuation directions of $\mathbf{Q}_3^\times/(\mathbf{Q}_3^\times)^3$; the former is crystalline and the latter is semistable with nonzero monodromy. If $\overline{r}$ is irreducible, its wild inertia is trivial and its tame exponents are $\{1,3\}$ or $\{5,7\}$ modulo eight. A ramified quadratic twist interchanges the pairs. The first lifts by induction from a crystalline character with labelled weights $(1,0)$ over the unramified quadratic extension. An unramified scalar correction fixes the determinant. Twisting back gives the claimed semistable lift; compare [64, proof of Proposition 7.8.1].

Retain this lift if its monodromy is nonzero. Otherwise its inertial type is scalar quadratic, say $\kappa \oplus \kappa$. Its potentially crystalline compact type is $\kappa \circ \det$, with reduction 1 or det. The fixed-determinant formula of [70, Theorem 1.2], throughout with determinant χ_3, gives $\mu_1(\overline{r}) > 0$ or $\mu_{\det}(\overline{r}) > 0$.

The cuspidal type for τ_4 has dimension two and trivial central character. It is the two-dimensional representation of $\mathrm{PGL}_2(\mathbf{F}_3) \simeq S_4$ obtained from the standard representation of S_3. The integral augmentation lattice in $\mathbf{Z}_3^3$ reduces to a module with a constant line and sign quotient. Sign inflates to det, so its semisimplification is $1 \oplus \det$. Breuil–Mézard for Hodge type $(1,0)$ and τ_4 has multiplicity $\mu_1(\overline{r}) + \mu_{\det}(\overline{r}) > 0$. Its three-torsion-free deformation ring gives the required potentially crystalline lift, with determinant exactly χ_3.

For the semistable choice, the characteristic-two local member is a quadratic twist of a special representation with diagonal characters $\chi_2\nu$ and ν, where ν is unramified quadratic. Both reduce to one, as does the quadratic twist. The full residual image is therefore an elementary abelian two-group. For the tame choice, inertia and arithmetic Frobenius matrices T, Φ satisfy

$$T^2 = -I, \qquad \Phi T \Phi^{-1} = T^{-1}, \qquad \Phi^2 = -3I.$$

The last equality follows from trace zero and determinant three. Modulo two these matrices become commuting involutions, again giving an elementary abelian two-group for every stable lattice. The field E' is the maximal elementary quadratic extension of $\mathbf{Q}_3$, so it contains both residual kernel fields. It is the compositum of the ramified and unramified quadratic extensions; its tame ramification index two gives the stated different. □

Over a real quadratic field with a single dyadic place, Lemma 2.4 applies to every finitely ramified, totally odd lift of determinant χ_2 with nonsolvable residue that is crystalline nearly ordinary of regular labelled weights, or crystalline of weight two and potentially nonordinary. It gives potential modularity and a strongly compatible system, with the ordinary ordering of the actual source.

24 The characteristic-three base over $\mathbf{Q}(\sqrt{7})$

Throughout this section $K = \mathbf{Q}(\sqrt{7})$, $C = K(\zeta_3)$, and $\mathfrak{p}_1 = (2+\sqrt{7})$, $\mathfrak{p}_2 = (2-\sqrt{7})$.

Theorem 24.1. *Assume GRH. Every continuous irreducible representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with determinant $\overline{\chi}_3$, unramified outside 3, is modular.*

Lemma 24.2. *The ordinary class number of K is one and its narrow class number is two. The extension $K(i)/K$ is the unique nontrivial quadratic extension unramified at every finite place, and is inert at both places above 3. Every finite character $G_K \to \overline{\mathbf{F}}_3^\times$ unramified outside 3 has square one.*

Proof. The Minkowski bound is $\sqrt{7} < 3$. The prime above 2 is principal, generated by $3 + \sqrt{7}$ of norm two, so the ordinary class number is one. The equation $x^2 - 7y^2 = -1$ has no solution modulo seven. The least positive solution of $x^2 - 7y^2 = 1$ with $y > 0$ is $(8, 3)$, so $\epsilon = 8 + 3\sqrt{7}$ generates the positive units and the narrow class number is two.

The discriminant of the biquadratic field $K(i)$ is $28(-4)(-7) = 28^2$. Thus its relative discriminant over K is trivial. Narrow class field theory gives uniqueness, and its completion at either place above three is unramified quadratic because -1 is nonsquare in $\mathbf{F}_3$.

The character assertion follows from Lemma 23.1. □

Lemma 24.3. *Assume GRH. Retain only a proved number N_3 of primes of norm three, and put $c = 2N_3/[L : \mathbf{Q}]$. The following lower bounds hold when the degree and c are at least the indicated values:*

b	$[L : \mathbf{Q}]$ *at least*	c	$\mathrm{rd}(L)$ *greater than*
21/2	60000	0	37.33078
15/2	2880	0	31.93013
9	19656	2/9	48.00416
23	20 000 000 000	1/9	50.67997.

In particular, every totally imaginary K-Galois field unramified outside three whose completions at both places above three are E_0 satisfies

$$[L : \mathbf{Q}] < 20\,000\,000\,000. \tag{46}$$

Proof. Use the explicit formula of Section 22, discarding the real-place term and retaining only the specified primes of norm three. Its one-term bound for B_b gives

$$\log \mathrm{rd}(L) \geq \gamma + \log(8\pi) - B_b^* - \frac{C_b}{n} + c \log 3 \sum_{m \geq 1} 3^{-m/2} F_b(m \log 3). \tag{47}$$

The prime sum is finite. Outward evaluation gives the four rows; the bound increases with n and c.

For the last assertion, the local degree is eighteen and residue degree one at both primes. Thus $N_3 = n/18$, $c = 1/9$, and the discriminant tower gives

$$\mathrm{rd}(L) = \sqrt{28}\, 3^{37/18} < 50.62072. \tag{48}$$

The last row contradicts this equality for degree at least twenty billion. □

Lemma 24.4. *Assume GRH. A representation as in Theorem 24.1 is modular unless its projective group is*

$$\mathrm{PSL}_2(\mathbf{F}_{3^{2e}}) \quad (e \geq 2), \qquad \textit{or} \qquad \mathrm{PGL}_2(\mathbf{F}_{3^{2e+1}}) \quad (e \geq 1).$$

In every remaining case its projective completion at each place above three is exactly E_0. In the projective general linear case the unique quadratic subfield over K is C.

Proof. Let L and M be the linear and projective kernel fields, with groups G and H over K, and let Z be the scalar subgroup of G. Both fields are totally imaginary, since complex

conjugation has eigenvalues $1, -1$. Solvable images are modular. Dickson's classification leaves A_5, $\mathrm{PSL}_2(\mathbf{F}_q)$ and $\mathrm{PGL}_2(\mathbf{F}_q)$ with $q = 3^f \geq 9$.

We first consider $q \geq 27$. If H is perfect, then $|Z| \geq 4$: for $|Z| \leq 2$ the determinant would kill Z and factor through the perfect group H, contradicting oddness; scalar order is prime to three. For $H = \mathrm{PGL}_2(\mathbf{F}_q)$, Z is nontrivial. Otherwise its perfect subgroup $\mathrm{PSL}_2(\mathbf{F}_q)$ would embed in $\mathrm{SL}_2(\overline{\mathbf{F}}_3)$, which has only one nonidentity involution. Therefore $[L : \mathbf{Q}] \geq 78624$ in either case.

If one place above three is tame or has nontrivial unramified factor α, Lemma 23.2 gives

$$\mathrm{rd}(L) < \sqrt{28}\, 3^{(3/2+37/18)/2} = \sqrt{28}\, 3^{16/9} < 37.30737,$$

contrary to the first row of Lemma 24.3. Both diagonal quotients are therefore cyclotomic with nontrivial wild inertia. If one wild rank is one, the projective field satisfies

$$\mathrm{rd}(M) \leq \sqrt{28}\, 3^{35/18} < 44.80380.$$

Its degree is at least $2|\,\mathrm{PSL}_2(\mathbf{F}_{27})| = 19656$, and its local degrees are at most 6 and 18, both totally ramified. Thus $c = 1/e_1 + 1/e_2 \geq 2/9$, contradicting the third row. Both wild ranks are two, so both local fields are E_0.

For odd f, the tame inertia involution has Borel torus coordinate -1, nonsquare in $\mathbf{F}_q$, and cannot belong to $\mathrm{PSL}_2(\mathbf{F}_q)$. For even f, a full $\mathrm{PGL}_2(\mathbf{F}_q)$ image would have a nontrivial squareclass quotient trivial on both local decomposition groups: these equal inertia, and -1 is a square. Its quadratic field would be unramified at every finite place and split at three, contrary to Lemma 24.2. This gives the asserted parity restrictions.

In the remaining projective general linear case, the squareclass quotient agrees with $\overline{\chi}_3$ on both decomposition groups at three. Their quotient is unramified at all finite places and split at three, so is trivial by the same lemma. The quadratic field is therefore C.

It remains to consider full $\mathrm{PGL}_2(\mathbf{F}_9)$. If its quadratic squareclass character ramifies at a place above three, projective wild inertia there must vanish: otherwise the decomposition group is in a Borel, and the tame Frobenius relation makes its inertia torus coordinates ± 1, both squares in $\mathbf{F}_9$. Linear wild inertia then vanishes as well, since a scalar in characteristic three has order prime to three. We obtain

$$\mathrm{rd}(L) < \sqrt{28}\, 3^{55/36} < 28.34747.$$

If the squareclass character is unramified everywhere finite, it is the character of $K(i)$, nontrivial on Frobenius at both places above three. Neither local quotient can then be exactly cyclotomic with wild inertia, because its projective decomposition equals inertia. Hence $\mathrm{rd}(L) < \sqrt{28}\, 3^{3/2} < 27.49546$. But $[L : \mathbf{Q}] \geq 2880$, contradicting the second row of Lemma 24.3.

The remaining small nonsolvable groups are A_5 and $\mathrm{PSL}_2(\mathbf{F}_9)$. Their embeddings in $\mathrm{PGL}_2(\mathbf{F}_9)$ have trivial squareclass character, since they are perfect. Theorem 1.2(5) and Lemma 2.1 of [3] give modularity of the projective representation and of its given linear lift. □

24.1 A classical form of level eighty-four

Let $\eta = \chi_{28}$ be the primitive quadratic Dirichlet character attached to $K/\mathbf{Q}$.

Lemma 24.5. *There is a normalized newform $f \in S_2(\Gamma_0(84), \eta)$ with coefficient field $F = \mathbf{Q}(\alpha)$ such that*

$$\alpha^4 + 14\alpha^2 + 32 = 0, \qquad a_5(f) = \alpha, \qquad a_{11}(f) = \frac{\alpha^3 + 10\alpha}{4}. \tag{49}$$

Moreover, $a_3(f) = 1$. The prime three is inert in F. Complex conjugation sends α to $-\alpha$, and its fixed field is $\mathbf{Q}(\sqrt{17})$, with $\sqrt{17} = \alpha^2 + 7$.

Proof. Take the normalized newform with LMFDB label `84.2.b.b` [45]. Its nebentypus is $\chi_{84}(55,\cdot)$, induced by η. The tabulated coefficient field is $\mathbf{Q}(\nu)$, where $\nu^4 - \nu^3 - 2\nu + 4 = 0$, and the coefficients are

$$a_3(f) = 1, \qquad a_5(f) = \nu^3 + \nu - 2, \qquad a_{11}(f) = \nu - \nu^2.$$

Put $\alpha = \nu^3 + \nu - 2$. Reduction modulo the polynomial of ν gives

$$\alpha^4 + 14\alpha^2 + 32 = 0, \qquad 4(\nu - \nu^2) = \alpha^3 + 10\alpha,$$

which proves (49).

Modulo three, $P(X) = X^4 + 14X^2 + 32$ becomes $X^4 - X^2 - 1$. If $y^2 = y + 1$ in $\mathbf{F}_9$, then $y^4 = -1$, so y has order eight and is not a square in $\mathbf{F}_9$. Thus P is irreducible modulo three and over $\mathbf{Q}$. It follows that $F = \mathbf{Q}(\nu) = \mathbf{Q}(\alpha)$. The polynomial discriminant is $2^{13}17^2$, prime to three, so three is inert in the full ring of integers of F. Finally $\alpha^2 = -7 \pm \sqrt{17}$ is negative at both real embeddings. Hence F is CM, complex conjugation sends α to $-\alpha$, and its fixed field is $\mathbf{Q}(\sqrt{17})$. □

Proposition 24.6. *There exists a finite Galois extension B/K with*

$$\mathrm{Gal}(B/K) \simeq D := \{g \in \mathrm{GL}_2(\mathbf{F}_9) : \det g \in \{1, -1\}\}, \qquad [B : K] = 1440.$$

It is unramified outside three, and its two local completions at three are contained in E_0. Its maximal solvable quotient is C_2, with fixed field C.

Proof. Let λ be the unique prime of F above three. Restrict the representation attached to f to G_K, and let $\bar{r}$ be its semisimplified reduction at λ. Its determinant is $\overline{\chi}_3$, since the rational determinant is $\eta\chi_3$, and its initial coefficient field is $\mathbf{F}_{81}$.

Write $t = \bar{\alpha}$ and $x = t^2$, so $x^2 = x + 1$. The primes 5 and 11 are inert in K, and the rational residual determinants at both are one. Squaring their Frobenii gives elements of G_K with determinant one and traces

$$s_5 = x + 1, \qquad s_{11} = 1 - x.$$

Here $\bar{a}_{11}^2 = -x$, from (49). Their projective invariants are

$$s_5^2 = 2, \qquad s_{11}^2 = -x - 1, \qquad (-x - 1)^2 = -1. \tag{50}$$

If z is squared trace divided by determinant, the eigenvalue ratio u satisfies $u + u^{-1} = z - 2$. The first equality gives projective order four. In characteristic three, $z^2 + 1 = 0$ is equivalent to $u^4 + u^3 + u^2 + u + 1 = 0$, so the second element has projective order five. Both have distinct eigenvalues.

All traces on G_K lie in $\mathbf{F}_9$. For a good rational prime q, the Petersson adjoint identity gives $\overline{a_q(f)} = \eta(q)a_q(f)$. At a split prime of K the trace is $a_q(f)$, and at an inert prime it is $a_q(f)^2 - 2\eta(q)q$. Both belong to the real field $F^+ = \mathbf{Q}(\sqrt{17})$, so their reductions lie in its residue field $\mathbf{F}_9$. Chebotarev gives the assertion for every element of G_K.

This also proves that $\bar{r}$ is irreducible. If its semisimplification were $\psi_1 \oplus \psi_2$, trace invariance under the ninth-power coefficient automorphism would either fix both characters or interchange them. In the first case their quotient has order dividing eight; in the second its ninth power is its inverse, so its order divides ten. Neither alternative permits both projective orders in (50).

Absolute irreducibility, together with the traces and determinant in $\mathbf{F}_9$ and the triviality of its Brauer group, gives a model of $\bar{r}$ over $\mathbf{F}_9$.

Both 1 and -1 are squares in $\mathbf{F}_9$, so its projective image lies in $\mathrm{PSL}_2(\mathbf{F}_9)$. By Dickson's classification, a proper subgroup cannot contain elements of both orders four and five: the

Borel, subfield, A_4 and S_4 alternatives have no element of order five, the dihedral alternatives have orders dividing eight or ten, and A_5 has no element of order four. Hence its projective image is all of $\mathrm{PSL}_2(\mathbf{F}_9) \simeq A_6$.

The scalar subgroup of the linear image is contained in μ_4, since the determinant has order two. It cannot have order at most two: then the determinant would factor through the perfect projective group, contradicting total oddness. Thus it has order four. The linear image has order $4 \cdot 360 = 1440$ and equals D. Moreover $D = \mu_4 \mathrm{SL}_2(\mathbf{F}_9)$, with the scalar factor central and $\mathrm{SL}_2(\mathbf{F}_9)$ perfect. Therefore $D' = \mathrm{SL}_2(\mathbf{F}_9)$ and its maximal solvable quotient is its determinant quotient, which cuts out C.

We check the local kernels, not merely the projective conductor. At 2 and 7, local–global compatibility for the newform gives conductor exponents respectively two and one [61, equation (1)]. These equal the exponents of its determinant character η. For a two-dimensional Weil–Deligne representation with ramified determinant, equality $a(V, N) = a(\det V)$ forces its inertia to be $1 \oplus \det V$ and $N = 0$. Indeed, if $V^I = 0$, then

$$a(V, N) \geq 2 + \mathrm{Swan}(V) \geq 2 + \mathrm{Swan}(\det V) > a(\det V).$$

Thus V^I is a line. The other inertia character is the ramified determinant, and a commuting nilpotent operator is zero. Consequently the inertia at 2 and 7 is $1 \oplus \eta$, which becomes trivial over K. At every other prime away from three the representation is already unramified.

At three, $\eta|_{G_{\mathbf{Q}_3}} = 1$. Since $3 \| 84$ and $a_3(f) = 1$, the local representation is semistable of Steinberg type with unit eigenvalue one. Local–global compatibility and Lemma 4.2 give the trivial unramified ordinary quotient at both places above three. Its determinant is χ_3, so the reduction is an actual extension of 1 by $\overline{\chi}_3$.

Its extension class lies in $H^1(\mathbf{Q}_3, \overline{\mathbf{F}}_3(\overline{\chi}_3))$, the rational cube-class space tensored with $\overline{\mathbf{F}}_3$. That space is generated by 3 and 4, so its full linear local kernel field is contained in E_0. There is no additional unramified scalar factor. Taking $B = K^{\ker \bar{r}}$ proves the proposition. □

24.2 Auxiliary composita and the residual theorem

Proposition 24.7. *There exists a totally imaginary K-Galois extension $\mathcal{F}$ of degree* 38880 *over K, unramified outside three, with completion E_0 at both places above three. Its degree has prime factors only* $2, 3, 5$*, and it contains C.*

Proof. Put $\pi = 2 + \sqrt{7}$, $\pi' = 2 - \sqrt{7}$ and $\epsilon = 8 + 3\sqrt{7}$, and consider

$$E = K(\zeta_3, \sqrt[3]{\pi}, \sqrt[3]{\pi'}, \sqrt[3]{\epsilon}).$$

The three cube classes are independent over K: valuations at $\mathfrak{p}_1, \mathfrak{p}_2$ first give independence of the two prime generators, and the fundamental unit is not a cube. They remain independent over C, since taking norms across a degree-two extension is injective on cube classes from K. Thus $[E : C] = 27$ and $[E : K] = 54$, with solvable group $(C_3)^3 \rtimes C_2$, where the involution acts by inversion.

The radicands are units outside three, so their Kummer extensions are unramified there. At the two three-adic embeddings, $\sqrt{7}$ reduces to $4, 5$ modulo nine. At each place one of π, π' has valuation one and the other is a unit congruent to seven modulo nine, hence is not a cube. The first two radicands therefore span the rational local cube-class space, giving completion E_0 at both places. The fundamental unit enlarges the global degree without enlarging either completion.

Let B be the field of Proposition 24.6. Its maximal solvable quotient is C_2, whereas E/K is solvable and contains C. Therefore $B \cap E = C$. Set $\mathcal{F} = BE$. We have

$$[\mathcal{F} : K] = \frac{1440 \cdot 54}{2} = 38880 = 2^5 \cdot 3^5 \cdot 5. \tag{51}$$

Its local completions are still E_0, because those of B are contained in E_0 and those of E equal E_0. All relative extensions away from three are unramified, and $C \subset \mathcal{F}$ makes it totally imaginary. $\square$

Whenever two K-Galois fields have completion E_0 at the same place, their local compositum is again E_0. This is equality of subfields of $\overline{\mathbf{Q}}_3$: the field in (41) is normal and is the full Kummer extension of the rational cube-class space. We shall repeatedly apply the degree bound (46) after taking such composita.

Proof of Theorem 24.1. By Lemma 24.4, it suffices to treat the two displayed families there, with projective field M locally equal to E_0. Let $H = \mathrm{Gal}(M/K)$ and use the auxiliary field $\mathcal{F}$ of Proposition 24.7.

Suppose $H = \mathrm{PSL}_2(\mathbf{F}_{3^{2e}})$, $e \geq 2$. Its simplicity makes $M \cap \mathcal{F}$ either K or M. The latter is impossible because $|H| \geq |\mathrm{PSL}_2(\mathbf{F}_{81})| = 265680 > 38880$. Hence

$$[M\mathcal{F} : \mathbf{Q}] \geq 2 \cdot 265680 \cdot 38880 = 20\,659\,276\,800,$$

contradicting (46).

Suppose $H = \mathrm{PGL}_2(\mathbf{F}_{3^{2e+1}})$ with $e \geq 2$. Its only proper nontrivial normal quotient is C_2, and its order exceeds 38880. Since both fields contain the unique quadratic subfield C, we have $M \cap \mathcal{F} = C$. Therefore

$$[M\mathcal{F} : \mathbf{Q}] \geq |\mathrm{PGL}_2(\mathbf{F}_{243})| \cdot 38880 = 557\,876\,056\,320,$$

again impossible.

It remains to consider $H = \mathrm{PGL}_2(\mathbf{F}_{27})$, of order $h = 19656$. Let σ be the nontrivial automorphism of $K/\mathbf{Q}$. If $M \neq M^\sigma$, the two fields intersect exactly in C: their only proper nontrivial common quotient is quadratic, and C is invariant under rational conjugation. For $P = MM^\sigma$ we have

$$\mathrm{Gal}(P/K) \simeq H \times_{C_2} H, \qquad [P : K] = h^2/2.$$

Its derived subgroup is $\mathrm{PSL}_2(\mathbf{F}_{27})^2$, which is perfect. Any quotient that does not kill this subgroup has a composition factor of order divisible by 7 and 13. Such a quotient cannot be a quotient of $\mathrm{Gal}(\mathcal{F}/K)$, whose order has prime factors only $2, 3, 5$. Thus $P \cap \mathcal{F} = C$, and

$$[P\mathcal{F} : \mathbf{Q}] = h^2 \cdot 19440 = 7\,510\,806\,051\,840,$$

contradicting (46). Consequently $M = M^\sigma$ and $M/\mathbf{Q}$ is Galois.

We now prove modularity rather than nonexistence. Write $\Gamma = \mathrm{Gal}(M/\mathbf{Q})$. Conjugation induces a map $\Gamma \to \mathrm{Aut}(H)$. Its outer image is a quotient of $\Gamma/H = C_2$, whereas $\mathrm{Out}(\mathrm{PGL}_2(\mathbf{F}_{27})) \simeq C_3$ consists of field automorphisms [37, Section 13, Lemma 52]. The outer action is trivial. As H is centreless, the conjugation map lands in $\mathrm{Inn}(H) \simeq H$. Composing it with $G_{\mathbf{Q}} \to \Gamma$ gives a two-dimensional projective representation of $G_{\mathbf{Q}}$ extending the given projective representation over K.

Its complex conjugations are nontrivial projective involutions. Tate's lifting theorem gives a finite linear lift in characteristic three, necessarily odd and irreducible [3, Section 2.1]. Serre modularity over $\mathbf{Q}$ [40, 41] and cyclic base change give residual modularity over K. The base change is cuspidal: otherwise its Galois representation restricted to K, and hence its residual semisimplification, would be reducible. Finally, the given linear representation differs from this restricted lift by a finite character, which preserves modularity. This proves the residual theorem. $\square$

25 The characteristic-three base over $\mathbf{Q}(\sqrt{13})$

In this section $K = \mathbf{Q}(\sqrt{13})$. Both completions above three are $\mathbf{Q}_3$, while two is inert.

Lemma 25.1. *The narrow class number of K is one. Every finite character $G_K \to \overline{\mathbf{F}}_3^\times$ unramified outside 3 has square one.*

Proof. The Minkowski bound $\sqrt{13}/2 < 2$ gives ordinary class number one. The unit $\varepsilon = (3+\sqrt{13})/2$ has norm -1; together with -1, it realizes all four real signatures. Thus the narrow class number is also one.

The character assertion follows from Lemma 23.1. □

Lemma 25.2. *Under GRH, a totally imaginary number field of degree n has root discriminant greater than 39.01 if $n \geq 531360$, greater than 37.63 if $n \geq 78624$, and greater than 28.02 if $n \geq 1440$.*

Proof. Use the explicit formula of Section 22, with no Euler terms or real-place term retained, using $B_b \leq B_b^*$. The parameters give

b	n	$\mathrm{rd}(L) >$
45/4	531360	39.01373
21/2	78624	37.63959
8	1440	28.02230.

These are outward interval evaluations of the displayed finite expressions. Each bound increases with n, since $C_b > 0$. The exact calculation is supplied in [49, `classical`]. □

Proposition 25.3. *Assume GRH. Every continuous irreducible totally odd representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ unramified outside 3 is modular.*

Proof. Let L/K be the linear kernel field, let $G = \mathrm{Gal}(L/K)$, and let Z and $H = G/Z$ be its scalar subgroup and projective image. The field L is totally imaginary. The scalar subgroup has order prime to three. Since three splits in K, the discriminant tower and (43) give

$$\mathrm{rd}(L) \leq \sqrt{13}\, 3^{13/6} < 38.971.$$

The solvable case is already modular. Otherwise Dickson's classification gives $H \simeq A_5$, $\mathrm{PSL}_2(\mathbf{F}_{3^f})$ or $\mathrm{PGL}_2(\mathbf{F}_{3^f})$, with $f \geq 2$. If $3^f \geq 81$, then $[L : \mathbf{Q}] \geq 2|\,\mathrm{PSL}_2(\mathbf{F}_{81})| = 531360$, contradicting Lemma 25.2.

Suppose $3^f = 27$. The wild inertia order at either prime above three is at most 27, so

$$\mathrm{rd}(L) \leq \sqrt{13}\, 3^{13/6 - 1/27} < 37.417.$$

If $H = \mathrm{PSL}_2(\mathbf{F}_{27})$, then $|Z| \geq 4$. Indeed, if $|Z| \leq 2$, determinant kills Z and factors through the perfect group H, contradicting oddness; order three is impossible. If $H = \mathrm{PGL}_2(\mathbf{F}_{27})$, then $|Z| \geq 2$: otherwise its perfect subgroup $\mathrm{PSL}_2(\mathbf{F}_{27})$ embeds in $\mathrm{SL}_2(\overline{\mathbf{F}}_3)$, which has only one nonidentity involution, namely $-I$. In either case $[L : \mathbf{Q}] \geq 78624$, again contradicting Lemma 25.2.

Suppose now that $H = \mathrm{PGL}_2(\mathbf{F}_9)$. Its nontrivial determinant-squareclass character is unramified outside three, so narrow class number one forces it to ramify at some $\mathfrak{p}_i$. At a place with nontrivial projective wild inertia, that inertia has a unique fixed point, defined over $\mathbf{F}_9$, and the local projective group lies in the corresponding Borel. Its torus quotient is abelian, so the tame Frobenius relation gives $a^3 = a$ for every inertia torus coordinate. Thus $a = \pm 1$, both

squares in $\mathbf{F}_9^\times$. The squareclass character is unramified there. At the place where it ramifies, projective wild inertia, and hence linear wild inertia, is trivial. Consequently

$$\mathrm{rd}(L) < \sqrt{13}\,3^{(1+13/6)/2} = \sqrt{13}\,3^{19/12} < 20.532.$$

But $[L:\mathbf{Q}] \geq 2|\,\mathrm{PGL}_2(\mathbf{F}_9)| = 1440$, and (40) with $b = 16$ gives $\mathrm{rd}(L) > 21.15$. Thus this image is excluded without GRH.

The remaining nonsolvable groups are A_5 and $\mathrm{PSL}_2(\mathbf{F}_9)$. They embed in $\mathrm{PGL}_2(\mathbf{F}_9)$ with trivial determinant-squareclass character, since they are perfect. The projective representation is modular by [3, Theorem 1.2(5)], and its given linear lift is modular by [3, Lemma 2.1]. □

26 The characteristic-three base over $\mathbf{Q}(\sqrt{37})$

Put $K = \mathbf{Q}(\sqrt{37})$. We use one wild type and one bounded quadratic or quartic type at the two three-primes. The compatible system thereby has a finite-flat characteristic-two member whose local fields fit inside a fixed solvable auxiliary extension.

Theorem 26.1. *Assume GRH. Every continuous irreducible representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_3)$ with determinant $\overline{\chi}_3$, unramified outside three, is modular.*

Write

$$w = \frac{1+\sqrt{37}}{2}, \qquad \epsilon = 5 + 2w, \qquad \pi = 3 + w, \qquad \pi' = 4 - w.$$

Thus $w^2 - w - 9 = 0$, $N_{K/\mathbf{Q}}(\epsilon) = -1$ and $\pi\pi' = 3$. Put $\mathfrak{p} = (\pi)$ and $\mathfrak{p}' = (\pi')$. Both completions above three are $\mathbf{Q}_3$, and the unique dyadic completion is the unramified quadratic extension of $\mathbf{Q}_2$.

Lemma 26.2. *The ordinary and narrow class numbers of K are one. Every finite character $G_K \to \overline{\mathbf{F}}_3^\times$ unramified outside three has square one.*

Proof. The Minkowski bound is $\sqrt{37}/2 < 3.05$. There is no ideal of norm two, and the two ideals of norm three are principal. Thus the ordinary class number is one. The unit ϵ has norm -1, which also gives narrow class number one. The character assertion follows from Lemma 23.1. □

Proposition 26.3. *Let $\overline{\rho}$ satisfy the hypotheses of Theorem 26.1, and suppose its image is nonsolvable. It lifts to a totally odd, potentially modular, strongly compatible system $\mathcal{R} = (r_\lambda)$ with determinant χ and weights $\{0,1\}$. The system is unramified away from three, has type τ_+ at $\mathfrak{p}$, and has one of the two local conditions of Lemma 23.5 at $\mathfrak{p}'$.*

Proof. Proposition 23.4 and Lemma 23.5 supply the local lifts at $\mathfrak{p}$ and $\mathfrak{p}'$, respectively. At the second place prescribe the tame supercuspidal type or a component through the special lift supplied by the lemma; no global quadratic character is required. In the latter case, the closure of the special component fixes Weil Frobenius eigenvalue ratio three, so the purity observation in Section 2 retains nonzero monodromy. Nonsolvability implies that the restriction to $G_{K(\zeta_3)}$ is irreducible, since an irreducible representation becoming reducible over a quadratic extension has dihedral projective image.

Lemma 2.4(i) now globalizes these weight-two components with determinant χ_3, no additional ramification, and compatibility at every finite place. In particular its characteristic-two member is crystalline of weights $\{0,1\}$ at two. □

Choose a place above two in the coefficient field and write r_2 for this member and $\overline{r}_2$ for its reduction. Lemma 23.3 makes $\overline{r}_2$ irreducible, with determinant one. Let M be its kernel field. Its completion at $\mathfrak{p}$ is E_+, while Lemma 23.5 puts its completion at $\mathfrak{p}'$ inside E'.

26.1 Ramification at two

For a finite Galois extension L/F of dyadic local fields, $d(L/F)$ denotes the valuation of its different, normalized by $v_F(2) = 1$. Thus $d(L/F)$ is the local contribution to the exponent of two in the root discriminant. We use Serre's upper numbering for ramification groups.

Lemma 26.4. *Let $F/\mathbf{Q}_2$ be unramified and let J be a finite flat commutative group scheme over $\mathcal{O}_F$ killed by two. The field generated by $J(\overline{F})$ has no positive upper ramification break greater than one, and its normalized different is strictly less than two.*

Proof. Fontaine's Theorem A and its corollary [31, pp. 515–516] apply to finite flat group schemes of any rank. His notation is $G^{(u)} = G^{u-1}$, where the right side uses Serre's upper numbering. For a group scheme killed by p^n over a field of absolute ramification index e, his bound is

$$G^{(u)} \text{ acts trivially if } u > e\left(n + \frac{1}{p-1}\right).$$

Here $e = n = 1$ and $p = 2$. Thus G^v acts trivially for $v > 1$. The different bound is the cited corollary. □

Corollary 26.5. *A compositum of finite Galois extensions of F having no upper break greater than one has the same property, and its normalized different is strictly less than two.*

Proof. Upper-numbering ramification is compatible with quotients. If all quotient fields have trivial groups above one, so does their compositum. The integral formula for the different then gives $d < 2$; more precisely, for tame inertia order t and wild inertia order Q, one has $d \le 2 - (t+1)/(tQ)$. □

We also need to recognize ordinary lifts in the dihedral case.

Lemma 26.6. *Let $F/\mathbf{Q}_2$ be unramified and let $r : G_F \to \mathrm{GL}_2(E)$ be Barsotti–Tate with determinant χ_2. If its residual image is a two-group, then r is ordinary.*

Proof. A two-group has only trivial irreducible representations in characteristic two, so both residual constituents are unramified. Apply Lemma 4.10 with $p = 2$ and $e(F/\mathbf{Q}_2) = 1$. □

26.2 An explicit solvable auxiliary field

Put

$$Q(X) = X^4 - 2X^3 + (3-2w)X^2 + (4+2w)X - w - 2, \qquad C = K(\sqrt{\pi}, \sqrt{\pi'}, \sqrt{-\epsilon}),$$

and let H be the splitting field of Q over K. We use $N = CH$.

Proposition 26.7. *The extension N/K is solvable Galois of degree 96, is unramified outside two and three, and has no dyadic upper break greater than one. Its completions at $\mathfrak{p}$ and $\mathfrak{p}'$ are E_+ and E', respectively.*

Proof. The cubic resolvent and discriminant of Q are

$$R(T) = T^3 + (2w-3)T^2 + 4 - 4w, \qquad \mathrm{disc}(Q) = \mathrm{disc}(R) = 144\delta, \quad \delta = 99 - 28w = -\epsilon^{-3}\pi^3.$$

The quartic is irreducible modulo the prime above eleven with $w \equiv 7$: its reduction is $X^4 - 2X^3 + 7X + 2$. This polynomial is relatively prime to $X^{121} - X$ and divides $X^{14641} - X$ over $\mathbf{F}_{11}$, so is irreducible. Hence Q is irreducible over K.

At $\mathfrak{p}$ we have $w \equiv 18 \pmod{27}$. The shifted polynomial is

$$Q(Y-1) = Y^4 - 6Y^3 + (15-2w)Y^2 + (6w-12)Y - 5w.$$

Its Newton polygon has segments of slopes -1 and $-1/3$, of lengths one and three. It therefore has one root in $\mathbf{Q}_3$ and an irreducible totally ramified cubic factor. The cubic roots have valuation $1/3$, whereas the linear root has valuation one. The resultant of these factors has valuation one. Since $v_{\mathfrak{p}}(\mathrm{disc}(Q)) = 5$, the cubic factor has discriminant exponent three. Its root is a uniformizer, so this is also the field discriminant exponent. Its discriminant square class is that of $-\epsilon\pi$, namely 3. Thus its splitting field is totally ramified S_3, with normalized different $7/6$ and quadratic subfield $\mathbf{Q}_3(\sqrt{3})$. By Lemma 23.3 it is E_+. The quartic's linear factor shows that its full local splitting field is the same field.

The global quartic group is transitive and contains this point-stabilizer S_3, so it is S_4. Its unique quadratic subfield is $D = K(\sqrt{-\epsilon\pi})$. Valuations at $\mathfrak{p}, \mathfrak{p}'$ and the mixed real signs of $-\epsilon$ give $[C : K] = 8$. The intersection $H \cap C$ is an elementary abelian two-extension and hence is D, the maximal such quotient field of S_4. Therefore $[N : K] = 24 \cdot 8/2 = 96$.

At $\mathfrak{p}'$ the reduction of Q is $X^2(X-1)^2$. Hensel factorization gives two quadratic factors with unit resultant. Their discriminant valuations are each at least one and sum to $v_{\mathfrak{p}'}(\mathrm{disc}(Q)) = 2$, so each is one. Both define ramified quadratic extensions; the local splitting field lies in E'. The local square classes of $\pi, \pi', -\epsilon$ are respectively $3, 1, 1$ at $\mathfrak{p}$, and $1, 3, -1$ at $\mathfrak{p}'$. Thus adjoining C leaves E_+ unchanged and makes the other completion exactly E'.

At two put $h = X^2 + X + 1 - w$. Direct calculation gives

$$Q = h^2 + 2g, \qquad g = -2X^3 + (2w+1)X - 6.$$

Over the residue field $\mathbf{F}_4$, the polynomial $\bar{h}$ is irreducible, since $\mathrm{Tr}_{\mathbf{F}_4/\mathbf{F}_2}(1 - \bar{w}) = 1$, and $\bar{g} = X$ is relatively prime to it. Dedekind's index criterion therefore shows that the quartic order is maximal at two. Its local field has ramification index two and residue degree two, and its relative discriminant exponent is four. These statements also follow by passing to the unramified quadratic extension defined by h and using the Eisenstein factor of Q near a root of h.

The reduction of R is $T^2(T+1)$, so it has a local linear factor and its local splitting group has order at most two. The local splitting group of Q is consequently contained in a subgroup of order eight of S_4, and is a two-group. Thus its lower ramification groups satisfy $G_0 = G_1$. In the permutation action on the four roots, inertia has two orbits, since the quartic field has residue degree two. The terms for G_0 and G_1 in the Artin conductor are therefore two each. They exhaust its discriminant exponent four, so faithfulness of the permutation action gives $G_2 = 1$. The largest upper break is at most one.

Finally, the radicands defining C are dyadic units. Every corresponding quadratic character has conductor at most two and upper break at most one. Corollary 26.5 gives the same bound for N. The displayed relative discriminant and the radicands also show that there is no ramification away from two and three. □

26.3 The discriminant comparisons

Let L/K be finite Galois and unramified outside two and three. If its dyadic completion has no upper break greater than one, its dyadic different is less than two. If its completion at $\mathfrak{p}$ is E_+ and its completion at $\mathfrak{p}'$ is contained in E', the discriminant tower gives

$$\mathrm{rd}(L) < 4\sqrt{37}\,3^{5/6} < 60.78017. \tag{52}$$

Each three-adic exponent is multiplied by one half, since $[K : \mathbf{Q}] = 2$.

Proposition 26.8. *Assume GRH for the Dedekind zeta function of L, and put $n = [L : \mathbf{Q}]$.*

(i) If its completions above $\mathfrak{p}, \mathfrak{p}'$ are E_+, E' and $n \geq 96768$, then $\mathrm{rd}(L) > 60.80125$.
(ii) If L is totally real, its completion above $\mathfrak{p}$ is E_+, and $n = 120$, then $\mathrm{rd}(L) > 66.10839$.

Proof. Use the explicit formula of Section 22 with its one-term bound $B_b \leq B_b^*$ and the finite Euler sums. In (i), there are $n/12$ primes of norm three and $n/8$ of norm nine. Thus $c_3 = 1/6$, $c_9 = 1/4$, and we discard the real-place term. In (ii), retain only $c_3 = 1/6$, use $r_1/n = 1$, and bound the real-place integral below by its first four alternating terms. The parameters and directed lower bounds are

	b	n	retained prime norms	$\mathrm{rd}(L) >$
(i)	56/5	96768	$3, 9$	60.8012536506
(ii)	5	120	3	66.1083983139.

The first expression increases with n, since $C_b > 0$. □

Proposition 26.9. *Assume GRH. The image of $\overline{r}_2$ constructed in Proposition 26.3 is either solvable or conjugate to $\mathrm{SL}_2(\mathbf{F}_4) \simeq A_5$.*

Proof. Dickson's classification makes a nonsolvable finite subgroup of $\mathrm{SL}_2(\overline{\mathbf{F}}_2)$ conjugate to $\mathrm{SL}_2(\mathbf{F}_q)$, for $q = 2^r$, $r \geq 2$. Suppose $q \geq 8$. This group is nonabelian simple, so its kernel field M is linearly disjoint over K from the solvable extension N. The compositum $L = MN$ satisfies

$$[L : \mathbf{Q}] = 2 \cdot 96 \cdot q(q^2 - 1) \geq 96768.$$

By Lemma 26.4, Proposition 26.7 and Corollary 26.5, its dyadic different is less than two. Its completions at three are exactly E_+ and E'. Equation (52) contradicts Proposition 26.8(i). □

26.4 Modularity of the remaining images

The sign of the type τ_+ gives a direct treatment of the dihedral case. We make the required quadratic calculation explicit.

Lemma 26.10. *Let D/K be quadratic, unramified at two and at every finite place away from three. If it ramifies at $\mathfrak{p}$, its completion there is $\mathbf{Q}_3(\sqrt{-3})$.*

Proof. Class number one shows that $D = K(\sqrt{\alpha})$ with α a product of $-1, \epsilon, \pi, \pi'$ modulo squares. At the unramified dyadic place, a unit quadratic class is unramified precisely when it is a square modulo four. Calculation in $(\mathbf{Z}/4)[w]/(w^2 - w - 9)$ leaves exactly the four classes

$$1, \qquad \epsilon\pi, \qquad -\pi\pi' = -3, \qquad -\epsilon\pi'.$$

At $\mathfrak{p}$ we have $w = 18 \pmod{27}$, hence $\pi/3 = 1 \pmod 3$ and $\epsilon = -1 \pmod 3$. The two ramified classes in this list, $\epsilon\pi$ and -3, therefore both have local square class -3. □

Proposition 26.11. *Assume GRH. The characteristic-two member r_2 of Proposition 26.3 is modular.*

Proof. Suppose first that the residual image is solvable. Since $\overline{r}_2$ is irreducible and has determinant one, its image is an odd dihedral group D_{2m}. Let D/K be its quadratic inducing extension. Its local quadratic field at $\mathfrak{p}$ is $\mathbf{Q}_3(\sqrt{3})$, by Lemma 23.3. Lemma 26.10 shows that D/K ramifies at the unique dyadic place.

The dyadic wild inertia image is a nontrivial normal two-subgroup of the local decomposition image in D_{2m}. It is generated by a reflection. The normalizer of a reflection in an odd dihedral

group is its own subgroup, so the entire local residual image is C_2. Lemma 26.6 makes r_2 ordinary. Allen's nearly ordinary residually dihedral theorem [2, Theorem 5.2.1 and Lemma 5.1.2] now gives modularity. The extra condition when the inducing quadratic field is CM is satisfied because the dyadic place ramifies in D; in particular it does not split. Allen's dihedral argument supplies the ordinary modular witness required in that theorem.

Suppose now that the residual image is nonsolvable. Proposition 26.9 makes it $A_5 = \mathrm{SL}_2(\mathbf{F}_4)$. Its residual modularity follows from [3, Theorem 2.5]. We check the real-place hypothesis in Thorne's characteristic-two lifting theorem. If the kernel field M were totally real, then $[M : \mathbf{Q}] = 120$, while its finite ramification gives

$$\mathrm{rd}(M) < 4\sqrt{37}\,3^{5/6} < 60.78017.$$

This contradicts Proposition 26.8(ii). Hence $\overline{r}_2(c_v) \neq 1$ for at least one real place v of K. The representation r_2 is Barsotti–Tate at two, its residual image is nonsolvable and modular, and this real-place condition holds. Thorne's Theorem 6.1 [69] applies directly. It permits both ordinary and nonordinary local representations and imposes no condition $K_2 = \mathbf{Q}_2$. □

Proof of Theorem 26.1. Solvable projective images are modular by automorphic induction and Langlands–Tunnell, with the characteristic-three lifting and twisting argument used in [3, proof of Theorem 2.5]. We may therefore suppose that the image is nonsolvable. Proposition 26.3 constructs a compatible system lifting $\overline{\rho}$. Proposition 26.11 makes its characteristic-two member modular. Strong compatibility, Chebotarev and strong multiplicity one then identify the system with the system of that Hilbert modular form. Its characteristic-three residual representation is $\overline{\rho}$, which is consequently modular. □

27 The split-three field applications

For the rest of this section let $K = \mathbf{Q}(\sqrt{d})$ with $d \in \{7, 13, 37\}$, and denote its two places above three by $\mathfrak{p}_1, \mathfrak{p}_2$.

Theorem 27.1. *Assume GRH for Dedekind zeta functions. For $K = \mathbf{Q}(\sqrt{d})$, $d \in \{7, 13, 37\}$, every continuous, absolutely irreducible, totally odd representation $G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$, $\ell > 2$, is modular.*

Proof of Theorem 27.1. For $\ell = 3$, the supported irreducible residual assertion of Theorem 1.2 is Theorem 24.1, Proposition 25.3, or Theorem 26.1, according as $d = 7, 13, 37$. Lemma 23.1 makes both constituents of a reducible supported determinant-$\bar{\chi}_3$ residue quadratic, so their quotient is $\bar{\chi}_3$. It is totally odd and extends to $G_{\mathbf{Q}}$.

Solvable residual images are already modular. Otherwise the initial weight-two lift and Lemma 2.4 give a nondihedral potentially modular compatible system. Two does not split and three splits in all three fields, so Theorem 1.2 proves its modularity. Reduce at the original coefficient place. GRH is used only in the supported residual assertions. □

28 The ramified-dyadic field applications

Proof of Theorem 13.1. For $d = 2, 3$, Proposition 15.12 and Proposition 16.11 give the supported irreducible residual assertions, and Lemma 13.3 gives the constituent quotient. For $d = 6, 10, 11$, the supported irreducible residual assertions required by Proposition 13.2 are Proposition 17.18, Theorem 18.1 and Proposition 19.13. It remains to verify the constituent quotient in its second hypothesis.

For $d = 6$, write the constituents as ψ and $\omega_5\psi^{-1}$. The unit $5 + 2\sqrt{6}$ has residues $2, 3$ at the two five-places, so the two tame exponents of ψ agree. The character ψ^2 is even and, after correction by 1 or ω_5^2, is everywhere unramified. Since the class number is one, ψ^2 is 1 or ω_5^2. The quotient is therefore ω_5 or ω_5^{-1}.

For $d = 10$, Lemma 18.9 gives the quotient ω_3 or $\omega_3\nu$, where ν is the Hilbert class character. The lemma also gives its extension to $G_{\mathbf{Q}}$. These characters are totally odd.

For $d = 11$, Lemma 19.1 gives the finite ray group C_4 and the narrow ray group $C_4 \times C_2$ at five. The even square ψ^2 factors through the finite ray group and has order at most two. Its unique nontrivial quadratic character is ω_5^2, so the quotient is again $\omega_5^{\pm 1}$. Both characters are totally odd and extend to $G_{\mathbf{Q}}$.

All hypotheses of Proposition 13.2 hold. □

29 The split-dyadic theorem

Put $\mathcal{D} = \{17, 33, 41, 57, 65, 73, 89\}$. All seven fundamental discriminants are equal to d, and two splits in $K = \mathbf{Q}(\sqrt{d})$. The dyadic lifting and source theorems used below are recalled in Section 5.

Theorem 29.1. *Assume GRH for Dedekind zeta functions. Let $K = \mathbf{Q}(\sqrt{d})$, $d \in \mathcal{D}$. For every odd prime p, every continuous absolutely irreducible totally odd representation $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is modular. There is no restriction on its conductor.*

Corollary 29.2. *Theorem 29.1 holds without GRH for $d = 17$. It also holds without GRH for $d = 33$ if one assumes that every Dedekind zeta function has no zero in $\operatorname{Re} s > 31/32$, away from its pole, as in [15, Theorem 1.1].*

The general criterion is Theorem 10.1. The next two sections verify its arithmetic and residual hypotheses. The source theorems used for the terminal representation are already proved in the general part.

30 Arithmetic of the split-dyadic fields

Write w_0, w_1 for the dyadic places of K. We record the arithmetic needed in the scalar-residual field construction.

Lemma 30.1. *Let S_2 consist of the dyadic and real places of K. Every finite-order character $G_K \to \overline{\mathbf{F}}_2^{\times}$ unramified outside two is trivial. Moreover*

$$H^1(G_{K,S_2}, \mathbf{F}_2) = \langle [-1], [\epsilon], [2], [\alpha] \rangle_{\mathbf{F}_2}$$

has dimension four, with the choices in Table 2.

Proof. Minkowski's bound is $\sqrt{d}/2$. Except at 57 and 73, every prime ideal required by this bound is dyadic. At 41, 65 and 89, three is inert, so its prime ideal has norm nine and lies outside the bound. The primes above three in the two remaining cases are principal, since

$$N(15 + 2\sqrt{57}) = -3, \qquad N(17 + 2\sqrt{73}) = -3.$$

For $d \neq 65$, the displayed α has norm ± 2 and generates a dyadic prime. At 65, its norm is four and its trace is odd, so it generates the square of a dyadic prime. That prime is not principal: an element of norm ± 2 would yield $x^2 - 65y^2 = \pm 8$, impossible modulo five. Thus $h_{65} = 2$ and

d	ϵ	α	$N\epsilon$	$N\alpha$	h_K	h_K^+
17	$4+\sqrt{17}$	$(5+\sqrt{17})/2$	-1	2	1	1
33	$23+4\sqrt{33}$	$(5+\sqrt{33})/2$	1	-2	1	2
41	$32+5\sqrt{41}$	$(7+\sqrt{41})/2$	-1	2	1	1
57	$151+20\sqrt{57}$	$(7+\sqrt{57})/2$	1	-2	1	2
65	$8+\sqrt{65}$	$(9+\sqrt{65})/2$	-1	4	2	2
73	$1068+125\sqrt{73}$	$(9+\sqrt{73})/2$	-1	2	1	1
89	$500+53\sqrt{89}$	$(217+23\sqrt{89})/2$	-1	2	1	1

Table 2: Units, dyadic generators, and ordinary and narrow class numbers.

the class group is generated by a dyadic prime. This proves the ordinary class numbers and shows that the S_2-ideal class group is trivial in every case.

The unit squareclass group has dimension two. If $N\epsilon = -1$, the mixed signs of ϵ show that its class and $[-1]$ are independent. For $d = 33, 57$ the unit is totally positive and is not a square: a square root would have integral trace whose square is, respectively, 44 or 48, or 300 or 304. Thus $[-1]$ and $[\epsilon]$ are again independent and form a basis. No fundamental-unit calculation is needed. The norm-minus-one units give the narrow class numbers in the other five cases; for $33, 57$, norm minus one is impossible modulo three, giving the remaining entries in the table.

Kummer theory identifies the group in the lemma with $\mathcal{O}_{K,S_2}^\times/(\mathcal{O}_{K,S_2}^\times)^2$. For $d \neq 65$, the valuation vectors of 2 and α form a basis of the dyadic divisor lattice. For 65, that lattice consists of pairs whose sum is even, and its basis is supplied by the vectors $(1,1)$ and $(2,0)$ of 2 and α. Together with -1 and the displayed unit squareclass, this proves the asserted basis.

Finally, a characteristic-two finite character has odd-order image. At either dyadic completion $\mathbf{Q}_2$, it kills the pro-two unit group and is unramified. It also kills real sign groups. Thus it factors through the ordinary class group, whose order is a power of two. The character is trivial. □

For $i = 0, 1$ write $V_i(K) = V_{w_i}^+(K)$ for the space in (1). We verify its vanishing using the full quadratic Selmer group, including the ideal-class contribution.

Lemma 30.2 (Base scalar Selmer vanishing)**.** *For every $d \in \mathcal{D}$ and either dyadic choice, $V_i(K) = 0$.*

Proof. Let H^+ be the subspace of $H^1(G_{K,S_2}, \mathbf{F}_2)$ trivial at both real places. The displayed ϵ and α are positive in the embedding $\sqrt{d} > 0$. For $d = 17, 41, 65, 73, 89$, their norms have signs $-1, +1$; for $d = 33, 57$, the signs are $+1, -1$. The two real conditions therefore give

$$H^+ = \langle [2], [\beta] \rangle, \qquad \beta = \begin{cases} \alpha, & d = 17, 41, 65, 73, 89, \\ \epsilon, & d = 33, 57. \end{cases}$$

Index the dyadic places by $w = (1+\sqrt{d})/2 \equiv 0, 1 \pmod 2$. Their local square classes are as follows:

d	$[\beta]_{w_0}$	$[\beta]_{w_1}$
17	-2	-1
33	-5	-5
41	5	10
57	-5	-5
65	5	5
73	-10	-5
89	-5	-10

These entries follow by Hensel lifting $w^2 - w + (1-d)/4$ and reading the valuation modulo two and the odd unit modulo eight. Each entry is outside the subgroup $\{1, [2]\} \subset \mathbf{Q}_2^\times/(\mathbf{Q}_2^\times)^2$. Thus restriction of H^+ to either dyadic completion is injective. Its kernel at w_{1-i} is precisely $V_i(K)$. □

For $d = 17, 41, 73, 89$, Table 2 gives narrow class number one. The totally positive element α of norm two has trace $5, 7, 9, 217$, respectively. At the other dyadic place its unit root satisfies $u + 2u^{-1} = \mathrm{Tr}(\alpha)$ and is congruent to $7, 5, 3, 3$ modulo eight. Thus these four fields satisfy condition (S) of [21, Lemma 9.1].

31 Residual representations unramified outside two

For a finite Galois extension $E/\mathbf{Q}_2$, let $d(E)$ be its normalized different exponent, with $v_2(2) = 1$. Put $E_2 = \mathbf{Q}_2(\sqrt{-1}, \sqrt{2}, \sqrt{5})$.

Lemma 31.1. *Let $r : G_{\mathbf{Q}_2} \to \mathrm{SL}_2(\mathbf{F}_q)$, $q = 2^a$, have finite image and kernel field E. Then $d(E) \le 2$. More precisely, either $E \subset E_2$, or wild inertia is nontrivial of order $Q \le q$, all positive upper breaks are at most one and the local abelianization has odd order, or E is unramified or tame with ramification index three. In the wild case outside E_2, $d(E) \le 2(1 - 1/Q)$ and $E \cap \mathbf{Q}_2(\sqrt{-1}, \sqrt{5}) = \mathbf{Q}_2$. In the tame ramified case the image is S_3 and its quadratic subfield is unramified.*

Proof. If wild inertia is trivial, its tame generator has eigencharacters θ, θ^{-1}, and Frobenius squares them. The normalizer acts by identity or inversion, so the nontrivial tame order is three; Frobenius is the Weyl involution and the image is S_3. Its different exponent is $2/3$.

Otherwise the wild subgroup fixes a unique line, stable under the whole decomposition group. The diagonal characters are θ, θ^{-1} and are unramified, since $\mathbf{Z}_2^\times$ is pro-two. If $\theta = 1$, the image is elementary abelian unipotent and $E \subset E_2$. The seven quadratic characters of E_2 have conductors $0, 2, 2, 3, 3, 3, 3$, giving $d(E_2) = 2$. If $\theta \ne 1$, Frobenius acts on the translation subgroup P by the nontrivial scalar $\theta(\mathrm{Frob}_2)^2$. The operator obtained by subtracting one is invertible, so $[P, \mathrm{Frob}_2] = P$, P is wild inertia, and $E^P/\mathbf{Q}_2$ is unramified of odd degree. Its Kummer module has no Frobenius invariants. The equivariant valuation map to $\mathbf{Z}/2\mathbf{Z}$ is zero, so all its squareclasses are units and all its quadratic characters have conductor at most two. The different bound follows from conductor–discriminant. The commutator identity gives odd local abelianization and hence disjointness from the stated biquadratic field. □

Lemma 31.2. *Put $H = K(\sqrt{-1}, \sqrt{\epsilon})$. It is biquadratic over K, Galois over $\mathbf{Q}$ and unramified at odd places, and its dyadic completions lie in $\mathbf{Q}_2(\sqrt{-1}, \sqrt{5})$. If L/K has group $\mathrm{SL}_2(\mathbf{F}_q)$, $q \ge 8$, and is unramified outside two, put $M = LH$. Let k count the dyadic places whose local kernel fields lie in E_2. Then*

$$[M : \mathbf{Q}] = 8q(q^2 - 1), \qquad \mathrm{rd}(M) \le \sqrt{d}\, 2^{2 - 1/q + k/(2q)}. \tag{53}$$

Each of these k places supplies in the explicit formula the contribution of at least $[M : \mathbf{Q}]/16$ primes of norm four.

Proof. The classes $[-1], [\epsilon]$ are independent by their real signs and nonsquareness. Conjugation sends $\sqrt{\epsilon}$ to $1/\sqrt{\epsilon}$ or $\sqrt{-1}/\sqrt{\epsilon}$, so $H/\mathbf{Q}$ is Galois. Its defining polynomials have discriminants supported at two, and every unit squareclass over $\mathbf{Q}_2$ is generated by $[-1], [5]$. Simplicity gives $L \cap H = K$.

At a dyadic place with $E \subset E_2$, the compositum still lies in E_2 and has different exponent at most two, ramification index at most four and residue degree at most two. Otherwise, in the

wild case of Lemma 31.1, adjoining the ramified quadratic part of H enlarges its elementary wild group from P to at most $P \times C_2$. All characters still have conductor at most two. Its different exponent is at most $2(1 - 1/(2Q)) \le 2 - 1/q$. In the tame case the compositum has ramification index at most six, with only a possible quadratic positive interval of length one. Its different exponent is at most $(1 - 1/6) + (1 - 1/2) = 4/3 < 2 - 1/q$. Averaging the two dyadic exponents proves (53).

At a counted place, residue degree two gives at least $[M : \mathbf{Q}]/16$ norm-four primes. Residue degree one gives at least $[M : \mathbf{Q}]/8$ norm-two primes; keeping their even Euler powers supplies the same lower contribution. □

Use the functions F_b, J_b, B_b, C_b of Section 22. Put $a = \pi/(2b)$ and let

$$\log \mathcal{L}(n, b, k) = \gamma + \log(8\pi) - \sum_{j=0}^{3} \left\{ \frac{1}{j + 1/2} - J_b(j + 1/2) \right\} - \frac{2a^2}{49} - \frac{C_b}{n} + \frac{k \log 4}{8} \sum_{m \ge 1} 4^{-m/2} F_b(m \log 4).$$

The tail estimate in that theorem makes this a lower bound for the root discriminant, under GRH, when the indicated norm-four Euler contributions are present. It increases with n at fixed b, k. We shall also write $B(G), C(G)$ for the corresponding integrals with an even test function G.

Theorem 31.3. *Assume GRH. For $d \in \mathcal{D}$, every absolutely irreducible characteristic-two representation of G_K unramified outside two is modular.*

Proof. Twist by the inverse square root of the odd-order determinant to make the image determinant one. Linear and projective kernels then coincide. For a solvable irreducible image use [21, Proposition 10.3], since $K/\mathbf{Q}$ is abelian and two splits completely. The image $\mathrm{SL}_2(\mathbf{F}_4)$ is covered by [3, Theorem 1.2(1)] and its projective-to-linear modularity statement. Otherwise Dickson's classification gives $\mathrm{SL}_2(\mathbf{F}_q)$, $q \ge 8$.

For M in Lemma 31.2, the following strict comparisons hold with $n = 8q(q^2 - 1)$ and $d \le 89$:

q	k	b	$\mathcal{L}(n, b, k) >$	upper bound $<$
8	1	7.8	38.07399	36.13605
16	0	9.8	36.64088	36.13605
$q \ge 32$	0	12	38.91320	37.73593

The last row drops Euler terms and uses $n \ge 261888$ and $\mathrm{rd}(M) \le 4\sqrt{89}$. In the first two rows, raising k by one increases the logarithmic lower bound by $\frac{\log 4}{8} \sum_m 4^{-m/2} F_b(m \log 4)$, exceeding $\log 2/(2q)$ already in the first summand since $F_b(\log 4) > 1/2$. Thus the table excludes every case except $q = 8, k = 0$. The exact calculation is supplied in [49, `split_dyadic`].

Let σ be the nontrivial automorphism of $K/\mathbf{Q}$. If $L^\sigma \ne L$, simplicity and Goursat's lemma give $[LL^\sigma : \mathbf{Q}] = 2 \cdot 504^2 = 508032$. Each local constituent has upper breaks at most one, since $k = 0$. The same is true of the compositum by compatibility of upper numbering with quotients, so $\mathrm{rd}(LL^\sigma) \le 4\sqrt{89}$. The last table row, by degree monotonicity, contradicts this.

Hence $L/\mathbf{Q}$ is Galois. Since $\mathrm{Out}(\mathrm{SL}_2(\mathbf{F}_8)) = C_3$, its quadratic conjugation action is inner. The group is centerless, so correcting a lift of σ gives a centralizing involution and $\mathrm{Gal}(L/\mathbf{Q}) = \mathrm{SL}_2(\mathbf{F}_8) \times C_2$. Projection extends the residual representation to $G_{\mathbf{Q}}$. Serre's theorem [40, 41, 43] makes that extension modular: in characteristic two oddness is the determinant condition $-1 = 1$, and no nontrivial complex-conjugation action is required. Cyclic base change to K is cuspidal, since the restriction is absolutely irreducible. Undo the determinant twist. □

31.1 Comparisons without GRH

For an even test function G the unconditional explicit formula has, in addition to (39), the zero terms associated to $\Phi_G(s) = \int_{\mathbb{R}} G(x)e^{(s-1/2)x}\,dx$; see [47, equation (2.3)]. If the zeros lie in $|\operatorname{Re} s - 1/2| \le a$, these terms may be discarded for $G = F_b/\cosh(ax)$. Indeed, for $|h| < a$ the Fourier transform of $\cosh(hx)/\cosh(ax)$ is

$$\frac{\pi}{a}\,\frac{\cos(\pi h/(2a))\cosh(\pi t/(2a))}{\sinh^2(\pi t/(2a)) + \cos^2(\pi h/(2a))} \ge 0.$$

The endpoints follow by continuity, and multiplication by the positive definite function F_b preserves this property. Write $B(G), C(G)$ for the integrals used in (39).

First, a totally real kernel field with group $\mathrm{SL}_2(\mathbf{F}_q)$, $q \ge 8$, is excluded for all seven fields without GRH: (40), with $b = 24$ and $n \ge 1008$, gives $\mathrm{rd}(L) > 55.42$, whereas $\mathrm{rd}(L) \le 4\sqrt{89} < 37.736$. This does not treat the totally imaginary kernels.

For $d = 17$, the kernel field has degree at least 1008 and $\mathrm{rd}(L) \le 4\sqrt{17} < 16.50$. Formula (40) with $b = 10$, dropping prime and real terms, gives $\mathrm{rd}(L) > 20.57$. This proves the unconditional residual theorem for 17.

For $d = 33$, put $b = 24$. If $k \ge 1$, retain just the first norm-four term from Lemma 31.2. At degree at least 4032, (40) gives $\mathrm{rd}(M) > 23.31$, whereas (53) gives $\mathrm{rd}(M) \le 4\sqrt{33} < 22.979$. If $q = 8$ and $k = 0$, the same formula without prime terms gives $\mathrm{rd}(M) > 21.75 > 21.07115$, contradicting (53). For $k = 0$ use the original kernel field L: Lemma 31.1 gives $[L : \mathbf{Q}] = 2q(q^2-1)$ and $\mathrm{rd}(L) \le \sqrt{33}\,2^{2-2/q}$. The resulting unconditional comparisons are

q	$[L : \mathbf{Q}]$	lower $\mathrm{rd}(L)$	upper $\mathrm{rd}(L)$
16	8160	> 21.97	< 21.07115
32	65472	> 22.15	< 22.00405

Only $k = 0$, $q \ge 64$ remains without the additional zero-free input.

Assume now the additional zero-free statement in Corollary 29.2. The functional equation puts the zeros of every relevant zeta function in the strip with $a = 15/32$. For $G = F_b/\cosh(15x/32)$ one has

$$B(G) < \frac{13}{20} + \frac{3\pi^2}{4b^2}, \qquad C(G) \le 4J_b(-1/32).$$

Here the first bound follows from $B(G) \le B_\infty(a) + \pi^2 Q(a)/(8b^2)$, where

$$B_\infty(a) = \int_0^\infty \frac{1 - \operatorname{sech}(ax)}{2\sinh(x/2)}\,dx < 13/20, \qquad Q(a) = \int_0^\infty \frac{x^2}{2\sinh(x/2)\cosh(ax)}\,dx < 6.$$

For completeness these constants have finite rational bounds: expand $B_\infty(a) = \sum_{j\ge0}(-1)^j S_j$, with $S_j = \sum_{m\ge0} 2a^2/[s(s+a)(s+2a)]$ and $s = m + 1/2 + 2ja$. The decreasing alternating sums through $j = 80, 81$, truncating each inner sum at 500 with error at most $a^2/(499.5 + 2ja)^2$, give $0.64911 < B_\infty(15/32) < 0.64922$. Also $Q(15/32) \le 4\sum_{j=0}^{3}(j + 31/32)^{-3} + 2/(3 + 31/32)^2 < 6$. For the remaining $q \ge 64$, take $b = 30$ and use $n \ge 32640$. Then

$$\mathrm{rd}(M) \ge 8\pi e^\gamma \exp\left(-\frac{13}{20} - \frac{3\pi^2}{3600} - \frac{4J_{30}(-1/32)}{32640}\right) > 23.06000 > 4\sqrt{33}.$$

This proves the residual assertion needed for the second part of Corollary 29.2. All numerical inequalities in this section follow from the displayed finite expressions and tail bounds by directed interval arithmetic. The exact calculation is supplied in [49, `split_dyadic`].

Proof of Theorem 29.1 and Corollary 29.2. Lemma 30.1 gives the two-power ordinary class number, and Lemma 30.2 verifies (SV). Theorem 31.3 supplies the irreducible residual base. Apply Theorem 10.1. For $d = 17$ use the unconditional residual comparison above; for $d = 33$ use its stated zero-free variant. The scalar and conductor arguments introduce no additional analytic hypothesis. □

32 Proof of the main theorem

Proof of Theorem 1.1. Theorem 27.1 proves the cases $d = 7, 13, 37$, and Theorem 13.1 proves $d = 2, 3, 6, 10, 11$. The source arguments and conductor reductions in Sections 20 and 21 give $d = 5, 21$. The remaining seven fields are covered by Theorem 29.1. Every congruence returns modularity to the original residual representation, without a restriction on its conductor. □

A Depth-four normal ray calculation

We give the matrices used in Proposition 18.10. The fields K, A, the standard module W_0 and the binary ray coordinates are those in its proof. For the depth-four calculation write $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on W_0. In the seven binary ray coordinates the class killed by $[\mathfrak{q}_+]$ is $v = (1,0,0,1,1,0,1)^t$. The source matrices for the actions of S, U in those coordinates are

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The solution space to $Xv = 0$, $XC_2 = SX$, $XC_3 = UX$ has basis

$$X_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Chinese remainder lifts of local units, chosen to be one at the other dyadic prime, have binary ray logarithm columns

$$g = (0,1,1,0,0,0,0)^t, \quad h = (0,0,0,1,0,0,0)^t, \quad j = (0,0,0,1,0,1,1)^t.$$

The first prime supplies g, h and the second supplies g, j. For $\alpha^2 + \alpha + 1 = 0$, evaluation of the two equivariant maps followed by the left U-eigenrow $(1, \alpha)$ gives, respectively,

$$\begin{pmatrix} 0 & 1+\alpha \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \alpha \\ 1 & 1+\alpha \end{pmatrix}.$$

Both determinants are nonzero, and the conjugate eigenrow has the same property. Hence vanishing on local units in either one of the two C_3 eigencharacters, at either prime, kills the global depth-four class. These conclusions concern the characteristic-two normal error; the order of tame inertia after reduction in the joint trace order is checked separately. The matrices and local logarithms are reproduced in [49, `ramified/d6_d10_d11/d10`].

B Auxiliary fields over the field of discriminant twenty-one

The unit assertions involve only a number field of degree twelve. One defining polynomial for A is

$$P(X) = X^{12} - 3X^{11} + 8X^{10} - 9X^9 + 9X^8 + 6X^7 + X^6 \\ + 6X^5 + 9X^4 - 9X^3 + 8X^2 - 3X + 1.$$

The field defined by P has discriminant $21^6 5^8$ and signature $(0, 6)$. The certificate reconstructs the embeddings of u and z, checks their equations and the norm-one relations, and tests the sixteen products at the two dyadic primes. Integrality and unit status follow directly from their monic minimal polynomials with constant coefficient ± 1. It uses only `nfinit`, exact field arithmetic, Galois conjugates and local-power tests; no computation of a full unit group or class group is used in the proof. The exact calculation is supplied in [49, `d21`].

The arithmetic and interval certificates are:

- `check_small_unit_formulas.gp`: the two-orbit unit identities and local squareclass kernels in Lemma 21.11;
- `verify_split5_local.py`: the two compact-type Brauer-character identities, calculated in $\mathbf{Z}[\zeta_{24}]$;
- `certify_K21_SC_SC.py`: real-ball evaluations of (39), with each comparison asserted only when its lower and upper balls are disjoint;
- `K21_SC_SC_arb_certificate.json`: the full-precision interval enclosures for Table 1.

The smallest gap, in the distinct-conjugate row, is greater than 0.00974417509. The analytic calculation certifies the displayed inequalities; the group and local hypotheses are proved in the preceding sections. The exact calculation is supplied in [49, `d21/inherited_python`, `d21/split5`].

C Normal Selmer calculations over the field of discriminant twenty-one

The following scripts in [49, `d21`] print certificates for the normal Selmer calculations. All class and unit calculations have absolute degree at most twelve. The accompanying code includes output records, the certificate `s3_mixed_full_sunit_portable.gp`, and the diagnostic-checking runner `run_gp_checked.py`. The certificate uses relative discriminants and Hilbert symbols to test the stated invariant subspace conditions.

file	certified assertion
`k21_cm_split_s3_classification.gp`	H class number one; ray groups and four cubic quotients
`k21_cm_split_s3_galois.gp`	quadratic involution and residual fields
`s3_mixed_full_sunit_portable.gp`	full mixed normal space of dimension six; two necessary cuts intersect in zero
`anti_C3_conductors_A1.gp`	A_1 has class group C_3; all nonzero normal classes have dyadic conductor six somewhere

The displayed masks in Section 21.3.2 refer to the chosen unit and S-unit bases. A different PARI version or valid basis may change these coordinates. Invariantly, the nontrivial C_3-part of the S_5-unit squareclass space has dimension six over $\mathbf{F}_2$; its two cuts have dimensions two and four, and their intersection is zero. For A_1, only the zero normal class meets both dyadic conductor bounds. Reproduction should record the actual basis and masks and check these invariant assertions. Quadratic discriminant exponents from relative discriminants are the local quadratic conductors. Successful output records include `bnfcertify=1`, the full normal rank and every retained mask; a zero process exit status alone does not certify success in PARI/GP. The exact calculation is supplied in [49, `d21`].

The nonsolvable branch uses the discriminant comparisons above and the `certify_K21_SC_SC.py` and `K21_SC_SC_arb_certificate.json`. It is used only for the nonsolvable branch of the mixed theorem; Theorem 21.15 has no analytic hypothesis. The exact calculation is supplied in [49, `d21/split5`].

AI disclosure

During the preparation of this article, the authors used ChatGPT 6-Astra and Sol 6.1 as an assistant in mathematical research and manuscript preparation. Its use included exploring proof strategies, checking mathematical arguments, investigating possible simplifications, assisting with computations and verification scripts, checking references, and drafting and revising portions of the exposition. The authors take full responsibility for the mathematical content, proofs, computations, and conclusions of the article.

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