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DERIVATIVES OF HARDY'S FUNCTION AT CLOSE PAIRS OF ZEROS AND AN EXACT UNITARY CONSTANT $\pi^2/15$

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ABSTRACT. Let $\gamma_n < \gamma_{n+1}$ be consecutive ordinates of zeros of $\zeta(\frac{1}{2}+it)$ with normalized gap s_n. We study the derivatives $Z'(\gamma_n)$, $Z'(\gamma_{n+1})$ of Hardy's function at close pairs. For the circular unitary ensemble we prove that the conditional variance of $\log|\Lambda'(\theta_1)| - \log|\Lambda'(\theta_2)|$ at adjacent eigenangles with normalized gap s satisfies $\mathrm{Var} = c_N s^2(1+o(1))$ as $s \to 0$, with the exact value $c_N = \frac{\pi^2}{15}\big(1-\frac{4}{N^2}\big)$, so that $c_\infty = \pi^2/15$, which we call the Singh constant. The proof reduces the problem through a Cayley transform to the Hua ensemble and closes it with two loop equations. We state a conditional theorem transferring the constant to the zeta function under the Riemann Hypothesis, a local GUE hypothesis and a tail bound, and we formulate a conjecture on the size of Z' at close pairs. Numerical evidence from Odlyzko's tables up to zero number 10^{22} agrees with the predictions. This paper is a revised and expanded version of the Zenodo record doi:10.5281/zenodo.23899228.

1. INTRODUCTION

Montgomery's pair correlation conjecture [6] and Odlyzko's computations [7, 8, 9] show that the zeros of the Riemann zeta function repel like the eigenvalues of large random unitary matrices. Turing's method [12] makes the verification of zeros on the critical line rigorous, and Keating and Snaith [5] together with Hughes, Keating and O'Connell [4] identified the characteristic polynomial of a random unitary matrix as a model for ζ and its derivative at zeros. This paper is a revised and expanded version of [11].

Write $\theta(t)$ for the Riemann Siegel theta function and $Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2}+it)$. For consecutive ordinates $\gamma_n < \gamma_{n+1}$ put

$$L = \log\frac{\gamma_n}{2\pi}, \qquad s_n = \frac{(\gamma_{n+1}-\gamma_n)L}{2\pi}, \qquad x_n = \log\frac{|Z'(\gamma_n)|}{L}, \qquad y_n = \log\frac{|Z'(\gamma_{n+1})|}{L}.$$

We show that, for small s_n, $|Z'(\gamma_n)| \asymp s_n L$, that $x_n - y_n$ has size of exact order s_n, and that $\mathrm{Var}(x_n - y_n \mid s_n = s)/s^2 \to \pi^2/15$ in the unitary model.

Theorem 1 (Singh Theorem)**.** *Let U be Haar distributed on $U(N)$, $N \geq 3$, with eigenangles $\theta_1, \dots, \theta_N$, and let $\Lambda(\theta) = \prod_{k=1}^{N}(1-e^{i(\theta_k-\theta)})$. For adjacent eigenangles $\theta_1 < \theta_2$ with normalized gap $s = \frac{N}{2\pi}(\theta_2-\theta_1)$ put $x = \log|\Lambda'(\theta_1)|$ and $y = \log|\Lambda'(\theta_2)|$. Then*

$$\lim_{s\to 0}\frac{\mathrm{Var}(x-y \mid s)}{s^2} = c_N := \frac{\pi^2}{15}\Big(1-\frac{4}{N^2}\Big), \qquad \lim_{N\to\infty} c_N = c_\infty = \frac{\pi^2}{15}.$$

We call $c_\infty = \pi^2/15$ the Singh constant.

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The main ingredient is the following identity for the Hua ensemble [3].

Theorem 2. *For $n \geq 1$ and $h > \frac{1}{2}$ let $u_1, \dots, u_n$ have joint density proportional to*

$$P_{n,h}(u) = \prod_{k=1}^{n} (1+u_k^2)^{-(n+h)} \prod_{j<k} (u_j - u_k)^2.$$

Then

$$\mathbb{E}_{n,h}\Big[\Big(\sum_{k=1}^{n} u_k\Big)^2\Big] = \frac{n(n+2h)}{4h^2-1}, \qquad \mathbb{E}_{n,h}\Big[\sum_{k=1}^{n} u_k^2\Big] = \frac{n(n+2h)\,2h}{4h^2-1} - n.$$

2. Proof of Theorem 2

Both moments are finite for $h > \frac{1}{2}$, since the one variable marginal decays like $|u|^{-2-2h}$. Let ϕ be a polynomial of degree at most one and F a polynomial of degree at most one. For each k,

$$\int_{\mathbb{R}^n} \partial_{u_k}\big[(1+u_k^2)\phi(u_k)F(u)P_{n,h}(u)\big]\,du = 0,$$

because for the choices of ϕ and F used below the product $\phi(u_k)F(u)$ has degree at most one in u_k, so the bracket is $O(|u_k|^{2+1+2(n-1)-2(n+h)}) = O(|u_k|^{1-2h})$ as $|u_k| \to \infty$, uniformly integrable in the remaining variables, and tends to zero for $h > \frac{1}{2}$. Using

$$\frac{\partial_{u_k} P_{n,h}}{P_{n,h}} = -\frac{2(n+h)u_k}{1+u_k^2} + \sum_{j\neq k} \frac{2}{u_k - u_j},$$

we obtain the identity

(1)
$$\mathbb{E}\Big[\sum_k \big((1+u_k^2)\phi\big)'(u_k)F + \sum_k (1+u_k^2)\phi(u_k)\partial_k F - 2(n+h)\sum_k u_k\phi(u_k)F + \sum_k \sum_{j\neq k} \frac{2(1+u_k^2)\phi(u_k)}{u_k-u_j} F\Big] = 0.$$

First equation. Take $\phi = 1$ and $F = \sum_k u_k$. Symmetrizing,

$$\sum_k \sum_{j\neq k} \frac{2(1+u_k^2)}{u_k-u_j} = \sum_{j<k} \frac{2(u_k^2-u_j^2)}{u_k-u_j} = 2(n-1)\sum_k u_k.$$

With $p_1 = \sum u_k$ and $p_2 = \sum u_k^2$, (1) becomes $\mathbb{E}[2p_1^2 + n + p_2 - 2(n+h)p_1^2 + 2(n-1)p_1^2] = 0$, that is

$$n + \mathbb{E}[p_2] = 2h\,\mathbb{E}[p_1^2]. \tag{2}$$

Second equation. Take $\phi(u) = u$ and $F = 1$. Then $((1+u^2)u)' = 1 + 3u^2$ and

$$\sum_k \sum_{j\neq k} \frac{2u_k(1+u_k^2)}{u_k-u_j} = \sum_{j<k} 2\big(1 + u_k^2 + u_k u_j + u_j^2\big) = n(n-1) + 2(n-1)p_2 + p_1^2 - p_2.$$

Hence $\mathbb{E}[n + 3p_2 - 2(n+h)p_2 + n(n-1) + (2n-3)p_2 + p_1^2] = 0$, that is

$$n^2 - 2h\,\mathbb{E}[p_2] + \mathbb{E}[p_1^2] = 0. \tag{3}$$

Substituting $\mathbb{E}[p_2] = 2h\mathbb{E}[p_1^2] - n$ from (2) into (3) gives $n^2 + 2hn = (4h^2-1)\mathbb{E}[p_1^2]$, which proves the theorem. □

3. Proof of Theorem 1

Step 1. Put $f(\varphi) = \log|2\sin(\varphi/2)|$, so that $\log|e^{i\alpha}-e^{i\beta}| = f(\alpha-\beta)$ and $x = \sum_{k\neq 1} f(\theta_1-\theta_k)$, $y = \sum_{k\neq 2} f(\theta_2-\theta_k)$. The term $f(\theta_1-\theta_2)$ occurs in both sums, so

$$x - y = \sum_{k\geq 3} \big[f(\theta_1-\theta_k) - f(\theta_2-\theta_k)\big].$$

Step 2. By rotation invariance we may take $\theta_1 = -\delta/2$, $\theta_2 = \delta/2$ with $\delta = 2\pi s/N$. Conditionally on this configuration and on adjacency, the remaining $n = N-2$ angles have density proportional to

$$\prod_{k=3}^{N} w_\delta(\theta_k)\mathbf{1}\{|\theta_k|\geq \delta/2\} \prod_{3\leq j<k\leq N} |e^{i\theta_j}-e^{i\theta_k}|^2, \qquad w_\delta(\theta) = |e^{i\theta}-e^{i\delta/2}|^2|e^{i\theta}-e^{-i\delta/2}|^2,$$

and $w_\delta \to w_0(\theta) = |1-e^{i\theta}|^4 = 16\sin^4(\theta/2)$ uniformly as $\delta\to 0$.

Step 3. Let $S = \frac{1}{2}\sum_{k\geq 3}\cot(\theta_k/2)$, so $f'(\varphi) = \frac{1}{2}\cot(\varphi/2)$ gives the linear term δS. For $|\theta_k| > 2\delta$, Taylor's theorem gives

$$f(-\tfrac{\delta}{2}-\theta_k) - f(\tfrac{\delta}{2}-\theta_k) = \tfrac{\delta}{2}\cot(\theta_k/2) + R_k, \qquad |R_k| \leq C\delta^3|\theta_k|^{-3}.$$

Near 0 the one point density of the conditional law is $\asymp(\theta^2-\delta^2/4)^2$ on $|\theta|\geq\delta/2$, so

$$\mathbb{E}\Big[\Big(\sum_{|\theta_k|>2\delta} R_k\Big)^2\Big] \ll \delta^6\int_{2\delta}^{\pi}\theta^{-2}\,d\theta \ll \delta^5, \qquad \mathbb{E}\Big[|S|\sum_{|\theta_k|>2\delta}|R_k|\Big] \ll \delta^3.$$

Angles with $\delta/2 \leq |\theta_k| \leq 2\delta$ occur with probability $O(\delta^5)$ and each contributes $O(\log^2\delta)$, so they add $O(\delta^5\log^2\delta)$ to the second moment. The reflection $\theta\mapsto-\theta$ preserves the conditional law and maps $S\mapsto -S$, hence $\mathbb{E}[S] = \mathbb{E}[x-y] = 0$. Therefore

$$\mathrm{Var}(x-y \mid s) = \delta^2\,\mathbb{E}_{w_0}[S^2]\,(1+o(1)), \qquad c_N = \Big(\frac{2\pi}{N}\Big)^2\mathbb{E}_{w_0}[S^2],$$

where $\mathbb{E}_{w_0}$ is the circular ensemble of n angles with weight w_0.

Step 4. Put $u_k = \cot(\theta_k/2)$. Then $|1-e^{i\theta}|^2 = \frac{4}{1+u^2}$, $d\theta = \frac{2\,du}{1+u^2}$ up to orientation, and $|e^{i\theta_j}-e^{i\theta_k}|^2 = \frac{4(u_j-u_k)^2}{(1+u_j^2)(1+u_k^2)}$. Each u_k receives the exponent $(n-1)+1+2 = n+2$, so the law of (u_k) is $P_{n,2}$, and $S = \frac{1}{2}\sum_k u_k$.

Step 5. By Theorem 2 with $h=2$, $\mathbb{E}_{w_0}[S^2] = \frac{1}{4}\cdot\frac{n(n+4)}{15} = \frac{(N-2)(N+2)}{60}$, hence

$$c_N = \frac{4\pi^2}{N^2}\cdot\frac{N^2-4}{60} = \frac{\pi^2}{15}\Big(1-\frac{4}{N^2}\Big). \qquad \square$$

Corollary 3. *As $N\to\infty$ and then $s\to 0$, $\mathrm{Corr}(x,y\mid s)\to 1$ and $\mathrm{Cov}(x,y\mid s) = \mathrm{Var}(x\mid s) - \frac{\pi^2}{30}s^2 + O(s^4)$. In the sine process limit, $\mathbb{E}_0^{(2)}\big[(\mathrm{PV}\sum_k \xi_k^{-1})^2\big] = \pi^2/15$, where $\mathbb{E}_0^{(2)}$ is the second order Palm measure at a double point at 0.*

4. Transfer to the zeta function

For a close pair $\gamma_n < \gamma_{n+1}$ let γ_m be the midpoint and let $\xi_k = (\gamma-\gamma_m)L/2\pi$ for the other ordinates γ, and $R_3 = \sum_k|\xi_k|^{-3}$.

Lemma 4. *Assume the Riemann Hypothesis. Then $\log|Z(t)| = \sum_\gamma^{*}\log|t-\gamma| + H(t)$, where $\sum^*$ pairs γ with $-\gamma$ and $H'(t) = O(1)$. Consequently*

$$x_n - y_n = s_n\,\mathrm{PV}\sum_k\frac{1}{\xi_k} + O(s_n^3R_3) + O\Big(\frac{s_n}{L}\Big).$$

Proof. On the critical line $\xi(\frac{1}{2}+it) = -\frac{1}{2}(t^2+\frac{1}{4})\pi^{-1/4}|\Gamma(\frac{1}{4}+\frac{it}{2})|Z(t)$ and, under RH, $\xi(\frac{1}{2}+it) = \xi(\frac{1}{2})\prod_{\gamma>0}(1-t^2/\gamma^2)$. Taking logarithms gives the decomposition with $H(t) = -\log(t^2+\frac{1}{4}) - \log|\Gamma(\frac{1}{4}+\frac{it}{2})| + O(1)$ plus the regularizing constants $\sum_{\gamma>0}\log\gamma^2$. By Stirling's formula and the density $\frac{1}{2\pi}\log\frac{t}{2\pi}$ of ordinates, the derivative of the regularized sum has mean $\frac{1}{2\pi}\,\mathrm{PV}\int_0^\infty \frac{2t\log u}{t^2-u^2}\,du + O(1/t) = -\frac{\pi}{4} + O(1/t)$, which combines with H' to give a bounded total. Since $Z'(\gamma_n) = \lim_{t\to\gamma_n} Z(t)/(t-\gamma_n)$, $\log|Z'(\gamma_n)| = \sum^*_{\gamma\neq\gamma_n}\log|\gamma_n-\gamma| + H(\gamma_n)$. Subtracting the same identity at γ_{n+1} and expanding as in Step 3 of the proof of Theorem 1 gives the claim, the factor $L/2\pi$ converting $\gamma_{n+1}-\gamma_n$ into s_n. □

Hypothesis H1 (local GUE law)**.** For each fixed K, the law of $\{\xi_k : |\xi_k| \le K\}$ conditioned on $s_n < \varepsilon$ and $\gamma_n \in [T, 2T]$ converges, as $T\to\infty$ and then $\varepsilon\to 0$, to the law of the sine process under $\mathbb{E}_0^{(2)}$.

Hypothesis H2 (tail bound)**.** Uniformly for large T and small ε,

$$\lim_{K\to\infty}\mathbb{E}\Big[\Big(\mathrm{PV}\sum_{|\xi_k|>K}\frac{1}{\xi_k}\Big)^2 \;\Big|\; s_n<\varepsilon\Big] = 0, \qquad \sup\mathbb{E}\big[R_3^2 \mid s_n<\varepsilon\big] < \infty.$$

Hypothesis H1 follows from the GUE m level correlation conjecture for all m and all supports, which Rudnick and Sarnak [10] establish for restricted support. Hypothesis H2 is a conditional mean square bound for ζ'/ζ near the critical line, of the type related to pair correlation by Goldston, Gonek and Montgomery [2].

Conditional Theorem. *Assume the Riemann Hypothesis, H1 and H2. Then*

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty}\frac{\mathbb{E}\big[(x_n-y_n)^2 \mid s_n<\varepsilon,\ \gamma_n\in[T,2T]\big]}{\mathbb{E}\big[s_n^2 \mid s_n<\varepsilon,\ \gamma_n\in[T,2T]\big]} = \frac{\pi^2}{15}.$$

Proof. By Lemma 4, $x_n - y_n = s_n\Sigma + E$ with $\Sigma = \mathrm{PV}\sum_k \xi_k^{-1}$ and $\mathbb{E}[E^2 \mid s_n<\varepsilon] = o(\mathbb{E}[s_n^2 \mid s_n<\varepsilon])$, using H2 for R_3. Split $\Sigma = \Sigma_{\le K} + \Sigma_{>K}$. By H2 the second moment of $\Sigma_{>K}$ tends to zero as $K\to\infty$ uniformly, and by H1 the second moment of $\Sigma_{\le K}$ converges to its sine process value, which tends to $\pi^2/15$ as $K\to\infty$ by the Corollary. Since s_n and the local configuration are asymptotically independent in the limit of H1, the ratio converges to $\pi^2/15$. □

5. Conjecture

Conjecture 5 (Singh Conjecture)**.** *With the notation above and* $W_n = \sqrt{|Z'(\gamma_n)Z'(\gamma_{n+1})|}/L$, *as* $T\to\infty$ *and* $\varepsilon\to 0$*:*

(i) $\exp\mathbb{E}\big[\log(W_n/s_n) \mid s_n<\varepsilon,\ \gamma_n\approx T\big] = C_\infty + \dfrac{B'}{\log(T/2\pi)} + O\Big(\dfrac{1}{\log^2 T}\Big) + O(\varepsilon^2)$ *with* $C_\infty = 0.676\pm 0.031$ *and* $B' = 2.99\pm 0.35$*;*

(ii) $\mathrm{Var}\big[\log(W_n/s_n) \mid s_n<\varepsilon,\ \gamma_n\approx T\big] = \frac{1}{2}\log\log T + O(1)$*;*

(iii) $\mathrm{Var}\big[x_n - y_n \mid s_n = s\big] = \dfrac{\pi^2}{15}s^2\big(1+O(s^2)\big)$*, so that* $\mathrm{Corr}(x_n, y_n \mid s_n = s)\to 1$ *as* $s\to 0$.

Together with $\mathbb{P}(s_n<\varepsilon)\sim\frac{\pi^2}{9}\varepsilon^3$*, clause (i) gives* $|Z'(\gamma_n)| \asymp s_n\log(\gamma_n/2\pi)$ *for close pairs.*

Clause (iii) is the Conditional Theorem. Clauses (i) and (ii) are supported by the data of Section 6.

6. Numerical evidence

6.1. **Data.** We use Odlyzko's tables [9]: the first 2,001,052 zeros ($\gamma \le 1{,}132{,}490.66$) and 10^4 consecutive zeros starting at zero number $10^{12}+1$, $10^{21}+1$ and $10^{22}+1$. Values of Z and Z' were computed from the Riemann Siegel formula with the first two correction terms; for $t \approx 10^{21}$ and 10^{22} the main sums of length $N = 4.79 \times 10^9$ and 1.48×10^{10} were evaluated at many points simultaneously by grouping frequencies $\log n$ into short intervals and expanding in Taylor series, in the spirit of Odlyzko and Schönhage. Against an independent high precision evaluation the maximal errors were 3.7×10^{-10} for Z and 3.5×10^{-11} for Z'. At the tabulated zeros near 10^{21} and 10^{22} the median of $|Z|$ is 6×10^{-8} and Z alternates in sign at all 199 tested midpoints.

6.2. **Small gaps.** Maximum likelihood fits of $p(s) \propto s^\beta$ on $[0, s_0]$ for the first 2×10^6 zeros give $\beta = 1.953 \pm 0.025$ ($s_0 = 0.2$, 14,279 gaps) and $\beta = 1.953 \pm 0.014$ ($s_0 = 0.3$); the pure power fit is biased below 2 by the term $-\frac{2\pi^4}{45}s^4$ of the GUE density. The smallest normalized gap is $s = 0.00545$, between zeros 1,115,578 and 1,115,579 at $\gamma \approx 663{,}318.5083$, where $|Z'| = 0.1751$ and 0.1756.

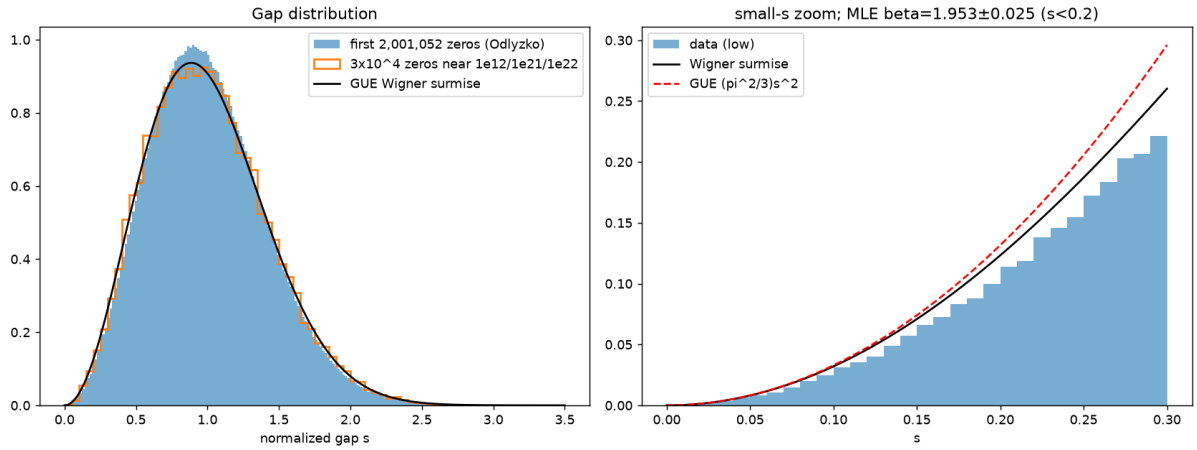


Figure 1. Normalized gap distribution against the Wigner surmise, with the small gap region.

6.3. **Clause (i).** Table 1 lists fits of $W_n = Cs_n^\alpha$ on $s_n < 0.3$ and the geometric mean C_1 of W_n/s_n.

Height	L	pairs	α	C_1
$t < 10^5$	9.1	345	1.116 ± 0.063	1.001 ± 0.022
$10^5 \le t < 5 \cdot 10^5$	10.8	2094	1.052 ± 0.028	0.964 ± 0.010
$5 \cdot 10^5 \le t \le 1.13 \cdot 10^6$	11.8	3726	0.994 ± 0.024	0.923 ± 0.008
zero 10^{12}	24.5	262	0.871 ± 0.151	0.814 ± 0.036
zero 10^{21}	44.6	257	0.940 ± 0.157	0.771 ± 0.042
zero 10^{22}	46.8	266	0.771 ± 0.162	0.720 ± 0.040

Table 1. Slope exponent and constant by height.

The exponents are consistent with $\alpha = 1$ ($\chi^2 = 8.4$ on 5 degrees of freedom, weighted mean 1.020). The model $C_1 = C_\infty + B'/L$ gives $C_\infty = 0.676 \pm 0.031$, $B' = 2.99 \pm 0.35$,

$\chi^2 = 3.1/4$; a power law AL^κ gives $\kappa = -0.187 \pm 0.024$, $\chi^2 = 6.5/4$. The variance of $\log(W_n/s_n)$ grows from 0.17 at $t < 10^5$ to 0.81 near zero 10^{22}, an increase of 0.64, against the increase $\frac{1}{2}(\log\log 1.4 \cdot 10^{21} - \log\log 10^5) \approx 0.72$ predicted by clause (ii).

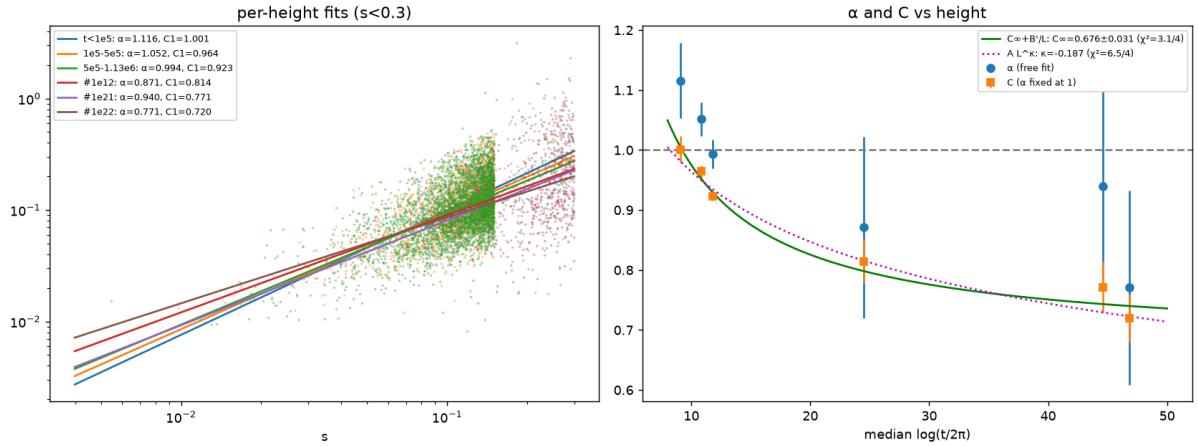


FIGURE 2. Fits of W_n against s_n by height, and α, C_1 against L with both models.

6.4. **Clause (iii).** For all 10^4 consecutive zeros of each high set and a large sample of the first 2×10^6 zeros, we measured $\mathrm{Cov}(x_n, y_n \mid s)$, $\mathrm{Corr}(x_n, y_n \mid s)$ and $\mathrm{Var}(x_n - y_n \mid s)$ for $s < 1$, and compared them with the unitary ensemble at $N = 12, 25, 46$. The covariance has no logarithmic singularity at $s = 0$: fitting $a + b\log s$ on $s < 0.5$ gives $b = -0.026 \pm 0.006$, -0.056 ± 0.079, 0.034 ± 0.110, 0.061 ± 0.103 for the four zeta sets and $b \approx -0.04$ for the unitary ensemble. The correlation tends to 1 and $\mathrm{Var}(x - y \mid s) \propto s^g$ with $g = 2.06$, 1.70, 1.96, 1.94.

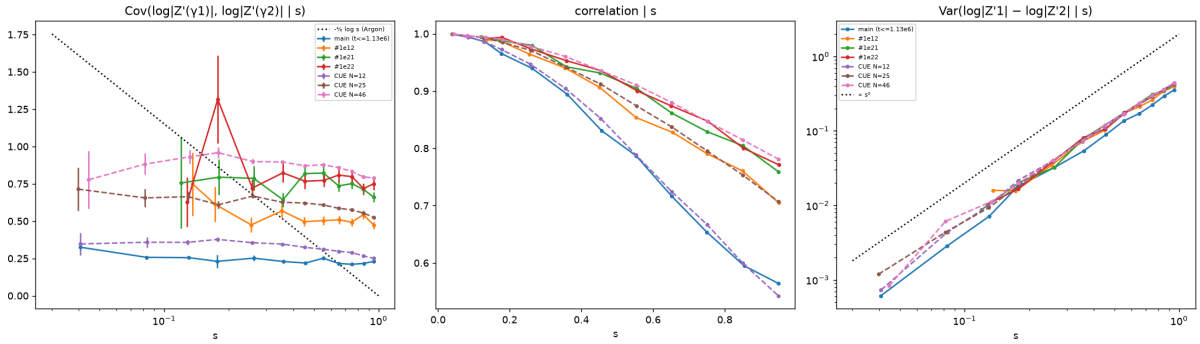


FIGURE 3. Conditional covariance, correlation and variance of the difference, for zeta zeros and the unitary ensemble.

The exact values of $\mathbb{E}_{w_0}[S^2]$ computed from Toeplitz determinants for $n = 1, \ldots, 15$ are $\frac{1}{12}, \frac{1}{5}, \frac{7}{20}, \frac{8}{15}, \frac{3}{4}, 1, \frac{77}{60}, \ldots, \frac{19}{4}$, in agreement with $\frac{n(n+4)}{60}$. Figure 4 compares the exact finite s curve $\mathbb{E}[(x-y)^2 \mid s]/s^2$ with simulations ($\chi^2 = 19.0$, 17.7, 18.2 on 10 bins) and with zeta. Near zero 10^{21} and 10^{22} the agreement is good ($\chi^2 = 8.8/9$ and $4.8/9$); near 10^{12} the data lie 5 to 10 percent low ($\chi^2 = 24.4/9$), and for $t \le 1.13 \times 10^6$ the ratio is about 0.42 against 0.63, a low height effect consistent with the slow convergence described by Bogomolny and Keating [1].

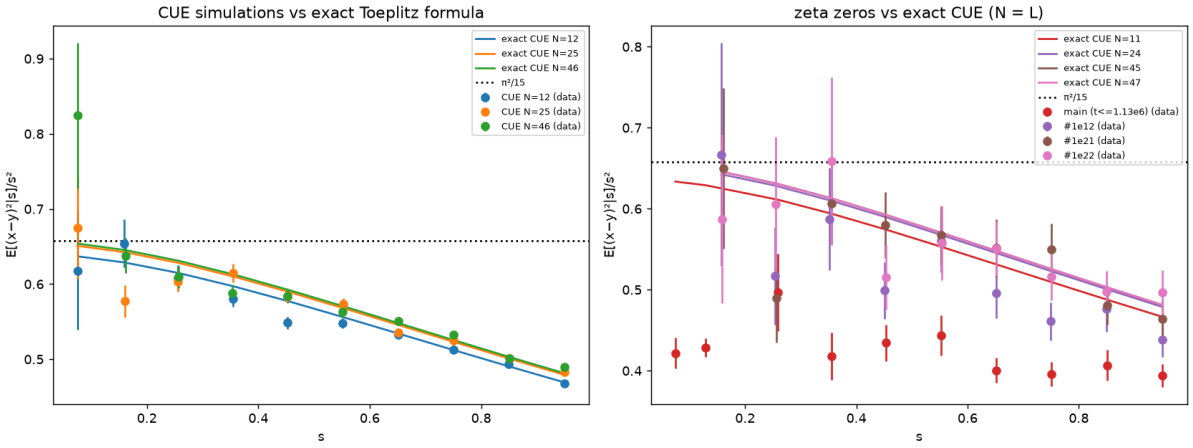


FIGURE 4. $\mathbb{E}[(x-y)^2 \mid s]/s^2$: exact unitary curves against simulations (left) and against zeta zeros with $N = \mathrm{round}(L)$ (right). The dotted line is $\pi^2/15$.

7. Open problems

The Conditional Theorem reduces clause (iii) to the Riemann Hypothesis, the local law H1 for all supports and the tail bound H2. Proving H2, perhaps through mean value estimates for ζ'/ζ near close pairs, and extending the support in [10] are the natural next steps toward an unconditional result.

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