Three-colour cuboids from four-colour cubes

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November 8, 2026

Abstract

We determine the least number of four-colour cubes that guarantees a cuboid with monochromatic faces using exactly three colours. Each source uses all four colours of a fixed palette; only proper rotations are allowed, and interior faces need not match. For integer side lengths $2 \le a \le b \le c$, the sharp threshold for the corner-and-edge frame is $\max\{16a-3, 4(a+b+c)-16, \lfloor(24c-1)/5\rfloor\}$, and the complete-cuboid threshold is $\max\{abc, 16a-3, \lfloor(24c-1)/5\rfloor\}$. The three frame bounds arise from colour-bucket capacity, the number of positions and balanced checkerboards. The upper bound combines corner containment, a weighted graph selecting edge directions, and Hall's theorem. For cubic frames the threshold is $16n-3$, with exactly two failing collections of size $16n-4$. For $n \ge 3$ we classify every failure of size at least $14n$ and count these collections and the inclusion-maximal ones. The case $n = 2$ answers the threshold question in Problem 2 of Berkove, Eickemeyer and Schulman (2018). All threshold, stability and counting results are formalised in Lean.

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1 Introduction

Berkove, Eickemeyer and Schulman (BES) [1, Problem 2] ask for the least number of four-colour cubes that guarantees a choice of eight cubes forming a three-colour corner solution: a $2 \times 2 \times 2$ cube whose exterior faces are monochromatic and use exactly three colours. Touching interior faces need not match. As in BES [1, Section 1], every source is a unit cube with one colour on each face and all four colours of one fixed palette present; every target uses exactly three of them. A *variety* is a face colouring up to proper rotation. Collections are multisets of varieties, so repeated copies are allowed.

We determine the thresholds for every cuboid with sides at least two. A frame consists of the corner and edge pieces, including each corner once. For ordered sides $2 \le a \le b \le c$, its cardinality is $4(a+b+c)-16$. Writing F and G for the least inventory sizes guaranteeing a frame and a complete box, respectively, our general result is

$$F(a,b,c) = \max\{16a-3,\ 4(a+b+c)-16,\ \lfloor(24c-1)/5\rfloor\},$$
$$G(a,b,c) = \max\{abc,\ 16a-3,\ \lfloor(24c-1)/5\rfloor\} \qquad \text{(Theorem 8)}.$$

The target may depend on the inventory. Three obstructions explain the frame formula: the shortest side controls a repeated-colour demand, the sum of the sides counts positions, and the longest side controls a checkerboard shortage. A frame needs no spare sources precisely when $b+c \ge 3a+4$ and $c \le 5(a+b)-19$.

Cubic targets admit a stronger structural description. Their frame threshold is $F(n,n,n) = 16n-3$ (Theorem 4). Exactly two collections of $16n-4$ cubes fail: $4n-1$ copies of each of four varieties, one with a monochromatic corner of each colour, all in the same orientation class. Reflection or any odd colour permutation exchanges these collections. Figure 1 shows the first case, $n = 2$. More generally, every failure of size at least $14n$, for $n \ge 3$, is confined to one ordinary class with bounded buckets (Theorem 5). Counting pair multiplicities and bucket slacks gives explicit binomial formulas for a band of failures (Theorem 6). This includes the 32

failures of size $16n-5$, twenty-four of which cannot be enlarged; the same counts hold separately at $n = 2$ (Theorem 7).

Filling the remaining positions gives the complete-cube threshold $\max\{n^3, 16n-3\}$ (Corollary 3). Thus any n^3 sources suffice for a complete cube exactly when $n \geq 4$: in particular, any 64 sources build a $4 \times 4 \times 4$ cube without spares.

For general complete boxes, the only shapes that can require spare cubes are rods $2 \times 2 \times c$ and the four boxes $2 \times 3 \times 3$, $2 \times 3 \times 4$, $3 \times 3 \times 3$ and $3 \times 3 \times 4$ (Corollary 6). Rods and strips $2 \times 3 \times c$ have the same frame threshold, despite their different cardinalities (Corollary 5). Every complete $2 \times 4 \times c$ box can be assembled from exactly its volume.

For comparison, BES [1, Theorem 5] show that any $12n-16$ four-colour cubes build a four-colour n-frame when $n \geq 3$. Requiring exactly three exterior colours raises the sharp threshold to $16n-3$.[1] Their completion argument [1, Section 1] gives a four-colour complete cube from any n^3 sources when $n \geq 3$; our result guarantees exactly three exterior colours when $n \geq 4$.

Sections 3–5 prove the corner threshold, cube stability and counting results. Section 6 proves the general cuboid thresholds from the common geometry and edge facts.

2 Model and geometric facts

Only proper rotations are allowed. Exterior faces must match the target; interior faces are unrestricted.

A corner type is a cyclically ordered triple of face colours, read counterclockwise when viewed from outside along the corner's body diagonal. Equivalently, order the three outward face normals to form a positively oriented basis. A trichromatic corner has three distinct colours and two orientations, abc and acb. BES [1, Section 2] use clockwise order; our convention reverses every triple, preserving containment and matching. A source cube fits a target corner by a proper rotation if and only if it has the same cyclic corner type: the cyclic alignments of two corner frames are exactly those realised by proper rotations of the cube. Only the presence of a corner type on a source matters: a physical copy occupies one position. Target occurrences remain separate requirements. For the corner problem ($n = 2$), a target is represented by its multiset of eight corner types. A collection builds a corner solution modelled on that target exactly when these eight occurrences can be matched to distinct cubes having the required types.

For distinct colours a, b, c, write $Q(a; b, c)$ for a target whose b and c faces are opposite and whose other four faces have colour a. Its corner multiset is $4aab + 4aac$. Write $R(a; b, c)$ for the analogous target with b and c adjacent; its corners are

$$2aaa + 2aab + 2aac + abc + acb.$$

Finally, write $D(a, b, c)$ for the target with opposite pairs aa, bb, cc; it has four corners of each orientation on $\{a, b, c\}$. These three families suffice for the corner argument. Section 5 introduces U for cube stability; Section 6 uses a different target H for directional cuboid demands.

Lemma 1. *In a cube using exactly three colours, the two orientations of a trichromatic corner occur equally often. If no trichromatic corner occurs, the cube is $Q(a; b, c)$ for some distinct a, b, c.*

Proof. For any cube coloured from a palette $\{a, b, c, d\}$, let $\nu(xyz)$ count corners of cyclic type xyz. Read the three directed transitions around each corner in the chosen cyclic order. Each edge between an a face and a b face contributes ab at one endpoint and ba at the other, so the total counts of these transitions agree. Corners with repeated colours contribute equally to both counts. The remaining contributions give

$$\nu(abc) - \nu(acb) + \nu(abd) - \nu(adb) = 0. \tag{1}$$

[1] BES's Table 7 also records the frame number $16n-17$, for $n \geq 4$, for three-colour source cubes building three-colour frames: a different problem with the same leading coefficient.

If only a, b, c occur, the last two terms vanish, proving the balance.

For the second assertion, if a colour c occurs on only one face, its four neighbours must all have the same colour a: a change from a to b around this face would produce a trichromatic corner. The opposite face is then the sole b face. Otherwise each colour occurs twice. If the two c faces are opposite, the four neighbours of either contain both a and b, forcing a change. If they are adjacent, the other three neighbours form a path containing both a and b, since each occurs only twice. Again a change is forced. Both cases produce a trichromatic corner. □

The target families test complementary constraints:

Target	Role
D	Detects the availability of both orientations
Q	Bounds the capacity of a colour bucket
R	Eliminates minority cubes in the corner case
U	Eliminates minority cubes in the cube stability proof
H	Supplies directional demands in cuboids

The two ordinary classes below supply opposite cyclic orientations. The balance lemma explains why confining a collection to one class obstructs all targets with trichromatic corners.

If every corner type of a variety v occurs on a variety w, replacing a copy of w by v cannot create a solution to any target. In particular, it preserves unsolvability. Table 1 records the reduction. Face tuples use the order $+x, -x, +y, -y, +z, -z$. The rows represent the seven orbits under colour permutations, with rotations already identified. They cover all 68 varieties; four orbits are retained.

Representative	Orbit size	Retained type	Replacement
001123	6	checkerboard C	—
010203	8	monochromatic-corner T	—
010213	12	pair type P	—
010223	12	pair type P	—
000123	12	—	001123
001213	12	—	001123
010123	6	—	001123

Table 1: The 68 varieties reduce to 38.

A reduced variety belongs to bucket a if it has at least two of the three corner types aab with $b \neq a$. The reduction, 38 minimal varieties, orientation classes, checkerboards and buckets below are from BES [1, Section 2, Table 8 and Definition 3]; we count each bucket's seven mirror pairs as fourteen individual varieties.

Lemma 2 (Finite geometry)**.** *Every non-retained variety in Table 1 contains the corner types of a checkerboard, so every variety reduces to one of the 38 retained varieties. Their bucket incidences are:*

- *There are eight T varieties. Each has a repeated colour a, corners aaa and aab for every $b \neq a$, and belongs only to bucket a.*
- *The 24 P varieties belong to two buckets, one for each repeated colour.*
- *The six checkerboards belong to no bucket. Their repeated colours occupy opposite face pairs; with repeated colours a, b, a checkerboard has both orientations on $\{a, b, c\}$ and $\{a, b, d\}$.*

The 32 ordinary varieties (T and P) split into two classes of 16, corresponding to BES's K_o and K_m (up to interchanging the names). Every member of a class has exactly the same four trichromatic corner types, one for each three-element colour set. The two classes have opposite orientations. In our convention one class has the trichromatic corner set

$$\Omega = \{012, 031, 023, 132\},$$

one set of BES's universal corners. *Each class contains exactly one T variety for each repeated colour. Even colour permutations preserve the classes; odd colour permutations and reflection exchange them.*

Proof. The face-colour multiplicities are $3+1+1+1$ or $2+2+1+1$. In the first case, the three equal faces either meet at a vertex or contain an opposite pair, giving two orbits. In the second case, both repeated pairs can be opposite, exactly one can be opposite, or both can be adjacent. In the last case, the two monochromatic edges either have perpendicular directions, giving two mirror configurations represented by 010213 and 010223, or are parallel, represented by 010123. These cases cover all varieties. The orbit sizes in Table 1 sum to 68, the total number of varieties [1], so the seven orbits are distinct. The three non-retained representatives contain the corner types of 001123; colour permutations transport these containments to their whole orbits. Table 2 gives the corner sets of the retained representatives. All three ordinary representatives in Table 2 belong to the class of Ω. Even colour permutations preserve Ω and odd ones reverse it, with triples read cyclically. Applying colour permutations to Table 2 gives the stated incidences and orientation classes. The eight T varieties split evenly between the classes, and even permutations act transitively on the four colours, so each class contains exactly one T variety for each repeated colour. Reflection reverses every trichromatic corner. □

Representative	Corner-type set
001123	$\{012, 013, 021, 031\}$
010203	$\{000, 001, 002, 003, 012, 023, 031, 132\}$
010213	$\{001, 003, 011, 012, 023, 031, 112, 132\}$
010223	$\{002, 003, 012, 022, 023, 031, 122, 132\}$

Table 2: Corner-type sets of the retained representatives, with cyclic triples written in lexicographically least form.

Lemma 3 (Bucket bound)**.** *Eight reduced cubes in bucket a can build a target $Q(a; b, c)$ for some b, c. Consequently, a reduced collection with no Q target has at most seven cubes in each bucket.*

Proof. Each of the eight cubes misses at most one of the three types aab. Thus at most one type is missed by five or more cubes. Choose two types, aab and aac, each missed by at most four cubes. Assign cubes missing aac to aab positions, and cubes missing aab to aac positions. These groups are disjoint and each has size at most four. Every remaining cube has both types, so they fill the remaining positions of $Q(a; b, c)$. □

3 The extremal corner collections

Theorem 1. *A collection of 28 four-colour cubes that builds no target of type Q, R or D consists of seven copies of each of the four T varieties in one ordinary orientation class.*

Proof. Replace each non-retained cube by a checkerboard using Lemma 2, leaving retained cubes unchanged. This preserves avoidance of all three target families. Let $A \geq B$ count

the cubes belonging to the two ordinary orientation classes, and let C count checkerboards, so $A + B + C = 28$. Call the class with A cubes the majority class (either one if equal). Call the remaining $M = B + C$ cubes *minority cubes*. A minority cube *supports* a three-element colour set E if it has the orientation on E opposite to the majority class. Smaller-class cubes support every E; a checkerboard supports two.

(i) At most six minority cubes: $M \le 6$. For $E = \{a, b, c\}$, let z_E count checkerboards supporting E. If $z_E \ge 8$, those eight checkerboards build the target $D(a, b, c)$. Thus $z_E \le 7$ for each E, and, since each checkerboard supports two triples, $2C \le 28$. It follows that $A + B \ge 14$, so $A \ge 7$. There are $m_E = B + z_E$ minority cubes supporting E. If $m_E \ge 4$, four such cubes and four majority cubes build $D(a, b, c)$. Hence $m_E \le 3$ for every E. Summing gives

$$4B + 2C \le 12, \qquad M \le 6, \qquad A = 28 - M \ge 22. \tag{2}$$

(ii) No minority cubes: $M = 0$. Let t_a and p_a count only majority-class T and P cubes in bucket a, and let n_T, n_P be their respective totals, so $A = n_T + n_P$. By Lemma 3, $t_a + p_a \le 7$. Each T cube belongs to one bucket and each P cube to two. This is BES's double-count [1, equation (1)]; summing the bucket bounds and using $n_T + n_P = 28 - M$ gives

$$n_T + 2n_P \le 28, \qquad n_P \le M. \tag{3}$$

We use the following construction. If six majority cubes supply $2aaa + 2aab + 2aac$ and a minority cube supports $\{a, b, c\}$, then the minority cube supplies the opposite trichromatic orientation and an unused majority cube supplies the other. Such a cube is available since $A \ge 22$. This builds $R(a; b, c)$.

Suppose $M > 0$. We claim that $t_a \ge 4$ forces $t_a + p_a \le 5$. Otherwise choose six majority cubes in bucket a, including at least four T cubes. Any minority cube supports some triple containing a, say $\{a, b, c\}$, since a checkerboard's two supported triples together contain every colour. Each of the at most two P cubes has two of the three types aax, so it has aab or aac; since $R(a; b, c)$ needs two of each, each P cube can take a position of a type it has. The T cubes have aaa and every type aax, so they fill the remaining positions of $2aaa + 2aab + 2aac$, and the construction above gives a solution, a contradiction. In particular, $t_a \le 5$ for every a.

If $t_a \le 3$, the bucket bound gives $2t_a + p_a \le t_a + 7 \le 10$. If $4 \le t_a \le 5$, the claim gives $2t_a + p_a \le t_a + 5 \le 10$. Thus $2t_a + p_a \le 10$ in all cases. Each majority T cube contributes two through its single bucket, and each majority P cube contributes one through each of its two buckets. Summing gives

$$2A = 2n_T + 2n_P = \sum_a (2t_a + p_a) \le 40,$$

contradicting $A \ge 22$ from (2). Therefore $M = 0$.

(iii) Equal multiplicities: $t_a = 7$. With $M = 0$, (3) forces $n_P = 0$. All 28 cubes are majority-class T cubes, and the four bucket bounds force each $t_a = 7$.

Since the reduced collection contains no checkerboards, no replacement occurred: the original collection was already reduced. □

Theorem 2. *Every collection of 29 four-colour cubes contains a three-colour corner solution.*

Proof. Otherwise every 28-cube subcollection would have the form in Theorem 1. Take one such subcollection S and let v be the remaining cube. Remove from S a cube of a variety different from v, and insert v. The multiplicity of v's variety is now 1 or 8, whereas every variety in a collection described by Theorem 1 has multiplicity 0 or 7, a contradiction. □

4 Sharpness

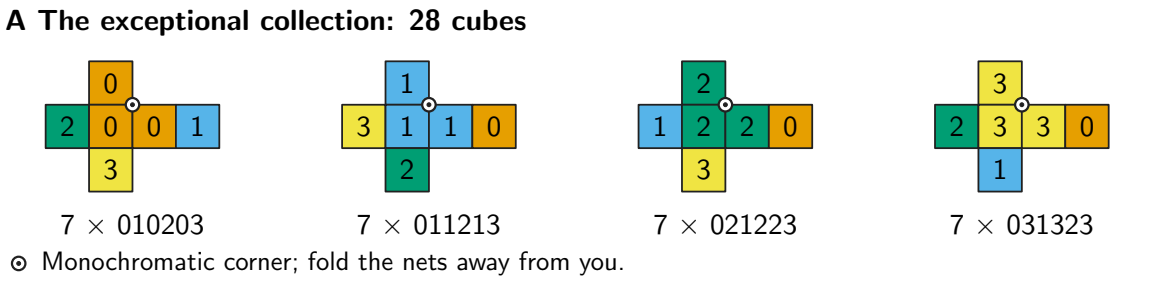


Figure 1: Why one exceptional 28-cube collection fails for three-colour targets. (A) Seven copies of each variety, with monochromatic corners marked. (B) Targets with a trichromatic corner need both orientations; the example's bottom face is hidden. (C) Targets without trichromatic corners require eight cubes with the same repeated colour, but only seven are available; a, b, c are distinct. Reflection gives the other obstruction.

Theorem 3. *The threshold 29 in Theorem 2 is best possible. For a fixed palette, exactly two 28-cube collections fail, and they are mirror images. Up to colour permutation, there is a single obstruction.*

Proof. Take seven copies of each of the following four face tuples (Figure 1):

$$010203, \qquad 011213, \qquad 021223, \qquad 031323.$$

These are T cubes with repeated colours $0, 1, 2, 3$, respectively, all in the same ordinary orientation class. A target with any trichromatic corner requires both orientations by Lemma 1, whereas the collection supplies only one. A target without one is $Q(a; b, c)$; every one of its eight corners has colour a twice, so only the seven T cubes of repeated colour a are usable. No three-colour target is possible. Reflection gives the analogous example in the other orientation class. Theorem 1 excludes all other collections. Every odd colour permutation exchanges the two orientation classes and hence these two collections. □

Remark (Allowing fewer colours). If targets may use at most three colours, 28 cubes suffice. In either exceptional collection, four T cubes of repeated colour a and four of repeated colour b supply $aaa + 3aab + bbb + 3bba$. These are the corners of a two-colour cube whose three a faces meet at one vertex and whose three b faces meet at the opposite vertex. Every other 28-cube collection builds a three-colour target by Theorem 3. This gives an upper bound, not an exact threshold, for that variant.

Corollary 1 (Restricted targets)**.** *Every collection of 29 four-colour cubes, and every solvable collection of 28, builds a target of type Q, R or D. These families contain just 28 of the 120 target varieties.*[2]

Proof. By Theorems 1 and 3, a 28-cube collection avoiding these families is unsolvable. The swap in the proof of Theorem 2 rules out a 29-cube collection avoiding them. There are $4\binom{3}{2} = 12$ varieties each of Q and R: choose a and the unordered pair $\{b, c\}$. There are $\binom{4}{3} = 4$ varieties of D, one for each three-colour set. □

Remark. For the palette $\{0, 1, 2, 3\}$, write $\delta(xyz) = \nu(xyz) - \nu(xzy)$. Applying (1) to the different colour pairs gives

$$\delta(012) = \delta(031) = \delta(023) = \delta(132).$$

This common integer is the cube's *degree*, relative to Ω. Reflection negates the degree, as does each transposition of colour labels; thus even colour permutations preserve it and odd ones negate it. Each ordinary representative in Table 2 has eight distinct corner types, so each occurs once; the two ordinary classes therefore have degrees $+1$ and -1. The checkerboard representative has neither 023 nor 032, so its degree, and that of every checkerboard, is zero.

5 Cubic frames: threshold and stability

An n-frame consists of the eight corner cubes and the $n - 2$ further cubes on each of the twelve edges of an $n \times n \times n$ cube. Thus it contains $L = 12n - 16$ cubes. Its six exterior faces are monochromatic and use exactly three colours; face interiors are not part of the frame. This is the frame model of BES [1, Section 1], with three target colours instead of four.

[2] For any fixed three colours, a rotation with k face cycles fixes $3^k - 3 \cdot 2^k + 3$ colourings using all three. The 24 rotations have $6, 4, 3, 2$ face cycles with multiplicities $1, 3, 12, 8$, so Burnside's lemma gives $(540 + 3 \cdot 36 + 12 \cdot 6)/24 = 30$ varieties. Four choices of omitted colour give 120.

Theorem 4 (Frame threshold and classification)**.** *For every $n \geq 2$, every collection of $16n-3$ four-colour cubes contains a three-colour n-frame. For a fixed palette, exactly two collections of $16n-4$ cubes fail: $4n-1$ copies of each of the four T varieties in one orientation class, or their reflections. Consequently the sharp frame threshold is $F(n) = 16n-3$.*

The corner-containment reduction also preserves edge availability: an edge occurs in a corner, so containment of corner types implies containment of edge types [1, Section 2, after Definition 2]. An edge type is an unordered pair of colours, allowing repetition. Every ordinary variety has all six bichromatic edge types. A checkerboard has five, missing precisely the pair of its singleton colours; different checkerboards miss different pairs. These facts follow from the representatives in Table 2, and also appear in BES's frame arguments [1, proofs of Lemmas 7–8 and Table 5]. A reduced cube has an aa edge if and only if it belongs to bucket a.

Lemma 4. *Let $n \geq 2$. A reduced collection with a D corner solution has an n-frame whenever its size N satisfies*

$$N \geq \max\{12n-16,\ 8n-1\}.$$

Proof. If a checkerboard variety occurs at least $4n$ times, use eight copies for a supported D corner solution. Its remaining copies supply at least $4(n-2)$ neighbours for each required edge type. Otherwise every checkerboard variety occurs at most $4n-1$ times. Remove any D corner solution. Each of its three bichromatic edge types then has at least

$$N - 8 - (4n-1) \geq 4(n-2)$$

available suppliers. In either case, the union of the supplier sets for two edge types is the entire residual collection, since no reduced cube misses two bichromatic types. There are at least $12(n-2)$ remaining cubes, so Hall's inequalities hold for all edge positions. □

One more target supplies the growing repeated-colour demands. For distinct a, b, c, let $U(a, b; c)$ have face tuple (a, b, a, b, a, c). Its corner and edge multisets are respectively

$$aaa + 2aab + aac + abb + bbc + abc + acb, \qquad 3aa + bb + 4ab + 2ac + 2bc.$$

Write T_a for the T variety with repeated colour a in a given ordinary class. Thus an ordinary class supplies a U-frame using $3n-2$ of its T_a cubes, n of its T_b cubes, and $1 + 8(n-2)$ other cubes, together with one minority cube supporting $\{a, b, c\}$. The T_a cubes fill four corners and the $3(n-2)$ monochromatic aa edge positions; the T_b cubes fill two corners and the $n-2$ monochromatic bb positions. The minority cube and one further cube from the same ordinary class fill the two trichromatic corners. Every remaining edge is bichromatic and can use any ordinary cube.

Theorem 5 (Stability)**.** *Let $n \geq 3$ and $N \geq 14n$. A collection of N four-colour cubes fails to build a three-colour n-frame if and only if all its cubes are ordinary varieties in one orientation class and each colour bucket contains at most $h = 4n-1$ cubes.*

Proof. Reduce a failing collection as before. Let $A \geq B$ count the two ordinary classes, let C count checkerboards, and put $M = B + C$. Write t_a, p_a, n_T, n_P for the majority-class counts. Lemma 4 excludes a D corner solution. The orientation count from Theorem 1 therefore gives

$$4B + 2C \leq 12, \qquad M \leq 6, \qquad A = N - M \geq 14n - 6.$$

A majority bucket of size $4n$ would build a Q-frame: eight of its cubes supply the corners, $4(n-2)$ supply the monochromatic edge positions, and $8(n-2)$ other majority cubes supply the bichromatic positions. There are enough since $A \geq 12n - 16$. Thus

$$t_a + p_a \leq h, \qquad n_T + 2n_P \leq 4h, \qquad n_T \geq 2N - 4h - 12 \geq 12n - 8.$$

Some $t_a \geq 3n-2$. At most one colour has fewer than n majority T cubes, since two such colours would give

$$n_T \leq 2(n-1) + 2h = 10n - 4 < 12n - 8.$$

If $M > 0$, a minority cube supports a triple containing a and another colour b with $t_b \geq n$. The $U(a, b; c)$ construction then applies. After its $4n-1$ majority corner and monochromatic-edge cubes have been selected, at least $A-(4n-1) \geq 10n-5 \geq 8(n-2)$ ordinary cubes remain. This contradiction gives $M = 0$. No checkerboards remain, so the reduction made no replacements in the original collection.

Conversely, one ordinary class cannot supply both orientations of a trichromatic corner. Lemma 1 leaves only Q targets. Their corners and monochromatic edges require $4n$ cubes from one bucket, exceeding its capacity h. □

Proof of Theorem 4. For $n = 2$, use Theorem 3. For $n \geq 3$, apply Theorem 5. At size $4h$, purity and $n_T + 2n_P \leq 4h = n_T + n_P$ force $n_P = 0$, and all four T multiplicities equal h. Both collections fail by the converse. At size $4h + 1$ the bucket inequalities are impossible, proving the upper bound. □

Theorem 6 (Counting stable failures)**.** *For $n \geq 3$ and $0 \leq d \leq 2n - 4$, exactly*

$$2\binom{d+15}{15}$$

fixed-palette collections of $16n - 4 - d$ cubes fail to build a three-colour n-frame. Exactly $2\binom{d+11}{11}$ of them are inclusion-maximal.

Proof. The size is at least $14n$, so Theorem 5 applies. Fix an orientation class. Write $x_1, \ldots, x_{12}$ for its P multiplicities, p_a for their incidences in bucket a, and $s_a = h - t_a - p_a$ for the four bucket slacks. Double counting gives

$$\sum_{j=1}^{12} x_j + \sum_a s_a = d.$$

Conversely, any nonnegative solution determines a failing collection by $t_a = h - p_a - s_a$. These values satisfy $t_a \geq h - d > 0$, so there are no additional feasibility constraints and the two orientation classes give disjoint collections. Stars and bars, followed by the two choices of orientation, gives the first count.

A positive slack permits adding the corresponding T_a. If all slacks vanish, adding any ordinary cube in the same class overfills a bucket; adding any other cube violates purity. The stability theorem applies to the enlarged collection, so these and only these failures are inclusion-maximal. Setting the four slacks to zero leaves twelve nonnegative variables summing to d, giving the second count. □

Theorem 7 (One cube below the extremal size)**.** *For every $n \geq 2$, exactly 32 fixed-palette collections of $16n - 5$ cubes fail to contain a three-colour n-frame. Put $h = 4n - 1$. They are:*

1. *the eight collections*

$$(h-1)T_a + hT_b + hT_c + hT_d,$$

one for each choice of ordinary orientation class and colour a;

2. *the twenty-four collections*

$$P + (h-1)T_a + (h-1)T_b + hT_c + hT_d,$$

one for each ordinary P variety, where a, b are its bucket colours and the four T varieties belong to its orientation class.

Each collection in the second family is inclusion-maximal among failures: adding any cube makes it solvable.

Proof. Let $N = 16n - 5$ and retain the majority notation from the stability proof. For $n \geq 3$, Theorem 5 gives $M = 0$ immediately.

For $n = 2$, the orientation count at $N = 27$ gives $M \leq 6$ and $A \geq 21$. If $M > 0$, the R argument in Theorem 1 still implies $2t_a + p_a \leq 10$ for every colour. Summing yields $2A \leq 40$, a contradiction. Thus $M = 0$ here too.

Now $n_P \leq 1$. If $n_P = 0$, four bucket capacities of h and total $4h - 1$ leave exactly one T multiplicity at $h - 1$, giving the eight deletion cases. If $n_P = 1$, every bucket must have size h, since $n_T + 2n_P = 4h$. The unique P cube fills two buckets, giving the stated multiplicities and twenty-four cases. No checkerboards remain, so the reduction made no replacements in the original collection.

Conversely, each listed collection lies in one ordinary orientation class. Orientation balance excludes every target with a trichromatic corner. A remaining target $Q(a; b, c)$ requires $4n$ distinct cubes with an aa edge for its corners and monochromatic edges, while bucket a has at most $h = 4n - 1$ cubes. Every listed collection therefore fails. Finally, adding any cube to a pair case produces a collection of size $16n - 4$ still containing a P cube; Theorem 4 excludes it from the failing collections of that size. □

Corollary 2. *For $n \geq 3$, every collection of at least $16n - 3$ source cubes has an n-frame modelled on one of the 40 targets of types Q, D, U.*

Proof. The frame counting argument uses only these three types: avoiding them forces purity and four bucket capacities of $4n-1$, hence at most $16n-4$ cubes. There are twelve Q targets and four D targets. The ordered choice of three distinct colours in $U(a, b; c)$ gives $4 \cdot 3 \cdot 2 = 24$ varieties: their face multiplicities $3, 2, 1$ recover a, b, c, respectively. These families are disjoint. □

Corollary 3 (Complete cubes)**.** *For every integer $n \geq 2$, the sharp threshold for a complete $n \times n \times n$ cube with exactly three exterior colours is*

$$G(n) = \max\{n^3, 16n - 3\}.$$

In particular, $G(2) = 29$, $G(3) = 45$, and $G(n) = n^3$ for $n \geq 4$.

Proof. Both terms are necessary: an assembly uses n^3 pieces, and contains an n-frame. Conversely, construct a frame using Theorem 4. Each face-interior position prescribes just one colour, which every unused source cube has; interior positions prescribe none. Thus arbitrary unused cubes complete the assembly whenever at least n^3 pieces are available. □

6 Sharp thresholds for all cuboids

For ordered side lengths $2 \leq a \leq b \leq c$, let $F(a, b, c)$ be the least inventory size guaranteeing a frame, and $G(a, b, c)$ the corresponding complete-box threshold. These extend the cube notation: $F(n, n, n) = F(n)$. A frame has eight corners and the noncorner positions on its twelve edges, so its cardinality is

$$L = 4(a + b + c) - 16.$$

The target may depend on the inventory. Every target has exactly three exterior colours, and internal faces remain unrestricted.

Theorem 8 (Cuboid thresholds)**.** *For integers $2 \leq a \leq b \leq c$, put $K(c) = \lfloor (24c - 1)/5 \rfloor$. Then*

$$F(a, b, c) = \max\{16a - 3,\ 4(a + b + c) - 16,\ K(c)\},$$
$$G(a, b, c) = \max\{abc,\ 16a - 3,\ K(c)\}.$$

The three frame bounds come from a colour bucket, the number of positions, and a balanced checkerboard collection. The shortest and longest sides control different obstructions. For the upper bound, a weighted graph chooses directions for a D target whenever such a corner solution exists. Otherwise the same bucket constraints provide a Q or H target. This proof uses the geometry of Section 2, including the bucket lemma, and the edge facts in Section 5.

6.1 Three lower bounds

Cardinality gives $F(a,b,c) \geq L$. Fix one ordinary orientation class and take $4a-1$ copies of each of its four T varieties. By Lemma 1, these $16a-4$ sources can only assemble a Q target. Its eight corners and four monochromatic edges parallel to an axis of length d require $8+4(d-2) = 4d \geq 4a$ sources from one bucket. That bucket has only $4a-1$ copies. Hence $F(a,b,c) \geq 16a-3$.

For the third bound, distribute $N_0 = K(c)-1$ sources as evenly as possible among the six checkerboards. Each bichromatic type then has at most

$$N_0 - \lfloor N_0/6 \rfloor = \lceil 5N_0/6 \rceil \leq 4c-1$$

suppliers, since $5N_0 \leq 24c-6$. All source corners are trichromatic. Thus every supported target must have equal-coloured opposite faces, of type D: equal colours on adjacent faces would produce a bicoloured corner. In every axis assignment, the eight corners and $4(c-2)$ positions on the longest edges require one common bichromatic pair. Their demand is $4c$, exceeding its supply. Therefore $F(a,b,c) \geq K(c)$.

6.2 Extending an opposite-face corner solution

Reduce the sources by corner containment. As in Section 5, this preserves edge availability and lifts every assembly to distinct original copies. Index the checkerboards by their missing colour pairs, viewing their multiplicities x_e as edge weights on K_4. The incident weight $z_v = \sum_{e \ni v} x_e$ counts checkerboards supporting both orientations of the triple omitting v. If A_0, A_1 count the two ordinary classes, that triple has a D corner solution exactly when

$$A_0 + z_v \geq 4, \qquad A_1 + z_v \geq 4, \qquad A_0 + A_1 + z_v \geq 8. \tag{4}$$

These are Hall's three inequalities for the two corner types, each with four occurrences. Call such a vertex v *good*; in particular, $z_v \geq 8$ makes v good.

Lemma 5 (Weighted triangle)**.** *Let $N \geq \max\{29, L, K(c)\}$ and $s_d = N-4d$ for $d \in \{a,b,c\}$. Give the edges of K_4 nonnegative integer weights with sum at most N. Suppose a nonempty set of vertices is designated good and includes every vertex of incident weight at least eight. Some good vertex has a complementary triangle whose edge weights, in increasing order, are at most s_c, s_b, s_a, respectively.*

Proof. We first record the numerical bounds

$$\begin{gathered} s_c \geq 5, \qquad N \leq 6s_c+5, \qquad s_b \geq \max\{7, 2b-1\}, \\ s_a \geq 4b-1, \qquad s_c \geq \lfloor (4b-1)/3 \rfloor. \end{gathered} \tag{5}$$

For $c \leq 6$, the first follows from $N \geq 29$; for $c \geq 7$, use $K(c) = 4c + \lfloor (4c-1)/5 \rfloor \geq 4c+5$. The inequality $5K(c) \geq 24c-5$ gives the second. For $b \leq 3$, $s_b \geq 17$; for $b \geq 4$, $s_b \geq 4(a+c)-16 \geq 4b-8 \geq \max\{7, 2b-1\}$. If $c \geq 4$, then $s_a \geq 4(b+c)-16 \geq 4b$; otherwise $s_a \geq 17 \geq 4b-1$. Finally, for $b \leq 3$ the last floor is at most three; for $b \geq 4$, $s_c \geq 4(a+b)-16 \geq 4b-8 \geq \lfloor (4b-1)/3 \rfloor$.

Suppose first that every edge weight is at most s_b. If no good vertex has a complementary edge of weight at most s_c, all three complementary edges of any good vertex have weight at least

s_c+1. Each other vertex then has incident weight at least $2(s_c+1) \geq 12$, so all four vertices are good. Every edge is complementary to one of them, forcing total weight at least $6s_c + 6 > N$. This contradiction gives the required light edge; the other two already fit s_b and s_a.

Otherwise an edge has weight at least $s_b + 1 \geq 8$. Either endpoint v is good. The other five edges have total weight at most $4b - 1$. Writing $y_1 \leq y_2 \leq y_3$ for the weights complementary to v, we have

$$y_1 \leq \lfloor (4b-1)/3 \rfloor \leq s_c, \qquad y_2 \leq 2b-1 \leq s_b, \qquad y_3 \leq 4b-1 \leq s_a.$$

□

Corollary 4. *If $N \geq \max\{29, L, K(c)\}$ reduced sources have any D corner solution, they have an $a \times b \times c$ frame.*

Proof. Apply Lemma 5 with goodness defined by (4). Assign the complementary colour pairs to the three edge directions according to the lemma. Every such assignment is realised by a D face colouring: permutations of the triangle's vertices induce all permutations of its edges. This chooses a target; each source is still placed by a proper rotation.

Reserve eight sources for its corners. Each possesses all three target pairs, so the residual supply for the pair assigned to a direction of length d is at least

$$N - x_e - 8 \geq 4(d-2).$$

Any two distinct pair neighbourhoods together contain every residual source, since no reduced source lacks two bichromatic types. The total edge demand is $L - 8 \leq N - 8$. Thus Hall's theorem fills the remaining positions. The chosen target and axes may differ from those of the initial corner solution. □

6.3 Reservations when no D corner solution exists

Take

$$N = \max\{16a - 3, L, K(c)\},$$

and suppose no D corner solution exists. Let $A \geq B$ count the ordinary classes and C the checkerboards. Equation (4) gives $z_v \leq 7$, hence $C \leq 14$ and $A \geq 8$. It then gives $B + z_v \leq 3$ for every v. Consequently

$$4B + 2C \leq 12, \qquad M := B + C \leq 6, \qquad A = N - M \geq N - 6 \geq 23. \tag{6}$$

Each individual checkerboard multiplicity is at most three. After reserving r sources, each bichromatic type has at least $N - r - 3$ suppliers; any two distinct types together see all $N - r$ remaining sources. Hall therefore completes any residual bichromatic requirements whose individual demands are at most $N - r - 3$ and whose total is at most $N - r$.

Use distinct colours u, v, w and put $m = a-2$, $t = b-2$, $q = c-2$. Table 3 gives two targets, with face tuples in the usual order and axis lengths (a, b, c).

Target and face tuple	Monochromatic demand	Bichromatic demands	r
$Q : (v, w, u, u, u, u)$	$uu : 4m$	$uv, uw : 2t + 2q$	$4a$
$H : (v, w, u, u, u, v)$	$uu : 2m,\ vv : t$	$uv : 2m + t + 2q$ $uw : t + 2q,\ vw : t$	$2a + b + 2$

Table 3: Noncorner cuboid demands. The reservation r supplies all corners and monochromatic edge positions.

A majority bucket with $4a$ sources supplies Q: use eight for its corners by Lemma 3, and the remaining $4a-8$ for its monochromatic edges. For H, the corner multiset is

$$2uuv + 2uuw + 2uvv + uvw + uwv.$$

It suffices to have $2a$ majority T_u sources, b majority T_v sources, a minority source supporting $\{u, v, w\}$, and one further majority source. Four T_u copies supply the uuv, uuw corners and $2(a-2)$ fill the uu edges; two T_v copies supply uvv and $b-2$ fill the vv edges. The minority and extra majority copies supply the two trichromatic orientations. The T_u, T_v and minority supplies are disjoint; the extra majority copy is chosen outside the reservations.

For either row, the largest bichromatic demand plus $r+3$ is $4a+2b+2c-5$. We have

$$N \geq 4a + 2b + 2c - 5 : \tag{7}$$

if $b+c \geq 6$ this follows from $N \geq L$, since the difference is $2b+2c-11 \geq 1$; the remaining shapes have $a = b = 2$, $c \leq 3$, and $N \geq 29$ suffices. Total demand is $L \leq N$. Thus both constructions extend to frames whenever the reserved sources are available.

Suppose no majority bucket has size $4a$. Put $h = 4a-1$, and write t_u, p_u for the T count and P incidence in bucket u. With n_T, n_P the majority totals, double counting gives

$$\begin{gathered} t_u + p_u \leq h, \qquad n_T + 2n_P \leq 4h, \qquad A = n_T + n_P \leq 4h, \\ N \leq 16a + 2, \qquad n_T \geq 2A - 4h \geq 16a - 14. \end{gathered} \tag{8}$$

There is a minority source, since otherwise $N = A \leq 4h$ contradicts $N \geq 16a - 3 = 4h + 1$. Every minority source supports a triple containing any prescribed colour. An ordinary minority source supplies the opposite orientation; a checkerboard supplies both.

6.4 Completing the upper bound

Since $L \leq N \leq 16a + 2$, integrality gives

$$b + c \leq 3a + 4, \qquad 2b \leq 3a + 4. \tag{9}$$

Some $t_u \geq 2a$, because $n_T \geq 16a - 14 > 4(2a-1)$ for every $a \geq 2$. If at most one colour has fewer than b copies of its T variety, a minority triple containing u has another colour v with $t_v \geq b$. These provide the T_u and T_v sources for H.

Otherwise two colours have fewer than b copies, giving

$$n_T \leq 2(b-1) + 2h = 8a + 2b - 4 \leq 11a.$$

For $a \geq 3$ this contradicts $n_T \geq 16a - 14$, so necessarily $a = 2$. Now $h = 7$, $n_T \geq 18$ and $L = 4(b+c) - 8 \geq 8b - 8$. Hence

$$16b - 56 \leq 2L - 40 \leq 2N - 40 \leq n_T \leq 2b + 12,$$

which forces $b \leq 4$. Thus $n_T \leq 20$, while $n_T \geq 2(N-M) - 28 \geq 30 - 2M$ forces $M \geq 5$. Since every $t_i \leq 7$, at least two colours u, v have $t_u, t_v \geq 4$; otherwise $n_T \leq 7 + 3 \cdot 3 = 16$. They satisfy $t_u \geq 2a$ and $t_v \geq b$. An ordinary minority source supports a triple containing both colours. If all minority sources are checkerboards, at most three miss the pair uv, by (6). Since $M \geq 5$, another checkerboard supports a triple containing u, v: one missing ij supports precisely the triples omitting i or j. Again the H reservation is available.

In both cases, $A \geq 16a - 9 > 2a + b$, by (9), supplies the extra majority copy. Equation (7) and the residual Hall matching complete the frame. Together with Corollary 4 and the three lower bounds, this proves the frame formula in Theorem 8.

Every position outside the frame exposes at most one face when all sides are at least two. Every unused source contains every palette colour, so it can fill any such position. Therefore $G = \max\{abc, F\}$. Writing $a = x + 2$, $b = y + 2$, $c = z + 2$ gives

$$abc - L = xyz + 2(xy + xz + yz) \geq 0,$$

proving the complete-box formula and finishing Theorem 8. A side of length one would let a source expose opposite exterior faces simultaneously and is outside this completion argument.

6.5 Consequences

The cube thresholds follow by setting $a = b = c = n$: $F(n, n, n) = 16n - 3$ and $G(n, n, n) = \max\{n^3, 16n - 3\}$. The cube stability theorem additionally classifies failures in its stated range. The two thinnest families also follow immediately.

Corollary 5 (Rods and strips)**.** *Put* $K_\ell = \max\{29, \lfloor (24\ell - 1)/5 \rfloor\}$. *Then*

$$F(2, 2, \ell) = G(2, 2, \ell) = K_\ell \qquad (\ell \geq 2),$$

and

$$F(2, 3, \ell) = K_\ell, \qquad G(2, 3, \ell) = \max\{29, 6\ell\} \qquad (\ell \geq 3).$$

Despite their four extra frame positions, strips have the same frame thresholds as rods. Shape matters even at fixed cardinality: $2 \times 2 \times 7$ and $2 \times 3 \times 6$ both have twenty-eight frame positions, but their thresholds are thirty-three and twenty-nine.

Corollary 6 (No spare sources)**.** *A frame has threshold equal to its cardinality if and only if*

$$b + c \geq 3a + 4 \qquad \text{and} \qquad c \leq 5(a + b) - 19.$$

Every complete cuboid can be assembled from any collection of exactly its volume, except for all rods $2 \times 2 \times c$ *and the four shapes*

$$2 \times 3 \times 3, \quad 2 \times 3 \times 4, \quad 3 \times 3 \times 3, \quad 3 \times 3 \times 4.$$

The latter thresholds are $29, 29, 45, 45$*, respectively.*

Proof. The frame conditions are precisely $L \geq 16a - 3$ and $L \geq K(c)$ after integer rounding. For the second, $24c - 1 < 5(L + 1)$ is equivalent to $c \leq 5(a + b) - 19$. Rods satisfy $K(c) > 4c$. For every other ordered shape, $ab \geq 6$, so $abc \geq 6c > K(c)$. The only remaining volume deficit is $abc < 16a - 3$. For $a \geq 4$, already $a^3 \geq 16a - 3$. For $a = 3$ it leaves $(b, c) = (3, 3), (3, 4)$; for $a = 2$, $b \geq 3$, it leaves the same two pairs. □

For example, the next family after strips satisfies

$$F(2, 4, c) = \begin{cases} 29, & c = 4, 5, \\ 4c + 8, & 6 \leq c \leq 11, \\ \lfloor (24c - 1)/5 \rfloor, & c \geq 12. \end{cases}$$

For $c \geq 4$, any collection of $8c$ source cubes assembles a complete $2 \times 4 \times c$ box.

7 Formal verification and computation

All threshold, stability, counting and restricted-target results are formalised in Lean 4 and mathlib [2, 3], both at `v4.35.0`. The threshold statements quantify over arbitrary indexed families of physical four-colour cubes. A solution selects distinct copies and proper rotations matching every exposed face. The classifications and counts concern multisets of rotation classes over a fixed palette. The forty-target result uses a checked canonical target index; the threshold and sharpness statements use no catalogue indices.

The kernel checks the finite geometry, including the equivalence between corner matching and proper-rotation placement. The allocation and counting proofs are symbolic in the dimensions. The cuboid lower bounds reuse the cube and rod obstructions through proved restriction maps, and the counting proof gives an explicit bijection with weak compositions. Separate geometric and inventory audits supplement these proofs; their outputs are not assumed by the formalisation. Appendix A distinguishes the library proofs from the narrower comparator challenge.

To find an assembly, use capacitated bipartite matching. Source varieties have capacities equal to their multiplicities; corner and edge types have capacities equal to their demands. Adjacency records whether a source has the required type. There are at most eight corner and twelve edge requirement nodes, regardless of the dimensions. An integral flow meeting all demands selects the copies; compatible proper rotations then place them. Unused sources fill the remaining positions when at least the volume is available.

For cubic frames in the guaranteed range, it suffices to test 28 targets of types Q, R, D when $n = 2$ (Corollary 1), or 40 of types Q, D, U when $n \geq 3$ (Corollary 2). Testing all 120 target varieties decides feasibility for arbitrary cube inventories. For a general cuboid, also test the at most six assignments of target axes to box directions. Proper rotations with the same axis assignment give the same corner multiset and directional edge demands, so these tests decide frame feasibility.

Further questions

The cuboid thresholds are determined, but their extremal inventories remain to be classified. In particular, the interaction between the bucket and checkerboard obstructions may give a cuboid counterpart of Theorem 5. For cubes, classifying failures below the stability range remains open.

Declaration of AI-generated research and review

GPT-6-Astra (OpenAI), used through Codex, developed the original research argument, computational search, Lean formalisation and manuscript under the author's direction. Claude Opus 5.5 (Anthropic), used through Claude Code, contributed the minority-cube argument, balance proof, further mathematical simplifications, illustrations and review. Subsequent Codex revisions incorporated and extended these contributions. Detailed attribution and review scope are recorded in the repository's `formalization.yaml`. The human author initiated and directed the research and accepts responsibility for the paper's content and the accuracy of this disclosure.

A Verification details

Fifty-eight results and their required definitions are restated in a standalone verification file (the comparator challenge) importing only mathlib. They include both cuboid thresholds, the no-spare criteria, the cube, rod and strip corollaries, position counts and frame completion, the stability and near-extremal classifications, their counts and the forty-target theorem. Failures are counted as vectors of rotation-class multiplicities, and a vector fails when some physical family with those multiplicities has no frame. The tables of ordinary classes and buckets are

identified with their corner-type definitions in Lemma 2. The comparator checks statements and their underlying definitions against the proof library, then replays proofs through the Lean, NanoDa and con-ron kernels.

The library's axiom guards permit only `propext`, `Classical.choice` and `Quot.sound`. No proof uses holes or `native_decide`; the challenge contains statement placeholders, which the proof library does not import. The repository[3] provides sources, reproduction commands and the comparator configuration. The physical model and palette-independent geometry come from `colour-cubes-core` [4]. The challenge restates its definitions locally; this checks agreement, not independent conception of the model.

`BoxDefs` gives one exposed-face position model for every cuboid. `BoxModel` proves its cardinalities and completion theorem. Cubes are equal-sided instances of that model. `CuboidResults` proves Theorem 8 and the no-spare criteria in Corollary 6. The upper proof follows the weighted-triangle and Q, H constructions in Section 6. A cuboid frame can be restricted and reassembled as an $a \times a \times a$ frame or a $2 \times 2 \times c$ rod. These restrictions transfer the bucket and balanced-checkerboard lower bounds. The cube, rod and strip thresholds are specialisations of this general theorem. The cube stability and obstruction classifications have their own counting arguments, since they describe failures below the threshold.

Independent rotation-based audits check the local geometry, coordinate positions, lower witnesses and constructive controls. Sampled inventories supplement the written and formal proofs; they are not an exhaustive upper-bound computation.

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