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TORUS FIBRATIONS ON 3-DIMENSIONAL MANIFOLDS WHICH ARE SMOOTH FUNCTIONS WITH FINITELY MANY CRITICAL VALUES AND STABLE FOLD MAPS

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Abstract. We present smooth functions which are torus fibrations with finitely many critical values on 3-dimensional closed manifolds, from (*stable*) fold maps into the plane. *Fold* maps are higher dimensional versions of Morse functions and introduced and studied first in the 1950s by Thom and Whitney. Saeki, Sakuma, and so on, are important researchers, studying related singularity theory and differential topology mainly, since the 1990s. 3-dimensional closed manifolds always admit ones into the plane.

Surface fibrations are important tools and objects in complex geometry, symplectic geometry, and 4-dimensional geometry. It is important to construct important examples of the fibrations. In our paper, we present a 3-dimensional related new study. Related to this, the author has succeeded in explicit construction of explicit functions with prescribed level sets and fold maps whose singular values sets are concentric on manifolds such as connected sums of sphere bundles over spheres.

1. Introduction.

Morse(-Bott) function theory has been fundamental and strong in geometry. In singularity theory of differentiable maps and related geometry, these functions are important objects to study. Morse functions have been generalized to higher dimensional situations and *fold* maps are formulated and studied by Thom and Whitney first in the 1950s ([33, 34]).

For Morse functions, refer to [23, 24], and [1], for Morse-Bott functions. Introduction to singularity theory of differentiable maps is systematically explained in the textbook [7]. We also refer to the paper [17] of the author for several fundamental terminologies and notions, for example.

Fibrations over manifolds are fundamental and important in various geometry. Especially, in complex geometry, symplectic geometry, and 4-dimensional geometry, surface fibrations are important tools. They are also important and interesting objects to construct and study in various ways.

In the present paper, we present smooth functions on 3-dimensional closed, connected, and orientable manifolds which are torus fibrations and which have finitely many singular values, induced naturally from generic (*stable*) fold maps into the plane, as Theorem 1.

For a differentiable map $c : X \to Y$, the *singular set* $s(c)$ is the set of all singular points of c and $c(S(c))$ is the *singular value set* of it.

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We use "*critical*" instead of "*singular*" in the case $Y := \mathbb{R}$, the case of real valued functions and the case Y being (at most) 1-dimensional.

For (classical) 3-dimensional manifold theory, see [8] for example, as a textbook.

In the next section, we introduce history on global singularity theory and several classes of smooth maps such as (*stable*) *fold* maps and *image simple* fold maps. The third section is devoted to our main result, Theorem 1. We also review related studies on smooth functions on 3-dimensional closed and connected manifolds such as Theorems 2 and 3 and some new related problems such as Problem 1. Theorem 4 is another result and obtained as a variant of Theorem 1.

2. Global singularity theory.

2.1. **Several classes of fold maps.** In global singualrity theory and related differential topology, global geometric properties of *fold* maps, especially, differential topological ones, are actively studied still now. [5, 28, 29] are of related pioneering studies.

A *fold* map $f : M \to N$ between manifolds with no boundary is a smooth map defined in the case $\dim M \geq \dim N \geq 1$ and as a smooth map into 1-dimensional manifold N which is locally Morse for suitable local coordinates, or a map locally represented as a submersion or the product of a Morse function f_p and the identity map on some open set in the $(\dim N - 1)$-dimensional Euclidean space $\mathbb{R}^{\dim N - 1}$ for $\dim N \geq 2$ and suitable local coordinates. The singular set $S(f)$ is a $(\dim N - 1)$-dimensional regular smooth submanifold of M and with no boundary and the restriction $f|_{S(f)}$ is an immersion. For each singular point (*fold* point) p of f, the *index* $i(p, f)$ *for* f is well-defined as the unique integer $0 \leq i(p, f) \leq \frac{\dim M - \dim N + 1}{2}$, the minimum between the index $0 \leq i(p, f_p, \mathrm{Morse}) \leq \dim M - \dim N + 1$ of the critical point of the local Morse function f_p and $\dim M - \dim N + 1 - i(p, f_p, \mathrm{Morse})$. The set of all singular points of each fixed index for f is also a $(\dim N - 1)$-dimensional regular smooth submanifold of M with no boundary. A closed manifold M of dimension at least 2 admits a fold map into $\mathbb{R}^2$ if and only if the Euler number $\chi(M)$ is even. In the odd dimensional case, M always admits a fold map into $\mathbb{R}^2$. By fundamental singularity theory, it is perturbed to be a *stable* one. A *stable* fold map $f : M \to N$ is a fold map such that all self-intersections of $f|_{S(f)}$ are normal-crossings.

Hereafter, $\pi_{k,k_1} : \mathbb{R}^k \to \mathbb{R}^{k_1}$ $(k > k_1 \geq 1)$ is the projection to the 1st k_1 components. The canonical projection of the unit sphere S^k to $\mathbb{R}^{k_1}$ is $\pi_{k,k_1}|_{S^k}$, whose image is the k-dimensional unit disk $D^{k_1} := \{x = (x_1, \ldots x_{k_1}) \in \mathbb{R}^{k_1} \mid \Sigma_{j=1}^{k_1} {x_j}^2 \leq 1\}$. We also have $S(\pi_{k,k_1}|_{S^k}) = \{x = (x_1, \ldots x_{k_1}, 0, \ldots) \in \mathbb{R}^{k+1} \mid \Sigma_{j=1}^{k} {x_j}^2 = 1\}$ and it is mapped to $S^{k_1 - 1}$ by π_{k,k_1}.

Several classes of fold maps have been studied well.

Special generic maps are fold maps the indices of singular points for which are always 0. They restrict the topologies and the differentiable structures of the spheres and the manifolds of the domains. See [29] for example.

Image simple fold maps are fold maps whose restrictions to the singular sets are embeddings. They are important in characterizing so-called *graphs manifolds* as ones admitting such maps into the plane among 3-dimensional closed, connected, and orientable manifolds. In the present paper, *graph manifolds* are defined to be 3-dimensional, closed, connected, and orientable manifolds obtained by gluing

finitely many circle bundles over compact and connected surfaces, along tori by diffeomorphisms. See [30] and see also [19].

Round fold maps are image simple fold maps whose singular value sets are concentric spheres. For example. a manifold M represented as a connected sum of finitely many S^{m-n}-bundles over S^n $(n \geq 2)$ taken in the smooth category admits a round fold map $f : M \to \mathbb{R}^n$ such that $f^{-1}(p)$ is a disjoint union of finitely many copies of S^{m-n} for each point $p \in \mathbb{R}^n - f(S(f))$. See [11, 12, 13, 14] and see also [19] again.

These fold maps are all generalizations of the canonical projections of the unit spheres (in the round fold map case, $n \geq 2$). They are also stable.

Note also that the name "image simple fold map" has first appeared in [27].

2.2. **Global singularity theory related to 4-dimensional theory.** We do not need to know global singularity theory related to 4-dimensional theory. However, surface (torus) fibrations are key objects in our study and in the related global singularity theory, they are also important. We explain related studies shortly.

First, for smooth maps of closed manifolds whose dimensions are at least 2 into $\mathbb{R}^2$, we can have stable maps by suitable slight perturbations. A *stable* map here is a smooth map explained as follows: its singular set is the disjoint union of smoothly embedded circles in the manifold of the domain and other than fold points, cusp points appear discretely as its singular points and these points are mapped injectively and apart from the image of the set of all fold points of the stable map.

[21] is on stable maps on 4-dimensional closed manifolds into the plane $\mathbb{R}^2$ and *torus fibrations* are defined and studied, as stable maps with preimages of points containing no critical point being disjoint unions of finitely many copies of S^2 and the torus $S^1 \times S^1$. The manifolds of the domains and deformations of the maps preserving the differentiable structures of the manifolds of the domains are studied.

Related to surface fibrations on 4-dimensional manifolds, [2] is important. This is also related to so-called *trisections*, certain nice decompositions of 4-dimensional closed and connected manifolds: they are variants of Heegard splittings of 3-dimensional closed and connected manifolds. [6] is important in related theory. See also [3]. These studies are also done for contributions to various geometry such as symplectic topology, in addition to 4-dimensional differential topology, for example.

3. Our main theorem, Theorem 1.

Theorem 1. *Let $f_{\mathrm{s}} : M \to \mathbb{R}^2$ be a stable fold map on a 3-dimensional closed, connected, and orientable manifold M.*

In this situation, there exists a smooth function $c_{f_{\mathrm{s}}(M)} : U_{f_{\mathrm{s}}(M)} \to \mathbb{R}$ on some small and connected open neighborhood $U_{f_{\mathrm{s}}(M)}$ of the image $f_{\mathrm{s}}(M) \subset \mathbb{R}^2$ and we have a smooth function $f := c_{f_{\mathrm{s}}(M)} \circ f_{\mathrm{s}}$ whose level sets with no critical point are always disjoint unions of finitely many copies of $S^1 \times S^1$, where the manifold of the target of f_{s} is restricted to $U_{f_{\mathrm{s}}(M)}$.

Furthermore, we can have them in such a way that the critical set $S(f)$ is the disjoint union of the following compact subsets in M.

(1) *The union of mutually disjoint finitely many circles embedded smoothly in $M - {f_{\mathrm{s}}}^{-1}(f_{\mathrm{s}}(S(f_{\mathrm{s}})))$, making $f|_{M-{f_{\mathrm{s}}}^{-1}(f_{\mathrm{s}}(S(f_{\mathrm{s}})))}$ Morse-Bott.*

(2) *The preimage* ${f_{\rm s}}^{-1}(f_{\rm s}(S(f_{\rm s})))$. *This is* 2*-dimensional subcomplex of* M*, seen as a* 3*-dimensional triangulated* (PL) *manifold if* $f_{\rm s}$ *is not special generic. This is* $S(f_{\rm s})$ *if* $f_{\rm s}$ *is special generic.*

Proof. We construct a smooth function $c_{f_{\rm s}(M)}$ on a suitable small connected open neighborhood $U_{f_{\rm s}(M)} \subset \mathbb{R}^2$ of the image $f_{\rm s}(M) \subset \mathbb{R}^2$, in STEP 1 and STEP 2.

STEP 1 Constructing a smooth function on a compact regular neighborhood $R(f_{\rm s}(S(f_{\rm s})))$ of $f_{\rm s}(S(f_{\rm s})) \subset \mathbb{R}^2$ in $\mathbb{R}^2$, chonsen in the smooth category.

Here, triangulations of smooth manifolds and generic maps (stable fold maps) are considered, where we do not need to know precise theory on them.

We also apply theory from [22], especially [22, Lemma 11].

Triangulations of M, $f_{\rm s}(M)$ and $f_{\rm s}(S(f_{\rm s}))$ are considered, explicitly. See [25] for hauptvermutung for polyhedra of dimension at most 2 and topological manifolds of dimension at most 3. They can be triangulated and their triangulations are unique, in the PL category (piecewise smooth) category. [32] is for existence of natural triangulations of generic smooth maps.

For a compact regular neighborhood $R(f_{\rm s}(S(f_{\rm s})))$ of $f_{\rm s}(S(f_{\rm s})) \subset \mathbb{R}^2$ in $\mathbb{R}^2$ chosen in the smooth category, or more generally, one of a subcomplex of a smooth manifold (equipped with a natural triangulation) with no boundary, see [9].

$R(f_{\rm s}(S(f_{\rm s})))$ is a compact smooth surface embedded smoothly in $\mathbb{R}^2$ and with non-empty boundary $\partial R(f_{\rm s}(S(f_{\rm s})))$. This also satisfies $f_{\rm s}(S(f_{\rm s})) \subset R(f_{\rm s}(S(f_{\rm s}))) - \partial R(f_{\rm s}(S(f_{\rm s})))$.

Thanks to [22, Lemma11], for an arbitrary positive number $\epsilon > 0$, we have a smooth function $c_{R(f_{\rm s}(S(f_{\rm s})))} : R(f_{\rm s}(S(f_{\rm s}))) \to \mathbb{R}$ whose image is $\{t \mid 0 \leq t \leq \epsilon\}$ satisfying the following.

- ${c_{R(f_{\rm s}(S(f_{\rm s})))}}^{-1}(0) = f_{\rm s}(S(f_{\rm s}))$.
- Except 0, $c_{R(f_{\rm s}(S(f_{\rm s})))} : R(f_{\rm s}(S(f_{\rm s}))) \to \mathbb{R}$ has no critical value.
- ${c_{R(f_{\rm s}(S(f_{\rm s})))}}^{-1}(\epsilon) = \partial R(f_{\rm s}(S(f_{\rm s})))$.

STEP 2 Extending $c_{R(f_{\rm s}(S(f_{\rm s})))}$ as a smooth real-valued function on $f_{\rm s}(M) \bigcup R(f_{\rm s}(S(f_{\rm s})))$.

By the fundamental existence theory of Morse functions, we can extend $c_{R(f_{\rm s}(S(f_{\rm s})))}$ as a smooth real-valued function on $f_{\rm s}(M) \bigcup R(f_{\rm s}(S(f_{\rm s})))$ which is Morse on $(f_{\rm s}(M) \bigcup R(f_{\rm s}(S(f_{\rm s}))) - f_{\rm s}(S(f_{\rm s}))$.

We also have a small and connected open neighborhood $U_{f_{\rm s}(M)}$ of the image $f_{\rm s}(M)$ in $\mathbb{R}^2$ which is also a proper subset of $f_{\rm s}(M) \bigcup R(f_{\rm s}(S(f_{\rm s})))$ and we define $c_{f_{\rm s}(M)}$ as the restriction of the function of STEP 2 there.

We check the topologies of the preimages $f_{\rm s}^{-1}(p)$. For this see also [31].

The preimage $f_{\rm s}^{-1}(p)$ is the disjoint union of finitely many smoothly embedded circles in M for $p \in \mathbb{R}^2 - f_{\rm s}(S(f_{\rm s}))$ (due to implicit function theorem each connected component of $f_{\rm s}^{-1}(p)$ is a smoothly embedded circle in M if the connected component contains no singular point of $f_{\rm s}$). For any point $p \in f_{\rm s}(M)$ being realized as the value of $f_{\rm s}$ at some singular point $p_{\rm s}$ of it with $i(p_{\rm s}, f_{\rm s}) = 1$, the preimage $f_{\rm s}^{-1}(p)$ is always regarded to be a 1-dimensional polyhedron embedded in M as a subcomplex which is not a topological manifold in general. For any point $p \in f_{\rm s}(M)$

being realized as the value of f_s at some singular point $p_{\rm s}$ of it with $i(p_{\rm s}, f_{\rm s}) = 0$, the preimage $f_{\rm s}^{-1}(p)$ is a one-point set if it is connected in addition. For any point $p \in f_{\rm s}(M)$ being realized as the value of f_s at some singular point of it $p_{\rm s}$ of it with $i(p_{\rm s}, f_{\rm s}) = 0$, the preimage $f_{\rm s}^{-1}(p)$ is a discrete set and of dimension 0 if and only if $f_{\rm s}^{-1}(p)$ contains only singular points of index 0 for f_s.

From this, we have a desired function $f := c_{f_{\rm s}(M)} \circ f_{\rm s} : M \to \mathbb{R}$: the manifold of the target of $f_{\rm s}$ is restricted to $U_{f_{\rm s}(M)}$. This completes the proof. □

Remark 1 is on another proof of Theorem 1, respecting [18], for example.

Hereafter, a *round* function means a smooth function whose critical set is the disjoint union of smoothly embedded circels. Round functions may not be Morse-Bott. For round functions, see [4, 20], for example.

Remark 1. We consider the case of an image simple fold map $f_{\rm s}$ in Theorem 1 and we have another way.

In STEP 1, we choose a small tubular neighborhood $T(f_{\rm s}(S(f_{\rm s})))$ of $f_{\rm s}(S(f_{\rm s}))$, regarded to be the product $f_{\rm s}(S(f_{\rm s})) \times D^1$ embedded smoothly in $\mathbb{R}^2$, $f_{\rm s}(S(f_{\rm s}))$ is regarded to be $f_{\rm s}(S(f_{\rm s})) \times \{0\}$, and we can also construct the projection of a trivial bundle $f_{\rm s}(S(f_{\rm s})) \times D^1$ over D^1. We can argue as in STEP 2 and the proof of Theorem 1. The resulting function f is a Morse-Bott function and it is also a round function.

This generalizes the round fold map case [18]. There, the function $c_{f_{\rm s}(M)}$ is defined as the natural height function mapping $x = (x_1, \ldots x_n)$ to $\Sigma_{j=1}^{n} {x_j}^2$, under the situation that the round fold map $f_{\rm s} : M \to \mathbb{R}^n$ satisfies $f_{\rm s}(S(f_{\rm s})) = \sqcup_{i=1}^{l} \{x = (x_1, \ldots x_n) \mid \Sigma_{j=1}^{n} {x_j}^2 = i^2\}$ with a positive integer l.

According to [30] ([19]), among 3-dimensional closed, connected and orientable manifolds, a graph manifold is characterized as a manifold admitting an image simple fold map into $\mathbb{R}^2$.

Related to Remark 1, we present a problem.

Problem 1. Among 3-dimensional closed, connected and orientable manifolds, is a graph manifold characterized as a manifold admitting a Morse-Bott function which is a round function and whose level sets with no critical point are always disjoint unions of finitely many copies of $S^1 \times S^1$?

In short, resulting functions in Remark 1 may form a small proper subclass of the class of functions of Problem 1.

Remark 2 is another remark on Theorem 1.

Remark 2. In Theorem 1, we drop the condition that M is orientable. Level sets of the resulting functions with no critical point may contain copies of the Klein Bottle K. Except this, the statement is same as the original one.

Theorem 2 is on the minimal number among the numbers of circles in critical sets of round functions on 3-dimensional closed and connected manifolds ([4, Theorem 2]).

Theorem 2 ([4])**.** *For every* 3*-dimensional closed and connected manifold* M*, there exists a round function* $f : M \to \mathbb{R}$ *the number of circles in whose critical set is smaller than or equal to* 4*.*

Theorem 3 is another result on smooth functions on 3-dimensional closed and connected manifolds ([22, Theorem 2]).

Theorem 3 ([22]). *The following are equivalent for a 3-dimensional closed and connected manifold M.*

(1) *M admits a smooth function $f : M \to \mathbb{R}$ with exactly two critical values p_1 and p_2 and with preimages $f^{-1}(p_1) \subset M$ and $f^{-1}(p_2) \subset M$ being 1-dimensional and connected polyhedra of Euler number $e_p \leq 1$.*
(2) *If M is orientable, then the Heegaard genus of M is at most $1 - e_p$. If M is non-orientable, then the Heegaard genus of M is at most $2 - 2e_p$.*

In short, natural generalizations of Morse functions with exactly two critical points on closed manifolds or Morse functions in Reeb's sphere theorem ([26]) are discussed, there.

By investigating [15, 16], we can see that a function of Theorem 3 can be also obtained as a smooth function which is, around each connected component of the critical set, represented as the composition of some fold map to $D^2 - S^1$, with the sum of the height function $h_{D^2-S^1}(x) = \pm({x_1}^2 + {x_2}^2)$ ($x := (x_1, x_2)$) and some constant real-valued function.

Theorem 4 is another result.

Theorem 4. *In Theorem 1, there exists another smooth function $c_{f_{\mathrm{s}},1}(M) : U_{f_{\mathrm{s}}(M)} \to \mathbb{R}$ on some small and connected open neighborhood $U_{f_{\mathrm{s}}(M)}$ of the image $f_{\mathrm{s}}(M) \subset \mathbb{R}^2$ and we have another smooth function $f_1 := c_{f_{\mathrm{s}},1}(M) \circ f_{\mathrm{s}}$ whose level sets with no critical point are always disjoint unions of finitely many copies of $S^1 \times S^1$, where the manifold of the target of f_{s} is restricted to $U_{f_{\mathrm{s}}(M)}$.*

Furthermore, we can have them in such a way that the critical set $S(f_1)$ is same as the case of Theorem 1 and around the preimage ${f_{\mathrm{s}}}^{-1}(f_{\mathrm{s}}(S(f_{\mathrm{s}})))$, f_1 is represented as the composition of some fold map into a surface with a Morse function there.

Proof. Except STEP 1, the proof is essentially same as that of Theorem 1.

It is important to find that f_{s} is a *two-colored* map in [35]. In other words, $\mathbb{R}^2 - f_{\mathrm{s}}(S(f_{\mathrm{s}}))$ is the disjoint union of two open sets $U_{\mathrm{red},f_{\mathrm{s}}}$ and $U_{\mathrm{blue},f_{\mathrm{s}}}$ of $\mathbb{R}^2$ the intersection of whose closures taken in $\mathbb{R}^2$ is $f_{\mathrm{s}}(S(f_{\mathrm{s}}))$. In short, adjacent connected components of $\mathbb{R}^2 - f_{\mathrm{s}}(S(f_{\mathrm{s}}))$ are colored in different colors from "red" and "blue".

Due to this, we have a Morse function $c_{R(f_{\mathrm{s}}(S(f_{\mathrm{s}}))),1} : R(f_{\mathrm{s}}(S(f_{\mathrm{s}}))) \to \mathbb{R}$ whose image is $\{t \mid -\epsilon \leq t \leq \epsilon\}$, which has exactly one critical value 0, and which satisfies the following, instead, in the situation of Theorem 1.

- The preimage ${c_{R(f_{\mathrm{s}}(S(f_{\mathrm{s}}))),1}}^{-1}(\{t \mid -\epsilon \leq t < 0\})$ is a subset of $U_{\mathrm{blue},f_{\mathrm{s}}}$.
- The preimage ${c_{R(f_{\mathrm{s}}(S(f_{\mathrm{s}}))),1}}^{-1}(\{t \mid 0 < t \leq \epsilon\})$ is a subset of $U_{\mathrm{red},f_{\mathrm{s}}}$.

Except this, we can argue as the proof of Theorem 1. This completes the proof. □

Remark 3. In [35], the notion of *two-colored* map is introduced for a generic (stable) map $f_{\mathrm{s}} : M \to N$ on a closed and connected manifold of dimension M to a smooth manifold N with no boundary, satisfying $\dim M = \dim N + 1$. Such a map may not be a two-colored map in general. However, in the case M is orientable or the case $N = \mathbb{R}^n$ (more generally the case $H^1(N; \mathbb{Z}_2)$ is trivial, with $\mathbb{Z}_2$ denoting the group of order 2), it is a two-colored map.

Remark 4. In Theorem 1, the space of the target $\mathbb{R}^2$ can be replaced by an arbitrary surface N with no boundary. In Theorem 4, we cannot do this in general, due to the fact presented in Remark 3.

Remark 5. The construction of $c_{f_{\mathrm{s}}(M)} : U_{f_{\mathrm{s}}(M)} \to \mathbb{R}$ in the proof of Theorem 1 can be generalized in the case of a nice smooth map $f_{\mathrm{s}} : M \to N$ whose codimension is non-positive and which is of a certain class such as one discussed in [32]. However, we omit related exposition explicitly.

4. Conflict of interest. Data availability. AI.

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We have no data other than the present file, for the present paper. We do not assume non-trivial mathematical arguments of formally unpublished preprints such as [18].

AI is not used for our mathematical arguments, essentially.

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