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Linear independence of even Dirichlet beta values

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Research note prepared with Codex

Working draft
3 November 2026

Abstract

Let $\beta(s) = 1 - 3^{-s} + 5^{-s} - \cdots$ denote the Dirichlet beta function. For all sufficiently large even integers s, we prove

$$\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{\beta(2), \beta(4), \ldots, \beta(s)\} \geq 0.21\sqrt{\frac{s}{\log s}}.$$

1 Introduction

The Dirichlet beta function is

$$\beta(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s}.$$

Its values at positive odd integers are nonzero rational multiples of powers of π. The arithmetic of its even values is much less understood. One way to measure progress without settling an individual value is to bound the dimension

$$D(s) = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{\beta(2), \beta(4), \ldots, \beta(s)\}, \qquad s \geq 2 \text{ even}. \tag{1.1}$$

A lower bound $D(s) \geq d$ means that at least d values in this finite list are linearly independent over the rationals. It implies that at least $d-1$ of them are irrational, but does not identify those values.

Rivoal and Zudilin [7, Theorem 1] proved a logarithmic lower bound with leading constant $1/(2+\log 2)$ for the span after adjoining 1. Fischler [2, Theorem 1] subsequently obtained

$$D(s) \geq \frac{1+o(1)}{1+\log 2}\log s \qquad (s \longrightarrow \infty,\ s \text{ even}), \tag{1.2}$$

as a special case of his result for periodic Dirichlet series. His later use of Siegel's lemma to construct rational functions with small coefficients gives a bound of order $\sqrt{s/\log s}$ for odd zeta values [3].

Mistiaen [5, Theorem 1.0.5 and Remark 3] extends that method to Dirichlet characters whose conductors are not divisible by 4. The restriction excludes the character of conductor 4 associated with β, so that theorem does not cover the even beta values. We obtain the following bound for the beta function.

Theorem 1.1. *For every sufficiently large even integer s,*

$$D(s) \geq 0.21\sqrt{\frac{s}{\log s}}.$$

The theorem concerns linear independence in a growing family. It does not, in particular, establish the irrationality of $\beta(4)$. A recent preprint of Kumar and Tosi [4] claims a bound of square-root order for the *number of irrational values* of real Dirichlet L-functions. The proof contains unresolved arithmetic errors,[1] so we do not regard that version as establishing the claimed bound. Our proof is independent of theirs. Theorem 1.1 concerns the stronger quantity (1.1): irrational numbers can still satisfy many rational linear relations.

Specializing general results on polylogarithms with periodic coefficients [6] gives bounds involving both even and odd beta values. However, the odd values $\beta(1), \beta(3), \dots$ are already linearly independent over the rationals: they are nonzero rational multiples of distinct powers of π, and π is transcendental. The lower bound supplied by those results can therefore be met by the odd values alone, without establishing the desired independence among the even values. Our construction avoids this issue by producing linear forms involving only 1 and the even beta values.

1.1 How the proof works

For a fixed even cutoff $s = 2v$, the proof constructs integer linear forms in 1 and the even beta values:

$$\lambda_0 + \lambda_1\beta(2) + \cdots + \lambda_v\beta(2v), \qquad \lambda_0, \dots, \lambda_v \in \mathbb{Z}.$$

If all these beta values were rational, any nonzero such form would have absolute value bounded below by the reciprocal of a common denominator. Linear-independence criteria refine this observation to give a lower bound on the dimension of their span.

With this set of beta values fixed, we use a criterion of Fischler, stated in Proposition 2.1. It requires a family of forms at each large index n, with controlled growth of their integer coefficients and exponential decay of their values. It also requires a rank condition on the coefficient matrix, which prevents the whole family from vanishing. Writing these bounds as $B^{n+o(n)}$ for the coefficients and $\alpha^{n+o(n)}$ for the values, with $B > 1$ and $0 < \alpha < 1$, the criterion gives $D(s) \geq -\log\alpha/\log B$ after removing the adjoined number 1. Thus constructing a small expression is only the first step: we must control its size after clearing denominators and establish the required rank condition.

The starting point is Fischler's rational function

$$F_n(t) = \sum_{i=1}^{a}\sum_{j=0}^{n} \frac{c_{i,j}}{(t+j)^i}, \qquad c_{i,j} \in \mathbb{Z},$$

whose coefficients have controlled size and whose many initial zero Laurent coefficients make it decay rapidly far from its poles [3, Theorem 4]. The coefficients are supplied by an existence theorem; no explicit formula for them is needed. We sum this rational function and its first h derivatives over half-integers. These give $h+1$ families of forms. The additional families provide the independent functions needed in the rank argument; their cost is to raise the largest possible beta argument from a to $a+h$.

Half-integer arguments already occur in classical approximations to beta values, including the hypergeometric construction of Rivoal and Zudilin [7, Section 2]. They are natural here because powers of $t + \frac{1}{2}$ are proportional to powers of the odd integers $2t+1$ in the beta series. Alternating sums therefore produce beta values. Combining the positive and negative half-integer sums removes the odd beta values. Section 3 carries this out using generating functions evaluated at $z = -1$.

[1] For example, in the notation of [4, Section 3.3], take $a = b = 1$ and $h = y = \ell = 0$. For sufficiently large admissible n, $K_0(0) = \prod_{j=0}^{(2r+T)n}(1 - rn + j) = 0$, because the factor $j = rn - 1$ vanishes, whereas $g_{0,1,1}(0) = 1$ by the empty-product convention. This contradicts the finite p_n-adic valuation asserted in Lemma 18. This objection concerns the proof as written and does not disprove the main theorem.

These generating functions satisfy a rational differential system whose coefficients have no pole at -1: it is an *ordinary point* of the system. This avoids the singular evaluation point identified in [5, Section 2]. At an ordinary point, the initial values determine an analytic solution, and its Taylor coefficients express derivatives in terms of those initial values. This is useful for the arithmetic of our linear forms. The sums first produce finite correction terms with rational coefficients. We remove those terms by differentiation *before* clearing denominators. Section 4 then bounds the denominators through the local differential system, and Section 5 compares the resulting coefficient growth and decay.

The remaining task is the rank condition. Section 6 follows Fischler's multiplicity method [3, Section 4.5]: failure of the rank condition would force certain polynomial combinations of solutions to have more vanishing than their degrees permit. Each of the $h+1$ families contributes one such combination, and the final count requires h to grow proportionally to a. Our system has an extra diagonal term, so we first make the change $z = w^2$ and rescale some components of the solution. This removes that term and allows the independence argument to work. The substitution doubles the relevant polynomial degrees, but the evaluation point now has two preimages, i and $-$i. Counting the vanishing at both recovers the required inequality.

The size of the final bound comes from balancing two costs. The rational-function construction has a parameter Ω that improves the decay exponent by a term proportional to Ω, but increases the coefficient-growth exponent by a term of order Ω^2/a. With the other auxiliary parameters fixed and h proportional to a, the denominator estimate contributes only a term of order $\log a$. Choosing Ω of order $\sqrt{a \log a}$ therefore keeps $\log B$ of order $\log a$ and makes $-\log \alpha$ of order $\sqrt{a \log a}$. Their quotient has order $\sqrt{a/\log a}$; the largest beta argument $a+h$ is proportional to a. Section 7 makes this balance precise and uses Fischler's numerical parameter choice to obtain the constant 0.21.

Section 2 states the criterion and the rational-function input precisely. The multiplicity estimate is stated in Section 6.1, alongside the rank argument that uses it.

1.2 Conventions

All logarithms are natural. We write $(i)_p = i(i+1)\cdots(i+p-1)$, with $(i)_0 = 1$, and $\mathsf{H}_d = \sum_{j=1}^{d} 1/j$ for the dth harmonic number. The symbol i denotes the imaginary unit. The auxiliary parameters of the construction are fixed while n tends to infinity through an indicated infinite set of integers. Every $o(n)$ in that stage of the proof is uniform in the finite indices and in the specified derivative range. Only in the last section do we let the parameter controlling the largest beta argument tend to infinity.

2 The criterion and the rational-function input

We first state the criterion that turns linear forms into a dimension bound, then the rational-function construction from which the forms are built. Both are due to Fischler. His third input, a multiplicity estimate, will be stated in Section 6.1 when the rank proof calls for it.

2.1 A refined linear-independence criterion

The rows of the matrix below are the coefficients of integer linear forms in the target numbers. The two size bounds measure the growth of those coefficients and the decay of the forms when evaluated at the target vector. The rank condition is the third hypothesis to be proved.

Proposition 2.1 ([3, Proposition 1], over $\mathbb{Q}$)**.** *Let $\theta = (\theta_0, \dots, \theta_v)^T$ be a real vector with $\theta_0 \neq 0$. For each n in an infinite set, let L_n be an integer matrix with $v+1$ columns and any*

finite number of rows. Suppose that, for fixed $B > 1$ and $\alpha < 1$ with $\alpha > 0$,

$$\max_{i,j} |(L_n)_{i,j}| \le B^{n+o(n)}, \qquad \max_i |(L_n\theta)_i| \le \alpha^{n+o(n)}.$$

Suppose also that, for sufficiently large n,

$$L_n x = 0, \quad x \in \mathbb{Q}^{v+1} \quad \Longrightarrow \quad x_0 = 0. \tag{2.1}$$

Then

$$\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{\theta_0, \dots, \theta_v\} \ge 1 - \frac{\log \alpha}{\log B}.$$

Condition (2.1) says that the first column is not a rational linear combination of the other columns. Full column rank is not required. This distinction is useful here because the rational function in Proposition 2.2 is obtained by an existence argument, and its coefficients need not give full rank. In our application, $\theta_0 = 1$ and the other coordinates are even beta values. We will bound the coefficients and values of the forms in Section 5, then prove (2.1) in Section 6. No lower bound on the value of each individual form is required.

2.2 A rational function with small coefficients

The construction supplies three properties: integer partial-fraction coefficients of controlled size, exact vanishing of the first Laurent coefficients at infinity, and extra smallness of subsequent Laurent coefficients. The first property controls the coefficients of our forms. The other two have different roles: the vanishing order, governed by ω, leaves enough decay to differentiate the sums and supplies zeros for the rank argument; the additional smallness, governed by Ω, makes the resulting forms exponentially small.

Proposition 2.2 ([3, Theorem 4])**.** *Let a be a positive integer and let ω, Ω, r be rational numbers with*

$$a > \Omega \ge \omega \ge 1, \qquad r \ge 1.$$

For every sufficiently large integer n such that $rn, \omega n, \Omega n$ are integers, there are integers $c_{i,j}$, not all zero, for $1 \le i \le a$ and $0 \le j \le n$, with the following properties. Set

$$F_n(t) = \sum_{i=1}^{a} \sum_{j=0}^{n} \frac{c_{i,j}}{(t+j)^i} = \sum_{\nu \ge 1} A_\nu t^{-\nu}.$$

Then $A_\nu = 0$ for $\nu < \omega n$,

$$|c_{i,j}| \le \exp(n \log \chi + o(n)), \quad \log \chi = \frac{\omega \log 2 + 3\omega^2 + \omega^2 \log(a+1) + \frac{1}{2}\Omega^2 \log r}{a - \omega}, \tag{2.2}$$

and, uniformly in ν,

$$|A_\nu| \le \exp(n \log \chi + o(n))\, n^\nu \nu^a \begin{cases} r^{\nu - \Omega n}, & \nu < \Omega n, \\ 1, & \nu \ge \Omega n. \end{cases} \tag{2.3}$$

Here a bounds the pole orders, and ωn is a lower bound for the order of vanishing at infinity. The parameter r will determine the summation cutoff rn. The larger threshold Ωn gives the stronger estimate in the first line of (2.3); it does not assert additional exact vanishing. In the tail estimate it will supply a factor $r^{-\Omega n}$, separate from the power decay controlled by ωn. All parameters are fixed as n grows. When we later increase a in Section 7, ω will remain fixed while Ω grows on the scale $\sqrt{a \log a}$.

Group the coefficients with the same pole order into polynomials

$$P_i(z) = \sum_{j=0}^{n} c_{i,j} z^j, \qquad b = \max\{i : P_i \not\equiv 0\}.$$

The P_i will multiply the special functions obtained by summing F_n. We retain the actual largest index b, rather than replacing it by a, because the rank proof needs a nonzero highest coefficient P_b. The integer b may depend on n, but always satisfies $1 \le b \le a$.

3 Half-integer sums and their differential system

We now turn the rational function of Proposition 2.2 into linear forms in beta values. First we identify the functions whose values at -1 give the required even arguments. We then express the sums of F_n in terms of those functions and remove the finite corrections by differentiation.

3.1 Parameters and the two differentiations

From now on $r \ge 2$ and $M = rn$. Choose an integer h with $0 \le h \le a$, and put $d = a + h$. We use $F_n, F_n', \dots, F_n^{(h)}$ to obtain $h + 1$ families for the rank argument. Differentiating a pole of order i raises its order, so these extra families require beta arguments up to d. The parameters and indices used in the construction have the following roles.

Symbol	Role
n	Index tending to infinity to produce increasingly small forms.
a, b	Allowed and actual largest pole orders of F_n; $1 \le b \le a$.
$h, d = a + h$	Largest derivative order of F_n and upper bound for the beta arguments.
$r, M = rn$	The sums start at the cutoff M, beyond the poles of F_n.
ω, Ω	Control exact Laurent vanishing and additional exponential smallness, respectively.
κ	Controls the upper limit $k \le \kappa n$ on the derivative rows; its constraints are imposed in Section 5.
p	Order of differentiation of $F_n(t)$ with respect to t, with $0 \le p \le h$.
$k - 1$	Order of differentiation of a generating function with respect to z; $k = 1$ means no differentiation.

Thus p selects a family, whereas k selects a derivative row within that family. The derivatives in z remove finite polynomial corrections and provide the rows needed for the linear-independence criterion. We keep the upper bound d fixed while n tends to infinity; the number of beta values is increased only in Section 7.

3.2 The functions at the evaluation point

The half-integer denominators are chosen so that alternating signs give the beta series: $(t + \frac{1}{2})^{-j} = 2^j (2t + 1)^{-j}$. For $j \ge 1$, put

$$\Phi_j(z) = \sum_{t=0}^{\infty} \frac{z^t}{(t + \frac{1}{2})^j} \qquad (|z| < 1).$$

The integral

$$\Phi_j(z) = \frac{1}{(j-1)!} \int_0^1 \frac{x^{-1/2} (-\log x)^{j-1}}{1 - zx} \, \mathrm{d}x \tag{3.1}$$

continues it to $\mathbb{C} \setminus [1, \infty)$. In particular, the series or its Abel limit gives $\Phi_j(-1) = 2^j \beta(j)$, including $j = 1$.

We still need to remove the odd arguments j. The positive and negative half-integer sums lead to functions at z and $1/z$; we combine them as

$$H_j(z) = z^{-1}\Phi_j(1/z) - (-1)^j \Phi_j(z).$$

Both terms are analytic near -1 by (3.1). At that point the combination supplies the factor $1 + (-1)^j$:

$$H_j(-1) = -(1 + (-1)^j) 2^j \beta(j). \tag{3.2}$$

Thus $H_j(-1)$ is zero for odd j and is $-2^{j+1}\beta(j)$ for even j.

Their derivatives are rational-function combinations of the same functions and the constant function. Let $\Phi_0(z) = 1/(1-z)$. For any positive integer d, the identity $z\Phi_j'(z) = \Phi_{j-1}(z) - \Phi_j(z)/2$ implies that $(1, H_1, \dots, H_d)$ is a solution of

$$\begin{aligned} y_0' &= 0, \\ y_1' &= \frac{2}{z(1-z)} y_0 - \frac{1}{2z} y_1, \\ y_j' &= -\frac{1}{z} y_{j-1} - \frac{1}{2z} y_j \qquad (2 \le j \le d). \end{aligned} \tag{3.3}$$

The coordinate y_0 represents the constant function 1, which appears in the derivative of H_1. Since the coefficients have no pole at -1, arbitrary initial values there determine an analytic solution. We will use the beta-valued initial vector for the linear forms, and arbitrary initial vectors for the rank proof.

Although the odd-indexed functions vanish at -1, their derivatives need not: the system gives $H_1(-1) = 0$ but $H_1'(-1) = -1$. We therefore keep all components during differentiation and use the zero odd values only when evaluating the resulting forms at -1.

3.3 Summation and finite tails

The sums start beyond the cutoff $M = rn$ so that the decay estimates for F_n apply. Since we use derivatives of F_n of orders $0 \le p \le h$, we take the system (3.3) with $d = a + h$.

The powers of z in the following definition align the two reflected sums with H_j. The factor z^M makes the finite corrections polynomial, as the next lemma shows. For $0 \le p \le h$ define, on the unit circle,

$$S_{n,p}(z) = z^M \sum_{t=M}^{\infty} \left\{ F_n^{(p)}(t + \tfrac{1}{2}) z^{-t-1} - F_n^{(p)}(-t - \tfrac{1}{2}) z^t \right\}. \tag{3.4}$$

Here $F_n^{(p)}$ is differentiated with respect to its argument. Near -1 we continue the two tails separately, using the identity below.

Lemma 3.1. *Near -1 there is an identity*

$$S_{n,p}(z) = V_{n,p}(z) + z^M \sum_{i=1}^{b} (-1)^p (i)_p P_i(z) H_{i+p}(z), \qquad \deg V_{n,p} \le 2M - 1. \tag{3.5}$$

Proof. Differentiate the partial fractions of F_n. For a fixed pair i, j write $s = i + p$ and change the summation index to $l = t + j$ in the positive tail, and to $l = t - j$ in the negative tail. The infinite parts give the corresponding term of (3.5). The finite correction is $(-1)^p (i)_p c_{i,j}$ times

$$-\sum_{l=0}^{M+j-1} \frac{z^{M+j-l-1}}{(l + \frac{1}{2})^s} + (-1)^s \sum_{l=0}^{M-j-1} \frac{z^{M+j+l}}{(l + \frac{1}{2})^s}. \tag{3.6}$$

Since $0 \le j \le n \le M$, all exponents are nonnegative and at most $2M-1$. Summing these corrections proves the assertion.

The positive and negative infinite tails are initially analytic on opposite sides of the circle. Each has the displayed continuation near -1 by (3.1). For the derivative orders used below, the differentiated series converge absolutely on the circle; their values agree with the derivatives of the continuations by Abel limits. □

3.4 Rows representing derivatives

Write the nonpolynomial part of (3.5) as $\sum_i C_i^{[p]} H_i$. Its coefficient polynomials are

$$C_0^{[p]} = 0, \qquad C_i^{[p]}(z) = \begin{cases} (-1)^p (i-p)_p z^M P_{i-p}(z), & p+1 \le i \le p+b, \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

These polynomials have integer coefficients and degree at most $M+n$. To track derivatives, write the system as $y' = \mathcal{A}y$ and temporarily suppress p in the row C. The product rule reads

$$(Cy)' = (C' + C\mathcal{A})y.$$

Differentiation replaces the coefficient row and keeps the same vector of functions. The new row depends only on C and the system, so it works for every solution y. Starting with $Q_{1,i}^{[p]} = C_i^{[p]}$, iteration gives

$$\begin{aligned} Q_{k+1,0}^{[p]} &= (Q_{k,0}^{[p]})' + \frac{2Q_{k,1}^{[p]}}{z(1-z)}, \\ Q_{k+1,i}^{[p]} &= (Q_{k,i}^{[p]})' - \frac{Q_{k,i}^{[p]}}{2z} - \frac{Q_{k,i+1}^{[p]}}{z} \qquad (1 \le i \le d), \end{aligned} \tag{3.8}$$

where $Q_{k,d+1}^{[p]} = 0$. The index $k=1$ corresponds to no differentiation; hence $Q_k^{[p]}$ represents the derivative of order $k-1$. For *every* solution of (3.3), differentiation gives

$$\left(\sum_{i=1}^{d} C_i^{[p]} y_i \right)^{(k-1)} = \sum_{i=0}^{d} Q_{k,i}^{[p]} y_i. \tag{3.9}$$

Since $\deg V_{n,p} \le 2M-1$, differentiation of order $2M$ already kills this correction. We use $k \ge 2M+2$, hence order $k-1 \ge 2M+1$: the extra derivative gives a convenient range for the estimates and the degree-$2M$ Taylor correction in the rank proof. In this range,

$$\frac{S_{n,p}^{(k-1)}(-1)}{(k-1)!} = \frac{Q_{k,0}^{[p]}(-1)}{(k-1)!} + \sum_{i=1}^{d} \frac{Q_{k,i}^{[p]}(-1)}{(k-1)!} H_i(-1). \tag{3.10}$$

Equation (3.2) makes the right side a rational linear form in 1 and the even beta values. Dividing by $(k-1)!$ turns the derivatives into Taylor coefficients, whose denominators we can control in the next section. The finite correction $V_{n,p}$ has disappeared, so its coefficient denominators do not have to be cleared. The same rows will connect the size estimates to the rank proof. For the solution $(1, H_1, \dots, H_d)$, they represent the derivatives that become small integer forms after factorial normalization and denominator clearing. For a solution whose initial vector lies in a putative rational kernel of these rows, identity (3.9) makes those derivatives vanish. This is the source of the high-order zeros used in Section 6.

4 Denominators of the local solution

Our aim is to turn the rational linear forms in (3.10) into integer linear forms without losing their smallness. Put $u = z + 1$ and let $U(u)$ be the fundamental matrix of (3.3), normalized by $U(0) = I$. Thus the solution with initial column x at -1 is $U(u)x$. The Taylor coefficients of U record how the derivatives of a solution depend on its initial values. Since the polynomial row $C^{[p]}$ has integer coefficients, denominators for these Taylor coefficients will suffice for our forms.

An integrating factor will turn the system into successive integrations. The essential denominator saving is already visible in two integrations: for nonnegative integers m_1, m_2,

$$\int_0^u x^{m_2} \left(\int_0^x t^{m_1} \, \mathrm{d}t \right) \mathrm{d}x = \frac{u^{m_1+m_2+2}}{(m_1+1)(m_1+m_2+2)}.$$

The denominator factors are the degrees reached at the two integration steps. The second is strictly larger than the first. Further integrations give a strictly increasing list of degrees, all at most the final degree. This motivates the distinct-product least common multiple from [3, Lemma 2]; here positive indices suffice.

For integers $d \geq 1$ and $k \geq 0$, define

$$\Delta_{d,k} = \operatorname{lcm} \left\{ \prod_{t \in I} t : I \subseteq \{1, \dots, k\}, \ |I| \leq d \right\}, \qquad \mathcal{D}_{d,k} = 4^k \Delta_{d,k}. \tag{4.1}$$

Empty products and $\operatorname{lcm}(1, \dots, 0)$ are taken to be 1.

Lemma 4.1. *One has*

$$\Delta_{d,k} = \prod_{j=1}^{d} \operatorname{lcm}(1, \dots, \lfloor k/j \rfloor). \tag{4.2}$$

In particular, $\mathcal{D}_{d,k}$ divides $\mathcal{D}_{d,k+1}$, and for fixed d,

$$\log \mathcal{D}_{d,k} = (\log 4 + \mathsf{H}_d)k + o(k). \tag{4.3}$$

Proof. For a prime p, the maximum valuation of a product of at most d distinct integers in $\{1, \dots, k\}$ is the sum of the $\min\{d, k\}$ largest valuations in that set. It equals

$$\sum_{e \geq 1} \min\{d, \lfloor k/p^e \rfloor\}.$$

The product on the right of (4.2) has the same valuation: for each e, there are exactly $\min\{d, \lfloor k/p^e \rfloor\}$ indices j with $p^e \leq \lfloor k/j \rfloor$. Divisibility follows either from the definition or from this formula. Finally, the prime number theorem gives $\log \operatorname{lcm}(1, \dots, K) = K + o(K)$. Sum this estimate over the fixed set $1 \leq j \leq d$ to obtain (4.3). □

The dependence on d is important. Allowing d arbitrary factors from $\{1, \dots, k\}$ would give the larger multiplier $\operatorname{lcm}(1, \dots, k)^d$, with logarithm asymptotic to dk. Distinct factors give $\mathsf{H}_d k$ instead. In our application d is proportional to a and k is of order n. First let n grow with a fixed, then compare the resulting exponential rates as a increases. The naive multiplier would cost an exponent of order an, overwhelming the intended decay on the scale $n\sqrt{a \log a}$. The distinct factors reduce this cost to order $n \log a$, which leaves that decay available for the dimension bound.

Lemma 4.2. *For the normalized fundamental matrix U defined above,*

$$\mathcal{D}_{d,k}[u^k]U(u) \in M_{d+1}(\mathbb{Z}) \qquad (k \geq 0).$$

Proof. The common diagonal term in the equations is removed by an integrating factor. Put $\eta(u) = (1-u)^{-1/2}$, set $y_i = \eta(u)v_i$ for $i \geq 1$, and leave $y_0 = v_0$ unchanged. Since $\eta(0) = 1$, this does not change initial values. The transformed equations are

$$v_0' = 0, \qquad v_1' = -\frac{\eta(u)}{1-u/2}v_0, \qquad v_i' = \frac{v_{i-1}}{1-u} \quad (i \geq 2). \tag{4.4}$$

Both $\eta(u)$ and $-\eta(u)/(1-u/2)$ have their degree-j coefficient in $4^{-j}\mathbb{Z}$. For η, this follows from

$$\eta(u) = \sum_{j\geq 0} \binom{2j}{j} \frac{u^j}{4^j};$$

multiplication by $(1-u/2)^{-1}$ preserves the asserted denominator bound. The coefficients of $(1-u)^{-1}$ are integers.

Each entry of the normalized solution of (4.4) is a finite sum of iterated integrals of depth at most d, and multiplication by η restores U. Expand one contribution to a coefficient of degree k. At each integration, the degree increases strictly. Its integration denominators are therefore distinct integers

$$1 \leq t_1 < \cdots < t_e \leq k, \qquad e \leq d.$$

Their product divides $\Delta_{d,k}$. The total degrees supplied by coefficients of the multiplied series, including the final factor η, are at most k; their powers of 4 divide 4^k. This controls each contribution and hence their sum. The argument includes the column with $v_0(0) = 1$, whose first integration uses the inhomogeneous term displayed in (4.4). □

Corollary 4.3. *For every p, k, i in the construction,*

$$\mathcal{D}_{d,k-1}\frac{Q_{k,i}^{[p]}(-1)}{(k-1)!} \in \mathbb{Z}.$$

For $k \geq 2M+2$ this gives an integer linear form in $1, \beta(2), \ldots, \beta(2\lfloor d/2 \rfloor)$ whose value is

$$\Lambda_{n,p,k} = \mathcal{D}_{d,k-1}\frac{S_{n,p}^{(k-1)}(-1)}{(k-1)!}. \tag{4.5}$$

Proof. The row $C^{[p]}(-1+u)$ has integral Taylor coefficients. The coefficient of u^{k-1} in $C^{[p]}(-1+u)U(u)$ is the row $Q_k^{[p]}(-1)/(k-1)!$, by (3.9). Use Lemma 4.2 and $\mathcal{D}_{d,j} \mid \mathcal{D}_{d,k-1}$ for $j \leq k-1$. Substitution of (3.2) multiplies even columns by integers and sets odd contributions to zero. □

5 Sizes of coefficients and of linear forms

We now supply the two size bounds in Proposition 2.1. Both include the multiplier from Section 4: the forms must remain small after their coefficients have been made integral. We use derivatives of order exceeding the degree of the finite correction, while leaving enough decay in the tail for its sum to converge. Accordingly, assume

$$2r < \kappa < \omega, \qquad 2M+2 \leq k \leq \kappa n, \qquad 0 \leq p \leq h. \tag{5.1}$$

In addition to the integer conditions in Proposition 2.2, we require κn to be integral. We call such n *admissible*: rn, ωn, Ωn and κn are all integers. We prove bounds uniform throughout the displayed range.

Proposition 5.1. *Every coefficient of the integer form $\Lambda_{n,p,k}$ is at most $B^{n+o(n)}$ in absolute value, where*

$$\log B = \log \chi + (r+1)\log(3/2) + \kappa(\log 8 + \mathsf{H}_{a+h}). \tag{5.2}$$

The values satisfy $|\Lambda_{n,p,k}| \le \alpha^{n+o(n)}$, where

$$\log \alpha = \log \chi - \Omega \log r + \kappa(1 + \log 4 + \mathsf{H}_{a+h}). \tag{5.3}$$

The term $-\Omega \log r$ in (5.3) supplies the decay; the other terms are costs from the coefficient bound, differentiation and denominators. Section 7 chooses the parameters so that $\log \alpha < 0$ and compares this decay with $\log B$.

Coefficient bound. On the disk $|z+1| \le 1/2$, the row-sum norm of the matrix in (3.3) is at most $11/3$, since $|z| \ge 1/2$ and $|1-z| \ge 3/2$. Integrating along a straight segment from -1 gives $\|U(z+1)\| \le e^{11/6}$. Equations (2.2) and (3.7) give

$$\max_i |C_i^{[p]}(z)| \le \exp(n \log \chi + o(n))(3/2)^{(r+1)n}$$

on the same disk. The factors $(i)_p$ depend only on the fixed parameters, and the number of coefficients is polynomial in n. Cauchy's coefficient estimate therefore bounds $Q_{k,i}^{[p]}(-1)/(k-1)!$ by this expression times 2^{k-1} and a fixed constant. Multiply by $\mathcal{D}_{d,k-1}$. By monotonicity it is enough to bound $\mathcal{D}_{d,\kappa n}$, and (4.3) yields (5.2). The factors 2^{i+1} in (3.2) are fixed while n tends to infinity. □

Value bound. For real $x \ge M + 1/2$ and $0 \le p \le h$, we first claim

$$|F_n^{(p)}(\pm x)| \le \exp(n \log \chi + o(n))\, r^{-\Omega n} (M/x)^{\omega n}. \tag{5.4}$$

Here $\exp(n \log \chi + o(n))$ is the coefficient cost and $r^{-\Omega n}$ is the extra smallness supplied by the Laurent bounds. The remaining factor $(M/x)^{\omega n}$ controls decay as x grows; it leaves enough power decay to sum after up to $\kappa n - 1$ derivatives because $\kappa < \omega$. The Laurent series converges for $|t| > n$, because all poles lie among $0, -1, \dots, -n$. Differentiation adds the factor $(\nu)_p x^{-p}$ to the bound for the νth term. For $\omega n \le \nu < \Omega n$, combine (2.3) as

$$r^{\nu - \Omega n}(n/x)^{\nu} = r^{-\Omega n}(M/x)^{\nu}.$$

There are $O(n)$ terms, with remaining factors bounded by a fixed power of n. For $\nu \ge \Omega n$, write $(n/x)^{\nu} = r^{-\nu}(M/x)^{\nu}$. The remaining tail is bounded by a polynomial in n times $r^{-\Omega n}$, because $r \ge 2$ and a, p are fixed. This proves (5.4), uniformly in x and p.

Write $\ell = k - 1$. After division by $\ell!$, the absolute derivatives at -1 of the monomials in (3.4) are

$$\binom{t - M + \ell}{\ell}, \qquad \binom{M + t}{\ell},$$

where the second is zero if $M + t < \ell$. For $x = t + 1/2$ and $\ell \ge 2M + 1$, both upper arguments are at most $(\ell/M)x$. The elementary bound $\binom{v}{\ell} \le (ev/\ell)^{\ell}$ gives

$$\left| \frac{S_{n,p}^{(\ell)}(-1)}{\ell!} \right| \le \exp(n \log \chi + o(n)) r^{-\Omega n} e^{\ell} \sum_{t=M}^{\infty} \left(\frac{M}{t + 1/2} \right)^{\omega n - \ell}.$$

The exponent $E = \omega n - \ell$ satisfies $E - 1 \ge (\omega - \kappa)n$. The sum is at most $1 + M/(E-1)$ by comparison with an integral, so it is uniformly bounded. We obtain

$$\left| \frac{S_{n,p}^{(\ell)}(-1)}{\ell!} \right| \le \exp\bigl(n(\log \chi - \Omega \log r + \kappa) + o(n)\bigr).$$

These bounds also justify the absolute convergence of the differentiated series used in Lemma 3.1. Multiplication by $\mathcal{D}_{d,\ell} \le \mathcal{D}_{d,\kappa n}$ proves (5.3). □

6 The weak rank condition

The size estimates are now available, but Proposition 2.1 also requires the weak rank condition (2.1). We prove it first for the derivative rows $Q_k^{[p]}(-1)$, before removing the odd columns. Section 7 then passes it to the integer forms in even beta values. The rows have zero entries for $i > b + h$: this holds for $C^{[p]}$ and is preserved by (3.8). We can therefore work with the system truncated at component $b + h$.

The proof adapts [3, Section 4.5]. A forbidden kernel vector would give solutions whose polynomial combinations have many vanishing derivatives at -1. We convert these into remainders for a single polynomial row, prove that the remainders are independent, and apply Proposition 6.2 to contradict their total vanishing. The inequality below is the one obtained from that final count.

Proposition 6.1. *Assume*

$$(h + 1)(\kappa - 2r) + \omega > a. \tag{6.1}$$

For all sufficiently large admissible n, if $x_0, \dots, x_{b+h} \in \mathbb{Q}$ satisfy

$$\sum_{i=0}^{b+h} Q_{k,i}^{[p]}(-1)x_i = 0 \quad (0 \le p \le h, \quad 2M + 2 \le k \le \kappa n),$$

then $x_0 = 0$.

The factor $h + 1$ records the gain from using all the derivative families: each will supply an independent remainder at each point above -1. For fixed r, κ, ω, condition (6.1) forces h to grow proportionally to a. This is why the extra families, and their larger beta cutoff $a + h$, were introduced.

6.1 The multiplicity estimate and the six points

The multiplicity estimate bounds the total vanishing of polynomial combinations of solutions in terms of the polynomial degrees. We first state it, then describe the contributions we will need from the six points in our application.

Let $Y' = \mathcal{A}(w)Y$ be a fixed rational differential system of size q. For polynomials $S_0, \dots, S_{q-1}$ of degree at most m, write

$$\mathcal{R}(Y) = \sum_{j=0}^{q-1} S_j Y_j.$$

We call this scalar function the *remainder* associated with the solution Y and the fixed polynomial row $(S_0, \dots, S_{q-1})$. The selected remainders must satisfy a common scalar linear differential equation of some order μ, with rational coefficients. In our application such an equation will follow from the finite dimension of the system.

At a point σ, the Nilsson class consists of finite sums of holomorphic functions multiplied by complex powers and nonnegative integer powers of a logarithm of the local coordinate. At infinity the local coordinate is $1/w$. The order is the least real part of a power that occurs after expanding the holomorphic factors; logarithmic factors do not change it. For example, $\operatorname{ord}_0(w^e(\log w)^j) = \operatorname{Re} e$. In particular, w^N has order $-N$ at infinity. Negative orders will therefore occur when we apply the theorem.

Proposition 6.2 ([3, Theorem 3]; see also [1, Theorem 3.1])**.** *Let Σ be a fixed finite set of points of the Riemann sphere containing infinity. At each $\sigma \in \Sigma$, choose a family of local solutions $Y_{\sigma,j}$, indexed by $j \in J_\sigma$, whose remainders are in the Nilsson class and are linearly*

independent over $\mathbb{C}$*. If a nonzero rational differential operator of order* μ *annihilates all these remainders, then*

$$\sum_{\sigma\in\Sigma}\sum_{j\in J_\sigma} \operatorname{ord}_\sigma \mathcal{R}(Y_{\sigma,j}) \le (m+1)(\mu - |J_\infty|) + C.$$

The constant C *depends only on the differential system and on* Σ*.*

The solution families at different points need not agree by analytic continuation, and no independence between different families is required. These features are explicit in [1, Section 3.1]. We use the formulation in [3], which also allows negative orders at infinity, as in its proof of Proposition 3. The constant is uniform in the polynomial row and in the selected solutions. This uniformity is essential when that row and the solutions depend on n.

The six points will appear after the substitution $z = w^2$. The assumed kernel equations produce high-order zeros at the two points $w = \mathrm{i}$ and $w = -\mathrm{i}$ above $z = -1$. The decay of F_n at infinity will supply additional zeros at $w = 1$ and $w = -1$. We will also construct b logarithmic remainders at 0 and infinity, where their orders can be read from the polynomial degrees.

The remainders at infinity have a second purpose. In our application the common differential operator has order $\mu \le b + h + 1$. Choosing $|J_\infty| = b$ reduces the factor $\mu - |J_\infty|$ on the right-hand side of the theorem to at most $h + 1$. This is essential: the degree cost then involves the number of derivative families, rather than the full size of the system. The negative orders at infinity must still be included on the left. Paired with the orders at 0, their contribution will be $-2bn$.

We will obtain polynomial degrees at most $4rn$ and the inequality

$$2(h+1)\kappa n + 2\omega n - 2bn + O(1) \le 4rn(h+1) + O(1).$$

The first two terms count the zeros at $\pm\mathrm{i}$ and ± 1; the third is the combined contribution at 0 and infinity. After division by $2n$, this contradicts (6.1). The rest of the section constructs the required remainders, verifies their independence, and justifies this count.

6.2 Taylor corrections and a common polynomial row

Suppose $x_0 \neq 0$, and scale so that $x_0 = 1$. Let $g_0, \dots, g_{b+h}$ solve (3.3) near -1 with $g_i(-1) = x_i$. In particular $g_0 = 1$. The kernel equations say that certain high derivatives of $\sum_i C_i^{[p]} g_i$ vanish at -1. To turn this into a zero of high order, we must also remove the lower Taylor coefficients. Choose $T_p \in \mathbb{Q}[z]$ of degree at most $2M$ to cancel the Taylor polynomial through degree $2M$ of $\sum_i C_i^{[p]} g_i$ at -1. The universal identity (3.9) and the assumed vanishing of the higher derivatives give

$$f_p(z) := T_p(z) + \sum_{i=1}^{b+h} C_i^{[p]}(z) g_i(z) = O((z+1)^{\kappa n}). \tag{6.2}$$

The derivatives of orders $0, \dots, 2M$ vanish by the choice of T_p; those of orders $2M+1, \dots, \kappa n-1$ vanish by hypothesis.

The functions f_p still have different coefficient rows. The multiplicity theorem instead needs one fixed polynomial row applied to several solutions of the same system. Following Fischler, we arrange this by combining the f_p with powers of a local branch $L(z) = -\log z$ near -1. The choice is suggested by $L' = -1/z$, the same coefficient that links g_j to g_{j-1} for $j \ge 2$ in (3.3). Powers of L can therefore provide the additional chain of components needed to hold the corrections T_p.

First take $h = 1$ and set

$$\rho_0 = f_0, \qquad \rho_1 = Lf_0 + f_1.$$

Substituting (3.7) gives

$$\rho_0 = T_0 + z^M \sum_{j=1}^{b} P_j g_j,$$

$$\rho_1 = T_1 + LT_0 + z^M \sum_{j=1}^{b} P_j (Lg_j - jg_{j+1}).$$

Both expressions use the same polynomial row

$$S = (T_1, T_0, z^M P_1, \dots, z^M P_b).$$

Indeed, $\rho_0 = SY_0$ and $\rho_1 = SY_1$ for the columns

$$Y_0 = (0, 1, g_1, \dots, g_b)^T,$$
$$Y_1 = (1, L, Lg_1 - g_2, \dots, Lg_b - bg_{b+1})^T.$$

The dependence on the family index is now entirely in Y_0, Y_1: the multipliers of T_1, T_0 and each P_j change, while S stays fixed. In particular, the logarithm belongs to the solutions, so it does not spoil the polynomial row.

The same combinations also preserve the differential chain. Put $Z_j = Lg_j - jg_{j+1}$ for $1 \le j \le b$. For $j \ge 2$, the product rule and (3.3) give

$$\begin{aligned} Z_j' &= -\frac{g_j}{z} + L\left(-\frac{g_{j-1}}{z} - \frac{g_j}{2z}\right) - j\left(-\frac{g_j}{z} - \frac{g_{j+1}}{2z}\right) \\ &= -\frac{Lg_{j-1} - (j-1)g_j}{z} - \frac{Lg_j - jg_{j+1}}{2z} \\ &= -\frac{Z_{j-1}}{z} - \frac{Z_j}{2z}. \end{aligned}$$

The term $L'g_j = -g_j/z$ changes the coefficient j to $j-1$, producing the previous component Z_{j-1}. At the first link, the same calculation gives $Z_1' = 2L/[z(1-z)] - Z_1/(2z)$. Together with $L' = -1/z$ and the equations for g, these identities show that both columns solve the $h = 1$ case of (6.5) below.

For general h, set

$$\rho_q(z) = \sum_{p=0}^{q} \binom{q}{p} L(z)^{q-p} f_p(z) \qquad (0 \le q \le h).$$

Each ρ_q retains the zero of order at least κn at -1. The common polynomial row will have $h+1$ entries for the corrections T_p and b entries for the original polynomials P_i:

$$S_i(z) = \begin{cases} T_{h-i}(z)/(h-i)!, & 0 \le i \le h, \\ z^M P_{i-h}(z), & h+1 \le i \le b+h, \end{cases} \tag{6.3}$$

The dependence on q is placed in the solution vectors. Define

$$y_{i,q}(z) = \begin{cases} 0, & i < h-q, \\ \dfrac{q!}{(i+q-h)!} L(z)^{i+q-h}, & h-q \le i \le h, \\ \displaystyle\sum_{p=0}^{q} \binom{q}{p} L(z)^{q-p} (-1)^p (i-h)_p g_{i-h+p}(z), & i > h. \end{cases} \tag{6.4}$$

Then $\rho_q = \sum_i S_i y_{i,q}$. All columns $Y_q = (y_{0,q}, \dots, y_{b+h,q})^T$ solve the same system

$$\begin{aligned} y_0' &= 0, \\ y_i' &= -y_{i-1}/z && (1 \le i \le h), \\ y_{h+1}' &= \frac{2}{z(1-z)} y_h - \frac{1}{2z} y_{h+1}, \\ y_i' &= -y_{i-1}/z - y_i/(2z) && (i \ge h+2). \end{aligned} \tag{6.5}$$

The first $h+1$ components are powers of L; the components with $i > h$ contain the original solution g. On differentiating the latter components, terms from the derivatives of L^{q-p} combine with those from the derivatives of g_{i-h+p}. After reindexing, their coefficients simplify by the identity

$$(j)_p - p(j)_{p-1} = (j-1)_p \qquad (j \ge 2,\ p \ge 1).$$

The $p = 0$ factor is 1. This gives the equation at index $i = h + j$. When $j = 1$, the corresponding factors vanish for $p \ge 1$; the remaining source is $2L^q/[z(1-z)]$, giving the equation for y_{h+1}. The diagonal term remains $-y_i/(2z)$ in every summand. The equations for $i \le h$ follow by differentiating powers of L.

6.3 Quadratic change of variable

The integrating factor in Section 4 removed the diagonal term locally to control Taylor denominators. Here we need the same cancellation in a form that preserves a polynomial row and leaves a system suitable for the residue argument. Multiplying the components $i > h$ by $\sqrt{z}$ removes their common diagonal term $-y_i/(2z)$. We introduce w with $z = w^2$ so that this change is rational in the new variable and the row remains polynomial. Set

$$v_i(w) = \begin{cases} y_i(w^2), & i \le h, \\ w\, y_i(w^2), & i > h. \end{cases}$$

The diagonal terms cancel, leaving

$$\begin{aligned} v_0' &= 0, \\ v_i' &= -2v_{i-1}/w && (i \ge 1,\ i \ne h+1), \\ v_{h+1}' &= 4v_h/(1-w^2). \end{aligned} \tag{6.6}$$

The common coefficient polynomials become

$$\widetilde{S}_i(w) = \begin{cases} S_i(w^2), & i \le h, \\ w^{2M-1} P_{i-h}(w^2), & i > h. \end{cases} \tag{6.7}$$

Their degrees are at most $4M$: those with $i \le h$ have degree at most $4M$, and those with $i > h$ have degree at most $2(M+n) - 1 \le 4M - 1$. For the solutions obtained from Y_q, the remainder $\sum_i \widetilde{S}_i v_i$ equals $\rho_q(w^2)$. It vanishes to order at least κn at each of i and $-$i.

The change of variable increases the polynomial degrees, but it also gives two points at which to count the high-order zeros. The same polynomials (6.7) are used at both points; only the local solution families differ. This is why the Taylor corrections were constructed in z before making the change of variable.

6.4 Independence by residues

To count these zeros with Proposition 6.2, we must know that no nonzero constant combination of the $h+1$ remainders vanishes identically. Independence of the solution vectors alone would not suffice: a fixed row can annihilate a nonzero combination of vectors. We must rule out such annihilation for our row, whose highest coefficient is nonzero because $P_b \not\equiv 0$. The triangular form of the new system makes this possible. Each derivative link has a nonzero residue, whereas the derivative of a rational function has zero residue at every finite point. The following lemma uses this to rule out rational-function relations between successive components. It abstracts the argument in [3, Section 4.5, Step 4]. Different links of the system may use different finite poles.

Lemma 6.3. *Suppose v_0 is a nonzero constant and $v_j' = a_j(w)v_{j-1}$ for $1 \le j \le t$, where each rational function a_j has a nonzero residue at some finite point. Then $v_0, \dots, v_t$ are linearly independent over $\mathbb{C}(w)$.*

Proof. Clear the denominators of a putative rational relation and write it as $\sum_{i=0}^{\alpha} U_i v_i = 0$, with $U_\alpha \neq 0$ and $\alpha \ge 1$. Let $D = \deg U_\alpha$. Differentiating $D+1$ times eliminates v_α. The coefficient of $v_{\alpha-1}$ is

$$U_{\alpha-1}^{(D+1)} + \sum_{j=0}^{D} (a_\alpha U_\alpha^{(j)})^{(D-j)}. \tag{6.8}$$

This follows by differentiating $U_\alpha v_\alpha$ successively and retaining the two highest components. All summands except $j = D$ are derivatives of rational functions; the first term is a polynomial. They have zero residue at every finite point. The remaining residue is $U_\alpha^{(D)} \operatorname{Res}(a_\alpha)$, nonzero at a suitable point. Thus (6.8) is not the zero function. Clearing denominators gives a polynomial relation with nonzero highest coefficient at index $\alpha - 1$. Descent eventually gives a nonzero polynomial times the nonzero constant v_0 equal to zero, a contradiction. □

In (6.6), the link $-2/w$ has residue -2 at 0, and $4/(1-w^2)$ has residue -2 at 1. Consider a nonzero constant combination of the solutions Y_q. If q_0 is the largest index with nonzero coefficient, its first nonzero component is at index $h - q_0$ and is a nonzero constant, by (6.4). This component remains constant after the change of variable and rescaling. Lemma 6.3, starting at that index, shows that the remainder cannot vanish identically, because its highest coefficient $\widetilde{S}_{b+h} = w^{2M-1}P_b(w^2)$ is nonzero. Thus the $h+1$ high-order remainders are linearly independent at each of i and $-$i.

6.5 The other local remainders

We also need remainders at 0, infinity and ± 1. These will come from solutions whose first $h+1$ components vanish. The remaining equations then integrate to powers of a logarithm. Their combined orders at 0 and infinity will be independent of M, while one of them will inherit additional zeros at ± 1 from the decay of F_n.

For $j = 1, \dots, b$, system (6.6) has a solution whose components below $h+j$ vanish and whose remaining components are

$$\widetilde{v}_{i,j}(w) = \frac{(-\log(w^2))^{i-h-j}}{(i-h-j)!} \qquad (i \ge h+j).$$

Its remainder is

$$\widetilde{\rho}_j(w) = w^{2M-1} \sum_{i=j}^{b} P_i(w^2) \frac{(-\log(w^2))^{i-j}}{(i-j)!}. \tag{6.9}$$

These b remainders are independent. In a nonzero constant relation, the least index j with nonzero coefficient supplies the unique highest logarithmic degree, with a nonzero coefficient proportional to P_b. This is impossible because $\log(w^2)$ is transcendental over $\mathbb{C}(w)$.

At 0, each remainder has order at least $2M-1$; at infinity its order is at least $-2(M+n)+1$.

For the zeros at ± 1, consider

$$R_n(z) := \sum_{i=1}^{b} P_i(z) \frac{(-\log z)^{i-1}}{(i-1)!} = O((z-1)^{\omega n-1}) \tag{6.10}$$

near 1, on the branch $\log 1 = 0$. The asserted order is another form of the exact Laurent vanishing controlled by ω, rather than the additional smallness controlled by Ω. Indeed, the partial fractions give the Laplace-transform identity

$$F_n(t) = \int_0^\infty e^{-tu} \sum_{i=1}^{b} P_i(e^{-u}) \frac{u^{i-1}}{(i-1)!}\, \mathrm{d}u \qquad (t > 0).$$

Write the factor multiplying e^{-tu} as $G(u)$. The elementary identity

$$\int_0^\infty e^{-tu} \frac{u^k}{k!}\, \mathrm{d}u = \frac{1}{t^{k+1}} \qquad (t > 0,\ k \geq 0)$$

explains how Taylor coefficients at $u = 0$ become Laurent coefficients at infinity. Here G is a finite sum of polynomials times exponentials e^{-ju} with $j \geq 0$, so repeated integration by parts gives, for fixed n and every fixed integer $K \geq 0$,

$$F_n(t) = \sum_{k=0}^{K} \frac{G^{(k)}(0)}{t^{k+1}} + O(t^{-K-2}) \qquad (t \longrightarrow +\infty).$$

Comparing with the Laurent expansion yields $A_{k+1} = G^{(k)}(0)$. Thus the exact zeros $A_1, \ldots, A_{\omega n-1}$ in Proposition 2.2 force G to vanish to order at least $\omega n - 1$. Finally, $G(u) = R_n(e^{-u})$ and $e^{-u} = 1 - u + O(u^2)$, so this change of variable preserves the order of vanishing and proves (6.10). This is the observation used in [3, Sections 3.5 and 4.5].

Now $\widetilde{\rho}_1(w) = w^{2M-1} R_n(w^2)$. At each of $w = 1$ and $w = -1$, choose the local branch for which $\log(w^2) = 0$ at that point. The map $w \mapsto w^2$ has nonzero derivative there, so (6.10) gives order at least $\omega n - 1$ at each point. These two local branches need not be restrictions of a single global branch; Proposition 6.2 allows the different local solutions.

6.6 Counting the orders

Apply Proposition 6.2 with

$$\Sigma = \{0, \infty, 1, -1, \mathrm{i}, -\mathrm{i}\}, \qquad m = 4M$$

to (6.6) and (6.7). The system is regular singular: its matrix has only simple finite poles and is $O(1/w)$ at infinity. Its solutions therefore have the required Nilsson expansions. The constructed families give the following bounds.

Point	Independent remainders	Order of each, at least
Each of $\mathrm{i}, -\mathrm{i}$	$h+1$	κn
0	b	$2M-1$
∞	b	$-2(M+n)+1$
Each of $1, -1$	1	$\omega n - 1$

There is a common rational differential operator of order $\mu \le b+h+1$ annihilating all remainders. Differentiating a remainder iterates the operation $S \mapsto S' + S\mathcal{A}$ on a row of length $b+h+1$; its first $b+h+2$ iterates are rationally dependent. Since the b remainders at infinity are independent, $\mu \ge b$. The multiplicity estimate gives

$$\begin{aligned}(2(h+1)\kappa - 2b + 2\omega)n - 2 &\le (4rn+1)(\mu - b) + C \\ &\le (4rn+1)(h+1) + C.\end{aligned}$$

The constant depends only on the system and Σ. The finitely many possibilities $1 \le b \le a$ allow a uniform constant, even when b varies with n. Dividing the last inequality by $2n$ would give

$$(h+1)(\kappa - 2r) + \omega \le b + o(1) \le a + o(1).$$

This contradicts (6.1) for sufficiently large n and proves Proposition 6.1.

7 The dimension estimate and the choice of parameters

We can now assemble the argument. Section 4 made the forms integral, Section 5 bounded their coefficients and values, and Section 6 supplied the rank statement. We first apply the criterion with all auxiliary parameters fixed, then choose them as the number of beta values increases.

For each admissible n, form the integer matrix with rows the coefficients of $\Lambda_{n,p,k}$ in

$$1, \beta(2), \dots, \beta(2\lfloor (a+h)/2 \rfloor).$$

It satisfies (2.1). Indeed, a rational vector annihilating all these rows lifts to one for the rows $Q_{k,i}^{[p]}(-1)$ by putting zero in every odd coordinate and multiplying coordinate $i/2$ by -2^{i+1} for even $i \ge 2$. Its constant coordinate is unchanged. The row multipliers $\mathcal{D}_{d,k-1}/(k-1)!$ are nonzero, so Proposition 6.1 forces that constant coordinate to be zero.

If $\log \alpha < 0$, Proposition 2.1 gives

$$\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \{1, \beta(2), \dots, \beta(2\lfloor (a+h)/2 \rfloor)\} \ge 1 - \frac{\log \alpha}{\log B}.$$

Removing 1 loses at most one dimension. Thus

$$D(2\lfloor (a+h)/2 \rfloor) \ge -\frac{\log \alpha}{\log B}. \tag{7.1}$$

We choose the parameters in the order of the constraints they must meet. First fix $r \ge 2$, κ, ω and a proportion $0 < h/a \le 1$ such that

$$2r < \kappa < \omega, \qquad \frac{h}{a}(\kappa - 2r) > 1.$$

The first condition gives a nonempty range of derivative rows with summable tails; the second ensures the rank condition. Next choose Ω so that its coefficient cost Ω^2/a has the same order as the denominator cost $\log a$. Thus Ω has order $\sqrt{a \log a}$, and its decay contribution $\Omega \log r$ dominates both costs. These choices leave $\log B$ of order $\log a$ and $-\log \alpha$ of order $\sqrt{a \log a}$.

For the numerical constants, use the choice of [3, Section 4.6]. Let a tend to infinity through positive multiples of 25, and set

$$h = \frac{9a}{25}, \quad r = \frac{39}{10}, \quad \kappa = \frac{529}{50}, \quad \omega = 12, \quad \Omega = \left\lfloor r\sqrt{a \log a} \right\rfloor. \tag{7.2}$$

For sufficiently large a these satisfy $a > \Omega \geq \omega > \kappa > 2r$ and $0 \leq h \leq a$. The rank condition has a small strict margin, visible in its slope:

$$\frac{h}{a}(\kappa - 2r) = \frac{9}{25}\left(\frac{529}{50} - 2 \cdot \frac{39}{10}\right) = \frac{1251}{1250} > 1.$$

Take n through sufficiently large multiples of 50. For each fixed a, all needed integer conditions and the weak rank statement then hold.

As $a \to \infty$, the Ω^2 term supplies the leading part of $\log \chi$, and $\kappa \mathsf{H}_{a+h} \sim \kappa \log a$ supplies the denominator contribution to $\log B$. The term $-\Omega \log r$ dominates $\log \alpha$. Equations (2.2), (5.2) and (5.3) therefore give

$$\begin{aligned} \log \chi &\sim \frac{r^2 \log r}{2} \log a, \\ \log B &\sim \left(\frac{r^2 \log r}{2} + \kappa\right) \log a, \\ \log \alpha &\sim -r \log r \sqrt{a \log a}. \end{aligned} \tag{7.3}$$

In particular, $\log \alpha < 0$ for sufficiently large a. With $s = a + h = 34a/25$, (7.1) implies

$$\liminf_{\substack{s \to \infty \\ 34 \mid s}} \frac{D(s)}{\sqrt{s/\log s}} \geq c_* := \frac{r \log r}{(r^2 \log r/2 + \kappa)\sqrt{34/25}} = 0.2174562348\ldots. \tag{7.4}$$

The strict inequality $c_* > 0.21$ requires little numerical precision: $1.36 < \log(3.9)$ and $\sqrt{1.36} < 1.167$ already suffice, since $x/(r^2 x/2 + \kappa)$ increases with $x > 0$.

Finally $D(s)$ is nondecreasing, and successive multiples of 34 have bounded gaps. For an arbitrary large even s, apply (7.4) to $34\lfloor s/34 \rfloor$. The quotient of the corresponding factors $\sqrt{s/\log s}$ tends to 1. The strict slack between c_* and 0.21 proves Theorem 1.1.

Remark 7.1. The limiting assertion (7.4) extends to all even s in the same way. We retain the simpler constant 0.21 in the theorem. No explicit threshold for *sufficiently large* is asserted; in particular, the multiplicity constant has not been tracked.

Acknowledgements and preparation

The proof uses Fischler's rational-function construction, refined linear-independence criterion and multiplicity method. The numerical parameters and the organization of the rank argument are taken from his proof, with the adaptations identified in the text. Mistiaen's explanation of the conductor restriction suggested the evaluation-point question addressed here.

This research note was prepared with OpenAI Codex, using Astra-6-Max for the exposition revisions. An independent proof audit by a second model, reported on 3 November 2026, found no substantive mathematical gap in the 19-page revision. The audit checked the denominator and rank arguments, the application of Fischler's multiplicity estimate and the final bound; the supplied and additional computational checks were reported to pass. The audit used the cited results and was not formal verification.

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