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KERNEL DIMENSION OF THE $\mathbb{Z}/2$ CONNECTION LAPLACIAN ON A FINITE SIMPLE GRAPH (A FORMALISATION IN LEAN 4 / MATHLIB V4.11.0)

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Abstract. We present a fully formalised, machine-checked proof library in Lean 4 / Mathlib v4.11.0 for the kernel-dimension theory of the $\mathbb{Z}/2$ *connection Laplacian* of a finite simple graph G. A connection graph is a finite simple graph equipped with a designated set $W \subseteq E(G)$ of *wrap edges*; on each edge e we attach a 2×2 real matrix (the Pauli σ_x on wrap edges, $\mathbb{I}_2$ otherwise), yielding a block Laplacian L_{Mob}. Our principal results, each proved from the three standard Lean axioms (`propext`, `Classical.choice`, `Quot.sound`) and nothing else, are the following.

(i) A Hadamard–Kronecker conjugation decomposes L_{Mob} into the block-diagonal sum of the ordinary (*scalar*) Laplacian and a *signed* Laplacian, both on G.

(ii) The orientation double cover $\widetilde{G}$ of G inherits a trivial connection, and its scalar Laplacian splits similarly.

(iii) A base connected component $C \in \pi_0(G)$ is *balanced* iff the induced map $\pi_0(\widetilde{G}) \to \pi_0(G)$ has fibre of cardinality 2 over C.

(iv) The kernel dimension of the signed Laplacian equals the number of balanced components; the full $\mathbb{Z}/2$-connection Laplacian has kernel dimension $\#\pi_0(G)+\#\{\text{balanced components}\}$ (Möbius) and $2\,\#\pi_0(G)$ (flat).

(v) On the n-cycle ($n \geq 3$) the scalar and the one-wrap signed Laplacians have explicit cosine spectra: their characteristic polynomials and eigenvalue multisets, with multiplicities, are proved for the explicit cycle matrices.

The bespoke arguments are the Hadamard rotation of part (i)–(ii) (a Kronecker-conjugation computation) and the double-cover lift of part (iii) (a walk induction). Along the way we observe and correct an incorrect statement in our initial written plan: the identity $\dim\ker L^{\mathrm{sg}}(G) = \#\pi_0(G) - \#\{\text{components of odd total wrap}\}$ holds for every wrap assignment on a cycle, but fails for some wrap assignments on general graphs; the universally valid replacement is the cohomological formula $\dim\ker L^{\mathrm{sg}}(G) = \#\{\text{balanced components}\}$, which is the content of Theorem 6.2. The Lean sources and a numerical cross-check on every simple graph of order ≤ 5 (3,456 configurations, zero disagreements) – and an extended fuzzer sweep at orders $n \leq 7$ (456,950 configurations, also zero disagreements) accompany this paper.

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1. Introduction

Let $G = (V, E)$ be a finite simple graph and let $W \subseteq E$ be a distinguished set of *wrap* edges. Attach to each $e \in E$ a 2×2 real matrix $\Phi(e)$ by

$$\Phi(e) := \begin{cases} \sigma_x & \text{if } e \in W, \\ \mathbb{I}_2 & \text{if } e \notin W, \end{cases} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The $\mathbb{Z}/2$-*connection Laplacian* of (G, W) is the $V \times V$ block matrix with 2×2 blocks

$$L_{\mathrm{Mob}}[u, v] := \begin{cases} \deg(u)\, \mathbb{I}_2 & u = v, \\ -\Phi(\{u, v\}) & \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Equivalently, $L_{\mathrm{Mob}} = D \otimes \mathbb{I}_2 - \sum_{e \in E} \mathbf{1}_e \otimes \Phi(e)$, where, for $e = \{u, v\}$, $\mathbf{1}_e$ is the symmetric $V \times V$ edge-indicator with entries 1 at (u, v) and at (v, u) and 0 elsewhere (both Kronecker factors ordered vertex-major, matching the $V \times \mathrm{Fin}\, 2$ index set of Equation (1)). The *flat* Laplacian L_{flat} is obtained by the same recipe with $\Phi \equiv \mathbb{I}_2$, i.e. by ignoring the wrap structure; it agrees with two copies of the ordinary graph Laplacian. The guiding question answered in this paper is:

In terms of (G, W)*, what is* $\dim\ker L_{\mathrm{Mob}}$ *and when is it strictly smaller than* $\dim\ker L_{\mathrm{flat}}$*?*

This is the finite-dimensional, discrete analogue of the spectrum of the Laplacian on a rank-two flat vector bundle over a graph: the kernel detects global *parallel* sections, not arbitrary sections. The representation splits as a trivial line, spanned by $(1, 1)^\top$, and a signed line, spanned fibrewise by $(1, -1)^\top$. Thus the rank-two bundle always admits the nowhere-zero parallel section $s(v) = (1, 1)^\top$, regardless of the wrap data. The signed line admits a nonzero parallel section over a component exactly when the wrap holonomy around every cycle in that component is trivial. Equivalently, the rank-two bundle has two independent parallel sections there rather than just one; this is the condition of *balance* in the language of Harary, Cartwright and Zaslavsky [1, 2].

Contribution and prior work. The contribution is a Lean formalisation and a documented verification process for classical spectral graph theory [4]. We do not claim new mathematical discovery of the balance–nullity or lift-splitting formulas. For the signed Laplacian $D - A_{\mathrm{sg}}$, the standard nullity formula is stated explicitly in Zaslavsky's survey [3, Lemma IV.2]. Bilu and Linial's 2-lift result gives the adjacency spectrum as the base and signed spectra with multiplicities [5, preprint Lemma 3.1]. Our Laplacian splitting uses the same sheet-symmetric and sheet-antisymmetric subspaces: the degree block acts as D in both. This also works for

irregular finite simple graphs; it does not require replacing D by a scalar. The orientation double cover of Definition 4.1 is the *Gremban expansion* of the signed graph whose negative edges are W; Diaz-Diaz, Devriendt and Lambiotte prove that its Laplacian is similar to the direct sum of the unsigned and signed Laplacians [6, Def. 2.1, Cor. 3.8], which is the content of Theorem 4.4. The doubling goes back to Gremban's reduction of symmetric diagonally dominant systems to systems of twice the dimension with non-positive off-diagonal entries [7], recalled in [8, §2.2]. On the formal side, Mathlib's `SimpleGraph` library at the pinned revision provides the scalar Laplacian `lapMatrix`, its positive semidefiniteness, and the identification of its kernel dimension with the number of connected components [9]. It has no signed or connection Laplacian and no orientation double cover; those, and the results above, are developed here in Lean 4 [10]. The repository certifies the resulting identities and records the remaining specification and formalisation obligations. The numerical work is a cross-check, not an ML application or a scalability benchmark.

Organisation and results. Section 2 sets the notation and fixes the Lean-formalised data. Section 3 proves the Hadamard–Kronecker decomposition that reduces the block Laplacian to two ordinary Laplacians (one scalar, one signed). Section 4 constructs the orientation double cover and proves that its scalar Laplacian splits as the direct sum of the scalar and signed Laplacians of G. Section 5 formalises the balance notion as the vanishing of a $\mathbb{Z}/2$-cohomology class and computes the fibre cardinality of the induced map on π_0. Section 6 assembles everything into the *recognition theorems* which count kernel dimension by balanced components. Section 7 specialises to the n-cycle, where the Fourier basis diagonalises both Laplacians. Section 8 summarises the formal Lean development.

The main structural decomposition, cover-splitting, cohomological recognition, kernel-dimension, and auxiliary identities are machine-checked in Lean 4 against Mathlib v4.11.0. The *complete* flat and one-wrap signed cycle characteristic polynomials and real root multisets, including algebraic multiplicities for every $n \geq 3$, are also machine-checked in `CycleMultiset.lean`; see Remark 7.3. The claims concern the explicit cycle matrices used in the Lean project. A further graph-level identification with arbitrary `ConnGraph` cycle instances is not asserted. The complete proof scripts and independent numerical checks are included in the accompanying repository.

2. Preliminaries

2.1. **Connection graphs.** A *connection graph* is a quadruple $G = (V, E, \mathrm{Adj}, W)$ where V is a finite type, $\mathrm{Adj} \subseteq V \times V$ defines a (mathematically simple) graph on V, $E = \{\{u, v\} : \mathrm{Adj}(u, v)\}$ is its edge set and $W \subseteq E$ is an arbitrary subset, the *wrap* edges. In Lean we keep the wrap data as a predicate `wrap : G.graph.edgeSet → Prop` and carry explicit `Fintype`/`DecidableEq`/`DecidableRel` instances throughout.

2.2. **Pauli and Hadamard.** The 2×2 Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is involutive, symmetric and has eigenvalues ± 1, with symmetric eigenvector $(1, 1)^\top$ and antisymmetric $(1, -1)^\top$. The *Hadamard rotation*

$$\widehat{R} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \widehat{R}\,\widehat{R} = 2\,\mathbb{I}_2, \tag{2}$$

simultaneously diagonalises σ_x and $\mathbb{I}_2$ in the sense that $\widehat{R}\,\sigma_x\,\widehat{R} = 2\,\mathrm{diag}(1, -1)$ and $\widehat{R}\,\mathbb{I}_2\,\widehat{R} = 2\,\mathbb{I}_2$. In the Lean formalisation we keep two copies of $\widehat{R}$: one indexed by $\{0, 1\} = \mathrm{Fin}\,2$ (for the connection-block decomposition of Section 3) and one indexed by Bool (for the cover-sheet decomposition of Section 4). They are the same matrix under the obvious bijection $\mathrm{Bool} \cong \mathrm{Fin}\,2$.

2.3. **Scalar and signed Laplacians.** Let G be a connection graph. The ordinary *scalar* Laplacian $L^{\mathrm{sc}} \in \mathbb{R}^{V\times V}$ is

$$L^{\mathrm{sc}}[u,v] := \begin{cases} \deg(u) & u = v, \\ -1 & \{u,v\} \in E, \\ 0 & \text{otherwise}, \end{cases}$$

i.e. Mathlib's `SimpleGraph.lapMatrix`. The *signed Möbius* Laplacian $L^{\mathrm{sg}} \in \mathbb{R}^{V\times V}$ encodes the wrap sign on the off-diagonal:

$$L^{\mathrm{sg}}[u,v] := \begin{cases} \deg(u) & u = v, \\ +1 & \{u,v\} \in E,\ \{u,v\} \in W, \\ -1 & \{u,v\} \in E,\ \{u,v\} \notin W, \\ 0 & \text{otherwise}. \end{cases}$$

Both are real symmetric and positive semidefinite as quadratic forms on $\mathbb{R}^V$. For L^{sc} this is Mathlib's `SimpleGraph.posSemidef_lapMatrix`. For L^{sg} we use the direct sum-of-squares expression

$$\langle x, L^{\mathrm{sg}} x\rangle = \sum_{\{u,v\}\in E\setminus W} (x_u - x_v)^2 + \sum_{\{u,v\}\in W} (x_u + x_v)^2,$$

which Lemma 9.5 (§9) turns into the Lean fact `signedLaplacianMobius_posSemidef` (`L13_PSD.lean`). Equivalently — but through a heavier route not used by the Lean proof — PSD follows from the cover splitting (Theorem 4.4): the orientation double cover's scalar Laplacian is PSD, and similarity to $\mathrm{diag}(L^{\mathrm{sc}}, L^{\mathrm{sg}})$ preserves the PSD property of each diagonal block. The Fourier diagonalisation in §7 below regards both matrices as acting on $\mathbb{C}^V$, but the matrices themselves remain real; complexification is a computational convenience, not a domain change.

3. The Block Decomposition

The connection Laplacian L_{Mob} of (1) lives on the product index set $V \times \mathrm{Fin}\,2$. Reindexing via the canonical bijection

$$\varphi : V \times \mathrm{Fin}\,2 \xrightarrow{\sim} V \sqcup V, \qquad (u,0) \mapsto \iota_\ell u,\ (u,1) \mapsto \iota_r u,$$

we get a matrix on the sum index $V \sqcup V$; the block-diagonal decomposition then becomes a genuine *block-diagonal* form `Matrix.fromBlocks`.

Theorem 3.1. *Let G be a connection graph and $\epsilon \in \{\mathit{flat}, \mathit{Mob}\}$. There exists an invertible $P \in \mathbb{R}^{(V\times\mathrm{Fin}\,2)\times(V\times\mathrm{Fin}\,2)}$ such that*

$$\varphi_*\big(P\,L_\epsilon\,P^{-1}\big) = \begin{pmatrix} L^{\mathrm{sc}} & 0 \\ 0 & B_\epsilon \end{pmatrix}, \qquad B_\epsilon = \begin{cases} L^{\mathrm{sg}} & \epsilon = \mathit{Mob}, \\ L^{\mathrm{sc}} & \epsilon = \mathit{flat}. \end{cases}$$

Proof. Set $P = \mathbb{I}_V \otimes_K \widehat{R}$ (Kronecker product with the Hadamard $\widehat{R}$ of (2) acting on the $\mathrm{Fin}\,2$ fibre). By functoriality of the Kronecker product,

$$P \cdot P = (\mathbb{I}_V \cdot \mathbb{I}_V) \otimes_K (\widehat{R}\,\widehat{R}) = \mathbb{I}_V \otimes_K (2\mathbb{I}_2) = 2\,\mathbb{I}_{V\times\mathrm{Fin}\,2},$$

so $P^{-1} = \frac{1}{2}P$, and in particular $\det P \neq 0$.

A direct entrywise calculation using $\widehat{R}\,\sigma_x\,\widehat{R} = 2\,\mathrm{diag}(1,-1)$ and $\widehat{R}\,\mathbb{I}_2\,\widehat{R} = 2\,\mathbb{I}_2$ yields, for $u, v \in V$ and $i, j \in \mathrm{Fin}\,2$,

$$\tfrac{1}{2}\,\big(P\,L_\epsilon\,P\big)\big[(u,i),(v,j)\big] = \begin{cases} L^{\mathrm{sc}}[u,v] & i = j = 0, \\ B_\epsilon[u,v] & i = j = 1, \\ 0 & i \neq j. \end{cases}$$

Reindexing via φ and using $P^{-1} = \frac{1}{2}P$ turns the left-hand side into $\varphi_*\big(P\,L_\epsilon\,P^{-1}\big)$, giving the claim. The flat case is the special case $W = \emptyset$ of the Möbius case, or equivalently the fact that $\widehat{R}\,\mathbb{I}_2\,\widehat{R} = 2\mathbb{I}_2$ collapses both diagonal blocks to the scalar Laplacian. □

Remark 3.2. In the Lean formalisation `laplacian_decomposes` the two ϵ-branches are proved separately: the Möbius branch uses $P = \mathbb{I}_V \otimes_K \widehat{R}$ as above, while the flat branch takes the simpler $P = \mathbb{I}_{V \times \mathrm{Fin}\,2}$ (identity similarity) and reduces the claim to $L_{\mathrm{flat}} = \mathrm{diag}(L^{\mathrm{sc}}, L^{\mathrm{sc}})$ after reindex via φ. Mathematically the flat case is the specialisation of the Möbius case at $W = \emptyset$, but the two Lean branches are syntactically disjoint.

Remark 3.3. In the Lean script the bijection φ is `prodFinTwoEquivSum` and the Kronecker-product identity is the lemma `mul_kronecker_mul` from `Mathlib.Data.Matrix.Kronecker`. The entrywise calculation is made into a single-sum `Finset.sum_eq_single` reduction that kills V-off-diagonal terms using the $\mathbb{I}_V$ factor of P, and then reduces the remaining $\mathrm{Fin}\,2 \times \mathrm{Fin}\,2$ block to one of three cases by matching on equality of vertices and wrap status.

4. The Orientation Double Cover

The preceding decomposition separates the flat part of G from its wrap data inside a single graph. The cover perspective lifts the wrap data into a *branching* structure on a new, connection-free graph.

Definition 4.1. The *orientation double cover* of the connection graph (G, W) is the simple graph $\widetilde{G}$ on vertex set $\widetilde{V} := V \times \mathrm{Bool}$, with adjacency

$$(u, b) \sim_{\widetilde{G}} (v, c) \iff \{u, v\} \in E \text{ and } \big(\{u, v\} \in W \Leftrightarrow b \neq c\big).$$

We view $\widetilde{G}$ as a connection graph with $\widetilde{W} = \emptyset$: every edge of $\widetilde{G}$ is "interior", the cover has *trivial* connection.

Equivalently, $\widetilde{G}$ is the $\mathbb{Z}/2$-covering space determined by the 1-cochain $\sigma_W : E \to \mathbb{Z}/2,\ e \mapsto \mathbf{1}_{e \in W}$ (which is automatically a 1-cocycle since the graph has no 2-cells; see §5). The non-trivial *deck transformation* $\delta : \widetilde{V} \to \widetilde{V},\ (v, b) \mapsto (v, \neg b)$ is a free graph involution.

Lemma 4.2. *δ is a graph automorphism of $\widetilde{G}$ of order 2 with no fixed points.*

Proof. Involutivity is immediate from $\neg\neg b = b$. The adjacency $(\delta x) \sim_{\widetilde{G}} (\delta y) \Leftrightarrow x \sim_{\widetilde{G}} y$ follows from the identity $(\neg b) \neq (\neg c) \Leftrightarrow b \neq c$ and fixity of the underlying edge $\{u, v\}$. Absence of fixed points is trivial: $(v, b) \neq (v, \neg b)$. □

Lemma 4.3. *For every $(u, b) \in \widetilde{V}$, $\deg_{\widetilde{G}}(u, b) = \deg_G(u)$.*

Proof. We construct a bijection $\mathcal{N}_G(u) \to \mathcal{N}_{\widetilde{G}}(u, b)$ by

$$v \longmapsto \big(v,\ b \oplus \mathbf{1}_{\{u,v\} \in W}\big),$$

where $\oplus$ is the Bool XOR. Correctness is the definition of $\sim_{\widetilde{G}}$; injectivity follows by reading the first coordinate; surjectivity uses the biconditional $\{u, v\} \in W \Leftrightarrow b \neq c$ to recover the unique v with $(u, b) \sim_{\widetilde{G}} (v, c)$. □

We write $\widetilde{L} := L^{\mathrm{sc}}(\widetilde{G})$ for the scalar Laplacian of the cover.

The Hadamard rotation on the Bool**-fibre.** Replace the $\mathrm{Fin}\,2$-fibre of Section 3 by the Bool-fibre, viewing

$$\widehat{R} : \mathbb{R}^{\mathrm{Bool} \times \mathrm{Bool}}, \qquad \widehat{R}[i, j] = \begin{cases} -1 & i = j = \top, \\ 1 & \text{otherwise}, \end{cases}$$

so $\widehat{R}\,\widehat{R} = 2\,\mathbb{I}$ on $\mathbb{R}^{\mathrm{Bool} \times \mathrm{Bool}}$. Set

$$P_{\mathrm{cov}} := \mathbb{I}_V \otimes_K \widehat{R} \quad \in \mathbb{R}^{\widetilde{V} \times \widetilde{V}}.$$

Theorem 4.4. *Under the canonical bijection* $\varphi : \widetilde{V} \xrightarrow{\sim} V \sqcup V$, $(v, \bot) \mapsto \iota_\ell v$, $(v, \top) \mapsto \iota_r v$, *we have*

$$\varphi_*\big(P_{\mathrm{cov}}\, \widetilde{L}\, P_{\mathrm{cov}}^{-1}\big) = \begin{pmatrix} L^{\mathrm{sc}}(G) & 0 \\ 0 & L^{\mathrm{sg}}(G) \end{pmatrix}.$$

In particular, P_{cov} *is invertible and* $\widetilde{L} \sim L^{\mathrm{sc}}(G) \oplus L^{\mathrm{sg}}(G)$.

Proof. Write every 2×2 block

$$B_{u,v} := \widehat{R}\, \widetilde{L}[u, v]\, \widehat{R} \in \mathbb{R}^{\mathrm{Bool} \times \mathrm{Bool}}$$

where $\widetilde{L}[u, v]$ is the $\{u\} \times$ Bool-rows, $\{v\} \times$ Bool-columns sub-block of $\widetilde{L}$. A direct Kronecker calculation gives

$$\big(P_{\mathrm{cov}}\, \widetilde{L}\, P_{\mathrm{cov}}\big)\big[(u, i), (v, j)\big] = (\widehat{R}\, \widetilde{L}[u, v]\, \widehat{R})[i, j] = B_{u,v}[i, j]. \tag{3}$$

Split $B_{u,v}$ into four structural cases:

- **(a)** *Diagonal.* If $u = v$, then by Lemma 4.3, $\widetilde{L}[u, u] = \deg(u)\, \mathbb{I}$, whence $\widehat{R}(\deg(u)\, \mathbb{I})\widehat{R} = \deg(u) \cdot 2\mathbb{I}$, so $B_{u,u}[i, j] = 2 \deg(u) \cdot \delta_{ij}$.
- **(b)** *Adjacent, non-wrap.* If $\{u, v\} \in E \setminus W$, then $\widetilde{L}[u, v]$ has -1 on the *diagonal* of Bool (sheets match) and 0 off-diagonal: $\widetilde{L}[u, v] = (-1)\mathbb{I}$. So $B_{u,v}[i, j] = -2\delta_{ij}$.
- **(c)** *Adjacent, wrap.* If $\{u, v\} \in E \cap W$, then $\widetilde{L}[u, v]$ has -1 off the diagonal (sheets flip) and 0 on the diagonal: $\widetilde{L}[u, v] = -\sigma_x$. So $\widehat{R}(-\sigma_x)\widehat{R} = -2\,\mathrm{diag}(1, -1)$, i.e. $B_{u,v}[i, j] = -2\delta_{ij}\,(-1)^{\mathbf{1}_{i=\top}}$.
- **(d)** *Non-adjacent.* If $\{u, v\} \notin E$, then $\widetilde{L}[u, v] = 0$ and $B_{u,v} = 0$.

Collecting cases and matching against the defining formulas of L^{sc} and L^{sg}:

$$B_{u,v}[i, j] = 2 \cdot \delta_{ij} \cdot \begin{cases} L^{\mathrm{sc}}[u, v] & i = \bot, \\ L^{\mathrm{sg}}[u, v] & i = \top. \end{cases}$$

Because $P_{\mathrm{cov}}^{-1} = \frac{1}{2} P_{\mathrm{cov}}$ (by $P_{\mathrm{cov}}^2 = 2\mathbb{I}$, exactly as in Section 3), reindexing through φ gives the claimed block-diagonal form. □

Corollary 4.5. $\dim_{\mathbb{R}} \ker \widetilde{L} = \dim_{\mathbb{R}} \ker L^{\mathrm{sc}}(G) + \dim_{\mathbb{R}} \ker L^{\mathrm{sg}}(G)$.

Proof. Kernel dimension is invariant under similarity by an invertible matrix and under reindexing by a bijection of the index set; for a block-diagonal matrix it is additive in the blocks. Apply these three standard linear-algebra facts to the similarity of Theorem 4.4. □

Remark 4.6. In the Lean code these three standard facts are the lemmas

`finrank_ker_toLin'_conj`, `finrank_ker_toLin'_reindex`, `finrank_ker_toLin'_fromBlocks_diag`.

They are proved once, over an arbitrary field, in `KernelDimension.lean` and reused for both the block-decomposition Theorem 3.1 and the cover-splitting Corollary 4.5.

5. $\mathbb{Z}/2$ Cohomology and Balanced Components

The wrap predicate $W \subseteq E$ reifies canonically as a $\mathbb{Z}/2$-valued 1-cochain

$$\sigma_W : E \longrightarrow \mathbb{Z}/2, \qquad \sigma_W(e) := \mathbf{1}_{e \in W}.$$

It is automatically a 1-cocycle (there is no 2-skeleton to impose constraints on it). Its cohomology class $[\sigma_W] \in H^1(G; \mathbb{Z}/2)$ is the obstruction to W arising as the coboundary of a vertex $\mathbb{Z}/2$-colouring: $[\sigma_W] = 0$ iff there exists $\varepsilon : V \to \mathrm{Bool}$ with $W = \{\, \{u, v\} \in E : \varepsilon(u) \neq \varepsilon(v) \,\}$. This is the classical notion of *balance* for signed graphs.

Definition 5.1. A connected component $C \in \pi_0(G)$ is *balanced* iff there is a map $\varepsilon : V \to \mathrm{Bool}$ such that for every edge $\{u, v\} \in E$ with $[u]_{\sim_G} = C$,

$$\{u, v\} \in W \iff \varepsilon(u) \neq \varepsilon(v).$$

We write $\pi_0^{\mathrm{bal}}(G) \subseteq \pi_0(G)$ for the subset of balanced components, and $\beta(G) := \#\pi_0^{\mathrm{bal}}(G)$.

Remark 5.2 (Scope conventions for Definition 5.1). Two conventions deserve to be flagged.

(a) **Component-local vs. graph-global.** The definition quantifies only over edges *inside* the component C; the values $\varepsilon(w)$ for w outside C are unconstrained and play no role. For a *connected* graph the present definition coincides with Harary's original graph-global notion of balance ($[\sigma_W] = 0$ in $H^1(G; \mathbb{Z}/2)$); for disconnected graphs they differ, and *every theorem below uses the component-local version.* Throughout the paper $\beta(G)$ denotes the component-local count.

(b) **Isolated vertices are vacuously balanced.** If $C = \{v_0\}$ is a singleton, the edge quantifier is vacuous and the condition holds with any ε. Consequently every isolated vertex counts towards $\beta(G)$; this is load-bearing for Theorem 6.2 below, because an isolated vertex contributes a zero row to L^{sg} and hence $+1$ to the kernel dimension.

The main combinatorial result of this section relates balance to the fibres of the induced map on connected components.

Theorem 5.3. *Let $\pi : \pi_0(\widetilde{G}) \to \pi_0(G)$ be the map induced by the sheet projection $\widetilde{G} \to G$, $(v, b) \mapsto v$. For every $C \in \pi_0(G)$,*

$$\#\pi^{-1}(C) \;=\; \begin{cases} 2 & C \text{ balanced,} \\ 1 & \text{otherwise.} \end{cases}$$

The proof occupies the remainder of this section. Fix $C \in \pi_0(G)$ and choose a representative $v_0 \in V$ with $[v_0]_{\sim_G} = C$. Set the *candidate lifts*

$$D_\perp := [(v_0, \perp)]_{\pi_0(\widetilde{G})}, \qquad D_\top := [(v_0, \top)]_{\pi_0(\widetilde{G})}.$$

Step 1: every component in the fibre is a candidate lift.

Lemma 5.4. *For every $D \in \pi^{-1}(C)$, either $D = D_\perp$ or $D = D_\top$.*

Proof. Since $\pi([(w, b)]) = [w]_{\pi_0(G)}$ factors through the first coordinate, $D = [(w, b)]$ for some $(w, b) \in \widetilde{V}$ with $[w] = C$. By connectedness of C there is a walk $w \to v_0$ in G; lifting edge by edge and setting

$$\text{next sheet}\big(b,\ \{u, u'\}\big) := \begin{cases} \neg b & \{u, u'\} \in W, \\ b & \text{otherwise,} \end{cases}$$

we obtain a walk $(w, b) \to (v_0, b'')$ in $\widetilde{G}$ for some $b'' \in \mathrm{Bool}$ determined by the walk. So $[(w, b)] = [(v_0, b'')]$, and $D \in \{D_\perp, D_\top\}$. □

Step 2: the balanced case.

Lemma 5.5. *If C is balanced, then $D_\perp \neq D_\top$, whence $\#\pi^{-1}(C) = 2$.*

Proof. Let ε witness balance. Define the *XOR* invariant $\chi : \widetilde{V} \to \mathrm{Bool}$ by

$$\chi(v, b) := \varepsilon(v) \oplus b.$$

We claim χ is constant on every connected component of $\widetilde{G}$ whose base image is C. Indeed, for an edge $(u, b) \sim_{\widetilde{G}} (u', b')$ with $[u]_{\sim_G} = C$,

$$\chi(u, b) \oplus \chi(u', b') = \varepsilon(u) \oplus \varepsilon(u') \oplus b \oplus b' = \mathbf{1}_{\{u,u'\} \in W} \oplus \mathbf{1}_{\{u,u'\} \in W} = 0,$$

using the definition of $\sim_{\widetilde{G}}$ and of balance. So χ is a locally constant function on the fibre over C, hence constant on components. But

$$\chi(v_0, \perp) = \varepsilon(v_0) \oplus \perp \neq \varepsilon(v_0) \oplus \top = \chi(v_0, \top),$$

so $(v_0, \bot)$ and $(v_0, \top)$ lie in different components: $D_\bot \neq D_\top$. Together with Lemma 5.4, this forces $\pi^{-1}(C) = \{D_\bot, D_\top\}$, of cardinality 2. □

Step 3: the unbalanced case.

Lemma 5.6. *If C is not balanced, then $D_\bot = D_\top$, whence $\#\pi^{-1}(C) = 1$.*

Proof. We prove the contrapositive: if $D_\bot \neq D_\top$, then C is balanced. Define

$$\varepsilon_0(v) := \mathbf{1}_{(v,\bot)\in D_\bot} \qquad (v \in V),$$

read in Bool via $0 \mapsto \bot$, $1 \mapsto \top$. We show that ε_0 itself witnesses balance of C (Definition 5.1). Two facts are used, for every u with $[u] = C$. First, some sheet over u lies in $D_\bot$: lifting a walk $v_0 \to u$ in C edge by edge from $(v_0, \bot)$, as in the proof of Lemma 5.4, ends at (u, b) for some b. Second, not both sheets over u lie in $D_\bot$: the deck involution δ is a graph automorphism (Lemma 4.2), so it maps components to components, and $\delta(v_0, \bot) = (v_0, \top)$ gives $\delta(D_\bot) = D_\top$. If $(u, \bot)$ and $(u, \top)$ both lay in $D_\bot$, then $(u, \top) = \delta(u, \bot)$ would also lie in $D_\top$, so $D_\bot \cap D_\top \neq \emptyset$ and $D_\bot = D_\top$, contrary to assumption. Together,

$$(\dagger) \qquad (u, \top) \in D_\bot \iff (u, \bot) \notin D_\bot \qquad \text{whenever } [u] = C.$$

Now let $\{u, u'\} \in E$ with $[u] = C$. If $\{u, u'\} \notin W$, then $(u, \bot) \sim_{\widetilde{G}} (u', \bot)$, so $(u, \bot)$ and $(u', \bot)$ lie in the same component and $\varepsilon_0(u) = \varepsilon_0(u')$. If $\{u, u'\} \in W$, then $(u, \bot) \sim_{\widetilde{G}} (u', \top)$, so $(u, \bot) \in D_\bot \iff (u', \top) \in D_\bot \iff (u', \bot) \notin D_\bot$ by (†), that is, $\varepsilon_0(u) \neq \varepsilon_0(u')$. Hence $\{u, u'\} \in W \iff \varepsilon_0(u) \neq \varepsilon_0(u')$ on every edge of C, and C is balanced, contradicting the hypothesis. Therefore $D_\bot = D_\top$, and $\pi^{-1}(C) = \{D_\bot\}$ by Lemma 5.4.

The Lean proof follows the same route. `balanced_of_sheets_separated` (`L6_Cohomology.lean`) takes as witness the unique sheet $b(u)$ with $(u, b(u)) \in D_\bot$, which equals $\neg\varepsilon_0(u)$ and witnesses balance equally well; its uniqueness, `unique_sheet_above`, is the deck-involution argument above. `componentProj_fiber_card` combines it with Lemmas 5.4 and 5.5. □

Lemmas 5.4–5.6 together prove Theorem 5.3. An immediate corollary is:

Corollary 5.7. $\#\pi_0(\widetilde{G}) = \#\pi_0(G) + \beta(G)$.

Proof. Summing $\#\pi^{-1}(C)$ over $C \in \pi_0(G)$: each balanced component contributes 2, each unbalanced contributes 1. So $\#\pi_0(\widetilde{G}) = 2\beta(G) + (\#\pi_0(G) - \beta(G)) = \#\pi_0(G) + \beta(G)$. □

6. Recognition Theorems

We now reap the kernel-dimension formulas. Throughout, G is a finite simple connection graph.

Theorem 6.1 (Scalar kernel dimension, Mathlib)**.** $\dim_{\mathbb{R}} \ker L^{\mathrm{sc}}(G) = \#\pi_0(G)$.

This is Mathlib's `SimpleGraph.card_ConnectedComponent_eq_rank_ker_lapMatrix` applied to the ordinary graph Laplacian and used as a primitive.

Theorem 6.2. $\dim_{\mathbb{R}} \ker L^{\mathrm{sg}}(G) = \beta(G)$.

Proof. By Corollary 4.5,

$$\dim\ker \widetilde{L} = \dim\ker L^{\mathrm{sc}}(G) + \dim\ker L^{\mathrm{sg}}(G),$$

where $\widetilde{L} = L^{\mathrm{sc}}(\widetilde{G})$. Applying Theorem 6.1 to $\widetilde{G}$ and to G, and invoking Corollary 5.7,

$$\#\pi_0(G)+\beta(G) = \#\pi_0(\widetilde{G}) = \dim\ker \widetilde{L} = \dim\ker L^{\mathrm{sc}}(G)+\dim\ker L^{\mathrm{sg}}(G) = \#\pi_0(G)+\dim\ker L^{\mathrm{sg}}(G).$$

Subtracting $\#\pi_0(G)$ yields the claim. □

Theorem 6.3. *For $\epsilon \in \{\mathrm{flat}, \mathrm{Mob}\}$,*

$$\dim_{\mathbb{R}} \ker L_\epsilon = \begin{cases} 2\,\#\pi_0(G) & \epsilon = \mathrm{flat}, \\ \#\pi_0(G) + \beta(G) & \epsilon = \mathrm{Mob}. \end{cases}$$

Proof. By Theorem 3.1, L_ϵ is similar (through the reindex φ) to $\mathrm{diag}(L^{\mathrm{sc}}, B_\epsilon)$ with $B_{\mathrm{flat}} = L^{\mathrm{sc}}$, $B_{\mathrm{Mob}} = L^{\mathrm{sg}}$. So

$$\dim\ker L_\epsilon = \dim\ker L^{\mathrm{sc}} + \dim\ker B_\epsilon.$$

Theorems 6.1 and 6.2 give the two diagonal-block kernel dimensions, yielding the claim. □

Theorem 6.4. $\dim\ker L_{\mathrm{Mob}} < \dim\ker L_{\mathrm{flat}}$ *iff* $\beta(G) < \#\pi_0(G)$*, equivalently, iff* not every *component of* G *is balanced.*

Proof. By Theorem 6.3, the strict inequality reduces to $\#\pi_0(G) + \beta(G) < 2\#\pi_0(G)$, i.e. $\beta(G) < \#\pi_0(G)$. The inequality $\beta(G) \le \#\pi_0(G)$ is immediate from the inclusion $\pi_0^{\mathrm{bal}}(G) \subseteq \pi_0(G)$ of Definition 5.1. □

Remark 6.5 (Initial written plan vs. correct statement). Our initial written plan contained the assertion

$$(\star) \qquad \dim\ker L^{\mathrm{sg}}(G) \stackrel{?}{=} \#\pi_0(G) - \#\{C \in \pi_0(G) : |W \cap E(C)| \text{ is odd}\}.$$

This is *false* in general. Example: let $G = K_4$ with $W = \{\{1,2\},\{3,4\}\}$, two disjoint wrap edges forming a matching. The whole graph is one component, the total wrap count is 2 (even), so $(\star)$ predicts $\dim\ker L^{\mathrm{sg}} = 1$. The signed matrix has determinant 48, so its kernel is zero; eigenvalue computation agrees. Thus $\dim\ker L^{\mathrm{sg}} = 0$, which is indeed $\beta(K_4) = 0$ from Theorem 6.2 (the two wrap edges together form a non-trivial cohomology class on K_4). Equation $(\star)$ is correct on $G = C_n$: there $\dim H^1(C_n;\mathbb{Z}/2) = 1$, no edge is a bridge, and total wrap-count mod 2 detects the class. Beyond C_n, however, the correctness of $(\star)$ depends on W, not on the component alone. Precisely, $(\star)$ is correct on a component C iff

$$C \text{ is balanced (Definition 5.1)} \iff |W \cap E(C)| \text{ is even.}$$

This characterises agreement component by component. On a disconnected graph the global identity $(\star)$ can also hold by cancellation: it holds exactly when the number of balanced components with odd wrap count equals the number of unbalanced components with even wrap count. A useful *sufficient* condition — which is *not* necessary — is: $\dim H^1(C;\mathbb{Z}/2) \le 1$ (i.e., by the Euler relation $\dim H^1(C) = |E(C)| - |V(C)| + 1$ on a connected component, C is at most unicyclic) *and* C carries an even number of wrapped bridges. This condition is not necessary: on K_4 with a single wrapped triangle, $(\star)$ holds although $\dim H^1(K_4;\mathbb{Z}/2) = 3$; with $W = \emptyset$ it holds trivially on every graph. Failure takes exactly two forms: an *unbalanced* component with even total wrap (K_4 with a wrapped matching; two triangles sharing a vertex with one wrap in each), and a *balanced* component with odd total wrap (a tree component with an odd number of wrapped bridges, e.g. P_3 with one wrapped edge — where $\dim H^1 = 0$, so a small cohomology alone guarantees nothing). All six example computations above are reproduced by the independent implementation in `findings/round8`, written from the manuscript alone without consulting Lean sources. The counterexample and corrected formula are already present in the initial public commit (2026-04-22). That record establishes neither which tool first found the error nor a causal order between computation and formalisation. We have not identified a timestamped preregistration. Exact rational checks for both this matching and the different adjacent-edge example in the Lean comment are in `findings/round10/test_k4_counterexamples.py`.

7. The Cycle Spectrum

We close with an explicit diagonalisation in the case $G = C_n$, the n-cycle. Identify $V(C_n) = \mathbb{Z}/n$, with edges $\{i, i+1 \pmod n\}$. Let $\mathbf{e}_k \in \mathbb{C}^n$ be the Fourier basis vector $\mathbf{e}_k[j] = e^{2\pi ikj/n}$ for $k \in \mathbb{Z}/n$. Every circulant has this complex Fourier eigenbasis. For a *real symmetric* circulant, the eigenvalues are real, so the nonzero vectors $\mathrm{Re}(\mathbf{e}_k)$ and $\mathrm{Im}(\mathbf{e}_k)$ are real eigenvectors. A real eigenbasis takes one sine–cosine pair for each $1 \le k < n/2$, together with $\mathbf{e}_0$ and, for even n, $\mathbf{e}_{n/2}$. For a general real circulant, a conjugate pair of non-real eigenvalues instead gives a real invariant plane; its sine and cosine vectors need not be eigenvectors.

Theorem 7.1 (Flat cycle spectrum)**.** *Let $n \geq 3$. As a multiset indexed by $k \in \mathbb{Z}/n$ (repeated values counted with multiplicity), the scalar Laplacian $L^{\mathrm{sc}}(C_n)$ has spectrum*

$$\operatorname{Spec} L^{\mathrm{sc}}(C_n) = \big\{\, 2 - 2\cos(2\pi k/n) \;:\; k \in \mathbb{Z}/n \,\big\},$$

with $\mathbf{e}_k$ an eigenvector for the eigenvalue $2 - 2\cos(2\pi k/n)$.

Theorem 7.2 (Möbius cycle spectrum)**.** *Let $n \geq 3$ and $W = \{\{0,1\}\}$ (one wrap edge). Then, as a multiset indexed by $k \in \mathbb{Z}/n$, $L^{\mathrm{sg}}(C_n)$ has spectrum*

$$\operatorname{Spec} L^{\mathrm{sg}}(C_n) = \big\{\, 2 - 2\cos((2k+1)\pi/n) \;:\; k \in \mathbb{Z}/n \,\big\}.$$

Sketch. The scalar Laplacian is the circulant $2\mathbb{I} - (S + S^{-1})$ for the shift $S : v \mapsto v(\cdot + 1)$, so it commutes with the $\mathbb{Z}/n$-action and is diagonalised by the Fourier basis with shift weight $+1$, giving eigenvalues $2 - (e^{2\pi ik/n} + e^{-2\pi ik/n})$. The signed Laplacian with a single wrap edge is *not* circulant before gauging; it commutes with the twisted shift S_{sg} satisfying $S_{\mathrm{sg}}^n = -\mathbb{I}$. Relabel cyclically so that the wrap edge is the closing edge $\{n-1, 0\}$, which is the convention of the Lean matrix. Conjugation by $U = \operatorname{diag}(\alpha^j)_{0 \leq j < n}$ with $\alpha = e^{i\pi/n}$ multiplies the interior off-diagonal entries by $\alpha^{\pm 1}$ and the closing-edge entries by $\alpha^{\mp(n-1)} = -\alpha^{\pm 1}$; since $\alpha^n = -1$, this cancels the wrap sign, and $U^{-1} L^{\mathrm{sg}} U = 2\mathbb{I} - (\alpha S + \alpha^{-1} S^{-1})$ is circulant, with eigenvalues $2 - (e^{(2k+1)\pi i/n} + e^{-(2k+1)\pi i/n})$. (For the wrap edge $\{0,1\}$ itself, use the exponents $(j-1) \bmod n$.) The Lean proof in `CycleMultiset.lean` checks the complete periodic and anti-periodic complex Fourier eigen-equations on the explicit cycle matrices, including the signed closing-edge sign. The distinct Fourier parameters yield an invertible Vandermonde eigenbasis, hence similarity to a diagonal matrix and the full characteristic-polynomial products. Euler's identity and injectivity of the coefficient inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ descend the products to real cosines; the polynomial-root theorems then certify the stated eigenvalue multisets with all algebraic multiplicities. The signed specialised matrix places its wrap on the closing edge; the single-edge convention $W = \{\{0,1\}\}$ above is obtained by cyclic relabelling, rather than by a separately formalised `ConnGraph` matrix-equivalence theorem. □

Remark 7.3. The earlier `CycleSpectrum.lean` theorems `scalarCycle_eigenvalue` and `signedCycle_eigenvalue` produce one real eigenvector per frequency; these witnesses alone do *not* establish multiplicities because distinct frequencies can yield the same cosine vector. This gap is now closed in `CycleMultiset.lean`. The named theorems `scalarCycle_charpoly_real` and `signedCycle_charpoly_real` prove both entire cosine characteristic-polynomial products for every $n \geq 3$; the theorems `scalarCycle_roots_real` and `signedCycle_roots_real` establish the corresponding *multisets* of roots, including repeated eigenvalues. The completeness argument uses the nonzero Vandermonde determinant of the distinct *complex* Fourier parameters and is itself formalised as `charpoly_of_vandermonde_eigenbasis`.

Corollary 7.4. *Let $G = C_n$ with $n \geq 3$ and let $W = \{\{0,1\}\}$ consist of a single wrap edge. Then $\ker L^{\mathrm{sg}}(C_n) = 0$, equivalently $\beta(C_n) = 0$, and hence*

$$\dim \ker L_{\mathrm{Mob}}(C_n, W) = 1 \;<\; 2 = \dim \ker L_{\mathrm{flat}}(C_n)$$

for every $n \geq 3$.

Proof. By Theorem 7.2 the signed eigenvalues are $2 - 2\cos\big((2k+1)\pi/n\big)$ for $k \in \mathbb{Z}/n$. Such an eigenvalue can vanish only if $(2k+1)/n$ is an even integer, which is impossible since $2k+1$ is odd. Hence the signed kernel is zero. The single wrap edge gives odd holonomy around the fundamental cycle, so the unique component is unbalanced, confirming $\beta(C_n) = 0$. The kernel dimensions then follow from Theorem 6.3. □

Remark 7.5. The parity-in-n variant arises for the all-edges-wrap pattern $W = E(C_n)$: the total wrap parity around the fundamental cycle is then $n \bmod 2$, so C_n is balanced exactly when n is even.

8. The Lean Formalisation

The central kernel-dimension and decomposition development is formalised in Lean 4 against Mathlib v4.11.0. The flat and one-wrap signed cycle characteristic polynomials and real root multisets, for all $n \geq 3$, are formalised in `CycleMultiset.lean`, beyond the individual eigenpairs from `CycleSpectrum.lean`. The `CycleGraphTransfer.lean` theorem `mobius_connection_roots_of_cycle_reindex` extends the resulting union of scalar and signed spectral multisets to a general `ConnGraph` *provided* a vertex equivalence and *both* explicit scalar/signed matrix-reindexing equalities are supplied. It does not derive those equalities from a cycle-graph hypothesis: that separate graph-encoding proof remains open. The dependency graph is linear:

$$\texttt{Basic} \to \texttt{KernelDimension} \to \texttt{L5_Cover} \to \cdots \to \texttt{L17_TracesAndLipschitz},$$

with the `CycleSpectrum` file depending on `Basic` and `KernelDimension`, and `CycleMultiset` extending the cycle argument by Fourier-basis completeness (the full file list is tabulated in Appendix A).

Axiomatic footprint. The formal development's source audit excludes `sorry`, `admit`, `native_decide`, and project-local axiom declarations. A selected set of representative end theorems, including all nine stages of the cycle-spectrum proof, is checked by `#print axioms` against only the three standard Lean axioms `propext`, `Classical.choice`, and `Quot.sound`. The main paper-to-Lean correspondences are listed below; this is not the list of explicit `#print axioms` targets. The exact 25 selected targets are recorded in `ConnectionLaplacian/AxiomAudit.lean`; their axiom reports also check transitive proof dependencies. Inclusion in the correspondence list alone does not establish coverage by that selected audit.

- Theorem 3.1 is `ConnGraph.laplacian_decomposes`;
- Theorem 4.4 is `ConnGraph.scalarLap_cover_splits`;
- Theorem 5.3 is `ConnGraph.componentProj_fiber_card`;
- Theorem 6.2 is `ConnGraph.signedLaplacian_kernel_dim_general`;
- Theorem 6.3 is `ConnGraph.connectionLaplacian_kernel_dim_general`;
- Theorem 6.4 is `ConnGraph.mobius_kernel_strict_iff_general`;
- Theorems 7.1–7.2 are certified with *all algebraic multiplicities* by `Cycle.scalarCycle_roots_real` and `Cycle.signedCycle_roots_real`; see Remark 7.3.

Numerical cross-check. As an independent safety net against the formal statements being merely *provable* but not intended, the accompanying Python script `findings/round2/fuzzer/fuzz.py` enumerates every simple graph on $n \in \{2, 3, 4, 5\}$ vertices up to isomorphism (51 graphs, matching OEIS A000088) and every $2^{|E|}$ wrap subset, totalling 3,456 configurations. For each (G, W) it checks:

- Theorem 4.4 via spectrum equality of $\widetilde{L}$ and $L^{\mathrm{sc}}(G) \oplus L^{\mathrm{sg}}(G)$;
- Corollary 4.5 via SVD-based integer kernel-dimension arithmetic;
- Theorem 5.3 via brute-force $\mathbb{Z}/2$-colouring search and connected-component counting.

All 3,456 configurations agreed with the formal theorem statements; zero disagreements were found.

Summary of effort distribution. The "mathematical" content, in descending order of novelty:

(1) The fibre-cardinality theorem (Theorem 5.3) has no counterpart in Mathlib; it requires the walk-lift construction, XOR-parity invariant, and deck-action freeness argument, all of which are built from scratch against Mathlib's `SimpleGraph.ConnectedComponent` eliminators.

(2) The cover-splitting Theorem 4.4 is an entrywise matrix identity. Mathlib's Kronecker library (`Matrix.mul_kronecker_mul`, etc.) mechanises $\sim$15%; the remaining work is a four-case matching on wrap/non-wrap adjacency and a 2x2 block calculation.

(3) The block-decomposition Theorem 3.1 reuses the Kronecker pattern from (2) on the Fin 2 fibre instead of the Bool fibre.
(4) Recognition Theorems 6.2 and 6.3 are a few lines of arithmetic conclusion once (1)–(3) are in place.

Lessons. The development corrected the false equation ($\star$) in the initial written plan. Numerical counterexamples and the formal balanced-component theorem supply different checks of that correction. The public history already records both in its initial commit, so it does not determine the first detector or the order of discovery. The methodological lesson is to retain explicit counterexamples and verified replacement statements.

Acknowledgements and AI provenance. This work was carried out as an independent, AI-assisted formal-verification project on combinatorial spectral identities. The human author directed the research questions, mathematical scope, specification checks, and adversarial review. Language models, including OpenAI ChatGPT and Anthropic Claude, contributed substantially to exposition, Lean formalization, proof construction, testing, and review. The author does not claim to have independently reconstructed every formal proof. The source repository records the development, corrections, and evidence boundaries in `docs/methods-note.md` and `Hardening/EVIDENCE-MAP.md`. Lean kernel acceptance establishes the formalized statements under explicit assumptions, not their originality or their match to every informal interpretation.

9. Further Identities: Traces, Bounds, and the Characteristic Polynomial

The block decomposition (Theorem 3.1) and cover splitting (Theorem 4.4) imply a family of exact spectral identities that sit one inference-step away from the recognition theorems. We collect the ones that admit a one-paragraph proof and are stated at the level of generality on which the formalised library already operates. Together they close the "trace-and-inequality gap" left open in earlier versions of this manuscript. The scalar and signed trace formulas and the kernel inequalities are formalised in `ConnectionLaplacian/L9_Bounds.lean`; the flat and Möbius traces and the quadratic-trace and Frobenius identities in `ConnectionLaplacian/L17_TracesAndLipschitz.lean`; positive semidefiniteness in `ConnectionLaplacian/L13_PSD.lean`; and the multiplicative spectral partition (Theorem 9.6) in `ConnectionLaplacian/L10_CoverCharpoly.lean`.

Proposition 9.1 (Trace formulas)**.** *For every connection graph $G = (V, E, W)$,*

$$\begin{aligned} \operatorname{tr} L^{\mathrm{sc}}(G) &= 2\,\#E, & \operatorname{tr} L^{\mathrm{sg}}(G) &= 2\,\#E, \\ \operatorname{tr} L_{\mathrm{flat}}(G) &= 4\,\#E, & \operatorname{tr} L_{\mathrm{Mob}}(G) &= 4\,\#E. \end{aligned}$$

Proof. The diagonal entry of L^{sc} at v is $\deg(v)$; irreflexivity of the underlying simple graph kills the wrap off-diagonal contribution at the diagonal, so the diagonal of L^{sg} is also $\deg(v)$; and the diagonals of $L_{\mathrm{flat}}, L_{\mathrm{Mob}}$ are each $\deg(v)\cdot\mathbb{I}_2$, contributing $2\deg(v)$. Summing over v and applying the degree-sum identity $\sum_v \deg(v) = 2\,\#E$ (Mathlib's `SimpleGraph.sum_degrees_eq_twice_card_edges`) proves all four equalities. The Lean names are `trace_scalarLaplacian` and `trace_signedLaplacianMobius` (`L9_Bounds.lean`), and `trace_laplacian_flat` and `trace_laplacian_mobius` (`L17_TracesAndLipschitz.lean`). □

Proposition 9.2 (Quadratic trace and Frobenius identities)**.** *For every connection graph $G = (V, E, W)$,*

$$\begin{aligned} \operatorname{tr}\big(L_{\mathrm{Mob}}(G)^2\big) &= \operatorname{tr}\big(L^{\mathrm{sc}}(G)^2\big) + \operatorname{tr}\big(L^{\mathrm{sg}}(G)^2\big), \\ \|L_{\mathrm{Mob}}(G)\|_F^2 &= \|L^{\mathrm{sc}}(G)\|_F^2 + \|L^{\mathrm{sg}}(G)\|_F^2, \end{aligned}$$

and the cross term obeys the polarisation identity

$$2\,\operatorname{tr}\big(L^{\mathrm{sc}}(G)\cdot L^{\mathrm{sg}}(G)\big) = \operatorname{tr}\big((L^{\mathrm{sc}}(G)+L^{\mathrm{sg}}(G))^2\big) - \operatorname{tr}\big(L^{\mathrm{sc}}(G)^2\big) - \operatorname{tr}\big(L^{\mathrm{sg}}(G)^2\big).$$

Proof. Starting from the Kronecker-conjugation similarity of Theorem 3.1 in its Möbius form, the trace-of-square and Frobenius identities follow from invariance of $\operatorname{tr}(M^2)$ and of $\operatorname{tr}(M^\top M)$ under orthogonal similarity and simultaneous row and column reindexing. Here $P = \mathbb{I}_V \otimes_K \widehat{R}$ satisfies $P^2 = 2\mathbb{I}$ and $P^\top = P$, so $Q = P/\sqrt{2}$ is an orthogonal involution and $QLQ^{-1} = PLP^{-1}$. Alternatively, as in Lean, symmetry gives $\|L\|_F^2 = \operatorname{tr}(L^2)$ for each of the three Laplacians, reducing the Frobenius identity to the trace-of-square identity. The cross-term identity is pure algebra: expand $(L^{\mathrm{sc}} + L^{\mathrm{sg}})^2$ and use symmetry of the scalar and signed Laplacians together with the cyclic property $\operatorname{tr}(L^{\mathrm{sc}}L^{\mathrm{sg}}) = \operatorname{tr}(L^{\mathrm{sg}}L^{\mathrm{sc}})$. The Lean names are `trace_sq_laplacian_decomposes`, `frobenius_laplacian_decomposes` and `trace_mul_scalar_signed_eq`. □

Remark 9.3 (Polarisation versus combinatorial form). The cross-term identity above is stated in *polarisation form* and is pure matrix algebra. For these graph Laplacians the additional combinatorial identity $\operatorname{tr}(L^{\mathrm{sc}} \cdot L^{\mathrm{sg}}) = \operatorname{tr}\big((L^{\mathrm{sc}})^2\big) - 4\,\#W$, which can be read off the entrywise formula for $L^{\mathrm{sc}} \cdot L^{\mathrm{sg}}$ by summing contributions at diagonal and off-diagonal positions, is confirmed numerically by the R6 Stage-6 fuzzer (zero deviation across 300 random connected graphs at $n = 7$) but is not yet a separately named Lean theorem. Its proof needs the graph-entry calculation and finite edge counting; the polarisation theorem alone does not establish it. This remains an open formalisation obligation.

Proposition 9.4 (Kernel inequalities)**.** *For every connection graph G,*

$$\begin{aligned} \dim\ker L^{\mathrm{sg}}(G) &\le \dim\ker L^{\mathrm{sc}}(G), \\ \dim\ker L_{\mathrm{flat}}(G) - \dim\ker L_{\mathrm{Mob}}(G) &= \#\pi_0(G) - \beta(G), \end{aligned}$$

and in particular the difference is bounded by $\#\pi_0(G)$.

Proof. By Theorems 6.1, 6.2, 6.3, the LHS of the first line equals $\beta(G)$, the RHS equals $\#\pi_0(G)$, and $\beta(G) \le \#\pi_0(G)$ because $\pi_0^{\mathrm{bal}}(G) \subseteq \pi_0(G)$. For the second line, $\dim\ker L_{\mathrm{flat}} = 2\,\#\pi_0(G)$ and $\dim\ker L_{\mathrm{Mob}} = \#\pi_0(G) + \beta(G)$, whose difference is $\#\pi_0(G) - \beta(G)$, itself bounded by $\#\pi_0(G)$. □

Lemma 9.5 (Positive semidefiniteness of the signed Laplacian)**.** $L^{\mathrm{sg}}(G) \succeq 0$*. Indeed*

$$\langle x, L^{\mathrm{sg}}(G)\,x\rangle = \sum_{e=\{u,v\}\in E\setminus W} (x_u - x_v)^2 + \sum_{e=\{u,v\}\in W} (x_u + x_v)^2 \qquad \text{for all } x \in \mathbb{R}^V.$$

Proof. Fix $x \in \mathbb{R}^V$. The off-diagonal entry of L^{sg} at (u, v) equals $+1$ when $\{u, v\} \in W$ and -1 when $\{u, v\} \in E \setminus W$; the diagonal is $\deg(u)$. Since L^{sg} is symmetric, each unordered edge $\{u, v\}$ contributes the pair of entries (u, v) and (v, u) to the quadratic form, yielding a factor of 2:

$$\langle x, L^{\mathrm{sg}}x\rangle \;=\; \sum_u \deg(u)\,x_u^2 \;+\; 2\sum_{\{u,v\}\in W} x_u x_v \;-\; 2\sum_{\{u,v\}\in E\setminus W} x_u x_v.$$

Rewriting $\sum_u \deg(u)\,x_u^2 = \sum_{\{u,v\}\in E}\big(x_u^2 + x_v^2\big)$ (each edge contributes to the degree of both endpoints) and splitting the degree sum along $E = (E \setminus W) \sqcup W$:

$$\langle x, L^{\mathrm{sg}}x\rangle \;=\; \sum_{\{u,v\}\in E\setminus W} (x_u^2 + x_v^2 - 2x_u x_v) \;+\; \sum_{\{u,v\}\in W} (x_u^2 + x_v^2 + 2x_u x_v),$$

which is exactly the announced sum of squares. Every summand is ≥ 0, so $\langle x, L^{\mathrm{sg}}x\rangle \ge 0$. □

Theorem 9.6 (Multiplicative spectral partition: cover characteristic polynomial)**.** *Let $\widetilde{L} = L^{\mathrm{sc}}(\widetilde{G})$ be the scalar Laplacian of the orientation double cover of G. Then the characteristic polynomials factor as*

$$\det\big(\lambda\,\mathbb{I}_{\widetilde{V}} - \widetilde{L}\big) \;=\; \det\big(\lambda\,\mathbb{I}_V - L^{\mathrm{sc}}(G)\big)\cdot\det\big(\lambda\,\mathbb{I}_V - L^{\mathrm{sg}}(G)\big).$$

Equivalently, as multisets the spectra satisfy $\operatorname{Spec}(\widetilde{L}) = \operatorname{Spec}(L^{\mathrm{sc}}(G)) \uplus \operatorname{Spec}(L^{\mathrm{sg}}(G))$.

Remark 9.7 (Direct-bundle factorisation). Composing Theorem 9.6 with the similarity of Theorem 3.1 gives the same factorisation directly for L_{Mob} without invoking the cover:

$$\det\big(\lambda\,\mathbb{I}_{V\times\mathrm{Fin}\,2} - L_{\mathrm{Mob}}(G)\big) \;=\; \det\big(\lambda\,\mathbb{I}_V - L^{\mathrm{sc}}(G)\big)\cdot\det\big(\lambda\,\mathbb{I}_V - L^{\mathrm{sg}}(G)\big),$$

and for every $\alpha\in\mathbb{R}$, $\det(L_{\mathrm{Mob}}+\alpha\mathbb{I}_{V\times\mathrm{Fin}\,2}) = \det(L^{\mathrm{sc}}+\alpha\mathbb{I}_V)\cdot\det(L^{\mathrm{sg}}+\alpha\mathbb{I}_V)$. As multisets, $\mathrm{Spec}(L_{\mathrm{Mob}}) = \mathrm{Spec}(L^{\mathrm{sc}})\uplus\mathrm{Spec}(L^{\mathrm{sg}})$. These are Lean theorems `mobius_charpoly_eq_scalar_times_signed`, `shiftedDet_factorises` and `mobius_charpoly_roots_eq_union`.

Proof. By Theorem 4.4, $\widetilde{L}$ is similar (through an invertible Hadamard–Kronecker change of basis composed with the reindex $V\times\mathrm{Bool}\cong V\sqcup V$) to the block matrix $\mathrm{diag}(L^{\mathrm{sc}}(G), L^{\mathrm{sg}}(G))$. Similarity preserves the characteristic polynomial, and the characteristic polynomial of a block-diagonal matrix factors into the product of its diagonal blocks' characteristic polynomials. The multiset equality follows by equating roots with multiplicity. □

Remark 9.8 (Interlacing as a corollary). Theorem 9.6 strictly strengthens the Cauchy interlacing one might expect between the spectra of L^{sg} and $\widetilde{L}$: interlacing would merely bound each eigenvalue of L^{sg} between two eigenvalues of $\widetilde{L}$, whereas Theorem 9.6 pins down the spectrum of $\widetilde{L}$ exactly as the disjoint union of the two smaller spectra. In particular $\det\big(\lambda\mathbb{I} - L_{\mathrm{Mob}}(G)\big) = \det\big(\lambda\mathbb{I} - L_{\mathrm{flat}}(G)\big)$ whenever every component of G is balanced.

Corollary 9.9 (Pointwise multiplicity)**.** *For every $\lambda\in\mathbb{R}$,*

$$m_\lambda(\widetilde{L}) \;=\; m_\lambda(L^{\mathrm{sc}}(G)) + m_\lambda(L^{\mathrm{sg}}(G)),$$

where $m_\lambda(M)$ denotes the multiplicity of λ as an eigenvalue of M.

Proof. Immediate from Theorem 9.6: the multiplicity of λ in a product of polynomials is the sum of multiplicities in the factors. □

10. Structure Theorems for Trees, Forests, and Cycles

We record the "topological" side of the recognition theorems: on graphs where H^1 is understood, the content of Theorem 6.2 becomes explicit.

Theorem 10.1 (Tree balance)**.** *If the underlying simple graph of G is a tree (connected and acyclic), then its unique component is balanced for* every *choice of wrap set $W\subseteq E(G)$. Consequently*

$$\beta(G) = 1, \qquad \dim\ker L^{\mathrm{sg}}(G) = 1, \qquad \dim\ker L_{\mathrm{Mob}}(G) = \dim\ker L_{\mathrm{flat}}(G) = 2,$$

and the strict inequality of Theorem 6.4 fails.

Proof. Fix a root r. Because the tree is connected and acyclic, for every vertex v there is exactly one path $p_{r\to v}$: any walk $r\to v$ shortens to a path, and two paths with the same endpoints coincide (Mathlib's `SimpleGraph.IsAcyclic.path_unique`). Define $\varepsilon(v) := \sum_{e\in p_{r\to v}} \mathbf{1}_{e\in W} \pmod 2$, with $\varepsilon(r) = \bot$ by the empty sum. Let $\{u,v\}\in E$. If v does not lie on $p_{r\to u}$, then $p_{r\to u}$ followed by the edge $\{u,v\}$ is a path $r\to v$, so it is $p_{r\to v}$. Otherwise the part of $p_{r\to u}$ up to v is a path $r\to v$, so it is $p_{r\to v}$, and the remaining part is a path $v\to u$, so it is the single edge $\{v,u\}$. In both cases $\varepsilon(u)\oplus\varepsilon(v) = \mathbf{1}_{\{u,v\}\in W}$, so ε witnesses balance (Definition 5.1). With $\#\pi_0(G) = 1$, the kernel dimensions follow from Theorems 6.2 and 6.3.

The Lean proof (`L11_Trees.lean`) takes a different route: on an acyclic graph the wrap parity of a walk depends only on its endpoints (`walkWrapParity_acyclic_endpoints`), so the two sheets over a vertex lie in different components of the double cover (`candidateLift_ne_of_isTree`), and balance follows from `balanced_of_sheets_separated` of Section 5. □

Corollary 10.2. *Paths P_n and stars $K_{1,n}$ have* $\dim\ker L_{\mathrm{Mob}}(G) = 2$ *independently of W. If the underlying graph is a forest (acyclic, possibly disconnected), each component is a tree; running the proof of Theorem 10.1 inside each component (balance is component-local, Remark 5.2) gives $\beta(G) = \#\pi_0(G)$,* $\dim\ker L^{\mathrm{sg}}(G) = \#\pi_0(G)$ *and* $\dim\ker L_{\mathrm{Mob}}(G) =$

$2\,\#\pi_0(G)$. *The forest statement is not a separately named Lean theorem; only the connected* `IsTree` *case is formalised.*

Theorem 10.3 (Walk-sum detector for balance)**.** *Let $C \in \pi_0(G)$. Then C is balanced iff every closed walk $w : v \to v$ with $v \in C$ satisfies $\sum_{e \in w} \mathbf{1}_{e \in W} \equiv 0 \pmod 2$.*

Proof. ($\Rightarrow$) If ε witnesses balance then for each edge $e = \{u, u'\}$ in C, $\varepsilon(u) \oplus \varepsilon(u') = \mathbf{1}_{e \in W}$. Summing along a closed walk $v \to v$, the ε-terms telescope, leaving $\sum_e \mathbf{1}_{e \in W} \equiv 0$. ($\Leftarrow$) Pick $v_0 \in C$ and define $\varepsilon(v) := \sum_{e \in w} \mathbf{1}_{e \in W} \pmod 2$ for any walk $w : v_0 \to v$, and $\varepsilon(u) := \bot$ for $u \notin C$ (Definition 5.1 leaves those values unconstrained); the closed-walk-even hypothesis makes ε walk-independent. By construction $\varepsilon(u) \oplus \varepsilon(v) \equiv \mathbf{1}_{e \in W}$ on every edge $e = \{u, v\}$ of C, so ε witnesses balance. □

Corollary 10.4 (Cycle even-wrap formula)**.** *If $G = C_n$ is a single cycle of length $n \geq 3$, then*

$$\beta(C_n, W) = \begin{cases} 1 & \#W \equiv 0 \pmod 2, \\ 0 & \#W \equiv 1 \pmod 2, \end{cases} \qquad \dim\ker L^{\mathrm{sg}}(C_n, W) = \begin{cases} 1 & \#W \text{ even}, \\ 0 & \#W \text{ odd}. \end{cases}$$

In particular the initial plan's formula $\dim\ker L^{\mathrm{sg}} = \#\pi_0(G) - \#\{\text{odd-wrap components}\}$ *holds for* every wrap assignment *on a connected cycle graph. It is not a valid formula for arbitrary graphs. For a* particular *wrap assignment, however, it can also hold on non-cycle graphs, as discussed in Remark 6.5; thus this is not a pointwise characterisation of cycles. The Lean certificate for this corollary is the weak, conditional form described in Remark 10.5.*

Proof. The first homology of an n-cycle with $\mathbb{Z}/2$-coefficients is one-dimensional, $H_1(C_n; \mathbb{Z}/2) \cong \mathbb{Z}/2$, generated by the fundamental cycle γ. Every closed walk w at a basepoint decomposes, modulo back-tracking (which contributes edges twice and hence 0 to the $\mathbb{Z}/2$-sum of wrap indicators), into an integer multiple of γ; its wrap-parity therefore equals $[w : \gamma] \cdot \#W \pmod 2$. The walk-sum detector of Theorem 10.3 therefore reduces to the parity of $\#W$ and Theorem 6.2 converts $\beta \in \{0, 1\}$ to $\dim\ker L^{\mathrm{sg}} \in \{0, 1\}$. □

Remark 10.5. The Lean certificate `cycle_isBalanced_iff_even_wrap_weak` (`L14_CycleEw.lean`) names this a *weak* form: it takes an Eulerian closed walk and the reverse implication as explicit inputs. Its signature quantifies over a general `ConnGraph`; a cycle-graph identification is not constructed there. Only balancedness implying even total wrap is proved from the Eulerian walk. The reverse implication needs a cycle-space argument, together with an explicit `fundamentalCycleWalk`; both remain open.

11. Counter-claims: What the Formalisation Does *Not* Prove

The extended fuzzer (§8) and the `NEGATOR` pass of our discovery swarm refuted four naive statements that were, at various points, attempted drafts of the identities above. We record them here both for historical honesty and as explicit boundary markers for the theorems that *did* survive.

Example 11.1 (Naive equality of Möbius and scalar kernels is false)**.** A tempting but wrong reformulation of Theorem 6.4 is: "$\dim\ker L_{\mathrm{Mob}} = \dim\ker L^{\mathrm{sc}}$ iff every component of G is balanced." This fails already on the empty graph on $n = 3$ vertices with $W = \emptyset$: every component is trivially balanced (vacuously so, cf. Remark 5.2), yet $\dim\ker L_{\mathrm{Mob}} = 6$ (a kernel inside $\mathbb{R}^{V \times \mathrm{Fin}\,2}$) while $\dim\ker L^{\mathrm{sc}} = 3$ (a kernel inside $\mathbb{R}^V$): the two kernels sit in different ambient spaces, so the attempted identity already fails on dimensional grounds. The correct equivalence uses the *flat* Laplacian: $\dim\ker L_{\mathrm{Mob}} = \dim\ker L_{\mathrm{flat}} = 2\,\#\pi_0(G)$ iff every component is balanced, which is Theorem 6.3 combined with Theorem 6.1.

Example 11.2 (A single wrap edge can leave the graph balanced)**.** The implication "G connected and $\#W = 1 \Rightarrow G$ is not balanced" is false. Counterexample: the path P_3 with vertices v_0-v_1-v_2 and the single wrap edge $\{v_0, v_1\}$ (a bridge) is balanced: the assignment $\varepsilon(v_0) = \top,\ \varepsilon(v_1) = \bot,\ \varepsilon(v_2) = \bot$ witnesses balance, since $\varepsilon(v_0) \oplus \varepsilon(v_1) = 1 = \mathbf{1}_{\{v_0, v_1\} \in W}$ and

$\varepsilon(v_1) \oplus \varepsilon(v_2) = 0 = \mathbf{1}_{\{v_1,v_2\}\in W}$. The correct form is: a single wrap edge unbalances the graph iff the wrap edge lies on a cycle.

Example 11.3 (Adding a non-wrap edge between components can *decrease* β)**.** The fuzzy-logic sweep (discovery shorthand: claims are labelled `C1`, `C2`, ... in the order the swarm proposed them, and $\tau \in [0,1]$ is the empirical satisfaction rate of a claim over its tested configurations, so $\tau = 0$ refutes and $\tau = 1$ confirms; §8, claim `C2`, $\tau = 0.0000$) refutes the naive monotonicity "adding a non-wrap edge can only preserve or increase β". On three isolated vertices ($W = \emptyset$, so $\beta = 3$), adding the non-wrap edge $\{v_0, v_1\}$ merges two balanced components into one still-balanced component and drops β to 2. The correct law is bidirectional: for a non-wrap edge e joining two distinct components, $\beta(G + e) \in \{\beta(G) - 1, \beta(G)\}$, with -1 when *at least one* endpoint lies in a balanced component and 0 when both endpoints lie in unbalanced components. (In the mixed balanced-plus-unbalanced case, the merged component is unbalanced, so the pre-existing balanced component is lost and β drops by 1.)

The correct monotonicity law concerns *deleting* edges, and it is sensitive to bridges rather than to wrap status.

Theorem 11.4 (Bridge-deletion monotonicity)**.** *Deleting a non-bridge edge cannot decrease β: for any edge $e \in E(G)$ that is not a bridge of the underlying graph,*

$$\beta(G) \;\le\; \beta(G - e).$$

Proof (formalised). The Lean theorem `numBalancedComponents_monotone_remove_nonbridge` (`L15_BridgeMonotone.lean`) has exactly this statement. Deleting a non-bridge edge preserves reachability, so the components of G and of $G - e$ correspond bijectively. A balanced component of G stays balanced in $G - e$, because the same ε witnesses balance: every edge of $G - e$ is an edge of G. This injects the balanced components of G into those of $G - e$. The earlier declaration `numBalancedComponents_monotone_remove_nonwrap_nonbridge` carries an additional, unused non-wrap hypothesis inherited from the discovery sweep; it is kept unchanged because the archived comparator challenge replays it. □

Example 11.5 (Non-wrap edge contraction is not kernel-stable)**.** On K_3 with $W = \{\{v_0, v_1\}\}$ the signed Laplacian has kernel dimension 0. Contracting the non-wrap edge $\{v_0, v_2\}$ identifies v_0 and v_2, leaving the single wrap edge $\{v_0, v_1\}$, a tree on 2 vertices: the signed Laplacian now has kernel dimension 1. So non-wrap edge contraction does *not* preserve $\dim\ker L^{\mathrm{sg}}$ in general. Contraction is taken here in the category of simple graphs: the two parallel edges between the merged vertex and v_1 (one wrap, one non-wrap) collapse to a single edge, and the resulting two-vertex tree is balanced whichever wrap status the surviving edge carries. In a signed multigraph the two parallel edges would instead form an unbalanced 2-cycle, and β would stay 0.

Appendix A. Formalisation index

The principal named theorems, propositions, corollaries, and lemmas of the main text are paired below with corresponding Lean 4 declarations in the `connection_laplacian_lean` sandbox. (A handful of subsidiary lemmas used only in intermediate proofs—e.g. the deck-transformation Lemma 4.2, the cover-degree Lemma 4.3, and the fiber-subset/separation Lemmas 5.4, 5.5, 5.6—are implicitly witnessed inside the proofs of theorems listed below and are not tabulated separately.) The table below pairs them explicitly. The production source audit forbids `sorry`, `admit`, `native_decide` and local axiom declarations. The separate CI axiom audit selects 25 declarations (18 before the five conditional graph-transfer results; the two latest, added on 2026-10-18, are `numBalancedComponents_monotone_remove_nonbridge` and `trace_laplacian_flat`), requiring each requested output to contain no axioms outside {`propext`, `Classical.choice`, `Quot.sound`}. This selected audit does not print every helper or every table row. Comparator replay covers seven historical and five later cycle statements

in two separate runs; the evidence map records their exact scope. The file paths are relative to `connection_laplacian_lean/ConnectionLaplacian/`.

Paper statement	Lean declaration	Source file
Eq. (1), §2 (data)	`ConnGraph`, `laplacian`	`Basic.lean`
Thm. 6.1	`scalarLaplacian_kernel_dim`	`KernelDim.lean`
Thm. 3.1 (direct sum)	`laplacian_decomposes`	`KernelDim.lean`
Thm. 3.1 (ker dim)	`laplacian_kernel_dim_decomposes`	`L8_Recognition.lean`
Thm. 4.4	`scalarLap_cover_splits`	`L5_Cover.lean`
Def. 5.1	`isBalanced`	`L6_Cohomology.lean`
Thm. 5.3	`componentProj_fiber_card`	`L6_Cohomology.lean`
Thm. 6.2	`signedLaplacian_kernel_dim_general`	`L8_Recognition.lean`
Thm. 6.3	`connectionLaplacian_kernel_dim_general`	`L8_Recognition.lean`
Thm. 6.4	`mobius_kernel_strict_iff_general`	`L8_Recognition.lean`
Prop. 9.1 (sc)	`trace_scalarLaplacian`	`L9_Bounds.lean`
Prop. 9.1 (sg)	`trace_signedLaplacianMobius`	`L9_Bounds.lean`
Prop. 9.4 (mono)	`signed_kernel_le_scalar_kernel`	`L9_Bounds.lean`
Prop. 9.4 (drop $= \#\pi_0 - \beta$)	`kernel_drop_eq_unbalanced`	`L9_Bounds.lean`
Prop. 9.4 (drop $\le \#\pi_0$)	`kernel_drop_le_numComponents`	`L9_Bounds.lean`
Lem. 9.5	`signedLaplacianMobius_posSemidef`	`L13_PSD.lean`
Thm. 9.6	`cover_charpoly_eq_scalar_times_mobius`	`L10_CoverCharpoly.lean`
Cor. 9.9	*corollary of* `cover_charpoly_eq_scalar_times_mobius`	
Thm. 10.1 (balance)	`tree_isBalanced_of_isTree`	`L11_Trees.lean`
($\beta = 1$)	`numBalancedComponents_of_isTree`	`L11_Trees.lean`
(ker dim)	`connectionLaplacian_kernel_dim_tree_mobius`	`L11_Trees.lean`
(gap zero)	`tree_kernel_gap_zero`	`L11_Trees.lean`
Cor. 10.2	*paths and stars specialise the tree theorem; the forest case is not a named Lean theorem*	
Thm. 10.3	`isBalanced_iff_closedWalk_wrap_even`	`L12_WalkH1.lean`
Rem. 7.3 (per-mode, flat)	`scalarCycle_eigenvalue`	`CycleSpectrum.lean`
Rem. 7.3 (per-mode, signed)	`signedCycle_eigenvalue`	`CycleSpectrum.lean`
Thm. 7.1 (charpoly)	`scalarCycle_charpoly_real`	`CycleMultiset.lean`
Thm. 7.1 (root multiset)	`scalarCycle_roots_real`	`CycleMultiset.lean`
Thm. 7.2 (charpoly)	`signedCycle_charpoly_real`	`CycleMultiset.lean`
Thm. 7.2 (root multiset)	`signedCycle_roots_real`	`CycleMultiset.lean`
(graph-conditional transfer)	`mobius_connection_roots_of_cycle_reindex`	`CycleGraphTransfer.lean`
Cor. 10.4 (weak, conditional; Rem. 10.5)	`cycle_isBalanced_iff_even_wrap_weak`	`L14_CycleEw.lean`
Thm. 11.4	`numBalancedComponents_monotone_remove_nonbridge`	`L15_BridgeMonotone.lean`
(historical, unused non-wrap hypothesis)	`numBalancedComponents_monotone_remove_nonwrap_nonbridge`	`L15_BridgeMonotone.lean`
Rem. 9.7 (direct charpoly)	`mobius_charpoly_eq_scalar_times_signed`	`L16_SpectrumUnion.lean`
Rem. 9.7 (spec union)	`mobius_charpoly_roots_eq_union`	`L16_SpectrumUnion.lean`
Rem. 9.7 (shifted det)	`shiftedDet_factorises`	`L17_TracesAndLipschitz.lean`
Prop. 9.1 ($\operatorname{tr} L_{\mathrm{flat}}$)	`trace_laplacian_flat`	`L17_TracesAndLipschitz.lean`
Prop. 9.1 ($\operatorname{tr} L_{\mathrm{Mob}}$)	`trace_laplacian_mobius`	`L17_TracesAndLipschitz.lean`
Prop. 9.2 ($\operatorname{tr}(\cdot^2)$)	`trace_sq_laplacian_decomposes`	`L17_TracesAndLipschitz.lean`
Prop. 9.2 (Frobenius)	`frobenius_laplacian_decomposes`	`L17_TracesAndLipschitz.lean`
Prop. 9.2 (polarisation)	`trace_mul_scalar_signed_eq`	`L17_TracesAndLipschitz.lean`

(File paths: `KernelDim.lean` abbreviates `KernelDimension.lean`.)

Reproducibility. The complete Lean and Python sources live at `https://github.com/jesusvilela/connection_laplacian_lean` (concept DOI 10.5281/zenodo.19695396, which resolves to the latest archived release). Each archived release is a snapshot of the repository at its tag. Release v.0.0.5 (2026-10-15; version DOI 10.5281/zenodo.23401896) predates the full cycle-spectrum proofs and their comparator replay; `.github/release-notes/` lists what each later release adds. Release v.0.0.6 (18 October 2026; commit `58fe64b`) is the historical archived verification baseline. This manuscript also incorporates editorial and explanatory corrections made on the original repository's `main` branch through commit `ae64b5e` (2 November 2026). These subsequent corrections should not be mistaken for changes to the archived release's proof artifacts. The Lean 4.11.0 formalization remains pinned; the newer Mathlib port is separate research work. From the project root,

```
lake build
```

builds the pinned development (Mathlib v4.11.0 + `ConnectionLaplacian`). Invoking `#print axioms` on a listed name reports its axiom dependencies; it does not regenerate the statement-to-file correspondence table above. Continuous fuzz-sweeps now span five rounds of the discovery swarm. The R3/R4 baseline exhausts 176,169 connection-graph configurations ($n \leq 5$ exhaustive; $n \in \{6, 7, 8\}$ wrap-subset / random-graph); R5 extends this to a 456,950-configuration sweep at $n \leq 7$ targeting the cover-charpoly and kernel-dimension identities; R6 Stage-2 `FUZZER-A` (`findings/round6/stage2_fuzzer_A/`) certifies the M6 multiset-union identity at $1.5 \cdot 10^{-15}$ relative precision on 3,727 iso-configurations at $n = 6$ plus a random $n = 7$ sample; R6 Stage-6 `FUZZER-B` (`findings/round6/stage6_fuzzer_B/`) cross-checks the shifted-determinant, trace-square, Frobenius-norm and cross-polarisation identities of L17 at $n = 7$ (300 graphs, 14,011 probes, zero counterexamples). Across all rounds, the only non-integer numerical deviations are 20 out of 24,971 configurations at $n = 8$ that show a 10^{-3}-order relative error in evaluating charpoly($\widetilde{L}^{\mathrm{cov}}$) through the companion matrix (attributable to condition-number growth, bounded by standard float-polynomial perturbation theory; max deviation $7.3 \cdot 10^{-3}$). The underlying Lean identity `cover_charpoly_eq_scalar_times_mobius` is an exact statement over $\mathbb{Z}[\lambda]$ and is unaffected. Fuzzer scripts and JSON outputs live under `findings/round{3,4,5,6}/`.

Counter-claims are also machine-shadowed. Each counter-example in §11 (Examples 11.1, 11.2, 11.3, 11.5) is emitted by the `NEGATOR` / `NEGATOR-FUZZY` passes of the discovery swarm; the generating Python scripts live under `findings/round3/negator/` and `findings/round4/negator_fuzzy/` and can be re-run for any $n \leq 5$ (and $n = 6$ with a wrap-subset cap) to regenerate the counter-witnesses from scratch.

References

[1] F. Harary. *On the notion of balance of a signed graph.* Michigan Math. J. **2** (1953/54), 143–146.

[2] T. Zaslavsky. *Signed graphs.* Discrete Appl. Math. **4** (1982), 47–74.

[3] T. Zaslavsky. *Matrices in the Theory of Signed Simple Graphs.* arXiv:1303.3083, 2013, Section IV.D, Lemma IV.2.

[4] F. R. K. Chung. *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics **92**, American Mathematical Society, Providence, 1997.

[5] Y. Bilu and N. Linial. *Lifts, discrepancy and nearly optimal spectral gap.* Combinatorica **26** (2006), 495–519. Preprint: *Constructing expander graphs by 2-lifts and discrepancy vs. spectral gap*, arXiv:math/0312022.

[6] F. Diaz-Diaz, K. Devriendt and R. Lambiotte. *Gremban expansion for signed networks: algebraic and combinatorial foundations for community–faction detection.* arXiv:2509.14193, 2025.

[7] K. D. Gremban. *Combinatorial Preconditioners for Sparse, Symmetric, Diagonally Dominant Linear Systems.* PhD thesis, Carnegie Mellon University, 1996. Technical report CMU-CS-96-123.

[8] D. A. Spielman and S.-H. Teng. *Nearly-linear time algorithms for preconditioning and solving symmetric, diagonally dominant linear systems.* arXiv:cs/0607105.

[9] The Mathlib Community. *The Lean Mathematical Library.* In: *CPP 2020 Proceedings*, 367–381, 2020.

[10] L. de Moura and S. Ullrich. *The Lean 4 theorem prover and programming language.* In: *CADE 28 Proceedings*, LNCS 12699, 625–635, 2021.