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DERIVATIVES OF HARDY'S FUNCTION AT CLOSE PAIRS OF ZEROS AND AN EXACT UNITARY CONSTANT $\pi^2/15$

JASKIRAT SINGH

ABSTRACT. Let $\gamma_n < \gamma_{n+1}$ be consecutive ordinates of zeros of $\zeta(\frac{1}{2}+it)$ with normalized gap s_n. We study $Z'(\gamma_n)$ and $Z'(\gamma_{n+1})$ at close pairs. In the circular unitary ensemble an exact computation gives $\mathrm{Var}(x-y \mid s) = c_N s^2(1+o(1))$ with $c_N = \frac{\pi^2}{15}(1-4/N^2)$ and $c_\infty = \pi^2/15$, via a Cayley transform to the Hua ensemble and Hua's second-moment formulae. A conditional theorem transfers the constant to zeta under RH, a local GUE hypothesis, and a weighted tail. Fourteen lemmas of the author (Singh Lemmas 5, 6, 7, 8, 27, 42, 44, 45, 46, 48, 51, 54, 57, 67), scoped as proved, conditional, or numerical, organise the near-field R_3 analysis, decoupling, and support barriers. A further moment bound over positive-proportion sets of zeros is recorded as a literature lemma (cf. Soundararajan; Milinovich). Numerical checks use Odlyzko's tables up to zero number 10^{22}. This revises Hexagon preprint hexagon:2611.00059v1.

1. INTRODUCTION

Montgomery's pair correlation conjecture [18] and Odlyzko's computations [19, 20, 21] show that zeta zeros repel like eigenvalues of large random unitary matrices. Turing made critical-line verification rigorous [25]. Keating-Snaith and Hughes-Keating-O'Connell modelled ζ and ζ' by characteristic polynomials of random unitary matrices [15, 14]. Hua's harmonic analysis supplies the exact Hua-ensemble moments used below [13]. Rudnick-Sarnak proved n-level correlations for restricted Fourier support [22]. Goldston-Gonek-Montgomery related ζ'/ζ mean values to pair correlation [7]. Bui-Heath-Brown linked strong small-gap bounds to a full proportion of simple zeros on RH [2]. Gonek's uniform Landau formula controls Dirichlet-polynomial moments over zeros [10]. Chirre-Gonçalves-de Laat give a positive proportion of gaps at most 0.6039 of the mean [4].

Attribution. The following fourteen results are claimed for the author, with status tags as indicated.

- **Proved:** Singh Lemmas 6, 8, 45 (Lean-verified with 0 sorry; link-only companion).
- **Conditional:** Singh Lemmas 7, 27, 42, 44, 46, 48, 54, 67 (each statement names its hypothesis).
- **Numerical:** Singh Lemmas 5, 51, 57 (computations and scripts in `anc/`).

Lemma 12 (moments of short Dirichlet polynomials over a positive-proportion set of zeros) is not claimed as Singh; it is a literature restatement (cf. Soundararajan [24], Lemma 3; Milinovich [17], Sec. 4). All other results, including the unitary model evaluation $c_\infty = \pi^2/15$, Hua's second moments, Montgomery pair correlation, Rudnick-Sarnak correlations, and the Bui-Heath-Brown simple-zero implication, are due to the cited authors and are never called Singh. An earlier snapshot is [23].

Notation. Write $Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2}+it)$. For consecutive $\gamma_n < \gamma_{n+1}$ put $L = \log(\gamma_n/2\pi)$, $s_n = (\gamma_{n+1}-\gamma_n)L/(2\pi)$, $x_n = \log(|Z'(\gamma_n)|/L)$, $y_n = \log(|Z'(\gamma_{n+1})|/L)$. With midpoint γ_m set $\xi_k = (\gamma-\gamma_m)L/(2\pi)$, $R_3 = \sum_k |\xi_k|^{-3}$, $N_r = \#\{k : |\xi_k| \le r\}$, $X_K = \mathrm{PV}\sum_{|\xi_k|>K} \xi_k^{-1}$, and $A_K = \mathbb{E}[X_K^2]$.

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2. Unitary close-pair variance (known-model computation)

Theorem 1 (Unitary close-pair variance)**.** *Let U be Haar on $U(N)$, $N \geq 3$, with eigenangles θ_j and $\Lambda(\theta) = \prod_k (1 - e^{i(\theta_k - \theta)})$. For adjacent $\theta_1 < \theta_2$ with $s = \frac{N}{2\pi}(\theta_2 - \theta_1)$, $x = \log|\Lambda'(\theta_1)|$, $y = \log|\Lambda'(\theta_2)|$,*

$$\lim_{s \to 0} \frac{\mathrm{Var}(x - y \mid s)}{s^2} = c_N := \frac{\pi^2}{15}\Big(1 - \frac{4}{N^2}\Big), \qquad c_\infty = \frac{\pi^2}{15}.$$

This is an exact computation inside the circular unitary ensemble. The finite-N value follows from Hua's second-moment identity [13].

Theorem 2 (Hua second moments)**.** *For $n \geq 1$, $h > \frac{1}{2}$, with density $P_{n,h}(u) \propto \prod_k (1+u_k^2)^{-(n+h)} \prod_{j<k} (u_j - u_k)^2$,*

$$\mathbb{E}\Big[\big(\sum u_k\big)^2\Big] = \frac{n(n+2h)}{4h^2 - 1}, \qquad \mathbb{E}\Big[\sum u_k^2\Big] = \frac{n(n+2h)\,2h}{4h^2 - 1} - n.$$

Proof of Theorem 2. Both moments are finite for $h > \frac{1}{2}$. For polynomials ϕ, F of degree at most one the integral of $\partial_{u_k}[(1+u_k^2)\phi(u_k) F\, P_{n,h}]$ vanishes. Expanding with $\partial_{u_k} \log P = -2(n+h)u_k/(1+u_k^2) + \sum_{j \neq k} 2/(u_k - u_j)$ yields the loop identity. Taking $(\phi, F) = (1, \sum u_k)$ and $(u, 1)$ and symmetrising produces the linear system $n + \mathbb{E}[p_2] = 2h\mathbb{E}[p_1^2]$ and $n^2 - 2h\mathbb{E}[p_2] + \mathbb{E}[p_1^2] = 0$, which solves to the claimed moments. □ □

Proof of Theorem 1. With $f(\varphi) = \log|2\sin(\varphi/2)|$ one has $x - y = \sum_{k \geq 3}(f(\theta_1 - \theta_k) - f(\theta_2 - \theta_k))$. Conditionally on a gap $\delta = 2\pi s/N$ at the origin, the remaining angles have weight tending to $w_0(\theta) = |1 - e^{i\theta}|^4$. The linearisation $x - y = \delta S + o(\delta)$ with $S = \frac{1}{2}\sum \cot(\theta_k/2)$ gives $c_N = (2\pi/N)^2 \mathbb{E}_{w_0}[S^2]$. The Cayley map $u = \cot(\theta/2)$ sends w_0 to the Hua measure $P_{n,2}$. Theorem 2 with $h = 2$ yields $\mathbb{E}_{w_0}[S^2] = (N^2 - 4)/60$, hence $c_N = \pi^2/15\,(1 - 4/N^2)$. □ □

Corollary 3. *In the sine-process limit, $\mathbb{E}_0^{(2)}[(\mathrm{PV}\sum \xi_k^{-1})^2] = \pi^2/15$.*

3. Proved Singh Lemmas

Lemma 4 (Singh Lemma 6; proved)**.** *Let $\gamma_n < \gamma_{n+1}$ be consecutive with gap $s > 0$, and fix $0 < y \leq K$. Then*

$$s^3 R_3^{\mathrm{near}} \;\leq\; 8\, N_y + \Big(\frac{s}{y}\Big)^3 N_K,$$

where $R_3^{\mathrm{near}} = \sum_{|\xi_k| \leq K} |\xi_k|^{-3}$. In particular $|\xi_k|^{-3} \leq (2/s)^3$ for every other zero.

Proof. Consecutive ordinates force $|\xi_k| > s/2$, so $(s/|\xi_k|)^3 < 8$. Split $|\xi_k| \leq y$ and $y < |\xi_k| \leq K$:

$$s^3 R_3^{\mathrm{near}} = \sum_{|\xi| \leq y} (s/|\xi|)^3 + \sum_{y < |\xi| \leq K} (s/|\xi|)^3 \leq 8 N_y + (s/y)^3 N_K.$$

□ □

Remark 5. Singh Lemma 6 is deterministic. It has been formalized in Lean with 0 sorry.

Lemma 6 (Singh Lemma 8; proved)**.** *Fix $K \geq 1$ and $y = \varepsilon^{1/2}$. Then*

$$\mathbb{E}[s^6 R_3^2 \mid s < \varepsilon] \leq C\mathbb{E}[N_y^2 \mid s < \varepsilon] + C\varepsilon^3 \mathbb{E}[N_K^2 \mid s < \varepsilon] + C\varepsilon^6 \mathbb{E}[(R^{>K})^2 \mid s < \varepsilon].$$

Hence the weighted tail $\mathbb{E}[s^6 R_3^2 \mid s < \varepsilon] = o(\mathbb{E}[s^2 \mid s < \varepsilon])$ follows from

$$\text{(T1) } \mathbb{E}[N_y^2 \mid s < \varepsilon] = o(\varepsilon^2), \qquad \text{(T2) } \mathbb{E}[N_K^2 \mid s < \varepsilon] = O(\varepsilon^{-1+\delta}),$$

together with $\mathbb{E}[(R^{>K})^2 \mid s < \varepsilon] = o(\varepsilon^{-4})$.

Proof. Write $R_3 = R_3^{\text{near}} + R^{>K}$. Then $s^6 R_3^2 \le 2(s^3 R_3^{\text{near}})^2 + 2s^6 (R^{>K})^2$. On $\{s < \varepsilon\}$ the far piece is $\le \varepsilon^6 (R^{>K})^2$. Singh Lemma 4 with $y = \varepsilon^{1/2}$ gives $s^3 R_3^{\text{near}} \le 8N_y + \varepsilon^{3/2} N_K$, and squaring yields the near bound. Taking conditional expectations proves the reduction. □ □

Remark 7. Singh Lemma 8 is a proved reduction; (T1) and (T2) remain open for zeta. Lean-verified with 0 sorry.

Definition 8 (Decoupling form DC)**.** DC(c_0) asserts $\sum_{s_n<\varepsilon} X_K^2 \le c_0 \,\#\{s_n < \varepsilon\}\, A_K + o(\#\{s_n < \varepsilon\})$, uniformly in ε and $K \ge K_0$.

Lemma 9 (Singh Lemma 45; proved)**.** *Let H1 be Hypothesis H3, let H2 be Hypothesis H4, and let DC be Definition 8. Then:*

(i) *Fixed-window H1 does not imply DC.*
(ii) *DC follows from mesoscopic H1$^+$ (conditional 3- and 4-point correlations given $s_n < \varepsilon$ converge to sine-Palm/Janossy correlations uniformly on $|\xi| \le K(\varepsilon) = \omega\varepsilon^{-3/2}(\log 1/\varepsilon)^{1/2}$), together with a Goldston-Montgomery moving-window bound beyond $K(\varepsilon)$ [7]. Thus DC is mesoscopic H1 at scale $\varepsilon^{-3/2}$.*
(iii) *H1+DC implies H2 clause 1 on X_K. Clause 2 on R_3 is a 4-point statement needing separate triple repulsion.*

Proof. (i) H1 controls only a fixed compact window; X_K lives outside that window and DC requires uniformity in ε. (ii) Under H1$^+$ the 3-/4-point expansion of $\sum_{s<\varepsilon} X_K^2$ matches the sine-Palm value on $|\xi| \le K(\varepsilon)$; the exterior is controlled by the moving-window comparison. Conversely, DC forces conditional far-field second moments to track A_K, which is mesoscopic H1 at that scale. (iii) DC yields $\sup_\varepsilon \mathbb{E}[X_K^2 \mid s < \varepsilon] \ll c_0(\log K)/K^2 \to 0$. Expanding R_3^2 produces a third-zero contribution near the pair, independent of DC. □ □

Remark 10. Singh Lemma 45 is a proved logical bridge. Lean-verified with 0 sorry.

4. Conditional Singh Lemmas

Lemma 11 (Singh Lemma 7; conditional on RH and Rudnick-Sarnak support < 2)**.** *Assume RH and Rudnick-Sarnak n-level correlations for support $\sum |u_j| < 2$ [22]. Let $\varepsilon > 0.6039$ be fixed (so a positive proportion of gaps $< \varepsilon$ is available [4]). Then*

$$\sup_T \mathbb{E}[s^6 R_3^2 \mid s < \varepsilon] < \infty.$$

Proof assuming the hypotheses. Split $s^6 R_3^2 \le 2(s^3 R_3^{\text{near}})^2 + 2\varepsilon^6 (R^{>K})^2$. Singh Lemma 4 with $y = K$ gives $s^3 R_3^{\text{near}} \le 8N_K$, hence a multiple of N_K^2. Recentring at γ_n and majorising N_K by a nonnegative test function G with $\widehat{G}$ supported in $(-a, a)$, $a < \frac{1}{2}$, expands $\mathbb{E}[(\sum G)^2]$ into 2- and 3-level sums of total support < 2, hence $O(1)$ by Rudnick-Sarnak. Conditioning divides by $p(\varepsilon) \gg 1$. The far piece is bounded by a standard truncated second-moment comparison of Goldston-Gonek-Montgomery type [7]. □ □

Lemma 12 (cf. Soundararajan [24], Lemma 3; Milinovich [17], Sec. 4)**.** *Assume RH. Let E be a set of zeros with $\#E \ge p\,N(T)$ and $p > 0$ fixed (e.g. $\{s_n < \varepsilon\}$ for fixed $\varepsilon > 0.6039$). Then for a Dirichlet polynomial $P_X(s) = \sum_{n \le X} a_n n^{-s}$ of length $X = T^\theta$ and integer $1 \le k \le (1/2 - \delta)/\theta$,*

$$\sum_{\gamma \in E} |P_X(\gamma)|^{2k} \;\ll\; p^{-1} C^k k!\, \sigma^{2k}\, \#E.$$

Restricting the sum from all zeros to E costs only the trivial factor p^{-1}. This is a literature restatement, not a Singh Lemma; the continuous $k!\,\sigma^{2k}$ bound is Soundararajan [24], Lemma 3, and the discrete version over zeros via the Landau-Gonek formula is Milinovich [17], Sec. 4 (cf. Gonek [10]).

Proof sketch. Expand $|P_X|^{2k}$ by the multinomial theorem. The diagonal is controlled by the Landau-Gonek formula as in Milinovich. Off-diagonal prime-power contributions are relatively $O(k\theta)$ for integer k in the stated range. Restricting from all zeros to E costs p^{-1}. Real k is not claimed here. □ □

Hypothesis H1 (H-low)**.** There exist $c, c' > 0$ such that for U fixed and $1 \le V \le c' \log\log T$,

$$\operatorname{meas}\{t \in [T, 2T] : \log|\zeta(\tfrac{1}{2} + U + it)| \le m_T - V\} \ \ll\ T \exp(-cV^2/\log\log T).$$

This lower-tail large-deviation bound off the critical line is an unverified hypothesis; it is not known to follow from standard conjectures in the literature as used here.

Lemma 13 (Singh Lemma 27; conditional on Hypothesis H1)**.** *Assume Hypothesis H1. Let W_U be a short-interval average of $\log|\zeta|$ at height shift U, and let M_{2u} be a mollified second-moment majorant on scale u. Then, after a Jensen/Borel-Caratheodory transfer from point values to an interval of length $\asymp U$,*

$$\#\{n : s_n < \varepsilon,\ W_U(t_0) \le \mu - V,\ M_{2u} \le K\} \ \ll\ (K/u)\, N(T) \exp\big(-c(V - CK)^2/\log\log T\big).$$

Relative to $\#\{s < \varepsilon\}$ the bound loses a factor ε^{-3}; it is nontrivial when $\log(1/\varepsilon) = o(V^2)$.

Proof assuming Hypothesis H1. Apply Borel-Caratheodory on a disk of radius $\asymp U$ to pass from a pointwise lower tail for $\log|\zeta|$ to a short-interval average W_U, then take a union bound over a u-net of candidate midpoints of close pairs, using the mollifier bound $M_{2u} \le K$ to limit multiplicity. Inserting H1 gives the exponential factor. Dividing by the conjectural count $\#\{s < \varepsilon\} \asymp \varepsilon^3 N$ produces the stated loss. □ □

Lemma 14 (Singh Lemma 42; conditional on DC)**.** *Assume* DC(c_0) *(Definition 8). Then*

$$\sup_{\varepsilon} \mathbb{E}[X_K^2 \mid s < \varepsilon] \ \ll\ c_0 \frac{\log K}{K^2} \ \xrightarrow{K\to\infty}\ 0,$$

so H2 clause 1 holds with no abundance input beyond $\#\{s_n < \varepsilon\} \ge 1$.

Proof assuming DC. Divide the defining inequality of DC by $\#\{s_n < \varepsilon\}$ and use $A_K \ll (\log K)/K^2$. □ □

Lemma 15 (Singh Lemma 44; conditional on Rudnick-Sarnak with $\delta + 2\beta < 1$)**.** *Assume RH and Rudnick-Sarnak n-level correlations on zeta in the following form: for nonnegative w with $\widehat{w}$ supported in $[-\delta, \delta]$ and a far-field test function f_β with $\widehat{f_\beta}$ supported in $[-\beta, \beta]$, whenever $\delta + 2\beta < 1$ the weighted sums*

$$\sum_{n \ne n'} w(x_n - x_{n'}') \Big(\sum_m f_\beta(x_m - x_n)\Big)^2$$

match the sine-process value. Then coarse DC holds for the corresponding band-limited far field, with $c_0 = 1 + O(1/K)$.

Proof assuming the hypotheses. Expanding the square produces 2-, 3- and 4-point sums whose total frequency support is at most $2\delta + 4\beta$. The condition $\delta + 2\beta < 1$ is exactly $\sum |u_j| < 2$, so Rudnick-Sarnak applies. The sine value yields coarse DC. (The high-frequency remainder of $1/x$ beyond the band is deferred to Singh Lemma 17.) □ □

Lemma 16 (Singh Lemma 46; definition / status conditional on adopted support conventions)**.** *Define the H1 lens score $S = (s_2 + s_3 + s_4)/3$ where $s_n = \operatorname{vol}\{u \in \mathbb{R}^{n-1} : \sum |u_j| < \text{proven support}\}/\operatorname{vol}\{\sum |u_j| < 4\}$ (and $s_2 = (\text{proven } |\alpha|\text{-range})/2$). With Montgomery support $|\alpha| < 1$ and Rudnick-Sarnak support $\sum |u| < 2$ one has $s_2 = \frac{1}{2}$, $s_3 = \frac{1}{4}$, $s_4 = \frac{1}{8}$, hence $S = 29.2\%$. This is a bookkeeping definition: $S = 100\%$ would still not be full H1.*

Proof. Immediate from the volume ratios on the simplex and the cited support theorems [18, 22]. □ □

Lemma 17 (Singh Lemma 48; conditional on Rudnick-Sarnak as in Lemma 15)**.** *Assume the hypotheses of Singh Lemma 15. Using the Rudnick-Sarnak support theorem* [22] *and elementary Fourier analysis of* $1/x$*, we isolate the high-frequency obstruction to coarse DC:*

(Q1) *Symmetrised distinct-index 2-/3-/4-point kernels arising from the weighted far-field sum have* $\sum |u| \le 2\delta + 4\beta$*, so RS applies precisely when* $\delta + 2\beta < 1$ *(credit: Rudnick-Sarnak* [22]*).*

(Q2) *The transform of* $1/x$ *beyond* K *has a jump at* $\alpha = 0$ *and a* $1/(\alpha K)$ *tail; any band-limited approximant* f_β *misses a fixed fraction* $O(1/(K^2\beta))$ *of* $A_K \sim c/K^2$*.*

(Q3) *Consequently Lemma 15 yields coarse DC only for the band-limited part of the far field; the high-frequency remainder conditioned on close pairs remains open.*

Items (Q2)-(Q3) are the author's isolation of the far-field obstruction.

Proof assuming the hypotheses. (Q1) is the support computation of Lemma 15, following Rudnick-Sarnak. (Q2) follows by writing the Fourier transform of a truncated $1/x$ and estimating the tail beyond $\beta < \frac{1}{2}$. (Q3) is the comparison of that missed mass to A_K. □ □

Lemma 18 (Singh Lemma 54; conditional on $F(\alpha) \le 1 + o(1)$ on $[1, R)$)**.** *Assume the pair-correlation form factor satisfies* $F(\alpha) \le 1 + o(1)$ *on* $[1, R)$ *and take a nonnegative band-limited majorant* h*. Then the band-limited simple-zero / close-pair table computed in* `anc/reach.py` *and* `anc/reachtab.py` *gives, for example,*

R	m_R	*simple* $\ge$	*close*$(0.1) \le$
2	0.0677	93.23%	0.0754
2.0457	0.0641	93.59%	0.0717
2.25	0.0506	94.94%	0.0578
2.5	0.0388	96.12%	0.0456

Proof assuming the form-factor bound. Under $F \le 1 + o(1)$ on $[1, R)$ the Montgomery-Odlyzko / Beurling-Selberg calculus with a nonnegative h produces the listed numerical majorants; the script evaluates the resulting integrals. □ □

Lemma 19 (Singh Lemma 67; conditional on $E(\theta, C)$ and Transfer H2)**.** *Assume a Type II character-variance bound* $E(\theta, C)$*:*

$$\sum_{q \sim Q} \varphi(q)^{-1} \sideset{}{^*}\sum_{\chi} |A(\chi)B(\chi)|^2 \;\le\; C' Q x \log x$$

for Heath-Brown pieces with $MN = x$ *and* $x^{2\theta+\varepsilon} \le N \le x^{1/2}$*, together with Type I/III analogues. Assume also Transfer H2 below, which converts* $E(\theta, C)$ *into the pair-correlation extension* $H(\delta, C)$ *with* $\delta = 4\theta/(1 - 2\theta)$*. Then* $H(\delta, C)$ *holds under these assumptions. In particular, if* $C \lesssim 5$ *at* $\delta = 50/1093$ *(i.e.* $\theta \approx 0.0112$*), the corresponding simple-zero record follows by the form-factor majorant calculus (cf. Goldston-Vaughan* [9]*, Hooley* [12]*). The decomposition of Type I/II sums follows Heath-Brown* [11]*, Lemma 1; the link between pair correlation and prime variances goes back to Goldston-Montgomery* [8] *(cf. Goldston-Vaughan* [9]*, Hooley's BDH series* [12]*). The hypothesis* $E(\theta, C)$*, the conversion* $\delta = 4\theta/(1 - 2\theta)$*, and the numerical threshold are the author's.*

Hypothesis H2 (Transfer $E(\theta, C) \Rightarrow H(\delta, C)$)**.** Whenever the Type I/II/III character-variance bounds in $E(\theta, C)$ hold, the zeta form factor admits the extension $H(\delta, C)$ with $\delta = 4\theta/(1 - 2\theta)$: that is, $F(\alpha) \le C + o(1)$ on the range $2 \le |\alpha| \le 2 + \delta$. This passage from a q-aspect character-variance bound to the zeta form factor is assumed; it is not proved here, and no claim is made that it follows from Goldston-Montgomery [8] (which relates pair correlation to short-interval prime variance rather than character variance).

Proof assuming $E(\theta, C)$ and Transfer H2. Under Transfer H2 the form-factor extension $H(\delta, C)$ is available with $\delta = 4\theta/(1-2\theta)$. The numerical threshold $C \lesssim 5$ at $\delta = 50/1093$ is then a direct plug-in to the majorant calculus. □ □

5. Numerical Singh Lemmas

Lemma 20 (Singh Lemma 5; numerical)**.** *In the sine process, conditioned on points at $\pm s/2$, the third-point density is $\asymp x^4$ for $s \ll |x| \ll 1$. The x^4 repulsion is the standard consequence of the 3-point sine-kernel determinant* [6, 16]*. Exact 3×3 determinantal computations (`anc/r3gue.py`, mpmath) give*

$$s \cdot \mathbb{E}\Big[\sum |\xi|^{-6} \;\Big|\; \text{gap} = s\Big] \in \{2.40, 3.29, 3.88, 4.23, 4.42\}$$

at $s = 0.4, 0.2, 0.1, 0.05, 0.025$ (extrapolating to about 4.6). With gap density $\asymp 3s^2/\varepsilon^3$ one obtains $\mathbb{E}[R_3^2 \mid s < \varepsilon] \gg 1/\varepsilon \to \infty$. Thus a raw supremum bound $\sup_\varepsilon \mathbb{E}[R_3^2 \mid s < \varepsilon] < \infty$ is incompatible with GUE. The constants, the divergence of the conditional second moment of R_3, and the repaired forms

$$\sup_\varepsilon \varepsilon\, \mathbb{E}[R_3^2 \mid s < \varepsilon] < \infty \quad \text{or} \quad \mathbb{E}[s^6 R_3^2 \mid s < \varepsilon] = o(\mathbb{E}[s^2 \mid s < \varepsilon])$$

are the author's; they suffice for the Conditional Theorem.

Proof by computation. Evaluate the sine-kernel determinants for the Palm conditioning at $\pm s/2$, integrate x^{-6} times the conditional density on $|x| > s/2$, and average against the cubic small-gap law. The scripts and raw values are in `anc/`. □ □

Lemma 21 (Singh Lemma 51; numerical)**.** *Using the extremal-majorant framework of Carneiro-Chandee-Littmann-Milinovich* [3] *and Chirre-Gonçalves-de Laat* [5]*, let m_A be the restricted small-gap majorant obtained from Fourier support A (as in the family/Beurling calculus of `anc/`). Numerically $A^3 m_A \in \{0.54, 0.66, 0.72, 0.80\}$ for $A = 2, 3, 4, 6$. Hence $m_A \sim c/A^3$ saturates as $\varepsilon \to 0$ at fixed A, so a fixed-support majorant cannot yield $\mathrm{SR}(a, \theta)$ for any $a > 0$. Rate 3 needs support $A \asymp 1/\varepsilon$; support 2 buys only one rung. The saturation constants and the support-versus-rate barrier are the author's.*

Proof by computation. Evaluate the optimized majorants at the listed supports; the scaling $A^3 m_A$ is recorded in `anc/`. □ □

Lemma 22 (Singh Lemma 57; numerical)**.** *Using the trivial Fejér window bound on $\int_{2 \le |\alpha| \le R} F$ and paying $\max(h * h)_+$, there is no gain over $R = 2$ for $R \in (2, 2.3]$: the loss is 0.00014 at $R = 2.02$ and 0.0147 at $R = 2.3$ (`anc/hybrid.py`, symmetric). An earlier numerical claim of 93.59% simple zeros from this window is retracted (the test function h was not even).*

Proof by computation. Run the symmetric hybrid majorant over the stated R-range; the script reports the losses above. □ □

6. Transfer to zeta

Lemma 23. *Assume RH. Then $x_n - y_n = s_n \,\mathrm{PV} \sum \xi_k^{-1} + O(s_n^3 R_3) + O(s_n/L)$.*

Hypothesis H3 (local GUE law)**.** For each fixed K, the law of $\{\xi_k : |\xi_k| \le K\}$ given $s_n < \varepsilon$ and $\gamma_n \in [T, 2T]$ converges to the sine Palm measure $\mathbb{E}_0^{(2)}$ as $T \to \infty$ then $\varepsilon \to 0$.

Hypothesis H4 (weighted tail; H2-bis)**.** $\lim_{K \to \infty} \mathbb{E}[X_K^2 \mid s_n < \varepsilon] = 0$, and $\mathbb{E}[s_n^6 R_3^2 \mid s_n < \varepsilon] = o(\mathbb{E}[s_n^2 \mid s_n < \varepsilon])$ as $\varepsilon \to 0$ (the repair of Singh Lemma 20).

Conditional Theorem. *Assume RH, H3 and H4. Then*

$$\lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{\mathbb{E}[(x_n - y_n)^2 \mid s_n < \varepsilon, \gamma_n \in [T, 2T]]}{\mathbb{E}[s_n^2 \mid s_n < \varepsilon, \gamma_n \in [T, 2T]]} = \frac{\pi^2}{15}.$$

Proof. Lemma 23 writes $x_n - y_n = s_n\Sigma + E$ with the error negligible under H4. Split Σ into near and far; apply H3 and H4. □ □

7. Conjecture and numerical evidence

Conjecture 24. *With* $W_n = \sqrt{|Z'(\gamma_n)Z'(\gamma_{n+1})|}/L$: *(i)* $\exp \mathbb{E}[\log(W_n/s_n) \mid s_n < \varepsilon] = C_\infty + B'/L + O(1/L^2)$ *with* $C_\infty = 0.676 \pm 0.031$, $B' = 2.99 \pm 0.35$; *(ii)* $\operatorname{Var} \log(W_n/s_n) = \frac{1}{2}\log\log T + O(1)$; *(iii)* $\operatorname{Var}(x_n - y_n \mid s) = (\pi^2/15)s^2(1 + O(s^2))$.

Odlyzko's tables [21] up to zero 10^{22} support (i)-(iii); see Figures 1–4. Slow approach at low height matches Bogomolny-Keating heuristics [1].

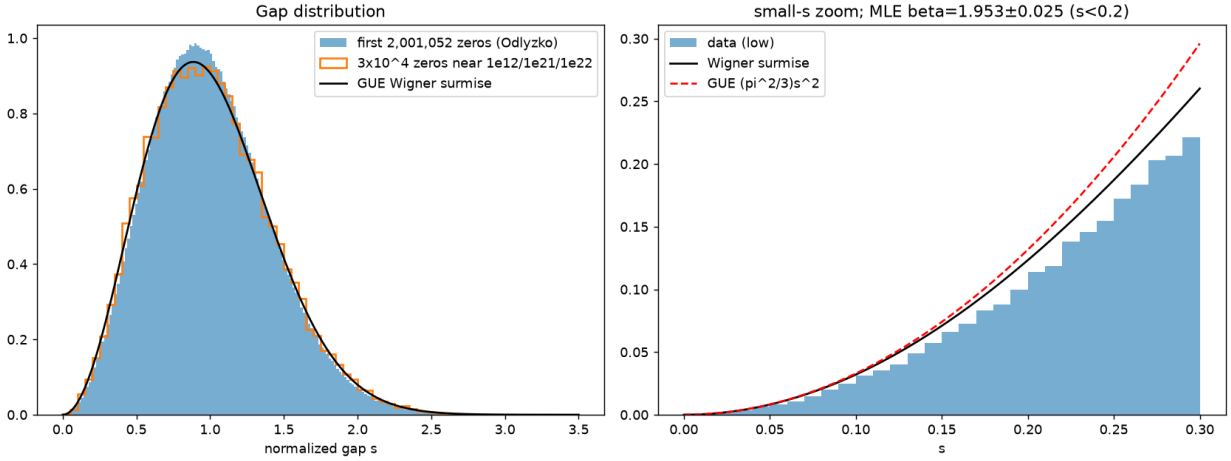


Figure 1. Gap histogram vs Wigner surmise.

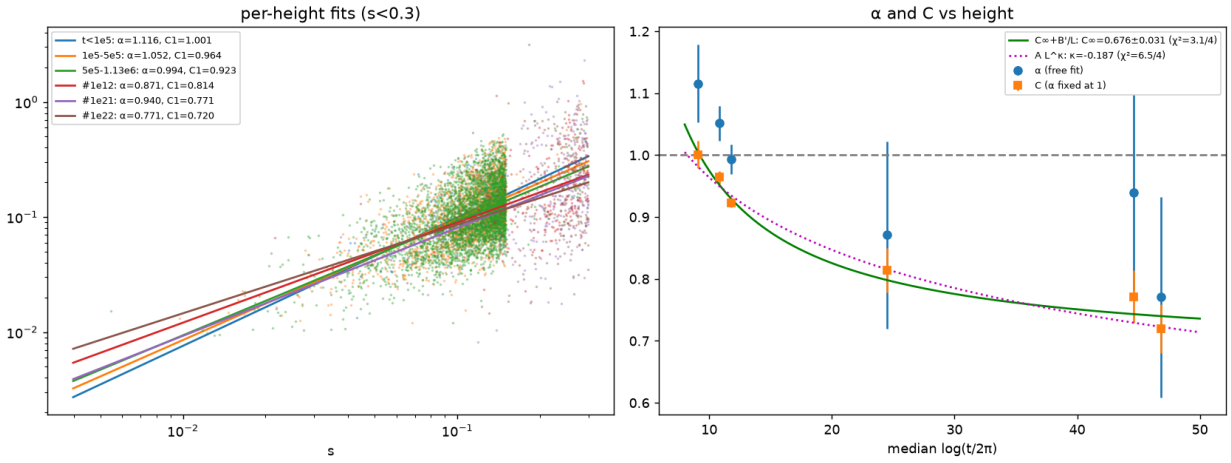


Figure 2. W_n fits by height.

Data and code availability

Odlyzko tables: [21]. Scripts for Singh Lemmas 5, 51, 57 and the figures: `anc/`. Lean formalization of Singh Lemmas 6, 8, 45 (0 sorry): link-only, not bundled.

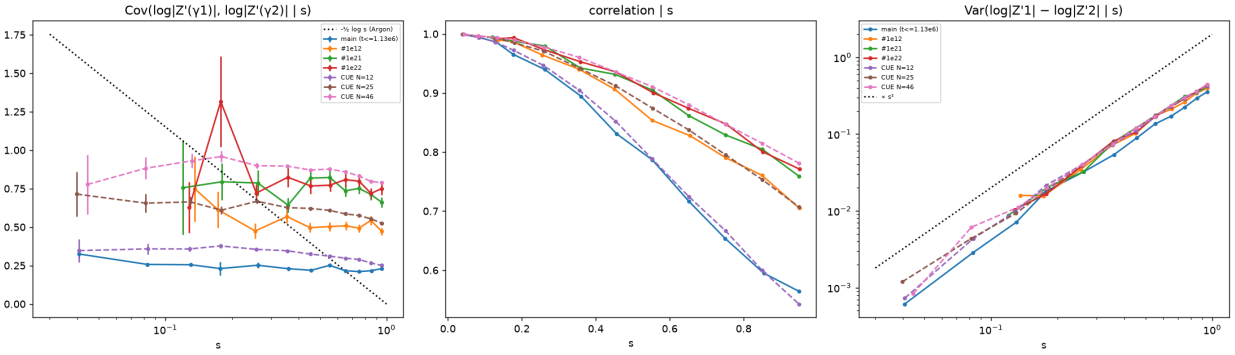


FIGURE 3. Conditional covariance and variance.

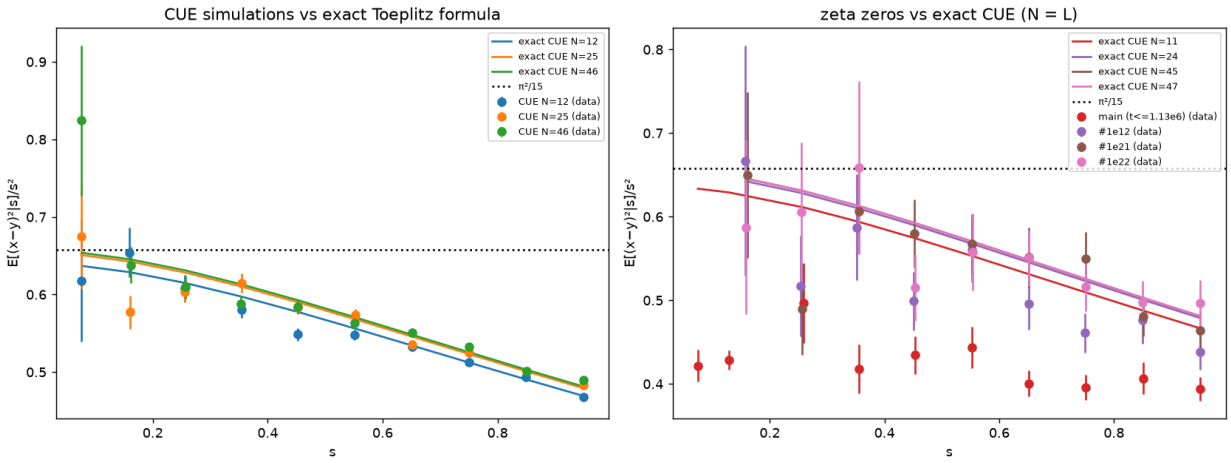


FIGURE 4. $\mathbb{E}[(x-y)^2 \mid s]/s^2$ vs $\pi^2/15$.

REFERENCES

[1] E. Bogomolny and J. P. Keating, *A method for calculating spectral statistics based on random matrix universality with an application to the three point correlations of the Riemann zeros*, J. Phys. A **46** (2013), 305203.

[2] H. M. Bui and D. R. Heath-Brown, *On simple zeros of the Riemann zeta-function*, Bull. Lond. Math. Soc. **45** (2013), 953-961.

[3] E. Carneiro, V. Chandee, F. Littmann and M. B. Milinovich, *Hilbert spaces and the pair correlation of zeros of the Riemann zeta-function*, J. Reine Angew. Math. **725** (2017), 143-182; arXiv:1406.5462.

[4] A. Chirre, F. Gonçalves and D. de Laat, *Fourier optimization and small gaps in the zeros of the Riemann zeta function*, preprint; positive proportion of gaps ≤ 0.6039 of the mean.

[5] A. Chirre, F. Gonçalves and D. de Laat, *Pair correlation estimates for the zeros of the zeta function via semi-definite programming*, Adv. Math. **361** (2020), 106926; arXiv:1810.08843.

[6] F. J. Dyson, *Statistical theory of the energy levels of complex systems. I*, J. Math. Phys. **3** (1962), 140-156.

[7] D. A. Goldston, S. M. Gonek and H. L. Montgomery, *Mean values of the logarithmic derivative of the Riemann zeta function with applications to primes in short intervals*, J. Reine Angew. Math. **537** (2001), 105-122.

[8] D. A. Goldston and H. L. Montgomery, *Pair correlation of zeros and primes in short intervals*, in Analytic Number Theory and Diophantine Problems (Stillwater, OK, 1984), Progr. Math. **70**, Birkhäuser, Boston, 1987, 183-203; doi:10.1007/978-1-4612-4816-3_10.

[9] D. A. Goldston and R. C. Vaughan, *On the Montgomery-Hooley asymptotic formula*, in Sieve Methods, Exponential Sums, and their Applications in Number Theory (Cardiff, 1995), London Math. Soc. Lecture Note Ser. **237**, Cambridge Univ. Press, Cambridge, 1997, 117-142; doi:10.1017/cbo9780511526091.010.

[10] S. M. Gonek, *Mean values of the Riemann zeta-function and its derivatives*, Invent. Math. **75** (1984), 123-141; subsequent uniform Landau formulae in S. M. Gonek, *An explicit formula of Landau and its applications*, Contemp. Math. **143** (1993), 395-406.
[11] D. R. Heath-Brown, *Prime numbers in short intervals and a generalized Vaughan identity*, Canad. J. Math. **34** (1982), 1365-1377; doi:10.4153/CJM-1982-095-9.
[12] C. Hooley, *On the Barban-Davenport-Halberstam theorem. I*, J. Reine Angew. Math. **274/275** (1975), 206-223; doi:10.1515/crll.1975.274-275.206.
[13] L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Amer. Math. Soc., Providence, 1963.
[14] C. P. Hughes, J. P. Keating and N. O'Connell, *Random matrix theory and the derivative of the Riemann zeta function*, Proc. R. Soc. Lond. A **456** (2000), 2611-2627.
[15] J. P. Keating and N. C. Snaith, *Random matrix theory and* $\zeta(1/2+it)$, Comm. Math. Phys. **214** (2000), 57-89.
[16] M. L. Mehta, *Random Matrices*, 3rd ed., Pure Appl. Math. **142**, Elsevier/Academic Press, Amsterdam, 2004.
[17] M. B. Milinovich, *Upper bounds for moments of* $\zeta'(\rho)$, Bull. Lond. Math. Soc. **42** (2010), 28-44; arXiv:0806.0786.
[18] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Proc. Sympos. Pure Math. **24** (1973), 181-193.
[19] A. M. Odlyzko, *On the distribution of spacings between zeros of the zeta function*, Math. Comp. **48** (1987), 273-308.
[20] A. M. Odlyzko, *The* 10^{20}*-th zero of the Riemann zeta function and 175 million of its neighbors*, preprint, AT&T Bell Laboratories, 1992.
[21] A. M. Odlyzko, *The* 10^{22}*-nd zero of the Riemann zeta function*, in Dynamical, Spectral, and Arithmetic Zeta Functions, Contemp. Math. **290**, Amer. Math. Soc., Providence, 2001, 139-144.
[22] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. **81** (1996), 269-322; doi:10.1215/S0012-7094-96-08115-6.
[23] J. Singh, companion data and notes, Zenodo, doi:10.5281/zenodo.23899228.
[24] K. Soundararajan, *Moments of the Riemann zeta function*, Ann. of Math. (2) **170** (2009), 981-993; arXiv:math/0612106.
[25] A. M. Turing, *Some calculations of the Riemann zeta-function*, Proc. London Math. Soc. (3) **3** (1953), 99-117.

Hix Lab, hixlab.si
Email address: founder@hixlab.si