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Non-Archimedean Riesz-Frechet Representation

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Abstract: We derive a Riesz-Frechet representation for bounded linear functionals defined on the p-adic Hilbert spaces introduced by Kalisch [*Ann. of Math. (2), 1947*]. We also notice the surprising difference between the Archimedean case and the non-Archimedean case (exact non-Archimedean version of Riesz-Frechet representation fails).
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1. Introduction

The century-old Riesz-Frechet representation (theorem) states that every bounded linear functional on a real/complex Hilbert space is completely determined by a single element in that Hilbert space.

Theorem 1.1. *[1, 2, 4, 8–14, 18, 19, 19–21, 23, 26, 28, 31–36, 39, 43, 44, 46] (**Riesz-Frechet Representation**) Let $\mathcal{H}$ be a complex Hilbert space. Let $f : \mathcal{H} \to \mathbb{C}$ be a bounded linear functional. Then there is a unique element $\tau \in \mathcal{H}$ satisfying the following conditions.*

(i) $f(h) = \langle h, \tau \rangle$ *for all* $h \in \mathcal{H}$.
(ii) $\|f\| = \|\tau\|$.

Over time, Riesz-Frechet representation has been extended to Lebesgue spaces [30, 37, 42] and spaces of continuous functionals on compact Hausdorff spaces (celebrated Riesz-Markov-Kakutani representation) [2, 3, 7, 8, 15, 24, 29, 35, 38, 45]. It is known that Riesz-Frechet representation also holds for quaternionic Hilbert spaces [16]. Riesz-Frechet representation has also been derived for certain classes of Banach spaces by Giles [17] and James [22]. The traditional proof of the Riesz-Frechet representation uses the orthogonal decomposition theorem, which rests on the parallelogram law [19, 32]. If the Hilbert space is separable, then one can also give a proof of the Riesz-Frechet representation for Hilbert spaces using orthonormal bases, which relies on the Gram-Schmidt orthonormalization process [20]. We are fundamentally motivated by the following question.

Question 1.2. *What is the non-Archimedean Riesz-Frechet representation?*

We begin by recalling the definition of non-Archimedean Banach and p-adic Hilbert spaces.

Definition 1.3. *[40] Let $\mathbb{K}$ be a field. A map $|\cdot| : \mathbb{K} \to [0, \infty)$ is said to be a **non-Archimedean valuation** if the following conditions hold:*

(i) *If $\lambda \in \mathbb{K}$ satisfies $|\lambda| = 0$, then $\lambda = 0$.*

(ii) $|\lambda\mu| = |\lambda||\mu|$ *for all* $\lambda, \mu \in \mathbb{K}$.
(iii) *(Ultratriangle inequality)* $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ *for all* $\lambda, \mu \in \mathbb{K}$.

We say $\mathbb{K}$ *is* ***non-Archimedean complete valued field*** *if it is complete w.r.t. the ultrametric*

$$d(\lambda, \mu) \coloneqq |\lambda - \mu|, \quad \forall \lambda, \mu \in \mathbb{K}.$$

Definition 1.4. *[41] Let* $\mathbb{K}$ *be a non-Archimedean complete valued field. A vector space* $\mathcal{X}$ *over* $\mathbb{K}$ *is said to be* ***non-Archimedean Banach space*** *if there is a map (called as ultra-norm)* $\|\cdot\| : \mathcal{X} \to [0, \infty)$ *satisfying the following conditions:*

(i) *If* $x \in \mathcal{X}$ *is such that* $\|x\| = 0$*, then* $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in \mathbb{K}$*, for all* $x \in \mathcal{X}$.
(iii) *(Ultranorm inequality)* $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ *for all* $x, y \in \mathcal{X}$.
(iv) $\mathcal{X}$ *is complete w.r.t. the ultrametric*

$$d(x, y) \coloneqq \|x - y\|, \quad \forall x, y \in \mathcal{X}.$$

Definition 1.5. *[25] Let* $\mathbb{K}$ *be a non-Archimedean valued field (with valuation* $|\cdot|$*) and* $\mathcal{X}$ *be a non-Archimedean Banach space (with norm* $\|\cdot\|$*) over* $\mathbb{K}$*. We say that* $\mathcal{X}$ *is a* ***p-adic Hilbert space*** *if there is a map (called as* ***p-adic inner product****)* $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ *satisfying the following conditions:*

(i) *If* $x \in \mathcal{X}$ *is such that* $\langle x, y \rangle = 0$ *for all* $y \in \mathcal{X}$*, then* $x = 0$.
(ii) $\langle x, y \rangle = \langle y, x \rangle$ *for all* $x, y \in \mathcal{X}$.
(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ *for all* $\alpha \in \mathbb{K}$*, for all* $x, y \in \mathcal{X}$.
(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ *for all* $x, y, z \in \mathcal{X}$.
(v) $|\langle x, y \rangle| \leq \|x\|\|y\|$ *for all* $x, y \in \mathcal{X}$.

Definition 1.5 is more general than the definition of p-adic Hilbert spaces used by Kalisch, which imposes further conditions [25]. We note that the Definition 1.5 is different than the p-adic Hilbert spaces defined recently by Claussnitzer and Thom [6] where an exact Riesz-Frechet representation is known [5]. We derive two different versions of non-Archimedean Riesz-Frechet representation with different assumptions and different conclusions.

2. Non-Archimedean Riesz-Frechet Representation

In the paper, we use the following notion of Schauder basis for non-Archimedean Banach spaces and orthonormal basis for p-adic Hilbert spaces.

Definition 2.1. *Let* $\mathcal{X}$ *be a non-Archimedean Banach space over* $\mathbb{K}$*. A collection* $\{\tau_n\}_n$ *in* $\mathcal{X}$ *is said to be a* ***Schauder basis*** *for* $\mathcal{X}$ *if for every* $x \in \mathcal{X}$*, there is a unique sequence* $\{\alpha_n(x)\}_n$ *in* $\mathbb{K}$ *such that*

$$x = \sum_{n=1}^{\infty} \alpha_n(x) \tau_n,$$

series converges in the ultra-norm on $\mathcal{X}$.

Definition 2.2. *Let* $\mathcal{X}$ *be a p-adic Hilbert space over* $\mathbb{K}$*. A Schauder basis* $\{\tau_n\}_n$ *for* $\mathcal{X}$ *is said to be a* ***p-adic orthonormal basis*** *for* $\mathcal{X}$ *if the following conditions hold:*

(i) $\|\tau_n\| \leq 1$ *for all* $n \in \mathbb{N}$.
(ii) $\langle \tau_n, \tau_m \rangle = \delta_{n,m}$ *for all* $n, m \in \mathbb{N}$.

Note that given a p-adic orthonormal basis $\{\tau_n\}_n$ for $\mathcal{X}$, condition (ii) in Definition 2.2 says that

$$1 = |\langle \tau_n, \tau_n \rangle| \leq \|\tau_n\|^2, \quad \forall n \in \mathbb{N}.$$

By using condition (i) in Definition 2.2 we then get $\|\tau_n\| = 1$ for all $n \in \mathbb{N}$. Thus all elements of p-adic orthonormal basis lie in the unit sphere. Following is the standard example of p-adic Hilbert space we keep in mind.

Example 2.3. *Let $\mathbb{K}$ be a non-Archimedean valued field. Define*

$$c_0(\mathbb{N}, \mathbb{K}) := \left\{ \{a_n\}_n : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \lim_{n\to\infty} a_n = 0 \right\}.$$

We define

$$\|\{a_n\}_n\| := \max_{n\in\mathbb{N}} |a_n|, \quad \forall \{a_n\}_n \in c_0(\mathbb{N}, \mathbb{K})$$

and

$$\langle \{a_n\}_n, \{b_n\}_n \rangle := \sum_{n=1}^{\infty} a_n b_n, \quad \forall \{a_n\}_n, \{b_n\}_n \in c_0(\mathbb{N}, \mathbb{K}).$$

Then $c_0(\mathbb{N}, \mathbb{K})$ is a p-adic Hilbert space. The canonical Schauder basis $\{e_n\}_n$ is a p-adic orthonormal basis for $c_0(\mathbb{N}, \mathbb{K})$.

Gram-Schmidt orthonormalization says that every separable Hilbert space has an orthonormal basis [27]. Eventhough we know from Example 2.3 that we have examples of p-adic orthonormal bases, we don't know existence result of p-adic orthonormal basis in every p-adic Hilbert space.
Following is our first version of Riesz-Frechet representation.

Theorem 2.4. *(**Non-Archimedean Riesz-Frechet Representation**) Let $\mathcal{X}$ be a p-adic Hilbert space over a complete non-Archimedean valued field $\mathbb{K}$. Let $f : \mathcal{X} \to \mathbb{K}$ be a bounded linear functional. Assume that there is a p-adic orthonormal basis $\{\omega_n\}_n$ for $\mathcal{X}$ such that*

$$\lim_{n\to\infty} f(\omega_n)\omega_n = 0.$$

Then there is a unique element $\tau \in \mathcal{X}$ satisfying the following conditions:

(i) $f(x) = \langle x, \tau \rangle$ *for all* $x \in \mathcal{X}$.
(ii) $\|f\| = \|\tau\|$.

Proof. Since $\{\omega_n\}_n$ is orthonormal,

$$\left\| \sum_{k=1}^{n} f(\omega_k)\omega_k \right\| = \max_{1\leq k\leq n} |f(\omega_k)|, \quad \forall n \in \mathbb{N}.$$

Since $\mathcal{X}$ is complete and $\lim_{n\to\infty} f(\omega_n)\omega_n = 0$, the series $\sum_{n=1}^{\infty} f(\omega_n)\omega_n$ converges in $\mathcal{X}$. Now define

$$\tau := \sum_{n=1}^{\infty} f(\omega_n)\omega_n.$$

We then have

$$\begin{aligned} f(x) = f\left(\sum_{n=1}^{\infty} \langle x, \omega_n \rangle \omega_n \right) &= \sum_{n=1}^{\infty} \langle x, \omega_n \rangle f(\omega_n) \\ &= \left\langle x, \sum_{n=1}^{\infty} f(\omega_n)\omega_n \right\rangle = \langle x, \tau \rangle, \quad \forall x \in \mathcal{X}. \end{aligned}$$

Now

$$\|f\| = \sup\left\{\frac{|f(x)|}{\|x\|} : x \in \mathcal{X}, x \neq 0\right\} = \sup\left\{\frac{|\langle x, \tau\rangle|}{\|x\|} : x \in \mathcal{X}, x \neq 0\right\} \leq \|\tau\|.$$

We now observe that

$$\|\tau\| = \left\|\sum_{n=1}^{\infty} f(\omega_n)\omega_n\right\| = \max_{n\in\mathbb{N}} |f(\omega_n)| \leq \|f\| \max_{n\in\mathbb{N}} \|\omega_n\| = \|f\|.$$

Uniqueness easily follows using the definition of p-adic inner product. □

Example 2.5. *Let p be a prime number. Define*

$$f : c_0(\mathbb{N}, \mathbb{Q}_p) \ni \{a_n\}_n \mapsto f(\{a_n\}_n) \coloneqq \sum_{n=1}^{\infty} p^n a_n \in \mathbb{Q}_p.$$

Let $\{a_n\}_n \in c_0(\mathbb{N}, \mathbb{Q}_p)$. Then

$$|p^n a_n|_p = \frac{1}{p^n}|a_n|_p \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} p^n a_n = 0, \quad \forall \{a_n\}_n \in c_0(\mathbb{N}, \mathbb{Q}_p).$$

So the map f is well-defined. Define

$$\tau \coloneqq \{p^n\}_n \in c_0(\mathbb{N}, \mathbb{Q}_p).$$

Then

$$f(x) = \langle x, \tau\rangle, \ \forall x \in c_0(\mathbb{N}, \mathbb{Q}_p) \quad \text{and} \quad \|f\| = \|\tau\| = \max_{n\in\mathbb{N}} |p^n|_p = \max_{n\in\mathbb{N}}\left\{\frac{1}{p^n}\right\} = \frac{1}{p}.$$

In the next example we show that every bounded linear functional on a p-adic Hilbert space cannot be represented by an element in the space.

Example 2.6. *Let p be a prime number. Define*

$$f : c_0(\mathbb{N}, \mathbb{Q}_p) \ni \{a_n\}_n \mapsto f(\{a_n\}_n) \coloneqq \sum_{n=1}^{\infty} n a_n \in \mathbb{Q}_p.$$

Let $\{a_n\}_n \in c_0(\mathbb{N}, \mathbb{Q}_p)$. Then

$$|na_n|_p = |n||a_n|_p \leq |a_n|_p \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} n a_n = 0, \quad \forall \{a_n\}_n \in c_0(\mathbb{N}, \mathbb{Q}_p).$$

So the map f is well-defined. We claim that there is no $\tau \in c_0(\mathbb{N}, \mathbb{Q}_p)$ such that $f(x) = \langle x, \tau\rangle$ for all $x \in c_0(\mathbb{N}, \mathbb{Q}_p)$. Let us suppose that this claim fails. Then thee exists a $\tau \in c_0(\mathbb{N}, \mathbb{Q}_p)$ such that $f(x) = \langle x, \tau\rangle$ for all $x \in c_0(\mathbb{N}, \mathbb{Q}_p)$. Set $\tau \coloneqq \{a_n\}_n \in c_0(\mathbb{N}, \mathbb{Q}_p)$. Let $\{e_n\}$ be the standard p-adic orthonormal basis for $c_0(\mathbb{N}, \mathbb{Q}_p)$. Now we see that

$$n = f(e_n) = \langle e_n, \tau\rangle = a_n, \quad \forall n \in \mathbb{N}.$$

But then $\tau = \{n\}_n \notin c_0(\mathbb{N}, \mathbb{Q}_p)$ because $|n|_p \nrightarrow 0$ as $n \to \infty$. Hence Riesz-Frechet representation fails.

Let $\mathcal{X}$ be a p-adic Hilbert space. A closed subspace $\mathcal{Y}$ of $\mathcal{X}$ is said to be orthogonally complementable if there exists a unique closed subspace $\mathcal{Z}$ of $\mathcal{X}$ satisfying the following: for every $x \in \mathcal{X}$, there exist unique $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$ such that

(i) $x = y + z$.
(ii) $\langle y, z \rangle = 0$.
(iii) $\|x\| = \max\{\|y\|, \|z\|\}$.

In this case, we write $x = y \oplus z$, $\mathcal{Z} = \mathcal{Y}^{\perp}$ and $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^{\perp}$.
Following is our second version of Riesz-Frechet representation.

Theorem 2.7. *(**Non-Archimedean Riesz-Frechet Representation**) Let $\mathcal{X}$ be a p-adic Hilbert space over a complete non-Archimedean valued field $\mathbb{K}$. Let $f : \mathcal{X} \to \mathbb{K}$ be a non-zero bounded linear functional. Assume that $\mathrm{Ker}(f)$ is orthogonally complementable in $\mathcal{X}$ and there exists a (nonzero) vector $\omega \in \mathrm{Ker}(f)^{\perp}$ such that $\langle \omega, \omega \rangle \neq 0$. Then there is a unique element $\tau \in \mathcal{X}$ satisfying the following conditions:*

(i) *$f(x) = \langle x, \tau \rangle$ for all $x \in \mathcal{X}$.*
(ii)

$$\frac{|\langle \tau, \tau \rangle|}{\|\tau\|} \leq \|f\| \leq \|\tau\|.$$

Moreover,

$$\|\tau\| = \frac{|f(\omega)|}{|\langle \omega, \omega \rangle|}\|\omega\| \leq \frac{\|f\|\|\omega\|^2}{|\langle \omega, \omega \rangle|}.$$

Proof. Define

$$\tau \coloneqq \frac{f(\omega)}{\langle \omega, \omega \rangle}\omega.$$

Let $x \in \mathcal{X}$. Then

$$f\left(x - \frac{f(x)}{f(\omega)}\omega\right) = f(x) - f(x) = 0.$$

Hence

$$x - \frac{f(x)}{f(\omega)}\omega \in \mathrm{Ker}(f).$$

We then have

$$\begin{aligned} 0 &= \left\langle x - \frac{f(x)}{f(\omega)}\omega, \tau \right\rangle = \langle x, \tau \rangle - \left\langle \frac{f(x)}{f(\omega)}\omega, \tau \right\rangle \\ &= \langle x, \tau \rangle - \frac{f(x)}{f(\omega)}\langle \omega, \tau \rangle = \langle x, \tau \rangle - \frac{f(x)}{f(\omega)}\left\langle \omega, \frac{f(\omega)}{\langle \omega, \omega \rangle}\omega \right\rangle \\ &= \langle x, \tau \rangle - f(x). \end{aligned}$$

Note that

$$|\langle \tau, \tau \rangle| = |f(\tau)| \leq \|f\|\|\tau\|.$$

□

We next extend the non-Archimedean Riesz-Frechet representation to non-Archimedean fields admitting conjugation (involution). We use the following definition of conjugation.

Definition 2.8. *Let $\mathbb{K}$ be a non-Archimedean complete valued field (with valuation $|\cdot|$). A map $\sigma : \mathbb{K} \to \mathbb{K}$ is said to be a* ***conjugation*** *or* ***involution*** *if the following conditions hold:*

(i) $\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta)$ *for all* $\alpha, \beta \in \mathbb{K}$.
(ii) $\sigma(\alpha\beta) = \sigma(\alpha)\sigma(\beta)$ *for all* $\alpha, \beta \in \mathbb{K}$.
(iii) $\sigma(\sigma(\alpha)) = \alpha$ *for all* $\alpha \in \mathbb{K}$.
(iv) $|\sigma(\alpha)| = |\alpha|$ *for all* $\alpha \in \mathbb{K}$.

In this case, we say that $\mathbb{K}$ is σ-conjugated.

Definition 2.9. *Let $\mathbb{K}$ be a σ-conjugated non-Archimedean complete valued field (with valuation $|\cdot|$) and $\mathcal{X}$ be a non-Archimedean Banach space (with norm $\|\cdot\|$) over $\mathbb{K}$. We say that $\mathcal{X}$ is a* ***p-adic σ-Hilbert space*** *if there is a map (called as* ***p-adic σ-inner product****) $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ satisfying the following:*

(i) *If $x \in \mathcal{X}$ is such that $\langle x, y \rangle = 0$ for all $y \in \mathcal{X}$, then $x = 0$.*
(ii) $\langle x, y \rangle = \sigma(\langle y, x \rangle)$ *for all* $x, y \in \mathcal{X}$.
(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ *for all* $\alpha \in \mathbb{K}$, *for all* $x, y \in \mathcal{X}$.
(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ *for all* $x, y, z \in \mathcal{X}$.
(v) $|\langle x, y \rangle| \leq \|x\|\|y\|$ *for all* $x, y \in \mathcal{X}$.

We now have p-adic versions of Theorem 2.4 and Theorem 2.7.

Theorem 2.10. *(****Non-Archimedean Riesz-Frechet Representation****) Let $\mathcal{X}$ be a p-adic σ-Hilbert space over a σ-conjugated complete non-Archimedean valued field $\mathbb{K}$. Let $f : \mathcal{X} \to \mathbb{K}$ be a bounded linear functional. Assume that there is a p-adic orthonormal basis $\{\omega_n\}_n$ for $\mathcal{X}$ such that*

$$\lim_{n\to\infty} f(\omega_n)\omega_n = 0.$$

Then there is a unique element $\tau \in \mathcal{X}$ satisfying the following conditions:

(i) $f(x) = \langle x, \tau \rangle$ *for all* $x \in \mathcal{X}$.
(ii) $\|f\| = \|\tau\|$.

Proof. Define

$$\tau := \sum_{n=1}^{\infty} \sigma(f(\omega_n))\omega_n.$$

The other arguments are similar to those used in the proof of Theorem 2.4. □

Theorem 2.11. *(****Non-Archimedean Riesz-Frechet Representation****) Let $\mathcal{X}$ be a p-adic σ-Hilbert space over a σ-conjugated complete non-Archimedean valued field $\mathbb{K}$. Let $f : \mathcal{X} \to \mathbb{K}$ be a non-zero bounded linear functional. Assume that $\mathrm{Ker}(f)$ is orthogonally complementable in $\mathcal{X}$ and there exists a (nonzero) vector $\omega \in \mathrm{Ker}(f)^{\perp}$ such that $\langle \omega, \omega \rangle \neq 0$. Then there is a unique element $\tau \in \mathcal{X}$ satisfying the following conditions:*

(i) $f(x) = \langle x, \tau \rangle$ *for all* $x \in \mathcal{X}$.
(ii)

$$\frac{|\langle \tau, \tau \rangle|}{\|\tau\|} \leq \|f\| \leq \|\tau\|.$$

Moreover,

$$\|\tau\| = \frac{|f(\omega)|}{|\langle \omega, \omega \rangle|}\|\omega\| \leq \frac{\|f\|\|\omega\|^2}{|\langle \omega, \omega \rangle|}.$$

Proof. Define

$$\tau := \frac{\sigma(f(\omega))}{\langle \omega, \omega \rangle}\omega.$$

The other arguments are similar to those used in the proof of Theorem 2.7. □

3. Conclusions

(1) In 1907, Riesz [34] and Frechet [13] independently showed that every continuous linear functional on a Hilbert space is completely determined by a single element in that Hilbert space [4].
(2) In 1938, Markov [29] and in 1941, Kakutani [24] extended the Riesz-Frechet representation to function spaces.
(3) In 1947, James [22] and 1967, Giles [17] extended the Riesz-Frechet representation to certain classes of Banach spaces.
(4) In this article, we derived two non-Archimedean Riesz representation theorems under certain conditions for p-adic Hilbert spaces.

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