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Log-concavity in the Newton basis and support of rank-two scattering coefficients

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Abstract

We prove that polynomials with nonnegative, interval-supported log-concave Newton coefficients are closed under multiplication and, for integer Newton coefficients, under $D(m) \mapsto \binom{D(m)}{k}$ for integers $k \geq 0$. Coordinatewise Newton multiplication preserves two-variable Schur positivity. We also determine the exact positive Newton support of symmetric rank-two cluster scattering coefficients.

AI use, research objective and scholarly credit

The research began with a shared Codex session on Conjectures 4, 12 and 13 lasting about 2 hours 58 minutes, followed by eleven ChatGPT rounds on Conjecture 12 with at least 50 hours 42 minutes documented in the surviving records. Codex also reviewed arguments, prepared research prompts and refined the exposition. The main text was reviewed using Rethlas-OSV [22, 23, 24]; the appendices, which list statements without proofs, were excluded from that review.

We set out to resolve Elgin–Reading–Stella's Conjecture 12 and have paused further work to share the progress obtained. The available proofs, failed approaches and review records are archived on GitHub; the archival release contains the materials and source maps. We welcome further work on the conjecture and these results.

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [15] in connection with total positivity and canonical bases. Scattering diagrams encode compatible wall-crossing automorphisms for these algebras. Gross, Pandharipande and Siebert [16] related ordered factorizations in the tropical vertex group to relative Gromov–Witten invariants of toric surfaces. Gross, Hacking, Keel and Kontsevich [8] constructed cluster scattering diagrams and theta functions by broken lines, while Bridgeland [17] connected scattering diagrams with Hall algebras and stability conditions for quiver representations.

Rank-two diagrams have infinitely many outgoing directions in the wild case [5]. Elgin, Reading and Stella [6], first posted in 2024, conjectured properties of the wall coefficients as

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functions of the exchange parameters. We consider the symmetric exchange matrix $\left(\begin{smallmatrix} 0 & m \\ -m & 0 \end{smallmatrix}\right)$ with positive integer m; it is wild for $m \geq 3$. For positive integers A, B, let $\tau_m(A,B) = \tau^{m,m}(A,B)$ be their coefficient in the original cluster wall function. On a primitive positive ray (d,e), where $d, e \geq 1$ are coprime integers, the function is

$$1 + \sum_{k \geq 1} \tau_m(kd, ke) z^k, \qquad z = x^{md} y^{me},$$

where x, y are formal variables and the initial wall functions are $1 + x^m, 1 + y^m$. Section 5 specifies the crossing action and its relation to quiver invariants.

Akagi's ordered dilogarithm formulas [2] express the relevant exponents by binomial polynomials in the initial parameters. In [3, Proposition 4.3], he proves Elgin–Reading–Stella's Conjecture 11: $\tau_m(A,B)$ has degree $A + B - 1$ and nonnegative coefficients in the ordinary binomial basis. Remark 1.4 there attributes the complete proof to Burcroff, Lee and Mou. Their tight-grading formula [5] gives a combinatorial approach to positive scattering coefficients. Burcroff, Lee, Mou, Musiker and Reineke [18] further connect quiver moduli and tight gradings; their Remark 1.5 records Akagi's result on Conjecture 11 separately from their methods.

We use $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}$ for the integers. For a real variable m and $j \in \mathbb{Z}_{\geq 0}$, put $\binom{m}{0} = 1$ and $\binom{m}{j} = m(m-1)\cdots(m-j+1)/j!$ when $j \geq 1$. For a real polynomial $F(m)$, define $\Delta F(m) = F(m+1) - F(m)$. Its Newton expansion is

$$F(m) = \sum_{j \geq 0} f_j \binom{m}{j}, \qquad f_j = \Delta^j F(0). \tag{1}$$

The sum is finite, and $\Delta^0 F = F$. For a real polynomial $D(m)$ and $k \in \mathbb{Z}_{\geq 0}$, the same convention gives $\binom{D(m)}{k} = \prod_{h=0}^{k-1} (D(m) - h)/k!$, with empty product one. The *Newton symbol* of F is $\mathcal{N}F(x) = \sum_{j \geq 0} f_j x^j$, where x is a formal variable. For nonzero F, its degree is the largest index with $f_j \neq 0$; set $\deg 0 = -\infty$. We extend every finite coefficient row by zero to all integer indices. Its positive support is $\operatorname{supp}(f) = \{j \in \mathbb{Z} : f_j > 0\}$.

A nonnegative row has no internal zeros when its positive support is an integer interval or is empty. We call F *Newton log-concave* if its Newton row is nonnegative, has no internal zeros, and

$$f_j^2 \geq f_{j-1} f_{j+1} \qquad (j \in \mathbb{Z}). \tag{2}$$

The zero polynomial is included. A finite row is unimodal if it is nondecreasing up to some index and nonincreasing thereafter. A nonnegative log-concave row without internal zeros is unimodal, since its positive consecutive ratios are nonincreasing.

The known expansion is

$$\tau_m(A,B) = \sum_{j=0}^{A+B-1} a_j(A,B) \binom{m}{j}, \qquad a_j(A,B) = \Delta_m^j \tau_m(A,B)\Big|_{m=0} \geq 0, \tag{3}$$

where Δ_m acts on the parameter m and the value at zero belongs to the polynomial extension.

Conjecture 1.1 (Elgin–Reading–Stella, Conjecture 12)**.** *For every $A, B \geq 1$, the coefficients in (3) are unimodal and log-concave.*

Geometric positivity concerns the values $\tau_m(A,B)$ at fixed m; Newton log-concavity compares finite differences in m. The complete Euler factorization combines polynomial products and $\binom{D(m)}{k}$, for which we prove the following closure properties.

Theorem 1.2. *(i) If F and G are Newton log-concave, then so is FG. For nonzero inputs with positive Newton supports $[L_F, U_F]$ and $[L_G, U_G]$, the output support is $[\max(L_F, L_G), U_F + U_G] \cap \mathbb{Z}$.*

(ii) *Suppose F, G are nonzero Newton log-concave polynomials and L is the least index with a positive Newton coefficient of G. If λ is real and $0 \le \lambda \le F(L)$, then $(F-\lambda)G$ is Newton log-concave.*

(iii) *If D is Newton log-concave and its Newton coefficients are integers, then $\binom{D(m)}{k}$ is Newton log-concave with integer Newton coefficients for every $k \in \mathbb{Z}_{\ge 0}$. If D is nonconstant and $k \ge 1$, its positive support is*

$$[\min\{n \in \mathbb{Z}_{\ge 0} : D(n) \ge k\}, k \deg D] \cap \mathbb{Z}.$$

Multiplication in the Newton basis counts two subsets by the size of their union. Its kernel is the Heisenberg dimension convolution of Aguiar, Ferrer Santos and Moreira [1, Proposition 1.3]. Walkup [14] studied binomial convolution, total positivity and unions of random sets. Wagner [19] and Brändén [20, 4] developed real-rootedness preservation for the associated Newton transform. The present hypotheses allow Newton symbols such as $1 + x + x^2$, whose coefficients are log-concave although its zeros are nonreal. The support threshold in Theorem 1.2(ii) permits the induction $k\binom{D}{k} = (D-k+1)\binom{D}{k-1}$ used in part (iii).

Liu and Mao [11] study the numerator of $\sum_{n\ge 0} F(n)x^n$ over $(1-x)^{\deg F+1}$. For $F(m) = 1+m+\binom{m}{2}$, that numerator is $1-x+x^2$, whereas its Newton symbol is $1+x+x^2$. Bérczi and Fehér [21, Theorem 3.4] prove that a nonnegative interval-supported log-concave Newton row gives a log-concave sequence of polynomial values at nonnegative integers. Their Lemma 5.1 proves Newton nonnegativity after multiplication by an affine factor whose zero is bounded by the first index with a positive Newton coefficient. Theorem 1.2(ii) preserves log-concavity for polynomial multipliers $F - \lambda$.

The proof uses nonnegative two-by-two minors and decreasing shifted-input ratios. The two-variable covering kernel preserves Schur positivity (Theorem 4.1), yielding preservation of partial synchronicity under common Newton multiplication. Hu, Wang, Zhao and Zhao [9] introduced this comparison and proved its preservation under Cauchy convolution. Polarization bounds the sum of mixed defects from two product pairings.

Theorem 1.3. *For all positive integers A, B and all integers j, with the zero-extension convention,*

$$a_j(A,B) > 0 \quad \Longleftrightarrow \quad \left\lceil \frac{A^2+B^2-1}{AB} \right\rceil \le j \le A+B-1. \tag{4}$$

Fixed-parameter nonvanishing [12, 5] and Newton positivity determine the first positive index. The additional assertion in (4) is that every subsequent Newton index up to the degree occurs: the polynomial $\binom{m}{1}+\binom{m}{3}$, for example, is positive at every integer $m \ge 1$ but has a gap in its Newton row. We obtain interval support by splitting vertices of complete bipartite quivers, proving interval support for their complete integer Euler exponents, and assembling distinct selections along each ray.

Conjecture 1.1 remains open. Section 7 gives two sufficient conditions in the Euler approach: log-concavity of the exponent rows and a collective comparison of partition terms. Appendices A and B record partial parameter ranges and further sufficient criteria, with proofs in the companion notes.

2 Newton multiplication and its positive kernel

We write $[x^j]P$ for the coefficient of x^j in a polynomial or formal power series P, assigning zero to absent powers. For example, if $F(m) = 1+2m+3\binom{m}{2}$, then $\mathcal{N}F = 1+2x+3x^2$ and $\Delta^2 F(0) = 3$.

For $n, i, j \in \mathbb{Z}_{\geq 0}$, define

$$K_n(i,j) = \begin{cases} \dfrac{n!}{(n-i)!(n-j)!(i+j-n)!}, & \max(i,j) \leq n \leq i+j, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

It is also zero if any index is negative. For a fixed n-element set, $K_n(i,j)$ counts ordered pairs of subsets of sizes i, j whose union is that set. Consequently

$$\binom{m}{i}\binom{m}{j} = \sum_n K_n(i,j)\binom{m}{n}. \tag{6}$$

For finite rows a, b, their *Newton product* is the finite row

$$C_n(a,b) = \sum_{i,j\geq 0} K_n(i,j) a_i b_j. \tag{7}$$

For $A(x) = \sum_i a_i x^i$ and $B(x) = \sum_j b_j x^j$, write $A(x) \diamond B(x) = \sum_n C_n(a,b)x^n$. Equation (6) says $\mathcal{N}(FG) = (\mathcal{N}F) \diamond (\mathcal{N}G)$.

Lemma 2.1. *If a positive sequence u_k, $k \geq 0$, is log-concave, then $u_k + u_{k+1}$ is log-concave. For every $i \in \mathbb{Z}_{\geq 0}$ the row $B_i(k) = \sum_{j=0}^{i} \binom{i}{j} u_{k+j}$ is therefore log-concave.*

Proof. The ratios u_{k+1}/u_k are nonincreasing. For $k \geq 1$ the required defect equals

$$\begin{aligned}(u_k^2 - u_{k-1}u_{k+1}) + (u_{k+1}^2 - u_k u_{k+2}) \\ + u_k u_{k+1} - u_{k-1}u_{k+2},\end{aligned}$$

whose three terms are nonnegative by these ratio inequalities. Iterate the neighbor-sum operation using $B_{i+1}(k) = B_i(k) + B_i(k+1)$. □

A matrix indexed by ordered row and column sets is called TP_2 here if all its entries and all two-by-two minors with increasing indices are nonnegative.

Lemma 2.2. *If b is a nonnegative log-concave row with interval positive support, the matrix*

$$M_b(n,i) = \begin{cases} \dbinom{n}{i} \displaystyle\sum_{j=0}^{i} \binom{i}{j} b_{n-i+j}, & 0 \leq i \leq n, \\ 0, & i > n, \end{cases} \qquad n, i \in \mathbb{Z}_{\geq 0}, \tag{8}$$

is TP_2. Moreover $C_n(a,b) = \sum_i M_b(n,i) a_i$.

Proof. First let b be positive at every index; infinite rows are allowed since each entry involves finitely many terms. The expression for C follows by choosing the first subset and then its intersection with the second. In Lemma 2.1 take $u = b$, so $B_i(k) = \sum_{j=0}^{i} \binom{i}{j} b_{k+j}$. For $0 \leq i < n$, put $k = n - i$. The identities

$$B_{i+1}(k-1) = B_i(k-1) + B_i(k), \qquad B_{i+1}(k) = B_i(k) + B_i(k+1),$$

and

$$\frac{\binom{n}{i}\binom{n+1}{i+1}}{\binom{n}{i+1}\binom{n+1}{i}} = \frac{k+1}{k}$$

give the adjacent minor

$$M_b(n,i)M_b(n+1,i+1) - M_b(n,i+1)M_b(n+1,i)$$
$$= \frac{\binom{n}{i+1}\binom{n+1}{i}}{k}\big((k+1)B_i(k)^2 + B_i(k)B_i(k+1) - kB_i(k-1)B_i(k+1)\big) \geq 0. \tag{9}$$

Indeed Lemma 2.1 gives $B_i(k)^2 \geq B_i(k-1)B_i(k+1)$. For $i = n$ the upper-right entry is zero, so that minor is nonnegative too. It follows that $M_b(n+1,i)/M_b(n,i)$ is nondecreasing for $0 \leq i \leq n$. For arbitrary rows $n < m$ and columns $i < j \leq n$, all entries below are positive, and

$$\frac{M_b(m,j)}{M_b(n,j)} = \prod_{r=n}^{m-1} \frac{M_b(r+1,j)}{M_b(r,j)} \geq \prod_{r=n}^{m-1} \frac{M_b(r+1,i)}{M_b(r,i)} = \frac{M_b(m,i)}{M_b(n,i)}.$$

Multiplying by $M_b(n,i)M_b(n,j)$ proves the required minor. If $i \leq n < j \leq m$, the upper-right entry vanishes; if $i > n$ or $j > m$, the determinant is zero.

For a finite row with support $[L,U]$, approximate it, for sufficiently small $\varepsilon > 0$, by

$$b_k^{(\varepsilon)} = \begin{cases} b_L\varepsilon^{L-k}, & 0 \leq k < L, \\ b_k, & L \leq k \leq U, \\ b_U\varepsilon^{k-U}, & k > U. \end{cases} \tag{10}$$

Its consecutive ratios consist of ε^{-1}, the original nonincreasing ratios, and ε. Choosing ε small makes the whole sequence log-concave; a singleton support only requires $\varepsilon \leq 1$. Every matrix entry and minor converges to the required one. The zero row gives the zero matrix. □

Corollary 2.3. *Let c be a finite nonnegative log-concave row with interval support. If finite nonnegative rows a, b satisfy $a_ib_j \geq a_jb_i$ for all $0 \leq i < j$, then*

$$C_n(a,c)C_m(b,c) \geq C_m(a,c)C_n(b,c) \qquad (0 \leq n < m).$$

In particular, a nonincreasing input ratio is sent to a nonincreasing output ratio wherever its denominator is positive.

Proof. Using Lemma 2.2, the difference is

$$\sum_{i<j}(a_ib_j - a_jb_i)(M_c(n,i)M_c(m,j) - M_c(n,j)M_c(m,i)) \geq 0.$$ □

For any row a write $a_i^+ = a_{i+1}$ for $i \geq 0$; the corresponding polynomial is ΔF when a is the Newton row of F.

Lemma 2.4. *For nonzero nonnegative log-concave rows a, b with interval support, each ratio*

$$\frac{C_n(a^+,b)}{C_n(a,b)}, \qquad \frac{C_n(a,b^+)}{C_n(a,b)}, \qquad \frac{C_n(a^+,b^+)}{C_n(a,b)}$$

is nonnegative and nonincreasing on the positive support of the denominator.

Proof. First let both rows be positive at every nonnegative index. Log-concavity gives $a_{i+1}a_j \geq a_{j+1}a_i$ for $i < j$. Corollary 2.3, whose proof uses only finitely many input entries for each pair of output indices, applies to a^+, a and the fixed row b. This proves the first ratio inequality; symmetry gives the second. The third is the product

$$\frac{C_n(a^+,b^+)}{C_n(a,b^+)}\frac{C_n(a,b^+)}{C_n(a,b)},$$

of two nonnegative nonincreasing ratios. Apply (10) to both rows and pass to the limit in the cross-product inequalities. A vanishing shifted row gives a zero numerator. □

Proof of Theorem 1.2(i). If either input is zero, the conclusion is immediate. Otherwise let a, b be the Newton rows of F, G, with positive supports $[L_a, U_a] \cap \mathbb{Z}$ and $[L_b, U_b] \cap \mathbb{Z}$. For two subsets covering an $(n+1)$-element set, its last element belongs to the first subset only, the second only, or both. Removing it gives

$$C_{n+1}(a, b) = C_n(a^+, b) + C_n(a, b^+) + C_n(a^+, b^+). \tag{11}$$

For positive inputs, division by $C_n(a, b)$ expresses the output ratio as the sum of the three nonincreasing ratios in Lemma 2.4. Hence the output is log-concave. Approximation (10) gives the inequalities for interval-supported finite inputs.

The coefficient is zero outside $[\max(L_a, L_b), U_a + U_b]$. For n in that interval choose $i = \min(U_a, n)$ and $j = \min(U_b, n)$. Both belong to their respective positive supports and satisfy $\max(i, j) \le n \le i + j$, so their summand in (7) is positive. □

3 Supported multipliers and integer selections

Proof of Theorem 1.2(ii). Write $F(m) = \sum_i f_i \binom{m}{i}$ and $G(m) = \sum_i g_i \binom{m}{i}$, with $\operatorname{supp} g = [L, U] \cap \mathbb{Z}$. For $\lambda = 0$ use part (i). If F is constant, multiplication by $F - \lambda \ge 0$ proves the assertion. Assume henceforth $\lambda > 0$ and $\deg F \ge 1$. Set $c_n = C_n(f, g)$ and $h_n = c_n - \lambda g_n$. Since $F(L) \ge \lambda > 0$, the first index with a positive Newton coefficient of F is at most L. Thus $\operatorname{supp} c = [L, U + \deg F]$. The diagonal entry of (8) is $M_f(n, n) = F(n)$, whence, for $L \le n \le U$,

$$c_n \ge F(n) g_n \ge \lambda g_n. \tag{12}$$

All evaluations $F(n)$ are nondecreasing on $\mathbb{Z}_{\ge 0}$. In fact $F(n+1) > F(n)$ for $n \ge L$: if its first positive index is $s \ge 1$, then $s \le L$ and $f_s \binom{n}{s-1} > 0$ occurs in $\Delta F(n)$; if that index is zero, interval support and nonconstancy give $f_1 > 0$. For $L < n \le U$, (12) is strict because $F(n) > F(L) \ge \lambda$ and $g_n > 0$. For $U < n \le U + \deg F$, one has $g_n = 0$ and $h_n = c_n > 0$. At the first index,

$$h_L = (F(L) - \lambda) g_L. \tag{13}$$

Thus the positive support of h is $[L, U + \deg F]$ if $\lambda < F(L)$ and $[L+1, U + \deg F]$ if $\lambda = F(L)$.

Restrict rows to $n \ge L$ and put

$$N(n, i) = M_f(n, i) - \lambda \mathbf{1}_{n=i}, \qquad i \ge 0,$$

where $\mathbf{1}_{n=i}$ is one when $n = i$ and zero otherwise. This matrix is TP_2. Its entries are nonnegative by $F(n) \ge \lambda$. For rows $n < m$ and columns $i < j$, a changed upper-right entry ($j = n$) increases the determinant. If a changed entry is upper-left ($i = n$) or lower-right ($j = m$), lower triangularity makes the upper-right entry zero, and the determinant is a product of nonnegative entries. A changed lower-left entry ($i = m$) makes the entire upper row zero. Unchanged minors are those of M_f.

Set $P_n = \sum_i N(n, i) g_{i+1}$. For every $i < j$, log-concavity with interval support gives $g_{i+1} g_j - g_{j+1} g_i \ge 0$, including the zero boundaries. Therefore

$$\begin{aligned} P_n h_{n+1} - P_{n+1} h_n = \sum_{i<j} &(g_{i+1} g_j - g_{j+1} g_i) \\ &\cdot \big(N(n, i) N(n+1, j) - N(n, j) N(n+1, i)\big) \ge 0. \end{aligned} \tag{14}$$

Hence P_n / h_n is nonnegative and nonincreasing on the positive support of h.

The other boundary entry $M_g(n, 0) = g_n$ and Lemma 2.2 imply

$$g_n c_{n+1} - g_{n+1} c_n = \sum_i f_i \big(M_g(n, 0) M_g(n+1, i) - M_g(n, i) M_g(n+1, 0)\big) \ge 0.$$

Thus g_n/c_n is nonincreasing, as is

$$\frac{c_n}{h_n} = \frac{1}{1 - \lambda g_n/c_n} \tag{15}$$

on the positive support of h. Equation (11) now gives

$$h_{n+1} = P_n + C_n(f^+, g) + C_n(f^+, g^+).$$

After division by h_n, the first term decreases by (14); each remaining term is the product of (15) and one of the nonnegative decreasing ratios of Lemma 2.4. The output ratios decrease, proving log-concavity. Equation (13) and the support already established handle the boundaries. □

Proof of Theorem 1.2(iii). If D is a nonnegative integer constant c, the output is the constant $\binom{c}{k}$, possibly zero. Otherwise $D(n)$ is a nondecreasing unbounded sequence of nonnegative integers for $n \in \mathbb{Z}_{\geq 0}$. Induct on k, starting with $\binom{D}{0} = 1$ and $\binom{D}{1} = D$. For $k \geq 2$ let L be the first index with a positive Newton coefficient of $\binom{D}{k-1}$. For any polynomial with nonnegative Newton coefficients, its first index with a positive Newton coefficient is its first nonnegative integer argument with positive value: $\binom{n}{j} = 0$ when $n < j$ and $\binom{j}{j} = 1$. The induction hypothesis and integrality therefore give

$$L = \min\{n \in \mathbb{Z}_{\geq 0} : D(n) \geq k - 1\}.$$

Apply (ii) with multiplier D and $\lambda = k - 1 \leq D(L)$ to $k\binom{D}{k} = (D - k + 1)\binom{D}{k-1}$. The first positive value of this output occurs at $D(n) \geq k$, and its degree is $k \deg D$. This proves the support assertion as well. Finally the coefficient at Newton index r is

$$\sum_{n=0}^{r} (-1)^{r-n} \binom{r}{n} \binom{D(n)}{k} \in \mathbb{Z}.$$ □

Corollary 3.1. *Let P be a nonzero Newton log-concave polynomial with $P(0) = 0$ and positive support $[L, U]$. For every integer $v \geq 1$, the polynomial $\binom{m}{v}P(m)/m$ is Newton log-concave, with support $[\max(L, v), U + v - 1]$.*

Proof. The expression equals $P(m)\prod_{a=1}^{v-1}(m-a)/v!$. Apply Theorem 1.2(ii) successively with $F(m) = m$ and $\lambda = a$. Before this step the first positive index is $\max(L, a)$, so the threshold holds. If that index equals a, (13) moves it up by one. The top index increases by one at every step. □

Corollary 3.2. *Let a, b be finite nonnegative log-concave rows with interval support. For $t \geq 0$ define*

$$C_n^{(t)}(a, b) = \sum_{i,j \geq 0} a_i b_j t^{i+j-n} K_n(i, j),$$

where only terms with $K_n(i, j) \neq 0$ are included and $0^0 = 1$. This output is log-concave with interval support. For nonzero inputs write $\operatorname{supp}(a) = [L_a, U_a] \cap \mathbb{Z}$ and $\operatorname{supp}(b) = [L_b, U_b] \cap \mathbb{Z}$. Its support is $[\max(L_a, L_b), U_a + U_b]$ if $t > 0$, and $[L_a + L_b, U_a + U_b]$ if $t = 0$.

Proof. For $t > 0$ replace a_i, b_j by $t^i a_i, t^j b_j$ and multiply output index n by t^{-n}. These geometric scalings preserve log-concavity. Let t decrease to zero for the inequalities. At $t = 0$ the surviving terms have $i + j = n$; their positive support is the sum of the two input support intervals. □

Here are counterexamples to several extensions. If $F = 1+m+\binom{m}{2}$, then F^2 has Newton row $(1,3,9,12,6)$; dividing that row by $\binom{4}{j}$ destroys log-concavity. Allowing a negative subtraction parameter gives $(F+1)\cdot 1$ with row $(2,1,1)$. Dropping integrality in (iii) gives $D(m) = 3m/2$, for which the coefficient of $\binom{m}{1}$ in $\binom{D}{3}$ is $-1/16$. Distinct selection also differs from multiset selection: for $D = m + 2\binom{m}{2} + 3\binom{m}{3}$, the Newton row of $\binom{D+3}{4}$ begins $(0,1,33,1263,\ldots)$, with defect $33^2 - 1263 = -174$. Interval support is also necessary for the product statement: the locally log-concave rows $(0,0,0,0,1)$ and $(2,1,0,0,1)$ have product row $(0,0,0,0,7,25,90,140,70)$, with defect $25^2 - 7\cdot 90 = -5$.

4 Schur positivity in two variables

For integers $p \geq q \geq 0$ the two-variable Schur polynomial is

$$s_{(p,q)}(x,y) = \sum_{i=q}^{p} x^i y^{p+q-i}.$$

A symmetric polynomial is *Schur-positive* if it is a finite nonnegative real linear combination of these polynomials. For monomial indices $i,j,k,\ell \in \mathbb{Z}_{\geq 0}$, define the bilinear coordinatewise Newton product by

$$x^i y^j \diamond_2 x^k y^\ell = \sum_{n,m\geq 0} K_n(i,k) K_m(j,\ell) x^n y^m. \tag{16}$$

The sums in (16) are finite.

Theorem 4.1. *The coordinatewise Newton product of two Schur-positive symmetric polynomials in two variables is Schur-positive.*

Lemma 4.2. *Let $n \geq l \geq 0$, $u \geq v \geq 0$, and $T \geq 0$ be integers. If $J = [\alpha,\beta]\cap\mathbb{Z}$ with integer endpoints and $\alpha+\beta = T$, then*

$$\sum_{j\in J} \left(K_n(u,j)K_l(v,T-j) - K_n(v,j)K_l(u,T-j)\right) \geq 0. \tag{17}$$

The assertion remains true for $v = -1$ with the zero-kernel convention.

Proof. Intersect J with $[0,T]$; the intersection, if nonempty, is still centered at $T/2$. If $n+l-T$, $n+l-u-v$, or $u+v+T-n-l$ is negative, every product is zero by the kernel support inequalities. Otherwise put $c = u+v+T-n-l$, $d = u-v$, $D = n-l$ and

$$A = \binom{n}{u}\binom{l}{v}, \quad B = \binom{n}{v}\binom{l}{u}, \quad w_t = \binom{u}{t}\binom{v}{c-t}, \quad W = [\alpha-n+u, \beta-n+u]\cap\mathbb{Z}.$$

Binomials with nonnegative integer upper index are zero outside their usual range. If $B = 0$, (17) is immediate. If $B > 0$, all four binomials in A, B are positive and

$$A/B = \prod_{h=v}^{u-1} \frac{n-h}{l-h} \geq 1.$$

For the positive term set $t = u+j-n$; for the negative term set $t = u+T-j-l$. The second range is

$$[u+T-\beta-l, u+T-\alpha-l] = [u+\alpha-l, u+\beta-l] = W + D,$$

because $\alpha + \beta = T$. Therefore

$$\sum_{j\in J} K_n(u,j)K_l(v,T-j) = \binom{n}{u}\binom{l}{v}\sum_{t\in W}\binom{u}{t}\binom{v}{c-t},$$
$$\sum_{j\in J} K_n(v,j)K_l(u,T-j) = \binom{n}{v}\binom{l}{u}\sum_{t\in W+D}\binom{u}{t}\binom{v}{c-t}.$$

The left side of (17) is consequently

$$A\sum_{t\in W} w_t - B\sum_{t\in W+D} w_t. \tag{18}$$

The intervals W and $W+D$ are reflections under $t \mapsto c+d-t$. For integer $t \le (c+d)/2$ one has $w_t \ge w_{c+d-t}$. To check this when the latter is positive, its support and the displayed bound give $d \le t \le u$, $t \ge c-v$ and $t \le c$; hence $w_t > 0$ and

$$\frac{w_t}{w_{c+d-t}} = \frac{(c+d-t)!(t-d)!}{t!(c-t)!} = \prod_{h=1}^{d}\frac{c-t+h}{t-d+h} \ge 1.$$

Cancel the intersection of the two intervals and pair their exclusive points by reflection. The points exclusive to W lie to the left of $(c+d)/2$ since $D \ge 0$. Thus $\sum_W w_t \ge \sum_{W+D} w_t$, which proves (18) nonnegative. For $v = -1$ both kernel products are zero. □

Proof of Theorem 4.1. By bilinearity it suffices to consider $s_{(p,q)} \diamond_2 s_{(r,s)}$. Put $P = p+q$, $R = r+s$. Fix $N \in \mathbb{Z}_{\ge 0}$, and let H_n be the coefficient of $x^n y^{N-n}$ in the homogeneous part of degree N of this coordinatewise product. This part is symmetric and nonnegative. With $H_{N+1} = 0$, telescoping gives

$$\sum_{i=0}^{N} H_i x^i y^{N-i} = \sum_{n=\lceil N/2\rceil}^{N} (H_n - H_{n+1}) s_{(n,N-n)}(x,y).$$

Thus it suffices to prove $H_n - H_{n+1} \ge 0$ for $n \ge N-n$. If $n = N$, the difference is $H_N \ge 0$. Otherwise put $l = N-n-1$, so $n \ge l$.

Define the four interval indicators on $\mathbb{Z}$ by

$$A(i) = \mathbf{1}_{q\le i\le p}, \quad A'(i) = \mathbf{1}_{q-1\le i\le p-1}, \qquad B(j) = \mathbf{1}_{s\le j\le r}, \quad B'(j) = \mathbf{1}_{s-1\le j\le r-1}.$$

The coefficient H_n counts a covering pair on n red labels and a covering pair on $l+1$ blue labels, with the first input's red size in $[q,p]$ and the second's red size in $[s,r]$. In H_n remove a distinguished blue label; in H_{n+1} remove a distinguished red label. Each removed label belongs to the first subset only, the second only, or both. The kernel identity

$$K_{a+1}(i,j) = K_a(i-1,j) + K_a(i,j-1) + K_a(i-1,j-1)$$

and an index shift in each red-label term give

$$\begin{aligned} H_n - H_{n+1} = &\sum_{i,j\in\mathbb{Z}} K_n(i,j)K_l(P-1-i,R-j)(A(i)-A'(i))B(j) \\ &+ \sum_{i,j\in\mathbb{Z}} K_n(i,j)K_l(P-i,R-1-j)A(i)(B(j)-B'(j)) \\ &+ \sum_{i,j\in\mathbb{Z}} K_n(i,j)K_l(P-1-i,R-1-j)(A(i)B(j)-A'(i)B'(j)). \end{aligned}$$

All sums are finite under the zero-kernel convention. Since $A - A' = \mathbf{1}_{i=p} - \mathbf{1}_{i=q-1}$, the first line equals

$$\sum_{j=s}^{r} (K_n(p,j)K_l(q-1,R-j) - K_n(q-1,j)K_l(p,R-j)) \geq 0$$

by Lemma 4.2 with $u = p, v = q-1, T = R$ and $J = [s,r] \cap \mathbb{Z}$. The second line equals

$$\sum_{i=q}^{p} (K_n(i,r)K_l(P-i,s-1) - K_n(i,s-1)K_l(P-i,r)) \geq 0,$$

using symmetry of the two input arguments of K and the lemma with $u = r, v = s-1, T = P$ and $J = [q,p] \cap \mathbb{Z}$.

For the third line use

$$AB - A'B' = \tfrac{1}{2}((A-A')(B+B') + (A+A')(B-B')).$$

Here

$$B + B' = \mathbf{1}_{[s-1,r]} + \mathbf{1}_{[s,r-1]}, \qquad A + A' = \mathbf{1}_{[q-1,p]} + \mathbf{1}_{[q,p-1]}.$$

The first pair of intervals is centered at $(R-1)/2$ and the second at $(P-1)/2$. For an interval J in the first pair, its contribution to twice the third line is

$$\sum_{j \in J} (K_n(p,j)K_l(q-1,R-1-j) - K_n(q-1,j)K_l(p,R-1-j)) \geq 0.$$

For an interval I in the second pair, the other contribution is

$$\sum_{i \in I} (K_n(i,r)K_l(P-1-i,s-1) - K_n(i,s-1)K_l(P-1-i,r)) \geq 0.$$

These are the window lemma with $T = R-1$ and $T = P-1$, respectively. If $P = 0$ or $R = 0$, no label can belong to both input subsets and the third line is zero. Otherwise both required values of T are nonnegative. Empty intervals contribute zero; when $q = 0$ or $s = 0$, the relevant $v = -1$ term is zero by the kernel convention. Each of the three sums is nonnegative; hence $H_n - H_{n+1} \geq 0$ for every required index. □

Uppercase letters denote the ordinary generating polynomials of their corresponding rows: $A(x) = \sum_i a_i x^i$, $B(x) = \sum_i b_i x^i$, and likewise for C, D. For finite rows a, b, define the mixed defect, for integers $m \geq n \geq 0$, by

$$\mathcal{D}_{m,n}(a,b) = a_m b_n + a_n b_m - a_{m+1} b_{n-1} - a_{n-1} b_{m+1}. \tag{19}$$

They are *partially synchronous* [9, Definition 3.2] if all these defects are nonnegative. Equivalently $A(x)B(y) + A(y)B(x)$ is Schur-positive: its Schur coefficient at (m,n) is exactly (19). For a nonnegative log-concave interval-supported row c, $C(x)C(y)$ is Schur-positive because $c_m c_n \geq c_{m+1} c_{n-1}$ for $m \geq n$.

Corollary 4.3. *For finite nonnegative rows, multiplication by a common Newton log-concave factor preserves partial synchronicity. If a, b are partially synchronous and c, d are partially synchronous, then*

$$\mathcal{D}_{m,n}(a \diamond c, b \diamond d) + \mathcal{D}_{m,n}(a \diamond d, b \diamond c) \geq 0 \qquad (m \geq n \geq 0).$$

Here the same symbol $\diamond$ denotes the operation on coefficient rows.

Proof. For the first assertion apply Theorem 4.1 to $(A(x)B(y)+A(y)B(x)) \diamond_2 (C(x)C(y))$. For the second apply it to the Schur-positive polynomial

$$\begin{aligned}&(A(x)B(y)+A(y)B(x)) \diamond_2 (C(x)D(y)+C(y)D(x))\\ &= (A\diamond C)(x)(B\diamond D)(y)+(A\diamond C)(y)(B\diamond D)(x)\\ &\quad +(A\diamond D)(x)(B\diamond C)(y)+(A\diamond D)(y)(B\diamond C)(x).\end{aligned}$$

Taking the Schur coefficient at (m,n) gives the stated sum. □

The weighted version of Theorem 4.1 follows too. For homogeneous input degrees P, R and output degree N, the overlap weight multiplies every term by t^{P+R-N}; this is nonnegative for $t \geq 0$, with the zero-overlap convention of Corollary 3.2.

5 Complete wall functions and bipartite quivers

Fix a positive integer m. The rank-two diagram has initial walls with functions $1+x^m$ and $1+y^m$ in the original cluster character coordinates of [6]. An outgoing ray in a primitive positive direction (d,e) has function

$$T_m(z) = 1+\sum_{k\geq 1} \tau_m(kd,ke)z^k, \qquad z = x^{md}y^{me}. \tag{20}$$

For an integer vector $v=(v_1,v_2)$, its crossing action, in one consistent orientation, is $x^{v_1}y^{v_2} \mapsto x^{v_1}y^{v_2}T_m(z)^{-\det((d,e),v)}$. Here $\det((a,b),(c,d)) = ad-bc$. Wall functions are formal series with constant term one; statements about them are interpreted in each finite total-degree quotient, so only finitely many terms occur. An absent outgoing ray has function one.

To distinguish the diagram's parameter from quiver dimensions, let $b, c \geq 1$ be independent integers in the ordered product construction. In the Lie algebra over $\mathbb{Q}$ with basis X^n, $n=(n_1,n_2) \in \mathbb{Z}^2_{\geq 0}\setminus\{0\}$, completed with respect to the total degree n_1+n_2, use

$$[X^n,X^v] = -\det(n,v)X^{n+v}, \qquad L_+(z) = \sum_{r\geq 1} \frac{(-1)^{r+1}}{r^2}z^r, \qquad \Psi[n] = \exp(L_+(X^n)).$$

The exponential denotes an element of the associated Baker–Campbell–Hausdorff group. Its action on Laurent monomials X^v, $v\in\mathbb{Z}^2$, with multiplication $X^nX^v = X^{n+v}$, is the exponential of the adjoint derivation $\mathrm{ad}(H)(f)=[H,f]$. Write $e_1=(1,0)$ and $e_2=(0,1)$. Factor $\Psi[e_2]^b\Psi[e_1]^c$ into decreasing source-fraction slopes $n_1/(n_1+n_2)$, refining each ray into the dilogarithm factors $\Psi[n]^{u_n(c,b)}$. For the outgoing indices $n=(n_1,n_2)$ with $n_1,n_2\geq 1$, ordered factorization and the binomial expansion of its exponents give [2, Theorem 3.19 and Sections 5, 7]

$$u_n(c,b) = \frac{1}{g_n}\sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\alpha_n(p,q)\binom{c}{p}\binom{b}{q}, \qquad g_n=\gcd(n_1,n_2), \quad \alpha_n(p,q)\geq 0. \tag{21}$$

The coefficients $\alpha_n(p,q)$ are rational constants, independent of b,c; this identity defines their normalization. For these positive-coordinate indices, define the diagonal polynomial

$$q_n(m) = \frac{g_n}{m}u_n(m,m) = \frac{1}{m}\sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\alpha_n(p,q)\binom{m}{p}\binom{m}{q}. \tag{22}$$

Each product $\binom{m}{p}\binom{m}{q}$ is divisible by m^2. After division by m, every summand is still divisible by m, so $q_n(0) = 0$. If $n = k(d,e)$ with (d,e) primitive, comparison of the crossing actions gives

$$T_m(z) = \prod_{j\geq 1}(1+z^j)^{q_{j(d,e)}(m)}. \tag{23}$$

Restriction of the original crossing action to $X = x^m, Y = y^m$ multiplies its exponent by m. On each initial wall the same restriction gives $(1+X)^m$ or $(1+Y)^m$; on the ray it gives T_m^m. Restriction preserves every path-ordered product, hence consistency. It also preserves outgoing supports, so uniqueness identifies this restricted diagram with the tropical diagram for the initial functions $(1+X)^m, (1+Y)^m$. Here $z = X^dY^e$. For an integer $j \geq 1$ and $X^v = X^{v_1}Y^{v_2}$, the defining bracket gives

$$[L_+(z^j), X^v] = -j\det((d,e),v)\sum_{r\geq 1}\frac{(-1)^{r+1}}{r}z^{jr}X^v = -j\det((d,e),v)\log(1+z^j)X^v.$$

Since this adjoint derivation fixes z, its powers act by powers of the displayed multiplier. Therefore

$$\exp\big(\mathrm{ad}(u_{j(d,e)}(m,m)L_+(z^j))\big)(X^{v_1}Y^{v_2}) = X^{v_1}Y^{v_2}(1+z^j)^{-ju_{j(d,e)}(m,m)\det((d,e),v)}.$$

Uniqueness of the consistent completion therefore gives exponent $ju_{j(d,e)}(m,m)/m = q_{j(d,e)}$ in the original wall. This proves (23) for integer $m > 0$. Every coefficient of z on its right is a polynomial in m, so the identity extends coefficientwise to the polynomial parameter in (3).

We next describe the geometric meaning of α used for its support. For $p, q \geq 1$ let $K_{p,q}$ be the complete bipartite quiver with labeled source vertices $1,\dots,p$, labeled sink vertices $1,\dots,q$, and one arrow from each source to each sink. A representation assigns finite-dimensional complex vector spaces to vertices and complex linear maps to arrows. For integers $0 \leq a \leq A$, write $\mathrm{Gr}(a,A)$ for the Grassmannian of a-dimensional linear subspaces of $\mathbb{C}^A$. For a dimension vector $\mathbf{d} = (a_1,\dots,a_p;b_1,\dots,b_q)$ with entries in $\mathbb{Z}_{\geq 0}$, at least one nonzero, and another dimension vector $\mathbf{f}$ on the same vertices, put

$$A = \sum_i a_i, \quad B = \sum_j b_j, \qquad \mu(\mathbf{d}) = \frac{A}{A+B}, \qquad \chi(\mathbf{d},\mathbf{f}) = \sum_v d_v f_v - A_{\mathbf{d}}B_{\mathbf{f}}.$$

Subscripts on A, B specify the dimension vector whose source or sink total is taken. A nonzero representation is semistable if every nonzero subrepresentation has slope at most its own, and stable if this is strict for every proper nonzero subrepresentation. The corresponding moduli spaces of semistable and stable representations are denoted $M_{\mathbf{d}}^{ss}$ and $M_{\mathbf{d}}^{st}$. The representation space is irreducible, and its nonempty semistable open subset has an irreducible GIT quotient $M_{\mathbf{d}}^{ss}$. For this acyclic quiver the quotient is projective. If $M_{\mathbf{d}}^{st}$ is nonempty, it is an open dense smooth subset of complex dimension $1-\chi(\mathbf{d},\mathbf{d})$. A dimension vector has *full support* if its dimension is positive at every vertex. Write $\mathrm{IH}^i(M;\mathbb{Q})$ for the rational intersection cohomology of M in the usual unshifted degree i. Define

$$I_{\mathbf{d}}(t) = \sum_{r\geq 0}\dim_{\mathbb{Q}}\mathrm{IH}^{2r}(M_{\mathbf{d}}^{ss};\mathbb{Q})t^r, \qquad D_{\mathbf{d}} = I_{\mathbf{d}}(1), \tag{24}$$

when the stable locus is nonempty, and define both to be zero otherwise. The moduli are projective and their intersection cohomology is even in this setting [10]. Thus $D_{\mathbf{d}} > 0$ exactly when a stable representation exists. This convention specifies the stable-closure invariant even when a semistable moduli space exists without stable points.

Proposition 5.1. *For $n_1, n_2 \geq 1$, $1 \leq p \leq n_1$ and $1 \leq q \leq n_2$, the coefficients in (21) satisfy*

$$\frac{\alpha_n(p,q)}{g_n} = \sum_{\substack{a_i, b_j \geq 1 \\ \sum_i a_i = n_1,\ \sum_j b_j = n_2}} D_{(a;b)} + \sum_{\substack{s \geq 1 \\ 2^{s+1} \mid g_n}} 2^{-s} \sum_{\substack{a_i, b_j \geq 1 \\ \sum_i a_i = n_1/2^s,\ \sum_j b_j = n_2/2^s}} D_{(a;b)}. \tag{25}$$

In each inner sum there are exactly p source entries and q sink entries, all labeled.

Proof. Write

$$\langle \mathbf{d}, \mathbf{f} \rangle = \chi(\mathbf{d}, \mathbf{f}) - \chi(\mathbf{f}, \mathbf{d}) = B_{\mathbf{d}} A_{\mathbf{f}} - A_{\mathbf{d}} B_{\mathbf{f}}$$

for the antisymmetric Euler form. It vanishes for vectors of the same slope, so the stability is generic in the sense of [10, Section 2.2], without a coprimality condition. The intersection-cohomology identification [10, Theorem 4.6 and Corollary 1.2] and the plethystic factorization [7, Theorems 3.1–3.2] apply.

Let w be a formal cohomological weight variable and let $t^{\mathbf{d}}$ denote a formal dimension-vector monomial, with $t^{\mathbf{d}} t^{\mathbf{f}} = t^{\mathbf{d}+\mathbf{f}}$ before introducing the quantum product. In each fixed-slope sum below, $\mathbf{d}$ ranges over nonzero nonnegative integer dimension vectors of that slope. Franzen–Reineke's quantum DT polynomials $\widetilde{\Omega}_{\mathbf{d}}(w)$ are characterized, at each fixed slope, by the normalized semistable series

$$\operatorname{Exp}\left(\frac{1}{1-w} \sum_{\mathbf{d}} (-w^{1/2})^{\chi(\mathbf{d},\mathbf{d})} \widetilde{\Omega}_{\mathbf{d}}(w) t^{\mathbf{d}} \right). \tag{26}$$

Here $\operatorname{Exp}(f) = \exp(\sum_{r \geq 1} \psi_r(f)/r)$; the Adams operation ψ_r replaces $w^{1/2}$ by $w^{r/2}$ and $t^{\mathbf{d}}$ by $t^{r\mathbf{d}}$ and preserves rational scalars. The quantum product is

$$t^{\mathbf{d}} \circ t^{\mathbf{f}} = (-w^{-1/2})^{\langle \mathbf{d}, \mathbf{f} \rangle} t^{\mathbf{d}+\mathbf{f}}.$$

At one slope the exponent is zero, so the plethystic expression uses ordinary commutative multiplication.

For a complex variety Z, write $[Z]$ for its motivic class, as in [10]. We work in the Grothendieck ring localized by $\mathbb{L} = [\mathbb{A}^1]$ and all $\mathbb{L}^r - 1$, $r \geq 1$. Adjoin its odd square root, characterized by

$$\psi_r(\mathbb{L}^{1/2}) = (-1)^{r-1} \mathbb{L}^{r/2}.$$

Its realization $\mathbb{L}^{1/2} \mapsto -w^{-1/2}$ respects Adams operations. Here $\operatorname{Hom}(U, V)$ denotes complex linear maps, and $GL_d = GL_d(\mathbb{C})$. For the representation space $R_{\mathbf{d}} = \prod_{i,j} \operatorname{Hom}(\mathbb{C}^{a_i}, \mathbb{C}^{b_j})$ and the change-of-basis group $G_{\mathbf{d}} = \prod_v GL_{d_v}$, the normalized total-stack coefficient is

$$\mathbb{L}^{\chi(\mathbf{d},\mathbf{d})/2} \frac{[R_{\mathbf{d}}]}{[G_{\mathbf{d}}]} = \mathbb{L}^{-\chi(\mathbf{d},\mathbf{d})/2} \prod_v \prod_{\nu=1}^{d_v} (1 - \mathbb{L}^{-\nu})^{-1}.$$

Its realization is the total-series coefficient of [7]. Hall multiplication with subobject $\mathbf{d}$ on the left and quotient $\mathbf{f}$ on the right contributes $\mathbb{L}^{-\chi(\mathbf{f},\mathbf{d})}$. After normalization its twist is $\mathbb{L}^{\langle \mathbf{d}, \mathbf{f} \rangle/2}$, realizing the quantum product above. The equal total series and Harder–Narasimhan factorization therefore identify the semistable series at each slope.

When the stable locus is nonempty, put $h = 1 - \chi(\mathbf{d}, \mathbf{d})$. Meinhardt–Reineke identify the normalized motivic invariant with $\mathbb{L}^{-h/2} I_{\mathbf{d}}(\mathbb{L})$, entering its plethystic logarithm divided by $\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}$. Under the realization this becomes

$$\frac{(-1)^{h+1} w^{(h+1)/2} I_{\mathbf{d}}(w^{-1})}{1-w} = \frac{(-1)^{\chi(\mathbf{d},\mathbf{d})} w^{\chi(\mathbf{d},\mathbf{d})/2} I_{\mathbf{d}}(w)}{1-w},$$

where the equality uses $I_{\mathbf{d}}(w^{-1}) = w^{-h} I_{\mathbf{d}}(w)$. Uniqueness of the plethystic logarithm in (26) gives $\widetilde{\Omega}_{\mathbf{d}}(w) = I_{\mathbf{d}}(w)$. When the stable locus is empty, both invariants are zero by the stable-closure statement of [10].

Change monomials by $t^{\mathbf{d}} = (-1)^{\chi(\mathbf{d},\mathbf{d})} X^{\mathbf{d}}$ and set $w = e^{-\eta}$ for a formal variable η. The parity identity

$$\chi(\mathbf{d}+\mathbf{f},\mathbf{d}+\mathbf{f}) - \chi(\mathbf{d},\mathbf{d}) - \chi(\mathbf{f},\mathbf{f}) \equiv \langle \mathbf{d},\mathbf{f}\rangle \pmod 2$$

turns the quantum product into

$$X^{\mathbf{d}} \star X^{\mathbf{f}} = e^{\eta\langle \mathbf{d},\mathbf{f}\rangle/2} X^{\mathbf{d}+\mathbf{f}}.$$

Let I_η be its total-dimension-completed augmentation ideal with coefficients in $\mathbb{Q}[[\eta]]$. The commutator of two monomials is

$$[X^{\mathbf{d}}, X^{\mathbf{f}}]_\star = (e^{\eta\langle \mathbf{d},\mathbf{f}\rangle/2} - e^{-\eta\langle \mathbf{d},\mathbf{f}\rangle/2}) X^{\mathbf{d}+\mathbf{f}},$$

which is divisible by η, with quotient at $\eta = 0$ equal to $\langle \mathbf{d},\mathbf{f}\rangle X^{\mathbf{d}+\mathbf{f}}$. Thus $\eta^{-1} I_\eta$ is closed under the quantum commutator, and $\eta^{-1} f \mapsto f|_{\eta=0}$ is a homomorphism to the classical Lie algebra with this bracket. In every finite total-dimension quotient the augmentation ideal is nilpotent, so the finite Baker–Campbell–Hausdorff formula gives the corresponding group homomorphism.

For a fixed slope, expand the logarithm in (26). The contribution of a fixed $\mathbf{d}$ and Adams index r to the coefficient of $X^{r\mathbf{d}}$ in η times that logarithm is

$$\frac{\eta}{r(1-e^{-r\eta})} (-1)^{(r+1)\chi(\mathbf{d},\mathbf{d})} e^{-r\eta\chi(\mathbf{d},\mathbf{d})/2} I_{\mathbf{d}}(e^{-r\eta}).$$

This is regular at $\eta = 0$ and has value $D_{\mathbf{d}}(-1)^{(r+1)\chi(\mathbf{d},\mathbf{d})}/r^2$. The sign uses $\psi_r(-a) = -\psi_r(a)$ for a coefficient's additive inverse, together with $\chi(r\mathbf{d}, r\mathbf{d}) = r^2\chi(\mathbf{d},\mathbf{d})$. Consequently the classical contribution of $\mathbf{d}$ is

$$D_{\mathbf{d}} \sum_{r\geq 1} \frac{(-1)^{(r+1)\chi(\mathbf{d},\mathbf{d})}}{r^2} X^{r\mathbf{d}}.$$

Put $L_-(z) = \sum_{r\geq 1} z^r/r^2$. The contribution is $D_{\mathbf{d}} L_+(X^{\mathbf{d}})$ for odd Euler square, and $D_{\mathbf{d}} L_-(X^{\mathbf{d}})$ for even Euler square. Use the formal identity

$$L_-(z) = \sum_{s\geq 0} 2^{-s} L_+(z^{2^s}). \tag{27}$$

For an integer $h \geq 1$, write $h = 2^v u$ with $v \in \mathbb{Z}_{\geq 0}$ and u a positive odd integer. The coefficient of z^h on the right is

$$\frac{1}{h^2} \sum_{s=0}^{v} 2^s (-1)^{h/2^s+1} = \frac{-\sum_{s=0}^{v-1} 2^s + 2^v}{h^2} = \frac{1}{h^2},$$

where the sum to $v-1$ is empty when $v = 0$.

Projection $\mathbf{d} \mapsto (A, B)$ preserves the antisymmetric form and has finite fibers in each degree; hence it defines a homomorphism of the completed Lie algebras. The total Hall element is the product of the sink-supported and source-supported elements, since every representation has its full sink space as a subrepresentation with source-supported quotient. For a source-only or sink-only dimension vector all arrows vanish. The stable objects are exactly the one-dimensional vertex simples: a higher-dimensional vertex space or two nonzero vertices admits a proper subrepresentation of equal slope. Each simple has dimension vector

$\mathbf{d}$ equal to one at its vertex and zero elsewhere. It has Euler square one and a one-point moduli space, hence contributes $L_+(X^{\mathbf{d}})$. The source simples commute with one another, as do the sink simples. Projection of the sink-supported and source-supported elements is therefore $\Psi[e_2]^b$ and $\Psi[e_1]^c$ for ambient palettes of sizes c, b. Uniqueness of ordered factorization identifies the projected slope factors with those defining $u_n(c, b)$. Inside a ray, the series $L_+(z^j)$ are triangularly independent in j, so their exponents are identified as well.

Finally $\chi(\mathbf{d}, \mathbf{d}) \equiv A + B - AB \pmod 2$, which is even exactly when both A and B are even. The sth extra term of (27) can therefore contribute at total n exactly when $2^{s+1} \mid g_n$. Deleting zero-dimensional vertices leaves a full-support representation on p chosen sources and q chosen sinks. Its placements in the ambient palettes number $\binom{c}{p}\binom{b}{q}$. Comparing with (21) proves (25). The labels account for every positive dimension allocation once. □

6 Interval support and its first index

Lemma 6.1. *For complete bipartite quivers with source–sink slope as above, splitting a vertex into two vertices of positive dimensions with the same neighbors preserves stability. Moreover, if $E_1, \dots, E_t$ are pairwise nonisomorphic stable representations of the same full-support dimension vector, with $t \geq 2$, then a suitable splitting of any one vertex of their direct sum makes that sum stable.*

Proof. Write the old vertex space as $V = V_1 \oplus V_2$. At a source restrict every incident arrow, and at a sink take its two components. A subrepresentation of the split representation merges to a subrepresentation of the old one, with the same source and sink totals. This proves the first assertion.

For the second, the equal-slope semistable category is abelian, and its stable objects are simple. Here morphisms and endomorphisms are taken in the category of quiver representations. Since $\mathrm{Hom}(E_i, E_j) = 0$ for $i \neq j$ and $\mathrm{End}(E_i) = \mathbb{C}$, their direct sum is semisimple in the category of semistable representations of that slope and has endomorphism ring $\mathbb{C}^t$. Its subobjects in this fixed-slope category are the images of the coordinate idempotents, hence are exactly the finitely many direct sums indexed by subsets of $\{1, \dots, t\}$, including their embedded vertex subspaces. At the chosen vertex each nonempty proper subset gives a nonzero proper subspace $U \subset V$ because the dimension vector has full support. Splittings with fixed dimensions $a, N - a$, $1 \leq a < N = \dim V$, are parameterized by the irreducible variety $GL_N/(GL_a \times GL_{N-a})$ of rank-a idempotents P. Compatibility with U is the closed condition $P(U) \subset U$, and it is proper. Indeed choose $0 \neq u \in U$, $w \notin U$, prescribe $P(u) = u + w$, $P(w) = 0$ on their two-dimensional span, and complete with $a - 1$ identity directions on a complement. This is an idempotent of rank a not preserving U. Avoiding the finite union of these proper closed sets eliminates every proper equal-slope subrepresentation. A subrepresentation of larger slope would merge to one of the original semistable sum, which is impossible. The split representation is stable. □

Lemma 6.2. *For fixed $n = (n_1, n_2)$ with $n_1, n_2 \geq 1$, the positive support of $\alpha_n(p, q)$ in $1 \leq p \leq n_1$, $1 \leq q \leq n_2$ is upward closed in each coordinate, and contains (n_1, n_2).*

Proof. By (25), a positive coefficient has a positive stable contribution. For a direct contribution of total dimension n, if $p < n_1$ some source dimension is at least two. Split that vertex and apply Lemma 6.1. The result contributes at $(p + 1, q)$; the sink argument is identical. For a binary contribution at level $s \geq 1$, its base has even Euler square and a nonempty stable moduli space. Its dimension $1 - \chi(\mathbf{d}, \mathbf{d})$ is a positive odd integer, so it contains arbitrarily many distinct complex stable points. Choose 2^s such points. Their direct sum has total dimension n, and Lemma 6.1 supplies a stable split representation contributing to the

direct term at the next palette size. At the top corner take every vertex dimension equal to one and every arrow nonzero. A subrepresentation containing a source must contain all sinks; its source fraction is strictly smaller unless it is the whole representation. Thus this representation is stable. □

Lemma 6.3. *For every $n = (n_1, n_2)$ with $n_1, n_2 \geq 1$, q_n has nonnegative integer Newton coefficients and positive support*

$$[s_n, n_1 + n_2 - 1] \cap \mathbb{Z}, \qquad 1 \leq s_n \leq \max(n_1, n_2).$$

Proof. For $p, q \geq 1$, union counting on two subsets after removing a distinguished color gives

$$\frac{1}{m}\binom{m}{p}\binom{m}{q} = \sum_{r=\max(p,q)}^{p+q-1} \frac{r!}{pq(r-p)!(r-q)!(p+q-r-1)!}\binom{m}{r}. \tag{28}$$

One can obtain the formula directly from $\frac{m}{pq}\binom{m-1}{p-1}\binom{m-1}{q-1}$ and (6), using $m\binom{m-1}{r-1} = r\binom{m}{r}$. Each indicated coefficient is positive. Along a step $(p, q) \mapsto (p+1, q)$, the two positive intervals in (28) intersect or are adjacent, since $\max(p+1, q) \leq p+q$. Lemma 6.2 connects every positive palette to the same top corner. Their union is an interval ending at $n_1 + n_2 - 1$ and starting at most at $\max(n_1, n_2)$. Equation (22) proves coefficient nonnegativity and interval support.

For integrality, at a positive integer m use the lattice $N^\circ = m\mathbb{Z}^2$ in the negative-determinant convention of [2, Definition 3.2 and Theorem 3.19]. Multiplying n by the positive rational number m/g_n gives the primitive point of N° on this ray. That theorem gives $u_n(m, m) \in (m/g_n)\mathbb{Z}_{\geq 0}$, so $q_n(m) \in \mathbb{Z}_{\geq 0}$. Its polynomial value at zero is zero by (22). All finite differences at zero are therefore integers. Combined with the already proved Newton nonnegativity, this proves the assertion. □

Lemma 6.4. *Suppose $f(m) = \sum_{r=s}^{D} c_r\binom{m}{r}$ with $D \geq s \geq 1$ and $c_r \in \mathbb{Z}_{>0}$. For $k = 0$, $\binom{f(m)}{0} = 1$ has support $\{0\}$. For every integer $k \geq 1$, $\binom{f(m)}{k}$ has nonnegative integer Newton coefficients and positive support*

$$[\min\{b \in \mathbb{Z}_{\geq 0} : f(b) \geq k\}, kD] \cap \mathbb{Z}.$$

Proof. On a palette $[b] = \{1, \ldots, b\}$, an object consists of a subset of size r and one of c_r decorations. Then $f(b)$ counts objects, and $\binom{f(b)}{k}$ counts unordered sets of k distinct objects. Group them by the union of their subsets: the Newton coefficient at index h counts such sets using all h colors. This proves integer nonnegativity. Let b_0 be the first palette size with at least k objects. A choice there uses every color, since otherwise relabeling its union contradicts minimality. Now start with any valid choice using $h < kD$ colors. If a color occurs in two objects, replace it in one of them by a new color, retaining its decoration. The old color remains in use, and the new object is distinct from all unchanged objects. Otherwise the subsets are disjoint and one has size less than D. Enlarge it by a new color and choose any decoration of the next size, which exists by the full interval of sizes. Both operations increase the union size by exactly one. No choice uses more than kD colors, and k disjoint size-D subsets attain that bound. □

Proposition 6.5. *For every $A, B \geq 1$, the positive Newton support of $\tau_m(A, B)$ is an interval ending at $A + B - 1$.*

Proof. Write $(A,B) = k(d,e)$ with (d,e) primitive. For $\ell = (\ell_1, \dots, \ell_k) \in \mathbb{Z}^k_{\geq 0}$, equation (23) gives

$$\tau_m(kd, ke) = \sum_{\substack{\ell \in \mathbb{Z}^k_{\geq 0} \\ \sum_{j=1}^k j\ell_j = k}} H_\ell(m), \qquad H_\ell(m) = \prod_{j=1}^k \binom{q_{j(d,e)}(m)}{\ell_j}. \tag{29}$$

Lemmas 6.3 and 6.4 give interval support and nonnegative Newton coefficients for every nonconstant factor. Products of polynomials with nonnegative interval-supported Newton rows have interval support by (6): the union of the contributing intervals $[\max(i,j), i+j]$ is the interval with endpoints $\max(L_1, L_2)$ and $U_1 + U_2$. Constant factors have support $\{0\}$. Thus every partition term has interval support and degree

$$T_\ell = \sum_j \ell_j(j(d+e) - 1) = k(d+e) - \sum_j \ell_j.$$

The direct term $\ell_k = 1$ is $q_{k(d,e)}$, whose interval is $[s_{k(d,e)}, k(d+e)-1]$. Since $1 \leq \sum_j \ell_j \leq k$ and $d + e - 1 \geq \max(d,e)$,

$$s_{k(d,e)} \leq k\max(d,e) \leq k(d+e-1) \leq T_\ell \leq k(d+e) - 1.$$

Every partition interval therefore meets the direct interval at its own top index T_ℓ. Their union is an interval ending at $k(d+e)-1 = A+B-1$, and their coefficients cannot cancel. □

Proposition 6.6. *For integers $m, A, B \geq 1$,*

$$\tau_m(A,B) > 0 \quad \Longleftrightarrow \quad A^2 + B^2 - mAB \leq 1. \tag{30}$$

Proof. Use the m-arrow Kronecker quiver and dimension vector (A,B), with the same source-fraction stability. Write $\chi_m(A,B) = A^2 + B^2 - mAB$. The stable-closure invariant $D_{(A,B)}$ is defined as in (24); it is positive precisely when the stable locus is nonempty. A nonempty stable locus has dimension $1 - \chi_m(A,B)$, so $\chi_m \leq 1$ is necessary for stability.

If $\chi_m(A,B) < 0$, fix a proper nonzero destabilizing subdimension (a,c) with $Ba - Ac \geq 0$. Here $a > 0$ and $c < B$: otherwise $Ba - Ac \geq 0$ would force either the zero or the full subdimension. The incidence variety over $\mathrm{Gr}(a,A) \times \mathrm{Gr}(c,B)$ of arrows preserving these subspaces has codimension relative to the representation space

$$\begin{aligned} ma(B-c) - a(A-a) - c(B-c) = \frac{Ba - Ac}{AB}(Aa + B(B-c)) \\ - \chi_m(A,B)\frac{a(B-c)}{AB} > 0. \end{aligned}$$

Its projection is closed because the Grassmannians are proper, and proper because of this dimension inequality. The finitely many destabilizing subdimensions yield a finite union of proper closed subsets of the irreducible affine representation space. Its complement consists of stable representations.

If $\chi_m(A,B) = 1$, for $m = 1$ the only positive solution is $(1,1)$, realized by a nonzero arrow. For $m \geq 2$, dualize every arrow and exchange source and sink. For a subspace of a dual vector space, $\perp$ denotes its annihilator in the original space. If (W,Z) is a proper nonzero subrepresentation of the dual representation of dimensions (B,A), its annihilators $(Z^\perp, W^\perp)$ form a proper nonzero subrepresentation of the original one. The original inequality $B\dim Z^\perp - A\dim W^\perp < 0$ becomes $A\dim W - B\dim Z < 0$. Duality therefore preserves stability and permits $A \leq B$. Put $A' = mA - B$. The equation $B(mA - B) = A^2 - 1$ gives $A' \geq 0$ and $\chi_m(A', A) = 1$. Also $A' < A$: for $m = 2$ one has $B = A + 1$; for $m \geq 3$,

$A \le B \le (m-1)A$ would give $B(mA-B) \ge (m-1)A^2 > A^2-1$. Iterating decreases total dimension until the first coordinate is zero. The equation $\chi_m(0,A)=1$ then gives $A=1$, so the terminal vector is $(0,1)$, after exchanging coordinates when needed.

Reverse a step as follows. For a stable representation $U' \to V'$ of dimensions (A', A), the joint arrow map $U' \to (V')^m$ is injective, since a nonzero common kernel violates stability. Identify U' with its image in $(V')^m$. Set $U = V'$, $V = (V')^m/U'$ and take the m coordinate inclusions followed by the quotient. For a subrepresentation (W, Z) with $0 < a = \dim W < A$, let $d = \dim(U' \cap W^m)$. Old stability gives $Ad - A'a < 0$ when $A' > 0$; if $d = 0$ this is immediate. The endpoint $A' = 0$ has $A = 1$ and no such a. Since the new images span a space of dimension $ma-d$, we have $\dim Z \ge ma-d > (mA-A')a/A = Ba/A$. Subrepresentations with zero source have smaller slope, while a full source forces the full sink. Thus the new representation is stable. At the terminal step $U' = 0$ and $V' = \mathbb{C}$, so the reverse construction is the $(1, m)$ representation with the m coordinate arrows $\mathbb{C} \to \mathbb{C}^m$. Reversing the descent proves existence at every positive real root. Its zero-dimensional irreducible moduli space is one point, so $D_{(A,B)} = 1$.

If $\chi_m(A, B) = 0$, the rational root B/A of $x^2 - mx + 1$ forces $m^2 - 4$ to be an integer square. Factoring $(m-h)(m+h) = 4$ gives $m = 2$ and $A = B$. For dimension $(1, 1)$ the stable moduli space is $\mathbb{P}^1$, so $D_{(1,1)} = 2$. For (k, k) with $k > 1$, a nonempty stable open set would meet the dense open set where the first arrow is invertible. Normalize it to the identity. An eigenline of the second arrow gives a proper equal-slope subrepresentation, a contradiction. Thus $D_{(k,k)} = 0$ for $k > 1$.

The Hall normalization used in Proposition 5.1 applies also to this quiver, now with classical bracket $[X^u, X^v] = -m \det(u, v) X^{u+v}$. Its Lie isomorphism to the bracket without m sends X^n to mX^n; hence the initial word becomes $\Psi[e_2]^m \Psi[e_1]^m$. The odd and even Euler-square contributions are $mD_nL_+(X^n)$ and $mD_nL_-(X^n)$. Comparison with the original crossing action, as in (23), divides the tropical wall exponents by m. Therefore along a primitive ray $n_0 = (d, e)$ the complete original wall is

$$T_m(z) = \prod_{\substack{j\ge 1 \\ \chi_m(jn_0) \text{ odd}}} (1+z^j)^{jD_{jn_0}} \prod_{\substack{j\ge 1 \\ \chi_m(jn_0) \text{ even}}} (1-z^j)^{-jD_{jn_0}}. \tag{31}$$

All these formal factors have nonnegative coefficients. If $\chi_m(n_0) > 1$, every invariant vanishes and the wall is one. If $\chi_m(n_0) = 1$, the wall is $1 + z$. If $\chi_m(n_0) = 0$, it is $(1-z)^{-2}$ with all coefficients positive. If $\chi_m(n_0) < 0$, every $D_{jn_0} > 0$, and the factor at $j = k$ supplies a positive coefficient at z^k. Since $\chi_m(kn_0) = k^2\chi_m(n_0)$, these four cases prove (30). □

Proof of Theorem 1.3. For $(A, B) \ne (1, 1)$, $A^2 + B^2 - AB = (A-B)^2 + AB > 1$; Proposition 6.6 gives $\tau_1(A, B) = 0$. Newton nonnegativity in (3) forces $a_0 = a_1 = 0$. For $(1, 1)$ the m-arrow stable moduli space is $\mathbb{P}^{m-1}$, so the first ray coefficient in (31) is m; polynomial continuation gives $\tau_m(1, 1) = m$ and again $a_0 = 0$. For a polynomial with nonnegative Newton coefficients the first index with a positive Newton coefficient equals the first integer argument with positive value. By Proposition 6.6 it is exactly $\lceil (A^2 + B^2 - 1)/(AB) \rceil$. Proposition 6.5 supplies every index up to $A + B - 1$. The lower endpoint is at least one because $A^2 + B^2 - 1 - AB = (A-B)^2 + AB - 1 \ge 0$. For its upper bound,

$$AB(A+B-1) - (A^2+B^2-1) = (A-1)(B-1)(A+B+1) \ge 0. \qquad \square$$

7 Conditional log-concavity of complete Euler sums

Fix a primitive positive pair (d, e) and an integer charge $k \ge 1$. If each $q_{j(d,e)}$, $1 \le j \le k$, is Newton log-concave, Lemma 6.3 and Theorem 1.2 show that every partition term H_ℓ in (29) is Newton log-concave.

Proposition 7.1. *Let $h^{(1)}, \dots, h^{(t)}$ be finite nonnegative log-concave rows. If for every integer j,*

$$\sum_{\nu<\mu} (2h_j^{(\nu)} h_j^{(\mu)} - h_{j-1}^{(\nu)} h_{j+1}^{(\mu)} - h_{j-1}^{(\mu)} h_{j+1}^{(\nu)}) \geq 0, \tag{32}$$

their sum is log-concave.

Proof. For $a_j = \sum_\nu h_j^{(\nu)}$, expand $a_j^2 - a_{j-1}a_{j+1}$. The diagonal terms are the nonnegative individual log-concavity defects, and the remaining terms sum to (32). □

By Proposition 6.5, these two conditions imply Conjecture 1.1; establishing them for actual walls remains open. Corollary 4.3 supplies a collective comparison when both product pairings occur with their specified coefficients; identifying such pairings in the complete wall sum remains necessary for that application.

A Partial log-concavity results for scattering coefficients

The following statements concern the coefficients in (3). Proofs are in the companion notes [13], indexed by document title and section in SOURCE_GUIDE.md. Coefficient and dimension indices are integers; logarithms are natural.

A.1 A prefix uniform in the integer slope

There exists an absolute integer $A_\star \geq 3$ such that, for all integers $A \geq A_\star$ and $r \geq 8$, setting $B = rA + 1$ gives

$$a_0(A,B) = \cdots = a_r(A,B) = 0, \qquad a_{r+1}(A,B) > 0,$$

and

$$a_j(A,B)^2 > a_{j-1}(A,B)a_{j+1}(A,B) > 0 \quad \left(r+2 \leq j \leq r + \left\lfloor \frac{A}{96 \log A} \right\rfloor - 1\right).$$

No numerical value of $A_\star$ is specified. The source is [13, Uniform slope prefix, Sections 1–7], using its Euler/MPS identity and uniform complex saddle estimate.

A.2 A whole row at a fixed integer slope

For every fixed integer $r \geq 8$ there exists an integer $A_0(r) \geq 1$ such that, for all $A \geq A_0(r)$, the entire Newton row at $(A, rA+1)$ is nonnegative, log-concave and unimodal, with strict log-concavity at every index whose two neighbors belong to its positive support. Combine Appendices A.1 and A.4; see [13, Primitive bulk, Section 5].

A.3 A whole row with residue two

For every integer $A \geq 3$ there exists an integer $K_0(A) \geq 1$ such that, for every integer $k \geq K_0(A)$, the Newton row at $(A, Ak-2)$ is nonnegative, log-concave and unimodal. Its positive support is

$$[k, Ak + A - 3] \cap \mathbb{Z},$$

and its internal log-concavity is strict. For even A this includes the nonprimitive case $\gcd(A, Ak-2) = 2$. No numerical value of $K_0(A)$ is specified. The proof combines the charge-two polynomial positivity in [13, Residue-two polynomial, Sections 1–5], the fixed-surplus comparisons of [13, Residue-two bridge, Sections 3–7] with the three uniform high-index ranges in [13, Residue-two splice, Sections 2–5]. The remaining head contains only finitely many surplus indices for each fixed A, so their thresholds have a common maximum.

A.4 Primitive coefficients at bounded aspect ratio

For every real $\rho \geq 1$ there exists an integer $D_0(\rho) \geq 1$ such that if $A, B \geq 2$ are coprime,

$$\frac{\max(A,B)}{\min(A,B)} \leq \rho, \qquad D = A + B - 1 \geq D_0(\rho),$$

then

$$a_j(A,B)^2 > a_{j-1}(A,B)a_{j+1}(A,B) > 0 \qquad \left(80(\rho+1)\sqrt{D} + 1 \leq j \leq D - 1\right).$$

The source is [13, Primitive bulk, Sections 2–5]; its inputs include the primitive Euler/MPS formula, the positive-block expansion and the uniform estimates used to join the three index ranges.

A.5 A high-index estimate without coprimality

Fix integers $A \geq 2$ and $0 \leq c < A$, and define

$$C_A = 2^{16A} A^{12A}.$$

The high-index estimate recorded in [13, General high-tail review, Sections 1–6], with the complete framed-wall and localization identities specified there, asserts the following. For an integer

$$k \geq \max\{2A, (64C_A)^2\},$$

put $B = Ak - c$ and $D = A + B - 1$. For every integer s with $2A \leq s \leq D - k - 1$, the three coefficients $a_{k+s-1}(A,B)$, $a_{k+s}(A,B)$ and $a_{k+s+1}(A,B)$ are positive and satisfy

$$\log \frac{a_{k+s}(A,B)^2}{a_{k+s-1}(A,B)a_{k+s+1}(A,B)} \geq \frac{1}{8s} > 0.$$

There is no coprimality restriction on A, B.

B Sufficient criteria for Newton log-concavity

The first two criteria are proved in the companion notes [13]; the third follows from Theorem 1.2.

B.1 Fixed-degree hook coordinates

For an integer $k \geq 1$, put $(m)_k = m(m-1)\cdots(m-k+1)$. Fix $n \geq 1$. If $d_1, \ldots, d_n$ is a nonnegative real log-concave sequence without internal zeros, then

$$\sum_{k=1}^{n} d_k m^{n-k} (m)_k$$

is Newton log-concave. For a nonzero input whose first positive index is s, the output's positive Newton support is $[s, n] \cap \mathbb{Z}$. See [13, Hook kernel, Sections 1–4].

B.2 A region for the remaining zeros

Let $f \in \mathbb{R}[m]$ have positive leading coefficient and be divisible by $(m)_s$ for an integer $s \geq 1$. If every complex zero ζ of $f/(m)_s$, counted with multiplicity, satisfies

$$\Re\zeta \leq s, \qquad (\Im\zeta)^2 \leq (s + 1 - \Re\zeta)^2 - \tfrac{1}{2},$$

then f is Newton log-concave. See [13, Root-region criterion, Sections 1–3]. For the original wall polynomials, the corresponding universal hook-coordinate and root-region hypotheses remain open.

B.3 Positive Euler factors

Fix a primitive positive pair (d, e) and a charge $k \geq 1$. If every $q_{j(d,e)}$ with $1 \leq j \leq k$ in (23) is Newton log-concave, then every partition contribution H_ℓ in (29) is Newton log-concave with nonnegative integer Newton coefficients. Apply Theorem 1.2; see Section 7. Together with the collective comparison in Proposition 7.1, these hypotheses imply log-concavity of the complete wall coefficient.

References

[1] M. Aguiar, W. Ferrer Santos and W. Moreira, *The Heisenberg product of symmetric functions and related algebraic structures*, author manuscript, https://pi.math.cornell.edu/~maguiar/heisenberg.pdf.

[2] R. Akagi, *Explicit forms in lower degrees of rank 2 cluster scattering diagrams*, arXiv:2309.15470v3 (2024).

[3] R. Akagi, *Some coefficients of rank 2 cluster scattering diagrams*, arXiv:2507.21916v3 (2025).

[4] P. Brändén, *Unimodality, log-concavity, real-rootedness and beyond*, arXiv:1410.6601.

[5] A. Burcroff, K. Lee and L. Mou, *Positivity of generalized cluster scattering diagrams*, arXiv:2503.03719v1 (2025).

[6] T. Elgin, N. Reading and S. Stella, *Cluster scattering coefficients in rank 2*, arXiv:2404.11381v4 (2025).

[7] H. Franzen and M. Reineke, *Semi-stable Chow–Hall algebras of quivers and quantized Donaldson–Thomas invariants*, arXiv:1512.03748v2 (2016).

[8] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), 497–608.

[9] H. Hu, D. G. L. Wang, F. Zhao and T. Y. Zhao, *Convolution preserves partial synchronicity of log-concave sequences*, arXiv:1507.08430v2.

[10] S. Meinhardt and M. Reineke, *Donaldson–Thomas invariants versus intersection cohomology of quiver moduli*, arXiv:1411.4062v2 (2016).

[11] Y. Liu and J. Mao, *Preservation of log-concavity under Hadamard products*, arXiv:2609.11589v1 (2026).

[12] M. Reineke, *Wild quantum dilogarithm identities*, Annals of Representation Theory **1** (2024), 385–391, doi:10.5802/art.12.

[13] Q. Tang, *Supplementary research notes on Newton coefficients and rank-two scattering*, companion proof files and source-path index, version 0.1.0 (2026), https://github.com/tqyqiqi1509/c12-newton-research-archive, archival release of 3 November 2026.

[14] D. W. Walkup, *Pólya sequences, binomial convolution and the union of random sets*, J. Appl. Probab. **13** (1976), 76–85, doi:10.2307/3212667.

[15] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529, arXiv:math/0104151v1.

[16] M. Gross, R. Pandharipande and B. Siebert, *The tropical vertex*, Duke Math. J. **153** (2010), no. 2, 297–362, arXiv:0902.0779v2.

[17] T. Bridgeland, *Scattering diagrams, Hall algebras and stability conditions*, arXiv:1603.00416v4 (2020).

[18] A. Burcroff, K. Lee, L. Mou, G. Musiker and M. Reineke, *Cluster scattering diagrams via quiver moduli and tight gradings*, arXiv:2511.14672v1 (2025).

[19] D. G. Wagner, *Total positivity of Hadamard products*, J. Math. Anal. Appl. **163** (1992), no. 2, 459–483.

[20] P. Brändén, *On operators on polynomials preserving real-rootedness and the Neggers–Stanley Conjecture*, J. Algebraic Combin. **20** (2004), no. 2, 119–130, arXiv:math/0303147v1.

[21] G. Bérczi and L. M. Fehér, *Positivity in classical enumerative geometry: A Case Study in Synchronized AI-Assisted Mathematics*, arXiv:2605.25271v1 (2026).

[22] S. Liu, *Rethlas-OSV: Desktop math agents for mathematicians*, 29 September 2026, updated 20 October 2026, https://srliu.com/blog/rethlas-osv-guide/.

[23] H. Ju, G. Gao, J. Jiang, B. Wu, Z. Sun, S. Liu, L. Chen, Yutong Wang, Yuefeng Wang, Z. Wang, W. He, P. Wu, L. Xiao, R. Liu, B. Dai and B. Dong, *Automated conjecture resolution with formal verification*, arXiv:2604.03789v2 (2026).

[24] F. Nie, H. Ye, S. Liu, Z. Wang, S. Ermon and J. Zou, *One-sided verification under limited expert review*, arXiv:2610.18089v2 (2026).

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