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ENERGY SOLUTIONS AND HARNACK ESTIMATES FOR NONLOCAL DRIFT–DIFFUSION

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Abstract. We consider nonlocal drift–diffusion equations of order $2s \in [1,2)$ with bounded divergence-free drift. The symmetric measurable jump kernel satisfies local coercivity, a global upper form bound, and a scale-invariant bound on the intensity of long jumps. The transport form is bounded and skew-symmetric on $H^{1/2} \times H^{1/2}$. In the fractional energy space, distributional and variational solutions coincide and satisfy nonlinear testing identities. Viscous approximations converge strongly in $C_t L^2_x \cap L^2_t H^s_x$ to the unique solution of the Cauchy problem. For sources bounded in the cylinder, supersolutions nonnegative there satisfy a weak Harnack inequality involving the time integral of the negative exterior tail. Under this source bound, locally bounded solutions whose exterior tail belongs to L^q in time for some $q > 1$ are locally Hölder continuous. This also gives Hölder continuity at positive times for homogeneous global solutions with L^2 data. Under upper jumping smoothness, control of the positive tail on an earlier time interval yields the full parabolic Harnack inequality for solutions nonnegative on the full spatial slab. A kernel supported on alternating dyadic annuli satisfies all the kernel assumptions. The local estimates depend on the drift through $R^{2s-1}\|b\|_\infty$.

1. Introduction

Consider the equation

$$\partial_t u + b(t,x)\cdot\nabla u + \mathcal{L}_t u = f, \tag{1.1}$$

where b is a bounded divergence-free vector field and $\mathcal{L}_t$ is a nonlocal diffusion operator of order $\alpha = 2s \in [1,2)$. Its symmetric jump kernel may depend measurably on time and need not have a pointwise lower bound of fractional Laplacian type. At the critical order $\alpha = 1$, the drift and diffusion have the same scaling, while the energy controls only half a spatial derivative. To compare a solution on an earlier cylinder with its infimum on a later one, one must also account for jumps from outside the region where the equation holds.

In the absence of drift, Felsinger and Kassmann [10] proved a weak Harnack inequality and Hölder estimates under quadratic-form assumptions. They also discussed first-order terms for $\alpha > 1$. Strömqvist [14] obtained Harnack estimates that allow negative values outside the cylinder. Kassmann and Weidner [11] proved a full parabolic Harnack inequality using time-integrated tails and distinguished the roles of local coercivity and upper jumping smoothness. In the probabilistic setting, Bass and Levin [1] proved a Harnack inequality for jump processes. Chen, Kumagai, and Wang [5] identified upper jumping smoothness as a necessary condition for the parabolic Harnack inequality for autonomous symmetric jump forms.

For fractional diffusion with drift, Caffarelli and Vasseur [4] proved Hölder regularity at the critical order in their study of the quasi-geostrophic equation. Chen and Zhang [6] obtained Harnack estimates for the fractional Laplacian with critical and supercritical divergence-free drifts. Liao [12] proved weak and full Harnack inequalities and Hölder estimates for pointwise comparable kernels and divergence-free drifts in scale-invariant classes.

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The definition of a local solution there requires a locally square-integrable spatial gradient [12, Definition 1.1]. Ciani and Nakamura [7] derived regularity estimates for functions in nonlocal parabolic De Giorgi classes.

For bounded incompressible drift, interpolation gives a continuous transport form on $H^{1/2}\times H^{1/2}$. Hence the form acts on the diffusion energy space even at $s = 1/2$. This bound justifies nonlinear testing in Section 2. The energy identity then gives strong convergence of the viscous approximations, while a fractional Nash inequality gives positive-time smoothing (Section 3).

For the local estimates, we add the time integral of the negative exterior tail and the negative source bound to the solution before taking powers. The derivative of this shift absorbs both terms in the energy inequality. The localized energy bounds in Section 4 and a differential inequality for the weighted mean of the logarithm give the weak Harnack inequality in Section 5. In Section 6 we use it to reduce the oscillation, estimating the tail on successive annuli. For the full Harnack inequality, upper jumping smoothness bounds the earlier positive tail in terms of the later infimum, the source, and an arbitrarily small multiple of a supremum on a larger past cylinder. Iteration on nested cylinders absorbs the supremum (Section 7).

The examples in Section 8 distinguish the kernel assumptions and illustrate the time dependence of the exterior tail. In Example 8.1 the kernel is supported on a union of separated dyadic annuli. Each missing jump splits into two allowed jumps inside the same ball, giving local coercivity. This kernel also satisfies upper jumping smoothness, even after multiplication by a symmetric measurable factor bounded above and bounded away from zero. In Example 8.3, concentrated jumps preserve the form and jump-intensity bounds but violate upper jumping smoothness.

1.1. **Kernel assumptions and notation.** For a ball $D \subset \mathbb{R}^n$, write

$$[v]_{s,D}^2 = \iint_{D\times D} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y, \qquad [v]_s = [v]_{s,\mathbb{R}^n}.$$

The norm on $H^s(\mathbb{R}^n)$ is $(\|v\|_2^2 + [v]_s^2)^{1/2}$. Set

$$\mathcal{E}_{t,D}(v,w) = \frac{1}{2}\iint_{D\times D}(v(x)-v(y))(w(x)-w(y))K(t;x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad \mathcal{E}_t = \mathcal{E}_{t,\mathbb{R}^n}. \tag{1.2}$$

The kernel K is nonnegative, jointly measurable, and symmetric in x, y. Fix $0 < \lambda \le \Lambda$. On a common set of times of full measure assume

$$\mathcal{E}_{t,D}(v,v) \ge \lambda[v]_{s,D}^2 \qquad \text{for every ball } D \text{ and } v \in H^s(D), \tag{1.3}$$

$$\mathcal{E}_t(v,v) \le \Lambda[v]_s^2 \qquad \text{for every } v \in H^s(\mathbb{R}^n), \tag{1.4}$$

$$\operatorname*{ess\,sup}_x \int_{|x-y|\ge r} K(t;x,y)\,\mathrm{d}y \le \Lambda r^{-\alpha} \qquad \text{for every } r > 0. \tag{1.5}$$

In (1.5), it suffices to take positive rational radii on a common set of spatial points of full measure. Monotone convergence then gives the estimate at every radius. We assume

$$b \in L^\infty((0,T)\times\mathbb{R}^n;\mathbb{R}^n), \qquad \operatorname{div}_x b = 0 \quad \text{in distributions.} \tag{1.6}$$

All spatial suprema and infima of measurable functions are essential extrema.

For $z_0 = (t_0, x_0)$, define the ambient cylinder $\mathcal{C}_R$, the past and future cylinders P_R and F_R, and the smaller past cylinder H_R by

$$\begin{aligned}
\mathcal{C}_R(z_0) &= (t_0 - 4R^\alpha, t_0 + R^\alpha)\times B_{4R}(x_0),\\
P_R(z_0) &= (t_0 - 3R^\alpha, t_0 - R^\alpha)\times B_{2R}(x_0),\\
F_R(z_0) &= (t_0, t_0 + R^\alpha)\times B_R(x_0),\\
H_R(z_0) &= (t_0 - \tfrac{5}{2}R^\alpha, t_0 - 2R^\alpha)\times B_R(x_0).
\end{aligned} \tag{1.7}$$

We take cylinders whose closures lie in the solution domain, except possibly for the terminal time face. The rescaled drift has norm

$$M_R = R^{\alpha-1}\|b\|_{L^\infty(\mathcal{C}_R)}. \tag{1.8}$$

Structural constants depend only on n, s, λ, Λ and, in local estimates, on an upper bound for the rescaled drift norm. On $\mathcal{C}_R$ this norm is M_R, and on a backward cylinder Q_R it is $R^{\alpha-1}\|b\|_{L^\infty(Q_R)}$. We indicate any further dependence.

For concentric balls with $r < \rho$, define

$$\Theta(v; t, r, \rho) = \operatorname*{ess\,sup}_{x\in B_r(x_0)} \int_{\mathbb{R}^n\setminus B_\rho(x_0)} |v(t,y)|K(t;x,y)\,\mathrm{d}y. \tag{1.9}$$

Here and below we omit the common center when it is clear. The weak Harnack inequality involves the negative tail

$$T_R^-(u) = \int_{t_0-4R^\alpha}^{t_0+R^\alpha} \Theta(u_-; t, 3R, 4R)\,\mathrm{d}t. \tag{1.10}$$

Under parabolic rescaling, $T_R^-(u)$ has the same scaling as u. For a bounded function v, the row bound (1.5) gives

$$\Theta(v; t, r, \rho) \le \Lambda(\rho - r)^{-\alpha}\|v(t)\|_\infty. \tag{1.11}$$

1.2. **Principal results.** Write

$$\mathcal{X}_T = L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;H^s(\mathbb{R}^n)).$$

For $u_0 \in L^2$ and $f \in L^2(0,T;H^{-s}(\mathbb{R}^n))$, a distributional energy solution is a function $u \in \mathcal{X}_T$ such that

$$\begin{aligned} &-\int_0^T\int u\,\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int u\,b\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \mathcal{E}_t(u,\phi)\,\mathrm{d}t \\ &\qquad = \int_0^T \langle f,\phi\rangle\,\mathrm{d}t + \int u_0\phi(0)\,\mathrm{d}x \end{aligned} \tag{1.12}$$

for every $\phi \in C_c^\infty([0,T)\times\mathbb{R}^n)$. Definition 4.2 specifies the local variational inequalities. For distributional energy solutions the equation implies the local time regularity required there.

Theorem 1.1 (Energy solutions and approximation)**.** *Under* (1.3)–(1.6)*, for every $u_0 \in L^2$ and $f \in L^2(0,T;H^{-s}(\mathbb{R}^n))$ there is a unique distributional solution $u \in \mathcal{X}_T$ with initial trace u_0. It belongs to*

$$L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;H^{-s}(\mathbb{R}^n)) \cap C([0,T];L^2).$$

It satisfies the energy identity (2.6) *and the renormalized testing formulas of Section 2. If f is locally bounded, it also satisfies the truncation inequalities of Proposition 4.4. The solutions u_ε of*

$$\partial_t u_\varepsilon + b\cdot\nabla u_\varepsilon + \mathcal{L}_t u_\varepsilon - \varepsilon\Delta u_\varepsilon = f, \qquad u_\varepsilon(0) = u_0,$$

satisfy, as $\varepsilon \downarrow 0$,

$$u_\varepsilon \longrightarrow u \quad \text{in } C([0,T];L^2)\cap L^2(0,T;H^s(\mathbb{R}^n)), \qquad \varepsilon\int_0^T \|\nabla u_\varepsilon\|_2^2\,\mathrm{d}t \longrightarrow 0. \tag{1.13}$$

Theorem 1.2 (Weak Harnack inequality)**.** *Let u be a local variational supersolution in $\mathcal{C}_R(z_0)$, let $f \in L^\infty(\mathcal{C}_R)$, and suppose that $u \ge 0$ there and $T_R^-(u) < \infty$. Then*

$$\fint_{P_R(z_0)} u\,\mathrm{d}x\,\mathrm{d}t \le C\left(\operatorname*{ess\,inf}_{F_R(z_0)} u + R^\alpha\|f_-\|_{L^\infty(\mathcal{C}_R)} + T_R^-(u)\right). \tag{1.14}$$

Here C is structural. If u is nonnegative on the full spatial slab, then $T_R^-(u) = 0$.

For Hölder estimates, set $Q_R(t_*, x_*) = (t_* - R^\alpha, t_*) \times B_R(x_*)$ and

$$d_\alpha((t,x),(\tau,y)) = |x-y| + |t-\tau|^{1/\alpha}.$$

Theorem 1.3 (Hölder regularity)**.** *Let $u \in \mathcal{X}_T$ solve (1.1) in Q_R, with $U = \|u\|_{L^\infty(Q_R)}$ finite. Suppose $f \in L^\infty(Q_R)$ and, for some $q > 1$,*

$$A_q = R^\alpha \left(\fint_{t_*-R^\alpha}^{t_*} \Theta(u; t, R/2, R)^q \, \mathrm{d}t \right)^{1/q} < \infty.$$

There are $\gamma \in (0,1)$ and C depending only on $n, s, \lambda, \Lambda, q$ and $R^{\alpha-1}\|b\|_{L^\infty(Q_R)}$ such that u has a representative satisfying

$$|u(z) - u(\zeta)| \le C\,(U + A_q + R^\alpha \|f\|_\infty) \left(\frac{d_\alpha(z,\zeta)}{R} \right)^\gamma, \qquad z, \zeta \in Q_{R/4}. \tag{1.15}$$

If u is bounded on the full spatial slab, its L^∞ norm there can replace A_q in (1.15) after adjustment of C. In particular, a global solution with $f = 0$ and $u_0 \in L^2$ is locally Hölder continuous at every positive time.

Definition 1.4 (Upper jumping smoothness)**.** The kernel has upper jumping smoothness with constant C_J if

$$K(t; x, y) \le C_J \fint_{B_r(x)} K(t; z, y) \, \mathrm{d}z, \qquad 0 < r \le |x-y|/4, \tag{1.16}$$

for almost every t, x, y.

Theorem 1.5 (Parabolic Harnack inequality)**.** *Assume also (1.16). Let $u \in \mathcal{X}_T$ solve (1.1) in $\mathcal{C}_R(z_0)$ with $f \in L^\infty(\mathcal{C}_R)$, and suppose $u \ge 0$ on the corresponding full spatial slab. Then*

$$\operatorname*{ess\,sup}_{H_R(z_0)} u \le C \left(\operatorname*{ess\,inf}_{F_R(z_0)} u + R^\alpha \|f\|_{L^\infty(\mathcal{C}_R)} \right). \tag{1.17}$$

The constant C depends only on the structural parameters and C_J.

At the critical order $\alpha = 1$, the factor $R^{\alpha-1}$ in the rescaled drift norm is one. For $\alpha > 1$, this factor tends to zero as $R \downarrow 0$. The estimates hold for arbitrary M_R.

2. Energy solutions and nonlinear testing

2.1. **Energy forms and the transport operator.** Applying (1.3) on balls increasing to $\mathbb{R}^n$ and using (1.4), we obtain

$$\lambda [v]_s^2 \le \mathcal{E}_t(v,v) \le \Lambda [v]_s^2, \qquad |\mathcal{E}_t(v,w)| \le \Lambda [v]_s [w]_s. \tag{2.1}$$

The last inequality follows by Cauchy–Schwarz in $L^2(K(t;x,y)\,\mathrm{d}x\,\mathrm{d}y)$. Thus the operator defined by $\langle \mathcal{L}_t v, w \rangle = \mathcal{E}_t(v,w)$ is bounded from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$.

Lemma 2.1 (Half-order transport estimate)**.** *For $b \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\operatorname{div} b = 0$, the form*

$$\mathcal{B}_b(v,w) = -\int v\, b \cdot \nabla w \, \mathrm{d}x, \qquad v, w \in C_c^\infty(\mathbb{R}^n),$$

extends continuously to $H^{1-\theta} \times H^\theta$, $0 \le \theta \le 1$, and

$$|\mathcal{B}_b(v,w)| \le C \|b\|_\infty \|v\|_{H^{1-\theta}} \|w\|_{H^\theta}. \tag{2.2}$$

In particular, it is bounded and skew-symmetric on $H^{1/2} \times H^{1/2}$.

Proof. Integration by parts yields

$$\mathcal{B}_b(v,w) = \int w\, b\cdot\nabla v\,\mathrm{d}x = -\mathcal{B}_b(w,v).$$

This gives the two estimates

$$|\mathcal{B}_b(v,w)| \le \|b\|_\infty \|v\|_{H^1}\|w\|_2, \qquad |\mathcal{B}_b(v,w)| \le \|b\|_\infty \|v\|_2\|w\|_{H^1}.$$

The two realizations of $b\cdot\nabla$, from H^1 to L^2 and from L^2 to H^{-1}, agree on H^1 by density of C_c^∞. Interpolating these operators gives (2.2), since $[H^a, H^c]_\theta = H^{(1-\theta)a+\theta c}$ [2]. Skew-symmetry extends to $H^{1/2}\times H^{1/2}$ by density. □

Proposition 2.2 (The threshold for the energy pairing)**.** *Let $0 \le \sigma < 1/2$. Even for the constant field $b = e_1$, the transport form has no bounded extension to $H^\sigma(\mathbb{R}^n)\times H^\sigma(\mathbb{R}^n)$.*

Proof. Choose a nonzero real-valued $\chi \in C_c^\infty(\mathbb{R}^n)$ and set

$$v_N(x) = \chi(x)\cos(Nx_1), \qquad w_N(x) = \chi(x)\sin(Nx_1).$$

Direct differentiation and one integration by parts give

$$\begin{aligned}\mathcal{B}_{e_1}(v_N, w_N) &= -\frac{1}{4}\int \partial_1(\chi^2)\sin(2Nx_1)\,\mathrm{d}x - \frac{N}{2}\int \chi^2(1+\cos(2Nx_1))\,\mathrm{d}x\\ &= -\frac{N}{2}\|\chi\|_2^2.\end{aligned}$$

The Fourier transforms of these functions are linear combinations of translates of $\widehat{\chi}$ by $\pm Ne_1$. The inequality

$$(1+|\xi \pm Ne_1|^2)^\sigma \le C_\sigma(1+N^2)^\sigma(1+|\xi|^2)^\sigma$$

therefore implies $\|v_N\|_{H^\sigma} + \|w_N\|_{H^\sigma} \le C(1+N)^\sigma$. A bounded bilinear extension would give $N \le C(1+N)^{2\sigma}$ for all N, a contradiction. □

Set $V = H^s(\mathbb{R}^n)$, $H = L^2$, and $V' = H^{-s}(\mathbb{R}^n)$. Since $s \ge 1/2$, the embedding $V \hookrightarrow H^{1/2}$ and Lemma 2.1 give

$$|\mathcal{B}_{b(t)}(v,w)| \le C\|b\|_\infty\|v\|_V\|w\|_V, \qquad \mathcal{B}_{b(t)}(v,v) = 0. \tag{2.3}$$

The time-dependent form $a_t(v,w) = \mathcal{E}_t(v,w) + \mathcal{B}_{b(t)}(v,w)$ is measurable and bounded. Moreover,

$$a_t(v,v) + \|v\|_2^2 \ge \min(1,\lambda)\|v\|_V^2. \tag{2.4}$$

2.2. **Construction and uniqueness.**

Lemma 2.3 (Time traces and the quadratic chain rule)**.** *If $v \in L^2(0,T;V)$ and $v' \in L^2(0,T;V')$, then v has a representative in $C([0,T];H)$ and*

$$\|v(t)\|_2^2 - \|v(r)\|_2^2 = 2\int_r^t \langle v', v\rangle\,\mathrm{d}\tau. \tag{2.5}$$

The corresponding product formula holds for two such functions.

Proof. Time mollification on a compact subinterval gives the identity for the regularized functions. The limits in the integral follow from convergence in L^2V and L^2V'. Pairing with each element of a countable dense subset of V gives a weakly continuous H representative. The same calculation applied on overlapping subintervals shows that its squared norm is the absolutely continuous function whose derivative is $2\langle v', v\rangle$. Weak continuity and continuity of the norm give strong continuity. One-sided regularization gives the endpoint traces. Polarization proves the product formula. This is the energy-space trace argument of [13]. □

Proposition 2.4 (The Cauchy problem)**.** *For the data in Theorem 1.1, there is exactly one variational solution in $L^2V \cap H^1V'$. It satisfies, for every $t \in [0,T]$,*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \mathcal{E}_\tau(u,u)\,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle f, u\rangle\,\mathrm{d}\tau. \tag{2.6}$$

Every distributional solution in $\mathcal{X}_T$ with initial datum u_0 is this solution.

Proof. Choose a linearly independent sequence (ϕ_j) in $C_c^\infty(\mathbb{R}^n)$ whose span is dense in V, hence also in H. Let P_N be the H-orthogonal projection onto $V_N = \operatorname{span}(\phi_1, \dots, \phi_N)$. The finite-dimensional system

$$(u_N', w) + a_t(u_N, w) = \langle f, w\rangle \quad (w \in V_N), \qquad u_N(0) = P_N u_0,$$

has an absolutely continuous solution because its coefficients are bounded and measurable and its forcing is square integrable. Testing with u_N eliminates the transport term. Young's inequality, (2.4), and Gronwall's inequality give

$$\|u_N\|_{L^\infty H}^2 + \|u_N\|_{L^2V}^2 \le C_T(\|u_0\|_2^2 + \|f\|_{L^2V'}^2). \tag{2.7}$$

The constant C_T is independent of N and $\|b\|_\infty$.

Along a subsequence, $u_N \rightharpoonup u$ in L^2V and $u_N \overset{*}{\rightharpoonup} u$ in $L^\infty H$. Testing against $\chi(t)\phi_j$ for fixed j and using density gives

$$\partial_t u + A_t u = f \quad \text{in } \mathcal{D}'(0,T;V'), \qquad \langle A_t v, w\rangle = a_t(v,w).$$

Boundedness of a_t gives $u' = f - A_t u \in L^2V'$. Lemma 2.3 then gives a continuous H trace. Keeping the initial boundary term in the Galerkin equations yields $u(0) = u_0$. Pairing the limit equation with u and using skew-symmetry proves (2.6).

If a function in $\mathcal{X}_T$ satisfies the distributional equation, boundedness of A_t implies $u' = f - A_t u \in L^2V'$. Density extends the equation to all V tests and identifies its trace. The difference of two solutions satisfies (2.6) with zero data. Both terms on its left are nonnegative, so the difference vanishes. □

2.3. **Renormalized transport and convex tests.**

Lemma 2.5 (Composition and localized chain rules)**.** *Let $0 < s < 1$. A Lipschitz function β with $\beta(0) = 0$ maps H^s continuously into H^s. Multiplication by a bounded Lipschitz function is continuous on H^s. If, in addition, $s \ge 1/2$, $v \in H^s(\mathbb{R}^n)$, $\eta \in W_c^{1,\infty}$, and β is Lipschitz without a restriction on $\beta(0)$, then*

$$\mathcal{B}_b(v, \beta(v)\eta^2) = -\int \Gamma(v) b \cdot \nabla(\eta^2)\,\mathrm{d}x, \qquad \Gamma' = \beta. \tag{2.8}$$

For a global test with $\beta(0) = 0$ one has $\mathcal{B}_b(v, \beta(v)) = 0$.

Proof. Let $v_j \to v$ in H^s. From any subsequence we may extract one with $v_j \to v$ almost everywhere and $\sum_j \|v_j - v\|_{H^s} < \infty$. Its increments $v_j(x) - v_j(y)$ then have a common majorant in $L^2(|x-y|^{-n-2s}\,\mathrm{d}x\,\mathrm{d}y)$. The Lipschitz bound for β and dominated convergence give convergence of the increments of $\beta(v_j)$ in this space. The same bound gives $\|\beta(v_j) - \beta(v)\|_2 \le \operatorname{Lip}(\beta)\|v_j - v\|_2$. The subsequence criterion proves continuity on H^s. For multiplication, expand

$$\eta(x)v(x) - \eta(y)v(y) = \eta(x)(v(x) - v(y)) + v(y)(\eta(x) - \eta(y))$$

and estimate the second summand using the Lipschitz bound on $|x-y| < 1$ and boundedness of η on $|x-y| \ge 1$.

Approximate v strongly in H^s by smooth compactly supported v_j. For v_j, the ordinary chain rule and $\operatorname{div} b = 0$ give (2.8). The left side passes to the limit by Lemma 2.1 and the preceding composition result, applied after subtracting $\beta(0)$. The right side converges in L^1 on the cutoff support because Γ has at most quadratic growth and $v_j \to v$ locally in L^2.

For a global test, the same approximation gives $\mathcal{B}_b(v_j, \beta(v_j)) = 0$ and hence the claimed identity. □

Lemma 2.6 (Chain rule with an absolutely continuous shift)**.** *Let J be a compact interval in the local time domain, let $\eta \in W^{1,\infty}_c(D)$, and suppose that $u \in L^2(J; H^s_{\mathrm{loc}}(D))$ and $\chi u \in L^2(J;V) \cap H^1(J;V')$ for every smooth cutoff χ supported in D. Let $h \in W^{1,1}(J)$ and let $\Gamma \in C^1(\mathbb{R})$ have a globally Lipschitz derivative. Then*

$$t \longmapsto \int \Gamma(u(t,x) + h(t))\eta(x)^2 \,\mathrm{d}x$$

is absolutely continuous, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \Gamma(u+h)\eta^2 \,\mathrm{d}x = \langle \partial_t u, \Gamma'(u+h)\eta^2\rangle + h'(t)\int \Gamma'(u+h)\eta^2 \,\mathrm{d}x \tag{2.9}$$

for almost every $t \in J$. The time pairing is localized by a cutoff equal to one on $\operatorname{supp}\eta$.

Proof. Choose $\chi \in C^\infty_c(D)$ equal to one near $\operatorname{supp}\eta$, and put $U = \chi u$. Lemma 2.3 gives $U \in C(J;H)$. Reflecting U across the endpoints of J and mollifying in time gives functions U_j smooth in time with

$$U_j \to U \text{ in } L^2(J;V) \cap C(J;H), \qquad U_j' \to U' \text{ in } L^2(J;V').$$

Choose smooth h_j converging to h in $W^{1,1}(J)$ and uniformly. A smooth Lipschitz approximation to Γ' first proves the classical identity for these functions; convergence of that approximation gives the identity for the stated Γ.

The functions $\Gamma'(U_j + h_j)\eta^2$ converge to $\Gamma'(U+h)\eta^2$ in $L^2(J;V)$. To see this, subtract the value at zero, use Lemma 2.5 after a further spatial localization of the scalar shift, and apply its dominated-subsequence argument also in time. They are uniformly bounded in $C(J;L^2)$, and

$$\sup_{t\in J}\left|\int (\Gamma'(U_j+h_j) - \Gamma'(U+h))\eta^2 \,\mathrm{d}x\right| \le C\left(\|U_j - U\|_{C(J;H)} + \|h_j - h\|_{C(J)}\right) \to 0.$$

The two terms on the right of the regularized identity therefore converge in $L^1(J)$, the first by V', V duality and the second by the uniform convergence above and $h_j' \to h'$ in $L^1(J)$. The integrals of $\Gamma(U_j + h_j)\eta^2$ converge uniformly in time. Indeed, Γ' grows at most linearly, so the mean value theorem and Cauchy–Schwarz bound the difference by the local L^2 difference of the arguments times a uniformly bounded factor. Passing to the integrated identity proves (2.9) and absolute continuity. □

For a global function $u \in L^2(J;V) \cap H^1(J;V')$, the same proof with $h = 0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \Gamma(u)\,\mathrm{d}x = \langle u', \Gamma'(u)\rangle \tag{2.10}$$

whenever $\Gamma(0) = \Gamma'(0) = 0$ and Γ' is globally Lipschitz. The bounds $|\Gamma(u)| \le C|u|^2$ and $|\Gamma'(u)| \le C|u|$ ensure integrability, and convergence in $C(J;H)$ gives uniform convergence of the integrals of Γ.

If $u + h \ge \varepsilon > 0$ on the test support, it suffices that Γ' be Lipschitz on $[\varepsilon, \infty)$, since it can then be extended to a Lipschitz function on $\mathbb{R}$. This applies to $\Gamma(r) = \log r$ and to $\Gamma(r) = r^p/p$ for $p < 1$, $p \ne 0$. Taking $\Gamma(r) = r_+^2/2$ and $h = -k$ gives

$$\langle u', (u-k)_+\eta^2\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (u-k)_+^2\eta^2 \,\mathrm{d}x + k'(t)\int (u-k)_+\eta^2 \,\mathrm{d}x, \qquad k \in W^{1,1}(J). \tag{2.11}$$

Thus $k' \in L^1(J)$ suffices for testing the equation for u at a moving level. A spatially constant shift cancels from the nonlocal differences and, by $\operatorname{div} b = 0$, from the transport term.

3. Strong approximation and positive-time smoothing

3.1. **Strong vanishing-viscosity convergence.**

Proposition 3.1. *The convergence in* (1.13) *holds for the entire family as* $\varepsilon \downarrow 0$.

Proof. For fixed $\varepsilon > 0$, use the form

$$a_t^\varepsilon(v,w) = a_t(v,w) + \varepsilon \int \nabla v \cdot \nabla w \,\mathrm{d}x \quad \text{on } H^1(\mathbb{R}^n).$$

The construction of Proposition 2.4 applies on this space. Its energy identity is

$$\frac{1}{2}\|u_\varepsilon(t)\|_2^2 + \int_0^t \mathcal{E}_\tau(u_\varepsilon, u_\varepsilon)\,\mathrm{d}\tau + \varepsilon \int_0^t \|\nabla u_\varepsilon\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle f, u_\varepsilon\rangle \,\mathrm{d}\tau. \tag{3.1}$$

Consequently u_ε is bounded in $L^\infty H \cap L^2 V$ and $\sqrt{\varepsilon}\nabla u_\varepsilon$ is bounded in L^2. Moreover,

$$\|\varepsilon \Delta u_\varepsilon\|_{L^2 H^{-1}} \le \varepsilon \|\nabla u_\varepsilon\|_{L^2} = O(\sqrt{\varepsilon}).$$

The equation gives a uniform bound for u_ε' in $L^2 H^{-1}$. Pairing with a countable dense set in H^1 and applying the scalar Arzelà–Ascoli theorem gives convergence along a subsequence in $C([0,T]; H_{\text{weak}})$. Weak convergence in $L^2 V$ and the preceding bound on the viscous term identify the limit equation. Proposition 2.4 identifies its solution as u, so these weak convergences hold for the full family.

At $t = T$, weak convergence in $L^2 V$ gives convergence of the forcing integral in (3.1). The terminal squared norm and the integrated form energy have lower limits at least as large as their values for u. By (2.6), those two values already account for the limit of the whole left-hand side of (3.1). Hence

$$\begin{aligned} \int_0^T \mathcal{E}_t(u_\varepsilon, u_\varepsilon)\,\mathrm{d}t &\longrightarrow \int_0^T \mathcal{E}_t(u,u)\,\mathrm{d}t, \\ \varepsilon \int_0^T \|\nabla u_\varepsilon\|_2^2 \,\mathrm{d}t &\longrightarrow 0. \end{aligned} \tag{3.2}$$

Weak convergence in $L^2 V$ gives $\int_0^T \mathcal{E}_t(u_\varepsilon, u)\,\mathrm{d}t \to \int_0^T \mathcal{E}_t(u,u)\,\mathrm{d}t$. Expanding the form energy of $u_\varepsilon - u$ and using (3.2) and (2.1), we obtain

$$\int_0^T [u_\varepsilon - u]_s^2 \,\mathrm{d}t \longrightarrow 0. \tag{3.3}$$

Cauchy–Schwarz for the forms now gives convergence of $\mathcal{E}_t(u_\varepsilon, u_\varepsilon)$ to $\mathcal{E}_t(u,u)$ in $L^1(0,T)$.

The functions $t \mapsto \int_0^t \langle f, u_\varepsilon - u\rangle \,\mathrm{d}\tau$ converge pointwise to zero. They are uniformly equicontinuous, since their increment on (r,t) is at most

$$\|f\|_{L^2(r,t;V')} \|u_\varepsilon - u\|_{L^2(0,T;V)}.$$

The convergence is therefore uniform. Since the form energies converge in $L^1(0,T)$ and the total viscous dissipation tends to zero, subtraction of the two energy identities gives uniform convergence of $\|u_\varepsilon(t)\|_H^2$ to $\|u(t)\|_H^2$. The range $u([0,T])$ is compact in H. Approximating it by a finite net and using convergence in $C_t H_{\text{weak}}$ gives

$$\sup_{t\in[0,T]} |(u_\varepsilon(t) - u(t), u(t))| \longrightarrow 0.$$

The polarization identity proves convergence in $C_t H$. Combining it with (3.3) proves convergence in $L^2 V$. □

Corollary 3.2 (Stability of the viscous approximation under changes in the data)**.** *Suppose* $u_0^\varepsilon \to u_0$ *in* L^2 *and* $f_\varepsilon \to f$ *in* $L^2(0,T;V')$. *With these data, the viscous solutions still satisfy* (1.13).

Proof. Let $\widetilde{u}_\varepsilon$ be the viscous solution with the fixed data u_0, f, and put $z_\varepsilon = u_\varepsilon - \widetilde{u}_\varepsilon$. Testing its equation by z_ε, then using Young's inequality and Gronwall's inequality, gives

$$\begin{aligned}\|z_\varepsilon\|^2_{C([0,T];H)} + \|z_\varepsilon\|^2_{L^2(0,T;V)} + \varepsilon\|\nabla z_\varepsilon\|^2_{L^2((0,T)\times\mathbb{R}^n)} \\ \le C_T\left(\|u_0^\varepsilon - u_0\|_2^2 + \|f_\varepsilon - f\|^2_{L^2(0,T;V')}\right).\end{aligned}$$

The constant is independent of ε. The right-hand side tends to zero, so Proposition 3.1 and the triangle inequality give both assertions in (1.13). □

3.2. **Contractions and the evolution family.**

Proposition 3.3 (Comparison and contraction)**.** *The homogeneous evolution preserves nonnegativity and is contractive in L^1, L^2, and L^∞ on the intersections of these spaces with L^2. If $u_0 \in L^2 \cap L^\infty$ and $f \in L^2_t L^2_x \cap L^1_t L^\infty_x$, then*

$$\|u(t)\|_\infty \le \|u_0\|_\infty + \int_0^t \|f(\tau)\|_\infty \,\mathrm{d}\tau. \tag{3.4}$$

Proof. For every increasing Lipschitz β with $\beta(0) = 0$,

$$\mathcal{E}_t(u, \beta(u)) \ge 0, \qquad \mathcal{B}_{b(t)}(u, \beta(u)) = 0.$$

Testing the homogeneous equation for a difference v with $-v_-$ proves comparison. Testing the equations for v and $-v$ with $(v - M)_+$ and $(-v - M)_+$, respectively, at the level $M = \|v(0)\|_\infty$, proves L^∞ contraction. For L^1 contraction, take

$$\Gamma_\delta(r) = \sqrt{r^2 + \delta^2} - \delta, \qquad \beta_\delta(r) = r(r^2 + \delta^2)^{-1/2}.$$

The integral of $\Gamma_\delta(u)$ is finite for $u \in L^2$, since $\Gamma_\delta(r) \le r^2/(2\delta)$. Its derivative is nonpositive by (2.10). Monotone convergence as $\delta \downarrow 0$ gives L^1 contraction. The L^2 assertion is (2.6).

For (3.4), use the moving level

$$k_\delta(t) = \|u_0\|_\infty + \delta + \int_0^t \|f(\tau)\|_\infty \,\mathrm{d}\tau.$$

The forcing and level terms combine to $\int (f - k_\delta')(u - k_\delta)_+ \le 0$. These integrals are finite because $k_\delta \ge \delta$ and $\|(u - k_\delta)_+\|_1 \le \delta^{-1}\|u\|_2^2$. Time regularization as in Lemma 2.6 gives the global moving-level identity. Indeed,

$$\big\|(u-k)_+ - (v-k)_+\big\|_1 \le \delta^{-1}(\|u\|_2 + \|v\|_2)\|u - v\|_2$$

for $k \ge \delta$. For two levels $k, \ell \ge \delta$, the corresponding L^1 difference is at most $|k - \ell|\delta^{-2}\|u\|_2^2$, since the truncations agree on $\{u \le \delta\}$. These bounds give uniform convergence in time of the integrals paired with $k_\delta' \in L^1$. The nonlocal term is nonnegative, so the initially vanishing truncation remains zero. Applying the same argument to $-u$ and letting $\delta \downarrow 0$ proves (3.4). □

Let $U(t, r)$ denote the homogeneous evolution from time r to time t. Uniqueness gives $U(t, r)U(r, q) = U(t, q)$.

Lemma 3.4 (Fractional Nash inequality)**.** *For $g \in L^1 \cap H^s(\mathbb{R}^n)$,*

$$\|g\|_2^{2+2\alpha/n} \le C[g]_s^2 \|g\|_1^{2\alpha/n}. \tag{3.5}$$

Proof. By the Fourier characterization of H^s, splitting at frequency ρ gives

$$\|g\|_2^2 \le C\rho^n \|g\|_1^2 + C\rho^{-\alpha}[g]_s^2.$$

For $g \ne 0$, choose ρ to equate the two terms. This gives (3.5) in every dimension, and the inequality is immediate for $g = 0$. □

Proposition 3.5 (Positive-time smoothing)**.** *For $0 \le r < t \le T$,*

$$\|U(t,r)\|_{L^1\to L^2} \le C(t-r)^{-n/(2\alpha)}, \qquad \|U(t,r)\|_{L^2\to L^\infty} \le C(t-r)^{-n/(2\alpha)}. \tag{3.6}$$

The constants depend on n, s, λ and are independent of $\|b\|_\infty$.

Proof. For nonzero initial data in $L^1 \cap L^2$, set $y(t) = \|u(t)\|_2^2$ and $P = \|u(r)\|_1$. Energy dissipation, L^1 contraction, and (3.5) give

$$y'(t) \le -cP^{-2\alpha/n} y(t)^{1+\alpha/n}.$$

Integration gives $y(t) \le CP^2(t-r)^{-n/\alpha}$. Density gives the first bound.

The adjoint evolution is the backward problem with form $\mathcal{E}_t - \mathcal{B}_{b(t)}$. Time reversal converts it to a forward problem satisfying the same assumptions. Lemma 2.3, applied to a forward and a backward solution, proves the adjoint relation. The first estimate therefore applies to $U(t,r)^*$. For $g \in L^1 \cap L^2$,

$$|(U(t,r)v, g)| = |(v, U(t,r)^* g)| \le C(t-r)^{-n/(2\alpha)} \|v\|_2 \|g\|_1.$$

Duality proves the second estimate. In particular, $\|u(t)\|_\infty \le Ct^{-n/(4s)}\|u_0\|_2$. □

4. Localized energy and boundedness

4.1. Cutoffs and local solution spaces.

Lemma 4.1 (Cutoff bounds)**.** *Assumption* (1.5) *implies*

$$\operatorname*{ess\,sup}_x \int_{|x-y|<r} |x-y|^2 K(t;x,y)\,\mathrm{d}y \le \frac{2\Lambda}{2-\alpha} r^{2-\alpha}. \tag{4.1}$$

If $0 \le \eta \le 1$ and $\operatorname{Lip}(\eta) \le L$, then

$$G_\eta(t,x) := \int (\eta(x)-\eta(y))^2 K(t;x,y)\,\mathrm{d}y \le C\Lambda L^\alpha. \tag{4.2}$$

Proof. Layer integration and (1.5) give

$$\int_{|x-y|<r} |x-y|^2 K\,\mathrm{d}y = \int_0^r 2\rho \int_{\rho<|x-y|<r} K\,\mathrm{d}y\,\mathrm{d}\rho \le 2\Lambda \int_0^r \rho^{1-\alpha}\,\mathrm{d}\rho.$$

For (4.2), split at L^{-1}, using the Lipschitz bound in the inner region and $|\eta(x)-\eta(y)| \le 1$ in the outer region. Constant cutoffs follow directly. □

Definition 4.2 (Local variational inequalities)**.** Let $f \in L^2_{\mathrm{loc}}(I \times D)$. A function $u = w + c(t)$, where $w \in \mathcal{X}_T$ and $c \in H^1_{\mathrm{loc}}(I)$ is spatially constant, is a local variational supersolution on $I \times D$ if, for each $\chi \in C_c^\infty(D)$, one has $\chi u \in H^1_{\mathrm{loc}}(I; H^{-s}(\mathbb{R}^n))$ and

$$\langle \partial_t u, \phi\rangle + \mathcal{B}_{b(t)}(u,\phi) + \mathcal{E}_t(u,\phi) \ge \int f\phi\,\mathrm{d}x \tag{4.3}$$

for almost every t and all nonnegative $\phi \in H^s(\mathbb{R}^n)$ compactly supported in D. Here $\mathcal{E}_t(u,\phi) = \mathcal{E}_t(w,\phi)$ and $\mathcal{B}_{b(t)}(u,\phi) = \mathcal{B}_{b(t)}(w,\phi)$. The time pairing includes $c'(t)$ and is defined using a cutoff equal to one near $\operatorname{supp}\phi$. These expressions are independent of the representation of u. The reversed inequality defines a subsolution, and equality defines a solution.

If $u \in \mathcal{X}_T$ satisfies the distributional equation on $I \times D$ with $f \in L^2_{\mathrm{loc}}(I \times D)$, then (2.1) and (2.3) give $\partial_t(\chi u) \in L^2_{\mathrm{loc}}(I; H^{-s}(\mathbb{R}^n))$ for every such cutoff. Density of smooth tests gives (4.3) with equality. Adding $c \in H^1_{\mathrm{loc}}(I)$ preserves this regularity and replaces f by $f + c'$. For levels with only an L^1 time derivative, Lemma 2.6 applies to the original equation.

Lemma 4.3 (Local parabolic Sobolev inequality)**.** *Put $\kappa = 1 + \alpha/n$. For a ball B_R and an interval J, write $\|\cdot\|_2$ for the spatial norm on $L^2(B_R)$. Then*

$$\int_J \int_{B_R} |w|^{2\kappa} \,\mathrm{d}x \,\mathrm{d}t \le C \left(\operatorname*{ess\,sup}_{t\in J} \|w(t)\|_2^2 \right)^{\alpha/n} \int_J \left([w(t)]_{s,B_R}^2 + R^{-\alpha} \|w(t)\|_2^2 \right) \mathrm{d}t. \tag{4.4}$$

Proof. On $\mathbb{R}^n$, apply the homogeneous Sobolev inequality at the order $\sigma = s/\kappa < n/2$, followed by Fourier interpolation between orders 0 and s. Since $2n/(n-2\sigma) = 2\kappa$, this gives

$$\|w\|_{2\kappa}^{2\kappa} \le C [w]_s^2 \|w\|_2^{2\kappa-2}.$$

On the unit ball, reflection across the sphere in a boundary annulus followed by multiplication by a fixed cutoff gives an extension bounded in both L^2 and H^s. The reflection is bi-Lipschitz on the annulus. Scaling to B_R replaces $[w]_s^2$ on the right by $[w]_{s,B_R}^2 + R^{-\alpha}\|w\|_{L^2(B_R)}^2$. See [8] for the fractional Sobolev and extension theorems. Integrating in time proves (4.4), including the case $n \le 2s$. □

4.2. **Truncation inequalities.** For a bounded domain D containing the compact support of η, write

$$\Theta_{D,k}^{\pm}(t,x) = \int_{D^c} (u(t,y) - k(t))_{\pm} K(t;x,y) \,\mathrm{d}y.$$

Proposition 4.4 (Local truncation inequality)**.** *Let u be a subsolution, $w = (u-k(t))_+$, and k absolutely continuous. Let $0 \le \eta \le 1$ be a compactly supported Lipschitz spatial cutoff in D. Then*

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int w^2\eta^2 \,\mathrm{d}x + \mathcal{E}_{t,D}(w\eta, w\eta) + \int_D w\eta^2 \Theta_{D,k}^- \,\mathrm{d}x \\ &\qquad \le \frac{1}{2}\int_D w^2 G_\eta \,\mathrm{d}x + \int_D w^2\eta |b\cdot\nabla\eta| \,\mathrm{d}x + \int_D (f_+ - k')w\eta^2 \,\mathrm{d}x + \int_D w\eta^2 \Theta_{D,k}^+ \,\mathrm{d}x. \end{aligned} \tag{4.5}$$

The inequality holds after integration against nonnegative time cutoffs. Applying it to $-u$ gives the lower truncation inequality for supersolutions.

Proof. The admissibility of $w\eta^2$ follows from Lemma 2.5, local time regularity, and (2.11). The transport term is exactly

$$\mathcal{B}_{b(t)}(u, w\eta^2) = -\int w^2\eta b\cdot\nabla\eta \,\mathrm{d}x.$$

For the interior form, monotonicity of the truncation and the identity

$$\begin{aligned} &(w(x)-w(y))(w(x)\eta(x)^2 - w(y)\eta(y)^2) \\ &\qquad = (w(x)\eta(x) - w(y)\eta(y))^2 - w(x)w(y)(\eta(x)-\eta(y))^2 \end{aligned}$$

give a lower bound $\mathcal{E}_{t,D}(w\eta, w\eta) - \frac{1}{2}\int_D w^2 G_\eta$. For the cross interaction, $w(x) > 0$ implies

$$(u(x)-u(y))w(x) = w(x)^2 + w(x)(k-u(y))_+ - w(x)(u(y)-k)_+.$$

Discard the nonnegative term $w(x)^2$. By symmetry, the two cross integrals in $\mathcal{E}_t$ agree and cancel its factor $1/2$. This proves (4.5).

All separated cross terms are finite before any tail supremum is taken. Indeed, if $A = \operatorname{supp}\eta$ and $d = \operatorname{dist}(A, D^c) > 0$, Cauchy–Schwarz for $K\,\mathrm{d}x\,\mathrm{d}y$, symmetry, and (1.5) give

$$\int_A \int_{D^c} |w(x)u(y)| K \,\mathrm{d}y \,\mathrm{d}x \le C d^{-\alpha} \|w\|_{L^2(A)} \|u\|_2.$$

For an affine shift $u = u_* + c(t)$, apply this estimate to $u_* \in \mathcal{X}_T$ and use the row bound and the local boundedness in time of c. The terms involving $|k|$ are also finite by the row bound. The time integrals are finite by the energy bounds and boundedness of k on compact intervals. □

4.3. **A separated-tail local bound.** On $Q_R = (t_* - R^\alpha, t_*) \times B_R(x_*)$, define

$$\mathcal{D}_R^+(u) = \operatorname*{ess\,sup}_{t\in(t_*-R^\alpha,t_*)} \sup_{R/2\le r<\rho\le R} (\rho - r)^\alpha \Theta(u_+; t, r, \rho). \tag{4.6}$$

By (1.11), $\mathcal{D}_R^+(u) \le \Lambda \|u\|_{L^\infty((t_*-R^\alpha,t_*)\times\mathbb{R}^n)}$ whenever u is bounded on the full spatial slab.

Proposition 4.5 (Local boundedness with separated tails)**.** *For a subsolution in Q_R,*

$$\operatorname*{ess\,sup}_{Q_{R/2}} u_+ \le C \left[\left(\fint_{Q_R} u_+^2 \right)^{1/2} + R^\alpha \|f_+\|_\infty + \mathcal{D}_R^+(u) \right] \tag{4.7}$$

whenever the quantities on the right are finite.

Proof. Subtract $\|f_+\|_\infty(t - t_* + R^\alpha)$ from u. The shifted subsolution has nonpositive source, and its positive part and positive tail can only decrease. Rescale to $R = 1$, $t_* = 0$ and continue to write u for the shifted function. Set $D = \mathcal{D}_1^+(u)$, $B = \|b\|_{L^\infty(Q_1)}$, and

$$r_j = \tfrac{1}{2} + 2^{-j-2}, \quad Q_j = (-r_j^\alpha, 0) \times B_{r_j}, \quad k_j = k(1 - 2^{-j-1}), \quad w_j = (u - k_j)_+.$$

Choose a spatial cutoff η_j equal to one on $B_{r_{j+1}}$, supported in $B_{(r_j+r_{j+1})/2}$, with $|\nabla\eta_j| \le C2^j$. Split the form at B_{r_j}. Choose a time cutoff that vanishes at $-r_j^\alpha$, equals one on $(-r_{j+1}^\alpha, 0)$, and has derivative bounded by $C2^{\alpha j}$. Define

$$A_{-1} = \int_{(-1,0)\times B_1} u_+^2, \qquad A_j = \int_{Q_j} w_j^2 \quad (j \ge 0).$$

For every $j \ge 0$, including $j = 0$ with A_{-1} as defined above,

$$|\{w_j > 0\} \cap Q_j| \le C2^{2j} k^{-2} A_{j-1}, \qquad \int_{Q_j} w_j \le C2^j k^{-1} A_{j-1}. \tag{4.8}$$

The first estimate follows from the gap between consecutive levels (from 0 to k_0 when $j = 0$). Cauchy–Schwarz and $A_j \le A_{j-1}$ give the second.

For $x \in \operatorname{supp}\eta_j$, the exterior integral in (4.5) is bounded by $C2^{\alpha j} D$. The cutoff estimates, (4.8), and $\alpha \ge 1$ therefore give

$$\begin{aligned} S_j &:= \operatorname*{ess\,sup}_{t\in(-r_{j+1}^\alpha,0)} \int w_j^2\eta_j^2 + \int_{-r_{j+1}^\alpha}^0 \mathcal{E}_{t,B_{r_j}}(w_j\eta_j, w_j\eta_j)\,\mathrm{d}t \\ &\le C2^{(\alpha+1)j}(1 + B + D/k) A_{j-1}. \end{aligned} \tag{4.9}$$

The Sobolev estimate (4.4) applied on this inner time interval yields $\int_{Q_{j+1}} w_j^{2\kappa} \le C S_j^\kappa$. Since $w_j \ge k2^{-j-2}$ on $\{w_{j+1} > 0\}$,

$$A_{j+1} \le C2^{Nj} k^{-2(\kappa-1)} (1 + B + D/k)^\kappa A_{j-1}^\kappa \quad (j \ge 0). \tag{4.10}$$

Here one may take $N = (\alpha + 3)\kappa - 2$. In later estimates N denotes a finite structural exponent that may increase from line to line. Set $Y_j = A_j/k^2$ and $Z_m = \max(Y_{2m-1}, Y_{2m})$. Applying (4.10) separately to the even and odd indices gives $Z_{m+1} \le C_1 C_2^m Z_m^\kappa$. Lemma A.1 applies if $k = C_*(A_{-1}^{1/2} + D) > 0$ with C_* sufficiently large in terms of B and the structural constants. Thus $A_j \to 0$. If $A_{-1} = 0$, the shifted function is already nonpositive on Q_1. Restoring the scale and source proves (4.7). □

Lemma 4.6 (Initial local boundedness under a pointwise upper bound)**.** *Assume, in addition,*

$$K(t; x, y) \le L|x - y|^{-n-\alpha}. \tag{4.11}$$

A local variational subsolution with bounded source is locally bounded above.

Proof. Use the preceding iteration in a unit cylinder compactly contained in the equation domain. Split its exterior integral into $B_1 \setminus B_{r_j}$ and B_1^c. For x in the cutoff support,

$$\int_{B_1\setminus B_{r_j}} u_+(t,y)K(t;x,y)\,\mathrm{d}y \le C2^{(n+\alpha)j}\int_{B_1} u_+(t,y)\,\mathrm{d}y.$$

Write $u = w + c(t)$ as in Definition 4.2. On the compact time interval under consideration, the quantity

$$B_2 = \operatorname*{ess\,sup}_t \|w(t)\|_2 + \sup_t |c(t)|$$

is finite and controls the last local integral. For $x \in B_{3/4}$ and $y \notin B_1$, Cauchy–Schwarz and (4.11) give

$$\int_{B_1^c} |w(t,y)|K(t;x,y)\,\mathrm{d}y \le L\|w(t)\|_2\left(\int_{|z|>1/4} |z|^{-2n-2\alpha}\,\mathrm{d}z\right)^{1/2} \le C\|w(t)\|_2.$$

Assumption (1.5) controls the contribution of $c(t)$. The exterior coefficient at step j is therefore at most $C2^{(n+\alpha)j}B_2$. Inserting this bound into (4.5) and using (4.8) gives

$$S_j \le C2^{(n+\alpha+1)j}(1 + \|b\|_\infty + B_2/k)A_{j-1}.$$

The recurrence (4.10) follows with a larger finite exponent. Taking k to be a sufficiently large multiple of $A_{-1}^{1/2} + B_2$ makes the initial normalized energies small, and Lemma A.1 gives the local upper bound. □

5. The weak Harnack inequality

We work first in $\mathcal{C}_1 = (-4,1)\times B_4$, with center $(0,0)$. Set

$$h(t) = \varepsilon + \int_{-4}^{t}\Big(\|f_-\|_{L^\infty(\mathcal{C}_1)} + \Theta(u_-;\tau,3,4)\Big)\,\mathrm{d}\tau, \qquad v = u + h(t), \quad \varepsilon > 0. \tag{5.1}$$

The shift satisfies $h \in W^{1,1}(-4,1)$ and $v \ge \varepsilon$ on $\mathcal{C}_1$. Values of u outside B_4 enter through the cross interactions. We apply Lemma 2.6 to u with the shift h. The shift cancels in spatial differences and in the transport pairing, while its derivative compensates for the negative exterior values and the negative part of the source.

5.1. **Power inequalities and moment iteration.**

Lemma 5.1 (A weighted power inequality)**.** *Fix $p_* \in (0,1)$. There are $c_*, C_* > 0$, depending only on p_*, such that, for $p \in (-\infty,0)\cup(0,p_*]$, $a,b>0$, and $A,B \ge 0$,*

$$\begin{aligned}&(a-b)(A^2a^{p-1} - B^2b^{p-1})\\ &\quad\le -\frac{c_*}{|p|}(Aa^{p/2} - Bb^{p/2})^2 + C_*\frac{1+|p|}{|p|}(A-B)^2(a^p+b^p).\end{aligned} \tag{5.2}$$

Proof. By symmetry and homogeneity, assume $b = 1$ and $a = t \ge 1$. Put $q = 1-p > 0$, $d = A - B$,

$$D = (t-1)(1-t^{-q}), \qquad E = (t-1)t^{-q}.$$

For $t > 1$, expansion and Young's inequality give

$$(t-1)(A^2t^{-q} - B^2) = -B^2D + 2BdE + d^2E \le -\tfrac{1}{2}B^2D + d^2(E + 2E^2/D). \tag{5.3}$$

Since $1 - t^{-q} = q\int_1^t r^{-q-1}\,\mathrm{d}r \ge q(t-1)t^{-q-1}$,

$$E \le t^p, \qquad E^2/D \le q^{-1}t^p.$$

Cauchy–Schwarz applied to $t^{p/2} - 1 = (p/2)\int_1^t r^{p/2-1}\,\mathrm{d}r$ gives

$$D \ge \frac{4q}{p^2}(t^{p/2}-1)^2.$$

Throughout the stated range of p, we have $q/|p| \geq c_* := 1 - p_*$ and $q \geq 1 - p_*$. The right-hand side of (5.3) is therefore at most

$$-\frac{2c_*}{|p|}B^2(t^{p/2}-1)^2 + \left(1 + \frac{2}{1-p_*}\right) d^2 t^p.$$

The bound $(At^{p/2} - B)^2 \leq 2B^2(t^{p/2}-1)^2 + 2d^2t^p$ gives (5.2) after increasing C_*. The case $t = 1$ follows by continuity. □

Consider cylinders $J' \times B_r \subset J \times B_\rho \subset (-4, 1) \times B_3$ with radii between fixed positive bounds. Set $d = \rho - r$. Let $\delta > 0$ be the distance from J' to the initial endpoint of J for $p < 0$, or to its terminal endpoint for $p > 0$.

Proposition 5.2 (Power energy estimates)**.** *Let $\eta = 1$ on B_r, with support in $B_{(r+\rho)/2}$ and $|\nabla\eta| \leq C/d$. For the function in* (5.1)*,*

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in J'} \int_{B_r} v^p \,\mathrm{d}x + \int_{J'} \mathcal{E}_{t,B_\rho}(\eta v^{p/2}, \eta v^{p/2}) \,\mathrm{d}t \\ &\qquad \leq C(1+|p|)\left(d^{-\alpha} + \|b\|_\infty d^{-1} + \delta^{-1}\right) \int_J \int_{B_\rho} v^p \,\mathrm{d}x \,\mathrm{d}t. \end{aligned} \tag{5.4}$$

The constant is uniform as $p \to 0$ within $(-\infty, 0) \cup (0, p_*]$.

Proof. Test by $\eta^2 v^{p-1}$. On $B_\rho \times B_\rho$, apply Lemma 5.1 and then (4.2). For $x \in \operatorname{supp}\eta$ and $y \notin B_\rho$,

$$(u(x) - u(y))v(x)^{p-1} \leq v(x)^p + u_-(y)v(x)^{p-1}.$$

By (1.5), the first term contributes at most $Cd^{-\alpha}\int_{B_\rho} v^p$. The second vanishes for $y \in B_4$ because $u \geq 0$ there. For $y \notin B_4$, its contribution is bounded by $\Theta(u_-; t, 3, 4)\int \eta^2 v^{p-1}$. The choice of h' in (5.1) absorbs this term and the negative part of the source. For the drift, Lemma 2.5 gives

$$\mathcal{B}_b(u, \eta^2 v^{p-1}) = -\frac{1}{p}\int v^p b \cdot \nabla(\eta^2) \,\mathrm{d}x. \tag{5.5}$$

Its absolute value is at most $C|p|^{-1}\|b\|_\infty d^{-1}\int_{B_\rho} v^p$.

Write $Y(t) = \int \eta^2 v^p$. Multiplication by $|p|$ gives

$$Y'(t) + c\mathcal{E}_{t,B_\rho}(\eta v^{p/2}, \eta v^{p/2}) \leq C(1+|p|)(d^{-\alpha} + \|b\|_\infty d^{-1}) \int_{B_\rho} v^p$$

when $p < 0$. For $p > 0$ the same inequality holds with $-Y'$ in place of Y'. Integrate forward for $p < 0$, using a time cutoff that vanishes at the initial endpoint of J. For $p > 0$, integrate backward from the terminal endpoint with the corresponding cutoff. In either case the endpoint term has the required sign, and the derivative of the cutoff contributes at most $C\delta^{-1}\int_J \int_{B_\rho} v^p$. Taking the supremum in time and bounding the integrated energy gives (5.4).

To justify the test, extend r^{p-1} from $[\varepsilon, \infty)$ to a globally Lipschitz function and apply Lemma 2.6 to its primitive, which agrees with r^p/p on this interval up to a constant. The factor $v^{p-1} \leq \varepsilon^{p-1}$ on the test support makes the term containing $h' \in L^1$ integrable. The constants in (5.4) are independent of the regularization. □

Define, for $0 \leq a \leq 1$,

$$E_a = (-3 - a/4, -1 + a/4) \times B_{2+a/4}, \qquad G_a = (-a/4, 1) \times B_{1+a/4}. \tag{5.6}$$

Thus $E_0 = P_1$ and $G_0 = F_1$. All constants below include the fixed volume normalizations for these sets.

Lemma 5.3 (Moment estimates)**.** *Put $p_0 = 1/4$. For $0 \le a < c \le 1$ and $0 < p \le p_0/2$,*

$$\|v\|_{L^{p_0}(E_a)} \le \left(\frac{C}{(c-a)^N}\right)^{1/p} \|v\|_{L^p(E_c)}, \tag{5.7}$$

$$\|v^{-1}\|_{L^{p_0}(G_a)} \le \left(\frac{C}{(c-a)^N}\right)^{1/p} \|v^{-1}\|_{L^p(G_c)}. \tag{5.8}$$

Moreover,

$$\|v\|_{L^1(E_0)} \le C\|v\|_{L^{p_0}(E_{1/2})}, \tag{5.9}$$

$$\|v^{-1}\|_{L^\infty(G_0)} \le C\|v^{-1}\|_{L^{p_0}(G_{1/2})}. \tag{5.10}$$

Proof. Choose $p_* = (1 + 1/\kappa)/2 < 1$ in Lemma 5.1. Combining (5.4) with (4.4) gives the single-step bound

$$\|v\|_{L^{p\kappa}(E_a)} \le \left(\frac{C(1+p)}{(c-a)^N}\right)^{1/p} \|v\|_{L^p(E_c)}, \qquad 0 < p \le p_*. \tag{5.11}$$

On $G_a \subset G_c$, taking the power $p = -\beta$ in (5.4) gives the corresponding estimate for v^{-1} with input exponent β and output exponent $\beta\kappa$, for every $\beta > 0$. The time gap in (5.6) is at the initial endpoint for $G_a \subset G_c$ and at the terminal endpoint for $E_a \subset E_c$.

For the inverse iteration use $\beta_j = \kappa^j p$ and gaps proportional to $(c-a)2^{-j-1}$. The logarithm of the product of constants is bounded by

$$\sum_{j=0}^{\infty} \frac{\log C + \log(1+\kappa^j p) + Nj\log 2 + N\log(1/(c-a))}{\kappa^j p} \le \frac{C + N'\log(1/(c-a))}{p}$$

for $p \le p_0$. Letting the exponents tend to infinity bounds the L^∞ norm of v^{-1} on the smaller cylinder by its L^p norm on the larger one, with the factor estimated above. This gives (5.8) and (5.10).

For (5.7), stop when the exponent first reaches or exceeds p_0 and use Hölder's inequality on the smaller cylinder. The finite product satisfies the same bound as above. For (5.9), start at p_0, iterate to $1/\kappa$, and reduce the last exponent to $1/\kappa$ by Hölder's inequality if necessary. One more application of (5.11) reaches 1. All input exponents are at most $\max(p_0, 1/\kappa) < p_*$, and the product of constants in this last iteration depends only on the fixed exponents and the structural data. □

5.2. **A weighted logarithmic inequality.** Choose a radial Lipschitz cutoff η equal to one on $B_{5/2}$, zero outside B_3, and nonincreasing in the radius. Put $\theta = \eta^2$, $m_\theta = \int \theta$, and

$$D_\theta(w) = \frac{1}{2}\iint_{B_3\times B_3} \min(\theta(x),\theta(y))|w(x)-w(y)|^2 K(t;x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad w_\theta = m_\theta^{-1}\int \theta w.$$

Dyda and Kassmann [9] use radial level sets to derive weighted Poincaré inequalities from unweighted estimates. Here the level sets of θ are balls on which (1.3) applies directly.

Lemma 5.4 (Weighted Poincaré inequality)**.** *For this cutoff,*

$$\int \theta|w - w_\theta|^2\,\mathrm{d}x \le CD_\theta(w). \tag{5.12}$$

At scale R the right side is $CR^\alpha D_\theta(w)$.

Proof. Use

$$\min(\theta(x),\theta(y)) = \int_0^1 \mathbf{1}_{\{\theta>q\}}(x)\mathbf{1}_{\{\theta>q\}}(y)\,\mathrm{d}q.$$

The superlevel sets are balls. Applying (1.3) on each such ball and integrating in q gives

$$D_\theta(w) \ge \lambda \iint_{B_3\times B_3} \min(\theta(x),\theta(y))\frac{|w(x)-w(y)|^2}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y.$$

Restrict y to $B_{5/2}$ and x to B_3. On this set the minimum is $\theta(x)$ and $|x-y| \le 6$. Jensen's inequality yields a lower bound $c\int \theta|w-w_{B_{5/2}}|^2$. The weighted mean minimizes the integral among constants, giving (5.12). Truncation of w and monotone limiting arguments cover every finite-energy w. Scaling gives the last assertion. □

Lemma 5.5 (Weighted logarithmic mean)**.** *For v in* (5.1)*, let*

$$W(t) = m_\theta^{-1} \int \theta(x) \log v(t,x)\,\mathrm{d}x.$$

Then W is locally absolutely continuous and

$$W'(t) \ge cD_\theta(\log v) - C(1+\|b\|_\infty^2) \ge c' \int \theta |\log v - W(t)|^2\,\mathrm{d}x - C(1+\|b\|_\infty^2). \tag{5.13}$$

Proof. For $a,b>0$ and $A,B\ge 0$,

$$(a-b)(A^2/a - B^2/b) \le -\tfrac{1}{2}\min(A,B)^2 |\log(a/b)|^2 + 3(A-B)^2. \tag{5.14}$$

By symmetry we may assume $A \ge B$. Write $d = A-B$ and $t = a/b$. The left-hand side of (5.14) equals

$$-B^2(t+t^{-1}-2) + (2Bd+d^2)(1-t^{-1}).$$

Write $Z = \sqrt{t} - 1/\sqrt{t}$, so that $Z^2 = t + t^{-1} - 2 \ge |\log t|^2$. For $t \le 1$ the second term is nonpositive. For $t \ge 1$ use $1 - t^{-1} \le \min(Z,1)$ and $2BdZ \le B^2Z^2/2 + 2d^2$. These bounds give (5.14).

Test the supersolution by η^2/v. Integrating (5.14) over $B_4 \times B_4$ bounds the interior form above by $-cD_\theta(\log v) + C$. The cross interaction is bounded above by

$$C + \Theta(u_-; t, 3, 4) \int \eta^2 v^{-1}\,\mathrm{d}x.$$

The shift h' absorbs this last term and f_-. The transport contribution is

$$-\int \log v\, b\cdot\nabla\theta\,\mathrm{d}x = -2\int (\log v - W)\eta b\cdot\nabla\eta\,\mathrm{d}x.$$

The equality uses $\int b\cdot\nabla\theta = 0$. By (5.12), its absolute value is at most

$$C\|b\|_\infty D_\theta(\log v)^{1/2} \le \tfrac{c}{2} D_\theta(\log v) + C\|b\|_\infty^2.$$

Lemma 2.6 with a Lipschitz extension of $r \mapsto r^{-1}$ from $[\varepsilon,\infty)$ gives local absolute continuity of W and identifies the time derivative as $m_\theta W'$ after the h' term is included. Dividing by m_θ and applying (5.12) gives (5.13). □

Lemma 5.6 (One-sided logarithmic distribution estimates)**.** *There is $a \in \mathbb{R}$ such that, for every $\ell > 0$,*

$$|\{E_1 : \log v > a + \ell\}| + |\{G_1 : \log v < a - \ell\}| \le C/\ell. \tag{5.15}$$

Proof. Let $c_0 = C(1+\|b\|_\infty^2)$ be the constant in (5.13), and put

$$z(t,x) = \log v(t,x) + c_0 t, \qquad Z(t) = W(t) + c_0 t.$$

Then $Z' \ge c\int \theta|z-Z|^2 \ge 0$. Choose $\tau_* = -1/2$ and set $a = Z(\tau_*)$, using the continuous representative of Z. For $t < \tau_*$, put $H(t) = a - Z(t) \ge 0$. On the set $\{x \in B_{5/2} : z(t,x) > a+\ell\}$ one has $z - Z > \ell + H$. Thus

$$|\{x \in B_{5/2} : z(t,x) > a+\ell\}| \le \frac{-H'(t)}{c(\ell+H(t))^2}.$$

Since $-H'/(\ell+H)^2$ is the derivative of $(\ell+H)^{-1}$, integration over the past interval gives at most $1/(c\ell)$. On the future interval take $H = Z - a$. The set $\{x \in B_{5/2} : z(t,x) < a - \ell\}$ has measure at most $H'/(c(\ell+H)^2)$ for almost every time, which gives the same integral bound. The spatial sections of E_1 and G_1 lie in $B_{5/2}$, and their time intervals lie on opposite

sides of τ_*. To pass from z to $\log v$, use $|c_0 t| \le 4c_0$ on $\mathcal{C}_1$. For $\ell > 8c_0$ apply the bounds at level $\ell/2$. For smaller ℓ, the cylinder volumes give (5.15) after increasing C. □

5.3. **Proof of weak Harnack.**

Proof of Theorem 1.2. At unit scale, let a_0 be the level given by Lemma 5.6. Apply Lemma A.3 to $e^{-a_0}v$ on (E_r) and to $e^{a_0}v^{-1}$ on (G_r). The moment hypotheses follow from (5.7) and (5.8), respectively. The two level-set bounds in (5.15) give the logarithmic hypotheses. Hence

$$\|v\|_{L^{p_0}(E_{1/2})} \le Ce^{a_0}, \qquad \|v^{-1}\|_{L^{p_0}(G_{1/2})} \le Ce^{-a_0}.$$

Equations (5.9) and (5.10) imply

$$\fint_{P_1} v \le Ce^{a_0} \le C \operatorname*{ess\,inf}_{F_1} v.$$

Since $u \le v$ in $\mathcal{C}_1$ and $h(t) \le \varepsilon + 5\|f_-\|_\infty + T_1^-(u)$, letting $\varepsilon \downarrow 0$ proves the unit-scale assertion. The transformation

$$\widetilde{u}(\tau,\xi) = u(t_0 + R^\alpha\tau, x_0 + R\xi), \quad \widetilde{K} = R^{n+\alpha}K, \quad \widetilde{b} = R^{\alpha-1}b, \quad \widetilde{f} = R^\alpha f$$

preserves (1.3)–(1.5), with the coefficients on the right evaluated at the transformed arguments. Since $\int \widetilde{\Theta}_- \,\mathrm{d}\tau = T_R^-(u)$ and $\|\widetilde{b}\|_{L^\infty(\mathcal{C}_1)} = M_R$, the unit-scale estimate gives (1.14). □

Corollary 5.7 (Growth from a positive-measure set)**.** *In the setting of Theorem 1.2, if*

$$|\{u \ge k\} \cap P_R| \ge \nu|P_R|, \qquad k > 0, \quad 0 < \nu \le 1,$$

then

$$\operatorname*{ess\,inf}_{F_R} u \ge c\nu k - C\left(R^\alpha\|f_-\|_\infty + T_R^-(u)\right). \tag{5.16}$$

Proof. The average on the left of (1.14) is at least νk. □

6. Oscillation decay and Hölder continuity

Proof of Theorem 1.3. Translate and scale to $Q_1 = (-1,0)\times B_1$. Write

$$\Theta(t) = \Theta(u;t,1/2,1), \qquad F = \|f\|_{L^\infty(Q_1)}, \qquad A = 2U + \|\Theta\|_{L^q(-1,0)} + F.$$

When $A = 0$ the assertion is immediate. Let c be the growth constant in Corollary 5.7, reduced so that $c \le 1$. Set

$$a_0 = c/16, \qquad \theta = 1 - a_0.$$

We choose a radius ratio $\rho \in (0, 1/8)$ below. Put $r_j = \rho^j$ and $\omega_j = A\theta^j$. We construct nested intervals $[m_j, M_j] \subset [-U, U]$ such that

$$m_j \le u \le M_j \quad \text{on } Q_{r_j}, \qquad M_j - m_j \le \omega_j, \tag{6.1}$$

and such that each endpoint changes by at most $a_0\omega_j$ in passing from j to $j+1$. Start with $m_0 = -U$ and $M_0 = U$.

Suppose (6.1) holds through scale j. If $M_j - m_j \le \theta\omega_j$, retain both endpoints. Otherwise set $d_j = M_j - m_j > \theta\omega_j$ and apply Corollary 5.7 at scale $r = r_{j+1}$ with time center $-r^\alpha$. The outer cylinder is

$$(-5r^\alpha, 0)\times B_{4r} \subset Q_{r_j},$$

provided $5\rho^\alpha \le 1$. Its future cylinder is precisely Q_r. On its past cylinder, at least one of the sets

$$\{u - m_j \ge d_j/2\}, \qquad \{M_j - u \ge d_j/2\}$$

has relative measure at least $1/2$. For the corresponding function, the lower bound in (5.16) contains the term $cd_j/4 \ge 2a_0\omega_j$. We estimate the negative exterior tail for either choice of function.

Both functions have zero negative part on Q_{r_j}. Decompose $B_{r_j}^c$ as B_1^c and the annuli $B_{r_i} \setminus B_{r_{i+1}}$, $0 \le i < j$. On the ith annulus, throughout $(-5r^\alpha, 0)$, the negative part is at most

$$\max(m_j - m_i, M_i - M_j) \le a_0 \sum_{k=i}^{j-1} \omega_k \le \omega_i.$$

For $x \in B_{3r}$ and y on this annulus, $|x-y| \ge r_{i+1} - 3r \ge r_{i+1}/2$. The row bound therefore gives the annular contribution

$$\begin{aligned} T_j^{\mathrm{ann}} &\le C \sum_{i=0}^{j-1} (r/r_{i+1})^\alpha \omega_i \\ &\le C\omega_j \sum_{\ell=1}^{j} (\rho^\alpha/\theta)^\ell \le C\omega_j \frac{\rho^\alpha}{\theta - \rho^\alpha}. \end{aligned} \tag{6.2}$$

All time intervals used here are contained in the earlier Q_{r_i}, by the choice $5\rho^\alpha \le 1$.

For $y \notin B_1$, the negative part is at most $U + |u(t,y)|$, because every endpoint stays in $[-U, U]$. Consequently Hölder's inequality in time gives

$$T_j^{\mathrm{far}} \le CUr^\alpha + C\|\Theta\|_{L^q(-1,0)} r^{\alpha(1-1/q)}. \tag{6.3}$$

The source contributes at most $Cr^\alpha F$. Let $\beta = \alpha(1-1/q) > 0$. Choose ρ so small that

$$\rho^\alpha < \theta, \quad \rho^\beta < \theta, \qquad C\left(\frac{\rho^\alpha}{\theta - \rho^\alpha} + \rho^\alpha + \rho^\beta\right) \le a_0.$$

Since $U, \|\Theta\|_q, F \le A$, (6.2)–(6.3) and the source bound give a total error of at most $a_0\omega_j$ in (5.16). On Q_r the chosen function is therefore at least $a_0\omega_j$. We can move the corresponding endpoint inward by $d_j - \theta\omega_j \le a_0\omega_j$. This gives (6.1) at scale $j+1$.

Let

$$\gamma = \frac{-\log\theta}{\log(1/\rho)}.$$

Decreasing ρ once more gives $0 < \gamma < \min(1, \beta)$. For $r_{j+1} < r \le r_j$, (6.1) implies

$$\operatorname*{ess\,osc}_{Q_r} u \le \omega_j \le CAr^\gamma. \tag{6.4}$$

The constants in (5.16) are uniform in j because $r_j^{\alpha-1}\|b\|_\infty \le \|b\|_\infty$ for every $\alpha \ge 1$.

Repeat the argument on a backward cylinder of radius $1/8$ centered at any point of $Q_{1/4}$. The part of its exterior tail inside B_1 is bounded by CU by (1.5). The part outside B_1 is bounded by $\Theta(t)$ since the points at which the kernel is evaluated lie in $B_{1/2}$. Thus the time L^q norm of this tail is at most $C(U + \|\Theta\|_q)$, and (6.4) holds uniformly at these centers.

The oscillation bound makes averages on shrinking backward cylinders converge at each center. Their limits agree with u at Lebesgue points and hence define a representative. For two such points at parabolic distance $0 < d < 1/64$, let z be the later point (either point if their times agree). The cylinder $Q_{2d}(z)$ contains all sufficiently small backward cylinders ending at either point. Its essential oscillation therefore bounds the difference between their limiting averages by CAd^γ. For larger distances use the bound $2U$. The same argument applies to the limiting averages at every center in $Q_{1/4}$. Rescaling gives (1.15). □

Corollary 6.1 (Globally bounded solutions and positive times)**.** *If u is bounded on the full spatial slab underlying Q_R, the tail term in* (1.15) *is bounded by $C\|u\|_{L^\infty(I\times\mathbb{R}^n)}$. For the homogeneous Cauchy problem with $u_0 \in L^2$, the solution is locally Hölder continuous in $(0,T)\times\mathbb{R}^n$.*

Proof. The first assertion is (1.11). For every $\tau > 0$, Proposition 3.5 bounds the solution on $[\tau, T]\times\mathbb{R}^n$. Apply Theorem 1.3 on cylinders with lower time endpoint greater than τ. □

7. Positive tails and the parabolic Harnack inequality

7.1. The kernel upper bound and an earlier-tail estimate.

Lemma 7.1. *Symmetry,* (1.5)*, and* (1.16) *imply* (4.11)*, with L depending only on n, α, Λ, C_J.*

Proof. For $d = |x - y|$, apply (1.16) with $r = d/4$. By symmetry,

$$K(t; x, y) \le C_J |B_{d/4}|^{-1} \int_{B_{d/4}(x)} K(t; y, z)\, \mathrm{d}z \le C d^{-n} \int_{|y-z| \ge 3d/4} K(t; y, z)\, \mathrm{d}z \le C d^{-n-\alpha}.$$

□

Work at unit scale and let u solve (1.1) in $\mathcal{C}_1$, with $u \ge 0$ on $(-4, 1) \times \mathbb{R}^n$. Lemma 7.1 gives the pointwise upper bound needed in Lemma 4.6, so u is locally bounded in $\mathcal{C}_1$. Set $F = \|f\|_{L^\infty(\mathcal{C}_1)}$ and, for $\varepsilon > 0$, define

$$v = u + F(t + 4) + \varepsilon, \qquad H = \operatorname*{ess\,inf}_{F_1} u + 5F + \varepsilon.$$

Since $f + F \ge 0$, v is a positive supersolution with zero negative tail. Applying Theorem 1.2 and using $\operatorname{ess\,inf}_{F_1} v \le H$, we obtain

$$\int_{P_1} v \le CH. \tag{7.1}$$

For $0 \le a \le 1$, define nested past cylinders

$$U_a = J_a \times B_{R_a}, \qquad J_a = (-5/2 - a/2, -2 + a/2), \qquad R_a = 1 + a/2, \qquad M(a) = \operatorname*{ess\,sup}_{U_a} v.$$

These cylinders lie in P_1, with $U_0 = H_1$. Local boundedness and nesting imply that M is bounded and nondecreasing on $[0, 1]$. For $0 \le a < \xi \le 1$, define $R_m = (R_a + R_\xi)/2$ and

$$S_{a,\xi} = \int_{J_\xi} \Theta(u; t, R_m, R_\xi)\, \mathrm{d}t.$$

The spatial gap and (4.11) make this quantity finite by Cauchy–Schwarz and the global $L^\infty_t L^2_x$ bound on u.

Lemma 7.2 (Control of an earlier positive tail)**.** *For $0 \le a < \xi < c \le 1$ and each fixed $q \in (0, 1)$,*

$$S_{a,\xi} \le C\big((\xi - a)(c - \xi)\big)^{-N} M(c)^q H^{1-q}. \tag{7.2}$$

Consequently, for every $\delta > 0$,

$$S_{a,\xi} \le \delta M(c) + C_\delta\big((\xi - a)(c - \xi)\big)^{-N_\delta} H. \tag{7.3}$$

Proof. Put $d = (R_\xi - R_m)/4$. Choose $0 \le \eta \le 1$ in $C^{0,1}_c(B_{R_\xi - d})$, equal to one on $B_{R_m + d}$, with $|\nabla \eta| \le C/d$. Testing the supersolution with $\eta^2 v^{-q}$, we keep the exterior contribution

$$I(t) = \int_{B_{R_\xi}} \eta(x)^2 v(t, x)^{-q} \int_{B^c_{R_\xi}} u(t, y) K(t; x, y)\, \mathrm{d}y\, \mathrm{d}x.$$

The weighted power inequality with exponent $1 - q$ and (5.5) give, after discarding a nonnegative energy term,

$$-\frac{1}{1-q} \frac{\mathrm{d}}{\mathrm{d}t} \int \eta^2 v^{1-q}\, \mathrm{d}x + I(t) \le C_q(d^{-\alpha} + \|b\|_\infty d^{-1}) \int_{B_{R_\xi}} v^{1-q}\, \mathrm{d}x. \tag{7.4}$$

In the cross interaction, (1.5) bounds the term containing $v(x)$ by $Cd^{-\alpha} \int_{B_{R_\xi}} v^{1-q}$. The term containing $v(y)$ controls $I(t)$ because $v(y) \ge u(y)$. The shifted source $f + F$ is nonnegative.

Let $t_- = \inf J_\xi$, $t_+ = \sup J_c$, and choose a nonincreasing time cutoff ζ equal to one on J_ξ, vanishing at t_+, with $|\zeta'| \le C/(c-\xi)$. Write $Y(t) = \int \eta^2 v^{1-q}$. Integration by parts gives

$$\int_{t_-}^{t_+} -Y'(t)\zeta(t)\,\mathrm{d}t = Y(t_-) + \int_{t_-}^{t_+} Y(t)\zeta'(t)\,\mathrm{d}t.$$

The initial term $Y(t_-)$ is nonnegative, and the integral containing ζ' is bounded below by $-C(c-\xi)^{-1}\int_{t_-}^{t_+} Y(t)\,\mathrm{d}t$. The same bound follows by taking the initial trace from inside the interval. Integrating (7.4) against ζ and using $I \ge 0$, we obtain

$$\int_{J_\xi} I(t)\,\mathrm{d}t \le C_q\big((\xi-a)(c-\xi)\big)^{-N} \int_{U_c} v^{1-q}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.5}$$

For $z \in B_{R_m}$ and $y \notin B_{R_\xi}$, the radius d satisfies $d \le |z-y|/4$ and $\eta = 1$ on $B_d(z)$. Upper jumping smoothness gives

$$\begin{aligned} I(t) &\ge M(c)^{-q} \int_{B_d(z)} \int_{B^c_{R_\xi}} u(t,y)K(t;x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\ge C^{-1} d^n M(c)^{-q} \int_{B^c_{R_\xi}} u(t,y)K(t;z,y)\,\mathrm{d}y. \end{aligned}$$

Taking the essential supremum over z, integrating over J_ξ, and applying (7.5) give $S_{a,\xi} \le C((\xi-a)(c-\xi))^{-N} M(c)^q \int_{U_c} v^{1-q}$. Since $U_c \subset P_1$, Hölder's inequality and (7.1) bound the integral by $|U_c|^q(\int_{P_1} v)^{1-q} \le CH^{1-q}$. This gives (7.2), and Young's inequality yields (7.3). □

7.2. **Interpolation for the local supremum.**

Lemma 7.3 (Local interpolation with an integrated tail)**.** *For $0 \le a < \xi \le 1$ and every $\delta \in (0,1)$,*

$$\operatorname*{ess\,sup}_{U_a} u \le \delta M(\xi) + C_\delta(\xi-a)^{-N_\delta}\|u\|_{L^2(U_\xi)} + CS_{a,\xi} + CF. \tag{7.6}$$

Proof. On J_ξ, put

$$\ell(t) = \int_{\inf J_\xi}^{t} \big(F + \Theta(u;\tau,R_m,R_\xi)\big)\,\mathrm{d}\tau, \qquad z = u - \ell(t).$$

By Lemma 2.6, we may test u at the moving levels $\ell(t) + k_j$ to estimate the positive truncations of z. The derivative ℓ' absorbs the source term and the tail from $B^c_{R_\xi}$ in (4.5). Apply the iteration of Proposition 4.5 between R_m and R_a, shrinking the time intervals from J_ξ to J_a. Split the form at each outer radius and keep the cutoff support a fixed fraction of the radial gap inside it. On the remaining annulus inside B_{R_ξ}, we have $u \le M(\xi)$, so (1.5) bounds its contribution by $C2^{\alpha j}(\xi-a)^{-\alpha}M(\xi)$. Since $z_+ \le u$, the integrals A_j satisfy

$$A_{j+1} \le C(\xi-a)^{-N} 2^{Nj} k^{-2(\kappa-1)} (1 + M(\xi)/k)^\kappa A_{j-1}^\kappa.$$

The initial integral is at most $\|u\|^2_{L^2(U_\xi)}$. Choose

$$k = \delta M(\xi) + C_\delta(\xi-a)^{-N_\delta}\|u\|_{L^2(U_\xi)}.$$

Then $M(\xi)/k \le \delta^{-1}$. Increasing C_δ and N_δ makes A_{-1}/k^2 small enough for Lemma A.1, and hence $z \le k$ on U_a. Since $0 \le \ell(t) \le S_{a,\xi} + CF$ on J_a, adding ℓ gives (7.6) with a tail coefficient independent of k. □

7.3. **Absorption on nested cylinders.**

Proof of Theorem 1.5. At unit scale fix $0 \le a < c \le 1$ and set $\xi = (a+c)/2$. Because $u \le v \le M(c)$ on U_ξ, equation (7.1) gives

$$\|u\|_{L^2(U_\xi)} \le \left(M(c) \int_{U_\xi} u \right)^{1/2} \le C M(c)^{1/2} H^{1/2}.$$

Apply Lemma 7.3 and estimate $S_{a,\xi}$ by Lemma 7.2 with $q = 1/2$. The L^2 term and the tail term are then each bounded by a power of $(c-a)^{-1}$ times $C_\delta M(c)^{1/2} H^{1/2}$. Take $\delta = 1/4$, so $\delta M(\xi) \le M(c)/4$. Apply Young's inequality to each of the two terms, allowing $M(c)/8$ for each. Since $v - u \le 5F + \varepsilon \le H$, we obtain

$$M(a) \le \tfrac{1}{2} M(c) + C(c-a)^{-N} H \qquad (0 \le a < c \le 1). \tag{7.7}$$

By Lemma A.2, applied to the bounded function M on $[0,1]$, (7.7) implies $M(0) \le CH$. Letting $\varepsilon \downarrow 0$ proves

$$\operatorname*{ess\,sup}_{H_1} u \le C \left(\operatorname*{ess\,inf}_{F_1} u + \|f\|_\infty \right).$$

Rescaling as in the proof of Theorem 1.2 gives (1.17). The dependence on C_J enters through the pointwise upper bound in Lemma 7.1 and the lower bound for $I(t)$ in Lemma 7.2. □

8. Kernel examples and exterior time dependence

8.1. **A kernel supported on alternating annuli.**

Example 8.1. Let

$$\mathcal{A} = \bigcup_{j \in \mathbb{Z}} [2^j, \tfrac{3}{2}\, 2^j], \qquad K_{\mathcal{A}}(x,y) = \mathbf{1}_{\mathcal{A}}(|x-y|) |x-y|^{-n-\alpha}.$$

This kernel satisfies (1.3)–(1.5) and (1.16). For every $j \in \mathbb{Z}$ it vanishes on the annulus $\frac{3}{2}\, 2^j < |x-y| < 2^{j+1}$.

Proof. Symmetry, the upper form bound, and the row bound follow from $0 \le K_{\mathcal{A}}(x,y) \le |x-y|^{-n-\alpha}$ and

$$\int_{|z| \ge r} |z|^{-n-\alpha} \, \mathrm{d}z = \frac{|\mathbb{S}^{n-1}|}{\alpha} r^{-\alpha}.$$

For the lower form bound on a ball D, write

$$Q_{\mathcal{A}}(v; D) = \iint_{D \times D} \mathbf{1}_{\mathcal{A}}(|x-y|) \frac{|v(x) - v(y)|^2}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y.$$

Suppose $x, y \in D$ and $r = |y - x| \in (3a/2, 2a)$, with $a = 2^j$. Set $z = y - x$ and $w = x + 2z/3$. The ball D is convex, so $w \in D$. The two segments have lengths

$$|w - x| = 2r/3 \in (a, 4a/3) \subset \mathcal{A}, \qquad |y - w| = r/3 \in (a/2, 2a/3) \subset \mathcal{A}.$$

By $|v(x) - v(y)|^2 \le 2|v(x) - v(w)|^2 + 2|v(w) - v(y)|^2$, the missing pairs contribute at most the sum of two integrals. In the first, change (x, z) to (x, z_1) with $z_1 = 2z/3$. In the second, use (w, z_2) with $z_2 = z/3$. The Jacobian and the kernel weight give the factors $(2/3)^\alpha$ and $(1/3)^\alpha$, respectively. Each map is injective on the union of the missing annuli, and its image consists of pairs in $D \times D$ with separation in $\mathcal{A}$. Enlarging the two integration domains to all such pairs gives

$$[v]_{s,D}^2 \le \left(1 + 2(2/3)^\alpha + 2(1/3)^\alpha\right) Q_{\mathcal{A}}(v; D). \tag{8.1}$$

Since $\mathcal{E}_{K_{\mathcal{A}}, D}(v, v) = Q_{\mathcal{A}}(v; D)/2$, (1.3) holds with

$$\lambda = \left[2\left(1 + 2(2/3)^\alpha + 2(1/3)^\alpha\right)\right]^{-1}.$$

To verify (1.16), fix h with $d = |h| \in [a, 3a/2]$ and let $0 < r \le d/4$. Write $e = h/d$. Take $h_* = h + (r/4)e$ if $d \le 5a/4$, and $h_* = h - (r/4)e$ otherwise. In either case

$$B_{r/8}(h_*) \subset B_r(h) \cap \{z : a < |z| < 3a/2\}. \tag{8.2}$$

The shift moves h away from the nearer of the two annular boundaries by $r/4$. The distance from h_* to the other boundary is at least $a/4 - r/4 \ge 5a/32$, since $r \le 3a/8$. Both distances exceed $r/8$, and $|h_* - h| + r/8 = 3r/8 < r$. Thus (8.2) holds. On $B_{r/8}(h_*)$ we have $|z| \le d + r \le 5d/4$, so

$$⨍_{B_r(h)} \mathbf{1}_{\mathcal{A}}(|z|)|z|^{-n-\alpha}\,\mathrm{d}z \ge 8^{-n}(5/4)^{-n-\alpha}d^{-n-\alpha}.$$

For $d \notin \mathcal{A}$ the kernel vanishes. Together these cases give (1.16) with $C_J = 8^n(5/4)^{n+\alpha}$. □

Corollary 8.2 (Measurable coefficients on the annular kernel)**.** *Let $a(t; x, y) = a(t; y, x)$ be jointly measurable and satisfy $0 < a_0 \le a(t; x, y) \le a_1 < \infty$. The kernel*

$$K(t; x, y) = a(t; x, y)K_{\mathcal{A}}(x, y)$$

satisfies the hypotheses of Theorem 1.5. The constants in the kernel bounds depend only on n, s, a_0, a_1.

Proof. Multiplication by a preserves the lower form bound with constant $a_0\lambda$ and the upper form and row bounds with constant $a_1\Lambda$. For the radii in (1.16),

$$K(t; x, y) \le a_1 C_J ⨍_{B_r(x)} K_{\mathcal{A}}(z, y)\,\mathrm{d}z \le (a_1/a_0)C_J ⨍_{B_r(x)} K(t; z, y)\,\mathrm{d}z.$$

Thus $(a_1/a_0)C_J$ is an upper jumping smoothness constant. □

8.2. **Concentrated jumps.** The next kernel satisfies the form and row bounds but fails upper jumping smoothness. Its added jumps have bounded total intensity at each point and connect pairs of balls of shrinking radius.

Example 8.3. There is a time-independent symmetric kernel satisfying (1.3)–(1.5) for which (1.16) fails for every finite constant C_J.

Proof. Let e be a unit vector, put $z_j = 20je$, and choose $0 < \varepsilon_j < 1/32$ with $\varepsilon_j \downarrow 0$. Define

$$A_j = B_{\varepsilon_j}(z_j), \qquad D_j = B_{\varepsilon_j}(z_j + e), \qquad m_j = |A_j| = |D_j|,$$

and set

$$K(x, y) = |x - y|^{-n-\alpha} + \sum_{j=1}^{\infty} m_j^{-1}(\mathbf{1}_{A_j}(x)\mathbf{1}_{D_j}(y) + \mathbf{1}_{D_j}(x)\mathbf{1}_{A_j}(y)). \tag{8.3}$$

The added kernel is nonnegative and symmetric. The first term gives (1.3). Each row of the added kernel has total mass at most one, and all its jumps have length in $[15/16, 17/16]$. Consequently its contribution to (1.5) is bounded by $Cr^{-\alpha}$.

For the upper form bound, let $V_j = B_3(z_j)$ and write $c_j = ⨍_{V_j} v$. The added energy is

$$\begin{aligned}\sum_j m_j^{-1} \int_{A_j}\int_{D_j} |v(x) - v(y)|^2\,\mathrm{d}y\,\mathrm{d}x &\le 2\sum_j \int_{A_j \cup D_j} |v - c_j|^2\,\mathrm{d}x \\ &\le C\sum_j [v]^2_{s,V_j} \le C[v]^2_s.\end{aligned}$$

Here the fractional Poincaré inequality on the fixed-radius balls follows directly from

$$\int_{V_j} |v - c_j|^2\,\mathrm{d}x = \frac{1}{2|V_j|} \iint_{V_j \times V_j} |v(x) - v(y)|^2\,\mathrm{d}x\,\mathrm{d}y \le C[v]^2_{s,V_j}.$$

Disjointness of the V_j gives $\sum_j [v]^2_{s,V_j} \le [v]^2_s$, proving (1.4).

Fix $r_0 = 1/8$. For almost every $x \in A_j$ and $y \in D_j$, we have $r_0 < |x-y|/4$ and $K(x,y) \geq m_j^{-1}$. The average over $B_{r_0}(x)$ satisfies

$$\fint_{B_{r_0}(x)} K(z,y)\,\mathrm{d}z \leq C + |B_{r_0}|^{-1}.$$

The stable part is uniformly bounded on $B_{r_0}(x)$ for these y, and the integral of the added kernel over the ball is at most $m_j^{-1}|A_j| = 1$. As $m_j \to 0$, the ratio of $K(x,y)$ to its average is unbounded. □

The kernel in (8.3) defines a regular symmetric Dirichlet form on $H^s(\mathbb{R}^n)$ with core $C_c^\infty(\mathbb{R}^n)$, since the added finite-rate form is bounded on L^2. For such autonomous forms on Euclidean space, a uniform parabolic Harnack inequality with time scale r^α implies upper jumping smoothness [5, Proposition 3.3]. The kernel in Example 8.3 therefore rules out a uniform full Harnack inequality under (1.3)–(1.5) alone, even for $b = 0$.

8.3. **Time separation for exterior contributions.** In Lemma 7.2, the tail is integrated before the cylinder F_1 on which the infimum is taken. An exterior datum supported at later times shows why the order of these intervals matters.

Example 8.4. Even for $b = 0$ and $K(x,y) = |x-y|^{-n-\alpha}$, an exterior tail integrated over a later time interval cannot be bounded by the average of a nonnegative local solution on an earlier cylinder alone.

Proof. Let $D = B_8$ and choose a smooth nonnegative function $g(t,y)$, nonzero and compactly supported in $(\tau, \tau+1) \times (B_{11} \setminus \overline{B}_{10})$. Let L_D denote the nonnegative Dirichlet realization of the stable form on the subspace

$$V_D = \{w \in H^s(\mathbb{R}^n) : w = 0 \text{ a.e. on } D^c\}.$$

For $x \in D$, set

$$F_g(t,x) = \int_{D^c} g(t,y)|x-y|^{-n-\alpha}\,\mathrm{d}y, \qquad w(t) = \int_\tau^t e^{-(t-r)L_D} F_g(r)\,\mathrm{d}r$$

for $t > \tau$, and set $w(t) = 0$ for $t \leq \tau$. Extend w by zero outside D. The Dirichlet semigroup preserves positivity, so $w \geq 0$. The energy estimate for the equation with source F_g gives $w \in L_t^\infty L_x^2 \cap L_t^2 H_x^s$ on finite time intervals. Since g is smooth, nonnegative, and compactly supported, $u = w + g$ is nonnegative and belongs to the same energy space.

For a test function ϕ supported in D, the cross interaction is $\mathcal{E}_t(g,\phi) = -\int_D F_g \phi\,\mathrm{d}x$. This cancels the source in the equation for w, so u satisfies $\partial_t u + \mathcal{L}u = 0$ in D. The solution vanishes before τ, whereas

$$\int_\tau^{\tau+1} \operatorname*{ess\,sup}_{x \in B_2} \int_{\mathbb{R}^n \setminus B_4} u(t,y)|x-y|^{-n-\alpha}\,\mathrm{d}y\,\mathrm{d}t > 0,$$

because $u = g$ on the support of the nonzero exterior datum. □

Appendix A. Iteration lemmas

A.1. **Superlinear and interpolation iterations.**

Lemma A.1 (Geometric convergence)**.** *Suppose that $z_j \geq 0$ and*

$$z_{j+1} \leq AB^j z_j^{1+\delta}, \qquad j \geq 0, \qquad A > 0,\ B > 1,\ \delta > 0.$$

If $z_0 \leq A^{-1/\delta} B^{-1/\delta^2}$, then

$$z_j \leq A^{-1/\delta} B^{-j/\delta - 1/\delta^2} \quad (j \geq 0).$$

In particular $z_j \to 0$. A recurrence $Y_{j+1} \leq AB^j Y_{j-1}^{1+\delta}$ for $j \geq 0$ has the same conclusion when Y_{-1} and Y_0 are sufficiently small.

Proof. Substituting the proposed bound for z_j into the recurrence gives

$$AB^j\left(A^{-1/\delta}B^{-j/\delta-1/\delta^2}\right)^{1+\delta} = A^{-1/\delta}B^{-(j+1)/\delta-1/\delta^2}.$$

Induction proves the first assertion. For the two-step recurrence set $Z_m = \max\{Y_{2m-1}, Y_{2m}\}$. The inequalities at $j = 2m$ and $j = 2m+1$ imply

$$Z_{m+1} \le AB^{2m+1}Z_m^{1+\delta}.$$

Apply the first assertion with constants AB and B^2. □

Lemma A.2 (Absorption on nested sets)**.** *Let F be a bounded nonnegative function on $[a,b]$. Suppose that*

$$F(r) \le qF(R) + A(R-r)^{-N}, \qquad a \le r < R \le b,$$

where $0 < q < 1$, $A \ge 0$, and $N > 0$. Then

$$F(a) \le C(q,N)A(b-a)^{-N}.$$

Proof. Choose $\vartheta \in (q^{1/N}, 1)$ and put $r_j = a + (b-a)(1-\vartheta^j)$. Repeated substitution gives

$$F(a) \le q^m F(r_m) + A(b-a)^{-N}(1-\vartheta)^{-N}\sum_{j=0}^{m-1}(q\vartheta^{-N})^j.$$

The first term tends to zero because F is bounded. The geometric series converges by the choice of ϑ. □

A.2. **The logarithmic iteration.** The following form of the Bombieri–Giusti argument [3] combines a moment inequality with a bound on the upper level sets of the logarithm. Section 5 applies it separately to a positive function and its reciprocal.

Lemma A.3 (Moment control from logarithmic level sets)**.** *Let $(U_r)_{1/2\le r\le 1}$ be an increasing family of measurable sets in a finite measure space. Normalize the ambient measure μ so that $\mu(U_1) \le 1$, and suppose that $\inf_r \mu(U_r) > 0$. Let $g > 0$ and $g \in L^{p_0}(U_1)$ for some $p_0 > 0$. Assume that, for constants $A \ge 1$ and $\nu > 0$,*

$$\|g\|_{L^{p_0}(U_r)} \le \left(\frac{A}{(R-r)^\nu}\right)^{1/p}\|g\|_{L^p(U_R)}, \qquad \tfrac{1}{2} \le r < R \le 1, \quad 0 < p \le p_0/2, \tag{A.1}$$

$$\mu\{x \in U_1 : \log g(x) > \ell\} \le \frac{A}{\ell}, \qquad \ell > 0. \tag{A.2}$$

Then $\|g\|_{L^{p_0}(U_{1/2})} \le C(A,\nu,p_0)$.

Proof. Put

$$F(r) = \max\left\{1, \log\|g\|_{L^{p_0}(U_r)}\right\}.$$

This is a bounded nondecreasing function. Fix $r < R$ and write $J = \log\|g\|_{L^{p_0}(U_R)}$. When J is larger than a constant depending on A, p_0, choose

$$p = \frac{\log J}{2J} \le \frac{p_0}{2}.$$

Split the integral of g^p at $\log g = J/2$. By (A.2) and Hölder's inequality,

$$\begin{aligned}\int_{U_R} g^p \,\mathrm{d}\mu &\le e^{pJ/2} + \left(\int_{U_R} g^{p_0}\,\mathrm{d}\mu\right)^{p/p_0}\left(\frac{2A}{J}\right)^{1-p/p_0}\\ &= J^{1/4} + J^{1/2}\left(\frac{2A}{J}\right)^{1-p/p_0} \le 2J^{1/4}.\end{aligned}$$

Indeed, the second term equals $2AJ^{-1/2}\exp((p/p_0)\log(J/(2A)))$. The exponential factor stays bounded for large J because $(\log J)^2/J \to 0$.

Let $D = \log A + \nu \log(1/(R-r))$. Taking logarithms in (A.1) now gives

$$\log \|g\|_{L^{p_0}(U_r)} \le \frac{D+\log 2}{p} + \frac{\log J}{4p} = \frac{J}{2} + \frac{2J(D+\log 2)}{\log J}.$$

If $\log J \ge 8(D+\log 2)$, the right-hand side is at most $3J/4$. In the remaining case, or when J is below the fixed lower threshold, monotonicity gives

$$F(r) \le F(R) \le C\exp(8D) \le C(R-r)^{-8\nu}.$$

Combining the two cases, and increasing C to cover $F(r) = 1$, yields

$$F(r) \le \tfrac{3}{4}F(R) + C(R-r)^{-8\nu}.$$

Since $g \in L^{p_0}(U_1)$, the function F is bounded and Lemma A.2 applies. Exponentiating the resulting bound on $F(1/2)$ proves the claim. □

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