TURÁN'S PURE POWER SUM PROBLEM: CERTIFIED BOUNDS FOR FINITE n AND $\limsup R_n \le 0.68637$

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Abstract. For $n \ge 1$ let $R_n = \inf \max_{1\le k\le n} |z_1^k + \cdots + z_n^k|$, the infimum being over all complex $z_1 = 1, z_2, \ldots, z_n$. This is Turán's pure power sum problem (Erdős problem #519). The best published upper bound for the constant $\limsup R_n$ is 0.69368, from a computation of Harcos reported by Biró in 2000. We prove $\limsup_{n\to\infty} R_n \le 0.68637$. The construction is the Biró–Harcos linearization, with the first half of the power sums prescribed as $s_k = Te^{i\varphi(k/n)}$ for a piecewise-linear phase φ that varies from the start. We prove a limit theorem: the quantity that controls the second half converges to an explicit functional of the solution of a linear Volterra equation whose associated operator is a contraction. The resulting inequality is certified in ball arithmetic by an a posteriori error bound, with two separately written programs that follow the same mathematical argument. We also give rigorous upper bounds for R_n at 13 values of n between 300 and 10^6, from explicit exact certificates, each checked by one or two programs written independently of the generator. In an appendix, R. Pinho applies the limit theorem to a finer profile, with sharper error bounds, and certifies $\limsup R_n \le 0.6863607950$. The work was done with substantial help from AI models; it has not been refereed.

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1. Introduction

For complex numbers $z_1, \ldots, z_n$ write $s_k(z) = z_1^k + \cdots + z_n^k$, and put

$$R_n = \inf_{z_1=1,\ z_2,\ldots,z_n\in\mathbb{C}} \max_{1\le k\le n} |s_k(z)|.$$

Turán asked how small all of the first n power sums can be when one of the numbers is 1; see his book [18]. The question appears as Erdős problem #519 [9], and the constant $\limsup R_n$ is entry C42 of the Optimization Constants repository [17]. Many authors normalize by $1 = z_1 \ge |z_2| \ge \cdots \ge |z_n|$. This gives the same R_n (Lemma 2.1).

Known bounds. Atkinson proved $R_n > 1/6$ [1]; see also [2]. Biró improved this to $R_n > 1/2$ [5], and then to $R_n > q$ for an absolute constant $q > 1/2$ [6]. In the other direction, it is trivial that $R_n < 1$, and Komlós, Sárközy and Szemerédi [12] proved $R_n < 1 - 1/(250n)$ for large n. The breakthrough is due to Biró [7], who proved for the first time that $\limsup R_n < 1$, and showed that $R_n < 5/6$ for all large n. His linearization technique is the basis of the present paper. In the same paper he reports a computation of Harcos, based on his construction, which gives $\limsup R_n < 0.69368$. Harcos [11] also rewrote Biró's proof in a more concise form, starting from the inverse Newton–Girard formulae, and this form is our starting point. Cheer and Goldston [8] computed near-optimal configurations for $n \le 55$, and their values decrease with n.

In 2026 a sequence of unrefereed contributions to [17], most of them prepared with AI assistance, lowered the claimed value of $\limsup R_n$ to 0.6906538 (Griego [10]), then, through refined step profiles of Röhrig and Plata, to 0.6888863712 (Pinho [15]) and 0.6873358063 (Koshta [14]). These use a coefficient-cancellation construction in which the phase of the prescribed power sums is constant on a first block of length τn.

Results. Our main result is the following.

Theorem 1.1. $\limsup_{n\to\infty} R_n \le 0.68637$.

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The proof has three parts.

(1) The Biró–Harcos linearization (Lemma 2.2): if the first $K \geq \lfloor n/2 \rfloor$ power sums are prescribed, the remaining ones can be chosen with one common modulus F_n so that 1 is among the z_j. Hence $R_n \leq \max(T, F_n)$, where T bounds the prescribed moduli.

(2) A limit theorem (Theorem A, Section 3). We prescribe $s_k = Te^{i\varphi(k/n)}$ for $k \leq \lfloor n/2 \rfloor$, where φ is any piecewise-linear function on $[0, \frac{1}{2}]$. Then $F_n \to F$, and F is an explicit functional of the bounded solution of a linear Volterra equation. The analytic inputs are that the integral operator of this equation is a contraction and that $\operatorname{Re}\gamma \geq 0$, where $\gamma = Te^{i\varphi(0)}$. For a constant phase, Theorem A reduces to the formula behind the bound 0.69368 of Biró and Harcos.

(3) A certified inequality (Section 4). For an explicit profile with $T = 0.68637$ we show $F < T$. We use an a posteriori bound for the Volterra equation and ball arithmetic (Arb [13]), and two separately written programs give $F \leq 0.6863660$ and $F \leq 0.6863615$ (Section 6 explains what this does and does not guarantee).

Compared with the 2026 certificates, the phase varies from the start ("first-block cut-off $\tau = 0$"), and the correction terms of those works are replaced by the Volterra equation.

We also give rigorous upper bounds for R_n at specific n (Theorem 5.1 and Table 3). For example $R_{1\,000\,000} \leq 0.686368028005$. Each bound comes from one explicit n-tuple, stored as an exact certificate, and every $|s_k|$, $1 \leq k \leq n$, is checked by at least two programs. As far as we could find, no rigorous bounds for specific $n > 55$ were known before.

Subsequent work. After a first version of this work was made public (with the bound 0.6865), R. Pinho used Theorem A with 49 phases on the non-uniform knots $j^2/4608$, together with sharper bounds for the propagation of the residual, to certify $\limsup R_n \leq 0.6863607950$ [16]. His proof is given in Appendix B. We checked his derivation and ran his main verifier on smaller grids; an independent floating-point solution of the limit equation for his profile gives $F \approx 0.686360744$, consistent with his certified bounds (Section 6).

TABLE 1. Upper bounds for $\limsup R_n$, in chronological order. The unrefereed entries are listed in, or have been submitted to, [17].

upper bound for $\limsup R_n$	source	status
5/6	Biró [7]	refereed
0.69368	Biró's construction, computation of Harcos [7]	refereed
0.6906538	Griego [10]	unrefereed
0.6888863712	Pinho [15]	unrefereed
0.6873358063	Koshta [14]	unrefereed
0.68637	this paper (Theorem 1.1)	unrefereed
0.6863607950	Pinho, Appendix B, using Theorem A	unrefereed

Table 1 summarizes the known upper bounds.

Organization. Section 2 contains the normalization and the Biró–Harcos linearization. Section 3 states and proves the limit theorem (Theorem A). Section 4 contains the a posteriori bound, the certified computation and the proof of Theorem 1.1. Section 5 gives the bounds for finite n, and Section 6 discusses the results and some open questions. Appendix A lists the profile, and Appendix B, by R. Pinho, gives the certificate for 0.6863607950.

Status. The proofs are complete and the computations can be repeated from the published certificates [3, 4]. The work was carried out with substantial help from AI models (see the statement at the end). It has not been refereed, and no professional mathematician has checked it yet.

2. Normalization and the Biró–Harcos linearization

Lemma 2.1. *Let R_n' be the infimum of $\max_{k\le n}|s_k(z)|$ over all $z\in\mathbb{C}^n$ with $\max_j|z_j|=1$. Then $R_n'=R_n$.*

Proof. If $\max_j|z_j|=|z_{j_0}|=1$, put $w_j=z_j/z_{j_0}$. Then $w_{j_0}=1$ and $|s_k(w)|=|s_k(z)|$; since the power sums are symmetric, we may renumber so that $w_1=1$. So $R_n\le R_n'$. Conversely, if $z_1=1$, let $\lambda=\max_j|z_j|=|z_{j_0}|\ge 1$ and put $w_j=z_j/z_{j_0}$. Then $\max_j|w_j|=1$ and $|s_k(w)|=\lambda^{-k}|s_k(z)|\le|s_k(z)|$. So $R_n'\le R_n$. □

For $s_1,\dots,s_K\in\mathbb{C}$ define the power series

$$A(x)=\exp\Bigl(-\sum_{k=1}^{K}\frac{s_k}{k}x^k\Bigr)=\sum_{m\ge 0}\alpha_m x^m,\qquad P_m=\alpha_0+\cdots+\alpha_m,$$

so that $\sum_m P_m x^m=A(x)/(1-x)$.

The following lemma is based on the work of Biró [7] and Harcos [11]: the linearization is due to Biró, and the short proof below, via the inverse Newton–Girard formulae, follows Harcos.

Lemma 2.2. *Let $n\ge 2$, $\lfloor n/2\rfloor\le K<n$ and $s_1,\dots,s_K\in\mathbb{C}$. Put $D_n=\sum_{k=K+1}^{n}|P_{n-k}|/k$; note $D_n\ge 1/n$, since $P_0=1$. Then there are $z_1=1,z_2,\dots,z_n\in\mathbb{C}$ with $s_k(z)=s_k$ for $k\le K$ and $|s_k(z)|=|P_n|/D_n$ for $K<k\le n$. Consequently*

$$R_n\le\max\Bigl(\max_{k\le K}|s_k|,\ \frac{|P_n|}{D_n}\Bigr).$$

Proof. For $K<k\le n$ put $\sigma_k=\lambda u_k$, where $\lambda=P_n/D_n$ and $u_k=\overline{P_{n-k}}/|P_{n-k}|$ (or $u_k=1$ if $P_{n-k}=0$). Put $\sigma_k=s_k$ for $k\le K$. Then $\sum_{k=K+1}^{n}(\sigma_k/k)P_{n-k}=\lambda D_n=P_n$.

Let $W(x)=\sum_{m=0}^{n}w_m x^m$ be the truncation to degree n of $\exp(-\sum_{k=1}^{n}\sigma_k x^k/k)$. Since $w_0=1$, the polynomial $x^nW(1/x)$ is monic of degree n; let $z_1,\dots,z_n$ be its roots, so that $W(x)=\prod_j(1-z_jx)$. Taking logarithms of power series with constant term 1 gives $\sum_k s_k(z)x^k/k\equiv\sum_{k\le n}\sigma_k x^k/k\pmod{x^{n+1}}$, so $s_k(z)=\sigma_k$ for all $k\le n$.

Every product of two monomials $x^k,x^{k'}$ with $k,k'>K$ has degree at least $2K+2>n$. Hence, modulo x^{n+1},

$$\exp\Bigl(-\sum_{k\le n}\frac{\sigma_k}{k}x^k\Bigr)\equiv A(x)\Bigl(1-\sum_{k=K+1}^{n}\frac{\sigma_k}{k}x^k\Bigr),$$

so $w_m=\alpha_m-\sum_{K<k\le m}(\sigma_k/k)\alpha_{m-k}$. Summing over $m\le n$ gives

$$W(1)=P_n-\sum_{k=K+1}^{n}\frac{\sigma_k}{k}P_{n-k}=0.$$

So 1 is one of the z_j, and we renumber so that $z_1=1$. □

Biró's construction [7] corresponds to Lemma 2.2 with a constant first block $s_k=\gamma$, $|\gamma|=T$ (see [11]). With the parameter found by Harcos, the limit gives $\limsup R_n\le 0.6936763\ldots<0.69368$ (see Remark 3.1).

3. The limit theorem

The profile. Fix $T\in(0,1)$, knots $0=u_0<u_1<\cdots<u_m=\frac12$ and real numbers $\psi_0,\dots,\psi_m$. Let $\varphi:[0,\frac12]\to\mathbb{R}$ be the continuous piecewise-linear function with $\varphi(u_j)=\psi_j$, and let $\beta_j=(\psi_{j+1}-\psi_j)/(u_{j+1}-u_j)$ be its slopes. Put $g(u)=Te^{i\varphi(u)}$, $\gamma=g(0)$, $a=\operatorname{Re}\gamma$, and

$$d(u)=\gamma-g(u)\quad(0<u\le\tfrac12),\qquad d(u)=\gamma\quad(\tfrac12<u\le 1),\qquad d(0)=0,\qquad \kappa=\sup_{0<u\le1}|d(u)|.$$

For $n\ge 2$ we apply Lemma 2.2 with $K=\lfloor n/2\rfloor$ and $s_k=g(k/n)$ for $1\le k\le K$ (note $k/n\le\frac12$). Since $|s_k|=T$,

$$R_n\le\max(T,F_n),\qquad F_n=|P_n|/D_n.\tag{1}$$

Theorem A. *Assume $\kappa < 1$ and $a \geq 0$. (Then $a \leq |\gamma| = T \leq \kappa < 1$, because $|d| = T$ on $(\frac{1}{2}, 1]$.)*

(1) *The equation*

$$t\rho(t) = d(t) + \int_0^t d(u)\rho(t-u)\,du \qquad (0 < t \leq 1) \tag{2}$$

has exactly one bounded measurable solution ρ on $(0, 1]$.

(2) *Put*

$$p(t) = t^{-\gamma} + \int_0^t (t-v)^{-\gamma}\rho(v)\,dv, \qquad F = \frac{|p(1)|}{\int_0^{1/2} |p(t)|\,\frac{dt}{1-t}}.$$

The denominator of F is positive and finite, and $F_n \to F$ as $n \to \infty$.

(3) *Consequently* $\limsup_n R_n \leq \max(T, F)$.

Powers t^w with $t > 0$ and $w \in \mathbb{C}$ mean $e^{w \log t}$. Throughout the proof let $\Lambda = \max_j |\beta_j|$ and $\mu = \max(T\Lambda, 2T)$. For $0 < u \leq \frac{1}{2}$ we have $|d(u)| = T|1 - e^{i(\varphi(u)-\varphi(0))}| \leq T\Lambda u$, and for $u > \frac{1}{2}$ we have $|d(u)| = T \leq 2Tu$. Hence

$$|d(u)| \leq \mu u \quad \text{on } (0, 1], \qquad |d| \leq \kappa. \tag{3}$$

Step 1: exact discrete identities. Put $c_k = 1 - s_k$ for $k \leq K$ and $c_k = 1$ for $k > K$. Since $-\log(1-x) = \sum_k x^k/k$, we have $\sum_m P_m x^m = \exp(\sum_{k\geq 1} c_k x^k/k)$. Write $c_k = (1-\gamma) + d_k$. Then $d_k = \gamma - g(k/n) = d(k/n)$ for $k \leq K$, and $d_k = \gamma = d(k/n)$ for $K < k \leq n$, because $k > K$ implies $k/n > \frac{1}{2}$. Thus

$$d_k = d(k/n) \qquad (1 \leq k \leq n), \tag{4}$$

and

$$\sum_m P_m x^m = (1-x)^{\gamma-1}G(x), \qquad (1-x)^{\gamma-1} = \sum_{j\geq 0} a_j x^j, \quad a_j = \frac{\Gamma(j+1-\gamma)}{\Gamma(1-\gamma)\,j!},$$

where $G(x) = \exp(\sum_k d_k x^k/k) = \sum_m \delta_m x^m$. Here $\delta_0 = 1$ and $m\delta_m = \sum_{k=1}^m d_k \delta_{m-k}$, from $xG' = (\sum_k d_k x^k)G$. Therefore

$$P_j = \sum_{i=0}^{j} a_{j-i}\delta_i. \tag{5}$$

Step 2: the continuous problem. Let X be the Banach space of bounded Borel-measurable functions on $(0, 1]$ with the sup norm, and $(\mathcal{K}f)(t) = t^{-1}\int_0^t d(u)f(t-u)\,du$. Then $\|\mathcal{K}f\| \leq \kappa\|f\|$, and $\mathcal{K}f$ is continuous on $(0, 1]$, being the convolution of two bounded functions divided by t. Equation (2) reads $\rho = d(t)/t + \mathcal{K}\rho$, and $d(t)/t$ is bounded by (3). Since $\kappa < 1$, the Neumann series gives a unique solution in X, and $\|\rho\| \leq B := \mu/(1-\kappa)$.

Regularity. The function $d(t)/t$ is uniformly continuous on $(0, \frac{1}{2}]$ (it tends to $-iT\beta_0 e^{i\psi_0}$ as $t \to 0$) and on $(\frac{1}{2}, 1]$. Also $|\mathcal{K}\rho(t)| \leq \mu Bt/2$, and $\mathcal{K}\rho$ is continuous. Hence ρ is uniformly continuous on $(0, \frac{1}{2}]$ and on $(\frac{1}{2}, 1]$; it has a jump only at $\frac{1}{2}$, and it is left-continuous there.

The function p. The map $t \mapsto \int_0^t u^{-\gamma}\rho(t-u)\,du$ is continuous on $[0, 1]$, being the convolution of $u^{-\gamma} \in L^1$ with a bounded function, so p is continuous on $(0, 1]$. Moreover $|p(t)| \geq t^{-a} - Bt^{1-a}/(1-a) > 0$ for small t, and $|p(t)| \leq t^{-a} + B/(1-a)$. So the denominator of F is positive and finite.

Step 3: the discrete solution converges to ρ. Let $\rho_n(m) = n\delta_m$ and $t_m = m/n$. By (4), the recursion for δ_m reads

$$t_m\rho_n(m) = d(t_m) + \frac{1}{n}\sum_{k=1}^{m-1} d(t_k)\rho_n(m-k) \qquad (1 \leq m \leq n). \tag{6}$$

Since $|d(t_m)| \leq \mu t_m$ and $\frac{1}{n}\sum_{k<m} |d(t_k)| \leq \kappa t_m$, induction gives $|\rho_n(m)| \leq \mu + \kappa B = B$.

Let $e_m = \rho_n(m) - \rho(t_m)$. Subtracting (2) at $t = t_m$ from (6),

$$t_m e_m = \frac{1}{n}\sum_{k=1}^{m-1} d(t_k)e_{m-k} + \tau_m, \qquad \tau_m = \frac{1}{n}\sum_{k=1}^{m-1} f_m(t_k) - \int_0^{t_m} f_m(u)\,du,$$

where $f_m(u) = d(u)\rho(t_m - u)$. Hence $|e_m| \le \kappa \max_{l<m} |e_l| + |\tau_m|/t_m$, and so

$$\max_{m\le n} |e_m| \le \frac{\sup_m |\tau_m|/t_m}{1-\kappa}. \tag{7}$$

Write $\tau_m = \sum_{k=1}^{m-1} \int_{t_{k-1}}^{t_k} (f_m(t_k) - f_m(u))\, du - \int_{t_{m-1}}^{t_m} f_m$. The last integral is at most $\mu B t_m/n$ in modulus, by (3). The function f_m is the product of d, which is continuous on $(0, \frac12]$ and on $(\frac12, 1]$, and of $u \mapsto \rho(t_m - u)$, which has at most one jump. So at most two cells contain a discontinuity, and this happens only when $t_m > \frac12$; on these cells the oscillation of f_m is at most $2\kappa B$. On every other cell the oscillation is at most $B\omega_d(1/n) + \kappa\omega_\rho(1/n)$, where ω_d, ω_ρ are the moduli of uniform continuity of d and ρ on their two pieces. Dividing by t_m (and using $t_m > \frac12$ when there are discontinuous cells),

$$\sup_m \frac{|\tau_m|}{t_m} \le B\omega_d(1/n) + \kappa\omega_\rho(1/n) + \frac{\mu B + 8\kappa B}{n} \longrightarrow 0.$$

With (7),

$$\max_{1\le m\le n} |\rho_n(m) - \rho(m/n)| \to 0. \tag{8}$$

Step 4: asymptotics of P_j. Put $\eta_l = \Gamma(1-\gamma)a_l l^\gamma = \Gamma(l+1-\gamma)l^\gamma/\Gamma(l+1)$ for $l \ge 1$. Then $\eta_l \to 1$, by the classical asymptotics of $\Gamma(l+\alpha)/\Gamma(l+\beta)$. Let $C_\eta = \sup_l |\eta_l| < \infty$, and put $\Pi_n(j) = \Gamma(1-\gamma)n^\gamma P_j$. By (5),

$$\Pi_n(j) = \eta_j\Big(\frac{j}{n}\Big)^{-\gamma} + \frac{\Gamma(1-\gamma)n^\gamma}{n}\rho_n(j) + \frac1n\sum_{l=1}^{j-1}\eta_l\Big(\frac{l}{n}\Big)^{-\gamma}\rho_n(j-l) \qquad (1\le j\le n).$$

The middle term (the term $l = 0$) has modulus at most $|\Gamma(1-\gamma)|Bn^{a-1} \to 0$.

(B2) Domination. Since v^{-a} is non-increasing, $\frac1n\sum_{l=1}^{j-1}(l/n)^{-a} \le \int_0^{j/n} v^{-a}dv$. Hence, for $1 \le j \le n$ and $t = j/n$,

$$|\Pi_n(j)| \le Ct^{-a}, \qquad C = C_\eta + |\Gamma(1-\gamma)|B + \frac{C_\eta B}{1-a}.$$

(B1) Convergence. For every $\varepsilon > 0$, $\sup_{\varepsilon n\le j\le n} |\Pi_n(j) - p(j/n)| \to 0$.

Proof of (B1). Let $t = j/n \in [\varepsilon, 1]$ and fix $\delta \in (0, \varepsilon/2)$. Split $p(t) = t^{-\gamma} + I_{\le\delta}(t) + I_{>\delta}(t)$, where $I_{\le\delta} = \int_0^\delta u^{-\gamma}\rho(t-u)\,du$ and $I_{>\delta} = \int_\delta^t u^{-\gamma}\rho(t-u)\,du$. Split the last sum in $\Pi_n(j)$ in the same way, into $\Sigma_{\le\delta}$ ($1 \le l \le \delta n$) and $\Sigma_{>\delta}$ ($\delta n < l \le j-1$).

(1) $|\eta_j(j/n)^{-\gamma} - t^{-\gamma}| = |\eta_j - 1|t^{-a} \le |\eta_j - 1|\varepsilon^{-a} \to 0$ uniformly, since $j \ge \varepsilon n$.
(2) The middle term tends to 0, as shown above.
(3) $|\Sigma_{\le\delta}| \le C_\eta B\delta^{1-a}/(1-a)$ and $|I_{\le\delta}| \le B\delta^{1-a}/(1-a) \le C_\eta B\delta^{1-a}/(1-a)$; note $C_\eta \ge 1$.
(4) Let Σ' be $\Sigma_{>\delta}$ with $\rho_n(j-l)$ replaced by $\rho((j-l)/n)$. Then, by (8),
$$|\Sigma_{>\delta} - \Sigma'| \le C_\eta \max_m |e_m| \cdot \frac1n\sum_{l\le n}\Big(\frac{l}{n}\Big)^{-a} \le \frac{C_\eta \max_m |e_m|}{1-a} \to 0.$$
(5) Let Σ'' be Σ' with η_l replaced by 1. Then $|\Sigma' - \Sigma''| \le \sup_{l>\delta n} |\eta_l - 1| \cdot B/(1-a) \to 0$.
(6) $\Sigma'' = \frac1n\sum_{\delta n<l\le j-1}(l/n)^{-\gamma}\rho(t - l/n)$ is a right-endpoint Riemann sum, with cells of width $1/n$, for $I_{>\delta}$. Take $n \ge 2/\delta$, so that every cell lies in $[\delta/2, 1]$, where $|u^{-\gamma}| \le (\delta/2)^{-a}$. The two end cells contribute at most $2(\delta/2)^{-a}B/n$. On every other cell the oscillation of $u \mapsto u^{-\gamma}\rho(t-u)$ is at most $B\omega_1(1/n) + (\delta/2)^{-a}\omega_\rho(1/n)$, where ω_1 is the modulus of continuity of $u^{-\gamma}$ on $[\delta/2, 1]$, except on the at most one cell that contains the jump of $\rho(t-\cdot)$ (at $u = t - \frac12$); that cell contributes at most $2(\delta/2)^{-a}B/n$. Hence $|\Sigma'' - I_{>\delta}| \to 0$ uniformly in $t \in [\varepsilon, 1]$.

Therefore $\limsup_n \sup_{\varepsilon n\le j\le n} |\Pi_n(j) - p(j/n)| \le 2C_\eta B\delta^{1-a}/(1-a)$ for every small $\delta > 0$, and letting $\delta \to 0$ proves (B1). □

Step 5: conclusion. Since $|\Gamma(1-\gamma)n^{\gamma}| = |\Gamma(1-\gamma)|n^{a}$, (1) gives

$$F_n = \frac{|\Pi_n(n)|}{\frac{1}{n}\sum_{j=0}^{L}|\Pi_n(j)|/(1-j/n)}, \qquad L = n-K-1 = \lceil n/2\rceil - 1.$$

By (B1), $\Pi_n(n) \to p(1)$. In the denominator, the term $j=0$ equals $|\Gamma(1-\gamma)|n^{a-1} \to 0$, and the terms $j \geq 1$ equal $\int_0^{L/n} h_n(t)\,dt$, where $h_n(t) = |\Pi_n(\lceil nt\rceil)|/(1-\lceil nt\rceil/n)$. By (B1) and the continuity of p, $h_n(t) \to |p(t)|/(1-t)$ for every $t \in (0,\frac{1}{2})$. By (B2), $h_n(t) \leq 2Ct^{-a}$, which is integrable. Since $L/n \to \frac{1}{2}$, dominated convergence shows that the denominator converges to $\int_0^{1/2}|p(t)|\,dt/(1-t) > 0$. Hence $F_n \to F$, and (1) gives $\limsup R_n \leq \max(T,F)$. This proves Theorem A. □

Remark 3.1 (The constant-phase case). If $\varphi \equiv \psi_0$, then $d=0$ on $(0,\frac{1}{2}]$, so $\rho = 0$ there and $\rho(t) = \gamma/t$ on $(\frac{1}{2},1]$. With $w = -\gamma$,

$$p(1) = 1 - w\int_0^{1/2}\frac{t^w}{1-t}\,dt, \qquad \int_0^{1/2}|p|\frac{dt}{1-t} = \int_0^{1/2}\frac{|t^w|}{1-t}\,dt.$$

This is the limit formula of [11] for Biró's construction. For $w = -0.43246 + 0.54237i$, that is, $s_k = 1-\alpha$ with the parameter $\alpha = 0.56754 + 0.54237i$ found by Harcos (see the Addendum of [7]), it gives $F = 0.6936763\ldots$, so $\max(T,F) < 0.69368$. (The note [11] uses the complex conjugate $w = -0.43246 - 0.54237i$; conjugating the whole profile does not change F.)

Remark 3.2. The proof uses only that φ is continuous and piecewise linear with finitely many pieces: this gives (3), the limit of $d(t)/t$ at 0, and the uniform continuity of d on $(0,\frac{1}{2}]$. In particular it applies to any finite set of knots.

4. The certified inequality $F < T$

4.1. **The profile.** We take $T = 0.68637$, the 25 equally spaced knots $u_j = j/48$ ($j = 0,\ldots,24$), and the phases $\psi_0,\ldots,\psi_{24}$ of Table 4 in Appendix A. They are exact decimal numbers, stored in the file `c42_profile_0.68637.json` of [3] (SHA-256 `dcea4edb...ded395`). In ball arithmetic:

- $a = \operatorname{Re}\gamma = 0.1660999\ldots \in (0,1)$;
- $\max_j|\psi_j - \psi_0| = 0.7753 < \pi$, so on $(0,\frac{1}{2}]$ the function $|d| = 2T|\sin((\varphi-\psi_0)/2)|$ is largest at a knot, and $|d| \leq 0.5190$ there;
- $|d| = T$ on $(\frac{1}{2},1]$, so $\kappa = T = 0.68637 < 1$;
- all slopes satisfy $1.20 \leq |\beta_j| \leq 5.82$.

So Theorem A applies, and Theorem 1.1 follows once we show $F < T$.

4.2. **An a posteriori bound.**

Lemma 4.1. *Let $\tilde\rho$ be a bounded measurable function on $(0,1]$, and put*

$$R(t) = d(t) + \int_0^t d(u)\tilde\rho(t-u)\,du - t\tilde\rho(t).$$

If $\sup_t|R(t)/t| < \infty$, then $\|\rho-\tilde\rho\|_\infty \leq \sup_{0<t\leq 1}|R(t)/t|/(1-\kappa)$.

Proof. The difference $e = \rho - \tilde\rho$ satisfies $e = \mathcal{K}e + R/t$, so $\|e\| \leq \kappa\|e\| + \|R/t\|$. □

Let $\tilde p(t) = t^{-\gamma} + \int_0^t (t-v)^{-\gamma}\tilde\rho(v)\,dv$ and $\varepsilon = \|\rho-\tilde\rho\|_\infty$. Then $|p(t)-\tilde p(t)| \leq \varepsilon t^{1-a}/(1-a)$, and therefore, whenever the denominator below is positive,

$$F \leq \frac{|\tilde p(1)| + \varepsilon/(1-a)}{\int_0^{1/2}|\tilde p|\frac{dt}{1-t} - \varepsilon\int_0^{1/2}\frac{t^{1-a}}{(1-a)(1-t)}\,dt}. \tag{9}$$

4.3. **The computation.** The program `rigorous_limit.py` of [3] evaluates the right-hand side of (9) in Arb ball arithmetic [13] (through python-flint 0.9.0, at 192 bits), from the exact inputs.

- *Grid and trial function.* Let $N = 48M$ and $h = 1/N$, so that all knots and $\frac{1}{2}$ are grid points. The trial function $\tilde\rho$ is linear on each cell $[ih, (i+1)h]$, continuous except for a jump J at $\frac{1}{2}$ (where the true ρ jumps). Its nodal values come from a floating-point collocation and are then rounded to exact binary numbers. This is the only floating-point step, and it is harmless: Lemma 4.1 holds for every $\tilde\rho$.
- *Nodal residuals.* On each cell d is either $\gamma - Te^{i(\theta+\beta x)}$ or the constant γ, so its moments against the two hat functions have closed forms. Then $R(ih^{\pm})$ is an exact discrete convolution, evaluated with Arb polynomial products.
- *Inside the cells.* With σ_i the slope of $\tilde\rho$ on cell i and $\Delta_k = \sigma_k - \sigma_{k-1}$,
$$R''(t) = d''(t) + d'(t)\tilde\rho(0) + \sigma_0 d(t) + \sum_{1\le k\le i} \Delta_k d(t-kh) + \mathbf{1}_{i\ge N/2} J d'(t-\tfrac12) - 2\sigma_i$$
inside cell i, and $|R'''| \le \Theta = T\Lambda^3 + T\Lambda^2|\tilde\rho(0)| + T\Lambda(|\sigma_0| + \sum_k |\Delta_k|) + |J|T\Lambda^2$ with $\Lambda = \max_j |\beta_j|$ as in Section 3. Linear interpolation with the Peano-kernel error bound (valid for complex-valued functions) then bounds $\sup|R/t|$ on each cell, using both one-sided values at $\frac12$. On the first cell we use $R(0) = 0$ and $R'(0^+) = d'(0) - \tilde\rho(0)$.
- *Numerator and denominator.* $\tilde p(1)$ is computed exactly, cell by cell, from the moments of $u^{-\gamma}$ and $u^{1-\gamma}$. For the denominator we split $[0, \frac12]$ at $t_* \approx 10^{-3}$. On $[0, t_*]$ we drop the weight, using $1/(1-t) \ge 1$, and integrate exactly the lower bound $|\tilde p(t)| \ge t^{-a} - P_* t^{1-a}/(1-a)$, where P_* bounds $|\tilde\rho|$ there. On each cell $[t_k, t_{k+1}] \subset [t_*, \frac12]$ let L be the linear interpolant of $\tilde p$ at the end points. Then
$$\int_{t_k}^{t_{k+1}} \frac{|\tilde p(t)|}{1-t}\,dt \ge \left|\int_{t_k}^{t_{k+1}} \frac{L(t)}{1-t}\,dt\right| - \frac{1}{1-t_{k+1}}\int_{t_k}^{t_{k+1}} |\tilde p - L|\,dt,$$
where the first integral (of L itself, not of $|L|$) is in closed form, and $\int|\tilde p - L| \le \frac{h^2}{8}\int|\tilde p''|$, where
$$\tilde p''(t) = \gamma(\gamma+1)t^{-\gamma-2} - \gamma\tilde\rho(0)t^{-\gamma-1} + \sigma_0 t^{-\gamma} + \sum_{kh<t} \Delta_k (t-kh)^{-\gamma} \qquad (t < \tfrac12)$$
is integrated term by term in modulus. Finally the ε-term of (9) is subtracted, using
$$\int_0^{1/2} \frac{t^{1-a}}{(1-a)(1-t)}\,dt \le \frac{2\cdot 2^{a-2}}{(1-a)(2-a)}.$$

The program prints PASS only if the lower bound for the denominator is certainly positive and the upper bound for F is certainly smaller than T; all the hypotheses it uses ($0 < a < 1$, $|\psi_j - \psi_0| < 3$, $\kappa < 1$, the positivity on $[0, t_*]$, finiteness of all bounds, the exact format of the profile) are checked as well.

A second program, `independent_check.py`, was written from the mathematical description only, without access to the first. It uses the same principle (Lemma 4.1 and (9)) but a different implementation: closed-form antiderivatives of a "ramp" representation of $\tilde\rho$, an interval enclosure of R'' on each cell instead of a global bound for R''', and a smaller cut t_*. The results are in Table 2.

TABLE 2. Certified bounds for the profile with $T = 0.68637$. The bounds for $\sup|R/t|$ and $|p(1)|$ are rounded up and those for the denominator down; the last column is the quotient of the two printed bounds, rounded up. All four runs give $F < T$.

program	M	N	$\sup\lvert R/t\rvert \le$	$\lvert p(1)\rvert \le$	$\int_0^{1/2} \lvert p\rvert\frac{dt}{1-t} \ge$	$F \le$
`rigorous_limit.py`	1280	61 440	$4.22\cdot10^{-7}$	0.39130280	0.57010586	0.68636867
`rigorous_limit.py`	2560	122 880	$1.06\cdot10^{-7}$	0.39130160	0.57010638	0.68636594
`independent_check.py`	1280	61 440	$4.22\cdot10^{-7}$	0.39130280	0.57010915	0.68636471
`independent_check.py`	2560	122 880	$1.06\cdot10^{-7}$	0.39130160	0.57011009	0.68636148

Proof of Theorem 1.1. By Table 2, $F < 0.68637 = T$ for the profile of Table 4, and Theorem A gives $\limsup R_n \le \max(T, F) = 0.68637$. □

Cross-checks (floating point, not part of the proof). A direct product-integration solver of the equation $tp(t) = \int_0^t c(u)p(t-u)\,du$, with $c = 1-g$ on $[0, \frac{1}{2}]$ and $c = 1$ afterwards, gives $F = 0.68636008$ for this profile, and reproduces 0.693676 for a constant phase. The exact finite-n construction (1) gives $F_n = 0.6864165$, 0.6863783 and 0.6863660 for $n = 2.5 \cdot 10^5$, 10^6 and $4 \cdot 10^6$, approaching the same limit from above. Inside sampled cells, the true residual $|R/t|$, computed by adaptive quadrature, stays below the cell bounds of the checker.

5. Certified bounds for finite n

Theorem 5.1. *For each n in Table 3, R_n is at most the value given there.*

Construction. For each n we apply Lemma 2.2 with $K = n - \lfloor (n-1)/2 \rfloor = \lfloor n/2 \rfloor + 1$ (one more than in Section 3; Lemma 2.2 allows both) and $s_k = ce^{i\varphi(k/n)}$, where φ is piecewise linear with 24 pieces and knots that are finer near 0. The 26 parameters are optimized numerically for each n, so that both halves end up with nearly the same modulus. This floating-point step only guides the search. The certified object is the polynomial $W(x) = (1-x)Q(x)$, where the coefficients $b_0 = 1, b_1, \dots, b_{n-1}$ of Q are exact dyadic rationals (the midpoints of balls computed from the construction). The tuple consists of $z_1 = 1$, exactly because of the factor $1 - x$, and the $n-1$ roots of $Z^{n-1} + b_1 Z^{n-2} + \cdots + b_{n-1}$. The roots are never computed: every power sum follows from the coefficients w_i of W through Newton's identities $s_k = -kw_k - \sum_{i=1}^{k-1} w_i s_{k-i}$.

Certification. Every $|s_k|$, $1 \le k \le n$, is checked by at least two of three programs [4]:

- P the generator, which computes $s_k = -k[x^k] \log W(x)$ in ball arithmetic at 320 bits;
- V a separately written checker that reads only the certificate, checks $b_0 = 1$ and $W(1) = 0$ exactly, and computes all s_k in ball arithmetic either by Newton's identities or as the coefficients of $-xW'/W$, with its own power-series inverse;
- E a checker in exact integer arithmetic (no floating point and no external library), which verifies $|s_k| \le r$ for every k by comparing integers; it was run for $n \le 5000$.

The values are rounded up to 12 decimals. So for $n \le 5000$ every bound is confirmed by two programs written independently of the generator (V and E), and for $n \ge 10\,000$ by one (V).

Table 3. Rigorous upper bounds for R_n from explicit certified tuples.

n	$R_n \le$	checks	n	$R_n \le$	checks
300	0.691018908239	P V E	20 000	0.686540042277	P V
500	0.689542310941	P V E	30 000	0.686489957913	P V
1 000	0.688245154929	P V E	50 000	0.686446081102	P V
2 000	0.687468053119	P V E	100 000	0.686409189526	P V
3 000	0.687169029139	P V E	300 000	0.686380380702	P V
5 000	0.686902345661	P V E	1 000 000	0.686368028005	P V
10 000	0.686673342516	P V			

For small n we also certified, with checker E, tuples found by an unconstrained numerical search; they are stored as floating-point numbers, and their exact binary values are the certified tuples. For $n \le 55$ they reproduce the values of Cheer and Goldston [8] to 12 decimals, so they are not new. For larger n they give $R_{70} \le 0.699927441737$, $R_{100} \le 0.696818594535$, $R_{150} \le 0.694131485985$, $R_{200} \le 0.692648924528$ and $R_{300} \le 0.691018903643$. At $n = 300$ the unconstrained search beats the construction above by only $4.6 \cdot 10^{-9}$, which suggests that the construction is close to optimal there.

Figure 1 shows all these bounds. They do not by themselves bound $\limsup R_n$, since R_n is not known to be monotone in n. They are consistent with Theorem 1.1: the values decrease steadily and stay above 0.68636.

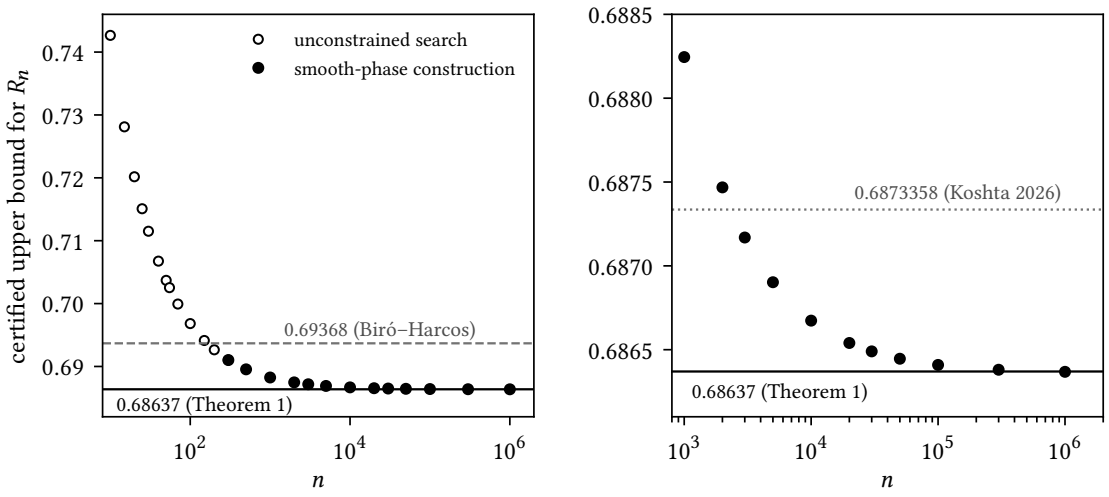


FIGURE 1. Certified upper bounds for R_n: the smooth-phase construction (Table 3, filled) and the unconstrained search (open; for $n \le 55$ these reproduce Cheer and Goldston [8]). Each point is a rigorous bound for one n; the horizontal lines are bounds for $\limsup R_n$. Left: all n, with the bound 0.69368 of Biró and Harcos and Theorem 1.1. Right: $n \ge 1000$, with the earlier claimed value 0.6873358063 (Koshta [14], unrefereed).

6. Discussion

The family of Theorem A. Minimizing $\max(T, F)$ over profiles in floating point, directly in the limit, gives about 0.6863655 with 24 equal pieces and about 0.686361 with knots that are finer near 0. Our certified value 0.68637 is within 10^{-5} of the first. Pinho's certificate (Appendix B) reaches 0.6863607950 with 49 phases on the knots $j^2/4608$, so the second optimum is essentially attained. For his profile our floating-point solver gives $F \approx 0.686360744$ (from $N = 73\,728$, with convergence of order h^2), below his certified upper bounds 0.6863607743 and 0.6863607774. We also ran his main verifier on our profile: it reproduces our value of $\sup |R/t|$ and gives $F \le 0.68636012$.

The gap. The best lower bound for $\liminf R_n$ is still a constant slightly above $1/2$ [6], while all constructions so far give values near 0.686. Natural questions are whether $\lim R_n$ exists, whether a different construction (for example, one not based on prescribing exactly half of the power sums) can go below the optimum of the family of Theorem A, and whether Theorem A can be made effective, that is, whether an explicit n_0 with $R_n \le 0.68637$ for all $n \ge n_0$ can be given.

How the error is controlled. The two programs of Section 4 are independent code, but they follow the same mathematical route, Lemma 4.1 and (9). A mistake shared by these two statements would not be caught by running both; their proofs are short and given above.

Use of AI

This work was carried out with substantial help from AI models, Claude Opus 5.5 and Claude Sonnet 5.5 (Anthropic). They were used for the literature search, for designing the constructions, for deriving and writing the proofs, for writing all of the code, and for drafting this text. The author planned and directed the work, ran all computations, and checked the results and the references. A separate instance of Claude reviewed the proof working only from the written note, and wrote the second checker of Section 4 without access to the first; its two reports are included in [3]. No professional mathematician has checked the arguments yet. Corrections are very welcome. Appendix B was written by R. Pinho; its last paragraph states how it was prepared and checked.

Data and code

The proof of Theorem 1.1, the profile and both checkers are in [3] (version 1.1.0). The finite-n certificates and their three checkers are in [4]. Both are archived on Zenodo; the archive of [4] also contains the large certificates ($n \geq 30\,000$, up to 218 MB). The certificate and the verifiers of Appendix B are in [16].

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I thank András Biró and Gergely Harcos for reading the first two sections and for their comments on the history of the linearization. I thank Ricardo Pinho for reading the code of [3] carefully, for reporting a bug in the first checker of its version 1.0.0 (a missing positivity check of the denominator bound, which did not affect the result), for sharing his certificate and verifiers, and for writing Appendix B.

References

[1] F. V. Atkinson, *On sums of powers of complex numbers*, Acta Math. Acad. Sci. Hungar. **12** (1961), 185–188.
[2] F. V. Atkinson, *Some further estimates concerning sums of powers of complex numbers*, Acta Math. Acad. Sci. Hungar. **20** (1969), 193–210.
[3] V. Bakeas, *Turán's pure power sum constant: $C_{42} \leq 0.68637$*, version 1.1.0 (2026), https://github.com/Bakeas/turan-c42-bound, doi:10.5281/zenodo.23836302.
[4] V. Bakeas, *Certified upper bounds in Turán's pure power sum problem (Erdős #519): certificates and checkers*, version 1.0.0 (2026), https://github.com/Bakeas/turan-power-sum-certificates, doi:10.5281/zenodo.23806909.
[5] A. Biró, *On a problem of Turán concerning sums of powers of complex numbers*, Acta Math. Hungar. **65** (1994), 209–216.
[6] A. Biró, *An improved estimate in a power sum problem of Turán*, Indag. Math. (N.S.) **11** (2000), 343–358.
[7] A. Biró, *An upper estimate in Turán's pure power sum problem*, Indag. Math. (N.S.) **11** (2000), 499–508.
[8] A. Y. Cheer and D. A. Goldston, *Turán's pure power sum problem*, Math. Comp. **65** (1996), 1349–1358.
[9] T. Bloom, *Erdős problem #519*, https://www.erdosproblems.com/519.
[10] S. Griego, *An improved asymptotic certificate for Turán's pure power sum constant C_{42}*, version 1.0.0 (2026), https://github.com/sebastian-griego/turan-c42-certificate.
[11] G. Harcos, *On a result of András Biró*, unpublished manuscript (2000), https://www.renyi.hu/~gharcos/biro.pdf.
[12] J. Komlós, A. Sárközy and E. Szemerédi, *On sums of powers of complex numbers* (in Hungarian), Mat. Lapok **15** (1964), 337–347.
[13] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), 1281–1292.
[14] D. Koshta, *Lowering the first-block cut-off in the coefficient-cancellation bound for Turán's pure power-sum constant*, unrefereed note (2026), https://github.com/DennisKoshta/c42-turan-power-sum-certificate.
[15] R. Pinho, *A 256-step certificate for Turán's pure power-sum constant*, unrefereed, Hexagon preprint hexagon:2610.00485v1 (2026); code at https://github.com/rpinho/c42-certificate/tree/a8a9bc309bd1992413fa07595bd60c5e3fa1b65b.
[16] R. Pinho, *Certificate for $C_{42} \leq 0.6863607950$ on Bakeas's Theorem A*, unrefereed (2026), https://github.com/rpinho/math-certificates/tree/2f49f2f05105d89515c969706600e05b88ae05cf/c42/volterra-quadratic-knots; Optimization Constants repository, pull request #305.
[17] T. Tao et al., *Optimization constants repository*, crowdsourced, constant C42, https://github.com/teorth/optimizationproblems.
[18] P. Turán, *On a New Method of Analysis and its Applications*, Wiley, New York, 1984.

Appendix A. The profile

Table 4. The phases $\psi_j = \varphi(j/48)$ of the profile with $T = 0.68637$ (exact decimals).

j	ψ_j	j	ψ_j	j	ψ_j	j	ψ_j
0	−1.3263721970070639	7	−1.0038358555840339	14	−0.8261837095643588	21	−0.6403292454488249
1	−1.2051941729697693	8	−0.9775022685808040	15	−0.8008543873844631	22	−0.6114334756718721
2	−1.1646625011528973	9	−0.9517710586610195	16	−0.7752453919165897	23	−0.5816917056214177
3	−1.1231259930522046	10	−0.9264376558164993	17	−0.7492783897382005	24	−0.5510810493931545
4	−1.0903014648736054	11	−0.9013360493727731	18	−0.7228792439055676		
5	−1.0595822516417037	12	−0.8763320557453986	19	−0.6959729461692964		
6	−1.0310889220729844	13	−0.8513131401702352	20	−0.6684830921710266		

Appendix B. A certificate with $C_{42} \le 0.6863607950$, by Ricardo Pinho

We apply the main paper's Theorem A, based on the Biró–Harcos linearization, with the notation $T, \psi_j, \varphi, \gamma, a, d, \rho, p, F, \kappa$ used there. The certificate `C42_compact_best.json` specifies 49 exact rational phases ψ_j at

$$u_j = \frac{j^2}{4608} \quad (0 \le j \le 48), \qquad T = 0.6863607950 = \frac{137272159}{200000000}.$$

The decimal strings in that file are exact, not floating-point inputs. Its SHA-256 digest is

`ac8d052d567069c6f2b32583791515628fde0fdfe3a1c15e8b2967d45e898765`

Interpolate the phases linearly, put $\gamma = Te^{i\psi_0}$, $a = \operatorname{Re}\gamma$, and retain the paper's definition $d(u) = \gamma - Te^{i\varphi(u)}$ for $0 < u \le 1/2$, $d(u) = \gamma$ for $u > 1/2$, with $d(0) = 0$. Here $-\pi/2 < \psi_0 < 0$, so $a > 0$; linear interpolation gives

$$\sup_{0 \le u \le 1/2} |\varphi(u) - \psi_0| = 0.8560993691766869 < 1.$$

Thus $|d(u)| \le T|\varphi(u) - \psi_0| < T$ on the first half, using $|1 - e^{ix}| \le |x|$, and $\kappa = T < 1$ since $|d| = T$ on the second half. Theorem A applies to these unequally spaced knots: by Remark 3.2 of the main paper its proof works for any finite set of knots. Here there are 48 linear pieces with a finite largest absolute slope Λ, giving $|d(u)| \le T\Lambda u$ near zero. Theorem A concludes $\limsup_n R_n \le \max(T, F)$ for the exact limit ratio

$$F = \frac{|p(1)|}{\int_0^{1/2} |p(t)|\, dt/(1-t)}, \qquad p(t) = t^{-\gamma} + \int_0^t (t-v)^{-\gamma}\rho(v)\, dv,$$

with ρ the unique bounded solution of its Volterra equation $t\rho(t) = d(t) + \int_0^t d(u)\rho(t-u)\, du$. So any certified upper bound on F that is below T gives $C_{42} \le T$.

We give the two error estimates used to sharpen the paper's a posteriori bound (Lemma 4.1 and bound (9)). Write $H = 1/2$, $b_r = 1 - a > 0$, and $\operatorname{Beta}(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1}\, ds$. For a bounded measurable trial $\widetilde{\rho}$, define

$$R(t) = d(t) + \int_0^t d(u)\widetilde{\rho}(t-u)\, du - t\widetilde{\rho}(t), \qquad \widetilde{p}(t) = t^{-\gamma} + \int_0^t (t-v)^{-\gamma}\widetilde{\rho}(v)\, dv.$$

Let r be a bounded nonnegative measurable majorant of $|R(t)/t|$, and set $e(t) = |\rho(t) - \widetilde{\rho}(t)|$, $E(t) = \int_0^t e(v)\, dv$. Values at isolated jumps do not affect these integrals. In the computation, r is constant on each grid cell.

Lemma B.1 (Integrated residual). *Put $\mathcal{A}_x = \int_0^x r(v)v^{-T}\, dv$ and $S_x = \int_0^x r(v)\, dv$. Then*

$$E(t) \le t^T\mathcal{A}_t, \qquad e(t) \le r(t) + Tt^{T-1}\mathcal{A}_t \quad \text{almost everywhere.}$$

The numerator error $|p(1) - \widetilde{p}(1)|$ is at most

$$\mathcal{E}_N = \int_0^1 r(v)(1-v)^{-a}\, dv + T\mathcal{A}_1 \operatorname{Beta}(T, b_r),$$

and the denominator error $\left|\int_0^H (|p(t)| - |\widetilde{p}(t)|)\, dt/(1-t)\right|$ is at most

$$\mathcal{E}_D = \frac{2}{b_r}\left[H^{b_r}S_H + T\mathcal{A}_H H^{T+b_r} \operatorname{Beta}(T, b_r + 1)\right].$$

Proof. Subtracting the trial equation from the Volterra equation and taking absolute values, with $|d| \le T$, gives $e(t) \le r(t) + TE(t)/t$. Consequently $(t^{-T}E(t))' \le t^{-T}r(t)$ almost everywhere. Integrate from ε to t and let $\varepsilon \downarrow 0$: ρ is bounded by Theorem A and $\widetilde{\rho}$ by assumption, so e is bounded and $\varepsilon^{-T}E(\varepsilon) = O(\varepsilon^{1-T}) \to 0$. This proves both assertions about E, e. The triangle inequality in the definition of p gives

$$|p(t) - \widetilde{p}(t)| \le \int_0^t (t-v)^{-a}e(v)\, dv.$$

At $t = 1$, insert the bound on e, use $\mathcal{A}_v \le \mathcal{A}_1$, and evaluate $\int_0^1 v^{T-1}(1-v)^{-a}\,dv = \mathrm{Beta}(T, b_r)$. This proves the numerator estimate. The reverse triangle inequality, $1/(1-t) \le 2$ for $t \le H$, and Tonelli's theorem for the nonnegative integrand give a denominator error at most

$$2\int_0^H \int_0^t (t-v)^{-a} e(v)\,dv\,dt = \frac{2}{b_r}\int_0^H e(v)(H-v)^{b_r}\,dv.$$

Insert the bound on e, bound $(H-v)^{b_r} \le H^{b_r}$ in its r-term and $\mathcal{A}_v \le \mathcal{A}_H$ in its other term, and substitute $v = Hs$ to evaluate the latter integral as $H^{T+b_r}\,\mathrm{Beta}(T, b_r+1)$. All endpoint integrals converge because $0 < T < 1$ and $0 \le a < 1$. □

Lemma B.2 (A smaller kernel cap near zero)**.** *Suppose* $0 < \delta \le H$ *and* $\sup_{0\le u\le\delta}|d(u)| \le k_\delta \le T$*, with* $k_\delta \ge 0$*. Define*

$$U_\delta = \delta^{k_\delta}\int_0^\delta r(v)v^{-k_\delta}\,dv, \qquad Q_x = \delta^{-T}U_\delta + \int_\delta^x r(v)v^{-T}\,dv \quad (x \ge \delta).$$

Then $E(\delta) \le U_\delta$*,* $E(t) \le t^T Q_t$ *for* $t \ge \delta$*, and valid replacements for the preceding errors are*

$$\mathcal{E}_{N,\delta} = (1-\delta)^{-a}U_\delta + \int_\delta^1 r(v)(1-v)^{-a}\,dv + TQ_1\,\mathrm{Beta}(T, b_r),$$

$$\mathcal{E}_{D,\delta} = \frac{2}{b_r}\left[H^{b_r}\left(U_\delta + \int_\delta^H r(v)\,dv\right) + TQ_H H^{T+b_r}\,\mathrm{Beta}(T, b_r+1)\right].$$

Proof. For $t \le \delta$, every argument of d in the residual equation lies in $[0,\delta]$, so $e \le r + k_\delta E/t$. The integrating-factor argument of Lemma B.1, now with $k_\delta < 1$, gives $E(\delta) \le U_\delta$; it also works when $k_\delta = 0$. For $t \ge \delta$, use the global cap again and integrate $(t^{-T}E)' \le t^{-T}r$ from δ. Replacing $E(\delta)$ by U_δ gives $E(t) \le t^T Q_t$, and hence $e(t) \le r(t) + Tt^{T-1}Q_t$ almost everywhere there.

Split the numerator error integral of Lemma B.1 at δ. On the early part, $(1-v)^{-a} \le (1-\delta)^{-a}$, so its contribution is at most $(1-\delta)^{-a}U_\delta$. On the late part insert the new bound on e, use $Q_v \le Q_1$, and enlarge the nonnegative beta integral from $[\delta, 1]$ to $[0, 1]$. For the denominator use the same Tonelli estimate $(2/b_r)\int_0^H e(v)(H-v)^{b_r}\,dv$. Its early part is at most $(2/b_r)H^{b_r}U_\delta$. On the late part, bound $(H-v)^{b_r} \le H^{b_r}$ in the r-term, use $Q_v \le Q_H$ in the other term, and enlarge that nonnegative integral to $[0, H]$. The beta evaluation in Lemma B.1 proves the formula. All integrals converge since $k_\delta \le T < 1$ and $a < 1$. □

For these phases one may take $k_\delta = T\sup_{0\le u\le\delta}|\varphi(u) - \psi_0|$: the exponential inequality used above proves the cap, and a linear phase attains its largest absolute excursion at a knot or the cutoff. The verifiers use $\delta = 1/256, 1/64, 1/16$. One may take the minimum of the valid numerator error bounds, and separately the minimum of the valid denominator error bounds, including those from Lemma 4.1. If $\mathcal{P}_0$ is a certified upper bound for $|\widetilde{p}(1)|$ and $\mathcal{D}_0$ a certified lower bound for $\int_0^H |\widetilde{p}(t)|\,dt/(1-t)$, then

$$F \le \frac{\mathcal{P}_0 + \mathcal{E}_N}{\mathcal{D}_0 - \mathcal{E}_D} \quad \text{provided } \mathcal{D}_0 - \mathcal{E}_D > 0,$$

by the triangle and reverse triangle inequalities; the same statement holds for any of the valid choices of errors above.

The two supplied PASS receipts use $N = 14155776 = 4608 \cdot 3072$ uniform cells on $[0, 1]$, so every phase knot and H is a grid point. The primary verifier uses 224-bit Arb ball arithmetic; the adapted Bakeas verifier uses 192 bits. Exact rational comparisons of their recorded upper bounds give, respectively,

$$F \le 0.6863607742194886, \qquad F \le 0.6863607773202807.$$

These are upward-rounded displays of the receipts' rational bounds. Both bounds are strictly below T, so Theorem A gives $C_{42} = \limsup_n R_n \le T$. The verifiers are separate implementations, one adapted from Bakeas's `independent_check.py`, but share Lemmas B.1 and B.2: a mistake in these bounds would not be caught by running both.

Lemma B.2 carries the result. With the primary receipt's numbers at this grid, the sup-norm bound (9) gives $F \approx 0.6863934$, Lemma B.1 alone gives $F \approx 0.6863610530$, and Lemma B.2 with $\delta = 1/256$ gives $F \approx 0.6863608171$, all above T. Only $\delta = 1/64$ ($F \approx 0.6863607838$) and $\delta = 1/16$ certify $F < T$, with a

margin $T - F \approx 2.08 \times 10^{-8}$ for $\delta = 1/16$. The local cap matters because the residual bound is largest near zero (worst cell at the first knot $t = 1/4608$), where the global cap T overstates the kernel.

This result is unrefereed. It was found and checked by AI agents under Ricardo Pinho's direction, and reviewed by AI, and by the author of the main paper with AI assistance, not by an independent human referee. The code, certificate, derivation, and receipts are available at commit `2f49f2f0` of the certificate repository:

https://github.com/rpinho/math-certificates/tree/2f49f2f05105d89515c969706600e05b88ae05cf/c42/volterra-quadratic-knots

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