Cartan determinants for almost laura, quasi-laura, semilinear clannish and piecewise Bäckström algebras

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Abstract

We prove the Cartan determinant conjecture for corners of finite global dimension in almost laura, quasi-laura and m-quasi-laura Artin algebras, finite-dimensional piecewise Bäckström algebras, and semilinear clannish Artin algebras of finite dimension over their defining division ring.

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1 Introduction

An Artin algebra is a unital associative algebra finite over a commutative Artin ring acting centrally. For a nonzero Artin algebra A, choose one representative S_i of each isomorphism class of simple right modules, and let P_i be its indecomposable projective cover, for $1 \le i \le n$, where $n \ge 1$. We use the composition Cartan matrix

$$C_A = ([P_j : S_i])_{1 \le i,j \le n},$$

where $[M : S_i]$ denotes the multiplicity of S_i in a composition series of the finite-length module M. Its determinant is independent of simultaneous relabelling. A corner is the ring eAe, with identity e, for an idempotent $e^2 = e$ in A. We write $\operatorname{gl.dim} A$ for the supremum of the projective dimensions of right A-modules.

Finite projective resolutions express the classes of simple modules as integral combinations of the classes of indecomposable projectives. Consequently, the Cartan matrix of an Artin algebra of finite global dimension has determinant 1 or -1. The classical finite-dimensional result goes back to Eilenberg [10]. The Cartan determinant conjecture asks whether the determinant is always 1. Zacharia established this for Artin algebras of global dimension at most two [22]; Chen and Guo prove it for representation-finite Artin algebras (those having finitely many indecomposable finite modules up to isomorphism) in [7, Theorem 1.1].

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The development of the conjecture has involved both projective resolutions and additional structure on the algebra. Burgess and Fuller studied generalizations of quasihereditary rings and the resulting constraints on the Cartan determinant [6]. Wilson's graded Cartan map supplies a complementary approach [21]: a positive grading records composition multiplicities by polynomials, and finite projective resolutions give a polynomial inverse. When the degree-zero algebra is semisimple, evaluation at zero fixes the sign. We give the required graded argument in Lemma 2.6, including its application to ordinary idempotent corners.

For algebras defined by paths and relations, explicit determinant formulas predate the present applications. Holm computed Cartan determinants of gentle algebras in 2005 [12]. Bessenrodt and Holm subsequently treated graded Cartan matrices and skewed-gentle algebras [5]. Their formulas express the determinant in terms of cycles with full zero relations and cover gentle and skewed-gentle algebras over fields. The semilinear clannish class considered here permits division-ring coefficients, automorphism twists, and quadratic special-loop relations; its degree-zero factors require a separate local calculation.

A second line of work decomposes a derived category. Han and Qin proved multiplicativity of Cartan determinants under suitable higher recollements and reduced the finite-dimensional conjecture to derived-simple algebras [17]. An idempotent corner also interacts with projective resolutions through its complementary idempotent. Ingalls and Paquette studied this interaction using Ext algebras and obtained determinant reduction under a finite-global-dimension hypothesis on the associated Yoneda algebra [13]. For the almost laura and quasi-laura classes, the graph of morphisms between indecomposable projectives gives a block triangular composition Cartan matrix.

Laura algebras arose in the study of module categories having only finitely many indecomposables outside their left and right parts. These parts are defined by projective dimensions of predecessors and injective dimensions of successors, respectively. Assem and Coelho studied two-sided gluings of tilted algebras [2]; Reiten and Skowroński developed generalized double tilted algebras [18]. These constructions organize the finite region between two homologically controlled parts. In 2004 Assem and Coelho studied endomorphism algebras of projective modules over laura algebras [3], establishing corner stability results for several classes in this setting.

Smith introduced almost laura algebras in 2008 [19]. Their defining condition is the vanishing of the infinite radical between indecomposables outside the left and right parts. This replaces a finiteness condition on the exceptional region by a condition on arbitrarily long chains of radical morphisms. The class contains representation-finite and quasi-tilted algebras. Smith's Proposition 4.13 identifies representation-finite members by finiteness of the left and right parts.

Trisections provide a related organization of the module category. De la Peña and Reiten studied trisections in 2007 [16]; Alvares, Assem, Coelho, Peña and Trepode developed compact trisections and quasi-directed components in 2011 [1]. A compact trisection has a finite middle part separating its left and right parts. The support algebras of these outer parts lead to quasi-laura algebras and to the recursively defined m-quasi-laura classes. Level zero consists of quasi-tilted algebras; at higher levels the left support has lower levels and the right support is quasi-tilted. The integer m measures this recursive structure, rather than the number of vertices. Already level zero contains hereditary algebras with arbitrarily many simple modules and hereditary algebras of wild representation type.

The other two applications use gradings of rings. Bennett-Tennenhaus and Crawley-Boevey's semilinear clannish algebras extend clannish presentations to division rings and twisted scalar actions [4]. Give ordinary arrows degree one and special loops degree zero. Finite dimension rules out a vertex carrying two independent quadratic special-loop factors. A remaining nonsemisimple degree-zero factor is a twisted dual-number ring, whose simple module has an infinite minimal resolution. Finite global dimension of a corner excludes precisely those factors that survive in that corner.

Bäckström pairs instead consist of an inclusion with a common Jacobson radical. Drozd's

piecewise Bäckström algebras allow the overring to be derived equivalent to a hereditary ring [9]. The triangularity theorem for piecewise hereditary algebras [11, 8] supplies an ordered Peirce decomposition of that overring. Assigning positive degrees to its off-diagonal blocks gives a grading that restricts to the original subring because the radicals coincide.

For almost laura and quasi-laura algebras, each strongly connected projective block is representation-finite or has global dimension at most two. Corners are treated blockwise, using the global-dimension-two corner theorem proved in Appendix A. The other two classes are treated by the graded Cartan calculation in Lemma 2.6.

Theorem A. Let R be an Artin algebra belonging to at least one of the following classes:

1. almost laura Artin algebras;
2. quasi-laura Artin algebras and m-quasi-laura Artin algebras, for any integer $m \geq 0$;
3. semilinear clannish rings of finite left dimension over their defining division ring;
4. finite-dimensional piecewise Bäckström algebras over an arbitrary field.

For every nonzero idempotent $e \in R$, if $\operatorname{gl.dim}(eRe) < \infty$, then

$$\det C_{eRe} = 1.$$

Taking $e = 1$ gives the conjecture for these classes. Their definitions are given in Sections 3–6.

2 Corners, strongly connected blocks and gradings

Modules are finitely generated right modules unless a side is specified. Write $J(A)$ for the Jacobson radical, $\operatorname{mod} A$ for the module category, and $\operatorname{proj} A$ for its projective objects. Projective and injective dimensions are denoted by pd_A and id_A. For Artin algebras the left and right global dimensions agree, and finite global dimension is detected on the finitely many simple modules. The operations Hom, End and Ext^q have their usual module-theoretic meanings, with $q \geq 0$; multiplication in an endomorphism ring is composition, with the rightmost map applied first. For a family of modules, add denotes closure under finite direct sums and direct summands. For auxiliary zero rings we use global dimension zero and the empty Cartan determinant 1.

An algebra is basic if its regular module has one copy of each indecomposable projective type. We may pass to a basic algebra by Morita equivalence. To transport a specified corner, transport the projective module eA, take one copy of every indecomposable summand of its image, and let f be their support idempotent in the basic algebra B. Then eAe is Morita equivalent to fBf. Morita equivalence preserves global dimension and the composition Cartan matrix up to relabelling. In a basic algebra every idempotent is conjugate to a sum of members of a fixed complete primitive orthogonal idempotent family.

For idempotents f, g, write $f \leq g$ when $fg = gf = f$. Fix such a family $1 = e_1 + \cdots + e_n$ in a basic algebra A, and put $P_i = e_i A$. Its projective Hom graph has vertices $1, \ldots, n$ and an edge $i \to j$, for $i \neq j$, when

$$\operatorname{Hom}_A(P_i, P_j) \cong e_j A e_i \neq 0.$$

A strongly connected component is a maximal set of vertices mutually reachable by directed paths. For a component Σ, write $f_\Sigma = \sum_{i \in \Sigma} e_i$ and $A_\Sigma = f_\Sigma A f_\Sigma$. Paths in this graph record chains of nonzero maps; their total composite may be zero.

Lemma 2.1. *For every component Σ, the map $A \to A_\Sigma$, $a \mapsto f_\Sigma a f_\Sigma$, is a unital ring epimorphism. For finite A_Σ-modules X, Y and $q \geq 0$, inflation F gives an injection*

$$\operatorname{Ext}^q_{A_\Sigma}(X, Y) \longrightarrow \operatorname{Ext}^q_A(FX, FY).$$

In particular, $\operatorname{gl.dim} A_\Sigma \le \operatorname{gl.dim} A$, *and*

$$\det C_A = \prod_{\Sigma} \det C_{A_\Sigma}. \tag{1}$$

Proof. If $i, j \in \Sigma$ and $k \notin \Sigma$, the two spaces $e_i A e_k$ and $e_k A e_j$ cannot both be nonzero: their edges and a return path inside Σ would place k in Σ. Hence

$$f_\Sigma A (1 - f_\Sigma) A f_\Sigma = 0,$$

which makes the compression multiplicative. Inflation is exact, as is restriction $E(M) = M f_\Sigma$, and EF is the identity. Applying these functors to Yoneda extensions proves the injection. Taking q above $\operatorname{gl.dim} A$ proves the dimension bound.

The exact functor $M \mapsto M e_i$ has nonzero value on precisely the simple module S_i. Thus $[P_j : S_i] > 0$ precisely when $e_j A e_i \neq 0$. Topologically ordering the components makes C_A block triangular. Restriction to f_Σ kills the other simples, preserves the component simples, and takes the component projectives to $e_j A_\Sigma$. The diagonal block is therefore exactly C_{A_Σ}, proving (1). □

Lemma 2.2. *Every corner of a representation-finite Artin algebra is representation-finite.*

Proof. For $B = eAe$, induction $- \otimes_B eA$ is left adjoint to restriction $N \mapsto Ne$. The adjunction unit is an isomorphism, since

$$(M \otimes_B eA)e \cong M \otimes_B B \cong M.$$

Induction is therefore fully faithful. If M is generated over B by $m_1, \dots, m_r$, then $M \otimes_B eA$ is generated over A by $m_1 \otimes e, \dots, m_r \otimes e$. Full faithfulness identifies its endomorphism ring with $\operatorname{End}_B(M)$, which is local when M is indecomposable. Thus distinct indecomposable finite B-modules give distinct indecomposable finite A-modules. There are only finitely many of the latter. □

Theorem 2.3. *Let R be an Artin algebra with* $\operatorname{gl.dim} R \le 2$. *Every nonzero corner $A = eRe$ of finite global dimension satisfies* $\det C_A = 1$.

This is the corner theorem of [20], proved in Appendix A. It implies the following result, also established in [7, Theorem 1.1].

Corollary 2.4. *A representation-finite Artin algebra of finite global dimension has Cartan determinant one.*

Proof. Choose an additive generator G of $\operatorname{mod} A$ containing A as a direct summand. Set $B = \operatorname{End}_A(G)$. The functor $\operatorname{Hom}_A(G, -)$ identifies $\operatorname{add} G$ with $\operatorname{proj} B$. For any presentation of a finite B-module N, the map between its projective terms comes from a map $X_1 \to X_0$ in $\operatorname{add} G$. Its module kernel X_2 belongs to $\operatorname{add} G$, giving an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(G, X_2) \longrightarrow \operatorname{Hom}_A(G, X_1) \longrightarrow \operatorname{Hom}_A(G, X_0) \longrightarrow N \longrightarrow 0.$$

Thus $\operatorname{gl.dim} B \le 2$. Projection onto the summand A gives a corner isomorphic to $\operatorname{End}_A(A) \cong A$, and Theorem 2.3 applies. □

Proposition 2.5. *Let R be an Artin algebra. Suppose that in a basic algebra Morita equivalent to R, every projective Hom component algebra is representation-finite or has global dimension at most two. Then every nonzero finite-global-dimension corner of R has Cartan determinant one.*

Proof. Work in a basic algebra and write e as a sum of selected primitive idempotents. The graph of $A = eRe$ is the induced subgraph on the selected vertices. Each component T of this graph lies inside a component Σ of the ambient graph, and

$$A_T = f_T R f_T = f_T R_\Sigma f_T.$$

Lemma 2.1, applied to A, gives finite global dimension of A_T. If R_Σ is representation-finite, apply Lemma 2.2 and Corollary 2.4. Otherwise apply Theorem 2.3. All determinants in (1) for A are one. □

A nonnegative grading of a ring A is a decomposition $A = \bigoplus_{d\geq 0} A_d$ into additive subgroups with $1 \in A_0$ and $A_d A_r \subseteq A_{d+r}$. All gradings below have finite support. If $A_d = 0$ for $d > N$, where $N \geq 0$, then the ideal $A_+ = \bigoplus_{d>0} A_d$ satisfies $(A_+)^{N+1} = 0$. A graded module M has a compatible integer grading. We use the shift $M\langle a\rangle_d = M_{d-a}$, so a zero-degree generator moves to degree a.

Lemma 2.6. *Let A be an Artin algebra with a finite nonnegative ring grading.*

1. *If* $\operatorname{gl.dim} A < \infty$, *then* $\operatorname{gl.dim} A_0 < \infty$.
2. *If A_0 is semisimple and* $\operatorname{gl.dim} A < \infty$, *then* $\det C_A = 1$.
3. *Every idempotent of A is conjugate to an idempotent of A_0.*

Proof. Since A_+ is nilpotent and $A/A_+ = A_0$, we have $J(A) = J(A_0) \oplus A_+$. The quotient A_0 is Artinian. Choose a complete primitive orthogonal idempotent family in A_0. Every finite A-module has finite length over A_0: the filtration by powers of A_+ has finite A-length quotients annihilated by A_+, and these are finite-length A_0-modules. Thus all homogeneous components used below have finite A_0-length.

For a finite graded module M, its top $M/MJ(A)$ is a finite direct sum of shifts of simple A_0-modules. Lift a homogeneous generator of each summand, multiplying by its primitive idempotent to preserve its support. These lifts define a surjection from a finite direct sum of shifts of modules $e_i A$ onto M: surjectivity follows from Nakayama's lemma. Its map on tops is an isomorphism, so its kernel lies in the radical of its source. This constructs a graded projective cover, and also proves that forgetting the grading gives an ordinary projective cover. Iterating on the kernels constructs the minimal graded resolution. Starting with a simple in degree zero, each kernel is concentrated in nonnegative degrees, so all shifts appearing in this construction are nonnegative. If A has finite global dimension, a sufficiently high syzygy is projective. Its projective cover is then an isomorphism, so the minimal resolution terminates. Taking degree zero is exact, since exactness of graded maps is degreewise. For $a \geq 0$,

$$(e_i A\langle a\rangle)_0 = \begin{cases} e_i A_0, & a = 0, \\ 0, & a > 0. \end{cases}$$

Thus the degree-zero complex is a finite projective A_0-resolution of the chosen simple. There are finitely many simple types, proving (1).

For (2), let $e_1, \ldots, e_n$ be one primitive representative of each projective type in the semisimple ring A_0. Put $P_i = e_i A$ and let $S_i = e_i A_0$, viewed as a simple A-module through the projection to A_0. Let t be a commuting indeterminate. For $1 \leq i, j \leq n$, define the polynomial matrix

$$C_A(t)_{ji} = \sum_{d\geq 0} [(P_i)_d : S_j]_{A_0}\, t^d.$$

Here the bracket is composition multiplicity as an A_0-module. Its constant matrix is the identity. If the q-th term of the finite graded resolution of S_i contains $P_j\langle a\rangle$ with multiplicity $b_{j,q,a,i}$, define

$$U(t)_{ji} = \sum_{q,a\geq 0} (-1)^q b_{j,q,a,i} t^a.$$

All multiplicities have indices $1 \le i, j \le n$ and $q, a \ge 0$; only finitely many are nonzero. For $1 \le r \le n$, the degreewise Euler characteristic of the resolution of S_i gives

$$\sum_{j=1}^{n} \sum_{q,a \ge 0} (-1)^q b_{j,q,a,i} t^a C_A(t)_{rj} = \begin{cases} 1, & r = i, \\ 0, & r \ne i. \end{cases}$$

Indeed, shifting P_j by a multiplies its composition polynomial by t^a, and S_i occurs only in degree zero. Thus

$$C_A(t) U(t) = I_n,$$

where I_n is the n-by-n identity matrix: its i-th column is the composition vector of S_i in degree zero. Thus $\det C_A(t)$ is a unit of $\mathbb{Z}[t]$ and has constant term one, so it is identically one. The finite degree filtration has quotients annihilated by A_+; their A-composition multiplicities are their A_0-composition multiplicities. Additivity therefore gives $C_A(1) = C_A$, proving (2).

For (3), let f be the degree-zero projection of e. Then $f^2 = f$ and $e - f \in A_+$. The element

$$u = ef + (1 - e)(1 - f)$$

is congruent to 1 modulo A_+ and hence invertible: for a positive integer h with $(1 - u)^h = 0$, its inverse is $\sum_{j=0}^{h-1} (1 - u)^j$. Since $eu = uf$, we have $u^{-1} eu = f$, and conjugation by u^{-1} identifies eAe with fAf. □

3 Almost laura algebras

Let $\operatorname{ind} A$ denote a full subcategory with one representative of each indecomposable finite module. A path from X to Y in $\operatorname{ind} A$ is a finite chain of nonzero morphisms, with length zero allowed. A predecessor of Y is an object admitting such a path to Y; successors are defined dually. Set

$$\mathcal{L}_A = \{X \in \operatorname{ind} A : \operatorname{pd}_A Y \le 1 \text{ for every predecessor } Y \text{ of } X\},$$
$$\mathcal{R}_A = \{X \in \operatorname{ind} A : \operatorname{id}_A Y \le 1 \text{ for every successor } Y \text{ of } X\}.$$

The radical $\operatorname{rad}_A(X, Y)$ is the categorical radical in $\operatorname{mod} A$; between indecomposables it consists of the nonisomorphisms. Its q-th power consists of sums of composites of q radical maps through finite modules. The infinite radical is $\operatorname{rad}_A^\infty(X, Y) = \bigcap_{q \ge 1} \operatorname{rad}_A^q(X, Y)$.

Definition 3.1. An algebra A is almost laura if $\operatorname{rad}_A^\infty(X, Y) = 0$ whenever $X, Y \in \operatorname{ind} A$ both lie outside $\mathcal{L}_A \cup \mathcal{R}_A$.

For disconnected algebras this is understood componentwise. The definition is invariant under Morita equivalence and under Artin duality, which interchanges the left and right parts. Thus Smith's left-module results apply with our right-module convention. We use [19, Proposition 4.13]: an almost laura algebra with finite left and right parts is representation-finite.

Lemma 3.2. *If the projective Hom graph of a basic Artin algebra B is strongly connected and $\mathcal{L}_B \cup \mathcal{R}_B \ne \varnothing$, then* $\operatorname{gl.dim} B \le 2$.

Proof. Suppose $X \in \mathcal{L}_B$. Some indecomposable summand P_i of a projective cover of X has a nonzero map to X. Strong connectivity makes every P_j a predecessor of X, so every P_j belongs to $\mathcal{L}_B$. For any finite module M, each indecomposable summand Y of the kernel of a projective cover of M has a nonzero component map to some P_j. It follows that $\operatorname{pd}_B Y \le 1$, and hence $\operatorname{pd}_B M \le 2$. This bounds global dimension. If instead $\mathcal{R}_B$ is nonempty, apply Artin duality and the same argument over B^{op}. □

Proposition 3.3. *Every projective Hom component algebra of an almost laura Artin algebra is representation-finite or has global dimension at most two.*

Proof. Let $B = A_\Sigma$ and let $F : \operatorname{mod} B \to \operatorname{mod} A$ be the inflation of Lemma 2.1. The Ext injection there gives $\operatorname{pd}_B Y \le \operatorname{pd}_A FY$ and $\operatorname{id}_B Y \le \operatorname{id}_A FY$. Inflation preserves indecomposable objects, nonzero maps and paths. Consequently

$$FX \in \mathcal{L}_A \implies X \in \mathcal{L}_B, \qquad FX \in \mathcal{R}_A \implies X \in \mathcal{R}_B.$$

If X, Y are outside $\mathcal{L}_B \cup \mathcal{R}_B$, their inflated modules are outside $\mathcal{L}_A \cup \mathcal{R}_A$. A radical map between indecomposables remains a nonisomorphism under the fully faithful functor F. Decomposing intermediate modules into indecomposables therefore gives

$$F\big(\operatorname{rad}_B^q(X,Y)\big) \subseteq \operatorname{rad}_A^q(FX, FY) \qquad (q \ge 1).$$

If a map lies in every power on the left, its image lies in the infinite radical on the right, which is zero. Faithfulness then makes the original map zero. Thus B is almost laura.

The graph of B is strongly connected. If one of its two parts is nonempty, Lemma 3.2 gives global dimension at most two. If both are empty, Smith's Proposition 4.13 gives representation-finiteness. □

4 Trisections and m-quasi-laura algebras

We give the recursive definition from [1, Definitions 2.1, 3.3 and 7.3]. A trisection is a partition

$$\operatorname{ind}\Lambda = \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{C}, \qquad \operatorname{Hom}(\mathcal{C}, \mathcal{A}) = \operatorname{Hom}(\mathcal{C}, \mathcal{B}) = \operatorname{Hom}(\mathcal{B}, \mathcal{A}) = 0.$$

For subcategories, $\operatorname{Hom}(\mathcal{U}, \mathcal{V}) = 0$ means that every morphism from an object of $\mathcal{U}$ to an object of $\mathcal{V}$ is zero. The middle part is convex: every path with endpoints there stays there. A trisection is separated if $\mathcal{B}$ is nonempty and every morphism from $\mathcal{A}$ to $\mathcal{C}$ factors through $\operatorname{add}\mathcal{B}$. It is compact if it is separated, $\mathcal{B}$ is finite and connected, and $\mathcal{B}$ contains all projectives and injectives in its Auslander–Reiten component. Here the Auslander–Reiten quiver has indecomposable modules as vertices and irreducible morphisms as arrows; connectedness of the middle part means connectedness by undirected walks of nonzero morphisms between its objects. For a finite convex middle part this places it in a single such component [1, Section 3.1].

Work in a basic algebra Λ with $1 = e_1 + \cdots + e_n$, and set $P_i = e_i\Lambda$ for $1 \le i \le n$. Let D be the usual Artin duality between finite right and left modules, obtained by dualizing over the commutative Artin base into a minimal injective cogenerator, and put $I_i = D(\Lambda e_i)$. The Nakayama equivalence $D\operatorname{Hom}_\Lambda(-, \Lambda)$ from finite projectives to finite injectives sends P_i to I_i; it is covariant and identifies $\operatorname{Hom}_\Lambda(P_i, P_j)$ with $\operatorname{Hom}_\Lambda(I_i, I_j)$. Define the support idempotents

$$a = \sum_{P_i \in \mathcal{A}} e_i, \qquad c = \sum_{I_i \in \mathcal{C}} e_i, \qquad b = \sum_{P_i, I_i \in \mathcal{B}} e_i. \tag{2}$$

The left and right support algebras are $a\Lambda a$ and $c\Lambda c$. The right support uses injective labels, as in the source definition.

A quasi-tilted Artin algebra has global dimension at most two and every indecomposable finite module has projective dimension at most one or injective dimension at most one. An algebra is 0-quasi-laura if it is quasi-tilted. For $m \ge 1$, it is m-quasi-laura if it admits a compact trisection for which the left support is a finite product of n-quasi-laura algebras with $0 \le n < m$, at least one connected factor having level $m - 1$, and the right support is a finite product of quasi-tilted algebras. This is the recursive convention of [1, Definition 7.3]. A quasi-laura algebra admits a compact trisection whose two support algebras are quasi-tilted [1, Definition 7.1]. Zero support algebras are allowed in this latter definition.

Lemma 4.1. *For a trisection with finite middle part, the corner $b\Lambda b$ defined in (2) is representation-finite.*

Proof. If i, j occur in b and both $e_i\Lambda e_k$ and $e_k\Lambda e_j$ are nonzero, there is a path $P_j \to P_k \to P_i$. Convexity places P_k in $\mathcal{B}$. The Nakayama equivalence gives the corresponding injective path and places I_k there too. Therefore k occurs in b, and

$$b\Lambda(1-b)\Lambda b = 0.$$

Compression to $b\Lambda b$ is a ring epimorphism with fully faithful exact inflation F. For an indecomposable nonzero $b\Lambda b$-module M, choose i with $Me_i \neq 0$. The identities $\mathrm{Hom}_\Lambda(P_i, FM) \cong Me_i$ and $\mathrm{Hom}_\Lambda(FM, I_i) \cong D(Me_i)$ give nonzero maps

$$P_i \longrightarrow FM \longrightarrow I_i.$$

Both endpoints are in $\mathcal{B}$, so FM is in $\mathcal{B}$. Inflation preserves indecomposability and reflects isomorphism. Finiteness of $\mathcal{B}$ proves the claim. For $b = 0$ there are no indecomposable modules to consider. □

Proposition 4.2. *Every projective Hom component algebra of a quasi-laura or m-quasi-laura Artin algebra is representation-finite or has global dimension at most two.*

Proof. We induct on m. For $m = 0$, the algebra has global dimension at most two, and Lemma 2.1 gives the assertion. Suppose $m \geq 1$ and fix a component Σ. All its projectives lie in the same part of the trisection, since a path cannot travel backwards through its three parts. The corresponding injectives also lie in one part, by the Nakayama equivalence. Since $\mathrm{Hom}_\Lambda(P_i, I_i) \cong D(e_i\Lambda e_i) \neq 0$, the projective part is no later than the injective part.

If the projectives lie in $\mathcal{A}$, then $f_\Sigma \leq a$. If this is not the case and the injectives lie in $\mathcal{C}$, then $f_\Sigma \leq c$. In the remaining case both lie in $\mathcal{B}$, so $f_\Sigma \leq b$. These alternatives exhaust the possibilities.

In the last case Lemmas 4.1 and 2.2 give representation-finiteness. In the first two cases Σ remains a full strongly connected component in the relevant support algebra: it is still strongly connected there, and any extra support vertex mutually reachable with it would already belong to its full ambient component. It lies in one connected support factor. Apply the induction hypothesis to a lower-level left factor, or the $m = 0$ case to a right factor. For a quasi-laura algebra the same trisection argument applies directly, since both nonzero support factors are quasi-tilted; an empty support contains no component to consider. □

5 Semilinear clannish algebras

Fix a division ring K, a finite quiver Q, and automorphisms σ_a of K indexed by its arrows. The semilinear path ring $K_\sigma Q$ is the free left K-space on paths, with the usual concatenation and the rule $a\lambda = \sigma_a(\lambda)a$. We multiply paths right to left, with noncomposable products equal to zero. For a vertex i, let e_i be its length-zero path; the sum of these vertex idempotents is the identity. A specified set S of loops is called special; all other arrows are ordinary. For each $s \in S$ at vertex i, fix $\beta_s, \gamma_s \in K$. Let Z be a set of paths of length at least two, none starting or ending with a special loop or containing two consecutive copies of the same special loop. The ring is

$$R = K_\sigma Q / \langle Z,\ s^2 - \beta_s s + \gamma_s e_i\ (s \in S)\rangle, \tag{3}$$

where brackets mean the two-sided ideal generated by the indicated elements. It is semilinear clannish if each vertex has at most two incoming and two outgoing arrows, and for each ordinary arrow a there is at most one arrow c with ca a path outside Z and at most one arrow c with ac

a path outside Z [4, Section 1]. Throughout this section R is an Artin algebra and has finite left dimension over K. The copy of K need not be central.

Give ordinary arrows degree one, special loops degree zero, and coefficients and vertex idempotents degree zero. The defining ideal is homogeneous: a relation in Z has positive degree, and each special-loop relation has degree zero. This defines a nonnegative grading on R. Each homogeneous component is a left K-subspace, so finite left dimension of R makes the grading have finite support.

Lemma 5.1. *Let $\sigma \in \mathrm{Aut}(K)$ and $\beta, \gamma \in K$. The two-sided quotient*

$$B = K[x;\sigma]/\langle x^2 - \beta x + \gamma \rangle, \qquad x\lambda = \sigma(\lambda)x,$$

is either zero, semisimple, or isomorphic to $K[y;\sigma]/(y^2)$. In the last case it is local and has infinite global dimension.

Proof. Reduction by the monic quadratic shows that B is spanned over K by $1, x$, where x also denotes its residue class, so its left dimension is at most two. A nonzero quotient contains K injectively. Every left ideal is a K-subspace, so B is left Artinian. If $J(B) = 0$, the Artin–Wedderburn theorem makes B semisimple. Dimension one gives $B \cong K$. In dimension two, expand the two sides of $x^2\lambda = x(x\lambda)$:

$$\beta\sigma(\lambda)x - \gamma\lambda = \sigma^2(\lambda)\beta x - \sigma^2(\lambda)\gamma.$$

Independence of $1, x$ gives

$$\beta\sigma(\lambda) = \sigma^2(\lambda)\beta, \qquad \gamma\lambda = \sigma^2(\lambda)\gamma \quad (\lambda \in K),$$

where $\sigma^2 = \sigma \circ \sigma$. Likewise, $x(x^2) = (x^2)x$ expands as

$$\big(\sigma(\beta)\beta - \sigma(\gamma)\big)x - \sigma(\beta)\gamma = (\beta^2 - \gamma)x - \beta\gamma,$$

so

$$(\sigma(\beta) - \beta)\gamma = 0, \qquad (\sigma(\beta) - \beta)\beta - \sigma(\gamma) + \gamma = 0.$$

If $\gamma \neq 0$, the first equality gives $\sigma(\beta) = \beta$, and the second gives $\sigma(\gamma) = \gamma$. If $\gamma = 0$, the second gives $(\sigma(\beta) - \beta)\beta = 0$; cancellation when $\beta \neq 0$, and direct substitution when $\beta = 0$, give the same conclusions.

If $J(B) \neq 0$, it is a one-dimensional left K-space, $B/J(B) \cong K$, and $J(B)^2 = 0$. Indeed, $J(B)$ is nilpotent and has trivial intersection with K, and its nonzero dimension is strictly less than two.

The image of x in $B/J(B)$ equals some $\eta \in K$. Put $y = x - \eta \in J(B)$. Then $y \neq 0$ and $y^2 = 0$. For $\lambda \in K$, the difference

$$y\lambda - \sigma(\lambda)y = \sigma(\lambda)\eta - \eta\lambda$$

lies in $J(B) \cap K$ and is zero. The resulting map from $K[y;\sigma]/(y^2)$ to B is an isomorphism by dimension. As left modules, repeated right multiplication by y gives an exact minimal resolution

$$\cdots \xrightarrow{\cdot y} B \xrightarrow{\cdot y} B \xrightarrow{\cdot y} B \longrightarrow B/(y) \longrightarrow 0.$$

For $a, c \in K$, right multiplication sends $a + cy$ to ay, so every differential has image and kernel Ky. Applying $\mathrm{Hom}_B(-, B/(y))$ gives zero differentials and a nonzero copy of $B/(y)$ in every degree. Consequently $\mathrm{Ext}^q_B(B/(y), B/(y)) \neq 0$ for every $q \geq 0$, and the left global dimension is infinite. For right modules, repeated left multiplication by y gives the same conclusion: $y(a + cy) = \sigma(a)y$, whose image and kernel are $yK = Ky$. □

Lemma 5.2. *The degree-zero ring R_0 of (3) is a finite product of semisimple rings and local rings $K[y;\sigma]/(y^2)$.*

Proof. The degree-zero relations are precisely the special-loop quadratics, so R_0 is the product of the vertex rings generated by special loops. A vertex with no such loop contributes K, and a vertex with one loop contributes the quotient in Lemma 5.1.

If a vertex has two special loops s, t, they exhaust both incidence bounds, so the vertex is isolated. Its presentation has just the two quadratic relations in s, t. If either one-loop quotient is zero, its relations already kill this component. If either has left dimension one, that loop is a scalar in K, and substituting its value leaves the other one-loop quotient. Suppose both factors have left dimension two. For each $r \in \{s, t\}$ write its parameters as $\sigma_r, \beta_r, \gamma_r$. The coefficient identities just derived hold in each factor. Let W be the left K-vector space with basis the alternating words in s, t, including the empty word 1. Define additive operators L_r on W by the following formulas, for $\lambda \in K$:

$$\begin{aligned} L_r(\lambda w) &= \sigma_r(\lambda) rw && \text{if } w \text{ does not begin with } r, \\ L_r(\lambda rv) &= \sigma_r(\lambda)(\beta_r rv - \gamma_r v) && \text{if } rv \text{ begins with } r. \end{aligned}$$

Here a scalar in an operator identity acts by left multiplication on W. The operators satisfy $L_r\lambda = \sigma_r(\lambda)L_r$. If w does not begin with r, then

$$L_r^2(w) = L_r(rw) = \beta_r rw - \gamma_r w = \beta_r L_r(w) - \gamma_r w.$$

On a word rv, the identities $\sigma_r(\beta_r) = \beta_r$ and $\sigma_r(\gamma_r) = \gamma_r$ give

$$L_r^2(rv) = (\beta_r^2 - \gamma_r)rv - \beta_r\gamma_r v = \beta_r L_r(rv) - \gamma_r rv.$$

For a scalar multiple λw, the identities $\beta_r\sigma_r(\lambda) = \sigma_r^2(\lambda)\beta_r$ and $\gamma_r\lambda = \sigma_r^2(\lambda)\gamma_r$ then give $L_r^2 = \beta_r L_r - \gamma_r$ on all of W. Thus scalar multiplication and L_s, L_t define a representation of the isolated vertex ring on the additive group of W. An alternating word acts on the empty word by producing that same basis word. Any left K-linear relation between alternating words would therefore give a relation between distinct basis vectors of W, and all its coefficients vanish. This contradicts finite dimension of R. The remaining vertex factors are covered by Lemma 5.1. □

Theorem 5.3. *Every nonzero finite-global-dimension corner of R satisfies the Cartan determinant conjecture.*

Proof. By Lemma 2.6(3), conjugate the chosen idempotent to $f \in R_0$ and put $A = fRf$. Its degree-zero ring is fR_0f. A nonzero corner of a local twisted dual-number factor is the whole factor, since its only idempotents are 0 and 1. Corners of semisimple factors are semisimple. If $\operatorname{gl.dim} A < \infty$, Lemma 2.6(1) gives finite global dimension of fR_0f, so Lemma 5.1 excludes all nonsemisimple factors. Therefore A_0 is semisimple, and Lemma 2.6(2) gives $\det C_A = 1$. □

6 Piecewise Bäckström algebras

A Bäckström pair is a unital inclusion $A \subseteq H$ of semiperfect rings with $J(A) = J(H)$ as subsets of H. A ring is semiperfect if its semisimple quotient is Artinian and idempotents lift modulo its radical. Following [9, Definition 2.1], A is piecewise Bäckström if it occurs in such a pair with H derived equivalent, as a ring, to a left hereditary ring. Here derived equivalence is understood in the usual tilting-complex sense. A ring is left hereditary if every submodule of a projective left module is projective. In the algebra version A and H are finite-dimensional over a field k; the proof below also allows an inclusion understood purely as a ring inclusion. The Bäckström class itself is obtained by requiring H hereditary.

We use the structural theorem that a finite-dimensional piecewise hereditary algebra is triangular [11, Theorem 1.1(i)]; see also [8, p. 797] for its formulation over a field. In ring terms, after grouping equivalent primitive idempotents, there are orthogonal idempotents $p_1, \dots, p_r$ with sum one such that $p_i H p_i$ is a full matrix ring over a division ring and the nonzero off-diagonal spaces $p_i H p_j$ follow a strict order.

Complexes below are cohomologically indexed. The shift $T[q]$ has $(T[q])^i = T^{i+q}$ and differential $(-1)^q d_T$. The perfect derived category consists of bounded complexes of finite projectives, up to quasi-isomorphism. A tilting generator T generates this category under shifts, cones and direct summands, and has no morphisms to its nonzero shifts. Morphisms between perfect complexes are computed up to chain homotopy; these agree with morphisms in the derived category. The Hom complex retains the graded maps before passage to homotopy.

Lemma 6.1. *If a finite-dimensional k-algebra H is derived equivalent as a ring to a hereditary ring, then H is triangular.*

Proof. Let L be the hereditary ring at the endpoint of the given derived equivalence. That equivalence provides a bounded complex T of finite projective H-modules which generates the perfect derived category, has $\mathrm{Hom}(T, T[q]) = 0$ for $q \neq 0$, and has endomorphism ring isomorphic to the hereditary endpoint: using right modules, the image T of the regular right module L_L satisfies $\mathrm{End}(T) \cong \mathrm{End}_L(L_L) \cong L$. Since the Hom complex of T is finite-dimensional over k, this identification equips L with the structure of a finite-dimensional hereditary k-algebra.

The induced tilting equivalence can be taken k-linear. Indeed, write $d_T^i : T^i \to T^{i+1}$ for the differential of T. The differential graded k-algebra $E = \mathrm{End}_H^\bullet(T)$ has degree-q term $\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_H(T^i, T^{i+q})$, for $q \in \mathbb{Z}$, multiplication induced by composition, and differential $d(f) = d_T f - (-1)^q f d_T$ on a degree-q map f. Its cohomology $H^q(E) = \ker(d : E^q \to E^{q+1}) / \mathrm{im}(d : E^{q-1} \to E^q)$ vanishes for $q \neq 0$. The differential graded subalgebra $\tau_{\leq 0} E$ has $\ker(d : E^0 \to E^1)$ in degree zero, E^q in negative degrees and zero in positive degrees. It gives quasi-isomorphisms of differential graded k-algebras

$$E \longleftarrow \tau_{\leq 0} E \longrightarrow H^0(E).$$

The left arrow is inclusion; the right arrow takes degree-zero cycles to their cohomology classes and is zero in negative degrees. Vanishing of $H^q(E)$ for $q \neq 0$ makes both arrows quasi-isomorphisms. Derived Morita theory for the tilting generator, in the form of [9, Theorem 1.1 and Corollary 1.2], therefore gives the required k-linear equivalence. Thus H is piecewise hereditary over k, and the triangularity theorem applies. □

Proposition 6.2. *A finite-dimensional piecewise Bäckström algebra A has a finite nonnegative ring grading with semisimple degree-zero ring and $A_+ = J(A)$. Every nonzero finite-global-dimension corner of A has Cartan determinant one.*

Proof. Choose its defining pair $A \subseteq H$. By Lemma 6.1, H is triangular. Choose integers w_i, for $1 \leq i \leq r$, in a strict order such that $p_i H p_j \neq 0$ and $i \neq j$ imply $w_i > w_j$. The Peirce decomposition $H = \bigoplus_{i,j=1}^r p_i H p_j$ gives a grading

$$H_d = \bigoplus_{w_i - w_j = d} p_i H p_j, \qquad H_0 = \prod_i p_i H p_i, \qquad H_+ = J(H).$$

Products add degrees because $p_j p_l = 0$ for $j \neq l$. All nonzero off-diagonal blocks have positive degree. The positive-degree ideal is nilpotent and its quotient is the displayed product of semisimple diagonal rings. It is therefore exactly $J(H)$.

Since $J(A) = J(H)$, all positive components belong to A. If $a = a_0 + a_+$ is the decomposition of an element of A in $H_0 \oplus J(H)$, then $a_0 = a - a_+ \in A$. Consequently the grading restricts to A, with

$$A_0 = A \cap H_0 \cong A/J(A), \qquad A_d = H_d \quad (d > 0).$$

Thus A_0 is semisimple. By Lemma 2.6(3), any corner is isomorphic to fAf for $f \in A_0$. Its zero component fA_0f is semisimple, so Lemma 2.6(2) proves the determinant assertion. □

Proof of Theorem A. For (1), combine Propositions 3.3 and 2.5. For (2), combine Propositions 4.2 and 2.5. Assertion (3) is Theorem 5.3, and (4) is Proposition 6.2. □

For a field k, consider the Bäckström pair

$$A = kI + kE_{12} \subseteq H = \mathrm{UT}_2(k).$$

Here I is the 2×2 identity, E_{12} the matrix unit, and $\mathrm{UT}_2(k)$ the upper triangular matrix algebra. Let ε be an indeterminate. The two rings share radical kE_{12}. The algebra H is hereditary, and the map $k[\varepsilon]/(\varepsilon^2) \to A$ sending ε to E_{12} and 1 to I is an isomorphism. Thus A has infinite global dimension.

A Corners of algebras of global dimension two

A.1 Kernels and Cartan factorization

In this appendix fix a commutative Artin base ring K; all algebras are finite over K. Write $\ell_K V$ for the composition length of a finite K-module V. We put $\mathrm{pd}_B 0 = -\infty$. For a category $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ denotes its morphisms from X to Y. All additive subcategories below are closed under isomorphisms and direct summands. We use K-linear, idempotent-complete Krull–Schmidt categories: every idempotent splits, and every object is a finite direct sum of indecomposable objects with local endomorphism rings, uniquely up to isomorphism and permutation. A local ring has a unique maximal right ideal, its Jacobson radical. The Hom spaces have finite K-length, and there are finitely many indecomposable isomorphism classes. A basic additive generator is the direct sum of one representative of each such class. The zero category is allowed as an auxiliary category, with the zero-ring conventions of Section 2.

If $\mathcal{C} = \operatorname{add} W$, the functor $\mathrm{Hom}_{\mathcal{C}}(W, -)$ sends Y to $\mathrm{Hom}_{\mathcal{C}}(W, Y)$ and a map $f : Y \to Y'$ to $t \mapsto ft$. Its values are right $\mathrm{End}_{\mathcal{C}}(W)$-modules by $t \cdot b = tb$. It identifies $\mathcal{C}$ with $\operatorname{proj} \mathrm{End}_{\mathcal{C}}(W)$. Indeed, it is fully faithful on finite sums of W, hence on their direct summands, and every finite projective module is such a summand of a finite free module. A categorical kernel of $f : X \to Y$ is a morphism $u : T \to X$ with $fu = 0$ such that every $v : V \to X$ with $fv = 0$ factors uniquely through u.

Lemma A.1. *If $\mathcal{C} = \operatorname{add} W$ has kernels of all its morphisms, then* $\operatorname{gl.dim} \mathrm{End}_{\mathcal{C}}(W) \le 2$.

Proof. Put $B = \mathrm{End}_{\mathcal{C}}(W)$. A finite right B-module N has a projective presentation induced by a morphism $f : X_1 \to X_0$ in $\mathcal{C}$. The kernel property gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}(W, \ker f) \longrightarrow \mathrm{Hom}(W, X_1) \longrightarrow \mathrm{Hom}(W, X_0) \longrightarrow N \longrightarrow 0.$$

Its first three modules are projective, so $\mathrm{pd}_B N \le 2$. In particular, every simple right B-module has projective dimension at most two. This also bounds arbitrary modules: the radical of B is nilpotent, every module has a finite filtration by its radical powers with semisimple factors, and direct sums of projective resolutions of simple modules resolve those factors in length at most two. Successive extensions preserve this bound. □

Cartan entries can be computed using lengths over K. Let $X_1, \dots, X_n$ represent the indecomposable isomorphism classes of $\mathcal{C}$, take $W = \bigoplus_i X_i$, and let L_i be the simple top of $\mathrm{Hom}(W, X_i)$ over $B = \mathrm{End}(W)$. Define the residual division ring

$$\Delta_i = \mathrm{End}(X_i)/\operatorname{rad} \mathrm{End}(X_i).$$

Let $\varepsilon_i \in B$ be the projection of W onto X_i followed by inclusion. The right projective $\operatorname{Hom}(W, X_j)$ is $\varepsilon_j B$; its simple top is $\varepsilon_j B/\varepsilon_j \operatorname{rad} B$. The corner $\varepsilon_i B \varepsilon_i$ is $\operatorname{End}(X_i)$, and the quotient $B/\operatorname{rad} B$ is the product of the division rings Δ_i. Thus $N \mapsto N\varepsilon_i$ is an exact component functor, with value Δ_i on L_i and zero on the other simples. Applying it to a composition series, and using $\operatorname{Hom}(W, X_j)\varepsilon_i \cong \operatorname{Hom}(X_i, X_j)$, gives

$$[\operatorname{Hom}(W, X_j) : L_i] = \frac{\ell_K \operatorname{Hom}(X_i, X_j)}{\ell_K \Delta_i}. \tag{4}$$

Thus these ratios, with sources indexing rows and targets indexing columns, give C_B. The same formula on a subset of the objects gives the Cartan matrix of their endomorphism algebra. Replacing an additive generator by its basic generator is a Morita equivalence and preserves the Cartan matrix up to simultaneous relabelling.

Let $\mathcal{L} \subseteq \mathcal{C}$ be an additive subcategory. A right $\mathcal{L}$-approximation of Y is a morphism $p_Y : U_Y \to Y$, with $U_Y \in \mathcal{L}$, through which every morphism from an object of $\mathcal{L}$ to Y factors. Such approximations exist here: take generators over K of the finitely many spaces $\operatorname{Hom}(V, Y)$, where V runs over the indecomposable objects of $\mathcal{L}$, and combine their maps into a morphism from a finite direct sum of these objects.

The object quotient $\mathcal{Q} = \mathcal{C}/[\mathcal{L}]$ has the same objects as $\mathcal{C}$, with

$$\mathcal{Q}(X, Y) = \mathcal{C}(X, Y)/[\mathcal{L}](X, Y),$$

where $[\mathcal{L}](X, Y)$ consists of maps factoring through objects of $\mathcal{L}$. Composition is induced from $\mathcal{C}$. The class of a map f will be denoted by $\bar{f}$. For $f : X \to Y$ and $p : U \to Y$, the map $(f, p) : X \oplus U \to Y$ has restrictions f and p on the two summands.

Lemma A.2. *Suppose each $Y \in \mathcal{C}$ has a right $\mathcal{L}$-approximation $p_Y : U_Y \to Y$ whose kernel exists and belongs to $\mathcal{L}$. Suppose also that $\ker(f, p_Y)$ exists in $\mathcal{C}$ for every $f : X \to Y$. Then $\mathcal{Q}$ has kernels and is equivalent to $\operatorname{proj} D$ for an Artin algebra D with $\operatorname{gl.dim} D \leq 2$. If Γ and $\Gamma_{\mathcal{L}}$ are endomorphism algebras of basic generators of $\mathcal{C}$ and $\mathcal{L}$, respectively, then*

$$\det C_\Gamma = \det C_{\Gamma_{\mathcal{L}}} \det C_D. \tag{5}$$

Proof. Fix $f : X \to Y$, and write $T = \ker(f, p_Y)$. Let $h : T \to X$ and $v : T \to U_Y$ be the components of its kernel morphism. Then $fh = -p_Y v$, so $\bar{f}\bar{h} = 0$.

For a map $g : V \to X$ with $\bar{f}\bar{g} = 0$, write $fg = ab$ with $b : V \to V'$ and $a : V' \to Y$, where $V' \in \mathcal{L}$. By approximation, $a = p_Y s$ for some $s : V' \to U_Y$. Thus $(g, -sb) : V \to X \oplus U_Y$ factors through T, giving a factorization of $\bar{g}$ through $\bar{h}$.

For uniqueness, suppose $k : V \to T$ and $hk = cb$ factor through $V' \in \mathcal{L}$, with $c : V' \to X$. Lift fc through p_Y, writing $fc = p_Y t$. The morphism $(c, -t)$ induces $j : V' \to T$, and $h(k - jb) = 0$. Let $\kappa : \ker p_Y \to U_Y$ be the kernel map. The pair $(0, \kappa)$ induces $\iota : \ker p_Y \to T$. If $w : V \to T$ satisfies $hw = 0$, then $p_Y vw = -fhw = 0$. There is a unique $s : V \to \ker p_Y$ with $\kappa s = vw$. The pairs $(h, v)w$ and $(h, v)\iota s$ are equal; since the kernel morphism (h, v) is monic, $w = \iota s$. Thus ι is the kernel of h. In particular, $k - jb$ factors through $\ker p_Y \in \mathcal{L}$. Both jb and $k - jb$ vanish in $\mathcal{Q}$, so $\bar{k} = 0$. Hence $\bar{h}$ is a categorical kernel of $\bar{f}$.

Decompose each object using its indecomposable summands in $\mathcal{C}$. The summands in $\mathcal{L}$ become zero in $\mathcal{Q}$. For a summand X outside $\mathcal{L}$, the ideal $[\mathcal{L}](X, X)$ contains no invertible map, since an invertible factorization would make X a summand of an object of $\mathcal{L}$. It is therefore contained in the radical of the local ring $\operatorname{End}_{\mathcal{C}}(X)$. The quotient endomorphism ring remains local, with the same residual division ring. Distinct surviving summands remain nonisomorphic: mutually inverse quotient maps lift to maps whose composites differ from identities by radical elements, and an invertible composite makes one indecomposable object a summand of the other.

Let V be the sum of the surviving representatives and set $D = \operatorname{End}_{\mathcal{Q}}(V)$. The projections onto these summands give a complete set of pairwise nonisomorphic primitive idempotents in the

Artin algebra D. Thus all finite projective D-modules are represented by direct sums of those objects. Full faithfulness of $\mathrm{Hom}_{\mathcal{Q}}(V,-)$ on these sums gives $\mathcal{Q} \simeq \operatorname{proj} D$, including idempotent completeness. The kernel assertion and Lemma A.1 imply $\operatorname{gl.dim} D \leq 2$.

For the determinant, let W be the basic generator of $\mathcal{C}$ and put $P_Y = \mathrm{Hom}_{\mathcal{C}}(W,Y)$. Let $I_Y = [\mathcal{L}](W,Y)$, the image of $\mathrm{Hom}(W, p_Y)$. The kernel gives

$$0 \longrightarrow P_{\ker p_Y} \longrightarrow P_{U_Y} \longrightarrow I_Y \longrightarrow 0.$$

For each simple L_i, additivity of composition multiplicities yields

$$[P_Y/I_Y : L_i] = [P_Y : L_i] - [P_{U_Y} : L_i] + [P_{\ker p_Y} : L_i].$$

Order the indecomposable objects with those of $\mathcal{L}$ first. Keep the corresponding Cartan columns and replace each remaining column by the displayed combination. Only columns indexed by $\mathcal{L}$ are added to it, so the transformation has determinant one. The new high columns vanish on the low rows, since every map from a low object factors through $\mathcal{L}$. The low diagonal block is $C_{\Gamma_{\mathcal{L}}}$ by (4). The high diagonal block is C_D: the surviving residual division rings are unchanged, and its component spaces are exactly $\mathcal{Q}(X_i, X_j)$. The transformed matrix is therefore

$$\begin{pmatrix} C_{\Gamma_{\mathcal{L}}} & 0 \\ * & C_D \end{pmatrix},$$

which proves (5). □

A.2 The filtration of a corner

Proof of Theorem 2.3. Put $A = eRe$ and $a = \operatorname{gl.dim} A < \infty$. For $\mathcal{E} = \operatorname{proj} R$, evaluation at e is the exact functor

$$F : \operatorname{mod} R \longrightarrow \operatorname{mod} A, \qquad F(N) = Ne.$$

It preserves kernels by restriction to the e-components. If a map is surjective and $y = ye$ is in its target, choose a preimage x; then xe maps to y. Hence F also preserves surjections and is exact.

The category $\mathcal{E}$ is closed under module kernels. Indeed, for a map $g : P_1 \to P_0$ between projectives, put $T = \ker g$. For any right R-module V, the short exact sequences with middle terms P_1 and P_0 give

$$\operatorname{Ext}^1_R(T,V) \cong \operatorname{Ext}^2_R(\operatorname{im} g, V) \cong \operatorname{Ext}^3_R(\operatorname{coker} g, V) = 0.$$

Thus T is projective; it is finitely generated because R is Artinian.

For integers $0 \leq m \leq a$, define

$$\mathcal{E}_m = \{P \in \operatorname{proj} R : \operatorname{pd}_A(Pe) \leq m\}, \qquad \mathcal{P} = \operatorname{add}(eR).$$

These are finite additive subcategories, $\mathcal{P} \subseteq \mathcal{E}_0$, and $\mathcal{E}_a = \mathcal{E}$. Let Γ_m be the endomorphism algebra of a basic generator of $\mathcal{E}_m$.

For $Q \in \operatorname{proj} A$, the induced module $Q \otimes_A eR$ belongs to $\mathcal{P}$: a splitting of Q from a finite free A-module tensors to a splitting from a finite sum of eR. For $Y \in \operatorname{proj} R$ we have

$$\mathrm{Hom}_R(Q \otimes_A eR, Y) \cong \mathrm{Hom}_A(Q, Ye), \qquad (Q \otimes_A eR)e \cong Q. \tag{6}$$

The first bijection sends u to $q \mapsto u(q \otimes e)$; its inverse sends v to $q \otimes er \mapsto v(q)r$. In the second isomorphism, $q \otimes ere$ maps to $qere$. Thus F restricts to an equivalence $\mathcal{P} \simeq \operatorname{proj} A$.

Positive layers. Fix $1 \leq m \leq a$ and $Y \in \mathcal{E}_m$. Choose a right $\mathcal{E}_{m-1}$-approximation $p_Y : U_Y \to Y$. Since $\mathcal{P} \subseteq \mathcal{E}_{m-1}$, (6) lifts a projective cover of FY to a map from an object of $\mathcal{P}$, which then factors through p_Y. Hence Fp_Y is surjective.

The module kernel $K_Y = \ker_R p_Y$ belongs to $\operatorname{proj} R$, and exactness of F gives

$$0 \longrightarrow FK_Y \longrightarrow FU_Y \longrightarrow FY \longrightarrow 0.$$

The projective-dimension bound for this sequence is

$$\operatorname{pd}_A FK_Y \le \max\{\operatorname{pd}_A FU_Y, \operatorname{pd}_A FY - 1\} \le m - 1. \tag{7}$$

For every right A-module V and integer $q \ge 1$, the long exact Ext sequence has a segment

$$\operatorname{Ext}^q_A(FU_Y, V) \longrightarrow \operatorname{Ext}^q_A(FK_Y, V) \longrightarrow \operatorname{Ext}^{q+1}_A(FY, V).$$

The outer terms vanish for $q > m - 1$, so $K_Y \in \mathcal{E}_{m-1}$.

For $f : X \to Y$ in $\mathcal{E}_m$, the kernel $T = \ker_R(f, p_Y)$ is again projective over R. Its evaluation fits into

$$0 \longrightarrow FT \longrightarrow FX \oplus FU_Y \xrightarrow{F(f,p_Y)} FY \longrightarrow 0.$$

The last map is surjective, and the same bound gives $\operatorname{pd}_A FT \le m$. Thus $T \in \mathcal{E}_m$. Lemma A.2 now applies to $\mathcal{E}_{m-1} \subseteq \mathcal{E}_m$. Write D_m for the endomorphism algebra of the surviving basic generator of $\mathcal{E}_m/[\mathcal{E}_{m-1}]$. We obtain

$$\operatorname{gl.dim} D_m \le 2, \qquad \det C_{\Gamma_m} = \det C_{\Gamma_{m-1}} \det C_{D_m}. \tag{8}$$

The initial layer. Let

$$\mathcal{Z} = \{P \in \operatorname{proj} R : Pe = 0\}.$$

The kernel of any map between its objects is projective over R and evaluates to zero. Lemma A.1 implies that the endomorphism algebra Z of a basic generator of $\mathcal{Z}$ has global dimension at most two.

Set $\mathcal{L}_0 = \operatorname{add}(\mathcal{P} \cup \mathcal{Z})$ inside $\mathcal{E}_0$. The two subcategories have no common nonzero indecomposable object, since F is an equivalence on $\mathcal{P}$ and vanishes on $\mathcal{Z}$. Moreover,

$$\operatorname{Hom}_R(eR, Z') \cong Z'e = 0 \qquad (Z' \in \mathcal{Z}).$$

Therefore the Cartan matrix of the endomorphism algebra of a basic generator of $\mathcal{L}_0$ is block triangular, with diagonal blocks C_A and C_Z. Here the first block uses $\operatorname{End}_R(eR) \cong eRe$, where $x \in eRe$ acts by left multiplication, and Morita invariance removes repeated summands of eR. Its determinant is $\det C_A \det C_Z$.

For $Y \in \mathcal{E}_0$, the module FY is projective over A. Define

$$p_Y : (FY) \otimes_A eR \longrightarrow Y, \qquad y \otimes er \longmapsto yr.$$

Every map $Q \otimes_A eR \to Y$ corresponds under (6) to $Q \to FY$ and therefore factors through p_Y by tensoring that map with eR. Also Fp_Y is an isomorphism. Choose a right $\mathcal{Z}$-approximation $z_Y : Z_Y \to Y$ and form

$$q_Y = (p_Y, z_Y) : (FY) \otimes_A eR \oplus Z_Y \longrightarrow Y.$$

This is a right $\mathcal{L}_0$-approximation. Its kernel is projective over R and evaluates to zero because Fq_Y is an isomorphism. Hence $\ker q_Y \in \mathcal{Z} \subseteq \mathcal{L}_0$.

For $f : X \to Y$ in $\mathcal{E}_0$, the kernel of (f, q_Y) is projective over R. After evaluation, projection onto the first component identifies the kernel of (Ff, Fq_Y) with FX; its inverse is $x \mapsto (x, -(Fq_Y)^{-1}Ff(x))$. It is therefore projective over A, so $\ker(f, q_Y) \in \mathcal{E}_0$. Applying Lemma A.2 to $\mathcal{L}_0 \subseteq \mathcal{E}_0$ gives an Artin algebra D_0 with

$$\operatorname{gl.dim} D_0 \le 2, \qquad \det C_{\Gamma_0} = \det C_A \det C_Z \det C_{D_0}. \tag{9}$$

Since $\mathcal{E}_a = \operatorname{proj} R$, its basic generator has endomorphism algebra Morita equivalent to R. Multiplying (8) and (9) yields

$$\det C_R = \det C_A \det C_Z \prod_{m=0}^{a} \det C_{D_m}. \tag{10}$$

Every nonzero algebra among $R, Z, D_0, \ldots, D_a$ has global dimension at most two. Zacharia's theorem [22] gives determinant one for each of them. Zero categories contribute the empty determinant one. Thus (10) gives $\det C_A = 1$. □

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