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# ON THE FONTAINE–MAZUR CONJECTURE FOR 2-ADIC REPRESENTATIONS

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Abstract. Let $F$ be a totally real field, abelian over $\mathbb{Q}$, in which 2 splits completely, and let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ be an odd irreducible representation, de Rham with distinct Hodge–Tate weights at the places above 2, whose residual representation is reducible with distinct characters at the places above 2. We prove that $\rho$ is modular in the following cases: if $\rho$ is non-split at every place above 2 at which it is reducible, which includes the case where $\rho$ is irreducible at all places above 2, and, without any condition on the splitness, if $\rho$ is of Steinberg type at some finite place not dividing 2, using an analogue at 2, which we prove, of Pan's finiteness theorem for fully ordinary deformation rings. Both results include the mixed case, in which $\rho$ is reducible at some places above 2 and irreducible at others. The pro-modularity input is obtained by extending to these fields the method of [OAI], which treats representations of $G_{\mathbb{Q}}$; this is where we use that the residual characters are distinct at 2. For $F = \mathbb{Q}(\sqrt{q})$ with $q \equiv 1 \pmod 8$ prime, we remove the condition at 2 from the first result in most cases; for $F = \mathbb{Q}(\sqrt{17})$ this includes every case where the residual semisimplification is scalar. The first result also holds when the residual representation is irreducible with solvable image, where residual modularity follows from an odd Artin lift. For the second result we combine the method of [Die26b] with Thorne's potential modularity theorem for ordinary representations of weight two and the local indecomposability theorem of Hida and Zhao.

## 1. Introduction

1.1. **Setting.** Let $F$ be a totally real field, abelian over $\mathbb{Q}$, in which 2 splits completely, so that $F_v = \mathbb{Q}_2$ for every place $v \mid 2$. Let $E/\mathbb{Q}_2$ be a finite extension with ring of integers $\mathcal{O}$, uniformizer $\varpi$ and residue field $\mathbb{F}$, which we enlarge when necessary. Let $\rho : G_F \to \mathrm{GL}_2(E)$ be a continuous, absolutely irreducible, totally odd representation, unramified outside finitely many places. Let $\Sigma_2$ be the set of places above 2, fix a partition $\Sigma_2 = \Sigma_{\mathrm{n}} \sqcup \Sigma_{\mathrm{o}}$, and consider, as in [Die26a, §1.2], the conditions:

- (N) for $v \in \Sigma_{\mathrm{n}}$, $\rho_v = \rho|_{G_{F_v}}$ is irreducible and de Rham with distinct Hodge–Tate weights;
- (O) for $v \in \Sigma_{\mathrm{o}}$, $\rho_v$ is reducible and de Rham with distinct Hodge–Tate weights;
- (R) $\bar{\rho}^{ss} = \bar{\chi}_1 \oplus \bar{\chi}_2$, the character $\bar{\chi} = \bar{\chi}_1\bar{\chi}_2^{-1}$ extends to a character $\tilde{\chi}$ of $G_{\mathbb{Q}}$, and $\bar{\chi}|_{G_{F_v}} \neq 1$ for every $v \mid 2$.

Since 2 splits completely, $\bar{\chi}|_{G_{F_v}} = \tilde{\chi}|_{G_{\mathbb{Q}_2}}$ for all $v \mid 2$. The character $\tilde{\chi}$ has odd order, hence is unramified at 2, and the last condition in (R) says that $\tilde{\chi}(\mathrm{Frob}_2) \neq 1$. Since the mod 2 cyclotomic character is trivial, (R) contains the analogue of the condition $\bar{\chi}|_{G_{F_v}} \neq \omega^{\pm 1}$ of [Die26b]. Under (O), the Jordan–Hölder character of $\rho_v$ with the smaller Hodge–Tate number is a subcharacter of $\rho_v$ (Lemma 2.1); it is the *classical character* $\psi_v^{\mathrm{cl}}$.

For odd $p$, the case $\Sigma_{\mathrm{o}} = \emptyset$ is due to Pan [Pan22] and Zhang [Zha25], the case $\Sigma_{\mathrm{n}} = \emptyset$ to Pan [Pan22, Theorem 5.1.2], and the mixed case is treated in [Die26a, Die26b]. Over $\mathbb{Q}$ there is a single place above 2, the mixed case does not occur, and the Fontaine–Mazur conjecture at 2 is proved in [OAI]. The purpose of this paper is to extend the methods of [Die26a, Die26b] to $p = 2$ over the fields $F$ above.

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1.2. **The pro-modularity input.** Pseudo-representations are understood as determinants in the sense of Chenevier [Che14]. Let $\Sigma$ be a finite set of places of $F$ containing the infinite places, the places above 2 and those at which $\rho$ is ramified. Put $\chi = \det\rho$ and, for a field $K \supset F$, let $R_K^{\mathrm{ps}}$ be the universal deformation ring of the determinant of $\bar\rho^{ss}|_{G_K}$ for the group $G_{K,\Sigma_K}$, with determinant $\chi|_{G_K}$, where $\Sigma_K$ is the set of places above $\Sigma$. For $K$ totally real of even degree, let $\mathbb{T}_K(U^p)$ be the Hecke algebra of the completed cohomology at 2 of the quaternion algebra over $K$ ramified exactly at the infinite places, of tame level $U^p$ and central character $\chi\varepsilon$, localized at the maximal ideal of $\bar\rho^{ss}$ (§3.1). It is a quotient of $R_K^{\mathrm{ps}}$.

**Hypothesis 1.1** (PM)**.** There are totally real fields $F \subset K \subset F_1$ of even degree, in which 2 splits completely, with $K/F$ Galois and solvable and $F_1/\mathbb{Q}$ Galois and solvable, such that for every irreducible component $C$ of $\mathrm{Spec}R_K^{\mathrm{ps}}$ containing the point defined by $\rho|_{G_K}$, the image of $C$ in $\mathrm{Spec}R_{F_1}^{\mathrm{ps}}$ under restriction to $G_{F_1}$ lies in the image of $\mathrm{Spec}\mathbb{T}_{F_1}(U^p)$ for some tame level $U^p$.

For odd $p$, [Zha25, Theorem 5.5.1] gives the stronger statement with $K = F_1$, for $p \geq 5$ the corresponding result of Pan, and its proof uses no local condition at $p$ [Die26a, Proposition 4.5]. For $p = 2$ and representations of $G_{\mathbb{Q}}$, the analogous statement is [OAI, Theorem 7.1]. In Section 8 we prove:

**Theorem 1.2** (Theorem 8.4)**.** *Assume* (R)*, that $\rho$ is de Rham with distinct Hodge–Tate weights at some place above* 2 *and not induced from a character of an index two subgroup, and that* $\det\rho = \varepsilon^w\chi_0$ *with $\chi_0$ of finite order. Then* (PM) *holds.*

The last condition holds under (N) and (O), by §2.2; it is needed because the propagation theorem and the Hecke algebras of [OAI] use a determinant of this form.

The proof follows [OAI, §§6–7]. The base field $\mathbb{Q}$ enters there through the construction of auxiliary fields with controlled Selmer groups [OAI, Proposition 6.1], which we carry out over $F$ when $\bar\chi|_{G_{\mathbb{Q}_2}} \neq 1$, replacing the conductor primes of cyclotomic fields by suitable primes of $F$ and ray class fields; through Leopoldt's conjecture for the abelian fields so constructed, which we replace by Waldschmidt's bound for the Leopoldt defect; and through the modular seed [OAI, Lemma 7.2], which we obtain by base change from $\mathbb{Q}$. Besides [OAI, §§5 and 7], which are used with these modifications, we use from [OAI] three results on completed cohomology at 2 over general totally real fields: the structure of completed forms, local–global compatibility for the blocks at 2, and the lower bound for the dimension of Hecke algebras [OAI, Propositions 4.1, 4.2 and 4.4].

1.3. **Main results.**

**Theorem A.** *Assume* (N), (O) *and* (R)*, and that $\rho_v$ is non-split for every $v \in \Sigma_{\mathrm{o}}$. Then $\rho$ is modular.*

This includes the case $\Sigma_{\mathrm{o}} = \emptyset$, the analogue at 2 of Pan's theorem in the generic case, and the case $\Sigma_{\mathrm{n}} = \emptyset$ with $\rho$ non-split at all places above 2. To remove the non-split hypothesis we assume:

(St) there is a finite place $v_0 \nmid 2$ of $F$ at which $\rho$ is of Steinberg type, that is, the Weil–Deligne representation of $\rho|_{G_{F_{v_0}}}$ has non-zero monodromy.

For odd $p$, the method of [Die26b] uses Pan's finiteness theorem for fully ordinary pseudo-deformation rings [Pan22, Theorem 5.1.2(1)]. For $p = 2$ we prove its analogue (Theorem 8.9), using the pro-modularity of Section 8 and the structure of the generic blocks of $\mathrm{GL}_2(\mathbb{Q}_2)$. For a totally real field $K$ in which 2 splits completely, let $R_K^{\mathrm{red}}$ be the quotient of $R_K^{\mathrm{ps}}$ on which the universal determinant is reducible on every decomposition group above 2, and $\Lambda_K = \mathcal{O}[[\prod_{v|2}(1+4\mathbb{Z}_2)]]$, acting through the characters $\psi_v^{(1)}$ of Definition 4.1. Let us write (OF) for the statement that $R_K^{\mathrm{red}}$ is finite over $\Lambda_K$; Theorem 8.9 proves it for the field $K$ used below.

**Theorem B.** *Assume* (N), (O), (R) *and* (St). *Then $\rho$ is modular.*

No condition is imposed on the splitness of $\rho_v$ for $v \in \Sigma_{\mathrm{o}}$ or on the Hodge–Tate weights beyond (N) and (O).

Over real quadratic fields we can remove the condition $\bar\chi|_{G_v} \neq 1$ from Theorem A in most cases. Let $q \equiv 1 \pmod 8$ be a prime, $F = \mathbb{Q}(\sqrt{q})$, and $w_q$ the place of $F$ above $q$. Consider the condition (S) that a totally positive generator of $w^{h^+}$, for a place $w \mid 2$ and $h^+$ the narrow class number, is not a square at the other place above 2 (Lemma 9.1). It holds for $q = 17$.

**Theorem C** (Theorem 9.4)**.** *Let $F = \mathbb{Q}(\sqrt{q})$ be as above. Assume* (N), (O), *that $\bar\rho^{ss} = \bar\chi_1 \oplus \bar\chi_2$ with $\bar\chi = \bar\chi_1\bar\chi_2^{-1}$ extending to $G_{\mathbb{Q}}$, and that $\rho_v$ is non-split for every $v \in \Sigma_{\mathrm{o}}$. Assume that $\bar\chi|_{G_v} \neq 1$ for $v \mid 2$; or that $\bar\chi = 1$ and $F$ satisfies* (S)*; or that $\bar\chi \neq 1$, $\rho$ is unramified at $w_q$ and $\bar\chi(\mathrm{Frob}_{w_q}) \neq 1$. Then $\rho$ is modular.*

For $F = \mathbb{Q}(\sqrt{17})$, this applies outside the case where $\bar\chi \neq 1$ is trivial at 2 and $\rho$ is ramified at $w_{17}$ or $\bar\chi(\mathrm{Frob}_{w_{17}}) = 1$; in particular it applies whenever $\bar\rho^{ss}$ is scalar. This is the case, for instance, if $\rho$ is unramified outside the places above 2, since then the extension of $\bar\chi$ to $G_{\mathbb{Q}}$ is ramified only at 2, hence trivial (Corollary 9.5). The proof uses the method of [OAI], with $F$ in the role of the auxiliary quadratic field of [OAI, §6] (Section 9).

Finally, when $\bar\rho$ is absolutely irreducible with solvable image, which in characteristic 2 means that it is dihedral, residual modularity is automatic, by an odd Artin lift, and the method applies without Selmer conditions.

**Theorem D** (Theorem 10.6)**.** *Let $F$ be as in §1.1. Assume* (N), (O), *that $\bar\rho$ is absolutely irreducible with solvable image, and that $\rho_v$ is non-split for every $v \in \Sigma_{\mathrm{o}}$. Then $\rho$ is modular.*

1.4. **The method.** Theorem A is proved as [Die26a, Theorem A]: (PM) places $\rho$ in the Hecke support of completed cohomology; Emerton's ordinary parts at the places of $\Sigma_{\mathrm{o}}$ give a point of the partially ordinary Hecke algebra carrying *a* subcharacter of $\rho_v$, which is the classical one when $\rho_v$ is non-split, and the classicality criterion of [Die26a, Theorem 4.2] applies. We check that the arguments of [Die26a, §§3–4] work for $p = 2$ (Section 3).

For Theorem B we follow [Die26b]. Let $C$ be an irreducible component through the point $x$ of $\rho$ of the deformation space $\mathrm{Spec}R^{\mathrm{o}}$ which is ordinary at $\Sigma_{\mathrm{o}}$. Cutting $C$ by the reducibility equations at the places of $\Sigma_{\mathrm{n}}$ gives a fully ordinary subfamily $D \subset C$ of dimension $1+d$. A regular modular point $y \in D$, together with a lower bound for the dimension of the partially ordinary Hecke algebra $\mathbb{T}^{\mathrm{o}}$, shows that $C$ is a component of $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}$, and the classicality criterion applies to $x$. For odd $p$ the modularity of the points of $D$ comes from Pan's theorem in the ordinary case. For $p = 2$ it is obtained as follows.

(1) By Theorem 8.9, $D$ is finite over the Iwasawa algebra $\Lambda$, so its points of weight two are dense. By (PM) and Emerton's adjunction, their restrictions to $G_{F_1}$ come from the fully ordinary part of completed cohomology over $F_1$, with *some* ordering of the characters at each place above 2.

(2) By (St), the ratio of the Frobenius eigenvalues at $v_0$ is $\mathrm{N}(v_0)^{\pm 1}$ along $C$. Points of $D$ of weight two are potentially modular by Thorne's theorem [Tho26, Theorem B], the corresponding Hilbert modular forms are of Steinberg type at $v_0$, hence square integrable there, and the theorem of Hida and Zhao [Zhao14, Theorem 1] shows that they are non-split at all places above 2. Hence the ordering in (1) is the classical one, and these points are modular and o-pro-modular (Proposition 6.1).

(3) Their regularity follows from the theorem of Newton and Thorne [NT23, Theorem B], which holds for $p = 2$, and from the full local–global compatibility at the places not above 2, which holds by Carayol's theorem since the forms are square integrable at $v_0$ (Proposition 6.2).

Condition (St) is also what excludes CM points from $D$ (Lemma 4.4): for odd $p$ this followed from (R), but the corresponding argument fails for $p = 2$, since quadratic characters are trivial modulo 2.

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**Notation.** We use the conventions of [Die26a]: $\mathrm{HT}(\varepsilon) = -1$, and characters of $G_{\mathbb{Q}_2}$ are identified with characters of $\mathbb{Q}_2^\times$ via the local Artin map. We write $G_v = G_{F_v}$, $I_v$ for the inertia subgroup, $\phi_v$ for a lift of the arithmetic Frobenius and $q_v = \mathrm{N}(v)$. For a determinant $D$ with coefficients in a ring $R$ and a prime $\mathfrak{r}$ of $R$, $\rho(\mathfrak{r})$ is the semisimple representation over an algebraic closure of $k(\mathfrak{r})$ with determinant $D \bmod \mathfrak{r}$ [Che14, Theorem 2.12]; for a point $z$ we also write $\rho_z$, and $T$ denotes the trace of the universal determinant. A point is *irreducible* if $\rho_z$ is absolutely irreducible. We write $h^i(V) = \dim H^i(V)$, and $d = [F : \mathbb{Q}]$ after the base change of §2.2.

## 2. Preliminaries

2.1. **Determinants and ordinary conditions.** For $p = 2$, traces do not determine two-dimensional determinants, and we use [Che14] throughout. Let $A$ be a complete local ring with residue field $k$ of characteristic 2 and $D$ a determinant of a profinite group $G$ with values in $A$, whose reduction is $\det(\bar\psi_1 \oplus \bar\psi_2)$ with $\bar\psi_1 \neq \bar\psi_2$. Then the Cayley–Hamilton quotient of $A[G]$ is a generalized matrix algebra $\left(\begin{smallmatrix} A & B \\ C & A \end{smallmatrix}\right)$ [Che14, Theorem 2.22 and §1.4], [BC09, §1.3]. A determinant with values in a field is reducible if and only if it factors through $A/BC$, and if $BC = 0$ the diagonal entries give characters $\psi_1, \psi_2 : G \to A^\times$ lifting $\bar\psi_1, \bar\psi_2$ with $D = \det(\psi_1 \oplus \psi_2)$. Following [Pan22, Definition 5.3.2], if $D|_H = \det(\psi_1 \oplus \psi_2)$ for a closed subgroup $H \subset G$, we say that $D$ is $\psi_1$-*ordinary* at $H$ if

$$T(\sigma\tau\eta) - \psi_1(\sigma)T(\tau\eta) - \psi_2(\tau)T(\sigma\eta) + \psi_1(\sigma)\psi_2(\tau)T(\eta) = 0 \qquad (\sigma, \tau \in H,\ \eta \in G).$$

The proof of [Die26a, Lemma 2.3] is characteristic free and gives: if $\rho_A : G \to \mathrm{GL}_2(A)$ has absolutely irreducible reduction and $\psi_1(\sigma_0) - \psi_2(\sigma_0) \in A^\times$ for some $\sigma_0 \in H$, then $\det \rho_A$ is $\psi_1$-ordinary at $H$ if and only if $\rho_A|_H$ has a stable direct summand of rank one with character $\psi_1$; such a summand is unique; and if $\det \rho_A$ is $\psi_1$- and $\psi_2$-ordinary, then $\rho_A|_H \simeq \psi_1 \oplus \psi_2$. We apply this to the completed local rings at irreducible points of characteristic 0, and to the residually distinct case above.

**Lemma 2.1.** *Let $\psi_1, \psi_2 : G_{\mathbb{Q}_2} \to E^\times$ be de Rham characters with $\mathrm{HT}(\psi_1) \neq \mathrm{HT}(\psi_2)$. If $\mathrm{HT}(\psi_1) < \mathrm{HT}(\psi_2)$, every extension of $\psi_2$ by $\psi_1$ is de Rham; if $\mathrm{HT}(\psi_1) > \mathrm{HT}(\psi_2)$, every de Rham extension of $\psi_2$ by $\psi_1$ is split.*

*Proof.* This is [Die26a, Lemma 2.1], whose proof uses the Bloch–Kato dimension formulas [BK90, Corollary 3.8.4], which hold for $p = 2$. □

**Lemma 2.2.** *Let $v \mid 2$ with $\bar\chi|_{G_v} \neq 1$, and let $R_v^{\mathrm{ps}}$ be the deformation ring of the determinant of $(\bar\chi_1 \oplus \bar\chi_2)|_{G_v}$ with determinant $\chi|_{G_v}$. There is $f_v \in R_v^{\mathrm{ps}}$ such that a deformation with values in a field is reducible if and only if it kills $f_v$.*

*Proof.* Since $\bar\chi|_{G_v} \neq 1$, the residual characters are distinct on $G_v$, so the Cayley–Hamilton algebra is a generalized matrix algebra with off-diagonal modules $B$, $C$, and the reducibility ideal is $BC$. The minimal number of generators of $B$ (resp. $C$) is at most $h^1(G_{\mathbb{Q}_2}, \bar\chi)$ (resp. $h^1(G_{\mathbb{Q}_2}, \bar\chi^{-1})$) [BC09, Theorem 1.5.5], [Deo, Lemma 2.5]; the argument only uses the generalized matrix algebra structure and applies to determinants. Since the mod 2 cyclotomic character is trivial, the local Euler characteristic formula gives $h^1(G_{\mathbb{Q}_2}, \bar\chi^{\pm 1}) = 1 + h^0(\bar\chi^{\pm 1}) + h^0(\bar\chi^{\mp 1}) = 1$. Hence $BC$ is principal, generated by $f_v = bc$. □

**Lemma 2.3.** *Let $G$ be a group, $k$ a field, $\alpha \neq \beta$ and $\eta \neq 1$ characters $G \to k^\times$, and $r = \left(\begin{smallmatrix} \alpha & * \\ 0 & \beta \end{smallmatrix}\right)$ a non-split extension. If $H^0(G, \mathrm{ad}^0 r \otimes \eta) \neq 0$, then $\alpha\beta^{-1} = \eta^{-1}$.*

*Proof.* This is [Die26b, Lemma 3.2], whose proof is characteristic free. □

**Lemma 2.4.** *Let $K$ be a totally real field and $r : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ an absolutely irreducible representation, de Rham with distinct Hodge–Tate weights at some place above 2, and not induced from a character of an index two subgroup. Then for every finite extension $K'/K$, $r|_{G_{K'}}$ is absolutely irreducible and not induced.*

*Proof.* Let $H$ be the Zariski closure of $r(G_K)$ and $H^\circ$ its identity component. If $H^\circ$ is solvable, then either it has a non-trivial unipotent radical, whose unique fixed line is $H$-stable, or it is a torus. If it is a central torus, the projective image of $r$ is finite and the Hodge–Tate weights are equal. If it is a non-central torus, $H$ permutes its two eigenlines and $r$ is induced. Hence $H^\circ \supset \mathrm{SL}_2$, and the same holds for $r|_{G_{K'}}$, whose Zariski closure contains $H^\circ$; this proves the claim (compare [OAI, Lemma 2.2]). □

2.2. **The base field and the reducible locus.** If $\rho$ is induced from a character $\theta$ of $G_K$ for a quadratic extension $K/F$, then, as in the proof of Lemma 4.4 below, $\theta$ is attached to an algebraic Hecke character, $K$ is CM because the Hodge–Tate weights are distinct, and $\rho$ is modular by automorphic induction. We assume from now on that $\rho$ is not induced. Let $K \subset F_1$ be as in Theorem 8.5; by Theorem 8.4 and Theorem 8.5, (PM) holds for them. The restrictions of $\rho$ to $G_K$ and $G_{F_1}$ are absolutely irreducible and not induced, by Lemma 2.4. Conditions (N), (O), (R) and (St) are preserved, since $F_{1,w} = F_v = \mathbb{Q}_2$ for $w \mid v \mid 2$ and Steinberg type is preserved by finite extensions; the non-splitness of the $\rho_v$, $v \in \Sigma_\mathrm{o}$, is preserved as well. Since $F_1/F$ and $K/F$ are solvable, $\rho$ is modular if $\rho|_{G_{F_1}}$ or $\rho|_{G_K}$ is, by solvable descent [BLGGT14, Lemma 2.2.2], [AC89]. We replace $F$ by $K$ and keep $F_1$, keep the notation $\Sigma_\mathrm{n}$, $\Sigma_\mathrm{o}$, $\Sigma$, $v_0$ for the corresponding sets and for a place above $v_0$, and put $d = [F : \mathbb{Q}]$, which is even.

Twisting $\rho$ by the Teichmüller lift of $\bar\chi_2^{-1}$, we may assume $\bar\rho^{ss} = 1 \oplus \bar\chi$. Let $\delta_F$ be the Leopoldt defect of $F$ at 2. By Waldschmidt's theorem [Wal81], the 2-adic rank of the units of $F$ is at least half of their rank, so $\delta_F \leq (d-1)/2$, and the pro-2 completion of $G_{F,\Sigma}^{\mathrm{ab}}$ has $\mathbb{Z}_2$-rank $1 + \delta_F \leq (d+1)/2$. Hence the reducible determinants, of the form $\det(\psi \oplus \chi\psi^{-1})$, form a closed subset of $\mathrm{Spec}R^{\mathrm{ps}}$ of dimension at most $(d+3)/2 < 1 + d$, where $R^{\mathrm{ps}} = R^{\mathrm{ps}}_F$; and every de Rham character of $G_F$ is of the form $\varepsilon^k\chi_0$ with $\chi_0$ of finite order.

## 3. Completed cohomology at 2 and ordinary parts

3.1. **Completed cohomology.** Let $B$ be the quaternion algebra over $F$ ramified exactly at the infinite places, $\psi = \chi\varepsilon$, and $U^p = \prod_{v\nmid 2} U_v$ a tame level, sufficiently small, with $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for $v \notin \Sigma$. Let $S = S_\psi(U^p)_\mathfrak{m}$ be the completed cohomology [Die26a, §3.1], [OAI, §4.1], localized at the maximal ideal $\mathfrak{m}$ of the Hecke algebra corresponding to $1 + \bar\chi$, and $\mathbb{T} = \mathbb{T}(U^p)_\mathfrak{m}$ its Hecke algebra, generated by the $T_w$, $w \notin \Sigma$. It carries a determinant $D_\mathfrak{m}$ of $G_{F,\Sigma}$ with $T(\mathrm{Frob}_w) = T_w$ and determinant $\chi$, so $\mathbb{T}$ is a quotient of $R^{\mathrm{ps}}$. Shrinking $U^p$ only enlarges the image of $\mathrm{Spec}\mathbb{T}$ in $\mathrm{Spec}R^{\mathrm{ps}}$, so the level in (PM) may be taken sufficiently small. The constructions of this section are used both over $F$ and over the field $F_1$ of (PM); we write $\mathbb{T}_{F_1}$, $\mathbb{T}^\mathrm{o}_{F_1}$, etc. for the latter. We use the following results of [OAI], which hold over any totally real field of even degree in which 2 splits completely.

(C1) [OAI, Proposition 4.1]. At sufficiently small levels the spaces of forms are free over the group algebras of the relevant finite 2-groups, the dual of $S$ is finite free over a sufficiently deep Iwasawa algebra, and locally algebraic vectors are classical forms on $B^\times$.

(C2) [OAI, Proposition 4.2]. Local–global compatibility. Put $S_{\mathrm{tor}} = S \otimes_{\mathcal{O}} E/\mathcal{O}$ and let $M = S_{\mathrm{tor}}^{\vee}$ be its Pontryagin dual. For the product $P$ of the projective generators of the blocks at the places above 2, the module $\mathbf{m} = \mathrm{Hom}(P, M)$ is faithful and finite over $\mathbb{T}$, and finite over $\widehat{\bigotimes}_{v|2} R_v^{\mathrm{ps}}$, whose action through the centres of the blocks agrees with the Galois action, with the following normalization, used throughout. The central character is $\psi = \chi\varepsilon$, and the block parameter at $v$ is $(\rho_x|_{G_v} \otimes \varepsilon)^{ss}$, of determinant $\chi_v\varepsilon^2$; we identify the local block base with $R_v^{\mathrm{ps}}$, of determinant $\chi$, by untwisting $\varepsilon$, and its action then agrees with the restriction of the global determinant. At a point $x$ of $\mathrm{Spec}\mathbb{T}[1/2]$, after a finite extension of $E$, the eigenspace $(S \otimes E)[x]$ contains $\widehat{\bigotimes}_{v|2} \Pi_v$, where $\Pi_v$ is an absolutely irreducible unitary Banach representation with the indicated block parameter; if $\rho_x|_{G_v}$ is absolutely irreducible, $\Pi_v$ is the representation attached to the correspondingly normalized parameter by the $p$-adic local Langlands correspondence [CDP14], which has dense locally algebraic vectors if $\rho_x|_{G_v}$ is de Rham with distinct Hodge–Tate weights [CDP14, Theorem 1.3].

(C3) [OAI, Proposition 4.4]. Every irreducible component of $\mathrm{Spec}\mathbb{T}$ has dimension at least $1 + 2d$.

We also use the classification of the absolutely irreducible objects of the blocks of $\mathrm{GL}_2(\mathbb{Q}_2)$ [PT21, Corollary 6.10 and Proposition 6.11]: if the local parameter at $x_v$ is reducible, every absolutely irreducible unitary Banach representation with this parameter is either a character $\eta \circ \det$ or a quotient of a unitary principal series $(\mathrm{Ind}_{\bar{B}}\mu)_{\mathrm{cont}}$. Characters do not occur in $(S \otimes E)[x]$ at irreducible points, by strong approximation [Die26a, Lemma 3.2].

3.2. **Ordinary parts.** Let $P = B_{\mathrm{o}} \times G_{\mathrm{n}} \subset \prod_{v|2} \mathrm{GL}_2(\mathbb{Q}_2)$ be the parabolic subgroup which is the upper Borel subgroup at the places of $\Sigma_{\mathrm{o}}$ and the whole group at those of $\Sigma_{\mathrm{n}}$, and let $S^{\mathrm{o}} = \mathrm{Ord}_P(S)$ be Emerton's ordinary part [Eme10a]. Properties (O1)–(O4) of [Die26a, §3.3] hold for any $p$, as does [Die26a, Lemma 3.3]: $S^{\mathrm{o}}/\varpi^n \simeq \varinjlim_c e\, S_\psi(U^p U(c), \mathcal{O}/\varpi^n)_{\mathfrak{m}}$, where $e$ is the ordinary idempotent at the places of $\Sigma_{\mathrm{o}}$. Let $\mathbb{T}^{\mathrm{o}}$ be the Hecke algebra of $S^{\mathrm{o}}$, generated by the $T_w$ and the diamond operators $\langle\gamma\rangle_v$, $v \in \Sigma_{\mathrm{o}}$, and $\psi_{v,1} : G_v \to (\mathbb{T}^{\mathrm{o}})^\times$ the character with $\psi_{v,1} \circ \mathrm{Art}(\gamma) = \langle\gamma\rangle_v$. When $\Sigma_{\mathrm{o}} = \Sigma_2$ we write $\mathbb{T}^{\mathrm{ord}}$ for $\mathbb{T}^{\mathrm{o}}$.

**Lemma 3.1.** *There is a determinant of $G_{F,\Sigma}$ over $\mathbb{T}^{\mathrm{o}}$ with $T(\mathrm{Frob}_w) = T_w$ and determinant $\chi$, which for every $v \in \Sigma_{\mathrm{o}}$ is $\det(\psi_{v,1} \oplus \chi\psi_{v,1}^{-1})$ on $G_v$ and $\psi_{v,1}$-ordinary at $G_v$.*

*Proof.* Write $\chi = \varepsilon^k\chi_0$ with $\chi_0$ of finite order. The central character $\psi = \chi\varepsilon$ need not be locally constant at 2, so we first normalize it. With the notation of §4.4 (the construction there does not depend on the parity of $k$), $\varepsilon = \eta_4\langle\varepsilon\rangle$, and $\nu$ is an even continuous character, unramified outside 2 and $\equiv 1 \pmod{\varpi}$, with $\nu^2 = \langle\varepsilon\rangle$. Put $\lambda = \nu^{-(k+1)}$. Then $\chi\lambda^2 = \varepsilon^{-1}\chi_0\eta_4^{k+1}$, so $(\chi\lambda^2)\varepsilon = \chi_0\eta_4^{k+1}$ has finite order. Multiplication by $\lambda \circ \mathrm{nrd}$ transports completed forms, ordinary parts and Hecke operators (refining the dyadic level at each torsion level if necessary), replacing $\psi_{v,1}$ by $\lambda_v\psi_{v,1}$; since the unipotent invariants are unchanged and $\lambda_v(\mathrm{Art}_v(2)) = 1$, the ordinary projector is preserved.

For the normalized determinant, finite-level forms are scalar-valued of parallel weight 2, and the proof of [Die26a, Lemma 3.4] applies, with the theorem of Wiles [Wil88, Theorem 2] replaced by local–global compatibility at 2 for Hilbert modular forms [Sai09, Ski09]: if $\pi$ is ordinary at $v$, the unit root of the Hecke polynomial gives a unit eigenvalue of Frobenius on $D_{\mathrm{pst}}(\rho_\pi|_{G_v})$, whose eigenline is a weakly admissible subobject, so that $\rho_\pi|_{G_v}$ has a subcharacter, which is the diamond character. The determinant and the ordinary identities hold over the reduced finite-level Hecke algebras $A_c[1/2]$ [Che14, Theorem 2.12], hence over their torsion-free integral models and in the completed limit. Undoing the twist gives the assertions for $\chi$ and $\psi_{v,1}$; no de Rham property of $\lambda$ is used. □

**Lemma 3.2.** *For $v \in \Sigma_{\mathrm{n}}$, the action of $R^{\mathrm{ps}}_v$ on the dual $M^{\mathrm{o}}$ of $S^{\mathrm{o}}$ through the centre of the block at $v$ agrees with its action through the restriction to $G_v$ of the determinant of Lemma 3.1. The module $\mathbf{m}^{\mathrm{o}} = \mathrm{Hom}(P_{\mathrm{n}}, M^{\mathrm{o}})$, for the product $P_{\mathrm{n}}$ of the projective generators at the places of $\Sigma_{\mathrm{n}}$, is faithful and finitely generated over $\mathbb{T}^{\mathrm{o}}$, and $(S^{\mathrm{o}} \otimes E)[\mathfrak{P}] \neq 0$ for every maximal ideal $\mathfrak{P}$ of $\mathbb{T}^{\mathrm{o}}[1/2]$.*

*Proof.* This is the proof of [Die26a, Lemma 3.5], with the local–global compatibility of Pan and Zhang replaced by (C2). □

**Proposition 3.3.** *Let $\mathfrak{p}$ be a maximal ideal of $\mathbb{T}[1/2]$ such that $\rho(\mathfrak{p})$ is irreducible and $\rho(\mathfrak{p})|_{G_v}$ is reducible for every $v \in \Sigma_{\mathrm{o}}$. Then there is a maximal ideal $\mathfrak{P}$ of $\mathbb{T}^{\mathrm{o}}[1/2]$ above $\mathfrak{p}$ such that $\psi_{v,1}(\mathfrak{P})$ is a subcharacter of $\rho(\mathfrak{p})|_{G_v}$ for every $v \in \Sigma_{\mathrm{o}}$.*

*Proof.* The proof of [Die26a, Proposition 4.1] applies: by (C2) the eigenspace $\Pi = (S \otimes E)[\mathfrak{p}]$ contains $\Pi' = \widehat{\bigotimes}_v \Pi'_v$ with $\Pi'_v$ absolutely irreducible; for $v \in \Sigma_{\mathrm{o}}$, $\Pi'_v$ is a quotient of a unitary principal series $(\mathrm{Ind}_{\bar{B}} \mu_v)_{\mathrm{cont}}$ by §3.1; Emerton's adjunction gives a non-zero map $\mu \otimes \widehat{\bigotimes}_{v \in \Sigma_{\mathrm{n}}} \Pi'_v \to \mathrm{Ord}_P(\Pi) = (S^{\mathrm{o}} \otimes E)[\mathfrak{p}]$, whence $\mathfrak{P}$; and Lemma 3.1 together with §2 shows that $\psi_{v,1}(\mathfrak{P})$ is a subcharacter. □

**Theorem 3.4.** *Let $\mathfrak{p}$ be a maximal ideal of $\mathbb{T}[1/2]$ and $\mathfrak{P}$ a maximal ideal of $\mathbb{T}^{\mathrm{o}}[1/2]$ above it. Assume that $\rho(\mathfrak{p})$ is irreducible, that $\rho(\mathfrak{p})|_{G_v}$ is absolutely irreducible and de Rham with distinct Hodge–Tate weights for $v \in \Sigma_{\mathrm{n}}$, and that for $v \in \Sigma_{\mathrm{o}}$ it is reducible and de Rham with distinct Hodge–Tate weights, with classical character $\psi_{v,1}(\mathfrak{P})$. Then $\rho(\mathfrak{p})$ is modular.*

*Proof.* This is the proof of [Die26a, Theorem 4.2], using Lemma 3.2, (C1), (C2) and the following density statement: if $\mu = \mu_1 \otimes \mu_2$ is a unitary character of $T(\mathbb{Q}_2)$ with $\mu_i|_{\mathbb{Z}_2^\times} = x^{a_i}\theta_i$, $\theta_i$ of finite order and $a_1 \geq a_2$, then $(\mathrm{Ind}_{\bar{B}} \mu)_{\mathrm{cont}}$ has dense locally algebraic vectors. For $a_1 = a_2$ this is proved directly in [Die26a, Lemma 2.2], and the argument is characteristic free. For $a_1 > a_2$ we use [BE10, Proposition 2.2.1]; see Remark 11.5. □

**Proposition 3.5.** *Every minimal prime of $\mathbb{T}^{\mathrm{o}}_{\mathfrak{m}'}$, for a maximal ideal $\mathfrak{m}'$ of $\mathbb{T}^{\mathrm{o}}$, has characteristic 0, and the corresponding irreducible component has dimension at least $1 + 2d - |\Sigma_{\mathrm{o}}|$.*

*Proof.* For $\Sigma_{\mathrm{o}} = \emptyset$ this is (C3), the analogue for $p = 2$ of [Pan22, Theorem 3.6.1]. In general, the proof of (C3) applies to $S^{\mathrm{o}}$ with the modifications of [Die26b, Proposition 6.1]: the freeness of $M^{\mathrm{o}}/\varpi$ over the Iwasawa algebra of a uniform pro-2 group of dimension $3|\Sigma_{\mathrm{n}}| + |\Sigma_{\mathrm{o}}|$, deduced from (C1) and the finite-level description of $S^{\mathrm{o}}$; the module $\mathbf{m}^{\mathrm{o}}$ of Lemma 3.2; and the dimension formula $\dim(N \widehat{\otimes} P_{\mathrm{n}}) \leq \dim_A N + |\Sigma_{\mathrm{n}}|$, with $A = \widehat{\bigotimes}_{v \in \Sigma_{\mathrm{n}}} R^{\mathrm{ps}}_v/\varpi \,\widehat{\otimes}\, \Lambda_T/\varpi$, whose base case is the bound $|\Sigma_{\mathrm{n}}|$ for the Gelfand–Kirillov dimension of smooth representations of finite length of $\prod_{v \in \Sigma_{\mathrm{n}}} \mathrm{GL}_2(\mathbb{Q}_2)$ [OAI, Lemma 4.3]. □

3.3. **Proof of Theorem A.** Let $x$ be the point of $\mathrm{Spec} R^{\mathrm{ps}}$ defined by $\rho$. By (PM), applied to an irreducible component through $x$, the restriction $\rho|_{G_{F_1}}$ defines a maximal ideal $\mathfrak{p}$ of $\mathbb{T}_{F_1}[1/2]$ with $\rho(\mathfrak{p}) \simeq \rho|_{G_{F_1}}$. Since 2 splits completely in $F_1$, the restrictions of $\rho|_{G_{F_1}}$ to the decomposition groups above $v \mid 2$ are the $\rho_v$. By Proposition 3.3 over $F_1$ there is $\mathfrak{P}$ above $\mathfrak{p}$ with $\psi_{w,1}(\mathfrak{P})$ a subcharacter of $\rho_v$ for $w \mid v \in \Sigma_{\mathrm{o}}$. As $\rho_v$ is non-split, its only subcharacter is the classical one (Lemma 2.1). By Theorem 3.4 over $F_1$, $\rho|_{G_{F_1}}$ is modular, hence so is $\rho$, by solvable descent. □

## 4. Setup for Theorem B

4.1. **Deformation rings.** For $v \mid 2$, on the locus where the universal determinant is reducible on $G_v$, it is $\det(\psi \oplus \chi\psi^{-1})$ with $\psi$ lifting 1 or $\bar{\chi}|_{G_v}$, uniquely since $\bar{\chi}|_{G_v} \neq 1$. For $v \in \Sigma_{\mathrm{o}}$ let $\bar{\psi}_v \in \{1, \bar{\chi}|_{G_v}\}$ be the reduction of $\psi_v^{\mathrm{cl}}$.

**Definition 4.1.** (1) $R^{\mathrm{o}}$ is the quotient of $R^{\mathrm{ps}}$ by the conditions that, for $v \in \Sigma_{\mathrm{o}}$, the universal determinant is $\det(\psi_v \oplus \chi\psi_v^{-1})$ on $G_v$ with $\psi_v$ lifting $\bar{\psi}_v$, and is $\psi_v$-ordinary at $G_v$.

(2) $R^{\mathrm{red}}$ is the quotient of $R^{\mathrm{ps}}$ by the condition that the universal determinant is reducible on $G_v$ for every $v \mid 2$. On $\mathrm{Spec}R^{\mathrm{red}}$ we write it as $\det(\psi_v^{(1)} \oplus \chi\psi_v^{(1),-1})$ on $G_v$, with $\psi_v^{(1)}$ lifting 1.

Let $\Lambda = \mathcal{O}[[\prod_{v|2}(1+4\mathbb{Z}_2)]]$, a power series ring over $\mathcal{O}$ in $d$ variables. Through the restrictions of the $\psi_v^{(1)}$ to $\mathrm{Art}(1+4\mathbb{Z}_2) \subset I_v$, $R^{\mathrm{red}}$ is a $\Lambda$-algebra. Since $\chi(1+4\mathbb{Z}_2) \subset 1+\varpi\mathcal{O}$ ($\bar{\chi}$ has odd order), $[u]_v \mapsto \chi(u)[u]_v^{-1}$ defines an automorphism of $\Lambda$, and replacing $\psi_v^{(1)}$ by $\chi\psi_v^{(1),-1}$ at some places changes the $\Lambda$-structure by such an automorphism.

The point $x$ of $\rho$ lies in $\mathrm{Spec}R^{\mathrm{o}}[1/2]$, with $\psi_v(x) = \psi_v^{\mathrm{cl}}$ for $v \in \Sigma_{\mathrm{o}}$, by Lemma 2.1 and §2. By (PM), the restriction to $G_{F_1}$ maps every irreducible component of $\mathrm{Spec}R^{\mathrm{ps}}$ through $x$ into the image of $\mathrm{Spec}\mathbb{T}_{F_1}(U_1^p)$, for a fixed sufficiently small tame level $U_1^p$ over $F_1$. Over $F$ we fix a sufficiently small tame level $U^p$, to be shrunk once in §7. For a maximal ideal $\mathfrak{m}'$ of $\mathbb{T}^{\mathrm{o}} = \mathbb{T}^{\mathrm{o}}(U^p)$ at which $\psi_{v,1}$ reduces to $\bar{\psi}_v$ for $v \in \Sigma_{\mathrm{o}}$, Lemma 3.1 gives a surjection $R^{\mathrm{o}} \twoheadrightarrow \mathbb{T}^{\mathrm{o}}_{\mathfrak{m}'}$ with $\psi_v \mapsto \psi_{v,1}$; a prime of $R^{\mathrm{o}}$ is *o-pro-modular* if it lies in the image of $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}_{\mathfrak{m}'}$ for such an $\mathfrak{m}'$.

4.2. **A lower bound for the component of $\rho$.**

**Proposition 4.2.** *There is an irreducible component $C$ of $\mathrm{Spec}R^{\mathrm{o}}$ containing $x$ with $\dim C \geq 1 + 2d - |\Sigma_{\mathrm{o}}|$. It is flat over $\mathcal{O}$, $C[1/2]$ is irreducible, and its image in $\mathrm{Spec}R^{\mathrm{ps}}_{F_1}$ lies in the image of $\mathrm{Spec}\mathbb{T}_{F_1}(U_1^p)$.*

*Proof.* Let $\mathfrak{p}_x$ be the prime of $x$ and $\widehat{R}_x$ the completion of $R^{\mathrm{o}}[1/2]$ at $\mathfrak{p}_x$. As in [Pan22, Proposition 2.2.2] and [Kis09b, §2.3], whose arguments only use Mazur's finiteness condition and apply to determinants [Che14, Theorem 2.22], $\widehat{R}_x$ pro-represents the deformations of $\rho_x$ to Artinian local $E$-algebras with determinant $\chi$ which admit, for $v \in \Sigma_{\mathrm{o}}$, a $G_v$-stable direct summand of rank one lifting the line of $\psi_v^{\mathrm{cl}}$ (§2). Let $V = \mathrm{ad}^0\rho_x$. We use the relative presentation of [OAI, Proposition 3.6], which holds for $p = 2$ in both coefficient characteristics: with one global frame and fixed determinant, over the completed product of the unrestricted framed local rings at the real places and at the $d$ places above 2, the global framed ring has a presentation with $g$ variables and $r$ relations, $g - r \geq 3 - 6d - h^0(G_F, V(1)) = 3 - 6d$, since $H^0(G_F, V(1)) = 0$ ($\rho_x \not\simeq \rho_x(1)$, by determinants); the local conditions at $\Sigma_{\mathrm{o}}$ are imposed by base change. The local rings have dimension at least 2 at each real place and at least 5 at $v \in \Sigma_{\mathrm{o}}$ [OAI, Corollary 3.7], and at least 6 at $v \in \Sigma_{\mathrm{n}}$, by the local Euler characteristic formula. Forgetting the global frame, which has smooth fibres of dimension 3 since $\rho_x$ is absolutely irreducible,

$$\dim \widehat{R}_x \geq 6|\Sigma_{\mathrm{n}}| + 5|\Sigma_{\mathrm{o}}| + 2d + (3 - 6d) - 3 = 2d - |\Sigma_{\mathrm{o}}|.$$

Choose a minimal prime $\mathfrak{q} \subset \mathfrak{p}_x$ of $R^{\mathrm{o}}$ with $\dim(R^{\mathrm{o}}/\mathfrak{q})_{\mathfrak{p}_x} = \dim R^{\mathrm{o}}_{\mathfrak{p}_x} = \dim \widehat{R}_x$. As $R^{\mathrm{o}}/\mathfrak{q}$ is a complete local domain, hence catenary, and $\mathfrak{p}_x$ has coheight 1, $C = V(\mathfrak{q})$ has dimension at least $1 + 2d - |\Sigma_{\mathrm{o}}|$. It is $\mathcal{O}$-flat since it contains a point of characteristic 0. Its image in $\mathrm{Spec}R^{\mathrm{ps}}$ is irreducible and contains $x$, so it lies in an irreducible component through $x$, whose restriction to $G_{F_1}$ lies in the image of $\mathrm{Spec}\mathbb{T}_{F_1}(U_1^p)$ by (PM). □

We fix such a component $C$ and write $A_C = R^{\mathrm{o}}/\mathfrak{q}$.

4.3. **The Steinberg place.** Let $v = v_0$, $q = q_{v_0}$ and $\phi = \phi_{v_0}$. By (St), $\rho|_{I_{v_0}} \simeq \mu \otimes u$ with $\mu$ a character of $I_{v_0}$ of finite order and $u$ unipotent and non-trivial. Put

$$P_{v_0} = (1+q)^2\chi(\phi) - q\,T(\phi)^2 \in R^{\mathrm{ps}}.$$

If $a, b$ are the eigenvalues of $\rho_z(\phi)$, then $P_{v_0}(z) = (a - qb)(b - qa)$.

**Lemma 4.3.** *(1) For every closed point $z$ of $C[1/2]$, $\rho_z|_{I_{v_0}} \simeq \mu \otimes u_z$ with $u_z$ unipotent, and $P_{v_0}$ vanishes on $C$.*

*(2) Let $K/F$ be a finite totally real extension, $w$ a place of $K$ above $v_0$, and $z$ a closed point of $C[1/2]$ such that $\rho_z|_{G_K}$ is the Galois representation of a cuspidal automorphic representation $\Pi$ of $\mathrm{GL}_2(\mathbb{A}_K)$. Then $\Pi_w$ is a twist of the Steinberg representation; in particular it is square integrable, and $\rho_z|_{G_{K_w}}$ is a non-split extension.*

*Proof.* (1) For $\sigma \in I_{v_0}$, the eigenvalues of $\rho_z(\sigma)$ are roots of unity of bounded order: for $\sigma$ in the wild inertia group because $\rho_z$ maps it to a finite group of bounded order, for $\sigma$ tame because its eigenvalues are permuted by the $q$-th power map. Hence $T(\sigma) \in A_C[1/2]$ takes finitely many values on the closed points, and as $A_C[1/2]$ is a Jacobson domain, $T(\sigma)$ is constant on $C[1/2]$, equal to $T_x(\sigma)$. The same applies to $\det \rho_z(\sigma) = \chi(\sigma)$ and to $T(\sigma\sigma')$ for $\sigma, \sigma' \in I_{v_0}$, so $\rho_z|_{I_{v_0}}$ and $\rho_x|_{I_{v_0}}$ have the same semisimplification, and $\rho_z|_{I_{v_0}} \simeq \mu \otimes u_z$ with $u_z$ unipotent. Let $\sigma_0 \in I_{v_0}$ map to a topological generator of the 2-part of the tame inertia group with $\mu(\sigma_0) = 1$, after replacing $\sigma_0$ by a power. Let $Z_N \subset C[1/2]$ be the closed subset where $T(\sigma_0 g) = T(g)$ for all $g$. At an irreducible point $z$, $z \in Z_N$ if and only if $\rho_z(\sigma_0) = 1$, that is, the monodromy of $\rho_z|_{G_{v_0}}$ vanishes. Hence $x \notin Z_N$, and the irreducible points outside $Z_N$ form a dense open subset of $C[1/2]$. At such a point, $\phi N \phi^{-1} = qN$ for the monodromy operator $N \neq 0$, so $\phi$ preserves $\ker N$ and the ratio of its eigenvalues is $q^{\pm 1}$; thus $P_{v_0}(z) = 0$. As $C[1/2]$ is reduced and Jacobson and $C$ is $\mathcal{O}$-flat, $P_{v_0}$ vanishes on $C$.

(2) By (1), the inertia group $I_w$ acts through the restriction of $\mu \otimes u_z$. A Frobenius lift $\phi_w$ equals $\phi^f$ up to inertia, where $f$ is the residue degree. If the monodromy of $\rho_z|_{G_{v_0}}$ is non-zero, Frobenius and inertia preserve its kernel, and inertia multiplies the two diagonal entries by the same scalar; if it is zero, inertia acts by scalars. In either case the ratio of the eigenvalues of $\rho_z(\phi_w)$ is $q_w^{\pm 1}$. By local–global compatibility up to semisimplification [Car86, Tay89, Jar97], the semisimplified Langlands parameter of $\Pi_w$ is $\nu \oplus \nu|\cdot|^{\pm 1}$ for a character $\nu$. As $\Pi_w$ is generic, it is a twist of the Steinberg representation. Since $\Pi$ is then square integrable at a finite place, full local–global compatibility at every finite place not dividing 2 holds by Carayol's theorem [Car86, Théorème (A)], and $\rho_z|_{G_{K_w}}$ has non-zero monodromy, hence is non-split. □

**Lemma 4.4.** *Let $z$ be a closed point of $C[1/2]$ such that $\rho_z$ is absolutely irreducible and de Rham at every place above 2, with distinct Hodge–Tate weights at at least one such place. Then $\rho_z$ is not induced from a character of an index two subgroup.*

*Proof.* Suppose $\rho_z \simeq \mathrm{Ind}_{G_K}^{G_F} \theta$ for a quadratic extension $K/F$. The character $\theta$ is de Rham at the places above 2, hence is attached to an algebraic Hecke character of $K$ [Ser68, Chapter III]. If $K$ is not CM, it contains no CM subfield, the infinity type of $\theta$ is parallel, $\theta = \varepsilon^n \theta_0$ with $\theta_0$ of finite order, and the Hodge–Tate weights of $\rho_z$ are equal at every place above 2, contradicting the assumption at one such place. If $K$ is CM, then $\theta \neq \theta^c$, and $\rho_z$ is the Galois representation of the cuspidal automorphic induction $\Pi$ of $\theta$, whose local components at the places not above 2 correspond to the Weil–Deligne representations of $\rho_z$, which have zero monodromy. By Lemma 4.3(2), applied with $K = F$, $\Pi_{v_0}$ is a twist of the Steinberg representation, whose parameter has non-zero monodromy. This is a contradiction. □

4.4. **Parity.** Write $\chi = \varepsilon^k \chi_0$ with $\chi_0$ of finite order. Points with Hodge–Tate weights $\{h, h+1\}$ at every place above 2 can only exist if $k$ is odd. If $k$ is even, let $\eta_4 : G_F \to \{\pm 1\}$ be the character with $\varepsilon \eta_4^{-1}$ valued in $1 + 4\mathbb{Z}_2$, put $\langle \varepsilon \rangle = \varepsilon \eta_4^{-1}$, enlarge $E$ to contain a square root $c$ of 5 with $c \equiv 1 \pmod{\varpi}$, and let $\nu$ be the continuous character $\nu = c^{\log_5 \langle \varepsilon \rangle}$, so that $\nu^2 = \langle \varepsilon \rangle$. It is unramified outside 2, $\nu \equiv 1 \pmod{\varpi}$, and $\nu(c_\infty) = 1$ for complex conjugations $c_\infty$. Twisting by $\nu$ gives compatible isomorphisms between $R^{\mathrm{o}}$, $R^{\mathrm{red}}$, $\mathbb{T}$, $\mathbb{T}^{\mathrm{o}}$ for $\chi$ and for the totally odd character $\chi' = \chi \langle \varepsilon \rangle$, of the same residual determinant, with $k$ replaced by $k+1$. They send $\psi_v$ to $\psi_v \nu$,

$P_{v_0}$ to $\nu(\phi)^2 P_{v_0}$, preserve (PM), irreducibility, splitness at the places above 2, regularity and o-pro-modularity, and send $C$ and $x$ to a component $C'$ and a point $x'$ with the properties of Proposition 4.2 and Lemma 4.3. The point $x'$ is not de Rham, but this property of $x$ is only used in the final step (§7). Hence, in Sections 5 and 6, we may and do assume that $k$ is odd, and put $h = (-k-1)/2$.

## 5. The fully ordinary subfamily

For $v \in \Sigma_{\mathrm{n}}$ let $f_v \in A_C$ be the image of the element of Lemma 2.2 under $R_v^{\mathrm{ps}} \to R^{\mathrm{ps}} \to A_C$, where the first map classifies the restriction of the universal determinant to $G_v$. A point $z$ of $C$ lies in $V(f_v)$ if and only if $\rho_z|_{G_v}$ is reducible, and $f_v$ lies in the maximal ideal of $A_C$. By Krull's height theorem, a minimal prime $\mathfrak{D}$ of $(f_v)_{v\in\Sigma_{\mathrm{n}}}$ satisfies $\dim A_C/\mathfrak{D} \geq 1+2d-|\Sigma_{\mathrm{o}}|-|\Sigma_{\mathrm{n}}| = 1+d$. Let $D = \mathrm{Spec} A_C/\mathfrak{D}$, with its reduced structure. Its points are reducible at every place above 2, so $D \subset \mathrm{Spec} R^{\mathrm{red}}$.

**Proposition 5.1.** *$D$ is finite over $\Lambda$, $\dim D = 1+d$, and $D \to \mathrm{Spec}\Lambda$ is surjective. In particular $D$ is flat over $\mathcal{O}$.*

*Proof.* $D$ is a closed subset of $\mathrm{Spec} R^{\mathrm{red}}$, which is finite over $\Lambda$ by Theorem 8.9. As $\dim D \geq 1+d = \dim \Lambda$ and $\Lambda$ is a domain, $\dim D = 1+d$ and $D \to \mathrm{Spec}\Lambda$ is surjective. Since the special fibre of $\mathrm{Spec}\Lambda$ has dimension $d$, $D$ is not contained in the special fibre. □

For odd $p$ this is the role of [Pan22, Theorem 5.1.2(1)] in [Die26b].

**Lemma 5.2.** *For every $v \mid 2$ there is $\psi_{v,\mathrm{s}} \in \{\psi_v^{(1)}, \chi\psi_v^{(1),-1}\}$ such that the universal determinant is $\psi_{v,\mathrm{s}}$-ordinary at $G_v$ on $D$; for $v \in \Sigma_{\mathrm{o}}$ we take $\psi_{v,\mathrm{s}} = \psi_v$.*

*Proof.* As in [Die26b, Lemma 5.1]: at irreducible points one of the two characters is a subcharacter, the two conditions are closed, and the irreducible locus of $D$ is irreducible. □

**Definition 5.3.** A closed point $y$ of $D[1/2]$ is *good* if $\rho_y$ is absolutely irreducible and, for every $v \mid 2$, $\psi_{v,\mathrm{s}}(y)|_{I_v} = \varepsilon^{-h}\theta_v$ and $\chi\psi_{v,\mathrm{s}}(y)^{-1}|_{I_v} = \varepsilon^{-h-1}\theta_v'$ with $\theta_v, \theta_v'$ of finite order and $\theta_v \neq \theta_v'$.

At a good point, the Jordan–Hölder characters of $\rho_y|_{G_v}$ have Hodge–Tate numbers $h < h+1$, and $\rho_y|_{G_v}$ is an extension of $\chi\psi_{v,\mathrm{s}}(y)^{-1}$ by $\psi_{v,\mathrm{s}}(y)$ (by Lemma 5.2 and §2), hence de Rham by Lemma 2.1. Thus $\psi_{v,\mathrm{s}}(y)$ is its classical character, and by Lemma 4.4, $\rho_y$ is not induced.

**Lemma 5.4.** *Good points are Zariski dense in $D$.*

*Proof.* The restrictions of the $\psi_{v,\mathrm{s}}$ to $\mathrm{Art}(1+4\mathbb{Z}_2)$ are, up to an automorphism of $\Lambda$, the coordinates of the finite surjective map $D \to \mathrm{Spec}\Lambda$ (Proposition 5.1). The weight conditions of Definition 5.3 hold at every point of $D[1/2]$ above a point of $\mathrm{Spec}\Lambda$ at which, for every $v$, $\psi_{v,\mathrm{s}} = \varepsilon^{-h}\zeta_v$ on $\mathrm{Art}(1+4\mathbb{Z}_2)$ for a character $\zeta_v$ of finite order, provided $\zeta_v$ avoids a finite set depending on $v$. Indeed, $\psi_{v,\mathrm{s}}|_{I_v}$ is then $\varepsilon^{-h}$ times a character of finite order, and $\chi\psi_{v,\mathrm{s}}^{-2}|_{I_v}$ is $\varepsilon^{-1}$ times a character of finite order whose restriction to $\mathrm{Art}(1+4\mathbb{Z}_2)$ is $\zeta_v^{-2}$ times a fixed character; it is non-trivial unless $\zeta_v$ lies in a finite set (compare [OAI, §8.4]). The coordinate of such a point at the generator $\mathrm{Art}(5)$ of the $v$-th factor is $\varepsilon^{-h}(\mathrm{Art}(5))\zeta_v(\mathrm{Art}(5))$, with $\zeta_v(\mathrm{Art}(5)) \in \mu_{2^\infty}$ arbitrary; it lies in the open unit disc around 1, so a non-zero power series in $d$ variables cannot vanish on the product of these infinite sets, by induction on $d$ and the Weierstrass preparation theorem. Hence these points are dense in $\mathrm{Spec}\Lambda$, and their inverse image is dense in $D$, as $D \to \mathrm{Spec}\Lambda$ is finite and surjective and $D$ is irreducible of the same dimension. Finally, the reducible points form a proper closed subset of $D$. □

## 6. Good points

**Proposition 6.1.** *Let $y$ be a good point. Then:*

*(1) $\rho_y|_{G_v}$ is non-split for every $v \mid 2$;*
*(2) there is a maximal ideal $\mathfrak{P}_y$ of $\mathbb{T}^{\mathrm{ord}}_{F_1}(U_1^p)[1/2]$ above the point of $\rho_y|_{G_{F_1}}$ with $\psi_{w,1}(\mathfrak{P}_y) = \psi_{v,\mathrm{s}}(y)$ for every place $w$ of $F_1$ above $v \mid 2$;*
*(3) $\rho_y$ is the Galois representation of a regular algebraic cuspidal automorphic representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$, not of CM type, ordinary at every $v \mid 2$, with $\pi_{v_0}$ square integrable;*
*(4) $y$ is o-pro-modular, with $\psi_{v,1} = \psi_v(y)$ for $v \in \Sigma_{\mathrm{o}}$, for a sufficiently small tame level $U^p$ over $F$.*

*Proof.* (1) Put $r = \rho_y \otimes \varepsilon^h$. It is absolutely irreducible, totally odd, unramified outside finitely many places, and for $v \mid 2$, $r|_{G_v}$ is an extension of $\varepsilon^{-1}\theta'_v$ by $\theta_v$, with Hodge–Tate weights $\{0,1\}$ and $\theta_v \neq \theta'_v$. It is potentially semistable, and since the two finite inertial characters of its Weil–Deligne representation are distinct and the monodromy commutes with inertia, the monodromy vanishes: $r|_{G_v}$ is potentially crystalline and ordinary in the sense of [Tho26, §1.1], which uses the convention $\mathrm{HT}(\varepsilon) = -1$ and asks for a subcharacter of Hodge–Tate weight 0 and a quotient of Hodge–Tate weight 1 (compare [OAI, Lemma 8.5]). By Thorne's theorem [Tho26, Theorem B], there are a finite totally real extension $F''/F$ and a cuspidal regular algebraic automorphic representation $\Pi$ of $\mathrm{GL}_2(\mathbb{A}_{F''})$ with $r|_{G_{F''}} \simeq r_{\Pi,\iota}$. Then $\Pi$ has parallel weight two. By Lemma 2.4 and since $\rho_y$ is not induced, $r|_{G_{F''}}$ is not induced, so $\Pi$ is not of CM type. For every place $w \mid 2$ of $F''$, $r|_{G_{F''_w}}$ is reducible with a subcharacter of Hodge–Tate weight 0, so by local–global compatibility at 2 [Sai09, Ski09], $\Pi$ is ordinary at $w$. By Lemma 4.3(2), $\Pi$ is square integrable at the places above $v_0$. Hence the theorem of Hida and Zhao [Zhao14, Theorem 1] applies to the Hilbert eigenform of $\Pi$: $r|_{G_{F''_w}}$ is indecomposable for every $w \mid 2$. A fortiori $\rho_y|_{G_v}$ is non-split for $v \mid 2$.

(2) Since $y \in C$, the restriction of $\rho_y$ to $G_{F_1}$ defines a maximal ideal $\mathfrak{p}_y$ of $\mathbb{T}_{F_1}(U_1^p)[1/2]$ (Proposition 4.2). It is absolutely irreducible, by Lemma 4.4 and Lemma 2.4, and reducible at every place above 2. By Proposition 3.3 over $F_1$, with all places above 2 ordinary, there is $\mathfrak{P}_y$ above $\mathfrak{p}_y$ such that $\psi_{w,1}(\mathfrak{P}_y)$ is a subcharacter of $\rho_y|_{G_v}$ for $w \mid v \mid 2$. By (1), the only subcharacter is $\psi_{v,\mathrm{s}}(y)$.

(3) By (2), $\psi_{w,1}(\mathfrak{P}_y)$ is the classical character of $\rho_y|_{G_v}$ for every $w \mid v \mid 2$. Theorem 3.4 over $F_1$ with all places above 2 ordinary, in which only the case $a_1 = a_2$ of the density statement occurs, shows that $\rho_y|_{G_{F_1}}$ is modular, and so is $\rho_y$ by solvable descent. The automorphic representation $\pi$ is not of CM type since $\rho_y$ is not induced, it is ordinary at the places above 2 by [Sai09, Ski09], and $\pi_{v_0}$ is square integrable by Lemma 4.3(2).

(4) This is the argument of [Die26b, Proposition 5.4(3)], with [Wil88, Theorem 2] replaced by [Sai09, Ski09]. For $v \in \Sigma_{\mathrm{o}}$, the Hodge–Tate numbers of $\rho_y|_{G_v}$ are distinct, so $\pi_v$ has a unique ordinary 2-stabilization, whose diamond character is a subcharacter of $\rho_y|_{G_v}$ by local–global compatibility at 2, with the smaller Hodge–Tate number; it is $\psi_v(y)$. As $d$ is even, the Jacquet–Langlands correspondence gives automorphic forms on $B^\times$ of some tame level $U^p$, and their ordinary stabilization at the places of $\Sigma_{\mathrm{o}}$ gives, by the finite-level description of $S^{\mathrm{o}}$ and (C1), an eigenvector in $S^{\mathrm{o}} \otimes E$ with Hecke eigenvalues $\mathrm{tr}\rho_y(\mathrm{Frob}_w)$ and diamond characters $\psi_v(y)$, $v \in \Sigma_{\mathrm{o}}$. □

**Proposition 6.2.** *Let $y$ be a good point and $\mathfrak{p}_y$ its prime in $R^{\mathrm{o}}$. Then $R^{\mathrm{o}}_{\mathfrak{p}_y}$ is regular of dimension $2d - |\Sigma_{\mathrm{o}}|$. Consequently $C$ is the unique irreducible component of $\mathrm{Spec}R^{\mathrm{o}}$ containing $y$, and $\dim C = 1 + 2d - |\Sigma_{\mathrm{o}}|$.*

*Proof.* We follow [Die26b, Proposition 5.5]. Let $V = \mathrm{ad}^0\rho_y$. As there, $\dim R^{\mathrm{o}}_{\mathfrak{p}_y} \geq 2d - |\Sigma_{\mathrm{o}}|$, and the tangent space $t_y$ is contained in the Selmer group $H^1_{\mathcal{L}}(F, V)$ with $\mathcal{L}_v$ the image of $H^1(G_v, \mathfrak{b}^0_v)$

for $v \in \Sigma_{\mathrm{o}}$ and $\mathcal{L}_v = H^1(G_v, V)$ for the other finite places of $\Sigma$; the identification of $t_y$ with deformations of $\rho_y$ uses [Che14, Theorem 2.22] and §2.

*Local terms at* 2. Let $v \mid 2$, and write $\rho_y|_{G_v}$ as an extension of $\psi'$ by $\psi$, with $\psi'\psi^{-1} = \varepsilon^{-1}\mu_v$, $\mu_v = \theta'_v\theta_v^{-1}$ non-trivial on $I_v$. The Jordan–Hölder factors of $V(1)$ are $\varepsilon$, $\varepsilon^2\mu_v^{-1}$ and $\mu_v$, all non-trivial, so $H^0(G_v, V(1)) = 0$. The argument of loc. cit. applies with $m = 1$: $V/\mathfrak{b}_v^0 \simeq \varepsilon^{-1}\mu_v$ has no invariants, the Jordan–Hölder factors $\varepsilon$ and $\mu_v$ of $(\mathfrak{b}_v^0)^\vee(1)$ are non-trivial, $\dim \mathcal{L}_v - h^0(G_v, V) = 2$ for $v \in \Sigma_{\mathrm{o}}$ and $h^1(G_v, V) - h^0(G_v, V) = 3$ for $v \in \Sigma_{\mathrm{n}}$, and $H^1_f(G_v, V) \subset \mathcal{L}_v$ since $H^1_g(G_v, \varepsilon^{-1}\mu_v) = 0$ [BK90, Corollary 3.8.4].

*Local terms away from* 2. Let $v \in \Sigma$ be finite, $v \nmid 2$. By Proposition 6.1(3) and Carayol's theorem [Car86, Théorème (A)], the Weil–Deligne representation of $\rho_y|_{G_v}$ corresponds to the generic representation $\pi_v$. If $\pi_v$ is supercuspidal, $\rho_y|_{G_v}$ is irreducible and $\rho_y|_{G_v} \not\simeq \rho_y|_{G_v}(1)$, since $\varepsilon|_{G_v}$ has infinite order; if $\pi_v$ is an irreducible principal series, the Jordan–Hölder characters of $\rho_y|_{G_v}$ do not have ratio $\varepsilon^{\pm 1}$; if $\pi_v$ is a twist of the Steinberg representation, $\rho_y|_{G_v}$ is a non-split extension of $\nu$ by $\nu\varepsilon$ and Lemma 2.3 applies. In all cases $H^0(G_v, V(1)) = 0$, so $h^1(G_v, V) - h^0(G_v, V) = h^2(G_v, V) = 0$.

*Global estimate.* As in loc. cit., the Greenberg–Wiles formula [DDT97, Theorem 2.19], valid for $p = 2$ with coefficients of characteristic 0, with $h^0(G_v, V) = 1$ at the $d$ real places, gives $h^1_{\mathcal{L}}(F, V) - h^1_{\mathcal{L}^\perp}(F, V(1)) = 2d - |\Sigma_{\mathrm{o}}|$ and $h^1_f(F, V) = h^1_f(F, V(1))$, and $H^1_{\mathcal{L}^\perp}(F, V(1)) \subset H^1_f(F, V(1))$. By [NT23, Theorem B], which holds for every prime, including $p = 2$, $H^1_f(F, \mathrm{ad}\rho_y) = 0$, as $\pi$ is regular algebraic and not of CM type. Hence $\dim t_y \leq 2d - |\Sigma_{\mathrm{o}}|$, and the conclusion follows as in loc. cit. □

## 7. Proof of Theorem B

We work with the data of §4.4, twisted if necessary. By Lemma 5.4 there is a good point $y \in D \subset C$, and by Proposition 6.1(4) it is the image of a prime of $\mathbb{T}^{\mathrm{o}}_{\mathfrak{m}'}$. Let $W$ be an irreducible component of $\mathrm{Spec}\mathbb{T}^{\mathrm{o}}_{\mathfrak{m}'}$ containing it, viewed in $\mathrm{Spec}R^{\mathrm{o}}$. As in [Die26b, §7], $W = V(\mathfrak{p}_W)$ for the inverse image $\mathfrak{p}_W$ of a minimal prime of $\mathbb{T}^{\mathrm{o}}_{\mathfrak{m}'}$, and a minimal prime $\mathfrak{q}' \subset \mathfrak{p}_W$ of $R^{\mathrm{o}}$ satisfies $\mathfrak{q}' \subset \mathfrak{p}_y$; by Proposition 6.2, $\mathfrak{q}' = \mathfrak{q}$, so $W \subset C$. By Propositions 3.5 and 6.2, $\dim W \geq 1 + 2d - |\Sigma_{\mathrm{o}}| = \dim C$, so $W = C$.

In particular $x \in W$ (after untwisting by $\nu$ if $k$ was even): there is a maximal ideal $\mathfrak{P}$ of $\mathbb{T}^{\mathrm{o}}_{\mathfrak{m}'}[1/2]$ above $x$ with $\psi_{v,1}(\mathfrak{P}) = \psi_v(x) = \psi_v^{\mathrm{cl}}$ for $v \in \Sigma_{\mathrm{o}}$. Let $\mathfrak{p}$ be its image in $\mathbb{T}[1/2]$; then $\rho(\mathfrak{p}) \simeq \rho$, and Theorem 3.4 applies by (N) and (O). Hence $\rho$ is modular over the field of §2.2, and over the original field by solvable descent. □

## 8. Pro-modularity at 2

In this section $F$ is the original field of §1.1: totally real, abelian over $\mathbb{Q}$, with 2 totally split. We keep (R), write $\alpha = \bar{\chi}$, a character of $G_F$ of odd order which extends to a character $\tilde{\alpha}$ of $G_{\mathbb{Q}}$, and assume, after the twist of §2.2, that $\bar{\rho}^{ss} = 1 \oplus \alpha$. Then $\tilde{\alpha}$ is unramified at 2, and $\alpha|_{G_v} = \tilde{\alpha}|_{G_{\mathbb{Q}_2}} \neq 1$ for every $v \mid 2$. Let $k_0 = \mathbb{F}_2(\alpha)$ and $q = |k_0| \geq 4$, and let $K_\alpha$ be the field cut out by $\alpha$, of odd degree over $F$. As in [OAI, §7.2], we first fix the seed $f$ of item (4) in the proof of Theorem 8.4, and enlarge $\Sigma$ by the places above the primes dividing its level. This is harmless for (PM): for the smaller set, $\mathrm{Spec}R^{\mathrm{ps}}_K$ is a closed subscheme of the one for the larger set, and its components through the point of $\rho|_{G_K}$ lie in components of the latter through that point. Let $e = [F : \mathbb{Q}]$, let $\Sigma_0$ be the set of finite places of $\Sigma$ not above 2, and for a number field $E$ and $\beta \in \{\alpha, \alpha^{-1}\}$ we use the Selmer dimension formula [OAI, (27)]; since $\beta$ has odd order it is trivial at the real places, and the Tate dual of $k_0(\beta)$ is $k_0(\beta^{-1})$.

8.1. **An auxiliary set of places.** By Chebotarev's theorem in the compositum of the Hilbert class field of $F$ and $K_\alpha$, which are linearly disjoint over $F$ in their 2-parts, we choose a finite set $P_0$ of places of $F$, not above 2 and not in $\Sigma_0$, with $\alpha(\mathrm{Frob}_v) \neq 1$ for $v \in P_0$, whose classes generate the 2-part of the class group. Let $S' = \Sigma_2 \cup P_0$ and $\mathcal{O}_{S'}^\times$ the group of $S'$-units. The $S'$-class group of $F$ has odd order. Hence for every place $v \notin S'$ we may fix $x_v \in F^\times$ with $(x_v) = v^{h_v}\mathfrak{a}_v$, where $h_v$ is odd and $\mathfrak{a}_v$ is supported on $S'$.

**Lemma 8.1.** *Let $n \geq 1$ and let $w \notin S' \cup \Sigma_0$ be a place of $F$ of degree one, which splits completely in $F(\mu_{2^{n+1}}, (\mathcal{O}_{S'}^\times)^{1/2^n})$. Then there is a cyclic extension $L_w/F$ of degree $2^n$, totally ramified at $w$, unramified at all other finite places, in which the real places and the places of $S'$ split completely. For a place $v \notin S' \cup \{w\}$, $\mathrm{Frob}_v$ generates $\mathrm{Gal}(L_w/F)$ if $x_v$ is not a square modulo $w$, and lies in $\mathrm{Gal}(L_w/F)^2$ if $x_v$ is a square modulo $w$.*

*Proof.* Let $\mathrm{Cl}_{w,S'}$ be the ray class group of modulus $w$ (without conditions at infinity) in which the places of $S'$ are trivial. There is an exact sequence

$$(\mathcal{O}_F/w)^\times/\mathrm{im}(\mathcal{O}_{S'}^\times) \to \mathrm{Cl}_{w,S'} \to \mathrm{Cl}_{S'} \to 1.$$

As $\mathrm{Cl}_{S'}$ has odd order, the 2-part of $\mathrm{Cl}_{w,S'}$ is the image of the 2-part of $A_w = (\mathcal{O}_F/w)^\times/\mathrm{im}(\mathcal{O}_{S'}^\times)$. The group $(\mathcal{O}_F/w)^\times$ is cyclic of order $\mathrm{N}(w) - 1$, divisible by $2^{n+1}$, and the image of $\mathcal{O}_{S'}^\times$ consists of $2^n$-th powers by the splitting assumption; hence the 2-part of $A_w$ is cyclic of order at least $2^n$. A character of order $2^n$ of it defines $L_w$ by class field theory: $w$ is totally ramified, since $(\mathcal{O}_F/w)^\times$ is the inertia group, the other finite places are unramified, the real places are unramified, hence split, and the places of $S'$ split. For $v \notin S' \cup \{w\}$, the class of $v^{h_v}$ is the image of $x_v$ in $A_w$, since $\mathfrak{a}_v$ is trivial in $\mathrm{Cl}_{w,S'}$. As $h_v$ is odd and the 2-part of $A_w$ is cyclic, the image of $\mathrm{Frob}_v$ in $\mathrm{Gal}(L_w/F)/\mathrm{Gal}(L_w/F)^2$ is non-trivial if and only if $x_v$ is not a square in $(\mathcal{O}_F/w)^\times$. □

We write $\gamma_w : \mathrm{Gal}(L_w/F) \xrightarrow{\sim} \mathbb{Z}/2^n\mathbb{Z}$, $\kappa_w = \gamma_w \bmod 2$ for the quadratic character, and $\delta_w$ for the ramified quadratic character of $G_{F_w}$ given by the restriction of $\kappa_w$.

**Lemma 8.2.** *Fix finitely many classes in $H^1(G_F, k_0(\beta))$, $\beta \in \{\alpha, \alpha^{-1}\}$, and a finite Galois extension $E'/F$ of 2-power degree. Then the Frobenius evaluations at places $w$ of degree one of $F$ which split completely in $K_\alpha E'$ satisfy the conclusions of* [OAI, Lemma 6.4(1) and (3)] *with $B$ replaced by $F$: for each fixed $\beta$, after choosing Frobenius representatives, their values on a $k_0$-linearly independent family of classes can be prescribed arbitrarily, and simultaneous prescriptions respect the compatibility between Frobenius-conjugate coefficient characters of loc. cit. There are infinitely many such places $w$.*

*Proof.* Put $N = G_{K_\alpha E'}$. As $K_\alpha/F$ has odd degree and $E'/F$ is Galois of 2-power degree, $\mathrm{Gal}(K_\alpha E'/F) = C \times P$, with $C$ of odd order acting non-trivially on $k_0(\beta)$ and $P$ a 2-group. So restriction to $N$ is injective, and the classes become independent homomorphisms on $N$, with image stable under $\beta(C)$, which generates $k_0$. The argument of [OAI, Lemma 6.4] then gives the surjectivity of the evaluations and the compatibility between coefficient isotypes, and Chebotarev's theorem realizes the corresponding conjugacy classes; the places of degree greater than one form a set of density zero and may be discarded. □

8.2. **The fields.**

**Proposition 8.3.** *There is a constant $U$, depending only on $F$, $\alpha$, $\Sigma_0$ and $P_0$, with the following property. For all integers $n > a \geq 1$ there are cyclic extensions $F \subset F_a \subset F_n$ of degrees $2^a$ and $2^n$, totally real, in which the places above 2 split completely, ramified exactly at a set $W$ of $t \leq U$ places of degree one, each totally ramified in $F_n$, and in which every place of $\Sigma_0$ has a unique place above it. Let $S_n$ be the set of places of $F_n$ above $\infty$, 2, $\Sigma_0$ and $W$, and $u_n \leq |\Sigma_0| + t$ the number of its non-dyadic finite places. Then for $\beta \in \{\alpha, \alpha^{-1}\}$:*

(1) $\{c \in H^1(G_{F_n,S_n}, k_0(\beta)) : \mathrm{loc}_v(c) = 0 \text{ for all } v \in S_n\} = 0$;
(2) $\dim H^1(G_{F_n,S_n}, k_0(\beta)) \le [F_n : \mathbb{Q}] + u_n$;
(3) restriction $H^1(G_{F_a,S_a}, k_0(\beta)) \to H^1(G_{F_n,S_n}, k_0(\beta))$ is injective, and $\alpha|_{G_{F_n}} \neq 1$, $\alpha|_{G_{F_n,w}} \neq 1$ for $w \mid 2$.

*Proof.* We follow the case $B = \mathbb{Q}$ of [OAI, §§6.4–6.5]. Given places $w_1, \dots, w_t$ as in Lemma 8.1, let $F_n$ be the field of $\gamma = \sum_j c_j \gamma_{w_j}$ with all $c_j$ odd, and $F_a$ its layer of degree $2^a$. For $\beta \in \{\alpha, \alpha^{-1}\}$ let $X^\beta$ be the Selmer group over $F$ with coefficients $k_0(\beta)$ and the conditions: zero at the real places and at the places above 2; the line $k_0\delta_{w_j}$ at $w_j$; unramified elsewhere. Let $Y^\beta$ be the Selmer group with coefficients $k_0(\beta^{-1})$ and the orthogonal conditions. By the Selmer dimension formula,

$$\dim X^\beta - \dim Y^\beta = -e,$$

since a real zero condition contributes $-1$, a dyadic zero condition contributes 0 because $\alpha|_{G_v} \neq 1$, a line $k_0\delta_{w_j}$ at a place where the coefficient is trivial contributes $1-1=0$, and the global invariants vanish. In particular $Y^\beta \neq 0$.

We choose the $w_j$ inductively. Each $w_j$ is of degree one, not in $S' \cup \Sigma_0 \cup \{w_1, \dots, w_{j-1}\}$, with $\mathrm{N}(w_j) \equiv 1 \pmod 4$, and splits completely in $K_\alpha$ and in $F(\mu_{2^{n+1}}, (\mathcal{O}_{S'}^\times)^{1/2^n})$. Moreover:

(i) if $j = 1$, $x_v$ is not a square modulo $w_1$ for every $v \in \Sigma_0$; if $j \ge 2$, $x_v$ is a square modulo $w_j$ for every $v \in \Sigma_0$;
(ii) for $i < j$, $x_{w_i}$ is a square modulo $w_j$, and $w_j$ splits in the quadratic subfield of $L_{w_i}$;
(iii) if $j \ge 2$, the Frobenius evaluation at $w_j$ is non-zero on every non-zero $X^\beta$ and on every $Y^\beta$, defined with the conductors $w_1, \dots, w_{j-1}$.

As for the first conductor in [OAI, §6.3], no evaluation condition is imposed at $w_1$. These conditions are compatible. Put $M_n = F(\mu_{2^{n+1}}, (\mathcal{O}_{S'}^\times)^{1/2^n})$, a finite 2-extension of $F$, Galois over $F$. For $j = 1$ we need a place of degree one which splits completely in $K_\alpha M_n$ and is inert in every $F(\sqrt{x_v})$, $v \in \Sigma_0$. Since $v \notin S'$ is odd, $M_n$ is unramified at $v$, and so is $F(\sqrt{x_{v'}})$ for $v' \in \Sigma_0 \setminus \{v\}$, as $x_{v'}$ is a unit at $v$; but $F(\sqrt{x_v})$ is ramified at $v$, since $x_v$ has odd valuation there. Hence $\mathrm{Gal}(M_n \prod_{v\in\Sigma_0} F(\sqrt{x_v})/F) = \mathrm{Gal}(M_n/F) \times (\mathbb{Z}/2)^{\Sigma_0}$, and, $K_\alpha$ having odd degree, Chebotarev's theorem gives infinitely many such $w_1$. For $j \ge 2$, conditions (i) and (ii) and the conditions preceding them say that $w_j$ splits completely in $K_\alpha E'_j$, where $E'_j$ is the compositum of $M_n$, the fields $F(\sqrt{x_v})$ for $v \in \Sigma_0$, and the fields $F(\sqrt{x_{w_i}})$ and the quadratic subfields of the $L_{w_i}$ for $i < j$. Thus $E'_j/F$ is a finite Galois 2-extension, and the odd-degree field $K_\alpha$ is kept separate. Lemma 8.2 with $E' = E'_j$ gives (iii) at infinitely many such places: for each coefficient isotype there are at most two required non-vanishings, on $X^\beta$ and on $Y^{\beta^{-1}}$, each failing with probability at most $1/q$, so the failure probability is at most $2/q < 1$; different isotypes are prescribed independently, and Frobenius-conjugate inverse characters impose the compatible conditions of [OAI, Lemma 6.4(3)]. Hence such $w_j$ exist.

Adding $w_j$ replaces the unramified condition at $w_j$ by the transverse ramified line $k_0\delta_{w_j}$; the conditions at the earlier conductors are unchanged, by (ii), since the new character $\gamma_{w_j}$ is unramified with even Frobenius value at $w_i$ (Lemma 8.1). For $j \ge 2$, by (iii) and the switching lemma [OAI, Lemma 6.3], which holds over any number field (the coefficient is trivial at $w_j$, which splits in $K_\alpha$, and $\mathrm{N}(w_j) \equiv 1 \pmod 4$), the switch introduces no new classes and strictly decreases every non-zero $X^\beta$. Let $C_0 = \sum_\beta \dim H^1(G_{F,\Sigma_0\cup\Sigma_2\cup\infty}, k_0(\beta))$. After the first step, each $X^\beta$ has dimension at most $\dim H^1(G_{F,\Sigma_0\cup\Sigma_2\cup\infty}, k_0(\beta)) + 1$, since the classes unramified at $w_1$ have codimension at most $\dim k_0\delta_{w_1} = 1$; so $\sum_\beta \dim X^\beta \le C_0 + 2$. This sum decreases at each step $j \ge 2$ while it is non-zero, and after $t \le C_0 + 3$ steps both $X^\beta$ vanish. This bound is independent of $n$ and $a$. We stop there.

We check the properties. The places of $S'$, in particular those above 2, and the real places split in every $L_{w_j}$, hence in $F_n$. At $v \in \Sigma_0$, $\gamma(\mathrm{Frob}_v) = c_1\gamma_{w_1}(\mathrm{Frob}_v) + \sum_{j\geq 2} c_j\gamma_{w_j}(\mathrm{Frob}_v)$ is odd by (i) and Lemma 8.1, so $v$ is inert in $F_n$. At $w_j$, the restriction of $\gamma$ to $G_{F_{w_j}}$ is $c_j\gamma_{w_j}$, totally ramified of degree $2^n$, plus a sum of unramified characters with even Frobenius values, by (ii); so $w_j$ is totally ramified in $F_n$, and the quadratic character of the local extension is $\delta_{w_j}$. Hence $u_n \leq |\Sigma_0| + t$.

For (1), let $Z$ be the group in (1). It is stable under the cyclic 2-group $\mathrm{Gal}(F_n/F)$; if it were non-zero it would have a non-zero invariant class, which descends to a class over $F$ since $k_0(\beta)^{G_{F_n}} = 0$, by inflation–restriction. As in [OAI, §6.4], the descended class lies in $X^\beta$: at the real places and the places above 2, which split, the local condition descends unchanged; at $v \in \Sigma_0$ and outside $S_n$ the extension $F_n/F$ is unramified, so the descended class is unramified; at $w_j$ the kernel of restriction to the totally ramified cyclic local extension is $k_0\delta_{w_j}$. As $X^\beta = 0$, $Z = 0$. For (2), the dual of the Selmer group with full conditions at $S_n$ is the group of (1) with coefficient $\beta^{-1}$, which vanishes; the Selmer dimension formula then gives the bound, a real place contributing 0, a dyadic place 1 (since $\alpha|_{G_v} \neq 1$), and a non-dyadic place of $S_n$ at most $h^2 \leq 1$, exactly as in [OAI, §6.5]. For (3), $\alpha$ has odd order and $F_n/F$ is a 2-extension, so $\alpha$ stays non-trivial on $G_{F_n}$ and on $G_{F_{n,w}} = G_{F_v}$ for $w \mid v \mid 2$, and inflation–restriction gives the injectivity. Finally $U = |\Sigma_0| + 3 + C_0$ works. □

### 8.3. **Proof of (PM).**

**Theorem 8.4.** *Assume* (R)*, that $\rho$ is de Rham with distinct Hodge–Tate weights at some place above* 2 *and not induced, and that* $\det\rho = \varepsilon^w\chi_0$ *with $\chi_0$ of finite order. Then* (PM) *holds.*

*Proof.* By Lemma 2.4, $\rho$ is not virtually solvable, and remains absolutely irreducible over every finite extension. We follow the proof of [OAI, Theorem 7.1] in the unmarked case, which is the case $\alpha|_{G_{\mathbb{Q}_2}} \neq 1$ of loc. cit., with $\mathbb{Q} \subset A \subset F$ replaced by $F \subset F_a \subset F_n$ of Proposition 8.3, and with the following changes.

(1) *Leopoldt's conjecture.* In [OAI, Lemma 3.8] and in the proof of [OAI, Lemma 7.3], abelian Leopoldt is used to bound by 1 the dimension of the family of diagonal characters with fixed product. The fields $F_a$, $F_n$ are not abelian over $\mathbb{Q}$ in general. Over a totally real field $K$ of degree $d_K$ we use instead the bound $1 + \delta_K \leq (d_K + 1)/2$ of §2.2. Thus the reducible locus in the characteristic two pseudo-deformation space has dimension at most $(d_K+1)/2$, and the reducible locus of Lemma 7.3 of loc. cit. over $F_n$ has dimension at most $d_n + u_n + 3 + (d_n - 1)/2$, where $d_n = [F_n : \mathbb{Q}] = 2^n e$: the three parameters of loc. cit. for the diagonal base, the global line and the coboundary become $3 + \delta_{F_n}$, and the extension classes are bounded by Proposition 8.3(2). The dihedral bound $d_K$ of [OAI, Lemma 3.8] does not use Leopoldt's conjecture. These are the only uses of Leopoldt's conjecture in [OAI].

(2) *Numerical conditions.* In place of [OAI, (39)] we choose $a$ and $n$ so large that $d_n > 2u_n + 10$ and $d_a > 2(U + C) + 10$, where $C$ is the constant cost of the trace cuts in [OAI, §7.4]. Then the reducible locus over $F_n$ has dimension less than $2d_n - 1$, the bound needed in Lemma 7.3 of loc. cit. (it suffices that $d_n > 2u_n + 7$). In the proof of its Lemma 7.4 the locus in which a curve is chosen has dimension at least $2d_n - 4 - u_n > d_n$, and in the proof of its Theorem 7.1 the loci over $F_a$ have dimension larger than $d_a$ after the trace cuts; since $d_K \geq (d_K + 1)/2$, these are larger than both the dihedral and the reducible bounds, so the weaker reducible bound changes nothing there.

(3) *Fields and Selmer bounds.* Proposition 8.3 replaces [OAI, Proposition 6.1]. Property (2) is the bound [OAI, (26)] in the unmarked case, and property (3) gives the residual data used in [OAI, Lemma 7.5 and §7.4]. There are no marks, so every point of the residual

fibre is transverse, and in [OAI, §7.4] any stable lattice of the seed and target curves gives a point of the residual scheme over $F_n$.

(4) *The seed.* By [OAI, Lemma 7.2], applied over $\mathbb{Q}$ to $1 \oplus \tilde{\alpha}$, there is a cuspidal eigenform $f$ whose residual representation is $1 \oplus \tilde{\alpha}$. Its level is supported on $\Sigma$, by the choice of $\Sigma$ at the beginning of this section. Its base change to $F_a$, through the tower $\mathbb{Q} \subset F \subset F_a$ of abelian and cyclic steps [AC89], is cuspidal by regularity, as in [OAI, §7.4], and has residual representation $\bar{\rho}^{ss}|_{G_{F_a}}$. This replaces the seed over $A$.

(5) *The target.* The target locus over $F_a$ is an irreducible component through the point of $\rho|_{G_{F_a}}$, of the unrestricted framed deformation space of the residual representations considered there, as in loc. cit.

The remaining arguments of [OAI, §7], together with the propagation theorem [OAI, Theorem 5.1], apply verbatim: they are formulated over totally real solvable fields of even degree in which 2 splits completely, and $F_a$, $F_n$ are such fields. They are not Galois over $\mathbb{Q}$ in general, but their Galois closures are totally real, solvable, and split completely at 2, since $F$ is Galois over $\mathbb{Q}$ and all places of $F$ above 2 split in $F_n$; this is all that [OAI, Lemma 4.6 and Theorem 5.1] use, and the extension $F_n/F_a$ is cyclic. Every irreducible component of $\mathrm{Spec}R^{\mathrm{ps}}_{F_a}$ through the point of $\rho|_{G_{F_a}}$ is the pseudo-image of an irreducible component of the framed deformation space through that point, since the point is absolutely irreducible; there are finitely many of them. Applied to each of them as the target locus, these arguments show that its image under restriction becomes pro-modular over a further totally real solvable extension, of even degree and split completely at 2, at one tame level. Taking the compositum of the Galois closures over $\mathbb{Q}$ of these finitely many extensions and a common deeper tame level as in loc. cit., we obtain (PM) with $K = F_a$, which is cyclic over $F$, and this compositum as $F_1$. The field $K$ is in general not Galois over $\mathbb{Q}$, and it cannot be replaced by its Galois closure $K'$, whose pseudo-deformation space through $\rho|_{G_{K'}}$ has components of larger dimension which are not restrictions from $K$; this is why (PM) only asks $K/F$ to be Galois. □

8.4. **Fully reducible loci.** The argument of Theorem 8.4 applies to many more loci than the components through the point of $\rho$.

**Theorem 8.5.** *Let $K = F_a$, with a sufficiently large, be as in the proof of Theorem 8.4. There is a totally real field $F_1 \supset K$, Galois and solvable over $\mathbb{Q}$, of even degree, in which 2 splits completely, and a tame level $U_1^p$, such that the image under restriction of every irreducible closed subset of $\mathrm{Spec}R^{\mathrm{ps}}_K$ whose generic point is absolutely irreducible lies in the image of $\mathrm{Spec}\mathbb{T}_{F_1}(U_1^p)$.*

*Proof.* Let $d = [K : \mathbb{Q}]$. Let $\mathcal{J}$ be the finite set of irreducible components of $\mathrm{Spec}R^{\mathrm{ps}}_K$ whose generic point is absolutely irreducible and whose dimension is at least $2d - 3$. Each $\widehat{C} \in \mathcal{J}$ can be used as the target locus in the proof of Theorem 8.4: by the choice of $a$, after the trace cuts of [OAI, §7.4] it still has dimension larger than the bounds for the dihedral and reducible loci, so it contains a non-virtually-solvable curve with finite local images at the allowed non-dyadic places. Hence its image becomes pro-modular after a further extension, and taking the compositum of these finitely many extensions and a common tame level gives $F_1$ and $U_1^p$ for all $\widehat{C} \in \mathcal{J}$.

Let $Z$ be an irreducible closed subset of $\mathrm{Spec}R^{\mathrm{ps}}_K$ with absolutely irreducible generic point. We claim that $Z$ is contained in some $\widehat{C} \in \mathcal{J}$. Let $\mathcal{C} \subset Z$ be an irreducible curve whose generic representation $\rho_{\mathcal{C}}$ is absolutely irreducible; such curves are dense in $Z$, since the reducible locus is a proper closed subset. Over the normalization of $\mathcal{C}$, a discrete valuation ring, choose a stable lattice whose reduction $\bar{\rho}_b$ is a non-split extension [SW99, Lemma 2.13]; this defines a point of the framed representation scheme with fixed determinant, in the completion of this scheme at the closed point $y$ defined by $\bar{\rho}_b$ (after a finite extension of the residue field). By [OAI, Proposition 3.6] over the real local base, which has dimension $2d$ [OAI, Lemma 3.12], every irreducible component

of this completion has dimension at least $2d+3-h^0((\mathrm{ad}^0\bar\rho_b)^\vee(1)) \geq 2d$. Let $Y_i$ be one containing the point of $\mathcal{C}$. Its generic representation is absolutely irreducible, and its pseudo-image $P_i$, the closure of its image in $\mathrm{Spec}R_K^{\mathrm{ps}}$, has dimension at least $2d-3$, since forgetting the frame has fibres of dimension 3 on the absolutely irreducible locus [OAI, Lemma 3.2]. Then $\mathcal{C} \subset P_i$, and $P_i$ is contained in an irreducible component of $\mathrm{Spec}R_K^{\mathrm{ps}}$, which belongs to $\mathcal{J}$. Hence the dense set of such curves in $Z$ is contained in the closed union of the elements of $\mathcal{J}$, and so is $Z$; as $Z$ is irreducible, it lies in one of them. □

In particular, Theorem 8.5 contains (PM) for the components through the point of $\rho$.

**Proposition 8.6.** *Let $K$ and $F_1$ be as in Theorem 8.5, and let $Z$ be an irreducible closed subset of $\mathrm{Spec}R_K^{\mathrm{red}}$ which is flat over $\mathcal{O}$. Then $Z$ is finite over $\Lambda_K$.*

*Proof.* Let $Z_1$ be the image of $Z$ in $\mathrm{Spec}R_{F_1}^{\mathrm{red}}$ under restriction; $Z$ is finite over $Z_1$ [OAI, Proposition 3.1], and $Z_1$ is irreducible, closed and $\mathcal{O}$-flat. On $Z_1$, the $\Lambda_{F_1}$-structure factors through the surjection $\Lambda_{F_1} \to \Lambda_K$ sending the factor of a place $w$ of $F_1$ to that of the place of $K$ below it, since $G_{F_{1,w}} = G_{K_v}$. It therefore suffices to show that $Z_1$ is finite over $\Lambda_{F_1}$.

If the generic point of $Z_1$ is reducible as a determinant of $G_{F_1}$, then $Z_1$ lies in the locus of determinants $\det(\psi \oplus \chi\psi^{-1})$, which is the closed image of a quotient of $\mathcal{O}[[G_{F_1,\Sigma}^{\mathrm{ab}}(2)]]$, with $\Lambda_{F_1}$ acting through the restrictions of $\psi$ to the inertia groups at the places above 2. By class field theory, the quotient of $G_{F_1,\Sigma}^{\mathrm{ab}}(2)$ by the closed subgroup generated by the images of $1+4\mathbb{Z}_2$ in these inertia groups is finite: it is a quotient of the pro-2 part of a ray class group whose conductor away from 2 is supported on the finitely many places of $\Sigma$, extended by finite groups. Hence this locus is finite over $\Lambda_{F_1}$.

Otherwise, the generic point of $Z$ is absolutely irreducible, so $Z_1$ lies in the image of $\mathrm{Spec}\mathbb{T}_{F_1}(U_1^p)$ by Theorem 8.5, and the closed points $z$ of $Z_1[1/2]$ with $\rho_z$ absolutely irreducible are dense in $Z_1$, as $Z_1$ is $\mathcal{O}$-flat. Each such $z$ defines a maximal ideal of $\mathbb{T}_{F_1}[1/2]$, reducible at every place above 2. By Proposition 3.3 over $F_1$, with all places above 2 ordinary, $z$ lies in the image of $\mathrm{Spec}\mathbb{T}_{F_1,\mathfrak{m}''}^{\mathrm{ord}}$ for some maximal ideal $\mathfrak{m}''$ of $\mathbb{T}_{F_1}^{\mathrm{ord}}$, which is a closed subset $Y_{\mathfrak{m}''}$ of $\mathrm{Spec}R_{F_1}^{\mathrm{red}}$: indeed $R_{F_1}^{\mathrm{red}} \to \mathbb{T}_{F_1,\mathfrak{m}''}^{\mathrm{ord}}$ is surjective, sending $\psi_w^{(1)}$ or $\chi\psi_w^{(1),-1}$ to $\psi_{w,1}$ according to the reduction of $\psi_{w,1}$, by Lemma 3.1. There are finitely many $\mathfrak{m}''$, so $Z_1 \subset Y_{\mathfrak{m}''}$ for one of them. Finally $\mathbb{T}_{F_1,\mathfrak{m}''}^{\mathrm{ord}}$ is finite over $\Lambda_{F_1}$, acting through the diamond operators: it acts faithfully on the dual of $\mathrm{Ord}_B(S_{F_1})$, which is finitely generated over $\Lambda_{F_1}$ by (O2) of [Die26a] and (C1). The two $\Lambda_{F_1}$-structures on $Y_{\mathfrak{m}''}$, through the $\psi_w^{(1)}$ and through the $\psi_{w,1}$, differ by an automorphism of $\Lambda_{F_1}$ (§4). Hence $Z_1$ is finite over $\Lambda_{F_1}$. □

8.5. **Ordinary parts at arbitrary primes.** We now remove the flatness assumption in Proposition 8.6, by an analogue of Proposition 3.3 for arbitrary primes of the Hecke algebra, including primes of characteristic 2, which are reducible at every place above 2. It uses the structure of the generic blocks of $\mathrm{GL}_2(\mathbb{Q}_2)$.

Fix a place $v \mid 2$ of a field $F'$ as in (C1)–(C3). The block $\mathcal{B}_v$ of $1 \oplus \alpha|_{G_v}$ is generic, since $\alpha|_{G_v} \neq 1$ and the mod 2 cyclotomic character is trivial: $\mathcal{B}_v = \{\pi_1, \pi_2\}$ with $\pi_i = (\mathrm{Ind}_{\bar B}\chi_i)^{\mathrm{sm}}$ irreducible principal series for two distinct characters $\chi_1, \chi_2$ of $T(\mathbb{Q}_2)$, and it contains no character [PT21, §4.1, type (ii)]. Let $J_i$ be an injective envelope of $\pi_i$ in the category of locally admissible torsion representations with the fixed central character, $J_v = J_1 \oplus J_2$, $P_v = J_v^\vee$, $E_v = \mathrm{End}(P_v)$, $e_1, e_2 \in E_v$ the idempotents of the two summands, and $Z_v$ the centre of $E_v$.

We distinguish the block and the global normalizations. The block deformation ring $R_v^{\mathrm{blk}}$ has determinant $\psi\varepsilon = \chi\varepsilon^2$, whereas $R_v^{\mathrm{ps}}$ has determinant $\chi$; cyclotomic twisting identifies the two deformation problems, with $t_v^{\mathrm{blk}} = \varepsilon t_v$, and we use this identification whenever the centre acts through the global Hecke algebra ((C2)). It preserves reducibility; we write $f_v$ for the image of the generator of Lemma 2.2 in either normalization, and $R_v^{\mathrm{red}} = R_v^{\mathrm{ps}}/(f_v)$.

We use the following inputs.

(P1) $Z_v \simeq R_v^{\mathrm{blk}}$ integrally, hence $Z_v \simeq R_v^{\mathrm{ps}}$ after untwisting [Paš16, Theorems 1.3 and 2.23]; the result is stated there for all $p$, in particular for $p = 2$, for generic blocks $\chi_1\chi_2^{-1} \neq 1, \omega^{\pm 1}$.

(P2) $E_v$ is a free $Z_v$-module of rank 4, $E_v = Z_v e_1 \oplus Z_v \tilde{\Phi}_{12} \oplus Z_v \tilde{\Phi}_{21} \oplus Z_v e_2$, with $e_i E_v e_i = Z_v e_i$, $\tilde{\Phi}_{12}\tilde{\Phi}_{21} = ce_1$ and $\tilde{\Phi}_{21}\tilde{\Phi}_{12} = ce_2$, where $(c) = (f_v)$ [Paš16, Theorem 2.23]. The unnormalized Colmez module satisfies $g^2 - t_v^{\mathrm{blk}}(g)g + \chi(g)\varepsilon(g)^2 = 0$, so the normalized module $\check{\mathbf{V}}_\chi(P_v) = \check{\mathbf{V}}(P_v) \otimes \varepsilon^{-1}$ satisfies $g^2 - t_v(g)g + \chi(g) = 0$ [Paš16, Theorem 2.23 and proof of Corollary 2.25]. For principal series we likewise multiply the Colmez character by $\varepsilon^{-1}$.

(P3) $\mathrm{Ord}_B(J_i)$ is an injective envelope $J_{\chi_i}$ of $\chi_i$ in the category of locally admissible representations of $T(\mathbb{Q}_2)$ with the fixed central character [PT21, Lemma 4.4]. Its endomorphism ring $\Lambda_i$ is the universal deformation ring of $\chi_i$, isomorphic to $\mathcal{O}[[\mathbb{Q}_2^\times(2)]]$, and $\mathrm{Hom}(J_{\chi_1}, J_{\chi_2}) = 0$ since $\chi_1 \neq \chi_2$.

**Lemma 8.7.** *The ring $E_v$ acts on $\mathrm{Ord}_B(J_v) = J_{\chi_1} \oplus J_{\chi_2}$ through a surjection $E_v \to \Lambda_1 \times \Lambda_2$, whose kernel $I_v$ contains $e_1E_ve_2$, $e_2E_ve_1$ and $f_v$, and satisfies $I_v \cap e_iE_ve_i = f_vZ_ve_i$. In particular $I_v^2 \subset f_vE_v$. If $N$ is a left (resp. right) $E_v$-module killed by $f_v$ with $N = I_vN$ (resp. $N = NI_v$), then $N = 0$.*

*Proof.* By functoriality $E_v$ acts on $\mathrm{Ord}_B(J_v)$, and the off-diagonal summands act by 0, by (P3). Let $K_i$ be the kernel of $Z_v \to \Lambda_i$. For a unitary character $\vartheta$ of $T(\mathbb{Q}_2)$ lifting $\chi_i$, with the fixed central character, let $\gamma$ be the normalized Colmez character of $(\mathrm{Ind}_{\bar{B}}\vartheta)_{\mathrm{cont}}$. As $\gamma(g) \neq 0$ is a root of the normalized relation of (P2), $t_v(g) \mapsto \gamma(g) + \chi(g)\gamma(g)^{-1}$. By Emerton's adjunction this specialization is the action of the centre on the corresponding specialization of $\Lambda_i$; these points are dense in $\mathrm{Spec}\Lambda_i$, which is reduced and $\mathcal{O}$-flat. Hence the centre acts on $\Lambda_i$ through $t_v \mapsto \gamma_i^{\mathrm{univ}} + \chi(\gamma_i^{\mathrm{univ}})^{-1}$, where $\gamma_i^{\mathrm{univ}}$ is the universal character lifting the appropriate one of the two distinct residual constituents. The induced map $R_v^{\mathrm{red}} \to \Lambda_i$ is an isomorphism, since both rings represent the deformations of that character, the other one being determined by the fixed product $\chi$ (§2). Thus $K_i = (f_v)$, in either ordering, and $E_v \to \Lambda_1 \times \Lambda_2$ is surjective. Modulo $f_v$, $I_v$ consists of the off-diagonal summands, whose products vanish by (P2); hence $I_v^2 \subset f_vE_v$, and iterating $N = I_vN$ or $N = NI_v$ gives $N = 0$. □

**Proposition 8.8.** *Let $F'$ be a totally real field of even degree in which 2 splits completely, $\mathbb{T} = \mathbb{T}_{F'}(U^p)_{\mathfrak{m}}$ and $\mathbb{T}^{\mathrm{ord}}$ as in §3, in the generic blocks above. Let $\mathfrak{q}$ be a prime of $\mathbb{T}$, of any characteristic, such that the determinant modulo $\mathfrak{q}$ is reducible on every decomposition group above 2. Then $\mathfrak{q}$ lies in the image of $\mathrm{Spec}\mathbb{T}^{\mathrm{ord}} \to \mathrm{Spec}\mathbb{T}$.*

*Proof.* Put $S_{\mathrm{tor}} = S \otimes_{\mathcal{O}} E/\mathcal{O}$ and $M = S_{\mathrm{tor}}^\vee$; we work in the smooth torsion category and its compact dual. Let $\mathcal{E} = \widehat{\bigotimes}_{v|2} E_v$, $P = \widehat{\bigotimes}_{v|2} P_v$, $J = P^\vee$ and $\mathbf{m} = \mathrm{Hom}(P, M)$, a right $\mathcal{E}$-module by precomposition. By (C2), $\mathbf{m}$ is faithful and finite over $\mathbb{T}$ and finite over the noetherian local central base; as $\mathcal{E}$ is finite over that base, $\mathbf{m}$ is finitely presented over $\mathcal{E}$. The compact block equivalence gives $M \simeq \mathbf{m}\widehat{\otimes}_{\mathcal{E}} P$ and $S_{\mathrm{tor}} \simeq \mathrm{Hom}_{\mathcal{E}}^{\mathrm{cts}}(\mathbf{m}, J)$. Applying the additive left exact functor $\mathrm{Ord}_B$ to a finite presentation of $\mathbf{m}$ gives $\mathrm{Ord}_B(S_{\mathrm{tor}}) \simeq \mathrm{Hom}_{\mathcal{E}}^{\mathrm{cts}}(\mathbf{m}, \mathrm{Ord}_B J)$.

Put $Q = (\mathrm{Ord}_B J)^\vee$. The socle of the product injective $J$ consists of the tensor products of the local principal series constituents, each with multiplicity one; since parabolic induction is exact, $\mathrm{Ord}_B J$ is injective, and by adjunction its torus socle consists of the corresponding tuples of local torus characters, each with multiplicity one. The proof of [PT21, Lemma 4.4], applied to the product category, gives

$$Q \simeq \bigoplus_{(i_v) \in \{1,2\}^{\{v|2\}}} \widehat{\bigotimes}_{v|2} \Lambda_{v,i_v} \simeq \mathcal{E}/I,$$

where $I$ is the two-sided ideal generated by the $I_v$, the $E_v$-actions being identified by naturality and Lemma 8.7. Dualizing the presentation,

$$X := (\mathrm{Ord}_B S_{\mathrm{tor}})^\vee \simeq \mathbf{m}\widehat{\otimes}_{\mathcal{E}} Q \simeq \mathbf{m}/\mathbf{m}I,$$

and by the finite-level description of §3 and the commutation of ordinary parts with filtered torsion limits, $X$ is the compact ordinary module defining $\mathbb{T}^{\mathrm{ord}}$.

The module $X$ is finite over $\mathbb{T}$, and $\mathbb{T}^{\mathrm{ord}}$ acts faithfully and $\mathbb{T}$-linearly on it, so $\mathbb{T}^{\mathrm{ord}} \subset \mathrm{End}_{\mathbb{T}}(X)$ is finite over the image of $\mathbb{T}$, and the image of $\mathrm{Spec}\mathbb{T}^{\mathrm{ord}}$ in $\mathrm{Spec}\mathbb{T}$ is $\mathrm{Supp}_{\mathbb{T}} X$. Let $N = \mathbf{m} \otimes_{\mathbb{T}} k(\mathfrak{q})$, a right $\mathcal{E}$-module, non-zero since $\mathbf{m}$ is faithful and finite over $\mathbb{T}$, and killed by every $f_v$, since $f_v$ acts through $\mathbb{T}$ and maps into $\mathfrak{q}$. By Lemma 8.7 each $I_v^2$ kills $N$; as the local factors commute, $NI^{s+1} = 0$, where $s$ is the number of places above 2. Hence $N = NI$ would give $N = 0$, and $X \otimes_{\mathbb{T}} k(\mathfrak{q}) \simeq N/NI \neq 0$. Thus $\mathfrak{q} \in \mathrm{Supp}_{\mathbb{T}} X$. □

**Theorem 8.9.** *Let $K$ and $F_1$ be as in Theorem 8.5. Then $R_K^{\mathrm{red}}$ is finite over $\Lambda_K$.*

*Proof.* It suffices to show that every irreducible component $Z$ of $\mathrm{Spec}R_K^{\mathrm{red}}$ is finite over $\Lambda_K$. If its generic point is reducible as a determinant of $G_K$, this is the second paragraph of the proof of Proposition 8.6, applied over $K$. Otherwise, by Theorem 8.5, the image $Z_1$ of $Z$ in $\mathrm{Spec}R_{F_1}^{\mathrm{red}}$ lies in the image of $\mathrm{Spec}\mathbb{T}_{F_1}(U_1^p)$. Its generic point defines a prime $\mathfrak{q}$ of $\mathbb{T}_{F_1}(U_1^p)$ which is reducible at every place above 2, so $\mathfrak{q}$ lies in the image of $\mathrm{Spec}\mathbb{T}_{F_1}^{\mathrm{ord}}$ by Proposition 8.8. Hence $Z_1 \subset Y_{\mathfrak{m}''}$ for a maximal ideal $\mathfrak{m}''$ of $\mathbb{T}_{F_1}^{\mathrm{ord}}$, and the last paragraph of the proof of Proposition 8.6 shows that $Z_1$, hence $Z$, is finite over $\Lambda_K$. □

In particular condition (OF) holds for the field $K$ used in Sections 4–7, and Proposition 5.1 holds unconditionally.

## 9. Removing the dyadic condition over real quadratic fields

In Theorem A, the condition $\bar{\chi}|_{G_v} \neq 1$ of (R) enters only through the pro-modularity input (PM): the arguments of Section 3 work at points of characteristic 0 and do not use it (Proposition 9.7). Over $\mathbb{Q}$, the construction of [OAI, §6] covers the cases $\bar{\chi}|_{G_{\mathbb{Q}_2}} = 1$ by means of an auxiliary real quadratic field $B$. When $F$ itself can play the role of $B$, the method of [OAI] applies to representations of $G_F$, and we obtain a version of Theorem A without this condition.

Throughout this section, $q$ is a prime with $q \equiv 1 \pmod 8$, $F = \mathbb{Q}(\sqrt{q})$, $w_1, w_2$ are the two places of $F$ above 2, and $w_q$ is the place above $q$. Then $F$ is real quadratic, abelian over $\mathbb{Q}$, and 2 splits in $F$, so $F$ is as in §1.1.

### 9.1. **The field.**

**Lemma 9.1.** *(1) The narrow class number $h^+$ of $F$ is odd, and every totally positive unit of $F$ is a square.*

*(2) There is a totally positive $\pi \in F$ with $(\pi) = w_1^{h^+}$. Its class in $F_{w_2}^\times/(F_{w_2}^\times)^2$ does not depend on the choice of $\pi$, and the condition*

*(S) $\pi$ is not a square in $F_{w_2} = \mathbb{Q}_2$*

*does not depend on the numbering of $w_1, w_2$.*

*(3) For $q = 17$, condition* (S) *holds.*

*Proof.* (1) The quadratic extension $F/\mathbb{Q}$ is ramified at exactly one finite prime, $q$, and $\mathbb{Q}$ has narrow class number one. By [OAI, Lemma 6.2], $h^+$ is odd and every totally positive unit of $F$ is a square.

(2) The ideal $w_1^{h^+}$ is trivial in the narrow class group, so it has a totally positive generator $\pi$. Two such generators differ by a totally positive unit, which is a square by (1). Let $\sigma$ be the non-trivial automorphism of $F$. Then $\sigma(\pi)$ is a totally positive generator of $w_2^{h^+}$, and $\sigma$ extends

to an isomorphism $F_{w_2} \to F_{w_1}$ sending $\pi$ to $\sigma(\pi)$; hence $\pi$ is a square in $F_{w_2}$ if and only if $\sigma(\pi)$ is a square in $F_{w_1}$.

(3) By [OAI, §6.1], for $q = 17$ we have $h^+ = 1$, and $\pi = (5 + \sqrt{17})/2$ is a totally positive generator of $w_1$. It is a root of $X^2 - 5X + 2$, and its image in $F_{w_2} = \mathbb{Q}_2$ is the unit root of this polynomial, which is congruent to 7 modulo 8. A unit of $\mathbb{Z}_2$ is a square if and only if it is congruent to 1 modulo 8. □

*Remark* 9.2. Suppose that $w_1 = (\pi_0)$ with $\pi_0$ totally positive, so that $\pi = \pi_0^{h^+}$ and $\pi_0$ have the same squareclass. Then $\pi_0$ is a root of $X^2 - tX + 2$, with $t = \mathrm{Tr}(\pi_0)$, and its image $u$ in $F_{w_2} = \mathbb{Q}_2$ is the unit root, so $u + 2u^{-1} = t$. Modulo 8, $u \equiv 1, 3, 5, 7$ gives $t \equiv 3, 1, 7, 5$ respectively. Hence (S) holds if and only if $t \not\equiv 3 \pmod 8$. For example $\pi_0 = (5 + \sqrt{17})/2$, $(7 + \sqrt{41})/2$, $(9 + \sqrt{73})/2$ give (S) for $q = 17, 41, 73$, while $\pi_0 = (11 + \sqrt{113})/2$ shows that (S) fails for $q = 113$.

**Lemma 9.3.** *Let $\bar\chi : G_F \to \mathbb{F}^\times$ be a character which extends to a character of $G_\mathbb{Q}$. Then the extension $\tilde\chi$ is unique, it has odd order, it is unramified at 2, and $\bar\chi|_{G_{w_i}} = \tilde\chi|_{G_{\mathbb{Q}_2}}$ for $i = 1, 2$. If $\bar\chi$ is unramified at $w_q$, then $\tilde\chi$ is unramified at $q$ and $\tilde\chi(\mathrm{Frob}_q) = \bar\chi(\mathrm{Frob}_{w_q})$.*

*Proof.* The group $\mathbb{F}^\times$ has odd order. Two extensions of $\bar\chi$ differ by a character of $\mathrm{Gal}(F/\mathbb{Q})$, which has order dividing 2 and values in $\mathbb{F}^\times$, hence is trivial. The extension $\tilde\chi$ has odd order, and its restriction to $G_{\mathbb{Q}_2}$ factors through the maximal abelian quotient; by local class field theory its restriction to inertia factors through $\mathbb{Z}_2^\times$, a pro-2 group, hence is trivial. As $F_{w_i} = \mathbb{Q}_2$, the restrictions agree. If $\bar\chi$ is unramified at $w_q$, then $\tilde\chi(I_q)$ is a group of odd order containing $\bar\chi(I_{w_q}) = 1$ as the image of a subgroup of index at most 2, so $\tilde\chi(I_q) = 1$. Since $F_{w_q}/\mathbb{Q}_q$ is totally ramified, $G_{F_{w_q}}$ surjects onto $G_{\mathbb{Q}_q}/I_q$, and $\tilde\chi(\mathrm{Frob}_q) = \bar\chi(\mathrm{Frob}_{w_q})$. □

### 9.2. **Statement.**

**Theorem 9.4.** *Let $\rho : G_F \to \mathrm{GL}_2(E)$ be as in §1.1 and assume* (N) *and* (O). *Assume that $\bar\rho^{ss} = \bar\chi_1 \oplus \bar\chi_2$, that $\bar\chi = \bar\chi_1\bar\chi_2^{-1}$ extends to a character $\tilde\chi$ of $G_\mathbb{Q}$, and that $\rho_v$ is non-split for every $v \in \Sigma_o$. Assume moreover one of the following:*

*(a) $\bar\chi|_{G_{w_1}} \neq 1$;*
*(b) $\bar\chi = 1$, and $F$ satisfies* (S)*;*
*(c) $\bar\chi \neq 1$, $\rho$ is unramified at $w_q$, and $\bar\chi(\mathrm{Frob}_{w_q}) \neq 1$.*

*Then $\rho$ is modular.*

By Lemma 9.3, $\bar\chi|_{G_{w_1}} = \bar\chi|_{G_{w_2}} = \tilde\chi|_{G_{\mathbb{Q}_2}}$, so (a) is condition (R), and case (a) is Theorem A. Among the representations satisfying the first assumptions, the cases not covered are $\bar\chi = 1$ when $F$ does not satisfy (S), and $\bar\chi \neq 1$ with $\tilde\chi|_{G_{\mathbb{Q}_2}} = 1$ when $\rho$ is ramified at $w_q$ or $\bar\chi(\mathrm{Frob}_{w_q}) = 1$. For $F = \mathbb{Q}(\sqrt{17})$, (S) holds by Lemma 9.1(3), so only the second case remains; in particular the theorem applies whenever $\bar\rho^{ss}$ is scalar, for instance trivial.

**Corollary 9.5.** *Let $\rho$ be as in §1.1, with $\bar\rho^{ss} = \bar\chi_1 \oplus \bar\chi_2$, and assume that $\bar\chi = \bar\chi_1\bar\chi_2^{-1}$ extends to a character $\tilde\chi$ of $G_\mathbb{Q}$ and is unramified at every finite place of $F$ not above 2; for instance, $\rho$ is unramified outside the places above 2. Then $\bar\chi = 1$, so $\bar\rho^{ss}$ is scalar, and if moreover $\det\bar\rho = 1$, then $\bar\rho^{ss}$ is trivial. Consequently, if $F$ satisfies* (S)*, for instance $F = \mathbb{Q}(\sqrt{17})$, and $\rho$ satisfies* (N), (O) *and is non-split at every $v \in \Sigma_o$, then $\rho$ is modular.*

*Proof.* Let $\ell$ be an odd prime. If $\ell \neq q$, then $F/\mathbb{Q}$ is unramified at $\ell$, so $\tilde\chi$ is unramified at $\ell$, since $\bar\chi$ is unramified at the places above $\ell$. If $\ell = q$, then $\tilde\chi$ is unramified at $q$ by Lemma 9.3. Hence $\tilde\chi$ is ramified at most at 2, so it factors through $\mathrm{Gal}(\mathbb{Q}(\mu_{2^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_2^\times$ and has 2-power order. As it takes values in $\mathbb{F}^\times$, which has odd order, $\tilde\chi = 1$, and $\bar\chi = 1$. If $\det\bar\rho = \bar\chi_1\bar\chi_2 = 1$, then $\bar\chi = \bar\chi_1^2 = 1$, so $\bar\chi_1 = 1$, since squaring is injective on $\mathbb{F}^\times$, and $\bar\chi_2 = 1$. The last assertion is case (b) of Theorem 9.4. □

The first part of the argument, that $\bar\chi = 1$, does not use that $F$ is quadratic. Over any field $F$ as in §1.1, it holds whenever $\bar\chi$ extends to $G_{\mathbb{Q}}$ and the extension is unramified at all odd primes.

9.3. **The arguments of Section 3 without the dyadic condition.**

**Lemma 9.6.** *Let $\Pi$ be an absolutely irreducible admissible unitary Banach representation of $\mathrm{GL}_2(\mathbb{Q}_2)$ with a central character, in some block, on which the centre of the block acts through a point whose pseudo-character $\psi_1 + \psi_2$ is reducible. Then $\Pi$ is either a character $\eta \circ \det$ or a quotient of a unitary principal series $(\mathrm{Ind}_{\bar B}\mu)_{\mathrm{cont}}$.*

*Proof.* Following [CDP14], $\Pi$ is *ordinary* if it is a subquotient of a unitary parabolic induction. If $\Pi$ is not ordinary, then by [CDP14, Theorem 1.1], which holds for every prime $p$, its Colmez representation is absolutely irreducible of dimension 2. By [PT21, Theorem 1.4], after inverting 2 the centre of the block is identified with the fixed-determinant pseudo-deformation ring, and it acts on $\Pi$ through the parameter of this Colmez representation, with the cyclotomic normalization of (C2); this characteristic-zero comparison applies to every block at $p = 2$, and no integral isomorphism is needed. The resulting parameter is irreducible, a contradiction.

Hence $\Pi$ is a subquotient of $I = (\mathrm{Ind}_{\bar B}\mu)_{\mathrm{cont}}$ for a unitary character $\mu$ of $T(\mathbb{Q}_2)$. The reduction modulo $\varpi$ of the unit ball of continuous functions is the smooth induction $\mathrm{Ind}_{\bar B}\bar\mu$. By Barthel–Livné [BL94], this has length at most 2, and when it is reducible its constituents are a character and a twist of the Steinberg representation. A closed $\mathrm{GL}_2(\mathbb{Q}_2)$-stable subspace $I_1 \subset I$ gives an exact sequence of reductions of unit balls, with non-zero terms if $I_1 \neq 0, I$; hence $I$ has length at most 2.

If $I$ is irreducible, $\Pi = I$. Otherwise there is an exact sequence $0 \to I_1 \to I \to I_2 \to 0$ with $I_1, I_2$ irreducible, and $\Pi \in \{I_1, I_2\}$. By Emerton's adjunction [Eme10a, Theorem 4.4.6], in its form for admissible unitary Banach representations ((O4) of [Die26a, §3.3]), $\mathrm{Hom}(I, \eta \circ \det) = \mathrm{Hom}_{T(\mathbb{Q}_2)}(\mu, \mathrm{Ord}_B(\eta \circ \det))$. Here $\mathrm{Ord}_B(\eta \circ \det) = 0$: on each $\mathcal{O}/\varpi^n$-lattice of $\eta \circ \det$, the Hecke operator of $t = \mathrm{diag}(2, 1)$ is $\sum_{n \in N_0/tN_0t^{-1}} nt$, which acts by $2\eta(2)$ and is not invertible. Hence $I_2$ is not a character. An admissible Banach representation whose unit ball has finite-dimensional reduction is finite-dimensional of the same dimension, and a one-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_2)$ factors through det. So the reduction of $I_2$ is not one-dimensional, and therefore that of $I_1$ is; then $I_1$ is a character. In both cases $\Pi$ is a character or a quotient of $I$. □

**Proposition 9.7.** *Let $F'$ be a totally real field of even degree in which 2 splits completely, and define $S$, $\mathbb{T}$, $S^{\mathrm{o}}$, $\mathbb{T}^{\mathrm{o}}$ over $F'$ as in Section 3, with $\mathfrak{m}$ the maximal ideal attached to a semisimple residual representation $\bar r'$ of $G_{F'}$, either of the form $1 \oplus \bar\chi'$ for a character $\bar\chi'$ of odd order, possibly trivial, or absolutely irreducible. Then:*

1. *(C1), (C2), (C3), the description of the absolutely irreducible objects with reducible parameter in §3.1, Lemmas 3.1 and 3.2, and Theorem 3.4 hold;*
2. *Proposition 3.3 holds for every maximal ideal $\mathfrak{p}$ of $\mathbb{T}[1/2]$ such that $\rho(\mathfrak{p})$ is irreducible and, for every $v \in \Sigma_{\mathrm{o}}$, $\rho(\mathfrak{p})|_{G_v}$ is reducible with distinct Jordan–Hölder characters.*

*Proof.* (1) Propositions 4.1, 4.2 and 4.4 of [OAI] are proved for every residual factor of the Hecke algebra; their block inputs, from [PT21], apply to every block, including the scalar residual block at 2 [OAI, proof of Proposition 4.2]. The description of the absolutely irreducible objects with reducible parameter is Lemma 9.6. The proof of Lemma 3.1 takes place at finite level and in characteristic 0, and uses no condition on $\bar r'$. The proof of Lemma 3.2 only uses (C2) and Emerton's functor. The proof of Theorem 3.4 uses Lemma 3.2, (C1), (C2), the correspondence of [CDP14] at the irreducible local parameters, the product structure of the blocks [OAI, proof of Proposition 4.2], and the density statement in its proof, none of which involves $\bar r'$. (For $a_1 > a_2$ the density statement is [BE10, Proposition 2.2.1] with $k = a_1 - a_2 + 2 \geq 3$; the case $k = 2$ excluded there is the case $a_1 = a_2$, treated directly.)

(2) In the proof of Proposition 3.3, (C2) gives $\Pi' = \widehat{\bigotimes}_v \Pi'_v$ in $(S \otimes E)[\mathfrak{p}]$, with $\Pi'_v$ absolutely irreducible. For $v \in \Sigma_o$, Lemma 9.6 and strong approximation [Die26a, Lemma 3.2] show that $\Pi'_v$ is a quotient of a unitary principal series. Emerton's adjunction then gives $\mathfrak{P}$. Modulo $\mathfrak{P}$, the determinant of Lemma 3.1 is $\psi_{v,1}(\mathfrak{P})$-ordinary at $G_v$ and equal to $\det(\psi_{v,1}(\mathfrak{P}) \oplus \chi\psi_{v,1}(\mathfrak{P})^{-1})$ there. These two characters are the Jordan–Hölder characters of $\rho(\mathfrak{p})|_{G_v}$, hence distinct, so some $\sigma_0 \in G_v$ separates them. The characteristic-free form of [Die26a, Lemma 2.3] recalled in §2, applied over the field $k(\mathfrak{p})$, shows that $\psi_{v,1}(\mathfrak{P})$ is a subcharacter of $\rho(\mathfrak{p})|_{G_v}$. □

### 9.4. **Pro-modularity.**

**Proposition 9.8.** *Let $\rho : G_F \to \mathrm{GL}_2(E)$ be as in §1.1, de Rham with distinct Hodge–Tate weights at some place above 2, not induced, with $\det\rho = \varepsilon^w\chi_0$ for $\chi_0$ of finite order, and with $\bar\rho^{ss} = \bar\chi_1 \oplus \bar\chi_2$, where $\bar\chi = \bar\chi_1\bar\chi_2^{-1}$ extends to $G_\mathbb{Q}$. Assume (b) or (c) of Theorem 9.4. Then* (PM) *holds. Moreover, if $\bar\chi|_{G_{w_1}} = 1$ (equivalently, at both places above 2), $K$ may be chosen abelian over $\mathbb{Q}$.*

*Proof.* If $\bar\chi|_{G_{w_1}} \neq 1$, then (R) holds by Lemma 9.3, and (PM) is Theorem 8.4; the assertion about $K$ concerns only the remaining case. We assume from now on that $\bar\chi|_{G_{w_1}} = 1$, so that $\tilde\chi|_{G_{\mathbb{Q}_2}} = 1$.

*Step 0: normalization.* Twisting by the Teichmüller lift of $\bar\chi_2^{-1}$, we may assume that $\bar\rho^{ss} = 1 \oplus \bar\chi$. In case (c), $\rho$ is unramified at $w_q$, so $\bar\chi_2$ is unramified there and the twist keeps $\rho$ unramified at $w_q$. The assumptions are otherwise unchanged. Let $\alpha = \tilde\chi$ be the character of Lemma 9.3, and put $\bar r = 1 \oplus \alpha$, a semisimple representation of $G_\mathbb{Q}$ with $\bar r|_{G_F} = \bar\rho^{ss}$. Then $\alpha$ is unramified at 2; in case (b), $\alpha = 1$; in the remaining part of case (c), $\alpha \neq 1$, $\alpha|_{G_{\mathbb{Q}_2}} = 1$, $\alpha$ is unramified at $q$, and $\alpha(\mathrm{Frob}_q) \neq 1$, by Lemma 9.3. By Lemma 2.4, $\rho|_{G_{F'}}$ is absolutely irreducible and not virtually solvable for every finite extension $F'/F$. Put $\chi = \det\rho$.

*Step 1: the seed and the allowed primes.* By [OAI, Lemma 7.2] there is a cuspidal eigenform $f$ of level $N\ell$ whose residual representation is $\bar r$. In its proof, $N$ is any odd integer $\geq 5$ divisible by the conductor of the Teichmüller lift of $\alpha$, and $\ell$ is any prime congruent to 1 modulo $N2^a$ for $a$ large. In case (c), the conductor of $\alpha$ is prime to $q$, so we take $N$ prime to $q$ and $\ell \neq q$; then the level of $f$ is prime to $q$. Let $\Sigma_\mathbb{Q}$ be the set of odd primes which divide $N\ell$, or lie below a finite place of $F$ at which $\rho$ is ramified. Then $\bar r$ is unramified outside $\Sigma_\mathbb{Q} \cup \{2\}$, and in case (c), $q \notin \Sigma_\mathbb{Q}$, since $w_q$ is the only place of $F$ above $q$. We enlarge $\Sigma$ to the set of places of $F$ above $\Sigma_\mathbb{Q} \cup \{2, \infty\}$; as observed at the beginning of Section 8, this is harmless for (PM).

*Step 2: the fields.* We apply [OAI, Proposition 6.1] to $\bar r$ and $\Sigma_\mathbb{Q}$, and denote its fields by $\mathbb{Q} \subset A \subset A'$. Its proof in the reducible cases with $\alpha|_{G_{\mathbb{Q}_2}} = 1$ uses an auxiliary real quadratic field $B$, with $A = BL_a$ and $A' = BL$ for cyclic 2-power subfields $L_a \subset L$ of cyclotomic fields [OAI, §§6.1, 6.4]. We show that one can take $B = F$ in both cases; then $F \subset A \subset A'$, and $A$, $A'$ are totally real, abelian over $\mathbb{Q}$, of even degree, with 2 split completely.

*Case (b).* In [OAI, §6.1], $B = \mathbb{Q}(\sqrt{17})$. The proof uses only the following properties of $B$:

- 2 splits in $B$;
- $B$ has odd narrow class number;
- there is a totally positive $\pi \in B$ generating a power $w_1^h$, $h$ odd, of one dyadic place, which is not a square at the other dyadic place;
- the condition that a conductor prime $r$ is inert in $B$ is compatible with the conditions (28) of loc. cit. and with the non-square conditions at the primes of $\Sigma_\mathbb{Q}$.

For $B = F$, the first three properties hold by Lemma 9.1 and (S), with $h = h^+$. For the fourth, inertness of $r$ in $F$ means that $q$ is a non-square modulo $r$, and the argument of loc. cit. applies verbatim: the squareclasses of distinct odd primes are independent, and all quadratic subfields of $E_n = \mathbb{Q}(\zeta_{2^{n+1}}, 2^{1/2^n})$ are unramified at odd primes. If $q \in \Sigma_\mathbb{Q}$, the condition repeats an existing one.

With these properties, the remaining argument of loc. cit. applies verbatim. Each quadratic step of $FL/F$ is ramified at exactly one finite prime, the prime above $r$. Hence, by [OAI, Lemma 6.2], $A'$ has odd narrow class number and its totally positive units are squares. The Kummer argument with the valuation-parity map then gives [OAI, (25)] for $\beta = 1$, with $\pi$ in place of $(5+\sqrt{17})/2$; its last step uses that $\pi$ is not a square at the places above $w_2$, whose completions are $F_{w_2} = \mathbb{Q}_2$. The restriction assertion holds as in loc. cit.

Every $\ell \in \Sigma_{\mathbb{Q}}$ is a non-square modulo $r$, so its Frobenius generates $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{Gal}(FL/F)$ (as $F \cap L = \mathbb{Q}$). A place of $F$ above $\ell$ has residue degree 1 or 2, so it has at most 2 places above it in $A'$. Hence $u_A, u_{A'} \le 2|\Sigma_{\mathbb{Q}}|$, independently of the degrees.

*Case (c).* In [OAI, §§6.3–6.5], $B = \mathbb{Q}(\sqrt{q_0})$ for a prime $q_0 \notin \Sigma_{\mathbb{Q}} \cup \{2\}$ with $q_0 \equiv 1 \pmod 8$ and $\alpha(\mathrm{Frob}_{q_0}) \neq 1$. Such a $q_0$ is chosen by Chebotarev's theorem, but the construction uses only these three properties, and they hold for $q_0 = q$ by Step 0 and Step 1. In particular, the vanishing of the local cohomology at $q_0$ in the proof of [OAI, Lemma 6.5] uses that $\alpha$ is unramified at $q_0$ with $\alpha(\mathrm{Frob}_{q_0}) \neq 1$, and the independence $B \cap E_2 = \mathbb{Q}$ in [OAI, §6.3] uses that $q_0 \notin \Sigma_{\mathbb{Q}}$. We take $q_0 = q$, so $B = F$.

In both cases, [OAI, Proposition 6.1] holds for $A \subset A'$, with the marked set $M$ of loc. cit. This gives the Selmer vanishing [OAI, (25)], the bounds [OAI, (26)] in the case $\alpha|_{G_{\mathbb{Q}_2}} = 1$, the restriction injectivity, and a bound $U$ for $u_A, u_{A'}$ independent of the degrees, so that the inequalities [OAI, (36)–(39)] can be achieved.

*Step 3: the target loci.* The point of $\rho|_{G_A}$ lies on $\mathrm{Spec} R_A^{\mathrm{ps}}$, the pseudo-deformation space of $\bar{r}|_{G_A}$ with determinant $\chi|_{G_A}$ for $G_{A,S_A}$, where $S_A$ is the set of places above $\Sigma_{\mathbb{Q}} \cup \{2, \infty\}$. Let $C$ be an irreducible component of $\mathrm{Spec} R_A^{\mathrm{ps}}$ through this point. Its absolutely irreducible locus is open, and it is non-empty, since it contains the point of $\rho|_{G_A}$. As in the proof of Theorem 8.5, $C$ is the pseudo-image of an irreducible component of the fixed-determinant framed representation scheme through a point above that of $\rho|_{G_A}$. By [OAI, Proposition 3.6 and Lemmas 3.12 and 3.2], $\dim C \ge 2[A:\mathbb{Q}] - 2$. This is the dimension bound for the target locus in [OAI, §7.4].

*Step 4: the argument of* [OAI, §7]. We follow [OAI, §§7.2–7.4] with the fields $A \subset A'$, the seed $f$, the determinant $\chi|_{G_A}$, the allowed set $S_A$ and the target locus $C$, and check that the target over $\mathbb{Q}$ enters there only through these data.

(i) [OAI, §7.2]: the allowed set contains the bad primes of the target and of the seed, by Step 1. The degree conditions [OAI, (39)] only involve $U$.

(ii) [OAI, Lemmas 7.3–7.5] concern the fixed-determinant framed representation scheme over $A'$ with residual representation $\bar{r}|_{G_{A'}}$ and the marks. They use:
- the relative presentation over the real local rings, which needs only that $\chi$ is totally odd;
- the bounds (25) and (26);
- Leopoldt's conjecture for $A'$, which holds since $A'$ is abelian over $\mathbb{Q}$;
- the propagation theorem [OAI, Theorem 5.1], which applies to the determinant $\chi|_{G_{A'}} = \varepsilon^w \chi_0|_{G_{A'}}$.

None of these refers to a representation of $G_{\mathbb{Q}}$.

(iii) [OAI, §7.4], seed locus: the base change of $f$ to the abelian field $A$ is cuspidal by regularity, and its residual representation is $\bar{r}|_{G_A}$. Its central character agrees with $\chi\varepsilon$ modulo $\varpi$, since both residual determinants equal $\alpha|_{G_A}$. Hence the argument of loc. cit. gives a pro-modular component $C_{\mathrm{seed}}$ of dimension at least $1 + 2[A:\mathbb{Q}]$.

(iv) [OAI, §7.4], target locus: we use $C$ in place of the component through $r|_{G_A}$. The trace cuts at the allowed non-dyadic places and, in the marked case, at the distinguished place $v_0$ cost at most $1+u_A$ plus a constant independent of the degrees. Since $\dim C \ge 2[A:\mathbb{Q}]-2$ and $[A:\mathbb{Q}]$ is large, the cut locus has dimension larger than $[A:\mathbb{Q}]$. By [OAI, Lemmas 3.8

and 3.9] over the abelian field $A$, it contains a non-virtually-solvable curve with finite local images at the allowed non-dyadic places.

(v) [OAI, §7.4], entering the transverse fibre over $A'$: the lattice argument uses the irreducibility of the curve representation, the constancy of its local pseudo-character at $v_0$ in the marked case, and the restriction injectivity of Step 2. All of these hold for the curve in $C$.

(vi) [OAI, §7.4], conclusion: Lemma 7.5 of loc. cit. makes the target curve potentially pro-modular, and [OAI, Theorem 5.1] over $A$ then makes $C$ potentially pro-modular.

Hence there are a totally real field $F_C$, Galois and solvable over $\mathbb{Q}$, of even degree and split completely at 2, and a tame level, such that the image of $C$ under restriction to $G_{F_C}$ is pro-modular.

*Step 5: conclusion.* There are finitely many components $C$ as in Step 3. Let $F_1$ be the compositum of the fields $F_C$; it is totally real, Galois and solvable over $\mathbb{Q}$, of even degree and split completely at 2. By [OAI, Lemma 4.6], pro-modularity is preserved under the restrictions to $F_1$, at one common tame level. Hence (PM) holds with $K = A$, which is abelian over $\mathbb{Q}$ and contains $F$, and with this $F_1$. □

9.5. **Proof of Theorem 9.4.** In case (a), condition (R) holds and the result is Theorem A. Assume (b) or (c). If $\rho$ is induced from a character of an index two subgroup, it is modular, as in §2.2. Otherwise, by (N), (O) and §2.2, $\det\rho = \varepsilon^w\chi_0$ with $\chi_0$ of finite order. Then Proposition 9.8 gives (PM) for fields $F \subset K \subset F_1$, after the twist of its Step 0, which does not affect the assumptions or the conclusion.

By (PM), the restriction $\rho|_{G_{F_1}}$ defines a maximal ideal $\mathfrak{p}$ of $\mathbb{T}_{F_1}[1/2]$ with $\rho(\mathfrak{p}) \simeq \rho|_{G_{F_1}}$, which is absolutely irreducible by Lemma 2.4. For a place $w$ of $F_1$ above $v \in \Sigma_{\mathrm{o}}$, we have $\rho(\mathfrak{p})|_{G_w} = \rho_v$, which is reducible and has distinct Jordan–Hölder characters, since its Hodge–Tate weights are distinct. By Proposition 9.7(2) there is $\mathfrak{P}$ above $\mathfrak{p}$ such that $\psi_{w,1}(\mathfrak{P})$ is a subcharacter of $\rho_v$ for every such $w$. As $\rho_v$ is non-split, this subcharacter is the classical character (Lemma 2.1). By Theorem 3.4 over $F_1$, which holds by Proposition 9.7(1), $\rho|_{G_{F_1}}$ is modular. Since $F_1/F$ is Galois and solvable, $\rho$ is modular by solvable descent [BLGGT14, Lemma 2.2.2], [AC89]. □

*Remark* 9.9. Over a general field $F$ as in §1.1, the same argument applies whenever $F$ is contained in the field $B$ used in [OAI, §6], which is then quadratic or $\mathbb{Q}$. This is why we restrict to $F = \mathbb{Q}(\sqrt{q})$. In the excluded case of Theorem 9.4 ($\bar\chi \neq 1$ trivial at 2, with $\rho$ ramified at $w_q$ or $\bar\chi(\mathrm{Frob}_{w_q}) = 1$), the choice $q_0 = q$ fails. If $\bar\chi(\mathrm{Frob}_{w_q}) = 1$, the proof of [OAI, Lemma 6.5] breaks down, since the local cohomology at $q$ no longer vanishes. If $\rho$ is ramified at $w_q$, then $q \in \Sigma_{\mathbb{Q}}$, and $\sqrt{q}$ lies in the field $E_2$ of [OAI, §6.3], which prevents the final inert switch. Condition (S) is used only in the scalar case (b).

## 10. Residually irreducible representations with solvable image

In this section $F$ is as in §1.1 and $\rho : G_F \to \mathrm{GL}_2(E)$ is as in §1.1, with $\bar\rho$ absolutely irreducible. For a residual image which is not solvable, modularity lifting theorems at 2 are available when $\bar\rho$ is modular [Tun]. We treat the case of solvable image. Then residual modularity is automatic, by an odd Artin lift, and the method of Sections 3 and 8 applies with the simplifications of the residually irreducible case.

### 10.1. **Residual representations.**

**Lemma 10.1.** *Let $\bar\rho : G_F \to \mathrm{GL}_2(\mathbb{F})$ be absolutely irreducible with solvable image. Then its projective image is dihedral of order $2n$ with $n \geq 3$ odd. After enlarging $\mathbb{F}$, there are a unique quadratic extension $L/F$ and a character $\bar\theta : G_L \to \mathbb{F}^\times$ with $\bar\theta \neq \bar\theta^c$ and $\bar\rho \simeq \mathrm{Ind}_{G_L}^{G_F}\bar\theta$, where $c$ is the non-trivial element of $\mathrm{Gal}(L/F)$; the character $\bar\theta/\bar\theta^c$ has order $n$.*

*Proof.* Let $k$ be an algebraic closure of $\mathbb{F}$. Since $k$ is perfect of characteristic 2, the map $\mathrm{SL}_2(k) \to \mathrm{PGL}_2(k)$ is an isomorphism, and we regard the projective image $G$ of $\bar\rho$ as a finite solvable subgroup of $\mathrm{SL}_2(k)$. An element of $\mathrm{SL}_2(k)$ of 2-power order $2^m$ satisfies $(x-1)^{2^m} = x^{2^m} - 1 = 0$, so it is unipotent. A non-trivial unipotent element fixes a unique line. By Kolchin's theorem, a non-trivial 2-subgroup of $\mathrm{SL}_2(k)$ fixes a unique line. If the largest normal 2-subgroup $O_2(G)$ were non-trivial, its fixed line would be $G$-stable, and $\bar\rho \otimes k$ would be reducible. Hence $O_2(G) = 1$, and the Fitting subgroup $N$ of $G$, which is non-trivial since $G$ is solvable and non-trivial, has odd order.

Let $H$ be a Sylow $p$-subgroup of $N$, $p$ odd, $H \neq 1$, and $z \neq 1$ in its centre. The element $z$ is semisimple, and not scalar since the only scalar in $\mathrm{SL}_2(k)$ is 1. Its centralizer is therefore a maximal torus $T \simeq k^\times$, which contains $H$. The other Sylow subgroups of the nilpotent group $N$ commute with $z$, so $N \subset T$, and $N$ is cyclic. As $N$ is normal in $G$ and consists of semisimple elements with the same two eigenlines, $G$ permutes these lines, so $G \subset N_{\mathrm{SL}_2(k)}(T) = T \rtimes \langle s \rangle$, where $s$ acts on $T$ by inversion. If $G \subset T$, then $\bar\rho \otimes k$ stabilizes the two eigenlines and is reducible. Hence $C = G \cap T$ is cyclic of index 2 in $G$, of odd order $n$, since the finite subgroups of $k^\times$ have odd order, and $G$ is dihedral of order $2n$. If $n = 1$, then $G$ is a 2-group, which is excluded; so $n \geq 3$.

Let $L$ be the quadratic extension of $F$ cut out by $G \to G/C$. Then $\bar\rho(G_L)$ has projective image $C \subset T$, so it stabilizes the two eigenlines of $T$, and $\bar\rho|_{G_L} \otimes k = \bar\theta \oplus \bar\theta'$. By Clifford theory and the irreducibility of $\bar\rho$, we have $\bar\theta' = \bar\theta^c \neq \bar\theta$ and $\bar\rho \otimes k \simeq \mathrm{Ind}_{G_L}^{G_F} \bar\theta$. The character $\bar\theta/\bar\theta^c$ maps $G_L$ onto the projective image $C$ of $\bar\rho(G_L)$, so it has order $n$. Finally, $L$ is unique: since $n$ is odd, $C$ is the commutator subgroup of $G$, so it is the unique subgroup of index 2. □

**Lemma 10.2.** *Let $\bar\rho = \mathrm{Ind}_{G_L}^{G_F} \bar\theta$ be as in Lemma 10.1. There is a character $\theta : G_L \to \mathcal{O}^\times$ of finite order, lifting $\bar\theta$, such that $r = \mathrm{Ind}_{G_L}^{G_F} \theta$ is irreducible and totally odd. Its reduction modulo $\varpi$ is isomorphic to $\bar\rho$.*

*Proof.* Let $\theta_0$ be the Teichmüller lift of $\bar\theta$; it has odd order, so $\theta_0(c_w) = 1$ for every complex conjugation $c_w \in G_L$. Let $v$ be a real place of $F$. If $v$ becomes complex in $L$, a complex conjugation $c_v$ at $v$ is not in $G_L$, so $\mathrm{Ind}\theta(c_v)$ interchanges the two lines of the induced representation and has trace 0. As $c_v^2 = 1$, it has eigenvalues $\pm 1$ and determinant $-1$, for every $\theta$. If $v$ splits into two real places $w, w'$ of $L$, then $\mathrm{Ind}\theta(c_w)$ is conjugate to $\mathrm{diag}(\theta(c_w), \theta(c_{w'}))$.

By weak approximation there is $x \in L^\times$ which is negative at one chosen place $w$ above each split real place $v$ and positive at all other real places of $L$. Let $\eta$ be the character of $L(\sqrt{x})/L$. Put $\theta = \theta_0\eta$. Then $\det r(c_v) = \eta(c_w)\eta(c_{w'}) = -1$ at the split real places, and $r$ is totally odd. Since $\eta$ takes values $\pm 1 \equiv 1 \pmod{\varpi}$, $\theta$ lifts $\bar\theta$, and $r \bmod \varpi \simeq \mathrm{Ind}\bar\theta = \bar\rho$. As $\bar\theta \neq \bar\theta^c$, also $\theta \neq \theta^c$, and $r$ is irreducible. □

In characteristic 2 the field $L$ need not be CM, since oddness is invisible modulo 2. The twist by $\eta$, which is trivial modulo 2, is what makes the Artin lift odd in all cases.

**Proposition 10.3.** *Let $\bar\rho$ be as in Lemma 10.1. There is a cuspidal Hilbert eigenform $g$ over $F$ of parallel weight $k \geq 2$ whose Galois representation $\rho_g$ satisfies $\bar\rho_g \simeq \bar\rho$.*

*Proof.* Let $r = \mathrm{Ind}_{G_L}^{G_F} \theta$ be as in Lemma 10.2. The automorphic induction $\pi$ of the Hecke character $\theta$ of $L$ [LL79], [AC89, Chapter 3] is cuspidal, since $\theta \neq \theta^c$. At a real place $v$ of $F$, its Langlands parameter is the restriction of $r$ to $W_{F_v}$, which factors through $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and is $1 \oplus \mathrm{sgn}$, since $r(c_v)$ has eigenvalues 1 and $-1$ by the oddness of $r$. Hence $\pi_v$ is the limit of discrete series of weight 1. Hence $\pi$ is generated by a cuspidal Hilbert eigenform $f$ of parallel weight 1, whose Galois representation is $r$ [RT83]. Its Hecke eigenvalues $c(v, f) = \mathrm{tr}\, r(\mathrm{Frob}_v)$, for $v \nmid N$, where $N$ is the level, lie in $\mathcal{O}$.

We now raise the weight, as in [Wil88]; see [All14, §5.1] for its use at $p = 2$. Since $F$ is abelian over $\mathbb{Q}$, Leopoldt's conjecture holds for $F$ at 2 [Bru67], and the 2-adic zeta function of

$F$ has a pole at $s=1$. We use Shimura's normalized Fourier coefficients $C(\mathfrak{m},\cdot)$ at the cusps at infinity of the $h^+$ connected components, with representatives $t_\lambda$ of the narrow class group prime to 2; for parallel weight $k$ they differ from the Fourier coefficients of the components by factors $N(t_\lambda)^{-k/2}$, which depend only on the component and are multiplicative in $k$. Let $l=2^m$ with $m\geq 2$, and let $G_l$ be the Eisenstein series of parallel weight $l$, level one and trivial character, with $C(\mathfrak{m},G_l)=\sum_{\mathfrak{a}|\mathfrak{m}}\mathrm{N}(\mathfrak{a})^{l-1}$ and constant terms $2^{-d}\zeta_F(1-l)$, up to rational 2-adic units, at the cusps at infinity of the components. Since 2 splits completely in $F$ and $l$ is even, the Kubota–Leopoldt interpolation gives $\zeta_{F,2}(1-l)=(1-2^{l-1})^d\zeta_F(1-l)$, where $\zeta_{F,2}=\prod_\chi L_2(s,\chi)$ over the even Dirichlet characters $\chi$ of $F$. As $\zeta_2$ has a pole at $s=1$ and $L_2(1,\chi)\neq 0$ for $\chi\neq 1$, by Leopoldt's formula and Brumer's theorem [Bru67], $v_2(\zeta_F(1-l))\to-\infty$ as $m\to\infty$. Let $E=2^dG_l/\zeta_F(1-l)$ with $m$ large. Its non-constant coefficients are divisible by an arbitrarily large power of 2, and its constant terms at the cusps at infinity are rational 2-adic units, hence congruent to 1 modulo $\varpi$ [Wil88]. Then $fE$ is a cusp form of parallel weight $l+1$, level $N$ and the character of $f$. Since the products are computed componentwise and the normalizing factors are multiplicative in the weight, $C(\mathfrak{m},fE)$ is the convolution of the coefficients of $f$ and $E$, and $C(\mathfrak{m},fE)\equiv C(\mathfrak{m},f) \pmod{\varpi}$ for every non-zero integral ideal $\mathfrak{m}$. Let $M$ be the lattice of cusp forms of parallel weight $l+1$, level $N$ and this character, with 2-integral Fourier coefficients; it is finite free over $\mathcal{O}$ and stable under the Hecke operators $T_v$, $v\nmid 2N$. The Hecke recursions for $T_v$ in weights 1 and $l+1$ differ by the factors $\mathrm{N}(v)^j$ with $j\geq 0$, which are odd integers, hence $\equiv 1 \pmod{\varpi}$. Hence the reduction of $fE$ in $M/\varpi M$ is a non-zero eigenvector of the $T_v$ with eigenvalues $c(v,f) \bmod \varpi$. By the lemma of Deligne and Serre [DS74, Lemme 6.11], after enlarging $\mathcal{O}$ there is an eigenform $g\in M$ with $c(v,g)\equiv c(v,f) \pmod{\varpi}$ for all $v\nmid 2N$. The characters of $f$ and $g$ agree, and $\mathrm{N}(v)^l\equiv 1 \pmod{\varpi}$, so the characteristic polynomials of $\rho_g(\mathrm{Frob}_v)$ and of $r(\mathrm{Frob}_v)$ agree modulo $\varpi$ for $v\nmid 2N$. By the Chebotarev density theorem and the Brauer–Nesbitt theorem, $\bar\rho_g^{ss}\simeq\bar\rho$, and $\bar\rho_g\simeq\bar\rho$ since $\bar\rho$ is irreducible. □

10.2. **Fields.** Let $\Sigma$ be as in §1.1, enlarged by the places above the primes dividing the level of the form $g$ of Proposition 10.3, chosen once and for all; as in Section 8, this is harmless for (PM). Let $\Sigma_0$ be the set of finite places of $\Sigma$ not above 2. Choose a finite set $P_0$ of places of $F$, not above 2 and not in $\Sigma_0$, whose classes generate the 2-part of the class group. Define $S'$, the elements $x_v$ and the fields $L_w$ as in §8.1; Lemma 8.1 holds without change, since it does not involve $\bar\chi$. Let $F(\bar\rho)$ be the field cut out by $\bar\rho$.

**Lemma 10.4.** *For all integers $n>a\geq 1$ there are cyclic extensions $F\subset F_a\subset F_n$ of degrees $2^a$ and $2^n$, totally real, in which the places above 2 split completely, ramified at exactly one place, of degree one, which is totally ramified in $F_n$, such that every place of $\Sigma_0$ has a unique place above it in $F_n$ and $F_n\cap F(\bar\rho)=F$. In particular $\bar\rho|_{G_{F_n}}$ is absolutely irreducible, and the number of non-dyadic places of $F_n$ above $\Sigma_0\cup\{w\}$ is $|\Sigma_0|+1$.*

*Proof.* Choose a place $w$ of $F$ of degree one, not in $S'\cup\Sigma_0$, unramified in $F(\bar\rho)$, which splits completely in $M_n=F(\mu_{2^{n+1}},(\mathcal{O}_{S'}^\times)^{1/2^n})$ and is inert in $F(\sqrt{x_v})$ for every $v\in\Sigma_0$. Such a $w$ exists by Chebotarev's theorem: as in the proof of Proposition 8.3, $\mathrm{Gal}(M_n\prod_{v\in\Sigma_0}F(\sqrt{x_v})/F)=\mathrm{Gal}(M_n/F)\times(\mathbb{Z}/2)^{\Sigma_0}$, and excluding the finitely many places ramified in $F(\bar\rho)$ is harmless. Let $F_n=L_w$ be the field of Lemma 8.1, and $F_a$ its layer of degree $2^a$.

By Lemma 8.1, $F_n$ is totally real, the places above 2 split, and $w$ is totally ramified. For $v\in\Sigma_0$, $x_v$ is not a square modulo $w$, so $\mathrm{Frob}_v$ generates $\mathrm{Gal}(F_n/F)$ and $v$ is inert. Every non-trivial subextension of $F_n/F$ is ramified at $w$, while $F(\bar\rho)/F$ is unramified at $w$; hence $F_n\cap F(\bar\rho)=F$, and $\bar\rho(G_{F_n})=\bar\rho(G_F)$. □

10.3. **Pro-modularity and modularity.**

**Proposition 10.5.** *Let $\rho$ be as in §1.1, with $\bar\rho$ absolutely irreducible with solvable image, de Rham with distinct Hodge–Tate weights at some place above 2, not induced, and with $\det\rho = \varepsilon^w\chi_0$ for $\chi_0$ of finite order. Then* (PM) *holds, for the pseudo-deformation rings of $\bar\rho|_{G_K}$, with $K/F$ cyclic.*

*Proof.* By Lemma 2.4, $\rho|_{G_{F'}}$ is absolutely irreducible and not virtually solvable for every finite extension $F'/F$. We follow the proof of [OAI, Theorem 7.1] in the case of absolutely irreducible residue, with $\mathbb{Q} \subset A \subset F$ replaced by $F \subset F_a \subset F_n$ of Lemma 10.4, as in the proof of Theorem 8.4, with the following simplifications and changes.

(1) *No reducible loci.* Since $\bar\rho|_{G_{F_n}}$ and $\bar\rho|_{G_{F_a}}$ are absolutely irreducible, the pseudo-deformation spaces over $F_n$ and $F_a$ contain no reducible points. Hence the reducible locus of [OAI, Lemma 7.3] is empty, there are no marks and no Selmer conditions, and Leopoldt's conjecture is not used. The only bad locus in [OAI, Lemma 3.8] is the dihedral one, of dimension at most $d_K$. By [OAI, Lemma 7.5], the residual fibre is a single conjugacy orbit, hence connected.

(2) *Numerical conditions.* With $u_n = |\Sigma_0|+1$, we choose $n$ and $a$ so large that $2d_n - 4 - u_n > d_n$ and $d_a > u_a + C + 4$, where $d_n = [F_n : \mathbb{Q}]$, $d_a = [F_a : \mathbb{Q}]$ and $C$ is the constant cost of the trace cuts in [OAI, §7.4].

(3) *The seed.* The base change of $g$ (Proposition 10.3) to $F_a$, through the cyclic extension $F_a/F$ [AC89], is cuspidal by regularity, as in [OAI, §7.4]. Its residual representation is $\bar\rho|_{G_{F_a}}$, and its level is supported on $\Sigma$. Its central character agrees with $\chi\varepsilon$ modulo $\varpi$, since $\det\bar\rho_g = \det\bar\rho$. By [OAI, Proposition 4.4] it gives a pro-modular component of dimension at least $1 + 2d_a$.

(4) *The target.* The target loci are the irreducible components of $\mathrm{Spec}R^{\mathrm{ps}}_{F_a}$ through the point of $\rho|_{G_{F_a}}$. As in Step 3 of the proof of Proposition 9.8, each is the pseudo-image of a component of the framed representation scheme and has dimension at least $2d_a - 2$.

The remaining arguments of [OAI, §7] and the propagation theorem [OAI, Theorem 5.1] apply as in the proof of Theorem 8.4. In [OAI, §7.4], any stable lattice of the seed and target curves gives a point of the residual scheme over $F_n$, since there are no marks and the residual representation over $F_n$ is absolutely irreducible. Taking Galois closures and a common level as there, we obtain (PM) with $K = F_a$, which is cyclic over $F$. □

**Theorem 10.6.** *Let $F$ be as in §1.1 and $\rho : G_F \to \mathrm{GL}_2(E)$ as in §1.1, satisfying* (N) *and* (O)*, with $\bar\rho$ absolutely irreducible with solvable image, and such that $\rho_v$ is non-split for every $v \in \Sigma_{\mathrm{o}}$. Then $\rho$ is modular.*

*Proof.* If $\rho$ is induced from a character of an index two subgroup, it is modular, as in §2.2. Otherwise, by (N), (O) and §2.2, $\det\rho = \varepsilon^w\chi_0$ with $\chi_0$ of finite order, and Proposition 10.5 gives (PM) for fields $F \subset K \subset F_1$. The restriction $\rho|_{G_{F_1}}$ defines a maximal ideal $\mathfrak{p}$ of the characteristic-zero Hecke algebra in the residual summand attached to $(\bar\rho|_{G_{F_1}})^{ss}$, and it is absolutely irreducible by Lemma 2.4. The solvable extensions in Proposition 10.5 need not preserve residual irreducibility: if $(\bar\rho|_{G_{F_1}})^{ss}$ is reducible, a Teichmüller twist of finite order puts it in the form $1 \oplus \bar\chi'$, possibly with $\bar\chi' = 1$; such a twist preserves regularity, the de Rham property and non-splitness, and transports Hecke support after enlarging the tame level if necessary. Hence Proposition 9.7 applies to every possible final residual summand. For $w \mid v \in \Sigma_{\mathrm{o}}$, $\rho|_{G_{F_{1,w}}} = \rho_v$ is reducible and non-split, with distinct Jordan–Hölder characters. By Proposition 9.7(2) there is $\mathfrak{P}$ above $\mathfrak{p}$ such that $\psi_{w,1}(\mathfrak{P})$ is a subcharacter of $\rho_v$, hence the classical character, by Lemma 2.1. By Theorem 3.4 over $F_1$, which holds by Proposition 9.7(1), $\rho|_{G_{F_1}}$ is modular. Undoing the auxiliary twist and applying solvable descent along $F_1/F$ [BLGGT14, Lemma 2.2.2], [AC89], $\rho$ is modular. □

*Remark* 10.7. By Lemma 10.1, in characteristic 2 the absolutely irreducible residual representations with solvable image are exactly the dihedral ones. There is no analogue of the bad dihedral

case: since $F(\zeta_2) = F$, the usual restriction to $G_{F(\zeta_p)}$ creates no exception. The difficulty of solvable residual image for the Taylor–Wiles method, which explains the hypothesis of non-solvable image in [Tun], is handled by the method of [OAI], whose patching takes place at a characteristic 2 curve with non-virtually-solvable generic image, and residual modularity is supplied by Proposition 10.3. In the nearly ordinary case, modularity of residually dihedral representations is due to Allen [All14], with no condition on the splitness and with an additional condition at 2 when the inducing field is CM.

Theorem 10.6 treats every absolutely irreducible residual representation with solvable image, under its stated hypotheses. For reducible residual semisimplifications, Theorem A requires the ratio to extend to $G_{\mathbb{Q}}$ and to be non-trivial at the places above 2, and for $F = \mathbb{Q}(\sqrt{q})$, $q \equiv 1 \pmod 8$ prime, Theorem 9.4 adds precisely the cases (b) and (c), still assuming that the ratio extends. The uncovered cases are therefore: non-extendible ratios; scalar ratios when (S) fails; and the non-trivial ratios which are trivial at 2 excluded in Theorem 9.4.

## 11. Remarks

*Remark* 11.1. Theorem 8.4 gives the potential form of pro-modularity stated in (PM): components over $K$ become pro-modular after restriction to $F_1$. This is the form produced by [OAI], whose propagation theorem is stated for potentially pro-modular loci. The deformation-theoretic arguments take place over $K$, and the Hecke algebras over $F_1$ are used at points of characteristic 0 for classicality and solvable descent. The proof of the ordinary finiteness theorem also uses primes of characteristic 2, through Proposition 8.8 and Theorem 8.9; it does not follow from points of characteristic 0 alone. The field construction of Proposition 8.3 treats only the case $\bar{\chi}|_{G_{\mathbb{Q}_2}} \neq 1$. The case $\bar{\chi}|_{G_{\mathbb{Q}_2}} = 1$ is treated over $\mathbb{Q}$ in [OAI] with marked places and an auxiliary quadratic field; for $F = \mathbb{Q}(\sqrt{q})$ it is treated in Section 9 under its explicit hypotheses, with $F$ itself as the auxiliary field. For general $F$ it would require the corresponding field construction and Selmer control over $F$, including an analogue of [OAI, Lemma 6.2].

*Remark* 11.2. For odd $p$, Pan proves that the fully ordinary pseudo-deformation ring is finite over $\Lambda$ [Pan22, Theorem 5.1.2(1)], by an $R = \mathbb{T}$ theorem for Hida families. Theorem 8.9 replaces it for $p = 2$ by two steps: potential pro-modularity of all the components (Theorem 8.5), and the fact that a prime of the big Hecke algebra which is reducible at every place above 2 comes from the fully ordinary Hecke algebra (Proposition 8.8). The second step cannot be obtained from Emerton's adjunction at points alone, since primes of characteristic 2 need not be limits of points of characteristic 0 which are reducible at 2; it uses instead the description of $\mathrm{Ord}_B$ on the projective generators of the generic blocks. Upper bounds by Selmer groups are not available in characteristic 2: the tangent space of the special fibre of the fully ordinary deformation space at a curve has dimension of the order of $2d$, because of the 2-torsion of $\mathbb{Q}_2^\times$, the invisibility of oddness modulo 2 at the real places, and the scalars in $\mathrm{ad}^0$.

*Remark* 11.3. Condition (St) is used in three places: to exclude CM good points (Lemma 4.4), which for odd $p$ followed from (R); to apply the theorem of Hida and Zhao at the potentially modular points of weight two, which for Hilbert modular forms over fields of even degree requires a square integrable local component [Zhao14]; and to have full local–global compatibility away from 2 at these points, by Carayol's theorem. Along $C$, Lemma 4.3 preserves the scalar inertial semisimplification and the equation $P_{v_0} = 0$ for the ratio of the Frobenius eigenvalues; non-zero monodromy is recovered at automorphic specializations by genericity and local–global compatibility, and is not asserted at every point of $C$. A Steinberg place above 2 would not serve this purpose: its type need not persist along $C$, Thorne's theorem is applied to potentially crystalline good points, and Steinberg type with consecutive weights at $v \mid 2$ forces scalar residual semisimplification at $v$, contrary to (R).

*Remark* 11.4. The present proof of Theorem B uses $\bar{\chi}|_{G_v} \neq 1$ to make the reducible locus at each place of $\Sigma_{\mathrm{n}}$ principal (Lemma 2.2), and in the generic-block argument of §8. This is a requirement of the proof; we do not claim that the conclusion fails without it. Since quadratic characters are trivial modulo 2, (R) excludes all cases in which $\bar{\rho}$ takes values in $\mathrm{GL}_2(\mathbb{F}_2)$.

*Remark* 11.5. We record the verification of the results used for $p = 2$.

(1) [BE10, Proposition 2.2.1] is stated for an arbitrary prime $p$, for characters of classical weight $k \geq 2$ satisfying the valuation conditions, which amount to unitarity, with $k = 2$ treated separately; we use it only for $a_1 > a_2$, that is $k \geq 3$.
(2) Saito's local–global compatibility at $p$ [Sai09, Theorem 1] does not exclude $p = 2$, and assumes that $[F : \mathbb{Q}]$ is odd or that $\pi$ is square integrable at a finite place; the remaining case is treated by Skinner [Ski09], whose argument covers $p = 2$ explicitly. This is what is used in Lemma 3.1 and in Proposition 6.1.
(3) The theorem of Hida and Zhao [Zhao14, Theorem 1] is stated for an arbitrary prime $p$, for Hilbert eigenforms of parallel weight two which are nearly ordinary at all places above $p$, not of CM type, and square integrable at a finite place if $[F : \mathbb{Q}]$ is even; no condition on $p$ appears in loc. cit. or in Hida's theorem [Hid13], apart from unramifiedness conditions in the latter, which Zhao removes.
(4) In [Tho26, §1.1], with $\mathrm{HT}(\varepsilon) = -1$, ordinary with Hodge–Tate weights $\{0, 1\}$ means isomorphic to $\left(\begin{smallmatrix} \psi_1 & * \\ 0 & \varepsilon^{-1}\psi_2 \end{smallmatrix}\right)$ with $\psi_1, \psi_2$ of finite order on inertia; this is the shape of $r|_{G_v}$ in Proposition 6.1(1), and [Tho26, Theorem B] only assumes that the determinant is totally odd.
(5) The presentation used in Proposition 4.2 is [OAI, Proposition 3.6 and Corollary 3.7], proved for $p = 2$ in both coefficient characteristics.
(6) The theorem of Newton and Thorne [NT23, Theorem B] is stated for every prime $p$, and its introduction explicitly allows $p = 2$.

Inputs (P1) and (P2) of Lemma 8.7 are [Paš16, Theorems 1.3 and 2.23], which treats generic blocks for $p = 2$ (the case $p > 2$ being [Paš13, §B.1]); the action of the centre on $\mathrm{Ord}_B(J_i)$ is deduced from the Cayley–Hamilton relation on $\check{\mathbf{V}}(P)$ used in the proof of [Paš16, Corollary 2.25], and (P3) is [PT21, Lemma 4.4], see also [Paš16, (2)] for its residual form. In the proof of Theorem 8.4, the modifications of [OAI, §7] were checked against loc. cit.: Leopoldt's conjecture is used there only in [OAI, Lemmas 3.8 and 7.3]; the numerical conditions of [OAI, (39)] are replaced by those of item (2); the seed must be fixed before the fields, as in [OAI, §7.2]; and $K = F_a$ is cyclic over $F$ but not Galois over $\mathbb{Q}$, which is why (PM) is stated with $K/F$ Galois. In the proof of Proposition 8.3, the Chebotarev conditions are compatible once no evaluation condition is imposed at the first conductor, whose inertness conditions are not of the form required by Lemma 8.2; this costs one more conductor in the bound for $t$.

*Remark* 11.6. In Theorem A, the condition $\bar{\chi}|_{G_v} \neq 1$ is used for the unique decomposition of reducible determinants at $v$, which identifies the classical character residually. Since the proof of Theorem A works at a point of characteristic 0, it is not essential there, except through Theorem 8.4, whose proof uses $\bar{\chi}|_{G_{\mathbb{Q}_2}} \neq 1$ in the field construction. For $F = \mathbb{Q}(\sqrt{q})$, Theorem 9.4 removes this condition outside one subcase.

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