A CERTIFIED LOWER BOUND FOR THE ESSENTIAL MINIMUM OF THE ZHANG–ZAGIER HEIGHT

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Abstract. Let h be the absolute logarithmic Weil height. We prove that the essential minimum of the Zhang–Zagier height $h(\alpha)+h(1-\alpha)$ is at least 0.2497815: for every $\delta>0$, only finitely many algebraic numbers α satisfy $h(\alpha)+h(1-\alpha)<0.2497815-\delta$. The previous best lower bound was 0.24874 (Flammang, 2018). The proof combines Smyth's auxiliary-function method, with the polynomials of Doche and of Flammang, and the condition that the discriminant of the minimal polynomial is a nonzero integer, in the potential-theoretic form used by Smith and by Orloski, Talebizadeh Sardari and Smith for the trace problem. The infimum of the resulting auxiliary function is bounded below rigorously by monotone interval bounds and adaptive bisection, from a stored certificate. The statement concerns the essential minimum and is weaker in nature than the pointwise bounds of Doche and Flammang.

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1. Introduction

For an algebraic number α of degree n, with primitive minimal polynomial $P=a_n\prod_{i=1}^n(x-\alpha_i)\in\mathbb{Z}[x]$, let

$$h(\alpha)=\frac{1}{n}\Big(\log|a_n|+\sum_{i=1}^n\log\max(1,|\alpha_i|)\Big),\qquad h_Z(\alpha)=h(\alpha)+h(1-\alpha).$$

Zhang [Zh92] and Zagier [Za93] showed that h_Z is bounded below by a positive constant outside $\{0,1,(1\pm i\sqrt{3})/2\}$; Zagier proved $h_Z(\alpha)\geq\frac{1}{2}\log\frac{1+\sqrt{5}}{2}$ outside the roots of $x(x-1)(x^2-x+1)$, with equality at the primitive tenth roots of unity. The *essential minimum* $\operatorname{ess}(h_Z)$ is the supremum of the real numbers H such that $\{\alpha\in\overline{\mathbb{Q}}:h_Z(\alpha)\leq H\}$ is finite. It is entry 82 of the list of optimization constants of Davis, Ivanisvili and Tao [OC]. The known bounds are

$$0.24874\text{ (Flammang [Fl18])}\ \leq\ \operatorname{ess}(h_Z)\ \leq\ 0.2536331\text{ (Grinsztajn [Gr26])},$$

after the lower bound 0.248247 and the upper bound 0.254437 of Doche [Do01a, Do01b], which Burgos Gil, Menares, Qu and Sombra [BMQS26] record as the best known; lower upper bounds have been announced in self-archived computations [OC, entry 82]. The lower bounds of Doche and Flammang are *pointwise*: they hold for every α outside an explicit finite set (Flammang states hers for algebraic integers; since the essential minimum is attained along sequences of algebraic integers [BMQS26], her bound also bounds $\operatorname{ess}(h_Z)$).

Theorem 1.1. $\operatorname{ess}(h_Z)\geq 0.2497815$. *Equivalently, for every $\delta>0$ only finitely many algebraic numbers satisfy $h(\alpha)+h(1-\alpha)<0.2497815-\delta$, and in multiplicative form* $\liminf\big(M(\alpha)M(1-\alpha)\big)^{1/\deg\alpha}\geq e^{0.2497815}=1.2837448\ldots$ *along any sequence of distinct algebraic numbers.*

More precisely, the proof gives, for α of degree n outside a finite explicit set,

$$h_Z(\alpha)\ \geq\ 0.2497815-\lambda_0\Big(\frac{\log n}{n}+\frac{599}{n^2}\Big),\qquad \lambda_0=0.0862985\ldots, \tag{1}$$

which is not a pointwise bound of the type of [Do01a, Fl18]; the loss is $8.0\cdot10^{-5}$ at $n=10^4$.

The method is that of Smyth [Sm81], as used by Doche and Flammang, to which one condition is added: the discriminant of P is a nonzero integer. In the limit this becomes a condition on the logarithmic energy of the limiting distribution of conjugates, which is the condition used by Smith [Smi24] and by Orloski, Talebizadeh Sardari and Smith [OTS25] for the Schur–Siegel–Smyth trace problem. In the framework of [BMQS26], where the essential minimum is

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characterised by strong duality between Smyth's method and measures satisfying $\int \log|Q|\,d\mu \geq 0$ for all $Q \in \mathbb{Z}[x]\setminus\{0\}$, this energy condition is implied by the full family of these linear conditions; its use here is computational: a single inequality replaces a large part of a search over polynomials of high degree. The contribution of this note is the numerical value and its certification.

2. The certificate inequality

Put $g(z) = \log^+|z| + \log^+|1-z|$. For α as above put $L = \frac{1}{n}\log|a_n| \geq 0$ and $\mu_\alpha = \frac{1}{n}\sum_i \delta_{\alpha_i}$. Since $1-\alpha$ has minimal polynomial $\pm P(1-x)$, with leading coefficient $\pm a_n$,

$$h_Z(\alpha) = 2L + \int g\,d\mu_\alpha. \tag{2}$$

Resultant conditions. If $Q \in \mathbb{Z}[x]$ does not vanish at α, then $\mathrm{Res}(P,Q) = a_n^{\deg Q}\prod_i Q(\alpha_i)$ is a nonzero integer, so

$$\deg(Q)\,L + \int \log|Q|\,d\mu_\alpha \;\geq\; 0. \tag{3}$$

Discriminant condition. $\mathrm{disc}(P) = a_n^{2n-2}\prod_{i<j}(\alpha_i-\alpha_j)^2$ is a nonzero integer, so

$$2\Big(1-\frac{1}{n}\Big)L + \frac{1}{n^2}\sum_{i\neq j}\log|\alpha_i-\alpha_j| \;\geq\; 0. \tag{4}$$

For a finite positive measure η with compact support put $U_\eta(z) = \int \log|z-w|\,d\eta(w)$ and $I(\eta) = \iint \log|z-w|\,d\eta\,d\eta$.

Lemma 2.1. *Let η be a probability measure with compact support and a density bounded by D. Then for every $\varepsilon > 0$ and every α of degree n,*

$$\frac{1}{n^2}\sum_{i\neq j}\log|\alpha_i-\alpha_j| \;\leq\; \frac{2}{n}\sum_i U_\eta(\alpha_i) - I(\eta) + \frac{1}{n}\log\frac{1}{\varepsilon} + D\pi\varepsilon^2.$$

Proof. Let σ be the average of the uniform probability measures on the circles of radius ε centred at the α_i. The potential of such a circle, centred at c, is $\log\max(|z-c|,\varepsilon) \geq \log|z-c|$, and its energy is $\log\varepsilon$; hence, with no assumption on the position of the circles, $I(\sigma) \geq \frac{1}{n^2}\big(\sum_{i\neq j}\log|\alpha_i-\alpha_j| + n\log\varepsilon\big)$. Next, $\int U_\eta\,d\sigma = \frac{1}{n}\sum_i\int\log\max(|\alpha_i-w|,\varepsilon)\,d\eta(w)$, and $\int\log\max(|a-w|,\varepsilon)\,d\eta(w) - U_\eta(a) = \int_{|w-a|<\varepsilon}\log\frac{\varepsilon}{|w-a|}\,d\eta(w) \leq D\int_{|u|<\varepsilon}\log\frac{\varepsilon}{|u|}\,dA(u) = \frac{D\pi\varepsilon^2}{2}$. Finally $\sigma-\eta$ is a compactly supported signed measure of total mass 0 and finite energy, so $I(\sigma-\eta)\leq 0$, that is $I(\sigma)\leq 2\int U_\eta\,d\sigma - I(\eta)$. Combining the three estimates gives the lemma. □

Proposition 2.2. *Let $Q_1,\dots,Q_m \in \mathbb{Z}[x]\setminus\{0\}$, $c_1,\dots,c_m \geq 0$, $\lambda_0 \geq 0$ with*

$$\sum_k c_k \deg Q_k + 2\lambda_0 \leq 2, \tag{5}$$

and let η be as in Lemma 2.1. Put

$$A(z) = g(z) - \sum_k c_k\log|Q_k(z)| - 2\lambda_0 U_\eta(z) + \lambda_0 I(\eta).$$

Then every algebraic α of degree n which is not a root of any Q_k satisfies $h_Z(\alpha) \geq \inf_{\mathbb{C}} A - \lambda_0\big(\frac{1}{n}\log n + \frac{D\pi}{n^2}\big)$. Consequently $\mathrm{ess}(h_Z) \geq \inf_{\mathbb{C}} A$.

Proof. Subtract from (2) the left-hand sides of (3) multiplied by c_k and the left-hand side of (4) multiplied by λ_0, and bound the double sum by Lemma 2.1 with $\varepsilon = 1/n$. The coefficient of L becomes $2 - \sum_k c_k\deg Q_k - 2\lambda_0(1-\frac{1}{n}) \geq 0$, and $L \geq 0$, so this term may be dropped; what remains is $\int A\,d\mu_\alpha$ minus the stated defect, and $\int A\,d\mu_\alpha \geq \inf A$. For the last assertion: given $\delta > 0$, the defect is below δ for $n \geq n_0(\delta)$, and by Northcott's theorem only finitely many α of degree $< n_0$ have $h_Z(\alpha) \leq \inf A$; the roots of the Q_k are finitely many. □

Lemma 2.3 (support)**.** *Under* (5) *with strict inequality,* $\inf_{\mathbb{C}} A$ *is attained on* $\{|z|=1\}\cup\{|z-1|=1\}$.

Proof. The two circles cut $\mathbb{C}$ into four open regions, on each of which g is harmonic (0, $\log|z|$, $\log|1-z|$ or $\log|z(1-z)|$). The functions $-\log|Q_k|$ and $-U_\eta$ are superharmonic on $\mathbb{C}$, so A is superharmonic on each region, and lower semicontinuous with values in $(-\infty,+\infty]$. As $|z|\to\infty$, $A(z)=(2-\sum_k c_k \deg Q_k - 2\lambda_0)\log|z|+O(1)\to+\infty$. By the minimum principle, the infimum of A over the closure of each region is attained on its boundary, which lies in the union of the two circles. □

This support property is also stated by Flammang [Fl18].

3. The certificate

The certificate, in the ancillary file `CERTIFICATE_C82_ZhangZagier_FINAL.json`, consists of:

- thirteen polynomials $Q_k\in\mathbb{Z}[z]$ with multipliers $c_k>0$: five built from the exceptional set of Doche [Do01a], namely z, $z(z-1)$, z^2-z+1, $\Phi_{10}(z)$ and $\Phi_{10}(z)\Phi_{10}(1-z)$, where $\Phi_{10}(z)=z^4-z^3+z^2-z+1$; and eight polynomials in $w=z(1-z)$, of degrees 6, 16, 16, 20, 22, 24, 32, 32 in z, selected by the optimisation from a list containing the polynomials of Flammang [Fl18, Table 1];
- $\lambda_0=0.08629858849600838$;
- a measure η which is a sum of uniform measures on 2157 pairwise disjoint closed discs centred on the two circles, of total mass 1 and density at most $D=190.582\ldots\le 190.59$. (The stored masses, which are binary floating-point numbers, sum to $1+7\cdot10^{-17}$; we take for η the stored measure divided by this total, which changes A by less than 10^{-15}, an amount absorbed by the margin of Section 4.)

The budget (5) equals $1.0204024<2$. All quantities involving η have closed forms: the potential of the uniform probability measure on the disc of centre c and radius r is $\log|z-c|$ for $|z-c|\ge r$ and $\log r+\frac12(|z-c|^2/r^2-1)$ otherwise; the mutual energy of two disjoint discs is $\log|c-c'|$, and the self-energy of a disc is $\log r-\frac14$. This gives $I(\eta)=4.0048\cdot10^{-5}$. Since $D\pi<599$, Proposition 2.2 gives (1) once $\inf A\ge0.2497815$ is established.

4. Rigorous bound for the infimum

By Lemma 2.3 it suffices to bound A from below on the circles $z=s+e^{i\theta}$, $s\in\{0,1\}$. On an interval $|\theta-t|\le H$ we have $|z(\theta)-z(t)|\le H$, and each term of A is bounded in the safe direction without any derivative of A:

(1) g is increasing in $|z|$ and $|1-z|$, so $g\ge\log^+\max(0,|z(t)|-H)+\log^+\max(0,|1-z(t)|-H)$;
(2) the disc potential is increasing and continuous in the distance ρ to the centre, so $U_\eta(z(\theta))\le\sum_j m_j u_j(|z(t)-c_j|+H)$;
(3) for each Q_k, by Taylor's formula with integral remainder on a segment, $|Q(\zeta)|\le|Q(x)|+|Q'(x)|d+\frac{d^2}{2}\max_{|\xi|\le|x|+d}|Q''(\xi)|$ for $|\zeta-x|\le d$, where $Q(x)$ and $Q'(x)$ are evaluated by Horner's rule, with the rounding error bound added, and $|Q''|$ is bounded by the sum of the absolute values of its coefficients. For the polynomials of large degree this is applied to the polynomial $\tilde Q_k$ of half degree with $Q_k(z)=\tilde Q_k(z-z^2)$ (an exact identity over $\mathbb{Z}$, checked by re-expansion), with $x=w(t)=z(t)-z(t)^2$ and $d=(|1-2z(t)|+2H)H$, since $w(\theta)-w(t)=(z(\theta)-z(t))(1-z(\theta)-z(t))$;
(4) $I(\eta)$ is computed in extended precision and decreased by 10^{-12}.

Floating-point errors are accounted for explicitly: half-widths are enlarged by 10^{-13}, which covers the errors on $z(t)$ and on all distances by the monotonicity of the bounds; the Horner error bound $\gamma_{4d}\sum_k|a_k||x|^k$, with $\gamma_m=mu/(1-mu)$ and $u=2^{-53}$, covers complex evaluation; and 10^{-9} is subtracted from every lower bound, several hundred times the total remaining rounding error. Adaptive bisection, starting from 2000 intervals per circle, then certifies $A\ge0.2497815$ on both circles. Before the bisection the program checks the hypotheses of Proposition 2.2 from the certificate file (signs, budget, total mass, disjointness of the discs, recomputed $I(\eta)$), and checks on random intervals and points that its lower bound never exceeds the directly evaluated value of A.

The program `prove_C82.py` (Python with NumPy) does this from the certificate file alone; its output is in `prove_C82_output.log`. An independent direct evaluation of A at $4 \cdot 10^5$ points per circle (`scan_C82.py`, output in `scan_C82_output.log`) gives the minimum value 0.24978197, so the margin of the certified bound is about $5 \cdot 10^{-7}$. This proves Theorem 1.1.

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