Off-Diagonal Commonality of a Shorter Even Cycle and a Longer Odd Cycle

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Abstract

Let $n \geq 4$ be even and let $\ell > n$ be odd. We determine the complete off-diagonal commonality region of the ordered pair (C_n, C_ℓ): it is the half-open interval $(0, a^\star_{n,\ell}]$ cut out by the unique root $a^\star_{n,\ell} \in (1/2, 1)$ of

$$\frac{a^{n-1}\big(n + (\ell - n)a\big)}{\ell} = 2^{1-n}.$$

The upper endpoint is witnessed by the balanced disjoint union of two cliques. For the lower bound, cycle densities are expressed as traces of graphon operators. Rank-one eigenvalue majorisation separates the Perron eigenvalue from the rest of the spectrum, while the Hilbert–Schmidt norm leaves room for at most one negative eigenvalue to contribute with the wrong sign. That eigenvalue is paid for by the Perron eigenvalue, leaving the elementary inequality $x^n + (1-x)^n \geq 2^{1-n}$.

1 Definitions and main theorem

A *graphon* is a symmetric measurable function $W\colon [0,1]^2 \to [0,1]$. If H is a finite graph, its homomorphism density in W is

$$t(H, W) := \int_{[0,1]^{V(H)}} \prod_{uv \in E(H)} W(x_u, x_v) \prod_{u \in V(H)} dx_u.$$

We write $e(H) = |E(H)|$. See Lovász [2] for background on graphons and homomorphism densities.

Following Behague, Morrison, and Noel [1], let $a, b \in (0,1)$ with $a + b = 1$. An ordered pair (H_1, H_2) is (a,b)-*common* if

$$\frac{t(H_1, W)}{e(H_1)a^{e(H_1)-1}} + \frac{t(H_2, 1-W)}{e(H_2)b^{e(H_2)-1}} \geq \frac{a}{e(H_1)} + \frac{b}{e(H_2)} \tag{1}$$

for every graphon W. Define

$$\pi(H_1, H_2) := \{a \in (0,1) : (H_1, H_2) \text{ is } (a, 1-a)\text{-common}\}.$$

Taking $a = 1/2$ and $H_1 = H_2 = H$ turns (1) into $t(H, W) + t(H, 1-W) \geq 2^{1-e(H)}$, so $1/2 \in \pi(H, H)$ says exactly that H is a common graph.

Our main result determines $\pi(C_n, C_\ell)$ when the shorter cycle is even and the longer one is odd.

Theorem 1.1 (Shorter even and longer odd cycles)**.** *Let $n \geq 4$ be even and let $\ell > n$ be odd. Then*

$$\pi(C_n, C_\ell) = \big(0, a^\star_{n,\ell}\big],$$

where $a^\star_{n,\ell}$ is the unique solution in $(1/2,1)$ of

$$\frac{a^{n-1}\big(n+(\ell-n)a\big)}{\ell} = 2^{1-n}. \tag{2}$$

That (2) has exactly one root in $(1/2,1)$ is elementary, since its left-hand side is strictly increasing in a; this is Lemma 2.1. The content of Theorem 1.1 is that $\pi(C_n, C_\ell)$ is precisely the half-open interval cut out by that root: commonality holds up to and including $a^\star_{n,\ell}$, and fails immediately afterwards.

Table 1 gives some numerical values of the critical point. The columns give the even length n and the rows give the odd length ℓ. An entry is replaced by 0 when $n \geq \ell$, which lies outside the shorter-even regime of Theorem 1.1.

$\ell\backslash n$	4	6	8	10	12	14
5	0.517213…	0	0	0	0	0
7	0.538147…	0.507351…	0	0	0	0
9	0.550283…	0.517832…	0.504065…	0	0	0
11	0.558163…	0.524929…	0.510344…	0.502577…	0	0
13	0.563680…	0.530044…	0.514968…	0.506759…	0.501779…	0
15	0.567753…	0.533903…	0.518513…	0.510006…	0.504764…	0.501302…

Table 1: The critical values $a^\star_{n,\ell}$ for shorter even cycles C_n and longer odd cycles C_ℓ.

The proof of Theorem 1.1 is self-contained apart from standard finite-dimensional spectral facts.

2 Proof of the main theorem

We begin with the scalar part of the proof, which establishes the existence and uniqueness of $a^\star_{n,\ell}$ and the numerical bounds on the quantities attached to a that are used later. From here to the end of the section, let

$$n \geq 4 \text{ be even}, \qquad \ell > n \text{ be odd}, \qquad d := \ell - n.$$

Thus d is a positive odd integer. For $a \in (0,1)$, set $b := 1-a$ and define

$$\kappa_{n,\ell}(a) := \frac{na^{n-1}}{\ell b^{\ell-1}}, \tag{3}$$

$$\rho_{n,\ell}(a) := a^n + \frac{n}{\ell}a^{n-1}b = \frac{a^{n-1}(n+da)}{\ell}. \tag{4}$$

Note that (C_n, C_ℓ) is (a,b)-common if and only if

$$t(C_n, W) + \kappa_{n,\ell}(a)\, t(C_\ell, 1-W) \geq \rho_{n,\ell}(a) \tag{5}$$

for every graphon W: this is (1) multiplied through by na^{n-1}.

Lemma 2.1 (The critical point)**.** *The function $\rho_{n,\ell}$ is strictly increasing on $(0,1)$, and there is a unique $a^\star_{n,\ell} \in (1/2,1)$ such that*

$$\rho_{n,\ell}(a^\star_{n,\ell}) = 2^{1-n}. \tag{6}$$

Proof. Differentiating (4) gives

$$\rho'_{n,\ell}(a) = \frac{n}{\ell}a^{n-2}(n-1+da) > 0.$$

Hence $\rho_{n,\ell}$ is strictly increasing. At the endpoints,

$$\rho_{n,\ell}\left(\frac{1}{2}\right) = 2^{1-n}\frac{n+\ell}{2\ell} < 2^{1-n}, \qquad \rho_{n,\ell}(1) = 1 > 2^{1-n}.$$

The intermediate value theorem proves existence, and strict monotonicity proves uniqueness. □

The next lemma gives bounds on $\kappa_{n,\ell}(a)b^d$ and on b, with constants not depending on n or ℓ.

Lemma 2.2 (Critical coefficient bounds)**.** *Let*

$$a^\star := a^\star_{n,\ell}, \qquad b^\star := 1 - a^\star, \qquad \kappa^\star := \kappa_{n,\ell}(a^\star).$$

Then

$$\kappa^\star (b^\star)^d < 1, \qquad b^\star > \frac{5}{14}.$$

Consequently, for every $0 < a \le a^\star$, with $b = 1 - a$ and $\kappa = \kappa_{n,\ell}(a)$,

$$\kappa b^d < 1, \qquad b > \frac{5}{14}. \tag{7}$$

Proof. Write

$$a^\star = \frac{1+\delta}{2}, \qquad b^\star = \frac{1-\delta}{2}, \qquad \delta \in (0,1).$$

Since $\ell = n + d$, the critical equation (6) is equivalent to

$$(1+\delta)^{n-1}(\ell + n + d\delta) = 2\ell. \tag{8}$$

It follows that

$$(1+\delta)^{n-1} < \frac{2\ell}{\ell+n} = 1 + \frac{d}{\ell+n}.$$

Bernoulli's inequality gives $(1+\delta)^{n-1} \ge 1 + (n-1)\delta$, so $(n-1)\delta < d/(\ell+n)$ and hence

$$0 < \delta < \frac{d}{(n-1)(\ell+n)}.$$

A second application of Bernoulli's inequality gives

$$(1-\delta)^{n-1} \ge 1 - (n-1)\delta > 1 - \frac{d}{\ell+n} = \frac{2n}{\ell+n}.$$

Combining these two estimates gives

$$\begin{aligned}\kappa^\star (b^\star)^d &= \frac{n}{\ell}\left(\frac{1+\delta}{1-\delta}\right)^{n-1} \\ &< \frac{n}{\ell}\frac{2\ell/(\ell+n)}{2n/(\ell+n)} = 1.\end{aligned}$$

Equation (8) also implies $(1+\delta)^{n-1} < 2$. Since $n - 1 \ge 3$ and $1 + \delta > 1$,

$$(1+\delta)^3 \le (1+\delta)^{n-1} < 2.$$

If $\delta \ge 2/7$, then

$$(1+\delta)^3 \ge \left(\frac{9}{7}\right)^3 = \frac{729}{343} > 2,$$

a contradiction. Thus $\delta < 2/7$, and hence

$$b^\star = \frac{1-\delta}{2} > \frac{5}{14}.$$

Finally,

$$\kappa_{n,\ell}(a)(1-a)^d = \frac{n}{\ell}\left(\frac{a}{1-a}\right)^{n-1}$$

is strictly increasing with respect to a on $(0,1)$. Therefore $\kappa b^d \le \kappa^\star (b^\star)^d < 1$ whenever $a \le a^\star$. Moreover, $b = 1-a \ge 1-a^\star = b^\star > 5/14$. □

Remark 2.3. The proof of Proposition 2.10 uses the bound $b > 5/14$ only through the inequality $2b^2 > 1/4$. Any explicit lower bound on b larger than $1/(2\sqrt{2}) = 0.35355\ldots$ would therefore serve, and $5/14 = 0.35714\ldots$ is a convenient rational one.

Turning to Theorem 1.1 itself, we first show that it is enough to prove the required inequalities for step graphons. This allows us to assume that the integral operator has finite rank and is represented by a real symmetric matrix, so the eigenvalue arguments below do not require the spectral theory of compact operators.

Lemma 2.4 (Step-graphon reduction)**.** *Fix integers $n, \ell \ge 3$ and a continuous function $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$. Suppose that*

$$\Phi\big(t(C_n, W), t(C_\ell, 1-W)\big) \ge 0$$

for every step graphon W. Then the same inequality holds for every graphon W.

Proof. For $j \ge 1$, partition $[0,1]$ into its 2^j dyadic intervals of equal length, and let $\mathcal{A}_j$ be the finite sigma-algebra generated by this partition. Set

$$W_j := \mathbb{E}[W \mid \mathcal{A}_j \otimes \mathcal{A}_j].$$

Conditional expectation preserves symmetry and the pointwise bounds $0 \le W \le 1$, so W_j is a step graphon.

The dyadic rectangle step functions form nested subspaces whose union is dense in $L^2([0,1]^2)$. Since conditional expectation is the orthogonal projection onto the corresponding subspace,

$$\|W_j - W\|_2 \longrightarrow 0.$$

The underlying space has measure one, so also $\|W_j - W\|_1 \to 0$.

For every fixed $m \ge 3$, the telescoping identity for products and the bounds $0 \le W_j, W \le 1$ give

$$|t(C_m, W_j) - t(C_m, W)| \le m\|W_j - W\|_1.$$

Indeed, after writing $x_{m+1} = x_1$, expand the difference of the two products into m terms, with the ith term containing $(W_j - W)(x_i, x_{i+1})$ and all other factors bounded in absolute value by one. Integration of each term gives $\|W_j - W\|_1$. The same estimate applies to $1 - W_j$ and $1 - W$. Both cycle densities therefore converge, and continuity of Φ proves the lemma. □

Let K be a symmetric step kernel, constant on every rectangle $B_i \times B_j$ of a finite measurable partition

$$[0,1] = B_1 \sqcup \cdots \sqcup B_N, \qquad |B_i| > 0.$$

Let T_K be its integral operator on $L^2([0,1])$,

$$(T_K f)(x) = \int_0^1 K(x,y) f(y)\, dy.$$

Write $\mathbf{1}_S$ for the indicator function of a measurable set S, let $\mathbf{1} := \mathbf{1}_{[0,1]}$, and set

$$E := \operatorname{span}\{\mathbf{1}_{B_1}, \dots, \mathbf{1}_{B_N}\}.$$

The operator T_K maps $L^2([0,1])$ into E and vanishes on $E^\perp$, so it is determined by its restriction $T_K|_E$. In the orthonormal basis

$$e_i := \frac{\mathbf{1}_{B_i}}{\sqrt{|B_i|}} \qquad (1 \le i \le N)$$

of E, that restriction is represented by the real symmetric matrix

$$\big(K_{ij}\sqrt{|B_i||B_j|}\big)_{i,j=1}^{N},$$

where K_{ij} is the value of K on $B_i \times B_j$. We call such an E a step space *associated with* K. Two partitions on which K is constant admit a common refinement, and refining enlarges E while appending zero eigenvalues to the matrix above; hence the nonzero spectrum of $T_K|_E$ does not depend on which associated step space is chosen.

Lemma 2.5 (Cycle densities are traces)**.** *For every integer $m \ge 3$,*

$$t(C_m, K) = \operatorname{Tr}(T_K^m),$$

where the trace is taken on E.

Proof. Expanding the matrix trace in the basis $(e_i)_{i=1}^N$ gives

$$\operatorname{Tr}(T_K^m) = \sum_{i_1,\dots,i_m=1}^{N} |B_{i_1}| \cdots |B_{i_m}| K_{i_1 i_2} K_{i_2 i_3} \cdots K_{i_m i_1}.$$

This is precisely the step-function expansion of $t(C_m, K)$. □

Let $J := T_1$ be the operator of the constant kernel 1. Since $[0,1]$ has measure one, $Jf = \langle \mathbf{1}, f\rangle \mathbf{1}$, so J is the rank-one orthogonal projection onto the constant functions. As $K \mapsto T_K$ is linear, a step graphon W and its complement $U := 1 - W$ satisfy

$$T_W = J - T_U. \tag{9}$$

Combined with Lemma 2.5, this expresses every cycle density of W as the trace of a power of $J - T_U$. We now bound such traces from below in terms of the eigenvalues of T_U alone.

For a vector $x \in \mathbb{R}^N$, write $x^\downarrow$ for its coordinates rearranged in nonincreasing order. Recall that x is *majorised* by y, written $x \prec y$, if

$$\sum_{i=1}^{j} x_i^\downarrow \le \sum_{i=1}^{j} y_i^\downarrow \quad (1 \le j < N), \qquad \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i.$$

Karamata's inequality [3, Chapter 4] states that, whenever $x \prec y$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is convex,

$$\sum_{i=1}^{N} \varphi(x_i) \le \sum_{i=1}^{N} \varphi(y_i). \tag{10}$$

We need the following rank-one case of Lidskii's additive majorisation theorem. A proof is included for completeness.

Lemma 2.6 (Rank-one lower majorisation)**.** *Let A be a real symmetric $N \times N$ matrix with eigenvalues*

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N,$$

and let P be a rank-one orthogonal projection. If

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$$

are the eigenvalues of $A + P$, then

$$(\alpha_1, \ldots, \alpha_{N-1}, \alpha_N + 1) \prec (\mu_1, \ldots, \mu_N).$$

Proof. We use the min–max formulation of the eigenvalues,

$$\alpha_i = \max_{\dim V = i} \min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle \qquad (1 \leq i \leq N), \tag{11}$$

where V runs over the subspaces of $\mathbb{R}^N$ of dimension i; the same formula with $A + P$ in place of A gives the μ_i. We first show that the two spectra interlace:

$$\mu_1 \geq \alpha_1 \geq \mu_2 \geq \alpha_2 \geq \cdots \geq \mu_N \geq \alpha_N. \tag{12}$$

Equivalently, $\mu_i \geq \alpha_i$ for $1 \leq i \leq N$ and $\alpha_i \geq \mu_{i+1}$ for $1 \leq i < N$.

For the first family of inequalities, note that P is positive semidefinite, so $\langle x, (A + P)x \rangle \geq \langle x, Ax \rangle$ for every unit vector x. Hence the inner minimum in (11) can only increase when A is replaced by $A + P$, and so can the maximum over V; therefore $\mu_i \geq \alpha_i$ for every $i \leq N$.

For the second family of inequalities, fix $1 \leq i < N$ and let V be a subspace of dimension $i+1$. The projection P has rank one, so $\ker P$ has dimension $N-1$, and $V + \ker P \subseteq \mathbb{R}^N$. Hence

$$\begin{aligned} \dim(V \cap \ker P) &= \dim V + \dim \ker P - \dim(V + \ker P) \\ &\geq (i+1) + (N-1) - N = i. \end{aligned}$$

Choose a subspace $V' \subseteq V \cap \ker P$ of dimension exactly i. Every $x \in \ker P$ satisfies $(A+P)x = Ax$, and minimising over the smaller space V' can only increase the value, so

$$\min_{\substack{x \in V \\ \|x\|=1}} \langle x, (A+P)x \rangle \leq \min_{\substack{x \in V' \\ \|x\|=1}} \langle x, (A+P)x \rangle = \min_{\substack{x \in V' \\ \|x\|=1}} \langle x, Ax \rangle \leq \alpha_i,$$

where the last step is (11) applied to the i-dimensional subspace V'. Taking the maximum over all subspaces V of dimension $i+1$ preserves the inequality, and by (11) for $A+P$ that maximum is μ_{i+1}. Hence $\mu_{i+1} \leq \alpha_i$, which proves (12).

A rank-one orthogonal projection has trace one, so

$$\sum_{i=1}^{N} \mu_i = \operatorname{Tr}(A+P) = \operatorname{Tr}(A) + 1 = \sum_{i=1}^{N} \alpha_i + 1, \tag{13}$$

which proves the equality of total sums required in the definition of majorisation.

Let

$$v := (\alpha_1, \ldots, \alpha_{N-1}, \alpha_N + 1).$$

Its coordinates also sum to $\sum_{i=1}^{N} \alpha_i + 1$. For the partial sums, fix $1 \leq j < N$. The sum of the j largest coordinates of v is the maximum of $\sum_{i \in S} v_i$ over all j-element subsets $S \subseteq \{1, \ldots, N\}$.

If $N \notin S$, then the bound is immediate:

$$\sum_{i \in S} v_i \leq \sum_{i=1}^{j} \alpha_i \leq \sum_{i=1}^{j} \mu_i.$$

If $N \in S$, we first have

$$\sum_{i \in S} v_i \le 1 + \alpha_N + \sum_{i=1}^{j-1} \alpha_i.$$

It remains to bound $\sum_{i=1}^{j} \mu_i$ from below by the same quantity. Applying $\mu_{i+1} \le \alpha_i$ from (12) to $i = j, \dots, N-1$ gives

$$\sum_{i=j+1}^{N} \mu_i = \sum_{i=j}^{N-1} \mu_{i+1} \le \sum_{i=j}^{N-1} \alpha_i,$$

so with (13),

$$\begin{aligned} \sum_{i=1}^{j} \mu_i &= \sum_{i=1}^{N} \mu_i - \sum_{i=j+1}^{N} \mu_i \\ &\ge \Big(\sum_{i=1}^{N} \alpha_i + 1 \Big) - \sum_{i=j}^{N-1} \alpha_i = 1 + \alpha_N + \sum_{i=1}^{j-1} \alpha_i. \end{aligned}$$

This settles the case $N \in S$, so every partial-sum inequality in the definition of majorisation holds. □

Corollary 2.7 (Even trace after subtracting a rank-one projection)**.** *Let T be a real symmetric $N \times N$ matrix with eigenvalues*

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N,$$

and let P be a rank-one orthogonal projection. If $m \ge 2$ is even, then

$$\operatorname{Tr}\big((P - T)^m\big) \ge (1 - \lambda_1)^m + \sum_{i=2}^{N} |\lambda_i|^m.$$

Proof. Apply Lemma 2.6 to $A = -T$. The eigenvalues of $-T$ in nonincreasing order are

$$-\lambda_N \ge \cdots \ge -\lambda_1.$$

Writing $\mu_1 \ge \cdots \ge \mu_N$ for the eigenvalues of $P - T$, the lemma gives

$$(-\lambda_N, \dots, -\lambda_2, 1 - \lambda_1) \prec (\mu_1, \dots, \mu_N).$$

Applying Karamata's inequality to the convex function $x \mapsto x^m$ and using $(-\lambda_i)^m = |\lambda_i|^m$, which holds because m is even, gives the result. □

The pointwise bounds $0 \le U \le 1$ enter the argument through the following consequences for the spectrum of T_U.

Lemma 2.8 (Perron domination and the tail bound)**.** *Let U be a step graphon and let $T = T_U|_E$ be its matrix on an associated step-function space. If*

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$$

are the eigenvalues of T, then

$$0 \le \lambda_1 \le 1, \tag{14}$$

$$|\lambda_i| \le \lambda_1 \quad (1 \le i \le N), \tag{15}$$

$$\sum_{i=2}^{N} \lambda_i^2 \le \lambda_1(1 - \lambda_1) \le \frac{1}{4}. \tag{16}$$

Proof. Let $f \in E$ be a unit vector. Then $|f| \in E$ is also a unit vector, so $\langle |f|, T|f| \rangle \le \lambda_1$. Pointwise nonnegativity of U gives $|\langle f, Tf \rangle| \le \langle |f|, T|f| \rangle$, and $U \le 1$ gives

$$\langle |f|, T|f| \rangle \le \left(\int_0^1 |f| \right)^2 \le \|f\|_2^2 = 1.$$

Taking f to be a unit eigenvector of λ_1,

$$\lambda_1 = \langle f, Tf \rangle \le |\langle f, Tf \rangle| \le \langle |f|, T|f| \rangle \le 1.$$

For an arbitrary unit vector f,

$$\lambda_1 \ge \langle |f|, T|f| \rangle \ge |\langle f, Tf \rangle| \ge 0.$$

Moreover, taking f to be a unit eigenvector of λ_i,

$$|\lambda_i| = |\langle f, Tf \rangle| \le \langle |f|, T|f| \rangle \le \lambda_1.$$

Finally, let

$$p := \int_{[0,1]^2} U(x,y)\, dx\, dy = \langle \mathbf{1}, T\mathbf{1} \rangle,$$

which gives $p \le \lambda_1$. Since $U^2 \le U$,

$$\sum_{i=1}^N \lambda_i^2 = \operatorname{Tr}(T^2) = \int_{[0,1]^2} U(x,y)^2\, dx\, dy \le p.$$

Therefore

$$\sum_{i=2}^N \lambda_i^2 \le p - \lambda_1^2 \le \lambda_1 - \lambda_1^2 \le \frac{1}{4}.$$

□

We can now assemble the proof.

Proposition 2.9 (Spectral reduction)**.** *Let W be a step graphon, set $U = 1 - W$, and let $T = T_U|_E$ on an associated step-function space. Order the eigenvalues of T as*

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N.$$

For every $\kappa > 0$,

$$t(C_n, W) + \kappa t(C_\ell, U) \ge f_\kappa(\lambda_1) + \sum_{i=2}^N \lambda_i^n (1 + \kappa \lambda_i^d), \tag{17}$$

where

$$f_\kappa(x) := (1-x)^n + \kappa x^\ell, \qquad 0 \le x \le 1.$$

Proof. Let $P := J|_E$. Since $\mathbf{1} \in E$ and $\|\mathbf{1}\|_2 = 1$, P is a rank-one orthogonal projection. By (9), $T_W|_E = P - T$. Lemma 2.5 and Corollary 2.7 give

$$t(C_n, W) = \operatorname{Tr}\big((P-T)^n\big) \ge (1-\lambda_1)^n + \sum_{i=2}^N |\lambda_i|^n.$$

Because n is even, $|\lambda_i|^n = \lambda_i^n$. This inequality together with

$$t(C_\ell, U) = \operatorname{Tr}(T^\ell) = \sum_{i=1}^N \lambda_i^\ell$$

gives (17). □

Proposition 2.10 (Lower bound up to the critical point)**.** *If $0 < a \le a^\star_{n,\ell}$, then every graphon W satisfies*

$$t(C_n, W) + \kappa_{n,\ell}(a)\, t(C_\ell, 1 - W) \ge \rho_{n,\ell}(a). \tag{18}$$

Proof. By Lemma 2.4 we may assume that W is a step graphon. Let $U = 1 - W$, let $T = T_U|_E$, and write the eigenvalues of T as

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N.$$

Proposition 2.9, applied with $\kappa := \kappa_{n,\ell}(a)$, gives

$$t(C_n, W) + \kappa t(C_\ell, U) \ge f_\kappa(\lambda_1) + \sum_{i=2}^{N} \lambda_i^n (1 + \kappa \lambda_i^d). \tag{19}$$

Suppose first that $1 + \kappa\lambda_i^d \ge 0$ for every $i \ge 2$. Then every summand of the sum in (19) is nonnegative, since n is even, and it is enough to bound $f_\kappa(\lambda_1)$ from below. Write $b := 1 - a$. Since $\kappa = \kappa_{n,\ell}(a)$,

$$\begin{aligned} f'_\kappa(x) &= -n(1-x)^{n-1} + \kappa\ell x^{\ell-1}, \\ f''_\kappa(x) &= n(n-1)(1-x)^{n-2} + \kappa\ell(\ell-1)x^{\ell-2} > 0, \end{aligned}$$

and

$$f'_\kappa(b) = -na^{n-1} + \kappa\ell b^{\ell-1} = 0.$$

Thus f_κ is strictly convex and has its unique minimum at b, with

$$f_\kappa(b) = a^n + \kappa b^\ell = \rho_{n,\ell}(a).$$

By (14) the value λ_1 lies in the domain $[0,1]$ of f_κ, so (19) gives

$$t(C_n, W) + \kappa t(C_\ell, U) \ge f_\kappa(\lambda_1) \ge f_\kappa(b) = \rho_{n,\ell}(a).$$

Call an eigenvalue λ_i, $i \ge 2$, *dangerous* if

$$1 + \kappa\lambda_i^d < 0,$$

that is, if its summand in (19) is negative. As shown above, if there is no dangerous eigenvalue then the claim follows. Assume therefore that there is at least one. Since d is odd and $\kappa > 0$, a dangerous eigenvalue has the form $\lambda_i = -\beta$ with $\beta > 0$. The bound $\kappa b^d < 1$ of (7) with the definition of a dangerous eigenvalue gives

$$\kappa\beta^d > 1 > \kappa b^d,$$

and since $x \mapsto x^d$ is strictly increasing, this yields

$$\beta > b > \frac{5}{14}, \tag{20}$$

where the second inequality is the second bound of (7).

We now show that there can be at most one dangerous eigenvalue. If two of $\lambda_2, \ldots, \lambda_N$ were dangerous, with magnitudes β and γ, then (16) and (20) would give a contradiction:

$$\frac{1}{4} \ge \sum_{i=2}^{N} \lambda_i^2 \ge \beta^2 + \gamma^2 > 2\left(\frac{5}{14}\right)^2 = \frac{25}{98} > \frac{1}{4}.$$

So let the unique dangerous eigenvalue be $-\beta$. Recall that (20), (15) and (14) give

$$b < \beta \le \lambda_1 \le 1.$$

The strictly convex function f_κ is increasing on $[b, 1]$, so $f_\kappa(\lambda_1) \ge f_\kappa(\beta)$. Since no other eigenvalue is dangerous, every summand of (19) apart from that of $-\beta$ is nonnegative, and therefore

$$\begin{aligned} t(C_n, W) + \kappa t(C_\ell, U) &\ge f_\kappa(\lambda_1) + \beta^n(1 - \kappa\beta^d) \\ &\ge f_\kappa(\beta) + \beta^n(1 - \kappa\beta^d) \\ &= (1-\beta)^n + \kappa\beta^\ell + \beta^n - \kappa\beta^{n+d} \\ &= (1-\beta)^n + \beta^n \\ &\ge 2^{1-n}, \end{aligned}$$

where the last inequality follows from convexity of $x \mapsto x^n$ applied to β and $1-\beta$. Finally, strict monotonicity of $\rho_{n,\ell}$ in Lemma 2.1 and $a \le a^\star_{n,\ell}$ imply

$$\rho_{n,\ell}(a) \le \rho_{n,\ell}(a^\star_{n,\ell}) = 2^{1-n}.$$

This proves (18) in the remaining case. □

The matching obstruction is the balanced disjoint union of two cliques,

$$W_{\mathrm{cl}}(x, y) := \begin{cases} 1, & x, y \in [0, 1/2] \text{ or } x, y \in (1/2, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.11 (Two-clique obstruction)**.** *If $a > a^\star_{n,\ell}$, then W_{cl} violates* (5)*.*

Proof. Every homomorphism from the connected graph C_n into W_{cl} maps all vertices into one of the two halves, so

$$t(C_n, W_{\mathrm{cl}}) = 2\left(\frac{1}{2}\right)^n = 2^{1-n}. \tag{21}$$

The complement $1 - W_{\mathrm{cl}}$ is the balanced complete bipartite graphon, and ℓ is odd, so

$$t(C_\ell, 1 - W_{\mathrm{cl}}) = 0. \tag{22}$$

By (21) and (22), the left-hand side of (5) at W_{cl} equals 2^{1-n}. Strict monotonicity of $\rho_{n,\ell}$ (Lemma 2.1) and $a > a^\star_{n,\ell}$ give

$$\rho_{n,\ell}(a) > \rho_{n,\ell}(a^\star_{n,\ell}) = 2^{1-n},$$

so (5) fails at W_{cl}. □

Proof of Theorem 1.1. By (4) the left-hand side of the critical equation (2) is $\rho_{n,\ell}(a)$, so that equation is (6); Lemma 2.1 shows that it has a unique solution $a^\star_{n,\ell}$ in $(1/2, 1)$, which is the number named in the theorem.

Now fix $a \in (0, 1)$. By (5), a belongs to $\pi(C_n, C_\ell)$ if and only if

$$t(C_n, W) + \kappa_{n,\ell}(a)\, t(C_\ell, 1 - W) \ge \rho_{n,\ell}(a)$$

holds for every graphon W. If $a \le a^\star_{n,\ell}$, then Proposition 2.10 establishes this inequality, so $a \in \pi(C_n, C_\ell)$. If $a > a^\star_{n,\ell}$, then the two-clique graphon W_{cl} of Proposition 2.11 violates it, so $a \notin \pi(C_n, C_\ell)$. The region therefore consists of exactly those a with $0 < a \le a^\star_{n,\ell}$, that is,

$$\pi(C_n, C_\ell) = \left(0, a^\star_{n,\ell}\right].$$

□

3 Remarks

The proof of Theorem 1.1 has been verified in the Lean 4 proof assistant [5], on top of the mathlib library [4]; see `https://github.com/mekty2012/Off-diagonal-Commonality-of-Even-and-Odd-Cycles`. The formal statement covers both directions and the existence and uniqueness of $a^\star_{n,\ell}$: the inequality holds for every graphon if and only if $a \le a^\star_{n,\ell}$. The formal proof differs from the one given here in four ways.

First, instead of the dyadic conditional expectations in Lemma 2.4, the formal proof approximates W directly by a step graphon in the L^1 norm; both cycle densities are Lipschitz in that norm, so the inequality transfers.

Second, the interlacing inequalities (12) are proved by an intersection argument instead of the min–max formulation: two spans of eigenvectors whose dimensions add up to more than N contain a common unit vector, and evaluating the quadratic forms of A and $A + P$ at that vector gives the inequality.

Third, the rearrangement in Lemma 2.6 is not obtained by sorting. Since the α_i are already ordered, rearranging $(\alpha_1, \dots, \alpha_{N-1}, \alpha_N + 1)$ only moves the entry $\alpha_N + 1$ to its correct position, and the formal proof writes this down explicitly, with the partial sums in closed form. Karamata's inequality (10) is proved in a form that requires only one of the two sequences to be monotone, so no second sorting argument is needed.

Fourth, the derivatives of f_κ are not computed. The paper locates the minimum of f_κ at b through f'_κ and f''_κ; the formal proof instead bounds $(1-x)^n$ and κx^ℓ from below by their supporting lines at $x = b$, both instances of Bernoulli's inequality, and the sum of the two lines is the constant $\rho_{n,\ell}(a)$ because $\kappa\ell b^{\ell-1} = na^{n-1}$. The same supporting-line estimate, together with monotonicity of powers on the nonnegative reals, proves that f_κ is increasing on $[b, 1]$. Strict monotonicity of $\rho_{n,\ell}$ is proved directly from its explicit formula, without differentiating. Finally, the formal proof obtains $x^n + (1-x)^n \ge 2^{1-n}$ by applying the supporting-line inequality at $1/2$ to both terms.

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