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Column rigidity in noncommutative L^1-spaces and completely isometric embeddings of Hilbertian operator spaces

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Abstract

We study exact embedding obstructions for Hilbertian operator spaces in noncommutative L^1-spaces. We prove a quantitative version of the column inequality, whose equality case forces the absolute values of the entries to be proportional. We deduce that an operator space embedded completely isometrically in a noncommutative L^1-space is a row Hilbert space whenever its first-column norms at the two-by-two level are Hilbert-Schmidt norms. This gives an exact embedding criterion for sums of weighted row and column spaces, including coordinate-dependent weights which are ordered in the same direction. In particular, $\mathrm{R}_n + \mathrm{C}_n$ does not embed completely isometrically into any noncommutative L^1-space when $n \geqslant 2$. For OH_n, whose complete isometric embeddability is established in a companion paper, we retain the common-support and commutative-target obstructions. We also show that distortion one is approached by matrix models but is attained by none of them, and that the matrix size must tend to infinity as the distortion tends to one.

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1 Introduction

Row and column Hilbert spaces play a central role in the structure of subspaces of noncommutative L^1-spaces. Noncommutative Khintchine inequalities describe many such subspaces by sums of row and column norms. They also provide completely isomorphic embeddings of quotients of subspaces of $\mathrm{R} \oplus \mathrm{C}$ into preduals of von Neumann algebras. We refer to Xu [Xu07,

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Theorem 6.8 p. 21 and Theorems 7.1–7.3 pp. 29–30] for this construction. Haagerup and Musat [HaM07, Theorem 3.1 p. 17] obtained an embedding of the dual of any subspace of $\mathrm{R} \oplus_\infty \mathrm{C}$ with cb-distortion at most $\sqrt{2}$. The question considered here is whether the distortion can be equal to one.

This question has different answers for closely related Hilbertian operator spaces. In [Arh26, Theorem 6.2 pp. 19–20], we construct a completely isometric embedding of Pisier's operator Hilbert space OH into the predual of a QWEP von Neumann algebra. Thus the completely isometric embedding problem for OH is no longer an open question. The present paper concerns the rigidity which prevents such embeddings for other spaces. In particular, we prove that $\mathrm{R}_n + \mathrm{C}_n$ does not embed completely isometrically into any noncommutative L^1-space when $n \geqslant 2$. The construction and the tensor norm results of [Arh26] are not repeated here.

Our main tool is the equality case in an inequality for columns. Let $K_1, \ldots, K_N \in \mathrm{L}^1(\mathcal{M})$, where $\mathcal{M}$ is a von Neumann algebra, and put

$$a \stackrel{\text{def}}{=} \left(\sum_{k=1}^N K_k^* K_k \right)^{\frac{1}{2}}, \quad t \stackrel{\text{def}}{=} \mathrm{Tr}(a), \quad \gamma \stackrel{\text{def}}{=} \left(\sum_{k=1}^N \|K_k\|_{\mathrm{L}^1(\mathcal{M})}^2 \right)^{\frac{1}{2}}. \tag{1.1}$$

We have $t \geqslant \gamma$. If $\gamma > 0$, we prove the estimate

$$\sum_{k=1}^N \left\| |K_k| - \frac{\|K_k\|_{\mathrm{L}^1(\mathcal{M})}}{\gamma} a \right\|_{\mathrm{L}^1(\mathcal{M})}^2 \leqslant 2t(t-\gamma). \tag{1.2}$$

Consequently, equality $t = \gamma$ holds if and only if the absolute values $|K_k|$ are proportional. The proof uses the polar decomposition of the whole column and the Cauchy–Schwarz inequality in L^2. It applies to Haagerup L^1-spaces without a semifiniteness assumption.

Testing a complete isometry on columns with two entries gives the following result, proved in Theorem 4.4. The notation $S_m^1(E)$ denotes Pisier's vector-valued Schatten space.

Theorem 1.1 *Let E be an operator space with a basis $(\xi_1, \ldots, \xi_n)$, and suppose that E embeds completely isometrically into a noncommutative L^1-space. Suppose that $c > 0$ and that for any $A = [a_{ik}] \in \mathrm{M}_{2,n}$ we have*

$$\left\| \sum_{i=1}^2 \sum_{k=1}^n a_{ik} e_{i1} \otimes \xi_k \right\|_{S_2^1(E)} = c \, \|A\|_{S^2} .$$

Then the map $\xi_k \mapsto c e_k$ is a complete isometry from the operator space E onto R_n.

The conclusion follows because equality in the column inequality forces the Gram relations $h_k^* h_l = \delta_{kl} a^2$ for the images of the basis vectors. These relations determine all matrix norms. Combining this rigidity theorem with Ozawa's compactness theorem gives a qualitative stability result, stated in Corollary 4.6.

As a consequence, we obtain the following classification.

Theorem 1.2 *Let $n \geqslant 2$ and $\alpha, \beta > 0$. The operator space $\alpha \mathrm{R}_n + \beta \mathrm{C}_n$ embeds completely isometrically into a noncommutative L^1-space if and only if*

$$\max\{\alpha, \beta\} \geqslant \sqrt{n} \min\{\alpha, \beta\}.$$

In these cases, the sum coincides completely isometrically with its smaller weighted row or column space.

Theorem 5.5 extends this statement to coordinate-dependent weights. For positive vectors $\alpha = (\alpha_k)_{k=1}^n$ and $\beta = (\beta_k)_{k=1}^n$, write $\mathrm{R}_n(\alpha)$ and $\mathrm{C}_n(\beta)$ for the spaces obtained by multiplying the k-th coordinate by α_k and β_k, respectively. If $\alpha_k \leqslant \beta_k$ for any k, then $\mathrm{R}_n(\alpha) + \mathrm{C}_n(\beta)$ embeds completely isometrically into a noncommutative L^1-space if and only if

$$\sum_{k=1}^{n} \frac{\alpha_k^2}{\beta_k^2} \leqslant 1.$$

There is a symmetric statement when all the inequalities between the weights are reversed. This gives a spectral criterion for the duals of subspaces of $\mathrm{R}_n \oplus_\infty \mathrm{C}_n$ whose associated positive contraction has spectrum on one side of $\frac{1}{2}$.

The obstruction for $\mathrm{R}_n + \mathrm{C}_n$ is independent of a particular probabilistic construction. By duality, $\mathrm{R}_n \cap \mathrm{C}_n$ is not a complete quotient of a C*-algebra. Moreover, the two constants in a noncommutative Khintchine inequality for the exact sum norm cannot both be one, even after changing the ambient von Neumann algebra and the realizing family.

Our use of column profiles is related to the fundamental functions of homogeneous Hilbertian operator spaces. Junge and Xu [JuX10, Theorem 0.1 pp. 77–78] recover the completely isomorphic structure of a regular space in their class from its row and column fundamental functions. Here the hypotheses are exact norm identities, and the conclusion is a complete isometry. The assumption that the space embeds in a predual is indispensable: $\mathrm{R}_n + \mathrm{C}_n$ and R_n have the same first-column norms, although they are not completely isometric for $n \geqslant 2$.

For OH_n, the first-column norm is $\|A\|_{S^{\frac{4}{3}}}$, so the Hilbert-Schmidt rigidity theorem does not apply. We retain the bilateral Gram identities, the common-support theorem, and the obstructions for atomic algebras with finite-dimensional summands and for commutative algebras. Combining the atomic obstruction with [Arh26, Corollary 6.3 p. 20] and the finite representability theorem of Effros, Junge and Ruan [EJR00, Corollary 7.3 p. 87], we obtain a precise distinction between finite-dimensional approximation and exact realization: for fixed $n \geqslant 2$, the optimal distortions into matrix trace classes tend to one as the matrix size tends to infinity, but remain strictly larger than one at each fixed size.

Approach of the paper A complete isometry preserves the norms in all spaces $S_m^1(E)$. For columns these norms become trace norms of square roots of Gram operators. The equality case in (1.2), applied to rank-one coefficient matrices and their rank-two sums, forces these Gram operators to be scalar multiples of one fixed positive operator. For OH_n, a different argument uses the absence of a linear term in $(1 + t^{\frac{4}{3}})^{\frac{3}{4}}$ at zero to identify the support projections.

Structure of the paper Section 2 contains the conventions and tensor product facts used below. Section 3 recalls duality and the attainment of the least predual distortion. Section 4 proves the quantitative column inequality and the rigidity theorem. Section 5 treats weighted row and column sums. Section 6 concerns the restrictions on realizations of OH_n.

2 Preliminaries

2.1 Vector-valued Schatten spaces

We refer to the books [BLM04], [EfR00] and [Pis03] for background on operator spaces. For any integers $m, n \geqslant 1$, we denote by $\mathrm{M}_{m,n}$ the space of complex $m \times n$ matrices, by $\mathrm{M}_m = \mathrm{M}_{m,m}$ and by e_{ij} the matrix units. We denote by $S_{m,n}^p$ the space $\mathrm{M}_{m,n}$ equipped with the Schatten

norm $\|A\|_{S^p} \overset{\text{def}}{=} \mathrm{Tr}(|A|^p)^{\frac{1}{p}}$. If E is an operator space and if $\alpha > 0$, we denote by αE the vector space E equipped with the matrix norms of E multiplied by α.

Let E be an operator space and $m \geqslant 1$ an integer. Following Pisier [Pis98], we denote by $S^1_m(E)$ the vector-valued Schatten space, that is the space $S^1_m \widehat{\otimes} E$. Recall that by [Pis98, (1.1) p. 18] we have a completely isometric identification

$$S^1_m(E) = \mathrm{R}_m \otimes_h E \otimes_h \mathrm{C}_m, \quad e_{ij} \otimes x \mapsto e_{1i} \otimes x \otimes e_{j1}, \tag{2.1}$$

where R_m and C_m denote the row and column spaces, see Subsection 2.3. We will use the following facts.

1. By [Pis98, Lemma 1.7, (1.7)' p. 23], a linear map $u\colon E \to F$ between operator spaces satisfies $\|u\|_{\mathrm{cb}} = \sup_{m \geqslant 1} \left\|\mathrm{Id}_{S^1_m} \otimes u\right\|_{S^1_m(E) \to S^1_m(F)}$. In particular, u is a complete isometry if and only if $\mathrm{Id}_{S^1_m} \otimes u\colon S^1_m(E) \to S^1_m(F)$ is an isometry for any integer $m \geqslant 1$.

2. By [Pis98, Corollary 1.2 p. 19], if $u\colon E \to F$ is a complete isometry then $\mathrm{Id}_{S^1_m} \otimes u\colon S^1_m(E) \to S^1_m(F)$ is an isometry.

3. If (E_0, E_1) is a compatible couple of operator spaces and if $0 < \theta < 1$, we have by [Pis98, (1.5) p. 20] an isometry $S^1_m((E_0, E_1)_\theta) = (S^1_m(E_0), S^1_m(E_1))_\theta$.

4. If F is a closed subspace of E, then the quotient map $E \to E/F$ induces an isometry $S^1_m(E)/S^1_m(F) \to S^1_m(E/F)$. Indeed, this follows from (2.1) and from the projectivity of the Haagerup tensor product [BLM04, Section 1.5.5 p. 31].

5. If E_0 and E_1 are finite-dimensional operator spaces, we have an isometry $S^1_m(E_0 \oplus_1 E_1) = S^1_m(E_0) \oplus_1 S^1_m(E_1)$. Indeed, recall that $E_0 \oplus_1 E_1$ is the operator space dual of $E_0^* \oplus_\infty E_1^*$. Since $S^1_m(F^*) = \mathrm{M}_m(F)^*$ isometrically for any finite-dimensional operator space F (see the proof of [Pis98, Lemma 1.7 p. 23]), we have

$$S^1_m(E_0 \oplus_1 E_1) = \mathrm{M}_m(E_0^* \oplus_\infty E_1^*)^* = \big(\mathrm{M}_m(E_0^*) \oplus_\infty \mathrm{M}_m(E_1^*)\big)^* = S^1_m(E_0) \oplus_1 S^1_m(E_1).$$

If E_0 and E_1 are two operator space structures on the same finite-dimensional vector space V, we consider as usual the intersection $E_0 \cap E_1$, that is the diagonal $\{(\xi, \xi) : \xi \in V\}$ of $E_0 \oplus_\infty E_1$, and the sum $E_0 + E_1$, that is the quotient of $E_0 \oplus_1 E_1$ by the subspace $\{(\xi, -\xi) : \xi \in V\}$. By duality [EfR00, Corollary 4.1.9 p. 73], we have a complete isometry $(E_0 \cap E_1)^* = E_0^* + E_1^*$.

2.2 Haagerup noncommutative L^1-spaces

Let $\mathcal{M}$ be a von Neumann algebra. We use the noncommutative L^p-spaces $\mathrm{L}^p(\mathcal{M})$ of Haagerup, see [Ter81] and [PiX03]. Recall that $\mathrm{L}^p(\mathcal{M})$ is a space of closed densely defined operators affiliated with the crossed product of $\mathcal{M}$ by the modular automorphism group of a normal semifinite faithful weight, that products of such operators are taken in the sense of the strong product and that the positive part $\mathrm{L}^p(\mathcal{M})_+$ is the set of positive elements of $\mathrm{L}^p(\mathcal{M})$. We denote by $\mathrm{Tr}\colon \mathrm{L}^1(\mathcal{M}) \to \mathbb{C}$ the trace functional. We will use the following classical properties, see [Ter81, Chapter II].

1. For any $h \in \mathrm{L}^1(\mathcal{M})$, let $\varphi_h\colon \mathcal{M} \to \mathbb{C}$, $a \mapsto \mathrm{Tr}(ha)$. The map $h \mapsto \varphi_h$ is an isometry from $\mathrm{L}^1(\mathcal{M})$ onto the predual $\mathcal{M}_*$, where $\|h\|_1 = \mathrm{Tr}(|h|)$. Moreover, for any $h \in \mathrm{L}^1(\mathcal{M})$ we have $\|h\|_1 = \sup\{|\mathrm{Tr}(ah)| : a \in \mathcal{M}, \|a\| \leqslant 1\}$.

2. (Hölder) If $0 < p, q, r \leqslant \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then for any $x \in \mathrm{L}^p(\mathcal{M})$ and any $y \in \mathrm{L}^q(\mathcal{M})$ we have $xy \in \mathrm{L}^r(\mathcal{M})$ and $\|xy\|_r \leqslant \|x\|_p \|y\|_q$. If $1 \leqslant p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$, then $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ for any $x \in \mathrm{L}^p(\mathcal{M})$ and any $y \in \mathrm{L}^{p^*}(\mathcal{M})$.

3. Any $x \in \mathrm{L}^p(\mathcal{M})$ admits a polar decomposition $x = w|x|$, where $|x| = (x^*x)^{\frac{1}{2}} \in \mathrm{L}^p(\mathcal{M})_+$ and where $w \in \mathcal{M}$ is a partial isometry. The right support $s_r(x) = w^*w$ and the left support $s_l(x) = ww^*$ are the smallest projections e, f of $\mathcal{M}$ satisfying $xe = x$ and $fx = x$. If $x \in \mathrm{L}^{\frac{1}{2}}(\mathcal{M})_+$ then $x^{\frac{1}{2}} \in \mathrm{L}^1(\mathcal{M})_+$.

4. The space $\mathrm{L}^2(\mathcal{M})$ is a Hilbert space for the inner product $\langle x, y\rangle = \mathrm{Tr}(y^*x)$.

For any integer $m \geqslant 1$, we identify matrices with entries in $\mathrm{L}^p(\mathcal{M})$ algebraically with $\mathrm{L}^p(\mathrm{M}_m \overline{\otimes} \mathcal{M})$. This is not an identification of the operator space norm on $\mathrm{M}_m(\mathrm{L}^p(\mathcal{M}))$ with the L^p-norm. We identify the trace functional of $\mathrm{L}^1(\mathrm{M}_m \overline{\otimes} \mathcal{M})$ with $\mathrm{Tr} \otimes \mathrm{Tr}$. If $A \in \mathrm{M}_m$ and $X \in \mathrm{L}^{\frac{1}{2}}(\mathcal{M})_+$ with $A \geqslant 0$, then the operators $A \otimes 1$ and $1 \otimes X$ are commuting positive operators, hence $(A \otimes X)^{\frac{1}{2}} = A^{\frac{1}{2}} \otimes X^{\frac{1}{2}}$ and

$$(\mathrm{Tr} \otimes \mathrm{Tr})\big((A \otimes X)^{\frac{1}{2}}\big) = \mathrm{Tr}(A^{\frac{1}{2}})\,\mathrm{Tr}(X^{\frac{1}{2}}). \tag{2.2}$$

Operator space structure Following Pisier [Pis03], we equip $\mathrm{L}^1(\mathcal{M})$ with the operator space structure of the predual $(\mathcal{M}^{\mathrm{op}})_*$ of the opposite von Neumann algebra $\mathcal{M}^{\mathrm{op}}$. With this convention, for any integer $m \geqslant 1$ the map $e_{ij} \otimes x \mapsto e_{ij} \otimes x$ induces an isometry from $S^1_m(\mathrm{L}^1(\mathcal{M}))$ onto $\mathrm{L}^1(\mathrm{M}_m \overline{\otimes} \mathcal{M})$, see [JuX10, p. 101] and [Pis98]. Since $\mathcal{M}^{\mathrm{op}}$ is again a von Neumann algebra, an operator space embeds completely isometrically into $\mathrm{L}^1(\mathcal{M})$ for some von Neumann algebra $\mathcal{M}$ if and only if it embeds completely isometrically into the predual $\mathcal{N}_*$ of some von Neumann algebra $\mathcal{N}$ equipped with its predual operator space structure. In particular, none of our statements depends on this convention. We will also use the fact that the canonical map $A^* \to (A^{**})_*$ is a complete isometry for any C*-algebra A, where A^* is equipped with its dual operator space structure.

Square functions Let $\mathcal{M}$ be a von Neumann algebra and let $x = (x_1, \ldots, x_n)$ be an n-tuple of elements of the Banach space $\mathrm{L}^1(\mathcal{M})$. We let

$$\sigma_c(x) \stackrel{\mathrm{def}}{=} \mathrm{Tr}\left(\left(\sum_{k=1}^n x_k^* x_k\right)^{\frac{1}{2}}\right) \quad \text{and} \quad \sigma_r(x) \stackrel{\mathrm{def}}{=} \mathrm{Tr}\left(\left(\sum_{k=1}^n x_k x_k^*\right)^{\frac{1}{2}}\right). \tag{2.3}$$

By [JMX06, p. 11], we have the equalities $\sigma_c(x) = \|\sum_{k=1}^n x_k \otimes e_{k1}\|_{\mathrm{L}^1(\mathcal{M}\overline{\otimes}\mathrm{M}_n)}$ and $\sigma_r(x) = \|\sum_{k=1}^n x_k \otimes e_{1k}\|_{\mathrm{L}^1(\mathcal{M}\overline{\otimes}\mathrm{M}_n)}$. In particular, σ_c and σ_r are norms on the space $\mathrm{L}^1(\mathcal{M})^n$. We refer to [JMX06, Section 2.B] for more information.

2.3 Row, column and operator Hilbert spaces

For any integer $n \geqslant 1$, we consider the row space $\mathrm{R}_n \stackrel{\mathrm{def}}{=} \mathrm{span}\{e_{1k} : 1 \leqslant k \leqslant n\}$ and the column space $\mathrm{C}_n \stackrel{\mathrm{def}}{=} \mathrm{span}\{e_{k1} : 1 \leqslant k \leqslant n\}$, which are subspaces of the algebra M_n. We denote by $(e_k)_{1 \leqslant k \leqslant n}$ their canonical bases. Recall that by [Pis03, Exercise 2.3.5 p. 41] we have the duality formulas $\mathrm{R}_n^* = \mathrm{C}_n$ and $\mathrm{C}_n^* = \mathrm{R}_n$ completely isometrically via the dual bases. We denote by OH_n the n-dimensional operator Hilbert space with its canonical basis $(e_k)_{1 \leqslant k \leqslant n}$. According to [Pis96, Theorem 1.1 p. 12], the dual operator space OH_n^* is completely isometric to $\overline{\mathrm{OH}_n}$. Moreover by [Pis96, Remark 1.3 p. 17] the conjugate $\overline{\mathrm{OH}_n}$ is completely isometric to OH_n via

$\overline{e_k} \mapsto e_k$. Any unitary of ℓ_n^2 induces by [Pis03, Theorem 7.2 (iii) p. 127] combined with [BLM04, (1.2.7) p. 6] a complete isometry of the operator space OH_n. Finally, by [Pis96, Corollary 2.6 p. 29] we have the complex interpolation formula $\mathrm{OH}_n = (\mathrm{R}_n, \mathrm{C}_n)_{\frac{1}{2}}$ completely isometrically. By [Pis03, Theorem 7.1 and (7.1) pp. 122–123], we have for any Hilbert space K and any finite family (x_k) of elements in $\mathrm{B}(K)$, the equality

$$\left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{\mathrm{B}(K)\otimes_{\min}\mathrm{OH}_n} = \left\| \sum_{k=1}^n x_k \otimes \overline{x_k} \right\|_{\mathrm{B}(K)\otimes_{\min}\overline{\mathrm{B}(K)}}^{\frac{1}{2}}.$$

We will use the following result.

Lemma 2.1 *Let $m, n \geqslant 1$ be integers and let $x_1, \ldots, x_n \in S_m^1$. We have*

$$\left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{S_m^1(\mathrm{R}_n)} = \sigma_c(x) \quad \text{and} \quad \left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{S_m^1(\mathrm{C}_n)} = \sigma_r(x). \tag{2.4}$$

Proof : For any $1 \leqslant k \leqslant m$, write $x_k = \sum_{i,j=1}^m x_k(i,j) e_{ij}$ for some complex numbers $x_k(i,j)$. Recall that by [EfR00, Proposition 9.3.5 p. 163], we have

$$\mathrm{R}_m \otimes_h \mathrm{R}_n = \mathrm{R}_{mn} \quad \text{and} \quad \mathrm{C}_n \otimes_h \mathrm{C}_m = \mathrm{C}_{nm} \tag{2.5}$$

completely isometrically. By the associativity of the Haagerup tensor product, we have isometrically

$$S_m^1(\mathrm{R}_n) \overset{(2.1)}{=} \mathrm{R}_m \otimes_h \mathrm{R}_n \otimes_h \mathrm{C}_m \overset{(2.5)}{=} \mathrm{R}_{mn} \otimes_h \mathrm{C}_m \overset{(2.1)}{=} S_{mn,m}^1.$$

The element $\sum_{k=1}^n x_k \otimes e_k$ of the space $S_m^1(\mathrm{R}_n)$ corresponds to the element

$$\sum_{k=1}^n \sum_{i,j=1}^m x_k(i,j) e_{(i,k)} \otimes e_j$$

of the space $\mathrm{R}_{mn} \otimes_h \mathrm{C}_m$. Its norm is the trace norm of the $mn \times m$ matrix whose entry at the position $((i,k),j)$ is $x_k(i,j)$. Up to a permutation of the rows, this matrix is the block column $[x_1 \ \cdots \ x_n]^\top$ whose blocks are $x_1, \ldots, x_n$. Hence its trace norm is

$$\mathrm{Tr}\left(\left(([x_1 \ \cdots \ x_n]^\top)^* [x_1 \ \cdots \ x_n]^\top \right)^{\frac{1}{2}} \right) = \mathrm{Tr}\left(\left(\sum_{k=1}^n x_k^* x_k \right)^{\frac{1}{2}} \right) \overset{(2.3)}{=} \sigma_c(x).$$

In the same way, the element $\sum_{k=1}^n x_k \otimes e_k$ of the space $S_m^1(\mathrm{C}_n) \overset{(2.1)}{=} \mathrm{R}_m \otimes_h \mathrm{C}_n \otimes_h \mathrm{C}_m \overset{(2.5)}{=} \mathrm{R}_m \otimes_h \mathrm{C}_{nm} \overset{(2.1)}{=} S_{mn,m}^1$ corresponds to the element $\sum_{k=1}^n \sum_{i,j=1}^m x_k(i,j) e_i \otimes e_{(k,j)}$ of the space $\mathrm{R}_m \otimes_h \mathrm{C}_{nm}$, whose norm is the trace norm of the block row $[x_1 \ \cdots \ x_n]$, that is $\sigma_r(x)$. ∎

3 Duality and least embedding distortion

The following observation is a direct consequence of the duality between complete isometries and complete quotient maps.

Proposition 3.1 *A finite-dimensional operator space F is a complete quotient of some C^*-algebra if and only if its dual F^* embeds completely isometrically into a noncommutative L^1-space.*

Proof : If $q\colon A \to F$ is a complete quotient map, then by [EfR00, Corollary 4.1.9 p. 73] the adjoint $q^*\colon F^* \to A^* = (A^{**})_*$ is a complete isometry and $(A^{**})_* = \mathrm{L}^1((A^{**})^{\mathrm{op}})$ completely isometrically. Conversely, if $J\colon F^* \to \mathrm{L}^1(\mathcal{M})$ is a complete isometry then by [EfR00, Corollary 4.1.9 p. 73] the adjoint $J^*\colon \mathcal{M}^{\mathrm{op}} \to F^{**} = F$ is a (exact) complete quotient map. ∎

For any finite-dimensional operator space E, we let

$$\mathrm{dist}_{\mathrm{L}^1}(E) \overset{\mathrm{def}}{=} \inf\left\{ \|u\|_{\mathrm{cb}} \left\|u^{-1}\right\|_{\mathrm{cb}} : u\colon E \to \mathrm{L}^1(\mathcal{M}) \text{ injective linear map, } \mathcal{M} \text{ von Neumann algebra} \right\}.$$

Here the map $u^{-1}\colon u(E) \to E$ denotes the inverse on the space $u(E)$. The number $\mathrm{dist}_{\mathrm{L}^1}(E)$ is finite since a finite-dimensional operator space E of dimension n is completely isomorphic to the row space $\mathrm{R}_n(1)$, which embeds completely isometrically into the trace class space S_n^1. If $\mathcal{OS}_n$ is the space of all n-dimensional subspaces of preduals of von Neumann algebras, equipped with their natural operator space structures is compact then we have

$$\mathrm{dist}_{\mathrm{L}^1}(E) = \mathrm{dist}_{\mathrm{cb}}(E, \mathcal{OS}_n).$$

With a standard use of ultraproducts we obtain the following result. In this paper, its role here is to turn in Corollary 5.6 the nonexistence of a completely isometric realization of the operator space $\mathrm{R}_n + \mathrm{C}_n$ in a noncommutative L^1-space into a strict lower bound for $\mathrm{dist}_{\mathrm{L}^1}(\mathrm{R}_n + \mathrm{C}_n)$. It also follows from a classical topology result [Mun00, Exercise 2 (b) p. 177] and Ozawa's compactness theorem [Oza00, Theorem 1 p. 310], which says that the space $\mathcal{OS}_n$ equipped with the Banach–Mazur distance $\delta_{\mathrm{cb}} \overset{\mathrm{def}}{=} \log \mathrm{dist}_{\mathrm{cb}}$ is compact.

Proposition 3.2 *Let E be a finite-dimensional operator space. The infimum defining $\mathrm{dist}_{\mathrm{L}^1}(E)$ is attained. In particular, $\mathrm{dist}_{\mathrm{L}^1}(E) = 1$ if and only if E embeds completely isometrically into a noncommutative L^1-space.*

Proof : Choose injective maps $u_j\colon E \to \mathrm{L}^1(\mathcal{M}_j)$ such that $\|u_j\|_{\mathrm{cb}} = 1$ and $\left\|u_j^{-1}\right\|_{\mathrm{cb}} \to \mathrm{dist}_{\mathrm{L}^1}(E)$. For a free ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$, put $\mathcal{M} \overset{\mathrm{def}}{=} \prod_{\mathcal{U}} \mathcal{M}_j^{\mathrm{op}}$. The ultraproduct norm identity [Arh26, Lemma 6.1 p. 19] gives a linear map $u\colon E \to \mathcal{M}^*$ satisfying, for any integer $m \geqslant 1$ and any $x \in S_m^1(E)$, the equality

$$\left\|(\mathrm{Id}_{S_m^1} \otimes u)(x)\right\|_{S_m^1(\mathcal{M}^*)} = \lim_{\mathcal{U}} \left\|(\mathrm{Id}_{S_m^1} \otimes u_j)(x)\right\|_{S_m^1(\mathrm{L}^1(\mathcal{M}_j))}.$$

For any integer j, note that $\mathrm{dist}_{\mathrm{L}^1}(E)^{-1}\|x\|_{S^1_m(E)} \leqslant \big\|(\mathrm{Id}_{S^1_m}\otimes u_j)(x)\big\| \leqslant \|x\|_{S^1_m(E)}$. Thus $\mathrm{dist}_{\mathrm{L}^1}(E)^{-1}\|x\|_{S^1_m(E)} \leqslant \big\|(\mathrm{Id}_{S^1_m}\otimes u)(x)\big\| \leqslant \|x\|_{S^1_m(E)}$. Recall that $\mathcal{M}^{**}$ is a von Neumann algebra, see [BLM04, Theorem A.5.6 p. 366]. Since $\mathcal{M}^* = (\mathcal{M}^{**})_*$ is a von Neumann algebra predual, the definition of $\mathrm{dist}_{\mathrm{L}^1}(E)$ gives $\|u\|_{\mathrm{cb}}\big\|u^{-1}\big\|_{\mathrm{cb}} = \mathrm{dist}_{\mathrm{L}^1}(E)$. If this product is one, multiplying u by a positive scalar gives a complete isometry. ∎

4 A rigidity theorem for columns in noncommutative L^1-spaces

The following inequality compares a column norm with the Euclidean norm of the individual trace norms. The estimate on the defect gives the equality case without squaring differences of unbounded operators.

Proposition 4.1 *Let $\mathcal{M}$ be a von Neumann algebra and let $K_1,\dots,K_N \in \mathrm{L}^1(\mathcal{M})$. Put $a \overset{\mathrm{def}}{=} (\sum_{k=1}^N K_k^*K_k)^{\frac12}$, $t \overset{\mathrm{def}}{=} \mathrm{Tr}(a)$ and $\gamma \overset{\mathrm{def}}{=} (\sum_{k=1}^N \|K_k\|^2_{\mathrm{L}^1(\mathcal{M})})^{\frac12}$. We have*

$$t \geqslant \gamma. \tag{4.1}$$

If $\gamma > 0$, then

$$\sum_{k=1}^N \Big\||K_k| - \frac{\|K_k\|_{\mathrm{L}^1(\mathcal{M})}}{\gamma}a\Big\|^2_{\mathrm{L}^1(\mathcal{M})} \leqslant 2t(t-\gamma). \tag{4.2}$$

Equality in (4.1) holds if and only if $\|K_l\|_{\mathrm{L}^1(\mathcal{M})}|K_k| = \|K_k\|_{\mathrm{L}^1(\mathcal{M})}|K_l|$ for any k,l. In this case, $|K_k| = \frac{\|K_k\|_{\mathrm{L}^1(\mathcal{M})}}{t}a$ whenever $t>0$.

Proof : The case $\gamma = 0$ is immediate. Suppose that $\gamma > 0$, and put $c_k \overset{\mathrm{def}}{=} \frac{\|K_k\|_{\mathrm{L}^1(\mathcal{M})}}{\gamma}$. Let $K_k = w_k|K_k|$ be the polar decompositions. In $\mathcal{P} \overset{\mathrm{def}}{=} \mathrm{M}_N\overline{\otimes}\mathcal{M}$, consider $K = \sum_k e_{k1}\otimes K_k$ and $x = \sum_k e_{1k}\otimes c_k w_k^*$. We have $|K| = e_{11}\otimes a$ and $xx^* \leqslant e_{11}\otimes 1$. Consequently,

$$\gamma = \sum_k c_k\|K_k\|_{\mathrm{L}^1(\mathcal{M})} = (\mathrm{Tr}\otimes\mathrm{Tr})(xK) \leqslant \|K\|_{\mathrm{L}^1(\mathcal{M})} = t.$$

Write the polar decomposition of the column as $K = V(e_{11}\otimes a)$, where $V = \sum_k e_{k1}\otimes v_k$ and $v_k \in \mathcal{N}$. If $e = s(a)$, then $K_k = v_k a$ and $\sum_k v_k^*v_k = e$. Put $b_k \overset{\mathrm{def}}{=} w_k^*v_k a^{\frac12} \in \mathrm{L}^2(\mathcal{N})$. Since $w_kw_k^* \leqslant 1$, we have

$$\sum_k \|b_k\|^2_{\mathrm{L}^2(\mathcal{M})} \leqslant \sum_k \mathrm{Tr}\left(a^{\frac12}v_k^*v_ka^{\frac12}\right) = \mathrm{Tr}(a) = t.$$

Moreover, cyclicity of the L^2 pairing gives

$$\langle b_k, a^{\frac12}\rangle = \mathrm{Tr}\left(a^{\frac12}w_k^*v_ka^{\frac12}\right) = \mathrm{Tr}(w_k^*K_k) = \|K_k\|_{\mathrm{L}^1(\mathcal{M})}.$$

Since $\sum_k c_k^2 = 1$, it follows that

$$\sum_k \Big\|b_k - c_ka^{\frac12}\Big\|^2_{\mathrm{L}^2(\mathcal{M})} \leqslant 2t - 2\sum_k c_k\|K_k\|_{\mathrm{L}^1(\mathcal{M})} = 2(t-\gamma).$$

Now $|K_k| - c_k a = (b_k - c_k a^{\frac12})a^{\frac12}$. Hölder's inequality therefore gives

$$\sum_k \| |K_k| - c_k a\|^2_{\mathrm{L}^1(\mathcal{M})} \leqslant t \sum_k \left\| b_k - c_k a^{\frac12} \right\|^2_{\mathrm{L}^2(\mathcal{M})} \leqslant 2t(t-\gamma).$$

This proves (4.2). If $t = \gamma$, then $|K_k| = c_k a$ for any k. Conversely, if the absolute values are proportional, choose l with $K_l \neq 0$. Then $|K_k| = \frac{\|K_k\|_{\mathrm{L}^1(\mathcal{M})}}{\|K_l\|_{\mathrm{L}^1(\mathcal{M})}}|K_l|$, and direct substitution in the definition of a gives $t = \gamma$. This proves the assertion. ∎

Remark 4.2 Applying Proposition 4.1 to the adjoints $K_1^*, \ldots, K_N^*$, we obtain

$$\mathrm{Tr}\left(\left(\sum_{k=1}^N K_k K_k^*\right)^{\frac12}\right)^2 \geqslant \sum_{k=1}^N \|K_k\|^2_{\mathrm{L}^1(\mathcal{M})},$$

with equality if and only if $\|K_l\|_{\mathrm{L}^1(\mathcal{M})}\,|K_k^*| = \|K_k\|_{\mathrm{L}^1(\mathcal{M})}\,|K_l^*|$ for any k, l. In the notation (2.3), we have for any $x \in \mathrm{L}^1(\mathcal{M})^n$

$$\sigma_c(x) \geqslant \left(\sum_{k=1}^n \|x_k\|^2_{\mathrm{L}^1(\mathcal{M})}\right)^{\frac12} \quad \text{and} \quad \sigma_r(x) \geqslant \left(\sum_{k=1}^n \|x_k\|^2_{\mathrm{L}^1(\mathcal{M})}\right)^{\frac12}.$$

For any von Neumann algebra $\mathcal{N}$, any $h_1, \ldots, h_n \in \mathrm{L}^1(\mathcal{M})$ and any matrix $Q = [q_{kl}]$ of M_n, we let

$$X_Q \overset{\text{def}}{=} \sum_{k,l=1}^n q_{kl} h_k^* h_l \quad \text{and} \quad Y_Q \overset{\text{def}}{=} \sum_{k,l=1}^n q_{kl} h_k h_l^*. \tag{4.3}$$

These are elements of $\mathrm{L}^{\frac12}(\mathcal{N})$ which depend linearly on Q. If $Q \geqslant 0$ and if $B = Q^{\frac12}$, then $X_Q = \sum_k y_k^* y_k$ with $y_k = \sum_l b_{kl} h_l$, so that $X_Q \geqslant 0$. In the same way $Y_Q \geqslant 0$ if $Q \geqslant 0$.

Proposition 4.3 *Let $\mathcal{M}$ be a von Neumann algebra, let $h_1, \ldots, h_n \in \mathrm{L}^1(\mathcal{M})$ and let $c > 0$. Suppose that for any positive matrix $Q \in \mathrm{M}_n$ of rank at most two we have*

$$\mathrm{Tr}\left(X_Q^{\frac12}\right) = c\,\mathrm{Tr}(Q)^{\frac12}. \tag{4.4}$$

Then there exists $X \in \mathrm{L}^{\frac12}(\mathcal{N})_+$ with $\mathrm{Tr}(X^{\frac12}) = c$ such that $h_k^* h_l = \delta_{kl} X$ for any $1 \leqslant k, l \leqslant n$.

Proof : Let Q_1, Q_2 be nonzero rank-one positive matrices. Write $Q_j = B_j^* B_j$ with a row $B_j = [b_{jk}] \in \mathrm{M}_{1,n}$, and put $K_j \overset{\text{def}}{=} \sum_k b_{jk} h_k$. Then $K_j^* K_j = X_{Q_j}$, and $Q_1 + Q_2$ has rank at most two. The assumption gives

$$\mathrm{Tr}\left(X_{Q_1+Q_2}^{\frac12}\right)^2 = c^2\,\mathrm{Tr}(Q_1+Q_2) = \|K_1\|^2_{\mathrm{L}^1(\mathcal{M})} + \|K_2\|^2_{\mathrm{L}^1(\mathcal{M})}.$$

By Proposition 4.1, the operators $X_{Q_1}^{\frac12}$ and $X_{Q_2}^{\frac12}$ are proportional. Their traces are $c\,\mathrm{Tr}(Q_j)^{\frac12}$, so $\mathrm{Tr}(Q_2)X_{Q_1} = \mathrm{Tr}(Q_1)X_{Q_2}$. Taking $Q_2 = e_{11}$ and putting $X = X_{e_{11}}$, we obtain $X_Q = \mathrm{Tr}(Q)X$ for any rank-one positive matrix Q. Such matrices linearly span M_n, hence the same identity holds for any $Q \in \mathrm{M}_n$. In particular, $h_k^* h_l = X_{e_{kl}} = \delta_{kl} X$, and $\mathrm{Tr}(X^{\frac12}) = c$. ∎

We can now state our rigidity theorem. Recall that if E is an operator space with a basis $(\xi_1, \ldots, \xi_n)$, the first-column tests of E are the elements $\sum_{i,k} a_{ik} e_{i1} \otimes \xi_k$ of $S_m^1(E)$ and the first-row tests are the elements $\sum_{j,k} b_{jk} e_{1j} \otimes \xi_k$, where $A = [a_{ik}]$ and $B = [b_{jk}]$ belong to $\mathrm{M}_{m,n}$.

Theorem 4.4 *Let E be an operator space with a basis $(\xi_1, \dots, \xi_n)$ and let $c > 0$. Suppose that there exists a complete isometry $J\colon E \to \mathrm{L}^1(\mathcal{M})$ for some von Neumann algebra $\mathcal{M}$.*

1. *If for any $A \in \mathrm{M}_{2,n}$ we have*

$$\left\| \sum_{i=1}^{2} \sum_{k=1}^{n} a_{ik} e_{i1} \otimes \xi_k \right\|_{S_2^1(E)} = c \, \|A\|_{S^2} \,, \tag{4.5}$$

   then the linear map $E \to \mathrm{R}_n$, $\xi_k \mapsto c e_k$ is a complete isometry.

2. *If for any $B \in \mathrm{M}_{2,n}$ we have $\left\| \sum_{j,k} b_{jk} e_{1j} \otimes \xi_k \right\|_{S_2^1(E)} = c \, \|B\|_{S^2}$, then the linear map $E \to \mathrm{C}_n$, $\xi_k \mapsto c e_k$ is a complete isometry.*

Proof : 1. Put $h_k \overset{\text{def}}{=} J(\xi_k)$. Let $A \in \mathrm{M}_{2,n}$. Since $\mathrm{Id}_{S_2^1} \otimes J$ is an isometry from $S_2^1(E)$ into $S_2^1(\mathrm{L}^1(\mathcal{N})) = \mathrm{L}^1(\mathrm{M}_2 \overline{\otimes} \mathcal{M})$, the norm of (4.5) is equal to the norm of the column $\sum_i e_{i1} \otimes y_i$ with $y_i = \sum_k a_{ik} h_k$, that is

$$\mathrm{Tr}\left(\left(\sum_{i=1}^{2} y_i^* y_i \right)^{\frac{1}{2}} \right) = \mathrm{Tr}\left(\left(\sum_{k,l=1}^{n} (A^*A)_{kl} h_k^* h_l \right)^{\frac{1}{2}} \right) = \mathrm{Tr}\left(X_{A^*A}^{\frac{1}{2}} \right).$$

Since any positive matrix $Q \in \mathrm{M}_n$ of rank at most two is of the form A^*A with $A \in \mathrm{M}_{2,n}$ and since $\|A\|_{S_2} = \mathrm{Tr}(A^*A)^{\frac{1}{2}}$, the assumption (4.5) gives (4.4). By Proposition 4.3, there exists $X \in \mathrm{L}^{\frac{1}{2}}(\mathcal{N})_+$ with $\mathrm{Tr}(X^{\frac{1}{2}}) = c$ and $h_k^* h_l = \delta_{kl} X$. Now, let $m \geqslant 1$ and $x_1, \dots, x_n \in S_m^1$. Consider an element $Z = \sum_k x_k \otimes h_k$ in $\mathrm{L}^1(\mathrm{M}_m \overline{\otimes} \mathcal{N})$. We have

$$Z^*Z = \sum_{k,l=1}^{n} x_k^* x_l \otimes h_k^* h_l = \left(\sum_{k=1}^{n} x_k^* x_k \right) \otimes X.$$

Hence

$$\left\| \sum_{k=1}^{n} x_k \otimes \xi_k \right\|_{S_m^1(E)} = \|Z\|_{\mathrm{L}^1(\mathrm{M}_m \overline{\otimes} \mathcal{N})} = (\mathrm{Tr} \otimes \mathrm{Tr})\big((Z^*Z)^{\frac{1}{2}}\big)$$
$$\overset{(2.2)}{=} c \, \mathrm{Tr}\left(\left(\sum_{k=1}^{n} x_k^* x_k \right)^{\frac{1}{2}} \right) \overset{(2.4)}{=} \left\| \sum_{k=1}^{n} x_k \otimes c e_k \right\|_{S_m^1(\mathrm{R}_n)}.$$

By [Pis98, Lemma 1.7, (1.7)' p. 23], the map $\xi_k \mapsto c e_k$ is a complete isometry.

2. The proof is identical, using the rows $\sum_j e_{1j} \otimes y_j$ with $y_j = \sum_k b_{jk} h_k$, the elements Y_Q instead of X_Q (that is, applying Proposition 4.3 to $h_1^*, \dots, h_n^*$) and the second equality of (2.4). Indeed, we have $\sum_j y_j y_j^* = Y_{B^T \overline{B}}$, and any positive matrix Q of rank at most two can be written $Q = B^T \overline{B}$ with $B \in \mathrm{M}_{2,n}$. ∎

Corollary 4.5 *Let $h_1, \dots, h_n \in \mathrm{L}^1(\mathcal{M})$. The map $e_k \mapsto h_k$ is a complete isometry from R_n into $\mathrm{L}^1(\mathcal{M})$ if and only if there exist $a \in \mathrm{L}^1(\mathcal{N})_+$ and $v_1, \dots, v_n \in \mathcal{M}$ such that*

$$\mathrm{Tr}(a) = 1, \quad h_k = v_k a, \quad v_k^* v_l = \delta_{kl} s(a). \tag{4.6}$$

In this case, the space spanned by the v_k in $\mathcal{M}$ is a column ternary ring of operators.

Proof : Theorem 4.4 and its proof give $h_k^* h_l = \delta_{kl} a^2$ with $a \geqslant 0$ and $\operatorname{Tr}(a) = 1$. Let $h_k = v_k a$ be the polar decompositions and put $e \overset{\text{def}}{=} s(a)$. Then $v_k^* v_k = e$ and $a(v_k^* v_l)a = 0$ for $k \neq l$. To cancel a, take its spectral projections $p_j = 1_{[\frac{1}{j},j]}(a)$ in the crossed product used to define the Haagerup spaces. Multiplication on both sides by the bounded truncated inverse $a^{-1}p_j$ gives $p_j v_k^* v_l p_j = 0$. Since $p_j \to e$ strongly, we obtain $v_k^* v_l = 0$. Conversely, (4.6) gives the norm computation in the proof of Theorem 4.4 with $c = 1$. Finally, $v_i v_j^* v_k = \delta_{jk} v_i$. Thus their span is closed under the ternary product and is a column space, in the terminology of [BLM04, Section 1.2.23 p. 11 and Section 4.4.1 p. 161]. ∎

The rigidity theorem also has a qualitative stability consequence. We use the compactness theorem of Ozawa [Oza00, pp. 309–312]: for fixed n, the set of n-dimensional subspaces of von Neumann algebra preduals, up to complete isometry, is compact for the cb-Banach–Mazur distance.

Corollary 4.6 *Let $n \geqslant 1$ be an integer. There exists a function $\omega_n \colon (0,1) \to [0,\infty)$ such that $\omega_n(\varepsilon) \to 0$ as $\varepsilon \to 0$ and with the following property. Let $0 < \varepsilon < 1$ and suppose that $E \subset \mathrm{L}^1(\mathcal{N})$ has a basis $(\xi_k)_{k=1}^n$ satisfying*

$$(1-\varepsilon)\,\|A\|_{S^2} \leqslant \left\| \sum_{i=1}^{2} \sum_{k=1}^{n} a_{ik} e_{i1} \otimes \xi_k \right\|_{S_2^1(E)} \leqslant (1+\varepsilon)\,\|A\|_{S^2} \tag{4.7}$$

for any $A \in \mathrm{M}_{2,n}$. Then the map $u \colon E \to \mathrm{R}_n$, $\xi_k \mapsto e_k$, satisfies $\|u\|_{\mathrm{cb}} \left\|u^{-1}\right\|_{\mathrm{cb}} \leqslant 1 + \omega_n(\varepsilon)$.

Proof : At the scalar level, (4.7) gives

$$(1-\varepsilon)|\lambda|_2 \leqslant \left\| \sum_{k=1}^{n} \lambda_k \xi_k \right\| \leqslant (1+\varepsilon)|\lambda|_2.$$

Consequently, the coordinate functionals have norm at most $(1-\varepsilon)^{-1}$ and $\|\xi_k\| \leqslant 1+\varepsilon$. Decomposing u and u^{-1} into rank-one maps shows that their cb-distortion is at most $n^2 \frac{1+\varepsilon}{1-\varepsilon}$. Thus the supremum of the distortions in the statement is finite for fixed ε.

Suppose that this supremum does not tend to one. There exist $\delta > 0$, a sequence $\varepsilon_j \to 0$, and spaces E_j with bases $(\xi_k^{(j)})$ satisfying (4.7), for which the coordinate maps $s_j \colon \mathrm{R}_n \to E_j$, $e_k \mapsto \xi_k^{(j)}$, have cb-distortion at least $1+\delta$. By Ozawa's compactness theorem, after passing to a subsequence there exist an n-dimensional predual subspace F and isomorphisms $v_j \colon E_j \to F$ with $\|v_j\|_{\mathrm{cb}} \to 1$ and $\left\|v_j^{-1}\right\|_{\mathrm{cb}} \to 1$. Here we rescale the isomorphisms so that their two cb-norms are equal.

The maps $w_j = v_j s_j \colon \mathrm{R}_n \to F$ are bounded at the scalar level. Passing to a further subsequence, we may suppose that they converge in operator norm to a map w. The scalar inequalities show that w is an isometry onto F. Passing to the limit in (4.7) gives the exact first-column identities for the basis $(w(e_k))$ of F. Theorem 4.4 therefore shows that w is a complete isometry. Moreover,

$$\|w_j - w\|_{\mathrm{cb}} \leqslant \sum_{k=1}^{n} \|w_j(e_k) - w(e_k)\|_F \to 0.$$

The Neumann series applied to $w^{-1}w_j$ gives $w_j^{-1} \to w^{-1}$ in cb-norm. Since $s_j = v_j^{-1} w_j$ and $s_j^{-1} = w_j^{-1} v_j$, the cb-distortions of s_j tend to one. This is a contradiction. Taking $\omega_n(\varepsilon)$ to be the supremum of the distortions minus one proves the assertion. ∎

5 Sums of weighted row and column spaces

Let $n \geqslant 1$ be an integer and let $\alpha, \beta > 0$. We consider the operator space $\alpha \mathrm{R}_n + \beta \mathrm{C}_n$, that is the quotient of $\alpha \mathrm{R}_n \oplus_1 \beta \mathrm{C}_n$ by the subspace $\{(\xi, -\xi) : \xi \in \mathbb{C}^n\}$, and we denote by (e_k) its canonical basis.

Lemma 5.1 *Let $m \geqslant 1$ be an integer. For any $x = (x_1, \ldots, x_n) \in (S^1_m)^n$ we have*

$$\left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{S^1_m(\alpha \mathrm{R}_n + \beta \mathrm{C}_n)} = \inf \big\{ \alpha \sigma_c(y) + \beta \sigma_r(z) : y, z \in (S^1_m)^n,\ y + z = x \big\}. \tag{5.1}$$

Proof : By the points 4 and 5 of Subsection 2.1, the space $S^1_m(\alpha \mathrm{R}_n + \beta \mathrm{C}_n)$ is isometric to the quotient of $S^1_m(\alpha \mathrm{R}_n) \oplus_1 S^1_m(\beta \mathrm{C}_n)$ by the subspace $\{(y, -y)\}$. It suffices to use Lemma 2.1. ∎

Lemma 5.2 *Let $m, n \geqslant 1$ be integers. For any $z \in (S^1_m)^n$ we have $\sigma_c(z) \leqslant \sqrt{n} \sigma_r(z)$ and $\sigma_r(z) \leqslant \sqrt{n} \sigma_c(z)$.*

Proof : Since $\sigma_c(z)$ is the trace norm of the column $\sum_{k=1}^n e_{k1} \otimes z_k$, the triangle inequality and the Cauchy-Schwarz inequality give

$$\sigma_c(z) \leqslant \sum_{k=1}^n \|z_k\|_{S^1_m} \leqslant \sqrt{n} \Big(\sum_{k=1}^n \|z_k\|^2_{S^1_m} \Big)^{\frac{1}{2}} \leqslant \sqrt{n} \sigma_r(z),$$

where we used Remark 4.2 in the last inequality. The second inequality is proved in the same way. ∎

Lemma 5.3 *Let $m, n \geqslant 1$ be integers and let $A = [a_{ik}], B = [b_{jk}] \in \mathrm{M}_{m,n}$. Consider the first-column test $T_A = \sum_{i,k} a_{ik} e_{i1} \otimes e_k$ and the first-row test $T'_B = \sum_{j,k} b_{jk} e_{1j} \otimes e_k$ of $S^1_m(\alpha \mathrm{R}_n + \beta \mathrm{C}_n)$.*

1. *If $\alpha \leqslant \beta$, then $\|T_A\| = \alpha \|A\|_{S^2}$ and $\|T'_B\| \leqslant \beta \|B\|_{S^2}$.*
2. *If $\beta \leqslant \alpha$, then $\|T'_B\| = \beta \|B\|_{S^2}$ and $\|T_A\| \leqslant \alpha \|A\|_{S^2}$.*

Proof : For the first-column test, we have $x_k = \sum_i a_{ik} e_{i1}$. Since $\sum_k x_k^* x_k = \|A\|^2_{S^2} e_{11}$ and $\sum_k x_k x_k^* = AA^*$, we have $\sigma_c(x) = \|A\|_{S^2}$ and $\sigma_r(x) = \|A\|_{S_1}$. For the first-row test, we have $x_k = \sum_j b_{jk} e_{1j}$, hence $\sum_k x_k x_k^* = \|B\|^2_{S^2} e_{11}$ and $\sum_k x_k^* x_k = (BB^*)^T$, so that $\sigma_r(x) = \|B\|_{S^2}$ and $\sigma_c(x) = \|B\|_{S^1}$. The upper estimates follow from (5.1) by choosing $(y, z) = (x, 0)$ or $(y, z) = (0, x)$.

1. Suppose $\alpha \leqslant \beta$ and let $y + z = x$ be a decomposition of the first-column test, so that $x_k = x_k e_{11}$. Put $y'_k \overset{\text{def}}{=} y_k e_{11}$ and $z'_k = z_k e_{11}$, so that $y' + z' = x$. For any positive matrix T, we have

$$\mathrm{Tr}((e_{11} T e_{11})^{\frac{1}{2}}) = \big\| T^{\frac{1}{2}} e_{11} \big\|_{S^1} \leqslant \mathrm{Tr}(T^{\frac{1}{2}}).$$

Hence $\sigma_c(y') \leqslant \sigma_c(y)$. Moreover, we have $\sum_k z_k e_{11} z_k^* \leqslant \sum_k z_k z_k^*$, so that $\sigma_r(z') \leqslant \sigma_r(z)$ by the operator monotonicity of the function $t \mapsto t^{\frac{1}{2}}$. Writing $y'_k = \sum_i b'_{ik} e_{i1}$ and $z'_k = \sum_i c_{ik} e_{i1}$ with $B' + C = A$, we obtain as above $\sigma_c(y') = \|B'\|_{S^2}$ and $\sigma_r(z') = \|C\|_{S^1}$. We conclude that

$$\alpha \sigma_c(y) + \beta \sigma_r(z) \geqslant \alpha \|B'\|_{S^2} + \beta \|C\|_{S^1} \geqslant \alpha \big(\|B'\|_{S^2} + \|C\|_{S^2} \big) \geqslant \alpha \|A\|_{S^2}.$$

Taking the infimum, we obtain $\|T_A\| \geqslant \alpha \|A\|_{S^2}$.

2. The proof is similar, using the left multiplication by e_{11} instead of the right multiplication. ∎

Theorem 5.4 *Let $n \geqslant 2$ be an integer and let $\alpha, \beta > 0$. The operator space $\alpha \mathrm{R}_n + \beta \mathrm{C}_n$ is completely isometric to a subspace of $\mathrm{L}^1(\mathcal{M})$ for some von Neumann algebra $\mathcal{M}$ if and only if $\max\{\alpha, \beta\} \geqslant \sqrt{n} \min\{\alpha, \beta\}$. Moreover, if $\beta \geqslant \sqrt{n}\alpha$ (resp. $\alpha \geqslant \sqrt{n}\beta$), then $\alpha \mathrm{R}_n + \beta \mathrm{C}_n = \alpha \mathrm{R}_n$ (resp. $\alpha \mathrm{R}_n + \beta \mathrm{C}_n = \beta \mathrm{C}_n$) completely isometrically.*

Proof : Put $E \stackrel{\text{def}}{=} \alpha \mathrm{R}_n + \beta \mathrm{C}_n$. Suppose first that $\beta \geqslant \sqrt{n}\alpha$. Let $m \geqslant 1$ and $x \in (S^1_m)^n$. For any decomposition $x = y + z$, we have by Lemma 5.2 and the triangle inequality for the norm σ_c

$$\alpha\sigma_c(y) + \beta\sigma_r(z) \geqslant \alpha\sigma_c(y) + \frac{\beta}{\sqrt{n}}\sigma_c(z) \geqslant \alpha\big(\sigma_c(y) + \sigma_c(z)\big) \geqslant \alpha\sigma_c(x).$$

Since the choice $(y, z) = (x, 0)$ gives the value $\alpha\sigma_c(x)$, we deduce from (5.1) and (2.4) that $\|\sum_k x_k \otimes e_k\|_{S^1_m(E)} = \|\sum_k x_k \otimes e_k\|_{S^1_m(\alpha \mathrm{R}_n)}$. By [Pis98, Lemma 1.7, (1.7)' p. 23], the identity map $\alpha \mathrm{R}_n \to E$ is a complete isometry. Moreover, the space $\alpha \mathrm{R}_n$ embeds completely isometrically into $\mathrm{L}^1(\mathrm{M}_n)$ via the map $e_k \mapsto \alpha e_{k1}$. Indeed, for any $x \in (S^1_m)^n$, the element $Z = \sum_k x_k \otimes e_{k1}$ of $\mathrm{L}^1(\mathrm{M}_m \otimes \mathrm{M}_n)$ satisfies $Z^*Z = (\sum_k x_k^* x_k) \otimes e_{11}$, hence $\|\alpha Z\|_{\mathrm{L}^1(\mathrm{M}_m \otimes \mathrm{M}_n)} = \alpha\sigma_c(x)$, and it suffices to use (2.4) and [Pis98, Lemma 1.7, (1.7)' p. 23]. The case $\alpha \geqslant \sqrt{n}\beta$ is similar, using the embedding $e_k \mapsto \beta e_{1k}$ of $\beta \mathrm{C}_n$ into $\mathrm{L}^1(\mathrm{M}_n)$.

Conversely, suppose that $J \colon E \to \mathrm{L}^1(\mathcal{M})$ is a complete isometry and that $\max\{\alpha, \beta\} < \sqrt{n} \min\{\alpha, \beta\}$. We first consider the case $\alpha \leqslant \beta$. By the first point of Lemma 5.3, the first-column tests of E satisfy (4.5) with $c = \alpha$. By the first point of Theorem 4.4, the identity map $E \to \alpha \mathrm{R}_n$ is a complete isometry. Consider the first-row test $T'_{\mathrm{Id}} = \sum_{k=1}^n e_{1k} \otimes e_k$ of $S^n_1(E)$. Its norm in $S^1_n(\alpha \mathrm{R}_n)$ is $\alpha\sigma_c(x)$ with $x_k = e_{1k}$, that is $\alpha \operatorname{Tr}((\sum_k e_{kk})^{\frac{1}{2}}) = \alpha n$. On the other hand, by the first point of Lemma 5.3, we have

$$\|T'_{\mathrm{Id}}\|_{S^1_n(E)} \leqslant \beta \, \|\mathrm{Id}_n\|_{S^2} = \beta\sqrt{n} < \alpha n.$$

This is a contradiction. If $\beta \leqslant \alpha$, the same argument with the second points of Lemma 5.3 and of Theorem 4.4 shows that the identity map $E \to \beta \mathrm{C}_n$ is a complete isometry, and the first-column test T_{Id} gives the contradiction $\beta n \leqslant \alpha\sqrt{n} < \beta n$. ∎

We now allow the weights to depend on the coordinate. For $\alpha = (\alpha_k)_{k=1}^n$ with $\alpha_k > 0$, we define $\mathrm{R}_n(\alpha)$ by

$$\left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{\mathrm{M}_m(\mathrm{R}_n(\alpha))} = \left\| \sum_{k=1}^n \alpha_k^2 x_k x_k^* \right\|^{\frac{1}{2}},$$

and define $\mathrm{C}_n(\beta)$ by the analogous formula with $\sum_k \beta_k^2 x_k^* x_k$. These are the finite-dimensional weighted row and column spaces which occur in the graph construction of [Xu07, Section 7 pp. 29–30].

Theorem 5.5 *Let $\alpha_k, \beta_k > 0$ for $1 \leqslant k \leqslant n$, and suppose that $\alpha_k \leqslant \beta_k$ for any k. The following assertions are equivalent.*

1. *$\mathrm{R}_n(\alpha) + \mathrm{C}_n(\beta)$ embeds completely isometrically into a noncommutative L^1-space.*
2. *The identity map $\mathrm{R}_n(\alpha) + \mathrm{C}_n(\beta) \to \mathrm{R}_n(\alpha)$ is a complete isometry.*
3. *$\sum_{k=1}^n \frac{\alpha_k^2}{\beta_k^2} \leqslant 1$.*

If $\beta_k \leqslant \alpha_k$ for any k, the symmetric assertions hold with $\mathrm{C}_n(\beta)$ in place of $\mathrm{R}_n(\alpha)$ and with $\sum_k \frac{\beta_k^2}{\alpha_k^2} \leqslant 1$.

Proof : Rescaling the k-th coordinate by α_k, it suffices to consider $E = \mathrm{R}_n + \mathrm{C}_n(d)$, where $d_k = \frac{\beta_k}{\alpha_k} \geqslant 1$. By the same quotient argument as in Lemma 5.1,

$$\left\| \sum_{k=1}^{n} x_k \otimes e_k \right\|_{S^1_m(E)} = \inf_{x=y+z} \left\{ \sigma_c(y) + \sigma_r((d_k z_k)_k) \right\}. \tag{5.2}$$

For a first-column test associated with $A \in \mathrm{M}_{m,n}$, the right multiplication by e_{11} contracts both terms in (5.2). We may therefore restrict the decompositions to first columns. Writing $A = B + C$ and $D = \mathrm{diag}(d_1, \ldots, d_n)$, the two terms become $\|B\|_{S^2}$ and $\|CD\|_{S^1}$. Since

$$\|CD\|_{S^1} \geqslant \|CD\|_{S^2} \geqslant \|C\|_{S^2} ,$$

the infimum is $\|A\|_{S^2}$, attained at $B = A$, $C = 0$. Assertion 1 therefore implies assertion 2 by Theorem 4.4.

Recall that the cb-norm of a map from a column Hilbert space to a row Hilbert space is its Hilbert-Schmidt norm, see [Jun96, Introduction p. II]. In the present notation, this gives

$$\|\mathrm{Id}\|_{\mathrm{cb},\mathrm{C}_n(d)\to\mathrm{R}_n} = \left(\sum_k d_k^{-2} \right)^{\frac{1}{2}}. \tag{5.3}$$

For completeness, at the matrix level we have

$$\left\| \sum_k x_k x_k^* \right\| \leqslant \sum_k \|x_k\|^2 \leqslant \left(\sum_k d_k^{-2} \right) \left\| \sum_k d_k^2 x_k^* x_k \right\|.$$

Equality in the resulting cb-norm estimate is obtained by taking $m = n$ and $x_k = d_k^{-1} e_{1k}$. This proves (5.3).

If assertion 2 holds, the canonical complete contraction $\mathrm{C}_n(d) \to E$ becomes the identity into R_n. Formula (5.3) implies assertion 3. Conversely, assertion 3 says that the identity $\mathrm{C}_n(d) \to \mathrm{R}_n$ is completely contractive. By the universal property of the operator space ℓ^1-sum, the sum map from $\mathrm{R}_n \oplus_1 \mathrm{C}_n(d)$ to R_n is completely contractive and factors through E. The reverse identity is the canonical complete contraction, so assertion 2 follows. Finally, a row space embeds completely isometrically into a matrix trace class, as shown in Theorem 5.4. Thus assertion 2 implies assertion 1. Interchanging rows and columns proves the last assertion. ∎

Corollary 5.6 *For any integer $n \geqslant 2$, the operator space $\mathrm{R}_n + \mathrm{C}_n$ is not completely isometric to a subspace of a noncommutative L^1-space. The same holds for the infinite-dimensional operator space $\mathrm{R} + \mathrm{C}$. Moreover, we have $\mathrm{dist}_{\mathrm{L}^1}(\mathrm{R}_n + \mathrm{C}_n) > 1$.*

Proof : The first assertion is the case $\alpha = \beta = 1$ of Theorem 5.4. The span of the first two basis vectors of $\mathrm{R} + \mathrm{C}$ is completely isometric to $\mathrm{R}_2 + \mathrm{C}_2$, since the canonical inclusions $\mathrm{R}_2 \to \mathrm{R}$, $\mathrm{C}_2 \to \mathrm{C}$ and the coordinate projections $\mathrm{R} \to \mathrm{R}_2$, $\mathrm{C} \to \mathrm{C}_2$ are complete contractions which induce complete contractions between the corresponding sums. The last assertion follows from Proposition 3.2. ∎

Corollary 5.7 *Let $n \geqslant 2$ be an integer and let $s, t > 0$. The operator space $s\mathrm{R}_n \cap t\mathrm{C}_n$ is completely isometric to a complete quotient of a C^*-algebra if and only if $\max\{s,t\} \geqslant \sqrt{n} \min\{s,t\}$. In particular, the operator spaces $\mathrm{R}_n \cap \mathrm{C}_n$ and $\mathrm{R} \cap \mathrm{C}$ are not complete quotients of C^*-algebras.*

Proof : By duality, we have completely isometrically $(s\mathrm{R}_n \cap t\mathrm{C}_n)^* = (s\mathrm{R}_n)^* + (t\mathrm{C}_n)^* = t^{-1}\mathrm{R}_n + s^{-1}\mathrm{C}_n$. By Proposition 3.1 and Theorem 5.4, $s\mathrm{R}_n \cap t\mathrm{C}_n$ is a complete quotient of a C*-algebra if and only if $\max\{t^{-1}, s^{-1}\} \geqslant \sqrt{n}\min\{t^{-1}, s^{-1}\}$, that is $\max\{s,t\} \geqslant \sqrt{n}\min\{s,t\}$. If $q\colon A \to \mathrm{R} \cap \mathrm{C}$ were a complete quotient map, then composing with the coordinate projection $\mathrm{R} \cap \mathrm{C} \to \mathrm{R}_2 \cap \mathrm{C}_2$, which is a complete quotient map since it admits a completely contractive right inverse, we would obtain a complete quotient map from A onto $\mathrm{R}_2 \cap \mathrm{C}_2$. ∎

We can reformulate Corollary 5.6 in terms of noncommutative Khintchine inequalities in L^1, see [HaM07] and the references therein.

Corollary 5.8 *Let $n \geqslant 2$ be an integer. For any $x \in (S^1_m)^n$, put*

$$|||x||| \stackrel{\mathrm{def}}{=} \inf\{\sigma_c(y) + \sigma_r(z) : y + z = x\}.$$

There exists $\delta_n > 0$ such that for any von Neumann algebra $\mathcal{N}$, any $h_1, \ldots, h_n \in \mathrm{L}^1(\mathcal{N})$ and any constants $0 < a \leqslant b$ satisfying

$$a|||x||| \leqslant \left\| \sum_{k=1}^n x_k \otimes h_k \right\|_{\mathrm{L}^1(\mathrm{M}_m \overline{\otimes} \mathcal{N})} \leqslant b|||x|||$$

for any integer $m \geqslant 1$ and any $x \in (S^1_m)^n$, we have $\frac{b}{a} \geqslant 1 + \delta_n$. In particular, the noncommutative Khintchine inequality in L^1 cannot hold with both constants equal to 1.

Proof : By Lemma 5.1, $|||x|||$ is the norm of $\sum_k x_k \otimes e_k$ in $S^1_m(\mathrm{R}_n + \mathrm{C}_n)$. By [Pis98, Lemma 1.7, (1.7)' p. 23], the assumption means that the linear map $u\colon \mathrm{R}_n + \mathrm{C}_n \to \mathrm{L}^1(\mathcal{N})$, $e_k \mapsto h_k$ satisfies $\|u\|_{\mathrm{cb}} \leqslant b$ and $\left\|u^{-1}\right\|_{\mathrm{cb}} \leqslant a^{-1}$. It suffices to take $\delta_n \stackrel{\mathrm{def}}{=} \mathrm{dist}_{\mathrm{L}^1}(\mathrm{R}_n + \mathrm{C}_n) - 1$, which is positive by Corollary 5.6. ∎

We apply Theorem 5.5 to the subspaces considered by Haagerup and Musat. Let $H \subset \mathrm{R}_n \oplus_\infty \mathrm{C}_n$ be n-dimensional, equipped with the Hilbert norm induced by the Hilbert direct sum. If U_1, U_2 are its coordinate maps, put $A \stackrel{\mathrm{def}}{=} U_2^* U_2$, so that $U_1^* U_1 = \mathrm{Id} - A$. For an orthonormal eigenbasis (ξ_k) of A with eigenvalues ν_k, formula (3.5) of [HaM07, p. 16] gives

$$H = \mathrm{R}_n\big((\sqrt{1-\nu_k})_k\big) \cap \mathrm{C}_n\big((\sqrt{\nu_k})_k\big)$$

completely isometrically via the basis (ξ_k).

Corollary 5.9 *Suppose that $0 < A < \mathrm{Id}$.*

1. *If $A \leqslant \frac{1}{2}\mathrm{Id}$, then H^* embeds completely isometrically into a noncommutative L^1-space if and only if $\mathrm{Tr}(A(\mathrm{Id} - A)^{-1}) \leqslant 1$.*
2. *If $A \geqslant \frac{1}{2}\mathrm{Id}$, then the corresponding criterion is $\mathrm{Tr}((\mathrm{Id} - A)A^{-1}) \leqslant 1$.*

In particular, if $A = \nu\mathrm{Id}$ with $0 < \nu < 1$, the criterion is $\nu \leqslant \frac{1}{n+1}$ or $\nu \geqslant \frac{n}{n+1}$.

Proof : Duality gives

$$H^* = \mathrm{R}_n\big((\nu_k^{-\frac{1}{2}})_k\big) + \mathrm{C}_n\big(((1-\nu_k)^{-\frac{1}{2}})_k\big).$$

If all $\nu_k \leqslant \frac{1}{2}$, the column weights are at most the row weights. The symmetric part of Theorem 5.5 gives the condition $\sum_k \frac{\nu_k}{1-\nu_k} \leqslant 1$. If all $\nu_k \geqslant \frac{1}{2}$, the condition is $\sum_k \frac{1-\nu_k}{\nu_k} \leqslant 1$. These are the stated trace conditions. For a scalar A, they reduce to the two inequalities in the last assertion. ∎

Remark 5.10 The general embedding theorem [HaM07, Theorem 3.1 p. 17] gives distortion at most $\sqrt{2}$ for every such dual H^*. Corollary 5.9 identifies exactly when distortion one is possible under the spectral assumption made there. It does not classify the case where A has eigenvalues on both sides of $\frac{1}{2}$. The complete isometry for OH in [Arh26] does not imply a complete isometry for all these graph duals.

6 The operator Hilbert space

We start by computing the first-column and first-row tests of OH_n.

Lemma 6.1 *Let $m, n, N \geqslant 1$ be integers with $n \leqslant N$ and let $\xi_1, \ldots, \xi_n$ be an orthonormal family of OH_N. For any $A = [a_{ik}], B = [b_{jk}] \in \mathrm{M}_{m,n}$ we have*

$$\left\| \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} e_{i1} \otimes \xi_k \right\|_{S^1_m(\mathrm{OH}_N)} = \|A\|_{S^{\frac{4}{3}}} \quad \text{and} \quad \left\| \sum_{j=1}^{m} \sum_{k=1}^{n} b_{jk} e_{1j} \otimes \xi_k \right\|_{S^1_m(\mathrm{OH}_N)} = \|B\|_{S^{\frac{4}{3}}} .$$

Proof : There exists a unitary of ℓ^2_N mapping e_k on ξ_k for any $1 \leqslant k \leqslant n$. This unitary is a complete isometry of OH_N and OH_n is a completely isometric subspace of OH_N. By the second point of Subsection 2.1, we can suppose that $N = n$ and $\xi_k = e_k$. Since $\mathrm{OH}_n = (\mathrm{R}_n, \mathrm{C}_n)_{\frac{1}{2}}$ completely isometrically, the third point of Subsection 2.1 and Lemma 2.1 show that $S^1_m(\mathrm{OH}_n)$ is isometric to the complex interpolation space $(S_0, S_1)_{\frac{1}{2}}$ where S_0 and S_1 denote the space $(S^1_m)^n$ equipped with the norms σ_c and σ_r.

Let V be the subspace of $(S^1_m)^n$ of the n-tuples x satisfying $x_k = x_k e_{11}$ for any k, and let $P\colon (S^1_m)^n \to V$ be the projection defined by $P(x) = (x_1 e_{11}, \ldots, x_n e_{11})$. We have seen in the proof of Lemma 5.3 that P is contractive for σ_c and for σ_r. Denote by V_0 and V_1 the space V equipped with the restrictions of σ_c and σ_r. The inclusion $\iota\colon V \to (S^1_m)^n$ and the projection P are contractive at both endpoints, hence define contractions $\iota\colon (V_0, V_1)_{\frac{1}{2}} \to (S_0, S_1)_{\frac{1}{2}}$ and $P\colon (S_0, S_1)_{\frac{1}{2}} \to (V_0, V_1)_{\frac{1}{2}}$. Since $P\iota = \mathrm{Id}_V$, we obtain for any $v \in V$

$$\|v\|_{(V_0,V_1)_{\frac{1}{2}}} = \|P\iota(v)\|_{(V_0,V_1)_{\frac{1}{2}}} \leqslant \|\iota(v)\|_{(S_0,S_1)_{\frac{1}{2}}} \leqslant \|v\|_{(V_0,V_1)_{\frac{1}{2}}} .$$

Now, the linear map $\Phi\colon \mathrm{M}_{m,n} \to V$, $A \mapsto (\sum_i a_{ik} e_{i1})_{1 \leqslant k \leqslant n}$ is a bijection which satisfies $\sigma_c(\Phi(A)) = \|A\|_{S^2}$ and $\sigma_r(\Phi(A)) = \|A\|_{S_1}$, see the proof of Lemma 5.3. Hence Φ induces an isometry from $(S^2_{m,n}, S^1_{m,n})_{\frac{1}{2}} = S^{\frac{4}{3}}_{m,n}$ onto $(V_0, V_1)_{\frac{1}{2}}$, by the interpolation identity [Pis98, Corollary 1.4 p. 19], restricted to rectangular corners. This proves the first equality. The second equality is proved in the same way, using the projection $x \mapsto (e_{11}x_1, \ldots, e_{11}x_n)$ and the equalities $\sigma_c = \|B\|_{S_1}$, $\sigma_r = \|B\|_{S_2}$ for the first-row tests. ∎

The column and row norms give the following restrictions on any complete isometric realization.

Proposition 6.2 *Let $n \geqslant 1$ be an integer and let $J\colon \mathrm{OH}_n \to \mathrm{L}^1(\mathcal{M})$ be a complete isometry. Put $h_k \overset{\text{def}}{=} J(e_k)$. For any positive matrix $Q \in \mathrm{M}_n$ we have, with the notations (4.3),*

$$\mathrm{Tr}\left(X_Q^{\frac{1}{2}}\right) = \mathrm{Tr}\left(Y_Q^{\frac{1}{2}}\right) = \left(\mathrm{Tr}\, Q^{\frac{2}{3}}\right)^{\frac{3}{4}}. \tag{6.1}$$

Equivalently, for any matrix $B = [b_{kl}] \in \mathrm{M}_n$, the n-tuple $Bh = (\sum_l b_{kl} h_l)_{1 \leqslant k \leqslant n}$ satisfies $\sigma_c(Bh) = \sigma_r(Bh) = \|B\|_{S^{\frac{4}{3}}}$.

Proof : Let $A \in \mathrm{M}_n$. As in the proof of Theorem 4.4, the image under $\mathrm{Id}_{S_1^n} \otimes J$ of the first-column test associated with A has norm $\mathrm{Tr}(X_{A^*A}^{\frac{1}{2}})$. By Lemma 6.1, this norm is equal to $\|A\|_{S^{\frac{4}{3}}} = (\mathrm{Tr}(A^*A)^{\frac{2}{3}})^{\frac{3}{4}}$. In the same way, the image of the first-row test associated with $B \in \mathrm{M}_n$ has norm $\mathrm{Tr}(Y_{B^TB}^{\frac{1}{2}})$, which is equal to $\|B\|_{S^{\frac{4}{3}}} = (\mathrm{Tr}(B^T\overline{B})^{\frac{2}{3}})^{\frac{3}{4}}$ since $B^T\overline{B} = (B^*B)^T$. Since any positive matrix Q can be written $Q = A^*A = B^T\overline{B}$, we obtain (6.1). For the last assertion, note that $\sum_k (Bh)_k^*(Bh)_k = X_{B^*B}$ and $\sum_k (Bh)_k(Bh)_k^* = Y_{B^T\overline{B}}$. ∎

Proposition 6.3 *Let $n \geqslant 2$ be an integer and let $\mathcal{M}$ be a von Neumann algebra. Let $u\colon \mathrm{OH}_n \to \mathrm{L}^1(\mathcal{M})$ be a linear map such that $\|u(\xi)\|_1 = \|\xi\|$ for any $\xi \in \mathrm{OH}_n$ and such that $\mathrm{Id}_{S_2^1} \otimes u\colon S_2^1(\mathrm{OH}_n) \to \mathrm{L}^1(\mathrm{M}_2\overline{\otimes}\mathcal{M})$ is contractive. Then all the nonzero elements of $u(\mathrm{OH}_n)$ have the same right support and the same left support.*

Proof : Let ξ, η be orthonormal vectors of OH_n. Put $h \overset{\text{def}}{=} u(\xi)$, $k \overset{\text{def}}{=} u(\eta)$ and $e \overset{\text{def}}{=} s_r(h)$. Let $t > 0$. By Lemma 6.1 applied with $A = \mathrm{diag}(1, t)$, the element $e_{11} \otimes \xi + te_{21} \otimes \eta$ of $S_1^2(\mathrm{OH}_n)$ has norm $(1 + t^{\frac{4}{3}})^{\frac{3}{4}}$. Its image is the column $Z = e_{11} \otimes h + te_{21} \otimes k$ of $\mathrm{L}^1(\mathrm{M}_2\overline{\otimes}\mathcal{M})$. Consider the polar decompositions $h = v|h|$ and $k(1-e) = w|k(1-e)|$. We have $v^*v = e$ and $w^*w = s_r(k(1-e)) \leqslant 1 - e$. Hence the row $x = e_{11} \otimes v^* + e_{12} \otimes (1-e)w^*$ satisfies $xx^* = e_{11} \otimes (e + w^*w) \leqslant e_{11} \otimes 1$, that is $\|x\| \leqslant 1$. Using $\mathrm{Tr}(ay) = \mathrm{Tr}(ya)$ for $a \in \mathcal{N}$ and $y \in \mathrm{L}^1(\mathcal{M})$, we obtain

$$(\mathrm{Tr} \otimes \mathrm{Tr})(xZ) = \mathrm{Tr}(v^*h) + t\,\mathrm{Tr}\left((1-e)w^*k\right) = \mathrm{Tr}(|h|) + t\,\mathrm{Tr}\left(w^*k(1-e)\right) = \|h\|_1 + t\,\|k(1-e)\|_{\mathrm{L}^1(\mathcal{M})}.$$

Since $\|h\|_{\mathrm{L}^1(\mathcal{M})} = 1$, we deduce that

$$1 + t\,\|k(1-e)\|_{\mathrm{L}^1(\mathcal{M})} \leqslant \|Z\|_{\mathrm{L}^1(\mathrm{M}_2\overline{\otimes}\mathcal{M})} \leqslant \left(1 + t^{\frac{4}{3}}\right)^{\frac{3}{4}} \leqslant 1 + \frac{3}{4}t^{\frac{4}{3}},$$

where we used the concavity of the function $s \mapsto (1+s)^{\frac{3}{4}}$ in the last inequality. Hence $\|k(1-e)\|_1 \leqslant \frac{3}{4}t^{\frac{1}{3}}$ for any $t > 0$, that is $k = ke$. Thus $s_r(k) \leqslant s_r(h)$ and, exchanging the roles of ξ and η, we obtain $s_r(h) = s_r(k)$. Applying the same argument to the adjoint of the image $e_{11} \otimes h + te_{12} \otimes k$ of the first-row test $e_{11} \otimes \xi + te_{12} \otimes \eta$, whose norm is also $(1 + t^{\frac{4}{3}})^{\frac{3}{4}}$ by Lemma 6.1, we obtain $s_l(h) = s_l(k)$.

Now, let ζ, ζ' be nonzero vectors of OH_n. We can suppose that they are linearly independent. Let ζ'' be a unit vector of $\mathrm{span}\{\zeta, \zeta'\}$ orthogonal to ζ. By the first part, we have $s_r(u(\zeta)) = s_r(u(\zeta''))$. Since $u(\zeta')$ is a linear combination of $u(\zeta)$ and $u(\zeta'')$, we deduce that $s_r(u(\zeta')) \leqslant s_r(u(\zeta))$. Exchanging the roles of ζ and ζ', we conclude that $s_r(u(\zeta)) = s_r(u(\zeta'))$. The case of left supports is similar. ∎

Corollary 6.4 *Let $n \geqslant 2$ be an integer and let $\mathcal{N} = \oplus_{i\in I}^{\infty} \mathrm{M}_{d_i}$ be an ℓ^∞-direct sum of matrix algebras. There exists no linear map $u\colon \mathrm{OH}_n \to \mathrm{L}^1(\mathcal{M})$ satisfying the assumptions of Proposition 6.3. In particular, OH_n does not embed completely isometrically into the predual of a finite-dimensional von Neumann algebra.*

Proof : Suppose that such a map u exists. Recall that $\mathrm{L}^1(\mathcal{M}) = \oplus_{i\in I}^1 S_{d_i}^1$ and that the supports of an element $y = (y^{(i)})$ of $\mathrm{L}^1(\mathcal{M})$ are computed componentwise. By Proposition 6.3, there exist projections $e = (e_i)$ and $f = (f_i)$ of $\mathcal{M}$ such that $s_r(u(\zeta)) = e$ and $s_l(u(\zeta)) = f$ for any nonzero $\zeta \in \mathrm{OH}_n$. Choose $i \in I$ with $e_i \neq 0$. For any nonzero ζ, the matrix $u(\zeta)^{(i)}$ has right support e_i and left support f_i. Hence it induces a linear bijection from $e_i\mathbb{C}^{d_i}$ onto $f_i\mathbb{C}^{d_i}$, and these

spaces have the same dimension $r \geqslant 1$. Fix orthonormal vectors ξ, η of OH_n and bases of $e_i\mathbb{C}^{d_i}$ and $f_i\mathbb{C}^{d_i}$. Let $p(\alpha,\beta)$ be the determinant of the matrix of the map $e_i\mathbb{C}^{d_i} \to f_i\mathbb{C}^{d_i}$ induced by $\alpha u(\xi)^{(i)} + \beta u(\eta)^{(i)}$. Then p is a homogeneous polynomial of degree $r \geqslant 1$ in $(\alpha,\beta) \in \mathbb{C}^2$. Hence p admits a nontrivial zero (α_0,β_0): if $\alpha \mapsto p(\alpha,1)$ is not constant, it has a root, and otherwise $p(\alpha,\beta) = p(0,1)\beta^r$ vanishes at $(1,0)$. The vector $\zeta = \alpha_0\xi + \beta_0\eta$ is nonzero but the restriction of $u(\zeta)^{(i)}$ to $e_i\mathbb{C}^{d_i}$ is not injective. Thus $s_r(u(\zeta)) \neq e$, which is a contradiction. The last assertion follows from the first point of Subsection 2.1. ∎

Remark 6.5 The argument of Corollary 6.4 does not extend to von Neumann algebras with a diffuse part. For example, if f is a function of $\mathrm{L}^\infty([0,1])$ whose distribution has no atoms, then the elements 1 and f of $\mathrm{L}^1([0,1])$ satisfy $s_r(\alpha + \beta f) = 1$ for any $(\alpha,\beta) \neq (0,0)$. However, the commutative case is excluded by the next result.

Proposition 6.6 *Let $n \geqslant 2$ be an integer and let $\mathcal{M}$ be a commutative von Neumann algebra. For any injective linear map $u\colon \mathrm{OH}_n \to \mathrm{L}^1(\mathcal{M})$ we have*

$$\|u\|_{\mathrm{cb}} \left\|u^{-1}\right\|_{\mathrm{cb}} \geqslant \kappa_n \overset{\mathrm{def}}{=} n^{\frac{3}{4}} \frac{\Gamma(\frac{3}{2})\Gamma(n)}{\Gamma(n+\frac{1}{2})} \geqslant \left(\frac{n+1}{2\sqrt{n}}\right)^{\frac{1}{2}} > 1. \tag{6.2}$$

Proof : Multiplying u by a positive constant, we can suppose that $\left\|u^{-1}\right\|_{\mathrm{cb}} = 1$. We can write $\mathcal{N} = \mathrm{L}^\infty(\Omega,\mu)$ for some localizable measure space (Ω,μ), so that $\mathrm{L}^1(\mathcal{N}) = \mathrm{L}^1(\Omega,\mu)$ and $\mathrm{L}^1(\mathrm{M}_n\overline{\otimes}\mathcal{M}) = \mathrm{L}^1(\Omega, S_n^1)$. Put $h_k \overset{\mathrm{def}}{=} u(e_k)$ and $h(\omega) \overset{\mathrm{def}}{=} (h_1(\omega),\dots,h_n(\omega)) \in \mathbb{C}^n$. Consider the first-column test $X = \sum_{k=1}^n e_{k1} \otimes e_k$ of $S_n^1(\mathrm{OH}_n)$, whose norm is $\|\mathrm{Id}_n\|_{S_{\frac{4}{3}}} = n^{\frac{3}{4}}$ by Lemma 6.1. Since the trace norm of a column matrix is the euclidean norm of the column, we have

$$n^{\frac{3}{4}} \leqslant \left\|(\mathrm{Id}_{S_1^n} \otimes u)(X)\right\|_{\mathrm{L}^1(\mathrm{M}_n\overline{\otimes}\mathcal{N})} = \int_\Omega \left\| \sum_{k=1}^n h_k(\omega) e_{k1} \right\|_{S_n^1} \mathrm{d}\mu(\omega) = \int_\Omega |h(\omega)|\,\mathrm{d}\mu(\omega).$$

Let σ be the normalized surface measure on the unit sphere $\mathbb{S}$ of $\mathbb{C}^n$ and put $c_n \overset{\mathrm{def}}{=} \int_{\mathbb{S}} |\xi_1|\,\mathrm{d}\sigma(\xi)$. By the unitary invariance of σ, we have $\int_{\mathbb{S}} |\sum_k \xi_k v_k|\,\mathrm{d}\sigma(\xi) = c_n|v|$ for any $v \in \mathbb{C}^n$. For any $\xi \in \mathbb{S}$, we have $\|u(\sum_k \xi_k e_k)\|_1 \leqslant \|u\|_{\mathrm{cb}}$. Integrating with respect to ξ and using Tonelli's theorem, we obtain

$$\|u\|_{\mathrm{cb}} \geqslant \int_{\mathbb{S}} \int_\Omega \left| \sum_{k=1}^n \xi_k h_k(\omega) \right| \mathrm{d}\mu(\omega)\,\mathrm{d}\sigma(\xi) = c_n \int_\Omega |h(\omega)|\,\mathrm{d}\mu(\omega) \geqslant c_n n^{\frac{3}{4}}.$$

Recall that the distribution of $|\xi_1|^2$ under σ is the beta distribution of parameters 1 and $n-1$. Hence $c_n = \frac{\Gamma(\frac{3}{2})\Gamma(n)}{\Gamma(n+\frac{1}{2})}$, $\int_{\mathbb{S}} |\xi_1|^2\,\mathrm{d}\sigma = \frac{1}{n}$ and $\int_{\mathbb{S}} |\xi_1|^4\,\mathrm{d}\sigma = \frac{2}{n(n+1)}$. By Hölder's inequality with the exponents $\frac{3}{2}$ and 3, we have

$$\frac{1}{n} = \int_{\mathbb{S}} |\xi_1|^{\frac{2}{3}} |\xi_1|^{\frac{4}{3}}\,\mathrm{d}\sigma \leqslant c_n^{\frac{2}{3}} \left(\frac{2}{n(n+1)}\right)^{\frac{1}{3}}.$$

Hence $c_n \geqslant n^{-\frac{3}{2}} (\frac{n(n+1)}{2})^{\frac{1}{2}}$ and $\kappa_n = n^{\frac{3}{4}} c_n \geqslant (\frac{n+1}{2\sqrt{n}})^{\frac{1}{2}}$. Finally, $n+1-2\sqrt{n} = (\sqrt{n}-1)^2 > 0$ for $n \geqslant 2$. ∎

Remark 6.7 We have $\kappa_2 = \frac{2^{\frac{7}{4}}}{3} \approx 1.1212$ and $\kappa_n \sim \frac{\sqrt{\pi}}{2} n^{\frac{1}{4}}$ as $n \to \infty$. The proof only uses the Banach space norm of u and the norm of one element of the space $S_1^n(\mathrm{OH}_n)$.

Proposition 6.8 *Let $n \geqslant 2$. For $N^2 \geqslant n$, let*

$$\mathrm{d}_N(\mathrm{OH}_n) \stackrel{\mathrm{def}}{=} \inf\left\{ \|u\|_{\mathrm{cb}} \left\|u^{-1}\right\|_{\mathrm{cb}} : u\colon \mathrm{OH}_n \to S_1^N \text{ injective}\right\}.$$

Then $\mathrm{d}_N(\mathrm{OH}_n) > 1$ for any fixed N, whereas

$$\lim_{N\to\infty} \mathrm{d}_N(\mathrm{OH}_n) = 1. \tag{6.3}$$

Consequently, if $u_j\colon \mathrm{OH}_n \to S_1^{N_j}$ have cb-distortions tending to one, then $N_j \to \infty$.

Proof : Fix N. Suppose that $\mathrm{d}_N(\mathrm{OH}_n) = 1$, and choose u_j with $\|u_j\|_{\mathrm{cb}} = 1$ and $\left\|u_j^{-1}\right\|_{\mathrm{cb}} \to 1$. Since the vector space of linear maps from OH_n to S_1^N is finite-dimensional, a subsequence converges to a linear map u. For any fixed m and any $x \in \mathrm{M}_m(\mathrm{OH}_n)$, the inequalities

$$\left\|u_j^{-1}\right\|_{\mathrm{cb}}^{-1} \|x\| \leqslant \|(\mathrm{Id}_{\mathrm{M}_m} \otimes u_j)(x)\| \leqslant \|x\|_{\mathrm{M}_m(\mathrm{OH}_n)}$$

pass to the limit. Hence u is a complete isometry, contrary to Corollary 6.4.

The existence of matrix embeddings with distortion arbitrarily close to one is [Arh26, Corollary 6.3 p. 20]. It follows there from the complete isometry into a QWEP predual and [EJR00, Corollary 7.3 p. 87]. Since the corner inclusion $S_1^N \to S_1^{N+1}$ is completely isometric, the numbers $\mathrm{d}_N(\mathrm{OH}_n)$ are nonincreasing. This proves (6.3). If (N_j) did not tend to infinity, a subsequence would be bounded by some N. Composing with corner inclusions would then give $\mathrm{d}_N(\mathrm{OH}_n) = 1$, a contradiction. ∎

Remark 6.9 Strong local reflexivity of von Neumann algebra preduals [EJR00, Theorem 6.7 p. 86] is an approximation property. Proposition 6.8 shows why finite representability with constants tending to one does not provide an exact finite-dimensional realization. Likewise, the norm-ultraproduct construction need not preserve semifiniteness of the realizing algebra.

Remark 6.10 For OH_n, the first-column profile $f(Q) = (\mathrm{Tr}\, Q^{\frac{2}{3}})^{\frac{3}{4}}$ of (6.1) satisfies $f(Q_1 + Q_2)^2 > f(Q_1)^2 + f(Q_2)^2$ for any nonzero positive matrices Q_1, Q_2 with orthogonal ranges, since $(a+b)^{\frac{3}{2}} > a^{\frac{3}{2}} + b^{\frac{3}{2}}$ for any $a, b > 0$. Thus the equality case of Proposition 4.1 is not reached in the situation of Proposition 4.3 and the mechanism of Theorem 4.4 does not apply. Note also that Propositions 6.2 and 6.3 describe restrictions on embeddings which now exist by [Arh26]: the identities (6.1) only involve the first-column and first-row tests and do not determine the operator space structure of $J(\mathrm{OH}_n)$.

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