COMPUTER-ASSISTED BOUNDS $0.556921 \le p_c \le 0.6463$ FOR SITE PERCOLATION ON THE SQUARE LATTICE

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Abstract. Let p_c be the critical probability of Bernoulli site percolation on $\mathbb{Z}^2$. We prove $0.556921 \le p_c \le 0.6463$. The previous best rigorous bounds known to us are $p_c \ge 0.5567$ (Ashton, 2026, extending van den Berg and Ermakov, 1996) and $p_c \le 0.666894$ (Wierman and Oberly, 2024). The lower bound uses the growth-process comparison of van den Berg and Ermakov with the exact posterior on the exploration tree. We reduce this posterior to a scalar interval recursion and explore the tree generation by generation, with a prescribed order among siblings. This passes the value $0.556788\ldots$ that the same scalar computation gives when the order is not used, and the value 0.556788090 reported by Ashton as the limit of his relaxation. The upper bound uses the substitution method of Wierman, in two stages with 4×4 regions, from the line graph of $\mathbb{Z}^2$ (site threshold $1/2$) to $\mathbb{Z}^2$. Both bounds are certified by short pure-Python programs in exact rational and integer arithmetic.

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1. Introduction

Each vertex of a graph G is open with probability p, independently. Let $\theta_G(p)$ be the probability that a fixed vertex lies in an infinite open cluster, and $p_c(G) = \sup\{p : \theta_G(p) = 0\}$. For G connected and periodic, $p_c(G)$ does not depend on the fixed vertex. For $p > p_c(G)$ an infinite open cluster exists almost surely, and for $p < p_c(G)$ almost surely none exists. Write S for the square lattice $\mathbb{Z}^2$ with nearest-neighbour edges and $p_c = p_c(S)$. Numerically $p_c \approx 0.592746$ [Ja15]; its exact value is unknown.

Theorem 1.1. $0.556921 \le p_c \le 0.6463$.

Previous bounds. Lower bounds: 0.503478 (Tóth [To85]), 0.541 (Menshikov–Pelikh [MP89]), 0.556 (van den Berg–Ermakov [BE96]), and 0.5567 (Ashton [As26], a computer-assisted proof posted on 18 October 2026). Ashton keeps the growth process of [BE96] and conditions on the exact posterior of the exploration tree, using a fixed exploration order. He reports that his cone relaxation of this posterior saturates at 0.556788090. Upper bounds: 0.679492 (Wierman [Wi95], by the substitution method) and 0.666894 (Wierman–Oberly [WO24], by a two-stage substitution from a self-matching lattice). A poster of Oberly and Wierman [OW24] reports 0.662869; we have not found it in published form.

What is new. For the lower bound (Section 2) we use the same comparison as [BE96, As26] with two changes. First, the posterior is reduced to one scalar per tree edge, and intervals for these scalars obey a two-variable recursion (Lemma 2.5). Second, the exploration order is chosen adaptively: generation by generation, and among the children of one vertex the child of the minority type first. The certificate needs only exact rational arithmetic and runs in under a second. When the order information is not used, the same computation has threshold $0.5567880904\ldots$, which agrees to ten digits with the value 0.556788090 of [As26] (Remark 2.8).

For the upper bound (Section 3) we compare the line graph of $\mathbb{Z}^2$ (site threshold exactly $1/2$, by Kesten's theorem) with an intermediate lattice H, and then H with S. Each stage uses 4×4 regions. The partition laws are computed exactly, the symmetry group of the region is used to reduce their size, and stochastic domination is certified by an integer maximum flow. As an internal test of the program we also run a 2×2 version of the second stage.

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Sources. We could not consult the articles [BE96], [Wi95] and [WO24]. For [BE96] we relied on the public account in [As26]. All arguments used below are therefore proved here, except the following classical facts, which we cite without having checked them against the original articles:

(F1) $p_c(S) + p_c(S^*) \geq 1$, where S^* is $\mathbb{Z}^2$ with nearest-neighbour and diagonal edges (Fisher [Fi61]; equality holds by Russo [Ru81]).

(F2) Bond percolation on $\mathbb{Z}^2$ has critical probability $1/2$ (Kesten [Ke80]).

We also use the standard fact that the existence of an infinite open cluster is a tail event, so it has probability 0 or 1.

Files. The two computational statements (Propositions 2.7 and 3.5) are certified by the files `anc/verif_site_Z2_minoration_0556921.py`, `anc/verif_site_Z2_majoration_06463.py`. They use only the Python standard library. Their outputs are in the corresponding `.log` files.

2. The lower bound

Throughout this section $p \in (0,1)$, $q = 1-p$, and β denotes the product Bernoulli measure with parameter $1-p$ (on any finite index set).

2.1. **The two-layer lattice.** Let L have vertex set $\mathbb{Z}^2 \times \{0,1\}$; call $(v,1)$ and $(v,0)$ the upper and lower vertex of the column v. For an edge $\{v,w\}$ of S, let min be the endpoint with the smaller coordinate sum and max the other. The edges of L are the vertical edges $(v,0)(v,1)$, and, for every edge $\{v,w\}$ of S, the three edges

$$(\min,1)(\max,1), \qquad (\min,0)(\max,0), \qquad (\min,1)(\max,0).$$

Lemma 2.1. *$p_c(L) \geq p_c(S^*)$.*

Proof. Let $\varphi(x_1,x_2,x_3) = (x_1 - x_2,\ x_1 + x_2 + x_3)$ for $(x_1,x_2) \in \mathbb{Z}^2$, $x_3 \in \{0,1\}$. It is a bijection onto $\mathbb{Z}^2$: given (a,b), necessarily $x_3 \equiv b - a \pmod 2$, and then $x_1 + x_2 = b - x_3$ and $x_1 - x_2 = a$ have the same parity. Edges of L are mapped to edges of S^*: in-layer edges along e_1 and e_2 become steps $(1,1)$ and $(-1,1)$; vertical edges become $(0,1)$; the edge $(\min,1)(\max,0)$ becomes $(1,0)$ if $\max = \min + e_1$ and $(-1,0)$ if $\max = \min + e_2$. So L is isomorphic to a subgraph of S^* with the same vertex set. With the same vertex states, every open path of L is mapped to an open path of S^*. □

Lemma 2.2. *If $\theta_S(1-p) > 0$ implies that L has an infinite open cluster with positive probability at parameter p, then $p_c(S) \geq 1-p$.*

Proof. If $\theta_S(1-p) = 0$ then $p_c(S) \geq 1-p$ by definition. Otherwise L percolates at p, so $p \geq p_c(L) \geq p_c(S^*)$ by Lemma 2.1, and (F1) gives $p_c(S) \geq 1 - p_c(S^*) \geq 1-p$. □

2.2. **The growth process.** Let $\varepsilon(v) = (u(v), l(v)) \in \{0,1\}^2$ record whether the upper and lower vertices of the column v are open. The $\varepsilon(v)$ are independent with law π, $\pi(ab) = p^{a+b}q^{2-a-b}$. We write states as $00, 01, 10, 11$, the first digit being u. For an edge $e = \{v,w\}$ of S put

$$\eta(e) = u(\min)u(\max) \ \vee \ l(\min)l(\max) \ \vee \ u(\min)l(\max),$$

so that $\eta(e) = 1$ if and only if some open vertex of the column min is adjacent in L to some open vertex of the column max.

The process assigns values $\xi(v) \in \{0,1\}$ to vertices of $\mathbb{Z}^2$ step by step. At step 0, $\xi(O) = 1$ if and only if $\varepsilon(O) \neq 00$. A vertex v with $\xi(v) = 1$ is a *candidate* if some neighbour of v has no value yet. At each later step a candidate b is selected by a rule depending only on the values revealed so far. Its neighbours without a value become its *children*, and each child w receives $\xi(w) = \eta(\{b,w\})$. The process stops when there is no candidate. The edges from vertices to their children form a tree T rooted at O; the *depth* of a vertex is its distance to O in T. A selected vertex has no unvalued neighbour afterwards, so it is selected only once.

Slots. A child w of b is a *P-child* if w is the max endpoint of $\{b,w\}$, and an *M-child* otherwise. The four neighbours of v are $v \pm e_1$, $v \pm e_2$: two would be P-children of v and two M-children. So a non-root vertex that is a P-child of its parent has slots (P,P,M), an M-child has slots

(P, M, M), and O has slots (P, P, M, M). For a non-root vertex we call the slot type opposite to its own type the *minority* type.

The rule. We use the following selection rule: select a candidate of minimal depth; among these, one whose parent was selected earliest; among children of the same parent, a minority child before the majority children; between two majority children, the one first in the fixed order $e_1, e_2, -e_1, -e_2$ of the directions from the parent. A vertex stops being a candidate only when it is selected or when all its neighbours have values, and it never becomes a candidate again. The depths of the selected vertices form a nondecreasing sequence: a selection creates candidates of depth one more than the selected vertex, and all other candidates have depth at least that of the selected vertex.

Lemma 2.3. *Say that a law μ on $\{0,1\}^k$ dominates β if $\mu(A) \geq \beta(A)$ for every up-set A. Assume that (i) $1 - q^2 \geq 1 - p$; (ii) given $\xi(O) = 1$, the law of the values of the four children of O dominates β; (iii) at every later step and for every possible history, the conditional law of the values given to the children dominates β. Then $\theta_S(1-p) > 0$ implies that L percolates with positive probability at p, and hence $p_c(S) \geq 1 - p$.*

Proof. Let $(U_t)_{t\geq 0}$ be independent uniform variables and $(Y(v))_{v\in\mathbb{Z}^2}$ independent $\mathrm{Ber}(1-p)$ variables, all independent of ε. Let $\mathcal{G}_t$ be the σ-field generated by the values revealed up to step t and by $U_0, \ldots, U_t$. At step t the set of new children C_t is $\mathcal{G}_{t-1}$-measurable, and, since the U's are independent of ε, the conditional law ν_t of $\xi|_{C_t}$ given $\mathcal{G}_{t-1}$ is the one in (ii) or (iii). By Lemma 2.4 below there is a kernel K_t with $\sum_x \nu_t(x)K_t(x,\cdot) = \beta$ and $K_t(x,z) > 0 \Rightarrow z \leq x$. Using U_t we draw $\zeta|_{C_t}$ from $K_t(\xi|_{C_t}, \cdot)$; at step 0 we draw $\zeta(O) \leq \xi(O)$ with $\zeta(O) \sim \mathrm{Ber}(1-p)$, which is possible by (i). Vertices that never receive a value get $\zeta(v) = Y(v)$. Then $\zeta \leq \xi$ on every valued vertex, and given $\mathcal{G}_{t-1}$ the vector $\zeta|_{C_t}$ has law β.

ζ is i.i.d. $\mathrm{Ber}(1-p)$. Let $\zeta^{(m)}$ equal ζ on vertices valued at steps $\leq m$, and Y elsewhere. Passing from $\zeta^{(m-1)}$ to $\zeta^{(m)}$ replaces $Y|_{C_m}$ by $\zeta|_{C_m}$. Given $\mathcal{G}_{m-1}$, both vectors have law β, and both are independent of Y off C_m. Since C_m is $\mathcal{G}_{m-1}$-measurable, $\zeta^{(m)}$ and $\zeta^{(m-1)}$ have the same law. By induction every $\zeta^{(m)}$ has the law of Y. As $m \to \infty$, $\zeta^{(m)} \to \zeta$ pointwise, so ζ has the law of Y.

Comparison. If the process stops, the set V_1 of valued vertices with $\xi = 1$ is finite and all neighbours of V_1 are valued. A ζ-open path from O starts in V_1, because $\zeta(O) \leq \xi(O)$. Each further vertex is a neighbour of V_1, hence valued, and has $\xi \geq \zeta = 1$, so it is again in V_1. Hence the ζ-cluster of O is finite. If the process never stops, infinitely many distinct vertices are selected, all with $\xi = 1$. By induction along T, every open vertex of a column v with $\xi(v) = 1$ is joined in L by an open path to an open vertex of the column O. The open vertices of one column are adjacent, and $\eta = 1$ on each tree edge with value 1. So L has an infinite open cluster. Since ζ is site percolation on S with density $1-p$, $\theta_S(1-p) > 0$ forces the process to survive with positive probability. Lemma 2.2 concludes. □

Lemma 2.4 (finite Strassen theorem)**.** *Let $(P, \leq)$ be a finite poset and μ, ν probability measures on P with $\mu(A) \geq \nu(A)$ for every up-set A. Then there is a probability measure Γ on $P \times P$ with marginals ν and μ, supported on $\{(x,y) : x \leq y\}$.*

Proof. Consider the network with arcs $s \to x$ of capacity $\nu(x)$, $x \to y'$ of infinite capacity for $x \leq y$, and $y' \to t$ of capacity $\mu(y)$, where y' is a copy of y. A cut of finite capacity is determined by the set A of x kept on the side of s. It must then cut all $y' \to t$ with $y \in \uparrow A$, so its capacity is at least $\nu(P \setminus A) + \mu(\uparrow A) \geq \nu(P \setminus A) + \nu(\uparrow A) \geq 1$. By the max-flow min-cut theorem there is a flow of value 1, and its values on the arcs $x \to y'$ define Γ. □

2.3. **The posterior.** Fix a step at which $b \neq O$ is selected, and let c be its parent. The history fixes T and the values $x_e = \eta(e)$ on its edges. An unvalued vertex is an endpoint of no tree edge. Hence, given the history, the variables ε of unvalued vertices are still i.i.d. with law π and independent of the rest. The law of $(\varepsilon(v))_{v \text{ valued}}$ is proportional to

$$\mathbf{1}[\varepsilon(O) \neq 00] \prod_v \pi(\varepsilon(v)) \prod_{e\in T} \mathbf{1}[\eta(e) = x_e], \tag{1}$$

because the history occurs exactly when the revealed values are the observed ones (the selection rule is a function of them). When b is selected its only tree neighbour is c.

Since T is a tree, (1) is computed by messages. For a tree edge $\{x, y\}$ let $m_{y\to x}(\gamma)$, for $\gamma \in \{0,1\}^2$, be the total weight of the branch of T at y (the component of $T \setminus \{x, y\}$ containing y), together with the factor of the edge $\{x, y\}$, when $\varepsilon(x) = \gamma$. For a vertex x and a tree neighbour y let $\mu_x^{(y)}$ be the *cavity law* of $\varepsilon(x)$: proportional to π times the messages into x from all tree neighbours except y, times $\mathbf{1}[\gamma \neq 00]$ if $x = O$. Then the posterior of $\varepsilon(b)$ is proportional to $w(\delta) = \pi(\delta) \sum_\gamma \mu_c^{(b)}(\gamma)\, \mathbf{1}[\eta = 1 \text{ on } \{c, b\}]$ (with $\varepsilon(c) = \gamma$, $\varepsilon(b) = \delta$), and the children of b have independent values given $\varepsilon(b)$.

The cavity laws give no mass to 00: for $x \neq O$ the message from the parent of x vanishes at 00, since $\eta = 1$ needs an open vertex in each column. A direct case check from the definition of η gives the following lemma, where λ and υ denote the probabilities that $l(\cdot) = 1$ and $u(\cdot) = 1$ under the indicated laws. Explicitly, $\eta = 0$ if one of the two columns is 00; otherwise $\eta = l(\max)$ if $\varepsilon(\min) = 01$, and $\eta = 1$ if $\varepsilon(\min) \in \{10, 11\}$.

Lemma 2.5. *Let y be a child of x in T, and normalise messages by their value at* 11*.*

1. *If y is a P-child, then $m_{y\to x}(10) = m_{y\to x}(11)$, and the quotient $m_{y\to x}(01)/m_{y\to x}(11)$ equals $1/q$ if $\eta(\{x, y\}) = 0$, and λ_y if $\eta(\{x, y\}) = 1$, where λ_y is computed from the law of $\varepsilon(y)$ proportional to $\pi(\cdot)\mathbf{1}[\cdot \neq 00] \prod_z m_{z\to y}$ over the children z of y.*
2. *If y is an M-child, the same holds with* 01 *and* 10 *exchanged and λ_y replaced by υ_y.*
3. *The message from x to y satisfies the following. If y is a P-child, then $m_{x\to y}(01) = m_{x\to y}(11)$ and $m_{x\to y}(10)/m_{x\to y}(11) = \upsilon$ computed under $\mu_x^{(y)}$; if y is an M-child, $m_{x\to y}(10) = m_{x\to y}(11)$ and $m_{x\to y}(01)/m_{x\to y}(11) = \lambda$ computed under $\mu_x^{(y)}$.*

Proof. If y is a P-child, then with $\varepsilon(x) = \gamma$ and $\varepsilon(y) = \delta$ we have $\eta = \mathbf{1}[\delta \neq 00]$ for $\gamma \in \{10, 11\}$ and $\eta = l(\delta)$ for $\gamma = 01$. Summing $\pi(\delta)$ against these gives (1): for $\eta = 0$ the values are q^2, q^2 at $10, 11$ and q at 01, and for $\eta = 1$ the ratio is $\sum_{l(\delta)=1} / \sum_{\delta\neq 00}$ of the weights of $\varepsilon(y)$. As a function of δ with γ summed out, $\eta = u(\gamma)$ for $\delta = 10$ and $\eta = \mathbf{1}[\gamma \neq 00]$ for $\delta \in \{01, 11\}$, which gives (3). The M-case is the mirror image: if y is an M-child, $\eta = \mathbf{1}[\delta \neq 00]$ for $\gamma \in \{01, 11\}$ and $\eta = u(\delta)$ for $\gamma = 10$. □

Reduction to one scalar. Put $\kappa = q/p = \pi(01)/\pi(11) = \pi(10)/\pi(11)$ and

$$f_l(X, Y) = \frac{X + 1}{X + Y + 1}, \qquad f_u(X, Y) = \frac{Y + 1}{X + Y + 1} = f_l(Y, X).$$

For a law with weights w on $\{01, 10, 11\}$, $X = w(01)/w(11)$ and $Y = w(10)/w(11)$, the probability of $l = 1$ is $f_l(X, Y)$ and that of $u = 1$ is $f_u(X, Y)$. The function f_l is nondecreasing in X and nonincreasing in Y. For a child y of x call the non-trivial ratio of Lemma 2.5(1)–(2) the *ratio* r_y of y. If y has $\eta = 1$, two slots of the same type as y and one of the other type, then by Lemma 2.5

$$r_y = f_l(\kappa\, r_1 r_2,\ \kappa\, r_3), \tag{2}$$

where r_1, r_2 are the ratios of the children of y in the two same-type slots and r_3 that of the other slot, an empty slot counting as ratio 1. (For an M-child, $\upsilon_y = f_u(\kappa r_3, \kappa r_1 r_2) = f_l(\kappa r_1 r_2, \kappa r_3)$.) An unselected child with $\eta = 1$ has ratio $f_l(\kappa, \kappa) = 1/(1 + q)$, and a child with $\eta = 0$ has ratio $1/q$.

Let $R_0 = [1/(1 + q),\ 1/q]$, which contains 1, and for $k \geq 1$ let R_k be the smallest interval containing R_0 and $f_l(\kappa R_{k-1}^2, \kappa R_{k-1})$. If $R_{k-1} = [a, b]$, its endpoints are $\min(\inf R_0,\ f_l(\kappa a^2, \kappa b))$ and $\max(\sup R_0,\ f_l(\kappa b^2, \kappa a))$. The R_k increase with k.

Lemma 2.6. *If every selected vertex in the branch at a child y has depth less than* $\operatorname{depth}(y) + k$*, then $r_y \in R_k$.*

Proof. Induction on k. If y is unselected (in particular if $k = 0$), then $r_y = 1/q$ when $\eta(\{x, y\}) = 0$ and $r_y = 1/(1 + q)$ otherwise, so $r_y \in R_0 \subseteq R_k$. If y is selected and $k \geq 1$, its children z have depth $\operatorname{depth}(y) + 1$ and all selected vertices in their branches have depth less than

depth$(z)+(k-1)$, so their ratios lie in R_{k-1} by induction; an empty slot has ratio $1 \in R_0 \subseteq R_{k-1}$. By (2) and the monotonicity of f_l, $r_y \in f_l(\kappa R_{k-1}^2, \kappa R_{k-1}) \subseteq R_k$. The inclusions $R_0 \subseteq R_1 \subseteq \cdots$ follow by induction from $R_0 \subseteq R_1$ and the monotonicity of f_l. □

The path scalar. For a vertex x and its child y on the path from O to b, let ψ_x be the ratio υ (if y is a P-child) or λ (if y is an M-child) of Lemma 2.5(3), computed under $\mu_x^{(y)}$. Let $x_0 = O, x_1, \ldots, x_{n-1} = c, x_n = b$ be the path. If x_i (with $i \geq 1$) has the same type as x_{i+1}, its remaining slots are one slot of its own type and one of the other type, and

$$\psi_{x_i} = f_u(\kappa r,\ \kappa r' \psi_{x_{i-1}}), \tag{3}$$

where r is the ratio of the same-type sibling of x_{i+1} and r' that of the other one. If x_{i+1} has the minority type of x_i, the two remaining slots are of the majority type, and

$$\psi_{x_i} = f_u(\kappa \psi_{x_{i-1}},\ \kappa r r'). \tag{4}$$

At the root, $\psi_O = f_u(\kappa r, \kappa r' r'')$, where r is the ratio of the other P-child of O and r', r'' those of its two M-children. These formulas are written for x_{i+1} a P-child; for an M-child, exchange 01 and 10 throughout, which leaves the formulas unchanged. Similarly, the posterior of $\varepsilon(b)$ is proportional to

$$\rho_\psi(01) = q, \quad \rho_\psi(10) = q\psi, \quad \rho_\psi(11) = p \qquad (\psi = \psi_c) \tag{5}$$

if b is a P-child, with 01 and 10 exchanged if b is an M-child.

Use of the order. When b (of depth n) is selected, no vertex of depth $> n$ has been selected, since depths of selected vertices are nondecreasing. So a sibling of x_{i+1} (depth $i+1$) has selected descendants only down to depth n, and its ratio lies in R_{n-i} by Lemma 2.6. For $i = n-1$ this is R_1. In case (4) at $i = n-1$, b is the minority child of c. It has been a candidate since it received its value, because candidacy is never regained. So the rule selects it before its two siblings, which are of majority type, and their ratios lie in R_0.

Let Ψ be the interval obtained by the following interval computation. For each $n \leq N+1$, start at the root with the siblings in R_n. Then apply, for $j = n-1, \ldots, 1$, the worse of (3) and (4) with siblings in R_j. For $j = 1$, use R_1 in (3) and R_0 in (4). For depths $n \geq N+2$, start instead at the vertex of depth $n-N-1$ with the trivial interval $[0,1]$ and apply the same N steps $j = N, \ldots, 1$. Take the hull over all cases. All bounds are rounded outward. By monotonicity of f_u, $\psi_c \in \Psi$ at every step.

2.4. **The finite check.** For a P-type b the children occupy slots (P, P, M). Given $\varepsilon(b) = \delta$ they are independent, and the probability that a child gets value 1 is

$$\text{P-slot:} \quad s_P(01) = p, \quad s_P(10) = s_P(11) = 1 - q^2,$$
$$\text{M-slot:} \quad s_M(10) = p, \quad s_M(01) = s_M(11) = 1 - q^2.$$

Neighbours of b that already have a value are replaced by imaginary children with fresh ε. The law of the real children is a marginal of the three-slot law, and marginals preserve stochastic domination. Let ν_ψ be the mixture over $\delta \sim \rho_\psi$ of the product laws on $\{0,1\}^3$. Condition (iii) of Lemma 2.3 holds if

$$M_A(\psi) := \nu_\psi(A) - \beta(A) \geq 0$$

for every $\psi \in \Psi$ and every up-set A of $\{0,1\}^3$. There are 18 nontrivial ones. The unnormalised form $\sum_\delta \rho_\psi(\delta)(\nu_\delta(A) - \beta(A))$ is affine in ψ. So it suffices to check the two endpoints of Ψ. M-type vertices give the same conditions after exchanging $01 \leftrightarrow 10$ and $P \leftrightarrow M$. For the root, the posterior of $\varepsilon(O)$ given $\xi(O) = 1$ is π restricted to $\{01, 10, 11\}$, the slots are (P, P, M, M), and the 166 nontrivial up-sets of $\{0,1\}^4$ are checked.

Proposition 2.7. *Let $p_0 = 443079/10^6$ and $N = 60$. In exact rational arithmetic, with outward rounding to the grid 10^{-40}, the computation above gives $\Psi \subseteq [0.3414794, 0.922660]$. Moreover: $\min_A M_A(\psi) \geq 5.01 \cdot 10^{-8}$ at both endpoints of Ψ; the root margin is at least 0.01456; and $1 - q^2 - (1 - p_0) \geq 0.1329$. The tail computation for depths $n \geq N+2$ gives the same interval as the depths $n \leq N+1$ to 15 digits.*

This is printed by the lower-bound program, which also certifies $p = 11077/25000$ (bound 0.55692) with margin $1.997 \cdot 10^{-6}$. Lemma 2.3 then gives $p_c \geq 1 - p_0 = 0.556921$.

The certified interval for the ratios is $R_{61} \subseteq [0.335962, 1.795587]$ at p_0. By bisection, the method with this order works up to $1 - p = 0.5569210257$.

Remark 2.8 (fixed order). If the order is not used, i.e. every R_j is replaced by R_{N+1}, the threshold of the same computation is $1 - p = 0.5567880904\ldots$. This agrees with the limit 0.556788090 reported in [As26]. Ashton propagates cones of messages along a fixed total order, while we use a scalar recursion; the two numbers agree to ten digits. We do not claim that $0.556788\ldots$ is a limit for every analysis with a fixed order. In the program this version serves only as a self-test, at $p = 0.4433$; it is not used in the proof.

3. The upper bound

3.1. **Lattices.** For a face $[i, i+1] \times [j, j+1]$ of $\mathbb{Z}^2$ we say "add its diagonals" for adding both edges $(i, j)(i+1, j+1)$ and $(i+1, j)(i, j+1)$. Let

- LG: add the diagonals of all faces with $i \equiv j \pmod 2$;
- H: add the diagonals of all faces with i and j odd;
- S: the square lattice.

Then $S \subset H \subset \mathrm{LG}$.

Lemma 3.1. LG *is isomorphic to the line graph of* $\mathbb{Z}^2$. *Hence* $p_c(\mathrm{LG}) = 1/2$.

Proof. Map the midpoint (a, b) of an edge of $\mathbb{Z}^2$ to $(a+b-\frac{1}{2},\ b-a+\frac{1}{2})$. The midpoints $(x+\frac{1}{2}, y)$ go to the points $(x+y, y-x)$ with even coordinate sum, and $(x, y+\frac{1}{2})$ to $(x+y, y-x+1)$ with odd coordinate sum. Two perpendicular edges sharing an endpoint become nearest neighbours. The four edges at a vertex (x, y) become the corners of the face with lower left corner $(x+y-1, y-x)$. The two collinear pairs among them become its diagonals. These faces are those with $i \not\equiv j \pmod 2$, and a translation by e_1 gives LG. Site percolation on a line graph is bond percolation on the original graph, so (F2) gives $p_c(\mathrm{LG}) = 1/2$. □

3.2. **Regions and ghost partitions.** Let G be one of these lattices and $\mathcal{R}$ a tiling of $\mathbb{Z}^2$ by translates $z + B + 4\mathbb{Z}^2$ of the box $B = \{0, 1, 2, 3\}^2$. A *crossing edge* is an edge of G whose endpoints lie in different tiles. A *ghost* of a tile R is a pair (v, e) with $v \in R$ and e a crossing edge at v. Given the states in R, the *ghost partition* Π_R of the ghosts of R puts (v, e) and (v', e') in the same block if v and v' are open and joined by an open path using edges of G inside R. Ghosts at closed vertices are singletons. The partitions of different tiles are independent. We order partitions by refinement: $\pi \leq \pi'$ if every block of π is contained in a block of π'.

Given partitions $\pi = (\pi_R)$, let $\mathcal{A}(\pi)$ be the graph whose vertices are the blocks of all tiles, with an edge between the block of (v, e) and the block of (w, e) for every crossing edge $e = \{v, w\}$.

Lemma 3.2. *G has an infinite open cluster if and only if $\mathcal{A}(\Pi)$ has an infinite connected component.*

Proof. If K is an infinite open cluster, take an infinite self-avoiding open path in K. It leaves every finite tile, so it splits into finite segments inside tiles joined by crossing edges. Entry and exit ghosts of a segment are in one block, so the path gives an infinite walk in $\mathcal{A}(\Pi)$ through blocks of infinitely many tiles. Conversely, $\mathcal{A}(\Pi)$ is locally finite, so an infinite component contains an infinite simple path $B_0, B_1, \ldots$, the edge B_iB_{i+1} coming from a crossing edge $e_i = \{v_i, w_i\}$ with $(v_i, e_i) \in B_i$ and $(w_i, e_i) \in B_{i+1}$. For $i \geq 1$ the ghosts (w_{i-1}, e_{i-1}) and (v_i, e_i) of B_i are distinct, since the path is simple. So both vertices are open and joined inside their tile. Then every e_i with $i \geq 1$ joins two open vertices, and $w_1, v_2, w_2, v_3, \ldots$ lie in one open cluster. Since the path is simple and each tile has finitely many blocks, it visits infinitely many tiles, so this cluster is infinite. □

Lemma 3.3. *Let G, G' be two of the lattices, with the same tiling and the same crossing edges. Suppose the law of Π_R for G' at p' dominates that for G at p in the refinement order, for every tile R. If G percolates at p, then G' percolates at p'.*

Proof. For each tile take a coupling (Π_R, Π'_R) with $\Pi_R \le \Pi'_R$, independently over tiles. The ghost sets and the edges of $\mathcal{A}$ are the same for G and G'. Coarsening only merges blocks within tiles, so an infinite component of $\mathcal{A}(\Pi)$, which meets infinitely many tiles, is mapped into an infinite component of $\mathcal{A}(\Pi')$. Lemma 3.2 applies to both lattices, since Π and Π' have the laws of the ghost-partition fields. □

The tile laws are the same for all tiles. The two stages are:

- *Stage 1*: tiles $B + 4\mathbb{Z}^2$, $G = \mathrm{LG}$ at $p = 0.50001$, $G' = H$ at $p' = 0.5755$. A face meeting two tiles has $i \equiv 3$ or $j \equiv 3 \pmod 4$. In LG it has diagonals only if $i \equiv j \pmod 2$, i.e. if i, j are both odd, which is exactly the condition in H. So the crossing edges coincide. The extra diagonals of LG (faces with i, j even) lie inside tiles.
- *Stage 2*: tiles $(1,1) + B + 4\mathbb{Z}^2$, $G = H$ at $p = 0.5755$, $G' = S$ at $p' = 0.6463$. A face meeting two tiles has $i \equiv 0$ or $j \equiv 0 \pmod 4$, so it has no diagonal in H. The crossing edges are the nearest-neighbour ones in both lattices.

3.3. **Symmetry and the certificate.** The dihedral group D_4 of the box acts on the ghosts of a tile, and in both stages it preserves the edges inside the tile and the crossing edges. So the partition laws are D_4-invariant (the program also checks this exactly). Let $\mathcal{O}$ and $\mathcal{O}'$ be the orbits of partitions for G and G', with masses $\nu(\mathcal{O})$ and $\mu(\mathcal{O}')$.

Lemma 3.4. *Suppose that there is a transport plan $f(\mathcal{O}, \mathcal{O}') \ge 0$ with marginals ν and μ, such that $f(\mathcal{O}, \mathcal{O}') > 0$ only if $x_0 \le y_0$ for some $x_0 \in \mathcal{O}$, $y_0 \in \mathcal{O}'$. Then the law of Π'_R dominates that of Π_R.*

Proof. For each such pair of orbits with $f > 0$, spread $f(\mathcal{O}, \mathcal{O}')$ uniformly over the pairs (gx_0, gy_0), $g \in D_4$. These pairs are comparable, since g preserves refinement. The point $x \in \mathcal{O}$ receives $f \cdot |\mathrm{Stab}(x_0)|/|D_4| = f/|\mathcal{O}|$, and similarly for $\mathcal{O}'$. By invariance, the laws are uniform on each orbit, so the marginals are the two partition laws. □

The program computes the partition laws as integer polynomials in p, by enumerating the 2^{16} states of a tile with union-find. It evaluates them exactly at the rational parameters, reduces to orbits, and joins two orbits by an arc when some element of the second is coarser than a fixed element of the first. It then computes an integer maximum flow (Dinic's algorithm) with capacities equal to the scaled masses.

Proposition 3.5. *In exact integer arithmetic:*

(1) *Stage 1:* 13823 *partitions for H and* 16977 *for* LG*, in* 1851 *and* 2265 *orbits, with* 300547 *comparability arcs. The maximum flow equals the total mass.*
(2) *Stage 2:* 8051 *partitions for S and* 12061 *for H, in* 1105 *and* 1625 *orbits, with* 135731 *arcs. The maximum flow equals the total mass.*

By Lemmas 3.4 and 3.3, LG percolates at $0.50001 > 1/2$ (Lemma 3.1), hence H percolates at 0.5755, hence S percolates at 0.6463. So $p_c \le 0.6463$. Both flows were also recomputed with the maximum-flow routine of the `networkx` library (version 3.7), with the same result.

Self-test. With 2×2 tiles $(1,1) + \{0,1\}^2 + 2\mathbb{Z}^2$, the domination of H at 0.586668 by S holds at 0.6795 and fails at 0.6794. This run is only a regression test of the program; it does not prove anything about [Wi95], whose chain we could not check.

Sharpness. Stage 2 fails with S at 0.6462 (deficit about $6 \cdot 10^{-5}$ of the total mass), while stage 1 still holds with H at 0.5754. So on the grid 10^{-4}, 0.6463 is the best value of this chain with 4×4 tiles.

Running the programs. On our machine the lower-bound program runs in under one second, and the upper-bound program in 15 to 25 seconds with about 110 MB of memory. Both print `CERTIFIE` on success.

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