hexagon:2611.00034v2

Haagerup symmetries of the E_8 theory and conformal nets at central charge eight

Cédric Arhancet

Abstract

A Haagerup algebra is a positive strongly rational vertex operator algebra U of central charge eight with $\mathrm{Rep}(U)$ braided equivalent to the Drinfeld center $Z(\mathcal{H}_3)$ of the Haagerup fusion category. We prove that U has exactly three holomorphic extensions, each isomorphic to V_{E_8}, and that the corresponding Lagrangian algebras have module categories $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. The extensions attached to $\mathcal{H}_2$ and $\mathcal{H}_3$ are the two lattice extensions of $V_{E_6\oplus A_2}$. We prove a recognition theorem for $\mathcal{H}_2$ and $\mathcal{H}_3$ among unitary fusion categories of rank six with three invertible objects, a noncommutative Grothendieck ring and trivial pointed associator. For simple regular unitary vertex operator algebras with bounded localized vertex operators, we construct the transport module of two ordinary unitary modules when the second integrates in Tener's annular sense. For a unitary conformal Haagerup subalgebra of $V_{E_6\oplus A_2}$, these results give a completely rational net with representation category $Z(\mathcal{H}_3)$, a realization of the Haagerup subfactor and the generalized orbifold of Bischoff. In particular, they apply to the strongly local explicit algebra S of levels $(3, 13)$, whose Haagerup property is established in [ArhIII, Theorem 6.1]. In [ArhIII], the net of S is recognized from strong rationality and lattice branching, and a diagonal extension of the net is then used to identify $\mathrm{Rep}(S)$. Here an exact construction of the compatible unitary structure on the diagonal lattice algebra $V_{E_8\oplus E_8}$, followed by passage to the twin algebra, proves complete unitarity and Condition I+ for every Haagerup algebra. This also gives the VOA–net comparison through smeared vertex operators for unitary conformal embeddings in $V_{E_6\oplus A_2}$.

Contents

1 Introduction **2**
2 Preliminaries **7**
2.1 The Haagerup fusion categories 7
2.2 The Drinfeld center 10
2.3 Étale and Lagrangian algebras 10
2.4 Haagerup algebras and their extensions 11
2.5 Unitary vertex operator algebras and strong locality 13
2.6 Conformal nets 14
3 Lagrangian algebras and holomorphic extensions **15**
3.1 A rank formula 15
3.2 The three Lagrangian algebras of the Haagerup double 16
3.3 Holomorphic extensions of a Haagerup algebra 19

2020 *Mathematics subject classification:* 46L37, 81T40, 17B69, 18M20.
Key words: Haagerup subfactor, conformal net, vertex operator algebra, Lagrangian algebra, Drinfeld center, generalized orbifold.

4 The conformal net of S 20
4.1 The lattice extensions and their unitary structures . . . 20
4.2 Strong locality of S . . . 23
4.3 Submodules and representations . . . 24
5 Unitarity of fusion products and complete rationality 25
5.1 Positivity of transport forms and unitarity of fusion products . . . 25
5.2 Unitarity of the modules and complete rationality . . . 29
6 A recognition theorem for the Haagerup categories 33
6.1 The fusion rules . . . 34
6.2 The Drinfeld center . . . 35
6.3 Sums of roots of unity . . . 37
6.4 The multiplicity . . . 39
6.5 Pointed parts and proof of the recognition theorem . . . 39
6.6 The associator of the pointed part and the twists . . . 40
6.7 Application to the conformal nets . . . 41
7 Consequences for the net of S, complete unitarity and (N) 43
7.1 The representation category and the holomorphic extensions . . . 43
7.2 The Haagerup subfactor and the generalized orbifold . . . 45
7.3 The third holomorphic extension . . . 46
7.4 A diagonal holomorphic extension of $S \otimes S$ . . . 47
7.5 Complete unitarity of S and the comparison (N) . . . 49
8 The algebra of case (I) and the twin algebra 55
Bibliography 62

1 Introduction

From a completely rational conformal net one obtains finite index subfactors, and the representation category of the net is a unitary modular tensor category [KLM01]. The connection with finite depth subfactors goes back to Jones [Bro20, Section 1.3 p. 2]. In the formulation of [Bis17a, Definition 4.1 p. 20], a subfactor $N \subset M$ arises from a completely rational net $\mathcal{A}$ if it is of the form $\beta(\mathcal{B}_a(I)) \subset \mathcal{B}_b(I)$ for two irreducible relatively local extensions $\mathcal{B}_a, \mathcal{B}_b \supset \mathcal{A}$ and a sector β contained in $\iota_b \rho \bar{\iota}_a$ for some $\rho \in \mathrm{Rep}(\mathcal{A})$, where ι_a and ι_b are the inclusions $\mathcal{A}(I) \subset \mathcal{B}_a(I)$ and $\mathcal{A}(I) \subset \mathcal{B}_b(I)$. He proved that $N \subset M$ arises from $\mathcal{A}$ as soon as $\mathrm{Rep}(\mathcal{A})$ is braided equivalent to the quantum double $D(N \subset M)$ [Bis17a, Proposition 4.7 p. 22], and he formulated, following [EvG11], the problem of realizing such quantum doubles by conformal nets [Bis17a, Problem 1.3 p. 16]. For all subfactors of index less than four this problem was solved in [Bis16]. The Haagerup subfactor of index $\frac{5+\sqrt{13}}{2}$ [AsH99], which has the smallest index above four among finite depth subfactors [JMS14, Section 4.2.1 p. 22], is the first case of index above four. Its quantum double is the Drinfeld center $Z(\mathcal{H}_3)$ of any of the three Morita equivalent Haagerup fusion categories $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ of Grossman and Snyder [GrS12].

Evans and Gannon predicted that $Z(\mathcal{H}_3)$ is realized by a chiral theory of central charge eight [EvG11], and numerical work suggests a Haagerup-symmetric theory with $c \approx 2$ [HL+22, V+22]. At central charge eight, explicit vertex operator algebras inside $V_{E_6 \oplus A_2}$ have recently been proposed by Jiao [Jia26], Riesen [Rie26a] and Huang [Hua26b]. Jiao gives a computer-assisted proof that his algebra realizes $Z(\mathcal{H}_3)$, in a preprint dated 4 October 2026, and Riesen and Huang announce the same property for their algebras, with outlines of proof. In [ArhI] we placed every Haagerup algebra in the chain $V_{E_6 \oplus A_2} \subset V_{E_8}$, determined its currents and constructed its twin. There are exactly two isomorphism classes as soon as one of the explicit candidates of

[Jia26, Rie26a] is a Haagerup algebra [ArhI, Theorems 12.3 and 12.4]. In [ArhIII, Theorem 6.1] we prove that the case (II) algebra S, of levels $(3, 13)$, is a Haagerup algebra, and in [ArhIII, Theorem 7.13(1)] we identify its case (I) twin, of levels $(1, 39)$, with the algebras of [Rie26a, Hua26b]. Albert, Honda, Kaidi and Zheng suggested that the chiral $(\mathfrak{e}_8)_1$ theory carries the symmetries $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, and listed the three Lagrangian algebras of $Z(\mathcal{H}_3)$ through the tube algebra [AHKZ26, Appendix A pp. 6–7], see also [BoS25]. Huang announced, with an outline of proof, that his algebra is strongly rational and completely unitary with $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$ as unitary ribbon categories, that its conformal net is completely rational, and that $\mathcal{H}_3$ acts by topological defects on a holomorphic extension isomorphic to V_{E_8} [Hua26b, Theorem 1 p. 1 and Sections 6 and 7 pp. 8–10].

We determine the three Lagrangian algebras of $Z(\mathcal{H}_3)$ and identify their module categories intrinsically. We realize them by the holomorphic extensions of an arbitrary Haagerup algebra. For unitary conformal subalgebras of $V_{E_6 \oplus A_2}$ we construct the associated nets, and for those which are Haagerup algebras we prove complete rationality and identify the net category. The finite-index step uses a general construction of unitary transport modules. The identification uses a recognition theorem for unitary fusion categories. We obtain the realization of the Haagerup subfactor and the generalized orbifold of Bischoff. Finally, the exact lattice computation in Section 7.5 gives a compatible unitary structure on a holomorphic diagonal extension for any Haagerup algebra of case (II). The coset theorems of Gui and the twin construction then give the complete unitarity theorem for arbitrary Haagerup algebras (Theorem 8.6).

Throughout, $d = \frac{3+\sqrt{13}}{2}$, so that $d^2 = 3d+1$. We use the labels $0, b, a, c_0, c_1, c_2, d_1, \ldots, d_6$ of [EvG11] for the simple objects of $Z(\mathcal{H}_3)$, whose quantum dimensions are 1, $3d+1$, $3d+2$ (four times) and $3d$ (six times). For a Haagerup algebra U we denote by M_z the simple U-module attached to z, and we put $V' = V_{E_6 \oplus A_2}$. By [ArhI, Proposition 4.1], U is contained in V' and $V' = U \oplus M_b$ as a U-module.

Theorem 1.1 *Let U be a Haagerup algebra.*

1. *Up to isomorphism of extensions, U has exactly three holomorphic extensions B_1, B_2, B_3, that is, simple vertex operator algebras B containing U with the same conformal vector and with $\dim B_0 = 1$, every B-module of which is a direct sum of copies of B. Each B_i is isomorphic to V_{E_8}. The corresponding Lagrangian algebras L_i of $\mathrm{Rep}(U)$ satisfy $\mathrm{Rep}(U)_{L_i} \cong \mathcal{H}_i$, and*

$$L_1 \cong 0 \oplus b \oplus a \oplus c_0, \quad L_2 \cong 0 \oplus b \oplus 2a, \quad L_3 \cong 0 \oplus b \oplus 2c_0$$

   as objects, in the labels of [EvG11].

2. *The extensions B_2 and B_3 are the two lattice vertex operator algebras $V' \oplus V'_w \oplus V'_{-w}$ attached to the two isotropic lines $\{0, \pm w\}$ of the discriminant group $\mathbb{Z}_3 \times \mathbb{Z}_3$ of $E_6 \oplus A_2$. Here V'_w denotes the simple V'-module attached to w, and B_2 is the extension for which the U-modules $V'_{\pm w}$ are isomorphic to M_a.*

3. *The subspace $U \oplus M_b$ of B_1 is not a vertex subalgebra, and U and B_1 are the only vertex subalgebras of B_1 containing U. For $i \in \{2, 3\}$, the simple vertex subalgebras of B_i containing U are U, V' and B_i.*

This is Theorem 3.9 with Propositions 3.3 and 3.6. The Lagrangian algebras attached to the two even parts $\mathcal{H}_2$ and $\mathcal{H}_1$ of the Haagerup subfactor were computed by Evans and Gannon in the language of Longo–Rehren inclusions [EvG11, pp. 18–19], the second one by a Frobenius

reciprocity argument of which Lemma 3.1 is a categorical version. The three Lagrangian algebras, with the categories $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ attached to them, are listed in [AHKZ26, (A8) p. 7 and pp. 3–4], in labels 1, π_1, π_2, σ_1 coming from the tube algebra of $\mathcal{H}_3$. We prove this identification in the labels of [EvG11], which are defined through the half-braidings on the even part $\mathcal{H}_2$ of the Haagerup subfactor, by means of the classification of module categories [GrS12] and of a formula of Ostrik for the dimensions of the spaces of morphisms between Lagrangian algebras [Ost15, Remark 2.19 pp. 7–8]. This proof does not use vertex algebras (Corollary 3.8), and it gives $\pi_2 = c_0$ and $\sigma_1 = a$ (Remark 3.7). The realization of these Lagrangian algebras by vertex operator algebras uses the restriction of the lattice modules computed in [ArhI, Lemma 4.3]. For the algebra of [Hua26b], which is a Haagerup algebra by [ArhIII, Theorem 7.13(1)], part 2 shows that the two types of lattice extension discussed in [Hua26b, Section 7 p. 10] are realized by the two isotropic lines, and that the extension attached in [Hua26b, Section 7 pp. 9–10] to the algebra $I_{\mathcal{H}_3}(1)$ is the lattice extension of type c_0. This agrees with [ArhI, Remark 2.1]. Part 3 shows that there is no vertex subalgebra strictly between U and the extension B_1 attached to $\mathcal{H}_1$.

We now turn to conformal nets. Let $S \subset V'$ be the vertex algebra generated by the weight-one and weight-two spaces of [ArhI, Theorem 1.3]. It is unitary for an explicit unitary structure of V' [ArhI, Theorem 10.10], and it is isomorphic to the algebra of Jiao [ArhI, Theorem 10.24]. We write ($\mathrm{H_J}$) for the assertion that Jiao's algebra is a Haagerup algebra [ArhI, Section 1], as stated in [Jia26], and ($\mathrm{H_U}$) for the assertion that every simple S-module admits a scalar product which is invariant with respect to the PCT operator of S induced by the scalar product of V'. Under ($\mathrm{H_J}$), all the simple S-modules except the six modules M_{d_l} are submodules of unitary V'-modules [ArhI, Proposition 4.1, Lemma 4.3 and Lemma 11.8], so that ($\mathrm{H_U}$) concerns these six modules. We prove ($\mathrm{H_U}$) under ($\mathrm{H_J}$) (Theorem 1.3). We write ($\mathrm{H_R}$) for the analogue of ($\mathrm{H_J}$) for the algebra of [Rie26a]. Let $\mathcal{A}_{E_8}$ be the conformal net of V_{E_8}, that is, the loop group net of E_8 at level one. For an S-submodule M of a lattice extension of V', we denote by π_M the representation of the conformal net $\mathcal{A}_S$ of S on the closure of M (Lemma 4.7), and we put $\pi_0 = \pi_S$, $\pi_b = \pi_{M_b}$, $\pi_a = \pi_{M_a}$, $\pi_{c_0} = \pi_{M_{c_0}}$, where M_a and M_{c_0} are taken in the two lattice extensions.

Theorem 1.2 *1. The unitary vertex operator algebra S is strongly local. Its conformal net $\mathcal{A}_S$ is a covariant subnet of the conformal net $\mathcal{A}_{V'}$ of V', which is itself a covariant subnet of index three of the net $\mathcal{A}_{H_\Lambda} \cong \mathcal{A}_{E_8}$ of each of the two lattice extensions $H_\Lambda = V' \oplus V'_w \oplus V'_{-w}$ attached to the isotropic lines $\Lambda = \{0, \pm w\}$ of the discriminant group of $E_6 \oplus A_2$. These inclusions are irreducible, that is, $\mathcal{A}_S(I)' \cap \mathcal{A}_{E_8}(I) = \mathbb{C}$ and $\mathcal{A}_{V'}(I)' \cap \mathcal{A}_{E_8}(I) = \mathbb{C}$ for any interval I.*

2. Assume ($\mathrm{H_J}$). Then $\mathcal{A}_S$ is completely rational. The vacuum representation of $\mathcal{A}_{E_8}$ restricts to $\mathcal{A}_S$ as $\pi_0 \oplus \pi_b \oplus 2\pi_x$, where $x = a$ for one lattice extension and $x = c_0$ for the other, and π_0, π_b, π_a, π_{c_0} are pairwise inequivalent irreducible representations with finite statistical dimensions 1, δ, $1+\delta$, $1+\delta$ for some $\delta > 1$. The indices are $[\mathcal{A}_{V'} : \mathcal{A}_S] = 1+\delta$ and $[\mathcal{A}_{E_8} : \mathcal{A}_S] = 3(1+\delta)$. Finally, $\mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to the Drinfeld center of a unitary fusion category of rank six with a pointed fusion subcategory of dimension three.

This is Theorems 4.5 and 5.6, Proposition 5.8 and Corollary 5.9. Theorem 4.5 applies to any unitary conformal subalgebra of V', and Theorem 5.6 to any such subalgebra which is a Haagerup algebra. The unitary structure on S used in part 1 is that of [ArhI, Theorem 10.10]. For part 2, the simple summands M of the lattice extensions and their conjugates are unitary and have annular representations. The next theorem constructs a transport module on $\overline{M} \boxtimes M$,

and the creation operators are localized by restriction from the lattice extensions. Tener's finite-index theorem [Ten24, Theorem 4.19] then applies to these sectors. The proof of Theorem 5.6 passes from the finite-index sectors to the subnet inclusion. It obtains complete rationality by [Lon03, Theorem 24 p. 20] and identifies Tener's net with $\mathcal{A}_S$ by [Ten19b, Lemma 7.6 p. 72].

Theorem 1.3 *Let V be a simple regular unitary vertex operator algebra with bounded localized vertex operators, and let M and N be ordinary unitary V-modules such that N integrates to a representation π^N of the conformal net of V in the sense of [Ten24, Definition 2.18 p. 20]. Then the fusion product $M \boxtimes N$ admits a unitary structure for which it is a transport module for M and N. Consequently, under* ($\mathrm{H_J}$)*, every simple S-module is unitarizable, that is,* ($\mathrm{H_U}$) *holds.*

This is Proposition 5.2 and Corollary 5.5. Tener proved that, when π^N exists, the transport form is positive semidefinite near the unit circle [Ten24, Proposition 4.2 p. 43], and that, if $M \boxtimes N$ admits a unitary structure, it is a transport module as soon as the transport form is positive semidefinite at one point (z, z) of $\mathcal{R}$ [Ten24, Proposition 3.16 p. 36], which gives [Ten24, Theorem 4.4 and Corollary 4.5 p. 44] under this hypothesis. He noted that the unitarizability of $M \boxtimes N$ is open in general [Ten24, Remark 3.18 p. 37]. Our proof replaces the density argument in a Hilbert space completion of $M \boxtimes N$ by an argument which does not use any Hilbert space completion of $M \boxtimes N$. We use a nondegenerate invariant Hermitian form on $M \boxtimes N$, the invariance of the relevant spaces under polynomials in L_0, and a moment argument (Lemma 5.1). In particular, the unitarity of $\overline{M} \boxtimes M$ used in the proofs of the finite index theorems [Ten24, Theorem 4.20 p. 55 and Corollaries 4.21 and 4.24 pp. 55–56] holds in their setting (Remark 5.3), and Remark 5.10 explains how this bears on the outline of [Hua26b, Section 6 p. 8].

Theorem 1.6 proves the VOA–net correspondence for S under ($\mathrm{H_J}$), in the setting of [Kaw18, Conjecture 7.6 p. 14] and [Gui26, Gui25, Ten24]. Denote by ($\mathrm{N_0}$) the statement that $\mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to $Z(\mathcal{H}_3)$, and by (N) the assertion that there is a braided equivalence $\mathrm{Rep}(\mathcal{A}_S) \to \mathrm{Rep}(S)$ which maps π_M to M for $M \in \{M_b, M_a, M_{c_0}\}$. Under ($\mathrm{H_J}$), (N) implies ($\mathrm{N_0}$). We first prove ($\mathrm{N_0}$) under ($\mathrm{H_J}$), through the following recognition theorem, which does not use vertex algebras.

Theorem 1.4 *Let $\mathcal{F}$ be a unitary fusion category with six simple objects up to isomorphism, three of which are invertible, and with noncommutative Grothendieck ring. Then $\mathcal{F}$ has the fusion rules of $\mathcal{H}_2$ and $\mathcal{H}_3$. If moreover the fusion subcategory generated by the invertible objects is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator, then $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$.*

This is Theorem 6.1. We follow the strategy outlined by Huang [Hua26b, Sections 5 and 6 pp. 7–9] and give complete proofs (Remark 6.12). The hypotheses force $\rho^2 \cong 1 \oplus m(\rho \oplus \alpha\rho \oplus \alpha^2\rho)$ for a non-invertible simple object ρ, an invertible object α of order three and some $m \geqslant 1$ (Lemma 6.2). The Frobenius–Schur indicators ν_1, ν_2, ν_3 [NgS07a, NgS07b] give three identities for the twists of $Z(\mathcal{F})$ (Lemma 6.4), and lower bounds for sums of roots of unity [Lar14], completed by an average over the Galois group of $\mathbb{Q}(\sqrt{10})$ when the squarefree part of $9m^2 + 4$ is 10 (Lemma 6.5), give $m = 1$ (Proposition 6.6). The classification of generalized Haagerup categories by Izumi [Izu18] then gives $\mathcal{H}_2$ or $\mathcal{H}_3$. The assumption on the associator is detected by the Drinfeld center. The third Frobenius–Schur indicator of α is equal to 1 if and only if this associator is trivial (Lemma 6.8), and $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or $\mathcal{H}_3$ if and only if no simple object of $Z(\mathcal{F})$ has a twist of order nine (Proposition 6.9). For unitary fusion categories this gives a recognition result for the Grothendieck ring studied in [GrS12, p. 5]. For the rank six categories of part 2 of Theorem 1.2, denoted $\mathcal{F}_x$ in part 4 of Theorem 5.6, one for each lattice

extension, the triviality of the associator follows from the existence of two lattice extensions of V' (Lemma 6.10), an input also used in [Hua26b, p. 7], and Theorem 1.4 gives (N_0) (Theorem 6.11).

Theorem 1.5 *Assume* (H_J).

1. $\mathrm{Rep}(\mathcal{A}_S)$ *is equivalent to* $Z(\mathcal{H}_3)$ *as a unitary modular tensor category, in particular* (N_0) *holds. Every braided equivalence maps* π_b *to* b*, and* $\delta = d^2$*.*

2. $\mathcal{A}_S$ *has exactly three irreducible local extensions with trivial representation category, up to isomorphism of extensions. Two of them are the nets* $\mathcal{A}_{H_\Lambda}$ *of the two lattice extensions, both isomorphic to* $\mathcal{A}_{E_8}$*, and the categories of modules over their dual canonical endomorphisms are* $\mathcal{H}_2$ *and* $\mathcal{H}_3$*. For the third one, this category is* $\mathcal{H}_1$*, and the only intermediate nets are* $\mathcal{A}_S$ *and the extension itself. For each lattice extension, the only intermediate nets are* $\mathcal{A}_S$*,* $\mathcal{A}_{V'}$ *and the extension itself. The net of the lattice extension of type a carries* $\mathcal{H}_2$*, as the vertex algebra does.*

3. *The Haagerup subfactor arises from* $\mathcal{A}_S$ *in the sense of [Bis17a, Definition 4.1 p. 20].*

4. $\mathcal{A}_S$ *is the generalized orbifold of* $\mathcal{A}_{V'}$ *by the hypergroup* $K = \{k_0, k_1\}$ *with* $k_1k_1 = d^{-2}k_0 + (1 - d^{-2})k_1$*. Hence, under these hypotheses,* $\mathcal{A}_S$ *solves [Bis17b, Problem 4.27 p. 27].*

This is Theorem 7.2 with Theorem 6.11 and Corollaries 7.4, 7.6 and 7.19. The last assertion of part 2 uses Theorem 1.6 below. Part 1 follows from Theorem 1.4 and the dimensions computed in Theorem 1.2 (Theorem 6.11 and Lemma 7.1). Part 4 rests on the fact, which does not use Theorem 1.4, that $\mathcal{A}_S$ is a generalized orbifold of $\mathcal{A}_{V'}$ by a two-element hypergroup with parameter δ (Proposition 7.5). By Theorem 1.6, the three extensions of part 2 are all isomorphic to $\mathcal{A}_{E_8}$, the third one being the net of the holomorphic extension of S attached to $\mathcal{H}_1$ (Proposition 7.10 and Corollary 7.19). For the algebra of Riesen in case (I), [ArhIII, Theorem 7.13(1)–(2)] gives the Haagerup property and complete unitarity. The assertions of Theorem 1.5 for its specified inclusion in V' use the compatible unitary structure asserted in [Rie26a, Theorem 1 p. 1 and p. 4] (Section 8). For the last assertion of part 2 and for the isomorphism of the third extension of its net with $\mathcal{A}_{E_8}$, the proof goes through the computer-assisted construction of Sections 7.4 and 7.5 applied to the twin algebra (Theorem 8.6).

Theorem 1.6 *Assume* (H_J).

1. *The algebra* $S \otimes S$ *has a holomorphic extension isomorphic to* $V_{E_8 \oplus E_8}$ *in which* $S \otimes 1$ *and* $1 \otimes S$ *are mutual commutants, and* $V_{E_8 \oplus E_8}$ *has a structure of unitary vertex operator algebra for which* $S \otimes 1$ *and* $1 \otimes S$ *are unitary subalgebras.*

2. *The algebra* S *is completely unitary, every unitary* S*-module is strongly integrable, and the functor of [Gui26, Section 3.4 p. 45] is an equivalence of braided* C^**-tensor categories from the category* $\mathrm{Rep}^u(S)$ *of unitary* S*-modules to the category* $\mathrm{Rep}^f(\mathcal{A}_S)$ *of representations of* $\mathcal{A}_S$ *with finite statistical dimension, which maps* M *to* π_M *for every* S*-submodule* M *of a lattice extension. In particular,* (N) *holds.*

3. *The twin algebra of* S *in case (I) and the common Heisenberg commutant of the two algebras are completely unitary.*

This is Propositions 7.12 and 7.16, Theorems 7.17 and 8.3 and Remark 8.4. Part 1 rests on exact computer calculations in the weight-one and weight-two spaces of $V_{E_8 \oplus E_8}$, which reduce,

in the first implementation, to spaces of dimensions 496 and 644, carried out with two separate implementations (Remark 7.18). We determine the embedding of the weight-one space of $S \otimes S$ up to automorphisms, the conformal vector of the coset of the affine subalgebra of $S \otimes 1$ up to four choices, and a positive rescaling of the standard antilinear involution of the lattice algebra which fixes these four vectors. Since $S \otimes 1$ and $1 \otimes S$ are commutants of explicit sets of vectors, they are then stable under this involution. Part 2 follows from the theorems of Gui on the strongly rational cosets of lattice vertex operator algebras [Gui26, Theorems 3.34 and 3.46 pp. 51 and 57], [Gui25, Theorem 6.18 p. 70]. It gives a second proof of ($\mathrm{H_U}$), independent of Theorem 1.3 but computer-assisted. Part 3 follows from part 2 and the twin construction (Theorem 8.3). In addition, Corollary 8.5 transfers Condition I+ of [Gui25, Definition 6.17 p. 70] from S to the twin algebra $S^\natural$ and to the common Heisenberg commutant Ω_S of S and $S^\natural$, and it proves the analogue of (N) for every Haagerup algebra realized as a unitary conformal subalgebra of V'. The proofs of Propositions 7.12 and 7.16, of Lemmas 7.14 and 7.15 and of parts 1 and 2 of Theorem 7.17 use no property of S beyond those of an arbitrary Haagerup algebra in case (II). Consequently, every Haagerup algebra is completely unitary, and the analogue of (N) holds for the algebra U^{R} of [Rie26a] under ($\mathrm{H_R}$) and the unitarity asserted in [Rie26a], without ($\mathrm{H_J}$) (Theorem 8.6). For the explicit twin of S, the identification with the algebra of [Hua26b] and its complete unitarity are given by [ArhIII, Theorem 7.13(1)–(2)].

Application to the explicit algebra. The inputs for the net of S are its unitary realization and weight-one data, denoted (I1) in [ArhIII, Theorem 2.1], and its strong rationality and lattice branching, (F1) and (F3) of [ArhIII, Theorem 2.2 and Appendix A]. From these, [ArhIII, Theorem 3.1(1)–(3)] proves finite index, complete rationality and $\mathrm{Rep}(\mathcal{A}_S) \cong Z(\mathcal{H}_3)$. The proof uses the lattice net and submodule results of Section 4, Proposition 5.2 for the transport modules of the lattice summands, and Theorem 6.1 for the unitary fusion categories of the net extensions.

For the vertex operator algebra, [ArhIII, Theorems 4.25, 5.1 and 6.1] constructs a compatible unitary diagonal extension from the net and uses the coset comparison. With (I1) and (F1)–(F3) this proves ($\mathrm{H_J}$), complete unitarity and Condition I+ for S. The hypothesis ($\mathrm{H_J}$) of Theorems 1.2, 1.3, 1.5 and 1.6 holds by [ArhIII, Theorem 6.1], whose proof uses no statement of this article which assumes ($\mathrm{H_J}$) (Section 7).

Table 1 records the hypotheses of the proofs and the sources for the application to S.

Structure of the paper. Section 2 collects the facts on Haagerup categories, Lagrangian algebras, vertex operator algebras and conformal nets that we use. Section 3 proves Theorem 1.1. Section 4 constructs the net of S. Section 5 proves the unitarity of fusion products, the unitarizability of the simple modules and the complete rationality. Section 6 proves Theorem 1.4 and deduces ($\mathrm{N_0}$). Section 7 derives the consequences of ($\mathrm{N_0}$), constructs the unitary structure on $V_{E_8 \oplus E_8}$ and proves the complete unitarity of S and (N). Section 8 discusses case (I) and the twin algebra.

2 Preliminaries

2.1 The Haagerup fusion categories

We follow the conventions of Grossman and Snyder [GrS12]. The two even parts of the Haagerup subfactor are unitary fusion categories $\mathcal{H}_1$ and $\mathcal{H}_2$ with four and six simple objects respectively [GrS12, p. 14].

Statement	Status
Three Lagrangian algebras of $Z(\mathcal{H}_3)$, their objects in the labels of [EvG11] and their categories of modules (Propositions 3.3 and 3.6, Corollary 3.8)	categorical proof
Holomorphic extensions of a Haagerup algebra, realization of L_2 and L_3 by the lattice extensions (Theorem 3.9, Propositions 3.5 and 3.6)	for any Haagerup algebra. For S, [ArhIII, Theorem 6.1] supplies ($\mathrm{H_J}$)
Strong locality of S, inclusions of nets (Theorem 4.5)	proved, using the computer-assisted [ArhI, Theorem 10.10]
Recognition theorem for $\mathcal{H}_2$ and $\mathcal{H}_3$, twists of order nine (Theorem 6.1, Proposition 6.9)	categorical proof
Transport modules for two ordinary unitary modules (Proposition 5.2)	regular simple unitary VOA with bounded localized vertex operators and an annular representation of the second module
Unitarizability of all simple modules (Corollary 5.5)	unitary conformal Haagerup subalgebra of V'. For S, also [ArhIII, Theorem 6.1]
Finite index, complete rationality, branching of sectors and $\delta > 1$ (Theorem 5.6 and Proposition 5.8)	unitary conformal Haagerup subalgebra of V'. For S, [ArhIII, Theorem 3.1(1)–(2)] uses (I1), (F1), (F3)
$\mathcal{F}_x \cong \mathcal{H}_2$ or $\mathcal{H}_3$, ($\mathrm{N_0}$), $\delta = d^2$, local extensions and the consequences of Bischoff (Theorem 6.11 and Section 7)	same hypotheses for Theorem 6.11. For S, [ArhIII, Theorem 3.1(2)–(4)] gives these net results
Agreement of the VOA and net labels, and the third holomorphic extension of $\mathcal{A}_S$ is $\mathcal{A}_{E_8}$ (Corollary 7.19)	uses (N) and complete unitarity from Theorem 7.17. For S, see [ArhIII, Corollary 6.2(1)–(2)]
Diagonal extension $V_{E_8\oplus E_8} \supset S \otimes S$ with compatible unitary structure (Propositions 7.12 and 7.16)	algebraic extension under ($\mathrm{H_J}$). Section 7.5 computes its PCT operator ($\mathrm{H_J}$) and the computed unitary diagonal. The analytic proof for S is in [ArhIII, Theorems 5.1 and 6.1 and Corollary 6.2(1)]
Complete unitarity of S and the comparison through $M \mapsto \pi_M$ (Theorem 7.17)	
Equivalence of complete unitarity for a Haagerup algebra, its twin and their Heisenberg commutant (Theorem 8.3(2))	uses the gluing data of [ArhI, Section 12]
Condition I+ for $S^\natural$ and Ω_S (Corollary 8.5(1))	transfer from S. For the explicit pair, see [ArhIII, Theorem 7.13(2)]
Complete unitarity and Condition I+ for every Haagerup algebra, and the analogue of (N) for unitary conformal embeddings (Theorem 8.6)	uses Section 7.5 in case (II) and transfers through the twin in case (I)
Complete unitarity of the explicit case (I) algebra and comparison for its lattice realization (Section 8)	complete unitarity by [ArhIII, Theorem 7.13(2)]. The comparison uses unitarity for the specified embedding

Table 1: Dependence of the main statements on the hypotheses.

The simple objects of $\mathcal{H}_1$ are 1, ν, η, μ, with Frobenius–Perron dimensions 1, $d+1$, d, $d-1$. They are self-dual and the fusion rules are given in [GrS12, Table 2 p. 15]. In particular

$$\mu \otimes \mu \cong 1 \oplus \nu, \quad \eta \otimes \eta \cong 1 \oplus \nu \oplus \eta \oplus \mu, \quad \nu \otimes \nu \cong 1 \oplus 2\nu \oplus 2\eta \oplus \mu. \tag{2.1}$$

The simple objects of $\mathcal{H}_2$ are 1, α, α^2, ξ, $\alpha\xi$, $\alpha^2\xi$, with Frobenius–Perron dimensions 1, 1, 1, d, d, d and fusion rules

$$\alpha^3 \cong 1, \quad \alpha\xi \cong \xi\alpha^2, \quad \xi^2 \cong 1 \oplus \xi \oplus \alpha\xi \oplus \alpha^2\xi. \tag{2.2}$$

The category $\mathcal{H}_3$ is defined as the category of $(1+\alpha+\alpha^2)$-$(1+\alpha+\alpha^2)$-bimodules in $\mathcal{H}_2$ [GrS12, Definition 3.19 p. 23]. It has six simple objects, three of dimension 1 and three of dimension d, and the same fusion ring as $\mathcal{H}_2$ [GrS12, Theorem 1.4 p. 4]. The three categories have the same global dimension

$$\dim \mathcal{H}_i = 3 + 3d^2 = 3(3d+2), \quad i \in \{1,2,3\}. \tag{2.3}$$

We will use the following facts.

Proposition 2.1 *1. The Morita equivalence class of $\mathcal{H}_1$ consists of exactly the three fusion categories $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, which are pairwise inequivalent.*

2. For any $i \in \{1,2,3\}$, the category $\mathcal{H}_i$ has exactly three indecomposable module categories up to equivalence, and their dual categories are $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$.

3. The category $\mathcal{H}_1$ has rank four, and $\mathcal{H}_2$, $\mathcal{H}_3$ have rank six.

4. The only fusion subcategories of $\mathcal{H}_1$ are Vec *and $\mathcal{H}_1$. For $i \in \{2,3\}$, the only fusion subcategories of $\mathcal{H}_i$ are* Vec*, the pointed subcategory $\mathcal{P}_i$ generated by the invertible objects, and $\mathcal{H}_i$. Moreover $\mathcal{P}_i$ has dimension three.*

Proof : 1. This is [GrS12, Theorem 1.4 p. 4 and Corollary 3.22 p. 24].

2. For $\mathcal{H}_2$, the three module categories are given by [GrS12, Corollary 3.16 p. 22], and their duals are described in [GrS12, Lemmas 3.17 and 3.18 p. 22, Definition 3.19 p. 23 and Corollary 3.22 p. 24]. For $\mathcal{H}_1$ and $\mathcal{H}_3$ this is [GrS12, Theorem 3.25 p. 24].

3. This was recalled above.

4. A fusion subcategory is determined by the set of its simple objects, which is closed under fusion and duality. By (2.1), any fusion subcategory of $\mathcal{H}_1$ containing μ contains ν, any fusion subcategory containing ν or η contains η, μ and ν. Hence a nontrivial fusion subcategory of $\mathcal{H}_1$ is $\mathcal{H}_1$. By (2.2), each object $\alpha^k\xi$ is self-dual and $\alpha^k\xi \otimes \alpha^k\xi \cong \alpha^{3k}\xi^2 \cong \xi^2$. Hence a fusion subcategory of $\mathcal{H}_2$ containing one of the objects $\alpha^k\xi$ contains the three objects of dimension d. It then contains $\xi \otimes \alpha\xi$, which contains α^2 by [GrS12, Table 1 p. 15], so that it is $\mathcal{H}_2$. The same argument applies to $\mathcal{H}_3$, which has the same fusion ring. The remaining subcategories are Vec and $\mathcal{P}_i$, and $\dim \mathcal{P}_i = 3$. ∎

Since the categories $\mathcal{H}_i$ are Morita equivalent, their Drinfeld centers $Z(\mathcal{H}_1)$, $Z(\mathcal{H}_2)$ and $Z(\mathcal{H}_3)$ are braided equivalent [ENO11, Theorem 3.1 p. 9], and we denote any of them by $Z(\mathcal{H}_3)$. By [GrS12, Theorem 1.5 p. 5], there is exactly one Morita equivalence between any two of the categories $\mathcal{H}_i$. Hence the Brauer–Picard group of $\mathcal{H}_3$ is trivial, so that every braided autoequivalence of $Z(\mathcal{H}_3)$ is isomorphic to the identity functor [ENO10, Theorem 1.1 p. 3]. Consequently, any two braided equivalences between a braided fusion category and $Z(\mathcal{H}_3)$ are monoidally naturally isomorphic, and the labels of the simple objects of $Z(\mathcal{H}_3)$ used below are intrinsic.

2.2 The Drinfeld center

We use the modular data of $Z(\mathcal{H}_3)$ in the form of [EvG11, equations (2.5a) and (2.5b) p. 20], as recalled in [ArhI, Section 2.1]. The simple objects are labelled

$$0,\ b,\ a,\ c_0,\ c_1,\ c_2,\ d_1,\dots,d_6,$$

with 0 the unit. Their quantum dimensions are

$$1,\quad 3d+1,\quad 3d+2,\quad 3d+2,\quad 3d+2,\quad 3d+2,\quad 3d,\dots,3d, \tag{2.4}$$

and their twists are

$$1,\ 1,\ 1,\ 1,\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}},\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}},\ \mathrm{e}^{\frac{2\pi\mathrm{i}6l^2}{13}}\quad (1\leqslant l\leqslant 6). \tag{2.5}$$

The charge conjugation S^2 is trivial, since the S-matrix of [EvG11, (2.5a)–(2.5b) p. 20] is real, symmetric and unitary, so that every simple object is self-dual, see also [ArhI, Section 2.1]. Since $Z(\mathcal{H}_3)$ is unitary, the quantum dimensions coincide with the Frobenius–Perron dimensions, and

$$\operatorname{FPdim} Z(\mathcal{H}_3) = (\dim \mathcal{H}_3)^2 = 9(3d+2)^2. \tag{2.6}$$

The twists $\mathrm{e}^{\frac{2\pi\mathrm{i}6l^2}{13}}$ are different from 1, since 13 does not divide $6l^2$ for $1\leqslant l\leqslant 6$. Hence the simple objects with trivial twist are exactly 0, b, a and c_0.

The modular data do not distinguish a from c_0. In [EvG11], the labels are defined through the half-braidings of [Izu01, Section 8] on Izumi's model $\{1,\alpha,\alpha^2,\xi,\alpha\xi,\alpha^2\xi\}$ of the six-object even part of the Haagerup subfactor [EvG11, p. 17], [Izu18, pp. 56–57]. This even part is $\mathcal{H}_2$, which is defined in [GrS12, p. 14] as the even part with six simple objects of the Haagerup subfactor. With $\mu=\xi\oplus\alpha\xi\oplus\alpha^2\xi$, the forgetful functor $F\colon Z(\mathcal{H}_2)\to\mathcal{H}_2$ maps b to $1\oplus\mu$, a to $2\cdot 1\oplus\mu$, the objects c_j to $\alpha\oplus\alpha^2\oplus\mu$ and the objects d_l to μ [EvG11, p. 18]. By Section 2.1, these labels are intrinsic. We use them throughout.

2.3 Étale and Lagrangian algebras

Let $\mathcal{C}$ be a non-degenerate braided fusion category. An étale algebra in $\mathcal{C}$ is a commutative separable algebra, and it is connected if $\dim \operatorname{Hom}_{\mathcal{C}}(1,A)=1$. For a connected étale algebra A, the category $\mathcal{C}_A$ of right A-modules is a fusion category and the full subcategory $\mathcal{C}_A^0$ of local (dyslectic) modules is a non-degenerate braided fusion category [DMNO13, Corollary 3.30 p. 15], with

$$\operatorname{FPdim}(\mathcal{C}_A)=\frac{\operatorname{FPdim}(\mathcal{C})}{\operatorname{FPdim}(A)},\quad \operatorname{FPdim}(\mathcal{C}_A^0)=\frac{\operatorname{FPdim}(\mathcal{C})}{\operatorname{FPdim}(A)^2}, \tag{2.7}$$

see [DMNO13, Lemma 3.11 p. 10 and Corollary 3.32 p. 15]. A connected étale algebra L is called Lagrangian if $\operatorname{FPdim}(L)^2=\operatorname{FPdim}(\mathcal{C})$ [DMNO13, Definition 4.6 p. 17], which is equivalent to $\mathcal{C}_L^0\cong\mathrm{Vec}$ [DMNO13, Remark 4.3 p. 16].

Proposition 2.2 *Let $\mathcal{C}$ be a non-degenerate braided fusion category.*

1. *Let L be a Lagrangian algebra in $\mathcal{C}$. The free module functor $F_L\colon\mathcal{C}\to\mathcal{C}_L$, $X\mapsto X\otimes L$, lifts to a braided equivalence $\mathcal{C}\to Z(\mathcal{C}_L)$, under which F_L corresponds to the forgetful functor $Z(\mathcal{C}_L)\to\mathcal{C}_L$.*

2. *Assume that $\mathcal{C} = Z(\mathcal{D})$ for a fusion category $\mathcal{D}$. Then the isomorphism classes of Lagrangian algebras of $\mathcal{C}$ are in bijection with the equivalence classes of indecomposable $\mathcal{D}$-module categories, and for the Lagrangian algebra L corresponding to $\mathcal{M}$ the fusion category $\mathcal{C}_L$ is equivalent to the dual category $\mathcal{D}^*_{\mathcal{M}}$.*

3. *Let $\mathcal{D}$ be a fusion category and $I\colon \mathcal{D} \to Z(\mathcal{D})$ the right adjoint of the forgetful functor. The étale subalgebras of the Lagrangian algebra $I(1)$ are in order-reversing bijection with the fusion subcategories of $\mathcal{D}$, and if $B \subset I(1)$ corresponds to $\mathcal{B} \subset \mathcal{D}$ then* $\mathrm{FPdim}(B)\,\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{D})$.

4. *Let A be a connected étale algebra in $\mathcal{C}$. The commutative algebras B in $\mathcal{C}$ equipped with a unital algebra morphism $A \to B$, which is automatically injective [DMNO13, p. 12], correspond to the commutative algebras in $\mathcal{C}_A^0$. Under this correspondence, connected étale algebras correspond to connected étale algebras.*

5. *Let $\mathcal{D}$ be a fusion category, let $\mathcal{B} \subset \mathcal{D}$ be a fusion subcategory and let $A_{\mathcal{B}} \subset I(1)$ be the étale subalgebra attached to $\mathcal{B}$ in part 3. Then the category $Z(\mathcal{D})^0_{A_{\mathcal{B}}}$ of local $A_{\mathcal{B}}$-modules is braided equivalent to $Z(\mathcal{B})$.*

Proof : 1. This is [DMNO13, Corollary 4.1 p. 16], with the central structure [DMNO13, (17) p. 10].

2. This is [DMNO13, Proposition 4.8 and Remark 4.9 p. 17], with the 2-equivalence of [DMNO13, Proposition 4.4 p. 16].

3. This is [DMNO13, Theorem 4.10 p. 18].

4. This is [DMNO13, Section 3.6 and Proposition 3.16 p. 12].

5. Put $\mathcal{C} = Z(\mathcal{D})$ and $A = A_{\mathcal{B}}$. In the proof of [DMNO13, Theorem 4.10 p. 18], the category $\mathcal{C}_A$ is identified with the relative center $Z_{\mathcal{B}}(\mathcal{D})$, whose objects are the objects of $\mathcal{D}$ equipped with a half-braiding with respect to the objects of $\mathcal{B}$, in such a way that the free module functor $\mathcal{C} \to \mathcal{C}_A$, with its central structure [DMNO13, (17) p. 10], corresponds to the forgetful functor $Z(\mathcal{D}) \to Z_{\mathcal{B}}(\mathcal{D})$ with the central structure given by the half-braidings. Since $\mathcal{C}$ is non-degenerate, there is a braided equivalence $Z(\mathcal{C}_A) \cong \mathcal{C} \boxtimes (\mathcal{C}_A^0)^{\mathrm{rev}}$ under which $\mathcal{C} \boxtimes 1$ corresponds to the image of the central structure of the free module functor [DMNO13, Corollary 3.30 and Remark 3.31 p. 15]. On the other hand, $Z_{\mathcal{B}}(\mathcal{D})$ is the dual of $\mathcal{D} \boxtimes \mathcal{B}^{\mathrm{op}}$ with respect to the module category $\mathcal{D}$ [DMNO13, p. 18], so that Schauenburg's equivalence [DMNO13, (12) p. 7] gives a braided equivalence $Z(Z_{\mathcal{B}}(\mathcal{D})) \cong Z(\mathcal{D}) \boxtimes Z(\mathcal{B})^{\mathrm{rev}}$, under which the central structure of the forgetful functor $Z(\mathcal{D}) \to Z_{\mathcal{B}}(\mathcal{D})$ corresponds to $Z(\mathcal{D}) \boxtimes 1$ [DMNO13, pp. 18–19]. The centralizer of $\mathcal{C} \boxtimes 1$ in $\mathcal{C} \boxtimes \mathcal{E}$ is $1 \boxtimes \mathcal{E}$ for any braided fusion category $\mathcal{E}$, since $\mathcal{C}$ is non-degenerate. Comparing the centralizers of $\mathcal{C} \boxtimes 1$ in the two descriptions of $Z(\mathcal{C}_A)$, we obtain a braided equivalence $(\mathcal{C}_A^0)^{\mathrm{rev}} \cong Z(\mathcal{B})^{\mathrm{rev}}$, hence $\mathcal{C}_A^0 \cong Z(\mathcal{B})$. This is the claim proved in [DMNO13, proof of Theorem 4.10 p. 18]. ■

2.4 Haagerup algebras and their extensions

A vertex operator algebra is strongly rational if it is simple, of CFT type, self-contragredient, rational and C_2-cofinite. It is positive if every irreducible module not isomorphic to the vertex operator algebra itself has positive lowest conformal weight. Following [ArhI], a Haagerup algebra is a positive strongly rational vertex operator algebra U of central charge eight together with a braided equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$. The modular tensor category $\mathrm{Rep}(U)$ is given by [Hua08]. Since $Z(\mathcal{H}_3)$ has no nontrivial invertible object, such an equivalence preserves the

ribbon structures [ArhI, Section 2.2 and Remark 12.6]. By Section 2.1, it is unique up to natural isomorphism, so that the simple U-modules M_z below are well defined.

Let U be a strongly rational vertex operator algebra. An extension of U is a vertex operator algebra B containing U as a vertex subalgebra with the same conformal vector and with $\dim B_0 = 1$. Two extensions are isomorphic if there is an isomorphism of vertex operator algebras between them which restricts to the identity of U. An extension B is holomorphic if it is simple and if every B-module is a direct sum of copies of B, that is, $\mathrm{Rep}(B) \cong \mathrm{Vec}$. By [HKL15, Theorem 3.2 p. 10], extensions of U correspond to haploid commutative associative algebras with trivial twist in $\mathrm{Rep}(U)$, and by [HKL15, Theorem 3.4 p. 16] the category of modules of an extension B is isomorphic to the category of local modules of the corresponding algebra. If the algebra is moreover rigid in the sense of [KiO02, Definition 1.11 p. 10], then its category of modules is semisimple [KiO02, Theorem 3.3 p. 17], so that it is separable [DMNO13, Proposition 2.7 p. 5], hence étale.

Lemma 2.3 *Let U be a positive strongly rational vertex operator algebra and B a simple extension of U with $B \cong B'$. Let W be a vertex subalgebra of B containing U. Then W is simple if and only if the invariant bilinear form of B restricts to a nondegenerate form on W. In this case, the algebra in $\mathrm{Rep}(U)$ corresponding to W is a connected étale subalgebra of the algebra corresponding to B.*

Proof : Since B is simple and isomorphic to B', it carries a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$, unique up to a scalar [FHL93], [Li94]. Let $R = \{w \in W : (w, W) = 0\}$ be the radical of its restriction to W. The vertex subalgebra W contains the conformal vector, hence it is stable under L_1. The invariance of the form expresses the adjoint of a mode $u_{(n)}$, for $u \in W$, as a finite combination of modes of the vectors $L_1^j u \in W$. Hence R is an ideal of W. Since $(\mathbf{1}, \mathbf{1}) \neq 0$, we have $R \neq W$. Consequently, if W is simple then $R = 0$.

Conversely, assume that $R = 0$, and let $J \subset W$ be a proper ideal. The space J does not contain $\mathbf{1}$, and since $\dim W_0 = 1$ and J is graded, we have $J \cap W_0 = 0$. For $j \in J$ and $w \in W$, the invariance of the form gives $(j, w) = (j_{(-1)}\mathbf{1}, w) = (\mathbf{1}, j^\dagger w)$, where $j^\dagger$ is a finite linear combination of modes of the vectors $L_1^k j$. These vectors lie in J, since J is an ideal containing j and the conformal vector belongs to W. Hence $j^\dagger w \in J$. Since $J \cap W_0 = 0$, J is graded and $(\mathbf{1}, W_n) = 0$ for $n \neq 0$, we obtain $(j, w) = 0$. Thus $J \subset R = 0$, and W is simple.

Finally, assume that W is simple. By [HKL15, Theorem 3.2 p. 10], W corresponds to a haploid commutative algebra A_W with trivial twist, which is a subalgebra of the algebra A_B attached to B. The multiplication of A_W corresponds to the vertex operator of W [HKL15, proof of Theorem 3.2 p. 10], so that the subobjects of A_W which are A_W-submodules correspond to the ideals of W. Since W is simple, A_W is simple as an A_W-module. The categorical dimension of A_W is a sum of quantum dimensions of simple U-modules, which are positive for positive strongly rational U [DJX13, Lemma 4.2 p. 13], [Hua08], so that it is nonzero. Since $\theta_{A_W} = \mathrm{id}$, [KiO02, Lemma 1.20 p. 15] shows that A_W is rigid in the sense of [KiO02, Definition 1.11 p. 10], its category of modules is semisimple [KiO02, Theorem 3.3 p. 17], and A_W is separable [DMNO13, Proposition 2.7 p. 5], hence étale. ∎

We recall the facts from [ArhI] that we use. Let U be a Haagerup algebra.

1. By [ArhI, Proposition 4.1], the subspace $V' = U \oplus M_b$ of the holomorphic extension of U attached to an algebra isomorphic to $0 \oplus b \oplus 2a$ is a vertex operator algebra isomorphic to $V_{E_6 \oplus A_2}$, and U is a conformal subalgebra of V'. We fix such an isomorphism. The discriminant group of $E_6 \oplus A_2$ is $\mathbb{Z}_3 \times \mathbb{Z}_3$, with quadratic form $q(i,j) = \frac{2i^2+j^2}{3}$ modulo one. We denote by V'_w, $w \in \mathbb{Z}_3 \times \mathbb{Z}_3$, the simple V'-modules.

2. Let A_0 be the algebra in $\mathrm{Rep}(U)$ attached to V'. By [ArhI, Lemma 4.3] and its proof, for $w \neq 0$ the restriction of V'_w to U is simple. This restriction is isomorphic to M_{c_1} if $w \in \{(0,1),(0,2)\}$, to M_{c_2} if $w \in \{(1,0),(2,0)\}$, and the four remaining classes, which are the nonzero elements of the two isotropic lines, restrict to M_a for the two classes of one line and to M_{c_0} for the two classes of the other line. Indeed, the proof of [ArhI, Lemma 4.3] shows that exactly two of these four classes restrict to M_{c_0}, and the set of such classes is stable under $w \mapsto -w$ since V'_{-w} is the contragredient of V'_w and c_0 is self-dual. In the coordinates chosen in [ArhI, Section 4.2], for which the holomorphic extension of [ArhI, Proposition 4.1], attached to an algebra isomorphic to $0 \oplus b \oplus 2a$, corresponds to the line generated by $(1,1)$, the classes restricting to M_a are $(1,1)$ and $(2,2)$.

We denote by Λ_a and Λ_{c_0} the isotropic lines whose nonzero classes restrict to M_a and M_{c_0} respectively, and we put

$$H_a \stackrel{\text{def}}{=} \bigoplus_{w \in \Lambda_a} V'_w, \quad H_{c_0} \stackrel{\text{def}}{=} \bigoplus_{w \in \Lambda_{c_0}} V'_w. \tag{2.8}$$

Each isotropic line Λ defines an even unimodular overlattice of $E_6 \oplus A_2$, which is isometric to E_8, and $\bigoplus_{w \in \Lambda} V'_w$ is the corresponding lattice vertex operator algebra, see Lemma 4.1 and the paragraph following it. Hence $H_a \cong H_{c_0} \cong V_{E_8}$, and as U-modules

$$H_a = U \oplus M_b \oplus M_a \oplus M_a, \quad H_{c_0} = U \oplus M_b \oplus M_{c_0} \oplus M_{c_0}. \tag{2.9}$$

For the explicit algebras we use the following external hypotheses, introduced in [ArhI, Section 1]. We write ($\mathrm{H_J}$) for the assertion that the explicit conformal algebra of [Jia26] is a Haagerup algebra, and ($\mathrm{H_R}$) for the same assertion for the algebra U^{R} of [Rie26a]. Positivity is part of these hypotheses. By [ArhI, Theorem 10.24], Jiao's algebra is isomorphic to the algebra S, so that ($\mathrm{H_J}$) implies that S is a Haagerup algebra. Both hypotheses are proved in [ArhIII, Theorems 6.1 and 7.13(1)], using results of this article, and we keep them as hypotheses below.

2.5 Unitary vertex operator algebras and strong locality

We use the framework of [CKLW18]. A unitary vertex operator algebra is a vertex operator algebra V with a normalized invariant scalar product, whose PCT operator θ is an antilinear involutive automorphism of V [CKLW18, Section 5]. A vertex subalgebra W of a unitary vertex operator algebra V is a unitary subalgebra if and only if $\theta W \subset W$ and $L_1 W \subset W$ [CKLW18, Proposition 5.23 p. 43]. A unitary vertex operator algebra with $V_0 = \mathbb{C}\mathbf{1}$ is simple [CKLW18, Proposition 5.3 p. 34].

A simple unitary vertex operator algebra is strongly local if its vertex operators satisfy energy bounds and the von Neumann algebras generated by the smeared vertex operators localized in disjoint intervals commute. A strongly local vertex operator algebra V defines an irreducible conformal net $\mathcal{A}_V$ on the Hilbert space completion $\mathcal{H}_V$ of V, by

$$\mathcal{A}_V(I) = W^*\big(\{Y(a,f) : a \in V,\ f \in C^\infty(S^1),\ \mathrm{supp}\, f \subset I\}\big),$$

the von Neumann algebra generated by the closures $Y(a,f)$ of the smeared vertex operators $Y_0(a,f)$ with domain V [CKLW18, (106) p. 49 and (114) p. 50], see [CKLW18, Theorem 6.8 p. 53]. We will use the following results.

Proposition 2.4 *Let V be a strongly local simple unitary vertex operator algebra.*

1. *If $\{\cdot|\cdot\}$ is another normalized invariant scalar product on V, then $(V, \{\cdot|\cdot\})$ is strongly local, and the two conformal nets are isomorphic [CKLW18, Theorem 6.8 p. 53].*

2. *A unitary subalgebra W of V is strongly local, and $\mathcal{A}_W$ embeds as a covariant subnet of $\mathcal{A}_V$ [CKLW18, Theorem 7.1 p. 55]. The map $W \mapsto \mathcal{A}_W$ is a bijection between unitary subalgebras of V and Möbius covariant subnets of $\mathcal{A}_V$ [CKLW18, Theorem 7.5 p. 58].*

3. *If G is a closed subgroup of the group of unitary automorphisms of V, then $\mathcal{A}_{V^G} = \mathcal{A}_V^G$ [CKLW18, Proposition 7.6 p. 58].*

4. *For a unitary subalgebra W of V, the coset subnet $\mathcal{A}_W^c$, given by $\mathcal{A}_W^c(I) = \mathcal{A}_W(S^1)' \cap \mathcal{A}_V(I)$, is the net of the coset W^c [CKLW18, Proposition 7.8 p. 58].*

5. *A tensor product $V^\alpha \otimes V^\beta$ of strongly local vertex operator algebras is strongly local, and $\mathcal{A}_{V^\alpha \otimes V^\beta} = \mathcal{A}_{V^\alpha} \otimes \mathcal{A}_{V^\beta}$ [CKLW18, Corollary 8.2 p. 62]. For a complex simple Lie algebra $\mathfrak{g}$ and a positive integer k, the simple affine vertex operator algebra $V_{\mathfrak{g},k}$ is strongly local and its net is the loop group net $\mathcal{A}_{G_k}$ [CKLW18, Example 8.7 p. 63].*

2.6 Conformal nets

Throughout, $\mathrm{Rep}(\mathcal{A})$ denotes the category of finite direct sums of irreducible representations of a completely rational net. Equivalently, it is the category of representations of finite statistical dimension, also denoted by $\mathrm{Rep}^f(\mathcal{A})$ below. This convention excludes infinite direct sums when we refer to a fusion category.

We refer to [KLM01, BKLR15] for conformal nets, their representations and Q-systems. A conformal net $\mathcal{A}$ is completely rational if it is split, strongly additive and has finite μ-index. In this case every representation on a separable Hilbert space is a direct sum of irreducible representations with finite statistical dimension [KLM01, Corollary 39 p. 27], there are finitely many irreducible sectors, $\mathrm{Rep}(\mathcal{A})$ is a unitary modular tensor category, and

$$\mu_{\mathcal{A}} = \sum_{\pi \in \mathrm{Irr}(\mathrm{Rep}(\mathcal{A}))} d(\pi)^2 \tag{2.10}$$

by [KLM01, Theorem 33 p. 24]. By [LoX04, MTW18], complete rationality is equivalent to rationality in the sense of [Ten24, Section 4.4 p. 54], namely finitely many irreducible representations, all with finite index.

For an inclusion $\mathcal{A} \subset \mathcal{B}$ of conformal nets, the index $[\mathcal{B} : \mathcal{A}]$ is the Jones index of $\mathcal{A}(I) \subset \mathcal{B}(I)$, which does not depend on I. We will use the following facts.

Proposition 2.5 *Let $\mathcal{A} \subset \mathcal{B}$ be a conformal subnet.*

1. *If $[\mathcal{B} : \mathcal{A}] < \infty$, then $\mathcal{A}$ is completely rational if and only if $\mathcal{B}$ is [Lon03, Theorem 24 p. 20]. In this case $\mu_{\mathcal{A}} = [\mathcal{B} : \mathcal{A}]^2 \mu_{\mathcal{B}}$ [KLM01, Proposition 24 p. 19].*

2. *If $\mathcal{A}$ is completely rational and $\mathcal{B}$ is an irreducible local extension of $\mathcal{A}$, in the sense that $\mathcal{A}(S^1)' \cap \mathcal{B}(I) = \mathbb{C}$ [KaL04, p. 9], then $[\mathcal{B} : \mathcal{A}] < \infty$ [KaL04, Proposition 2.3 p. 12]. The hypotheses of [KaL04, Proposition 2.3] on the sectors of $\mathcal{A}$ hold by [KLM01, Theorem 33 p. 24 and Corollary 39 p. 27].*

3. *If $\mathcal{A}$ is completely rational and $[\mathcal{B} : \mathcal{A}] < \infty$, the restriction of the vacuum representation of $\mathcal{B}$ to $\mathcal{A}$ is the dual canonical endomorphism θ of the inclusion, which carries the structure of a commutative* Q*-system in* $\mathrm{Rep}(\mathcal{A})$*, and $[\mathcal{B} : \mathcal{A}] = d(\theta)$ [LoR95, Theorem 4.9*

p. 25]. Conversely, a commutative Q*-system* Θ *in* $\mathrm{Rep}(\mathcal{A})$ *with* $\dim \mathrm{Hom}(\mathrm{id}, \Theta) = 1$ *defines an irreducible local extension of* $\mathcal{A}$*, unique up to isomorphism of extensions [LoR95, Theorem 4.9 p. 25], [BKLR15, Section 5.2.1 p. 77]. Intermediate extensions correspond to intermediate* Q*-systems [BKLR15, Proposition 4.11 p. 49].*

A commutative Q-system Θ with $\dim \mathrm{Hom}(\mathrm{id}, \Theta) = 1$ is a connected étale algebra in the braided fusion category $\mathrm{Rep}(\mathcal{A})$. Conversely, by [CGGH23, Theorem 3.2 p. 11], a haploid algebra in a C^*-tensor category is equivalent to a Q-system if and only if it is rigid. Since a connected étale algebra is rigid [DMNO13, Remark 3.4 p. 7], a connected étale algebra in $\mathrm{Rep}(\mathcal{A})$ is equivalent to a commutative Q-system. A braided equivalence between unitary braided fusion categories is monoidally naturally isomorphic to a unitary braided equivalence [Reu23, Theorem 5 p. 2].

We recall the results of Bischoff that we use. A conformal net $\mathcal{A}$ is called holomorphic if it is completely rational with $\mu_{\mathcal{A}} = 1$, that is, if $\mathrm{Rep}(\mathcal{A}) \cong \mathrm{Vec}$.

Proposition 2.6 *1. Let* $\mathcal{A}$ *be a completely rational net. Then* $\mathrm{Rep}(\mathcal{A})$ *is braided equivalent to* $Z(\mathcal{F})$ *for some unitary fusion category* $\mathcal{F}$ *if and only if* $\mathcal{A}$ *has a holomorphic finite index extension. In this case,* $\mathcal{F}$ *gives a holomorphic extension* $\mathcal{B}_{\mathcal{F}}$ *of* $\mathcal{A}$*, and there is a Galois correspondence between full fusion subcategories of* $\mathcal{F}$ *and intermediate nets* $\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}_{\mathcal{F}}$ *[Bis17a, Proposition 4.5 p. 21].*

2. Let $N \subset M$ *be a finite index, finite depth subfactor. If there is a completely rational net* $\mathcal{A}$ *such that* $\mathrm{Rep}(\mathcal{A})$ *is braided equivalent to* $D(N \subset M)$*, then* $N \subset M$ *arises from* $\mathcal{A}$ *in the sense of [Bis17a, Definition 4.1 p. 20], see [Bis17a, Proposition 4.7 p. 22].*

3. Let $\mathcal{A} \subset \mathcal{B}$ *be a finite index subnet. There is a canonical hypergroup* K *acting properly on* $\mathcal{B}$ *by vacuum preserving unital completely positive maps such that* $\mathcal{A} = \mathcal{B}^K$*, and* $[\mathcal{B} : \mathcal{A}]$ *is the weight of* K *[Bis17b, Theorem 1.3 p. 3 and Theorem 4.22 p. 24].*

3 Lagrangian algebras and holomorphic extensions

3.1 A rank formula

The following standard fact is the main tool of this section. In the language of Longo–Rehren inclusions, it is the Frobenius reciprocity argument used in [EvG11, p. 19] for the four-object even part of the Haagerup subfactor, and it is also used in [Hua26b, (23) p. 7]. We include a proof.

Lemma 3.1 *Let* $\mathcal{C}$ *be a non-degenerate braided fusion category and let* L *be a Lagrangian algebra in* $\mathcal{C}$*. Then* $\dim \mathrm{Hom}_{\mathcal{C}}(L, L)$ *is equal to the number of isomorphism classes of simple objects of the fusion category* $\mathcal{D} = \mathcal{C}_L$*. Moreover, the braided equivalence* $\Phi\colon \mathcal{C} \to Z(\mathcal{D})$ *of part 1 of Proposition 2.2 maps the algebra* L *to an algebra isomorphic to* $I(1)$*, where* I *is the right adjoint of the forgetful functor* $Z(\mathcal{D}) \to \mathcal{D}$*.*

Proof : Put $\mathcal{D} = \mathcal{C}_L$. By part 1 of Proposition 2.2, there is a braided equivalence $\Phi\colon \mathcal{C} \to Z(\mathcal{D})$ such that $F \circ \Phi$ is isomorphic to the free module functor $F_L\colon X \mapsto X \otimes L$, where $F\colon Z(\mathcal{D}) \to \mathcal{D}$ is the forgetful functor. The right adjoint of F_L is the functor $\mathcal{D} \to \mathcal{C}$ which forgets the module structure, and it maps the unit object L of $\mathcal{D}$ to L. Since Φ is an equivalence, the right adjoint of $F \circ \Phi$ is $\Phi^{-1} \circ I$, where I is the right adjoint of F. By uniqueness of adjoints, we obtain an isomorphism $\Phi(L) \cong I(1)$. It is an isomorphism of algebras, since the algebra structures of L

and $I(1)$ are induced by the lax monoidal structures of the right adjoints of the tensor functors F_L and F, and $F \circ \Phi \cong F_L$ as tensor functors [DMNO13, Lemma 3.9 p. 10]. Consequently, we have

$$\mathrm{Hom}_{\mathcal{C}}(L, L) \cong \mathrm{Hom}_{Z(\mathcal{D})}(I(1), I(1)) \cong \mathrm{Hom}_{\mathcal{D}}(F(I(1)), 1).$$

By [ENO05, Proposition 5.4 p. 21], [EGNO15, Proposition 9.2.2 p. 278], we have $F(I(1)) \cong \bigoplus_X X \otimes X^*$, where X runs through a set of representatives of the isomorphism classes of simple objects of $\mathcal{D}$. Since $\dim \mathrm{Hom}_{\mathcal{D}}(X \otimes X^*, 1) = \dim \mathrm{Hom}_{\mathcal{D}}(X, X) = 1$ for any simple object X, the conclusion follows. This is [ENO05, Proposition 5.6 p. 21], and the identification of L with $I(1)$ is also a consequence of [DMNO13, Proposition 4.4 p. 16]. ∎

3.2 The three Lagrangian algebras of the Haagerup double

Lemma 3.2 *Let L be a Lagrangian algebra in $Z(\mathcal{H}_3)$. Then L is isomorphic, as an object, to $0 \oplus b \oplus X$ with $X \in \{a \oplus a,\ c_0 \oplus c_0,\ a \oplus c_0\}$.*

Proof : The pairing $L \otimes L \to L \to 1$ is nondegenerate [DMNO13, Remark 3.4 p. 7], so that L is a rigid haploid algebra in the unitary braided fusion category $Z(\mathcal{H}_3)$. By [CGGH23, Theorem 3.2 p. 11], L is equivalent to a standard commutative Q-system, which has trivial twist by [CGGH23, Proposition 2.22 p. 9]. Hence L is a direct sum of simple objects with trivial twist. By Section 2.2, we can write $L \cong 0 \oplus n_b b \oplus n_a a \oplus n_{c_0} c_0$ with nonnegative integers n_b, n_a, n_{c_0}, where the multiplicity of 0 is one since L is connected. By (2.6), the Lagrangian condition reads

$$1 + n_b(3d+1) + (n_a + n_{c_0})(3d+2) = 3(3d+2) = 9d + 6.$$

Since d is irrational, we can compare the coefficients of 1 and d and obtain $1+n_b+2(n_a+n_{c_0}) = 6$ and $n_b + n_a + n_{c_0} = 3$. Hence $n_a + n_{c_0} = 2$ and $n_b = 1$. ∎

Proposition 3.3 *The braided fusion category $Z(\mathcal{H}_3)$ has exactly three Lagrangian algebras up to isomorphism, L_1, L_2 and L_3, characterized by $Z(\mathcal{H}_3)_{L_i} \cong \mathcal{H}_i$. As objects, $L_1 \cong 0 \oplus b \oplus a \oplus c_0$, and L_2, L_3 belong to $\{0 \oplus b \oplus 2a,\ 0 \oplus b \oplus 2c_0\}$.*

Proof : By part 2 of Proposition 2.2 applied to $\mathcal{D} = \mathcal{H}_3$, the Lagrangian algebras of $Z(\mathcal{H}_3)$ are in bijection with the indecomposable $\mathcal{H}_3$-module categories, and the category of modules over the Lagrangian algebra attached to $\mathcal{M}$ is the dual category $(\mathcal{H}_3)^*_{\mathcal{M}}$. By part 2 of Proposition 2.1, there are three such module categories and their duals are $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. Since these three categories are pairwise inequivalent, the three Lagrangian algebras are pairwise non-isomorphic, and we can label them so that $Z(\mathcal{H}_3)_{L_i} \cong \mathcal{H}_i$.

By Lemma 3.1 and part 3 of Proposition 2.1, we have

$$\dim \mathrm{Hom}(L_1, L_1) = 4, \quad \dim \mathrm{Hom}(L_2, L_2) = \dim \mathrm{Hom}(L_3, L_3) = 6. \tag{3.1}$$

For the three objects of Lemma 3.2, the dimensions of the endomorphism spaces are $1+1+4 = 6$ for $0 \oplus b \oplus 2a$ and $0 \oplus b \oplus 2c_0$, and $1+1+1+1 = 4$ for $0 \oplus b \oplus a \oplus c_0$, since 0, b, a, c_0 are pairwise non-isomorphic simple objects. This proves the assertion. ∎

Remark 3.4 The objects of L_1 and L_2 appear in [EvG11, pp. 18–19] as the dual canonical endomorphisms of the Longo–Rehren inclusions of the two even parts of the Haagerup subfactor, together with the three monomial modular invariants $|\chi_0 + \chi_b + 2\chi_a|^2$, $|\chi_0 + \chi_b + \chi_a + \chi_{c_0}|^2$ and $|\chi_0 + \chi_b + 2\chi_{c_0}|^2$ [EvG11, Proposition 4 p. 32]. Proposition 3.6 below attaches the third invariant to the category $\mathcal{H}_3$, which was found later [GrS12].

We now realize the Lagrangian algebras of objects $0 \oplus b \oplus 2a$ and $0 \oplus b \oplus 2c_0$ by means of a Haagerup algebra.

Proposition 3.5 *Let U be a Haagerup algebra, and let L_{Λ_a} and $L_{\Lambda_{c_0}}$ be the algebras in* $\mathrm{Rep}(U)$ *attached to the extensions H_a and H_{c_0} of* (2.8).

1. *L_{Λ_a} and $L_{\Lambda_{c_0}}$ are Lagrangian algebras, isomorphic as objects to $0 \oplus b \oplus 2a$ and $0 \oplus b \oplus 2c_0$ respectively.*
2. *Under the braided equivalence* $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$*, we have $\{L_{\Lambda_a}, L_{\Lambda_{c_0}}\} = \{L_2, L_3\}$ and the remaining Lagrangian algebra L_1 is isomorphic to $0 \oplus b \oplus a \oplus c_0$ as an object.*
3. *The algebra A_0 attached to $V' = U \oplus M_b$ is a connected étale algebra, and L_{Λ_a}, $L_{\Lambda_{c_0}}$ are the only Lagrangian algebras of* $\mathrm{Rep}(U)$ *containing A_0. In particular L_1 does not contain A_0 as a subalgebra.*

Proof : 1. The vertex operator algebra H_a is a simple extension of U isomorphic to V_{E_8}, hence holomorphic and self-contragredient. By [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16] and Lemma 2.3 applied to $W = B = H_a$, the algebra L_{Λ_a} is a connected étale algebra in $\mathrm{Rep}(U)$ whose category of local modules is equivalent to $\mathrm{Rep}(H_a) \cong \mathrm{Vec}$. By [DMNO13, Remark 4.3 p. 16], it is Lagrangian. Its object is read off from (2.9). The same argument applies to H_{c_0}.

2. Since a and c_0 are non-isomorphic, the objects of L_{Λ_a} and $L_{\Lambda_{c_0}}$ are non-isomorphic, so these two Lagrangian algebras are distinct. By Proposition 3.3, they have endomorphism spaces of dimension six, hence they are L_2 and L_3 in some order, and L_1 is the remaining one, with the stated object.

3. The algebra A_0 corresponds to the simple vertex subalgebra $V' \subset H_a$, so it is a connected étale subalgebra of L_{Λ_a} by Lemma 2.3. By [HKL15, Theorem 3.4 p. 16] and [CKM24, Theorem 3.65 p. 85], the braided category $\mathrm{Rep}(U)^0_{A_0}$ is braided equivalent to $\mathrm{Rep}(V')$, which is the pointed modular category of the discriminant form $(\mathbb{Z}_3 \times \mathbb{Z}_3, q)$. By part 4 of Proposition 2.2, the connected étale algebras L of $\mathrm{Rep}(U)$ containing A_0 correspond to the connected étale algebras B of $\mathrm{Rep}(U)^0_{A_0} \cong \mathrm{Rep}(V')$. If W is the corresponding extension of U and of V', then [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16], applied to $U \subset W$ and to $V' \subset W$, give isomorphisms of categories $\mathrm{Rep}(U)^0_L \cong \mathrm{Rep}(W) \cong \mathrm{Rep}(V')^0_B$. Hence L is Lagrangian if and only if B is, by [DMNO13, Remark 4.3 p. 16], and the Lagrangian algebras of $\mathrm{Rep}(U)$ containing A_0 correspond to the holomorphic extensions of V', by [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16] and Lemma 2.3. A holomorphic extension of the lattice vertex operator algebra V' is a direct sum of simple currents V'_w with w in a subgroup Λ of $\mathbb{Z}_3 \times \mathbb{Z}_3$ of order three on which q vanishes, since its Frobenius–Perron dimension is three. Indeed, the simple summands of a connected étale algebra in the pointed category $\mathrm{Rep}(V')$ have trivial twist, appear with multiplicity at most one and form a subgroup, by the stabilizer argument of the proof of [ArhI, Proposition 4.1]. Since q vanishes on Λ, the restriction of the pointed braided category $\mathrm{Rep}(V')$ to Λ is equivalent to $\mathrm{Rep}(\mathbb{Z}_3)$ as a symmetric category, with trivial associator and braiding, so that an algebra structure on $\bigoplus_{w \in \Lambda} V'_w$ is given by a normalized 2-cochain with values in $\mathbb{C}^\times$ and trivial coboundary, the components of the multiplication being nonzero since the pairing of a connected étale algebra is nondegenerate [DMNO13, Remark 3.4 p. 7]. Hence this algebra structure is unique up to isomorphism since $H^2(\mathbb{Z}_3, \mathbb{C}^\times) = 0$. The form $q(i,j) = \frac{2i^2+j^2}{3}$ vanishes exactly on the two lines generated by $(1,1)$ and $(1,2)$. Hence there are exactly two Lagrangian algebras of $\mathrm{Rep}(V')$, giving H_a and H_{c_0}. The last statement follows from part 2. ∎

Proposition 3.6 *In the labels of [EvG11] recalled in Section 2.2, we have $L_2 \cong 0 \oplus b \oplus 2a$ and $L_3 \cong 0 \oplus b \oplus 2c_0$ as objects. If U is a Haagerup algebra, then the extension H_a of U carries $\mathcal{H}_2$ while H_{c_0} carries $\mathcal{H}_3$, that is, $L_{\Lambda_a} \cong L_2$ and $L_{\Lambda_{c_0}} \cong L_3$.*

Proof : By Lemma 3.1 applied to $\mathcal{D} = \mathcal{H}_2$, the Lagrangian algebra L_2 corresponds to $I(1)$ under $Z(\mathcal{H}_3) \cong Z(\mathcal{H}_2)$. By adjunction, the multiplicity of a simple object z of $Z(\mathcal{H}_2)$ in $I(1)$ is $\dim \mathrm{Hom}_{\mathcal{H}_2}(F(z), 1)$, where F is the forgetful functor. By Section 2.2, this multiplicity is 1 for $z \in \{0, b\}$, 2 for $z = a$ and 0 for the other simple objects. Hence $L_2 \cong 0 \oplus b \oplus 2a$. This is the computation of [EvG11, p. 18].

We now compute L_3. Let $\Phi\colon Z(\mathcal{H}_3) \to Z(\mathcal{H}_2)$ be a braided equivalence with $\Phi(L_2) \cong I(1)$, given by Lemma 3.1. For $i \in \{1, 2, 3\}$, the algebra $\Phi(L_i)$ is a Lagrangian algebra of $Z(\mathcal{H}_2)$ with category of modules equivalent to $\mathcal{H}_i$. By part 2 of Proposition 2.2, it is the Lagrangian algebra attached in [DMNO13, Section 4.2 p. 17] to an indecomposable $\mathcal{H}_2$-module category $\mathcal{N}_i$, and $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ are pairwise inequivalent, since the algebras $\Phi(L_i)$ are pairwise non-isomorphic, and they are, up to equivalence, all the indecomposable $\mathcal{H}_2$-module categories (part 2 of Proposition 2.1). Here $\mathcal{N}_2$ is the regular module category, since $\Phi(L_2) \cong I(1)$ and $I(1)$ is the Lagrangian algebra attached to $\mathcal{H}_2$ as a module category over itself [DMNO13, Remark 4.9 p. 17]. We equip $\mathcal{H}_2$ with its spherical structure with positive dimensions [ENO05, Proposition 8.23 p. 34]. The induced dimensions of the simple objects of $Z(\mathcal{H}_2)$ are positive, so that the twists of the Lagrangian algebras $\Phi(L_i)$ are trivial [Ost15, Remark 2.19 p. 8]. Hence, by [Ost15, Remark 2.19 pp. 7–8], the number of simple objects of the category $\mathrm{Fun}_{\mathcal{H}_2}(\mathcal{N}_2, \mathcal{N}_i)$ of module functors is $\dim \mathrm{Hom}(\Phi(L_2), \Phi(L_i)) = \dim \mathrm{Hom}(L_2, L_i)$. Since a module functor from the regular module category is determined by the image of the unit object, $\mathrm{Fun}_{\mathcal{H}_2}(\mathcal{N}_2, \mathcal{N}_i)$ is equivalent to $\mathcal{N}_i$, and $\dim \mathrm{Hom}(L_2, L_i)$ is the rank of $\mathcal{N}_i$. By Proposition 3.3 and the first paragraph, $\dim \mathrm{Hom}(L_2, L_1) = \dim \mathrm{Hom}(0 \oplus b \oplus 2a, 0 \oplus b \oplus a \oplus c_0) = 4$, so that $\mathcal{N}_1$ has rank four.

Let $A = 1 \oplus \alpha \oplus \alpha^2$ with the structure of Q-system in $\mathcal{H}_2$ used to define $\mathcal{H}_3$ [GrS12, Lemma 3.14 p. 21 and Definition 3.19 p. 23], and let $\mathcal{M}$ be the category of right A-modules in $\mathcal{H}_2$, which is a semisimple left $\mathcal{H}_2$-module category since A is separable [DMNO13, Proposition 2.7 p. 5]. For objects Y, Y' of $\mathcal{H}_2$, we have $\mathrm{Hom}_A(Y \otimes A, Y' \otimes A) \cong \mathrm{Hom}(Y, Y' \otimes A)$, and since A is a Q-system, every simple right A-module is a summand of some $Y \otimes A$ with Y simple. Since $\xi\alpha \cong \alpha^2\xi$ and $\xi\alpha^2 \cong \alpha\xi$ by (2.2), we have $\alpha^j\xi \otimes A \cong \alpha^j\xi \oplus \alpha^{j+1}\xi \oplus \alpha^{j+2}\xi$ for $j \in \{0, 1, 2\}$. Hence $\dim \mathrm{Hom}_A(Y \otimes A, Y' \otimes A)$ is equal to 1 if Y and Y' belong both to $\{1, \alpha, \alpha^2\}$ or both to $\{\xi, \alpha\xi, \alpha^2\xi\}$, and to 0 otherwise. We obtain that the simple objects of $\mathcal{M}$ are $A \cong \alpha \otimes A \cong \alpha^2 \otimes A$ and $\xi \otimes A \cong \alpha\xi \otimes A \cong \alpha^2\xi \otimes A$, up to isomorphism. Since every simple object of $\mathcal{M}$ is a summand of $Y \otimes A$ for some Y, the module category $\mathcal{M}$ is generated by A, hence indecomposable. It has rank two, so that it is equivalent neither to $\mathcal{N}_2$, of rank six, nor to $\mathcal{N}_1$, of rank four. Hence $\mathcal{M} \cong \mathcal{N}_3$ and $\dim \mathrm{Hom}(L_2, L_3) = 2$. Since $\dim \mathrm{Hom}(0 \oplus b \oplus 2a, 0 \oplus b \oplus 2a) = 6$, Proposition 3.3 gives $L_3 \cong 0 \oplus b \oplus 2c_0$.

Finally, let U be a Haagerup algebra. By part 1 of Proposition 3.5, L_{Λ_a} and $L_{\Lambda_{c_0}}$ are Lagrangian algebras isomorphic to $0 \oplus b \oplus 2a$ and $0 \oplus b \oplus 2c_0$ as objects. By the above and Proposition 3.3, L_2 and L_3 are the only Lagrangian algebras with these objects, so that $L_{\Lambda_a} \cong L_2$ and $L_{\Lambda_{c_0}} \cong L_3$. ∎

Remark 3.7 The conventions in [ArhI, Remark 2.1] agree with Proposition 3.6. In the tube-algebra labelling of [AHKZ26, (A8) p. 7], the Lagrangian algebra of $\mathcal{H}_3$ is $1 \oplus \pi_1 \oplus 2\pi_2$. This is the category $\mathcal{H}_3$ of [GrS12], since the algebra $1 \oplus \rho \oplus \alpha\rho$ relates it to $\mathcal{H}_1$ at index $4 + \sqrt{13}$ [AHKZ26, p. 4], [GrS12, Theorems 1.3 and 1.4 pp. 3–4]. Comparing its Lagrangian algebra with $0 \oplus b \oplus 2c_0$ gives $\pi_2 = c_0$ and $\sigma_1 = a$, as in [ArhI, Remark 2.1]. The transposition of a and c_0 preserves the modular data of Section 2.2.

Corollary 3.8 *The three Lagrangian algebras of $Z(\mathcal{H}_3)$ are isomorphic, as objects, to*

$$0 \oplus b \oplus a \oplus c_0, \quad 0 \oplus b \oplus 2a, \quad 0 \oplus b \oplus 2c_0,$$

with categories of modules $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ respectively.

Proof : This is Proposition 3.3 with Proposition 3.6. ■

3.3 Holomorphic extensions of a Haagerup algebra

Theorem 3.9 *Let U be a Haagerup algebra.*

1. *Up to isomorphism of extensions, U has exactly three holomorphic extensions B_1, B_2, B_3. The algebra L_i in $\mathrm{Rep}(U)$ attached to B_i is the Lagrangian algebra of Proposition 3.3, so that $\mathrm{Rep}(U)_{L_i} \cong \mathcal{H}_i$. Each B_i is a strongly rational holomorphic vertex operator algebra isomorphic to V_{E_8}.*
2. *$B_2 = H_a$ and $B_3 = H_{c_0}$ as extensions of U.*
3. *The subspace $U \oplus M_b$ of B_1 is not a vertex subalgebra. The only vertex subalgebras of B_1 containing U are U and B_1. For $i \in \{2,3\}$, the only simple vertex subalgebras of B_i containing U are U, V' and B_i.*

Proof : 1. Let B be a holomorphic extension of U. Since $\mathrm{Rep}(B)$ is trivial, the contragredient module B' is isomorphic to B. By [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16] and Lemma 2.3, the algebra L_B attached to B is a connected étale algebra with $\mathrm{Rep}(U)^0_{L_B} \cong \mathrm{Rep}(B) \cong \mathrm{Vec}$, hence a Lagrangian algebra. Conversely, a Lagrangian algebra L of $\mathrm{Rep}(U)$ is a haploid commutative algebra with trivial twist (Lemma 3.2 and its proof), so it defines an extension B_L of U with $\mathrm{Rep}(B_L) \cong \mathrm{Rep}(U)^0_L \cong \mathrm{Vec}$ by [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16]. Since L is connected, L is a simple object of $\mathrm{Rep}(U)_L$ [DMNO13, p. 10], and the ideals of B_L correspond to the subobjects of L in $\mathrm{Rep}(U)_L$ [HKL15, proof of Theorem 3.2 p. 10]. Hence B_L is simple. These two constructions are inverse to each other up to isomorphism, by [HKL15, Theorem 3.2 p. 10]. By Proposition 3.3, there are exactly three Lagrangian algebras, which proves the first two assertions. The nonvacuum summands of L_i are b, a and c_0, whose lowest conformal weights are equal to one [ArhI, (4.1)]. Hence B_i is of CFT type. It is self-contragredient since $\mathrm{Rep}(B_i) \cong \mathrm{Vec}$, and it is C_2-cofinite and rational by the argument of the proof of [CKM22, Corollary 5.13 p. 53], since its category of modules is semisimple, see also [ArhI, proof of Proposition 4.1]. A strongly rational holomorphic vertex operator algebra of central charge eight is isomorphic to V_{E_8} [DoM04, Theorem 1 p. 2].

2. This is Proposition 3.6.

3. Let W be a simple vertex subalgebra of B_i containing U. By Lemma 2.3, the corresponding algebra A_W is a connected étale subalgebra of L_i. By Lemma 3.1, under the braided equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_i)$ given by part 1 of Proposition 2.2, the algebra L_i corresponds to $I_{\mathcal{H}_i}(1)$. By part 3 of Proposition 2.2, the connected étale subalgebras of L_i correspond to the fusion subcategories of $\mathcal{H}_i$, with $\mathrm{FPdim}(A_W)\,\mathrm{FPdim}(\mathcal{B}) = \dim \mathcal{H}_i$ if A_W corresponds to $\mathcal{B}$. By part 4 of Proposition 2.1, the connected étale subalgebras of L_1 are the unit object 0 and L_1, and for $i \in \{2,3\}$ the connected étale subalgebras of L_i are 0, L_i and a unique subalgebra of Frobenius–Perron dimension $\frac{3(3d+2)}{3} = 3d+2$, corresponding to $\mathcal{P}_i$. For $i \in \{2,3\}$, the extension B_i contains V', which is simple, so this subalgebra is A_0. This proves the statements on simple vertex subalgebras.

Let now W be any vertex subalgebra of B_1 containing U, and let $(\cdot,\cdot)$ be the invariant bilinear form of B_1. It is U-invariant, so it pairs the isotypic component of type z with that of type z^*, and its restriction to each isotypic component of a self-dual type is nondegenerate. Since $L_1 \cong 0 \oplus b \oplus a \oplus c_0$ is multiplicity free, W is a sum of isotypic components of B_1, all of

self-dual types. Hence the restriction of $(\cdot,\cdot)$ to W is nondegenerate, W is simple by Lemma 2.3, and the description of the connected étale subalgebras of L_1 obtained above gives $W \in \{U, B_1\}$. In particular, $U \oplus M_b$ is not a vertex subalgebra of B_1. ∎

Remark 3.10 By part 3 of Theorem 3.9, there is no vertex subalgebra strictly between U and $B_1 \cong V_{E_8}$. The weight-one space of B_1 is $U_1 \oplus R_b \oplus R_a \oplus R_{c_0}$, where R_z is the lowest weight space of M_z, and its dimension is $(6+13\gamma)+(80-13\gamma)+81+81 = 248 = \dim \mathfrak{e}_8$ by [ArhI, (4.1)], for both values $\gamma \in \{0,1\}$ of the parameter of loc. cit. In fact $\gamma = 0$ by [ArhI, Theorem 7.1].

Remark 3.11 Let L be a Lagrangian algebra in $\mathrm{Rep}(U)$ with holomorphic extension B. In the language of symmetry topological field theories, the fusion category $\mathrm{Rep}(U)_L \cong \mathcal{H}_i$ of L-modules is interpreted as the category of the topological defect lines of B which preserve U, see [AHKZ26, GaR26, Hua26b]. Theorem 3.9 thus gives a vertex algebraic version of the suggestion of [AHKZ26] that the chiral $(\mathfrak{e}_8)_1$ theory carries the three symmetries $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, under the assumption that a Haagerup algebra exists. An operator algebraic version is given in Section 7, under ($\mathrm{H_J}$).

4 The conformal net of S

Throughout this section, $L = E_6 \oplus A_2$ denotes the root lattice and $V' = V_L = M(1) \otimes \mathbb{C}_\varepsilon[L]$ is the lattice vertex operator algebra of [ArhI, Section 10], built as in [FLM88, Chapter 8] with a bimultiplicative cocycle $\varepsilon\colon L \times L \to \{\pm 1\}$ satisfying $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)^{-1} = (-1)^{\langle\alpha,\beta\rangle}$ and $\varepsilon(\alpha,\alpha) = (-1)^{\langle\alpha,\alpha\rangle/2}$ for any $\alpha,\beta \in L$. The second property holds for the cocycle of [ArhI, Section 10.1], which satisfies it on the simple roots, since $\alpha \mapsto \varepsilon(\alpha,\alpha)(-1)^{\langle\alpha,\alpha\rangle/2}$ is a group homomorphism $L \to \{\pm 1\}$ by the first property. We denote by L° the dual lattice, so that $L^\circ/L \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ with the quadratic form q of Section 2.4. No hypothesis on Haagerup algebras is made in this section.

4.1 The lattice extensions and their unitary structures

Let $\Lambda \subset L^\circ/L$ be one of the two isotropic lines and let $L_\Lambda \subset L^\circ$ be its preimage. Since q vanishes on Λ, the lattice L_Λ is even, and it is unimodular since $[L_\Lambda : L] = 3$ and $|L^\circ/L| = 9$. Hence L_Λ is isometric to E_8.

Lemma 4.1 *1. The cocycle ε extends uniquely to a bimultiplicative map $\varepsilon_\Lambda\colon L_\Lambda \times L_\Lambda \to \{\pm 1\}$. We have $\varepsilon_\Lambda(\alpha,\beta)\varepsilon_\Lambda(\beta,\alpha)^{-1} = (-1)^{\langle\alpha,\beta\rangle}$ and $\varepsilon_\Lambda(\alpha,\alpha) = (-1)^{\langle\alpha,\alpha\rangle/2}$ for any $\alpha,\beta \in L_\Lambda$.*

2. The lattice vertex operator algebra

$$H_\Lambda \overset{\text{def}}{=} M(1) \otimes \mathbb{C}_{\varepsilon_\Lambda}[L_\Lambda] = \bigoplus_{w \in \Lambda} V'_w, \quad V'_w \overset{\text{def}}{=} M(1) \otimes \mathbb{C}_{\varepsilon_\Lambda}[L + \lambda_w],$$

where $\lambda_w \in L^\circ$ is a representative of w, contains $V' = V'_0$ as a vertex subalgebra with the same conformal vector, and H_Λ is isomorphic to V_{E_8}.

Proof : 1. A bimultiplicative map $M \times M \to \{\pm 1\}$ on a lattice M is the same thing as a bilinear form on the $\mathbb{F}_2$-vector space $M/2M$. The inclusion $L \subset L_\Lambda$ induces an isomorphism $L/2L \to L_\Lambda/2L_\Lambda$. Indeed, both spaces have dimension eight, and if $x \in L$ satisfies $x = 2y$ with

$y \in L_\Lambda$, then $3y \in L$ and $2y \in L$, hence $y \in L$. This gives the existence and uniqueness of ε_Λ. The maps $(\alpha,\beta) \mapsto \varepsilon_\Lambda(\alpha,\beta)\varepsilon_\Lambda(\beta,\alpha)^{-1}$ and $(\alpha,\beta) \mapsto (-1)^{\langle\alpha,\beta\rangle}$ are bimultiplicative on L_Λ, since L_Λ is integral, and they coincide on L. By the same isomorphism, they coincide on L_Λ. Finally, the map $s(\alpha) = \varepsilon_\Lambda(\alpha,\alpha)(-1)^{\langle\alpha,\alpha\rangle/2}$ satisfies $s(\alpha+\beta) = s(\alpha)s(\beta)$ by the commutator identity, so it is a group homomorphism $L_\Lambda \to \{\pm 1\}$. It is trivial on L, and $[L_\Lambda : L] = 3$ is odd, so that s is trivial.

2. By part 1, the construction of lattice vertex operator algebras [FLM88, Chapter 8] applies to $(L_\Lambda, \varepsilon_\Lambda)$, and the subspace $M(1) \otimes \mathbb{C}_\varepsilon[L]$ is a vertex subalgebra equal to V' with the same conformal vector. Up to isomorphism, the lattice vertex operator algebra does not depend on the choice of the cocycle, and $L_\Lambda \cong E_8$, so that $H_\Lambda \cong V_{E_8}$. ∎

The subspace V'_w of H_Λ realizes the simple V'-module attached to w. An extension of V' whose underlying V'-module is $\bigoplus_{w\in\Lambda} V'_w$ is unique up to isomorphism of extensions, by the argument of the proof of part 3 of Proposition 3.5, since $H^2(\mathbb{Z}_3, \mathbb{C}^\times) = 0$.

Hence, in the situation of Section 2.4, the vertex operator algebras H_a and H_{c_0} of (2.8) are isomorphic, as extensions of V', to H_{Λ_a} and $H_{\Lambda_{c_0}}$. For a character $\chi\colon L \to \mathbb{R}_{>0}$, we write $\chi(\lambda) = \mathrm{e}^{\langle h_\chi,\lambda\rangle}$ with h_χ in the real span of L, and we extend χ to L° by the same formula. Let ϕ_1 be the antilinear map of H_Λ defined by [ArhI, (10.3)] for the trivial character, that is

$$\phi_1\big(ch_1(-n_1)\cdots h_k(-n_k)\mathrm{e}^\lambda\big) = (-1)^k\bar{c}h_1(-n_1)\cdots h_k(-n_k)\mathrm{e}^{-\lambda}, \quad c\in\mathbb{C},\ \lambda\in L_\Lambda,$$

where $h_1,\dots,h_k$ belong to the real span of L.

Lemma 4.2 *1. The map ϕ_1 is an antilinear involutive automorphism of H_Λ. If β denotes the normalized invariant bilinear form of H_Λ, then $H_1(u,v) \overset{\text{def}}{=} \beta(u,\phi_1(v))$ is a normalized invariant scalar product on H_Λ.*

2. Let $\{\cdot|\cdot\}$ be a normalized invariant scalar product on V'. There exists x in $(V')_1$ such that $\tilde{h} = \exp(x(0))$ is an automorphism of H_Λ preserving V', $\{u|v\} = H_1(\tilde{h}u,\tilde{h}v)$ for any $u,v\in V'$ and $H_1(u,\tilde{h}u) > 0$ for any nonzero $u\in V'$. Consequently, $\{u|v\}_\Lambda \overset{\text{def}}{=} H_1(\tilde{h}u,\tilde{h}v)$ is a normalized invariant scalar product on H_Λ which extends $\{\cdot|\cdot\}$, and V' is a unitary subalgebra of $(H_\Lambda,\{\cdot|\cdot\}_\Lambda)$. The scalar product $\{\cdot|\cdot\}_\Lambda$ does not depend on the choice of x.

3. For the unitary structure ϕ_χ of [ArhI, Theorem 10.10], one can take $x = \frac{1}{2}h_\chi(-1)\mathbf{1}$ in part 2. The corresponding PCT operator of H_Λ is $\phi_\chi = \phi_1 \circ \exp(h_\chi(0))$.

Proof : 1. The cocycle ε_Λ is bimultiplicative, so that $\varepsilon_\Lambda(-\alpha,-\beta) = \varepsilon_\Lambda(\alpha,\beta)$, and $\varepsilon_\Lambda(\alpha,\alpha) = (-1)^{\langle\alpha,\alpha\rangle/2}$ by Lemma 4.1. Hence $\mathrm{e}^\alpha\mathrm{e}^{-\alpha} = \varepsilon_\Lambda(\alpha,-\alpha)\mathrm{e}^0 = (-1)^{\langle\alpha,\alpha\rangle/2}\mathrm{e}^0$ for any $\alpha\in L_\Lambda$, which is the relation used in [DoL14, Lemma 4.11 p. 23], and ϕ_1 is the antilinear involution for which lattice vertex operator algebras are unitary [DoL14, Theorem 4.12 p. 24], [CKLW18, Example 5.9 p. 35], as in the proof of [ArhI, Theorem 10.10].

2. By part 1, the restriction of H_1 to V' is a normalized invariant scalar product on V'. Since V' is simple, there exists by [CKLW18, Proposition 5.19 p. 40] an automorphism h of V' with $\{u|v\} = H_1(hu,hv)$ for any $u,v\in V'$ and $H_1(u,hu) > 0$ for any nonzero $u\in V'$. In particular h preserves each $(V')_n$ and is diagonalizable with positive eigenvalues. Let D be the linear map acting by $\log\lambda$ on the eigenspace of h for the eigenvalue λ. Since $h(u_{(n)}v) = \lambda\mu u_{(n)}v$ if $hu = \lambda u$ and $hv = \mu v$, the map D is a derivation of V' preserving the grading. Its restriction to $(V')_1$ is a derivation of the semisimple Lie algebra $((V')_1, [u,v] = u_{(0)}v) \cong \mathfrak{e}_6 \oplus \mathfrak{sl}_3$, so that there is $x\in(V')_1$ with $D(u) = x_{(0)}u$ for $u\in(V')_1$. The derivations D and $x_{(0)} = x(0)$ of V' coincide on $(V')_1$ and vanish on the vacuum. The space $(V')_1$ contains the vectors $h(-1)\mathbf{1}$ and e^α for the roots α of L, the lattice L is spanned by its roots, and $\mathrm{e}^{\alpha+\beta}$ is a nonzero multiple of $\mathrm{e}^\alpha_{(-\langle\alpha,\beta\rangle-1)}\mathrm{e}^\beta$

[FLM88, Chapter 8]. Hence $(V')_1$ generates V', and $D = x(0)$. Now $x(0)$ is a derivation of H_Λ which preserves the finite-dimensional homogeneous spaces, so that $\tilde{h} = \exp(x(0))$ is an automorphism of H_Λ which extends $h = \exp(D)$. The transport of a normalized invariant scalar product by an automorphism is a normalized invariant scalar product, and the PCT operator of $\{\cdot|\cdot\}_\Lambda$ is $\tilde{h}^{-1}\phi_1\tilde{h}$, which preserves V'. Since $L_1V' \subset V'$, the last assertion follows from [CKLW18, Proposition 5.23 p. 43]. Two choices of x give automorphisms of H_Λ which coincide on V', since a positive square root is unique. They differ by an automorphism of H_Λ fixing V' pointwise, which acts on each simple V'-module V'_w, $w \in \Lambda$, by a scalar, hence by a power of the automorphism g_Λ of Proposition 4.3. This automorphism is unitary for H_1 and commutes with $\tilde{h}$, so that $\{\cdot|\cdot\}_\Lambda$ does not depend on the choice of x.

3. With $g = \exp(h_\chi(0))$, which acts by $\chi(\lambda)$ on $M(1) \otimes \mathrm{e}^\lambda$, we have $\phi_\chi = \phi_1 \circ g = g^{-1/2} \circ \phi_1 \circ g^{1/2}$ on V' by the proof of [ArhI, Theorem 10.10], and the same computation holds on H_Λ. We still denote by $H_\chi(u,v) = H_1(g^{1/2}u, g^{1/2}v)$ the corresponding scalar product on H_Λ, which extends the scalar product H_χ of V' of [ArhI, Theorem 10.10]. ■

Proposition 4.3 *Let Λ be an isotropic line and let $\{\cdot|\cdot\}$ be a normalized invariant scalar product on H_Λ.*

1. *The unitary vertex operator algebra $(H_\Lambda, \{\cdot|\cdot\})$ is strongly local and has bounded localized vertex operators in the sense of [Ten19b, Definition 4.12 p. 42]. The net of [Ten19b] and the net of [CKLW18] of $(H_\Lambda, \{\cdot|\cdot\})$ are isomorphic, and they are isomorphic to the loop group net $\mathcal{A}_{E_8}$ of E_8 at level one. This net is completely rational and $\mu_{\mathcal{A}_{E_8}} = 1$.*

2. *Assume that $\{\cdot|\cdot\}$ is one of the scalar products of part 2 of Lemma 4.2. Let $\beta_\Lambda \in L^\circ$ with $\langle \beta_\Lambda, \lambda_w \rangle \in \frac{1}{3} + \mathbb{Z}$ for a generator w of Λ, and put $g_\Lambda \overset{\text{def}}{=} \mathrm{e}^{2\pi\mathrm{i}\beta_\Lambda(0)}$. Then g_Λ is a unitary automorphism of order three of H_Λ with fixed point algebra V', and $\mathcal{A}_{V'} = \mathcal{A}_{H_\Lambda}^{\langle g_\Lambda \rangle}$. The net $\mathcal{A}_{V'}$ is completely rational, $\mu_{\mathcal{A}_{V'}} = 9$, $[\mathcal{A}_{H_\Lambda} : \mathcal{A}_{V'}] = 3$ and $\mathcal{A}_{V'}(I)' \cap \mathcal{A}_{H_\Lambda}(I) = \mathbb{C}$ for any interval I. Moreover $\mathcal{A}_{V'}$ is isomorphic to $\mathcal{A}_{E_6,1} \otimes \mathcal{A}_{SU(3),1}$.*

Proof : 1. The lattice vertex operator algebra of E_8 coincides with $V(E_8, 1)$ [Ten19b, Proof of Theorem 7.4 p. 71]. With its unitary structure, it has bounded localized vertex operators [Ten19b, Theorem 7.4 p. 71], it is strongly local with net $\mathcal{A}_{E_8}$ [CKLW18, Example 8.7 p. 63], and the nets of [Ten19b] and [CKLW18] are isomorphic [Ten19b, Corollary 7.7 p. 72]. Let $\psi\colon H_\Lambda \to V(E_8, 1)$ be an isomorphism of vertex operator algebras. The pull back by ψ of the scalar product of $V(E_8, 1)$ is a normalized invariant scalar product on H_Λ. By [CKLW18, Proposition 5.19 p. 40], it is of the form $\{hu|hv\}$ for some automorphism h of H_Λ. Then $\psi \circ h^{-1}$ is a unitary isomorphism from $(H_\Lambda, \{\cdot|\cdot\})$ onto $V(E_8, 1)$, and all the properties are transported. Here bounded localized vertex operators refer to the system of generalized annuli fixed in [Ten19b, Section 7]. A unitary isomorphism of vertex operator algebras preserves the conformal vectors, hence intertwines the representations of the Virasoro algebra and the operators attached to the annuli, so that it transports these properties and the nets. Finally, by [Ten19b, Proposition 7.5 p. 72], the net of [Ten19b] of $V(E_8, 1)$ is isomorphic to the lattice net of E_8 of Dong and Xu, since E_8 is the code lattice of the extended Hamming code [Ten19b, Example 7.2 p. 70]. By [DoX06, Corollary 3.19 p. 14], this net is completely rational with μ-index $|E_8^\circ/E_8| = 1$.

2. The map g_Λ acts on $M(1) \otimes \mathrm{e}^\lambda$ by the scalar $\mathrm{e}^{2\pi\mathrm{i}\langle \beta_\Lambda, \lambda \rangle}$, which is 1 on V' and a primitive cube root of unity on V'_w for $w \neq 0$. Such an element β_Λ exists since the pairing $L^\circ/L_\Lambda \times L_\Lambda/L \to \mathbb{Q}/\mathbb{Z}$ induced by $\langle \cdot, \cdot \rangle$ is perfect. Hence g_Λ is an automorphism of order three with fixed point algebra V'. The sectors $M(1) \otimes \mathrm{e}^\lambda$ are pairwise orthogonal for H_1, since β pairs the sector of λ with the sector of $-\lambda$ and ϕ_1 exchanges them. So g_Λ is unitary for H_1. The automorphism

$\tilde{h} = \exp(x(0))$, $x \in (V')_1$, of Lemma 4.2 commutes with g_Λ, since $g_\Lambda x(0) g_\Lambda^{-1} = (g_\Lambda x)(0) = x(0)$. Hence g_Λ is also unitary for $\{\cdot|\cdot\}$. By [CKLW18, Theorem 6.9 p. 53 and Proposition 7.6 p. 58], g_Λ defines a vacuum preserving automorphism of the net $\mathcal{A}_{H_\Lambda}$ and $\mathcal{A}_{V'} = \mathcal{A}_{H_\Lambda}^{\langle g_\Lambda \rangle}$. The group $\langle g_\Lambda \rangle$ acts faithfully and properly in the sense of [DoX06, Definition 2.1 p. 5]. Indeed, if $g \in \langle g_\Lambda \rangle$ acts trivially on $\mathcal{A}_{H_\Lambda}(I)$ for some interval I, then $gx\Omega = x\Omega$ for any $x \in \mathcal{A}_{H_\Lambda}(I)$, so that $g = 1$ since Ω is cyclic for $\mathcal{A}_{H_\Lambda}(I)$. By part 1 and [DoX06, Theorem 2.4 p. 6], the net $\mathcal{A}_{V'}$ is completely rational with $\mu_{\mathcal{A}_{V'}} = 3^2 \mu_{\mathcal{A}_{H_\Lambda}} = 9$.

The vertex operator algebra V' contains the conformal vector of H_Λ, so that $\mathcal{A}_{V'}(I)' \cap \mathcal{A}_{H_\Lambda}(I) = \mathbb{C}$ by Lemma 4.4 below. In particular $\mathcal{A}_{V'}(I) \subset \mathcal{A}_{H_\Lambda}(I)$ is an irreducible subfactor. By part 2 of Proposition 2.5, its index is finite, and by part 1 of the same proposition, $[\mathcal{A}_{H_\Lambda} : \mathcal{A}_{V'}]^2 = \mu_{\mathcal{A}_{V'}}/\mu_{\mathcal{A}_{H_\Lambda}} = 9$. Finally $V' \cong V_{E_6} \otimes V_{A_2}$, and the Frenkel–Kac construction gives $V_{E_6} \cong V_{\mathfrak{e}_6,1}$ and $V_{A_2} \cong V_{\mathfrak{sl}_3,1}$. Transporting the scalar product along this isomorphism as in part 1, the last assertion follows from parts 1 and 5 of Proposition 2.4. ∎

4.2 Strong locality of S

Lemma 4.4 *Let V be a strongly local simple unitary vertex operator algebra and let W be a unitary subalgebra of V containing the conformal vector of V. Then for any interval I we have*

$$\mathcal{A}_W(I)' \cap \mathcal{A}_V(I) = \mathcal{A}_W(S^1)' \cap \mathcal{A}_V(I) = \mathbb{C},$$

where $\mathcal{A}_W(S^1)$ is the von Neumann algebra generated by the algebras $\mathcal{A}_W(J)$.

Proof : The first equality holds for any covariant subnet of an irreducible conformal net [CKLW18, Example 3.7 p. 24], [Kös04]. By [CKLW18, Proposition 7.8 p. 58], the coset subnet $\mathcal{A}_W(S^1)' \cap \mathcal{A}_V(I)$ is $\mathcal{A}_{W^c}(I)$, where W^c is the coset of W in V. By [CKLW18, Proposition 5.31 p. 45], the conformal vector ω of V decomposes as $\omega = \omega^W + \omega^{W^c}$. Here $\omega^W = \omega$ since $\omega \in W$, so that $\omega^{W^c} = 0$ and $W^c = \mathbb{C}\Omega$ by [CKLW18, Remark 5.30 p. 45]. Hence $\mathcal{A}_{W^c}(I) = \mathbb{C}$. ∎

Theorem 4.5 *Let $W \subset V'$ be a vertex subalgebra containing the conformal vector of V' which is a unitary subalgebra for a normalized invariant scalar product $\{\cdot|\cdot\}$ on V'. Let Λ be an isotropic line and equip H_Λ with the scalar product $\{\cdot|\cdot\}_\Lambda$ of Lemma 4.2.*

1. *W is strongly local, and $\mathcal{A}_W \subset \mathcal{A}_{V'} \subset \mathcal{A}_{H_\Lambda}$ are covariant subnets, where $\mathcal{A}_{H_\Lambda} \cong \mathcal{A}_{E_8}$ and $[\mathcal{A}_{H_\Lambda} : \mathcal{A}_{V'}] = 3$.*
2. *The coset subnets of $\mathcal{A}_W$ in $\mathcal{A}_{V'}$ and in $\mathcal{A}_{H_\Lambda}$ are trivial, and the inclusions are irreducible: $\mathcal{A}_W(I)' \cap \mathcal{A}_{H_\Lambda}(I) = \mathbb{C}$ and $\mathcal{A}_{V'}(I)' \cap \mathcal{A}_{H_\Lambda}(I) = \mathbb{C}$ for any interval I.*

In particular, all these statements hold for the algebra S of [ArhI, Theorem 10.10], with the scalar product H_χ.

Proof : By Lemma 4.2, the vertex operator algebra V' is a unitary subalgebra of $(H_\Lambda, \{\cdot|\cdot\}_\Lambda)$. Since W contains the conformal vector, $L_1 W \subset W$, and the PCT operator of H_Λ restricts to that of V', which preserves W. Hence W is a unitary subalgebra of H_Λ by [CKLW18, Proposition 5.23 p. 43]. By Proposition 4.3, H_Λ is strongly local, and by part 2 of Proposition 2.4, the unitary subalgebras W and V' are strongly local with nets embedded as covariant subnets. The remaining assertions follow from Proposition 4.3 and Lemma 4.4. For S, the hypothesis holds by [ArhI, Theorem 10.10], and the scalar product $\{\cdot|\cdot\}_\Lambda$ is H_χ by part 3 of Lemma 4.2. ∎

Remark 4.6 By part 1 of Proposition 2.4, the net $\mathcal{A}_W$ does not depend, up to isomorphism, on the normalized invariant scalar product on W.

4.3 Submodules and representations

Let V be a strongly local simple unitary vertex operator algebra and W a unitary subalgebra containing the conformal vector of V. Let e_W be the orthogonal projection onto the closure $\mathcal{H}_W$ of W. By [CKLW18, Theorem 7.1 p. 55], the conformal net of W is $I \mapsto \mathcal{B}_W(I)e_W$, where $\mathcal{B}_W(I) = \mathcal{A}_V(I) \cap \{e_W\}'$. Let $\mathcal{C}_W(I)$ be the von Neumann algebra generated by the closed operators $Y(w,f)$, $w \in W$, $\operatorname{supp} f \subset I$. These operators are affiliated with $\mathcal{A}_V(I)$ and commute with e_W, and the restriction of $Y(w,f)$ to $\mathcal{H}_W$, with domain $D(Y(w,f)) \cap \mathcal{H}_W$, is the smeared vertex operator of W [CKLW18, proof of Theorem 7.1 p. 55]. Hence $\mathcal{C}_W(I) \subset \mathcal{B}_W(I)$. Since the reduction by e_W of the von Neumann algebra generated by closed operators commuting with e_W is the von Neumann algebra generated by their restrictions to $\mathcal{H}_W$, the algebra $\mathcal{C}_W(I)e_W$ is generated by the smeared vertex operators of W, so that $\mathcal{C}_W(I)e_W = \mathcal{B}_W(I)e_W$ [CKLW18, Theorem 7.1 p. 55]. Since Ω is separating for $\mathcal{A}_V(I)$, the map $x \mapsto xe_W$ is injective on $\mathcal{B}_W(I)$, and we obtain $\mathcal{C}_W(I) = \mathcal{B}_W(I)$. We denote this algebra by $\mathcal{A}_W(I)$. The restriction map $r_I \colon \mathcal{A}_W(I) \to \mathcal{A}_W(I)|_{\mathcal{H}_W}$ is an isomorphism of von Neumann algebras onto the local algebra of the net of W. Let $M \subset V$ be a W-submodule. Since $\omega \in W$, the subspace M is graded by the eigenvalues of L_0. We denote by $\mathcal{H}_M$ its closure in $\mathcal{H}_V$.

Lemma 4.7 *1. We have $V = M \oplus (M^\perp \cap V)$ as W-modules. For any interval I, the subspace $\mathcal{H}_M$ is invariant under $\mathcal{A}_W(I)$, and, for any $w \in W$ and any $f \in C^\infty(S^1)$, the restriction $Y_M(w,f) \stackrel{\text{def}}{=} Y(w,f)|_{\mathcal{H}_M}$, with domain $D(Y(w,f)) \cap \mathcal{H}_M$, is a closed operator on $\mathcal{H}_M$, and M is a core for it. The formula $\pi_M(x) \stackrel{\text{def}}{=} r_I^{-1}(x)|_{\mathcal{H}_M}$, for $x \in \mathcal{A}_W(I)|_{\mathcal{H}_W}$, defines a representation π_M of the conformal net of W on $\mathcal{H}_M$.*

2. If M is a simple W-module, then π_M is irreducible.

3. Let V_1 and V_2 be strongly local simple unitary vertex operator algebras which both contain W as a unitary subalgebra with the same conformal vector, the scalar products of V_1 and V_2 restricting to the same scalar product on W. Let $M_1 \subset V_1$ and $M_2 \subset V_2$ be simple W-submodules. Then the representations π_{M_1} and π_{M_2} are unitarily equivalent if and only if the W-modules M_1 and M_2 are isomorphic.

Proof : 1. Let P be the orthogonal projection onto $\mathcal{H}_M$. Since the homogeneous spaces are finite-dimensional and orthogonal, P maps V onto M and $(1-P)V = M^\perp \cap V$ is the sum of the orthogonal complements of the spaces $M(n) \stackrel{\text{def}}{=} M \cap V_n$ in V_n. Let $w \in W$ be homogeneous. By [CKLW18, (95) p. 43], the adjoint of a mode w_n on V is a finite linear combination of modes of the vectors $L_1^j \theta w$, which lie in W. Hence the modes of w preserve M and $M^\perp \cap V$. Consequently, the operator $Y_0(w,f)$ with domain V satisfies $PY_0(w,f)b = Y_0(w,f)Pb$ for any $b \in V$. Since $Y(w,f)$ is the closure of $Y_0(w,f)$ [CKLW18, p. 49] and $PV \subset V$, we obtain $PY(w,f) \subset Y(w,f)P$. The same holds for the adjoint, so that P commutes with the von Neumann algebra generated by the operators $Y(w,f)$, $\operatorname{supp} f \subset I$, that is, with $\mathcal{A}_W(I)$. If $\xi \in \mathcal{H}_M$ belongs to the domain of $Y(w,f)$, and if $b_k \in V$ satisfies $b_k \to \xi$ and $Y(w,f)b_k \to Y(w,f)\xi$, then $Pb_k \in M$, $Pb_k \to \xi$ and $Y(w,f)Pb_k = PY(w,f)b_k \to Y(w,f)\xi$. Hence M is a core for $Y_M(w,f)$. The compatibility of the maps π_M with the inclusions of intervals holds by construction.

Since the normal isomorphism r_I maps the operator $Y(w,f)$ affiliated with $\mathcal{A}_W(I)$ to its restriction to $\mathcal{H}_W$, the representation π_M maps the smeared vertex operator $Y(w,f)|_{\mathcal{H}_W}$ of W to $Y_M(w,f)$.

2. We first show that $A \subset L_0|_{\mathcal{H}_M}$ for the operator A defined below. Choose real functions $f_1, f_2 \in C^\infty(S^1)$ with support in proper intervals and $f_1 + f_2 = 1$. The vector ω is quasi-primary

and fixed by θ, so that $Y(\omega, f_j)$ is symmetric by [CKLW18, (113) p. 50]. Hence the operator $A = Y_M(\omega, f_1) + Y_M(\omega, f_2)$ with domain $D(Y_M(\omega, f_1)) \cap D(Y_M(\omega, f_2))$ is symmetric. On M, it coincides with $Y_0(\omega, 1) = L_0$. The restriction of L_0 to $\mathcal{H}_M$ is self-adjoint and M is a core for it. Thus the closure of A is a symmetric extension of a self-adjoint operator, so it is equal to it, and $A \subset L_0|_{\mathcal{H}_M}$.

Let $T \in \pi_M(\mathcal{A}_W)'$ be a projection. By part 1, $TY_M(w, f) \subset Y_M(w, f)T$ whenever $\operatorname{supp} f$ is contained in a proper interval. For $\xi \in M$, we obtain $T\xi \in D(A)$ and $AT\xi = TA\xi = TL_0\xi$. Hence $L_0T\xi = TL_0\xi$, and T maps each eigenspace $M(n)$ of $L_0|_{\mathcal{H}_M}$ into itself. Now let $w \in W$ be homogeneous and $\xi \in M(k)$. The vector $Y_0(w, f)\xi = \sum_n \hat{f}_n w_n \xi$ has component $\hat{f}_n w_n \xi$ in $M(k-n)$. Comparing the components of $TY_M(w, f)\xi = Y_M(w, f)T\xi$, we obtain $\hat{f}_n T w_n \xi = \hat{f}_n w_n T\xi$. For any n there exists f supported in a proper interval with $\hat{f}_n \neq 0$, for instance $f(z) = z^n g(z)$ with $g \geqslant 0$ nonzero and supported in a proper interval. Hence $T|_M$ is an endomorphism of the simple W-module M, so it is a scalar, and $T \in \{0, 1\}$.

3. The restrictions to W of the normalized invariant bilinear forms of V_1 and V_2 are normalized invariant bilinear forms on W, hence they coincide [Li94]. Since the scalar products also coincide on W, the PCT operators of V_1 and V_2 restrict to the same antilinear map θ of W, and the adjoint formula [CKLW18, (95) p. 43] is the same in V_1 and V_2. If $\phi\colon M_1 \to M_2$ is an isomorphism of W-modules, then $(u, v) \mapsto (\phi u|\phi v)$ is an invariant scalar product on M_1. The invariant sesquilinear forms on M_1 correspond to the module maps from the simple module $\overline{M_1}$, with conjugate scalars and the action of w given by that of θw, to the simple contragredient module M_1'. Hence they form a space of dimension at most one. Hence ϕ is a positive multiple of a unitary map $u\colon M_1 \to M_2$. It preserves the grading and extends to a unitary $u\colon \mathcal{H}_{M_1} \to \mathcal{H}_{M_2}$. Since M_k is a core for $Y_{M_k}(w, f)$, we obtain $uY_{M_1}(w, f)u^* = Y_{M_2}(w, f)$ for any $w \in W$ and any f with support in I. By part 1, the normal representations $x \mapsto u\pi_{M_1}(x)u^*$ and π_{M_2} of $\mathcal{A}_W(I)|_{\mathcal{H}_W}$ map the closed operator $Y(w, f)|_{\mathcal{H}_W}$ to the same closed operator. Hence they coincide on the partial isometries and on the spectral projections of the moduli of the polar decompositions of these operators, which generate $\mathcal{A}_W(I)|_{\mathcal{H}_W}$. Thus u intertwines π_{M_1} and π_{M_2}.

Conversely, let $T\colon \mathcal{H}_{M_1} \to \mathcal{H}_{M_2}$ be a unitary intertwiner. The operator $\left(\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right)$ commutes with $(\pi_{M_1} \oplus \pi_{M_2})(\mathcal{A}_W(I))$, with which the closed operator $Y_{M_1}(w, f) \oplus Y_{M_2}(w, f)$ is affiliated by part 1. Hence $TY_{M_1}(w, f) \subset Y_{M_2}(w, f)T$. The argument of part 2, applied to the operators $A_k = Y_{M_k}(\omega, f_1) + Y_{M_k}(\omega, f_2) \subset L_0|_{\mathcal{H}_{M_k}}$, shows that T maps $M_1(n)$ into $M_2(n)$ and that $T|_{M_1}\colon M_1 \to M_2$ is a nonzero morphism of W-modules. Since M_1 and M_2 are simple, it is an isomorphism. ∎

Remark 4.8 In the framework of bounded localized vertex operators, the statements of Lemma 4.7 are counterparts of [Ten19b, Proposition 5.4 p. 50, Proposition 5.5 p. 51 and Theorem 5.6 p. 52].

5 Unitarity of fusion products and complete rationality

In Section 5.1, V is a general simple regular unitary vertex operator algebra. From Section 5.2 on, $U \subset V'$ is a Haagerup algebra which is a vertex subalgebra of V' containing the conformal vector of V'. The main example is $U = S$ under ($\mathrm{H_J}$).

5.1 Positivity of transport forms and unitarity of fusion products

Let V be a simple regular unitary vertex operator algebra and let M, N be unitary V-modules. Throughout this subsection, modules are ordinary grading-restricted modules. By regularity

they are finite direct sums of irreducible ordinary modules. Their conformal weights are rational [DLM00, Theorem 11.3 p. 48], in particular real, and their L_0-eigenspaces are finite-dimensional. All forms and module homomorphisms are initially defined on the algebraic finite-energy subspaces. Tener defined the transport form $\langle\cdot,\cdot\rangle_{z,w}$ on $M\otimes N$, for (z,w) in the domain $\mathcal{R}$ of [Ten24, (3.5) p. 24], by means of the annihilation operator $\mathcal{Y}_M^-$ of M and of the vertex operator of N [Ten24, Definition 3.1 p. 25]. A transport module for M and N is a unitary V-module K with a dominant intertwining operator $\mathcal{Y}$ of type $\binom{K}{MN}$ belonging to the space $\mathcal{I}_{\mathrm{Hilb}}$ of [Ten24, Definition 2.8 p. 12], such that $\langle a_1\otimes b_1, a_2\otimes b_2\rangle_{z,w} = \langle \mathcal{Y}(a_1,z)b_1, \mathcal{Y}(a_2,w)b_2\rangle$ on $\mathcal{R}$ [Ten24, Definition 3.3 p. 26]. Tener proved that the fusion product $M\boxtimes N$ is a transport module as soon as the transport form is positive semidefinite at one point (z,z) of $\mathcal{R}$, under the hypothesis that $M\boxtimes N$ admits a unitary structure [Ten24, Proposition 3.16 p. 36], and he noted that the unitarizability of $M\boxtimes N$ is open in general [Ten24, Remark 3.18 p. 37]. We show that this hypothesis holds in the setting of bounded localized vertex operators when the representation π^N exists. We denote by $P = M\boxtimes N$ the fusion product, by $\mathcal{Y}_\boxtimes$ its universal intertwining operator, by $P(s)$ the eigenspace of L_0 for the eigenvalue s and by P_s the projection onto $P(s)$ defined on the algebraic completion of P.

Lemma 5.1 *Let Σ be a countable subset of $[s_0,\infty)$ for some $s_0\in\mathbb{R}$, let $\varepsilon>0$ and let $(c_s)_{s\in\Sigma}$ be complex numbers with $\sum_s |c_s|\mathrm{e}^{\varepsilon s}<\infty$. If $\sum_s |p(s)|^2 c_s \geqslant 0$ for every polynomial $p\in\mathbb{C}[X]$, then $c_s\geqslant 0$ for every $s\in\Sigma$.*

Proof : Let $\nu=\sum_s |c_s|\delta_s$, a finite positive measure on $[s_0,\infty)$. Let $f\in \mathrm{L}^2(\nu)$ be orthogonal to all polynomials. For $\operatorname{Im}\zeta > -\frac{\varepsilon}{2}$ and $s\geqslant s_0$, we have $|\mathrm{e}^{\mathrm{i}\zeta s}|\leqslant C_\zeta \mathrm{e}^{\frac{\varepsilon}{2}s}$ for a constant C_ζ which is locally bounded in ζ, and $\int |f|\mathrm{e}^{\frac{\varepsilon}{2}s}\,\mathrm{d}\nu \leqslant \|f\|_2(\int \mathrm{e}^{\varepsilon s}\,\mathrm{d}\nu)^{\frac12}<\infty$. Hence the function $F(\zeta)=\int f(s)\mathrm{e}^{\mathrm{i}\zeta s}\,\mathrm{d}\nu(s)$ is holomorphic on the half-plane $\operatorname{Im}\zeta>-\frac{\varepsilon}{2}$, and all its derivatives at 0 vanish. Thus $F=0$, the Fourier transform of the finite complex measure $f\nu$ vanishes, and $f=0$ in $\mathrm{L}^2(\nu)$. So the polynomials are dense in $\mathrm{L}^2(\nu)$. Fix $t\in\Sigma$ and choose polynomials p_n with $\|p_n-1_{\{t\}}\|_2\to 0$. Using $\big||\alpha|^2-|\beta|^2\big|\leqslant|\alpha-\beta|(|\alpha|+|\beta|)$, we obtain

$$\Big|\sum_s |p_n(s)|^2 c_s - c_t\Big| \leqslant \int \big||p_n|^2-1_{\{t\}}\big|\,\mathrm{d}\nu \leqslant \|p_n-1_{\{t\}}\|_2\big(\|p_n\|_2+\|1_{\{t\}}\|_2\big) \xrightarrow[n\to\infty]{} 0.$$

Since the sums on the left are nonnegative, $c_t\geqslant 0$. ∎

Proposition 5.2 *Let V be a simple regular unitary vertex operator algebra with bounded localized vertex operators, and let M and N be ordinary unitary V-modules such that the representation π^N of [Ten24, Definition 2.18 p. 20] exists. Then $M\boxtimes N$ admits an invariant scalar product for which $(M\boxtimes N,\mathcal{Y}_\boxtimes)$ is a transport module for M and N.*

Proof : By regularity the ordinary modules M and N are finite direct sums of simple modules. These decompositions can be chosen orthogonal by invariance of their scalar products. The annihilation operator $\mathcal{Y}_M^-$ pairs only matching summands of $\overline{M}$ and M, by its definition as the adjoint of the creation operator. The vertex operator of N preserves each summand, and π^N restricts to the representation of each summand by [Ten19b, Proposition 5.5]. Consequently the transport kernel is the orthogonal direct sum of the kernels for the pairs of summands. The tensor product distributes over these finite direct sums and the sum of the corresponding universal intertwining operators is universal. It therefore suffices to prove the proposition for simple M and N.

A nondegenerate invariant Hermitian form. The unitary forms identify $\overline{M}$ with M' and $\overline{N}$ with N'. For any ordinary module X, the complex conjugate space $\overline{X}$ is a module via

$Y_{\overline{X}}(v,x)\overline{a} = \overline{Y_X(\theta v, x)a}$, with coefficientwise conjugation and θ the PCT operator of V. Conjugation of intertwining operators [Gui22, (1.25)] acts on coefficients and uses no scalar product on the target. Hence the conjugate of $\mathcal{Y}_{\boxtimes}$ is universal and $\overline{P} \cong \overline{M} \boxtimes \overline{N}$. Rigidity identifies P' with $N' \boxtimes M'$, and braiding gives $M' \boxtimes N' \cong P'$ [Hua08].

The space $\mathcal{F} \cong \mathrm{Hom}_V(\overline{P}, P')$ of invariant sesquilinear forms on P therefore contains a nondegenerate form. Here a form B is invariant if

$$B(Y_P(v,x)a,b) = B(a, Y_P(\mathrm{e}^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1})b).$$

The involution $B \mapsto B^*$, with $B^*(a,b) = \overline{B(b,a)}$, preserves $\mathcal{F}$ and its fixed points are a real form of $\mathcal{F}$. Since L_0 is symmetric for B, its real eigenspaces are mutually orthogonal. The radical in the first variable is a submodule. If it is nonzero it contains a simple submodule whose lowest weight is among the finitely many lowest weights of the simple summands of P. It follows that nondegeneracy is detected by the product of the determinants of B on these finitely many eigenspaces. This polynomial is nonzero on $\mathcal{F}$, so it cannot vanish identically on its real form. Choose an invariant nondegenerate Hermitian form Q_0 on P in this way. It is nondegenerate on every $P(s)$.

An algebraic adjoint. By associativity and the universal property of $\mathcal{Y}_{\boxtimes}$, there is an intertwining operator $\mathcal{Z}$ of type $\binom{N}{\overline{M}\ P}$ such that

$$Y^N(\mathcal{Y}^-_{\overline{M}}(c, z_1 - z_2)a, z_2)b = \mathcal{Z}(c, z_1)\mathcal{Y}_{\boxtimes}(a, z_2)b$$

as germs of matrix coefficients in the overlap $|z_1| > |z_2| > |z_1 - z_2| > 0$. Take positive real base points with $0 < z_1 - z_2 < z_2 < z_1$ and real logarithms. Convergence and associativity here are for ordinary modules paired with their graded duals [Hua05]. The product uses P only as an ordinary module.

Write $Q_P = Q_0$ and $Q_N = \langle \cdot, \cdot \rangle_N$, linear in the first variable. The maps $h_X : X \to (\overline{X})'$ given by $h_X(x)(\overline{y}) = Q_X(x,y)$, for $X = P, N$, are isomorphisms of modules by invariance and graded nondegeneracy. For the conjugate intertwining operator $\overline{\mathcal{Z}}$ of type $\binom{\overline{N}}{M\ \overline{P}}$, let $A_r(\overline{\mathcal{Z}})$ denote its contragredient as in [HLZ12, (3.87) and Proposition 3.46]. It uses the factor $\mathrm{e}^{xL_1}\mathrm{e}^{(2r+1)\pi \mathrm{i} L_0}x^{-2L_0}$ on the first input. Define the intertwining operator of type $\binom{P}{M\ N}$ by

$$\mathcal{Z}^\dagger = h_P^{-1} A_{-1}(\overline{\mathcal{Z}})(\mathrm{id}_M, h_N).$$

This definition is graded and applies to the possibly indefinite form Q_0. It is a bijection of the corresponding spaces of intertwining operators. With conjugate logarithms at w and $\overline{w}$, put $c = \mathrm{e}^{\overline{w}L_1}\mathrm{e}^{\mathrm{i}\pi L_0}(\overline{w})^{-2L_0}\overline{a_2}$, as in [Ten24, (3.4) and (3.6)]. The plus sign in the rotation factor is obtained by conjugating the factor $\mathrm{e}^{-\mathrm{i}\pi L_0}$ in A_{-1}. For $\xi \in P(s)$ the defining graded pairing gives

$$\langle \mathcal{Z}(c, \overline{w}^{-1})\xi, b_2 \rangle_N = Q_0\big(\xi, P_s \mathcal{Z}^\dagger(a_2, w)b_2\big).$$

By universality $\mathcal{Z}^\dagger = \Gamma \mathcal{Y}_{\boxtimes}$ for a V-endomorphism Γ of P. Its adjoint $\Lambda = \Gamma^\sharp$ is taken grade by grade for Q_0. Invariance of Q_0 makes Λ a V-endomorphism, and the last pairing is $Q_0(\Lambda\xi, P_s \mathcal{Y}_{\boxtimes}(a_2, w)b_2)$.

Nondegeneracy. Let $u \in \mathrm{Hom}_V(\overline{M} \boxtimes P, N)$ correspond to the intertwining operator of which $\mathcal{Y}_{\boxtimes}$ is the algebraic adjoint just defined. By the graded pairing, $\mathcal{Z}$ corresponds to $u \circ (\mathrm{id}_{\overline{M}} \boxtimes \Lambda)$. At the positive real base points above, the associator of the $P(z)$-tensor products identifies the iterate with the product. Transporting the tensor products to $P(1)$ along the positive real axis gives, with the associativity convention of [Ten24, (3.21)–(3.22)],

$$u \circ (\mathrm{id}_{\overline{M}} \boxtimes \Lambda) = \ell_N \circ (y \boxtimes \mathrm{id}_N) \circ \alpha^{-1}.$$

Here $\alpha : (\overline{M} \boxtimes M) \boxtimes N \to \overline{M} \boxtimes P$ is the associator, ℓ_N is the left unitor corresponding to Y^N, and $y : \overline{M} \boxtimes M \to V$ corresponds to $\mathcal{Y}_M^-$. Since M is simple, rigidity gives $y = d_M \operatorname{ev}_M$, where ev_M is the evaluation for the dual $\overline{M} \cong M'$ and $d_M \in \mathbb{C}$. Creation is nonzero on the vacuum, so annihilation is nonzero and $d_M \neq 0$. Define

$$X = (\mathrm{id}_M \boxtimes u) \circ \alpha' \circ (\operatorname{coev}_M \boxtimes \mathrm{id}_P) \circ \ell_P^{-1},$$

where $\alpha' : (M \boxtimes \overline{M}) \boxtimes P \to M \boxtimes (\overline{M} \boxtimes P)$. Naturality of the associators and the triangle identity for evaluation and coevaluation give $X\Lambda = d_M \mathrm{id}_P$. Thus Λ is injective. It preserves the finite-dimensional spaces $P(s)$, so it is invertible. Therefore $Q = Q_0(\Lambda \cdot, \cdot)$ is invariant and nondegenerate and makes distinct eigenspaces of L_0 orthogonal.

The transport identity. Set $z_1 = \overline{w}^{-1}$ and $z_2 = z$. The product converges absolutely on $\mathcal{R}$ because $|z_1| > |z_2|$. Pairing with the algebraic vector b_2 uses the graded dual of N. Substituting $\xi = P_s \mathcal{Y}_{\boxtimes}(a_1, z) b_1$ into the adjoint identity and summing gives the product expansion

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{z,w} = \sum_s Q\big(P_s \mathcal{Y}_{\boxtimes}(a_1, z) b_1, P_s \mathcal{Y}_{\boxtimes}(a_2, w) b_2\big). \tag{5.1}$$

To see that the equality holds throughout $\mathcal{R}$, start near a point $z = w \in (2^{-1/2}, 1)$, where the product and the iterate agree as germs with the chosen real branches. For homogeneous inputs, the summand on the right is a coefficient times $z^{s-h_1} \overline{w}^{s-h_2}$, where h_i is the sum of the weights of a_i and b_i. The common branch convention of [Ten24, Section 3.1] makes the sum single-valued, since a common change of logarithm has phase $\mathrm{e}^{2\pi \mathrm{i}(h_2 - h_1)} = 1$ for the simple modules M and N. After factoring out the input powers, the series is in the variable $t = z\overline{w}$ with the logarithm on the right half-plane. Absolute convergence at $z = w = \sqrt{\rho}$, $1/2 < \rho < 1$, gives normal convergence on compact subsets with $|t| < \rho$, since the weights are bounded below. Finally $\mathcal{R}$ is connected: in coordinates (t, z) it is given by $\operatorname{Re} t > 1/2$ and $|t| < |z| < 1$. Thus the germ identity continues to the entire domain, with the single-valued iterate of [Ten24, Definition 3.1] on the left, which converges absolutely on $\mathcal{R}$ by [Ten24, Proposition 3.11 p. 33].

Positivity of the transport form. By [Ten24, Proposition 3.11 p. 33], $\mathcal{Y}_M^-$ belongs to the space $\mathcal{I}_{\mathrm{Hilb}}$ of [Ten24, Definition 2.8 p. 12]. By [Ten24, Proposition 4.2 p. 43], whose hypotheses do not involve P, there is $r < 1$ such that the form $\langle \cdot, \cdot \rangle_{z,z}$ on $M \otimes N$ is positive semidefinite whenever $(z, z) \in \mathcal{R}$ and $|z| > r$. Fix a real number z with $\max(r, 2^{-\frac{1}{2}}) < z < 1$, so that $(z, z) \in \mathcal{R}$. For $x \in M \otimes N$, put $\xi = \mathcal{Y}_{\boxtimes}(x, z)$, $\xi_s = P_s \xi$ and $c_s(x) = Q(\xi_s, \xi_s)$. By (5.1), $\sum_s c_s(x) = \langle x, x \rangle_{z,z} \geqslant 0$.

Invariance under L_0. For an intertwining operator $\mathcal{Y}$, we have $[L_0, \mathcal{Y}(a, x)] = x \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{Y}(a, x) + \mathcal{Y}(L_0 a, x)$ and $\frac{\mathrm{d}}{\mathrm{d}x} \mathcal{Y}(a, x) = \mathcal{Y}(L_{-1} a, x)$. Hence

$$L_0 \mathcal{Y}_{\boxtimes}(a, z) b = \mathcal{Y}_{\boxtimes}(L_0 a + z L_{-1} a, z) b + \mathcal{Y}_{\boxtimes}(a, z) L_0 b,$$

where L_0 acts componentwise on the algebraic completion $\prod_s P(s)$ of P, which contains the vectors $\mathcal{Y}_{\boxtimes}(x, z)$, $x \in M \otimes N$, defined by linearity. With the operator $T_z \overset{\text{def}}{=} (L_0 + z L_{-1}) \otimes \mathrm{id}_N + \mathrm{id}_M \otimes L_0$ of $M \otimes N$, we obtain $p(L_0) \mathcal{Y}_{\boxtimes}(x, z) = \mathcal{Y}_{\boxtimes}(p(T_z) x, z)$ for any polynomial p and any $x \in M \otimes N$. Put $x_p \overset{\text{def}}{=} p(T_z) x$. Since $P_s p(L_0) \xi = p(s) \xi_s$, we get $\sum_s |p(s)|^2 c_s(x) = \langle x_p, x_p \rangle_{z,z} \geqslant 0$.

Exponential decay. Let $1 < \lambda < z^{-1}$. If $x = \sum_k a_k \otimes b_k$ with homogeneous a_k and b_k of weights h_k and h'_k, put $x' = \sum_k \lambda^{h_k + h'_k} a_k \otimes b_k$. Then $P_s \mathcal{Y}_{\boxtimes}(x', \lambda z) = \lambda^s \xi_s$, since $\lambda^{L_0} \mathcal{Y}_{\boxtimes}(a, z) b = \mathcal{Y}_{\boxtimes}(\lambda^{L_0} a, \lambda z) \lambda^{L_0} b$. The absolute convergence of (5.1) at the point $(\lambda z, \lambda z) \in \mathcal{R}$ gives $\sum_s \lambda^{2s} |c_s(x)| < \infty$. The weights of P are bounded below.

Conclusion. By Lemma 5.1, $c_s(x) \geqslant 0$ for all s and all x. For homogeneous $a \in M$ and $b \in N$ of weights h and h', we have $P_s\mathcal{Y}_{\boxtimes}(a,z)b = z^{-n-1}a_{(n)}b$ with $n = h + h' - s - 1$. By the commutator formula, the coefficients $a_{(n)}b$ span a submodule of P, which is equal to P by uniqueness in the universal property of $(P, \mathcal{Y}_{\boxtimes})$ and semisimplicity. That is, $\mathcal{Y}_{\boxtimes}$ is dominant. Hence every $v \in P(s)$ can be written $v = \sum_k (a_k)_{(n_k)}b_k = P_s\mathcal{Y}_{\boxtimes}(x,z)$ with $x = \sum_k z^{n_k+1}a_k \otimes b_k$, and $Q(v,v) = c_s(x) \geqslant 0$. By polarization, Q is a positive semidefinite Hermitian form. It is invariant and nondegenerate, hence positive definite, and P is a unitary V-module for Q. Finally, (5.1) reads $\langle x, y\rangle_{z,w} = Q(\mathcal{Y}_{\boxtimes}(x,z), \mathcal{Y}_{\boxtimes}(y,w))$ on $\mathcal{R}$, and $\mathcal{Y}_{\boxtimes}$ is dominant and lies in $\mathcal{I}_{\mathrm{Hilb}}$ by [Ten24, Proposition 3.11 p. 33], so that $(P, \mathcal{Y}_{\boxtimes})$ is a transport module. ■

Remark 5.3 The proof of [Ten24, Proposition 3.16 p. 36] uses a Hilbert completion of an already unitary fusion product. Here nondegeneracy comes from the algebraic adjoint and rigidity, and positivity follows from Lemma 5.1. The only integration used in Proposition 5.2 is that of the second input, through [Ten24, Proposition 4.2]. In particular, if π^M exists, the proposition constructs the transport module for $(\overline{M}, M)$ in the order required by [Ten24, Theorem 4.19 p. 55]. This supplies directly the transport module that the proof of [Ten24, Theorem 4.20 p. 55] obtains from [Ten24, Corollary 4.5 p. 44], which assumes, through [Ten24, Theorem 4.4 p. 44], that $\overline{M} \boxtimes M$ admits a unitary structure. This assumption is not listed in the statement of [Ten24, Theorem 4.20], see Remark 5.10.

5.2 Unitarity of the modules and complete rationality

Lemma 5.4 *The U-module V' decomposes as $V' = U \oplus M$ with $M \cong M_b$. Consequently, for each isotropic line Λ, the vertex operator algebra H_Λ of Lemma 4.1 is a holomorphic extension of U containing $V' = U \oplus M_b$, and the facts recalled in Section 2.4 apply to $U \subset V' \subset H_\Lambda$. In particular, the isotropic lines Λ_a and Λ_{c_0} are defined, and for $w \in \Lambda_x \setminus \{0\}$, $x \in \{a, c_0\}$, the U-module V'_w is simple and isomorphic to M_x.*

Proof : The vertex operator algebra V' is a simple extension of U, and $V' \cong (V')'$. By Lemma 2.3 applied to $W = B = V'$ and by [HKL15, Theorem 3.4 p. 16], the algebra A in $\mathrm{Rep}(U)$ attached to V' is a connected étale algebra, and the category $\mathrm{Rep}(U)^0_A$ is isomorphic to $\mathrm{Rep}(V')$, which has nine simple objects [Don93]. Hence $\mathrm{FPdim}\,\mathrm{Rep}(U)^0_A \geqslant 9$, and by (2.7) and (2.6), we have $\mathrm{FPdim}(A)^2 \leqslant \frac{9(3d+2)^2}{9}$, that is, $\mathrm{FPdim}(A) \leqslant 3d+2$. As in the proof of Lemma 3.2, A is a direct sum of simple objects with trivial twist containing 0 once. By (2.4), the only such objects of Frobenius–Perron dimension at most $3d+2$ are 0 and $0 \oplus b$. Since $V' \neq U$, we obtain $A \cong 0 \oplus b$. The other assertions follow, since $H_\Lambda \cong V_{E_8}$ is holomorphic and contains V' by Lemma 4.1. Here the part of the proof of [ArhI, Lemma 4.3] which shows that exactly two nonzero classes restrict to M_{c_0} uses only the decomposition $V' = U \oplus M_b$, the twists and the modular transformation of the one-point functions, so that it applies to the given embedding $U \subset V'$. ■

Assume now that U is a unitary subalgebra of V' for a normalized invariant scalar product, and equip each H_Λ with the scalar product of part 2 of Lemma 4.2. Let N_b be the orthogonal complement of U in V', which is a U-submodule isomorphic to M_b by Lemma 5.4 and part 1 of Lemma 4.7. With the notation of Lemma 4.7, we put

$$\pi_0 \stackrel{\mathrm{def}}{=} \pi_U, \quad \pi_b \stackrel{\mathrm{def}}{=} \pi_{N_b}, \quad \pi_x \stackrel{\mathrm{def}}{=} \pi_{V'_w} \text{ for } w \in \Lambda_x \setminus \{0\},\ x \in \{a, c_0\},$$

where V'_w is viewed as a U-submodule of H_{Λ_x}. By part 3 of Lemma 4.7, π_x does not depend on the choice of w up to unitary equivalence, and π_0 is the vacuum representation of $\mathcal{A}_U$.

Corollary 5.5 *Let $U \subset V'$ be a Haagerup algebra containing the conformal vector of V' which is a unitary subalgebra for a normalized invariant scalar product on V'. Then every simple U-module admits a scalar product which is invariant with respect to the PCT operator of U induced by this scalar product. In particular, under* ($\mathrm{H_J}$)*, every simple S-module admits a scalar product which is invariant for the PCT operator of S induced by H_χ, which is the assertion* ($\mathrm{H_U}$) *of the introduction.*

Proof : Fix an isotropic line Λ and put $H = H_\Lambda$, with the scalar product of part 2 of Lemma 4.2. By part 1 of Proposition 4.3, H has bounded localized vertex operators. As in the proof of Theorem 4.5, U is a unitary subalgebra of H containing its conformal vector, so that U has bounded localized vertex operators, for the same system of generalized annuli [Ten19b, Proposition 4.16 p. 43]. The U-module N_b is a unitary U-submodule of H isomorphic to M_b, and the representation π^{N_b} exists by [Ten19b, Theorem 5.6 p. 52], applied to the H-module H, for which π^H is the vacuum representation. The strongly rational algebra U is regular [ABD04, Theorem 4.5]. By Proposition 5.2, $N_b \boxtimes N_b$ admits an invariant scalar product. In the normalized S-matrix of [EvG11, (2.5a)–(2.5b) p. 20], put $x = \frac{13-3\sqrt{13}}{26}$ and $y = \frac{3}{\sqrt{13}}$. Then $x(1-x) = \frac{1}{13}$ and $1-2x = y$. Outside the columns 0 and b, the squares of the entries in the rows 0 and b agree. Since all simple objects are self-dual, $S^2 = 1$, and the Verlinde formula gives

$$N_{bb}^k = \delta_{0k} + \frac{1-2x}{3}\left(\frac{S_{k0}}{x} - \frac{S_{kb}}{1-x}\right).$$

For $k = 0$ the added term vanishes. For $k \in \{b, a, c_0, c_1, c_2\}$ it is $\frac{(1-2x)^2}{9x(1-x)} = 1$, and for $k = d_l$ it is $\frac{(1-2x)y}{9x(1-x)} = 1$. Thus $b \otimes b$ is the direct sum of the twelve simple objects, each with multiplicity one. Hence every simple U-module is isomorphic to a submodule of $N_b \boxtimes N_b$, and the restriction of the scalar product to this submodule is invariant. The last assertion follows, since under ($\mathrm{H_J}$) the algebra S is a Haagerup algebra which contains the conformal vector of V' and is a unitary subalgebra of V' [ArhI, Theorem 10.10 and Theorem 10.24]. ■

Theorem 5.6 *Let $U \subset V'$ be a Haagerup algebra containing the conformal vector of V' which is a unitary subalgebra for a normalized invariant scalar product on V'.*

1. *The conformal net $\mathcal{A}_U$ is completely rational. For $x \in \{a, c_0\}$, the index $[\mathcal{A}_{H_{\Lambda_x}} : \mathcal{A}_U]$ is finite.*

2. *The representations π_0, π_b, π_a, π_{c_0} are irreducible and pairwise inequivalent. The vacuum representation of $\mathcal{A}_{V'}$ restricts to $\mathcal{A}_U$ as $\pi_0 \oplus \pi_b$, and the vacuum representation of $\mathcal{A}_{H_{\Lambda_x}}$ restricts to $\mathcal{A}_U$ as $\pi_0 \oplus \pi_b \oplus 2\pi_x$.*

3. *Put $\delta \overset{\text{def}}{=} d(\pi_b)$. Then $\delta \geqslant 1$, $[\mathcal{A}_{V'} : \mathcal{A}_U] = 1 + \delta$, $d(\pi_a) = d(\pi_{c_0}) = 1 + \delta$, $[\mathcal{A}_{H_{\Lambda_x}} : \mathcal{A}_U] = 3(1+\delta)$ and $\mu_{\mathcal{A}_U} = 9(1+\delta)^2$.*

4. *For $x \in \{a, c_0\}$, the dual canonical endomorphism Θ_x of $\mathcal{A}_U \subset \mathcal{A}_{H_{\Lambda_x}}$ is a Lagrangian algebra in $\mathrm{Rep}(\mathcal{A}_U)$, and the fusion category $\mathcal{F}_x \overset{\text{def}}{=} \mathrm{Rep}(\mathcal{A}_U)_{\Theta_x}$ is a unitary fusion category of rank six and Frobenius–Perron dimension $3(1+\delta)$ such that $\mathrm{Rep}(\mathcal{A}_U)$ is braided equivalent to $Z(\mathcal{F}_x)$. The category $\mathcal{F}_x$ contains a pointed fusion subcategory of dimension three, corresponding to the intermediate net $\mathcal{A}_{V'}$.*

Proof : 1. Fix x and put $H = H_{\Lambda_x}$, with the scalar product of Lemma 4.2. By Proposition 4.3, H is strongly local, has bounded localized vertex operators, and its nets in the sense of

[Ten19b] and [CKLW18] are isomorphic. By the proof of Theorem 4.5, U is a unitary conformal subalgebra of H. The algebra U is regular by [ABD04, Theorem 4.5] and inherits bounded localized vertex operators by [Ten19b, Proposition 4.16].

Let M be a simple U-summand of H. The representations π^M and $\pi^{\overline{M}}$ of $\mathcal{A}_U^{\mathrm{T}}$ in the sense of [Ten24, Definition 2.18] exist by [Ten19b, Theorem 5.6 and Lemma 5.7], applied to M and to its image under the PCT operator of H, which realizes $\overline{M}$. With p_M the orthogonal projection onto M, skew symmetry gives $p_M Y^H(m,z)v = \mathcal{Y}_M^+(m,z)v$ for $m \in M$, $v \in U$. The common conformal vector makes the power shift in [Ten19b, Theorem 5.15] zero, so $\mathcal{Y}_M^+ \in \mathcal{I}_{\mathrm{loc}}$. Proposition 5.2, with inputs $\overline{M}, M$, makes $\overline{M} \boxtimes M$ a transport module, and rigidity and semisimplicity give a copy of U in it. Now [Ten24, Theorem 4.19] gives finite index for each such sector π^M.

The restriction π_H of the vacuum of $\mathcal{A}_H^{\mathrm{T}}$ to $\mathcal{A}_U^{\mathrm{T}}$ is a finite sum of these sectors and has finite index. For a complementary interval I' the inclusion

$$\pi_H(\mathcal{A}_U^{\mathrm{T}}(I)) \subset \mathcal{A}_H^{\mathrm{T}}(I) \subset \pi_H(\mathcal{A}_U^{\mathrm{T}}(I'))'$$

then gives finite index for the subnet by multiplicativity of index, as in the last step of [Ten24, Corollary 4.24]. The ambient net is completely rational by part 1 of Proposition 4.3, so part 1 of Proposition 2.5 gives complete rationality of $\mathcal{A}_U^{\mathrm{T}}$. By [Ten19b, Lemma 7.6 p. 72], this net is isomorphic to the net $\mathcal{A}_U$ of [CKLW18], which is therefore completely rational.

Now consider the inclusion $\mathcal{A}_U \subset \mathcal{A}_H$ of nets of [CKLW18]. By Theorem 4.5, its coset is trivial and $\mathcal{A}_U(I)' \cap \mathcal{A}_H(I) = \mathbb{C}$. In particular, $\mathcal{A}_H$ is an irreducible local extension of $\mathcal{A}_U$ in the sense of [KaL04, p. 9]. Since $\mathcal{A}_U$ is completely rational, the index $[\mathcal{A}_H : \mathcal{A}_U]$ is finite by part 2 of Proposition 2.5. The same argument applies to the inclusion $\mathcal{A}_U \subset \mathcal{A}_{V'}$.

2. As a U-module, $H = U \oplus N_b \oplus V'_w \oplus V'_{-w}$ for $w \in \Lambda_x \setminus \{0\}$, and the summands are pairwise orthogonal. Indeed, the lattice sectors are orthogonal for H_1, the automorphism $\tilde{h} = \exp(y(0))$, $y \in (V')_1$, of Lemma 4.2 preserves each V'_w, and N_b is the orthogonal complement of U in V'. Hence $\mathcal{H}_H = \mathcal{H}_U \oplus \mathcal{H}_{N_b} \oplus \mathcal{H}_{V'_w} \oplus \mathcal{H}_{V'_{-w}}$, and by part 1 of Lemma 4.7, the vacuum representation of $\mathcal{A}_H$ restricts to $\pi_0 \oplus \pi_b \oplus \pi_{V'_w} \oplus \pi_{V'_{-w}}$. By Lemma 5.4, the summands are simple U-modules, isomorphic to U, M_b, M_x, M_x. These are pairwise non-isomorphic except for the last two. By parts 2 and 3 of Lemma 4.7, we obtain the decomposition $\pi_0 \oplus \pi_b \oplus 2\pi_x$ into irreducible representations, with π_0, π_b, π_x pairwise inequivalent. Applying part 3 of Lemma 4.7 to $V_1 = H_{\Lambda_a}$, $V_2 = H_{\Lambda_{c_0}}$ and to the non-isomorphic U-modules M_a and M_{c_0}, we see that π_a and π_{c_0} are inequivalent. The same argument applied to $\mathcal{H}_{V'} = \mathcal{H}_U \oplus \mathcal{H}_{N_b}$ gives the restriction of the vacuum representation of $\mathcal{A}_{V'}$.

3. By part 1 and part 3 of Proposition 2.5, the dual canonical endomorphism of $\mathcal{A}_U \subset \mathcal{A}_{V'}$ is $\theta = \pi_0 \oplus \pi_b$, so that $[\mathcal{A}_{V'} : \mathcal{A}_U] = d(\theta) = 1 + \delta$. Here $\delta \geqslant 1$, as for any statistical dimension. By part 1 of Proposition 2.5 applied to the two finite index inclusions $\mathcal{A}_U \subset \mathcal{A}_{V'}$ and $\mathcal{A}_U \subset \mathcal{A}_H$, and by Proposition 4.3, we have

$$\mu_{\mathcal{A}_U} = [\mathcal{A}_{V'} : \mathcal{A}_U]^2 \mu_{\mathcal{A}_{V'}} = 9(1+\delta)^2 \quad \text{and} \quad \mu_{\mathcal{A}_U} = [\mathcal{A}_H : \mathcal{A}_U]^2 \mu_{\mathcal{A}_H} = [\mathcal{A}_H : \mathcal{A}_U]^2.$$

Hence $1 + \delta + 2d(\pi_x) = d(\Theta_x) = [\mathcal{A}_H : \mathcal{A}_U] = 3(1+\delta)$, so that $d(\pi_x) = 1 + \delta$.

4. By part 3 of Proposition 2.5 and part 1, Θ_x is a commutative Q-system with

$$\dim \mathrm{Hom}(\mathrm{id}, \Theta_x) = \dim \mathcal{A}_U(I)' \cap \mathcal{A}_H(I) = 1,$$

hence a connected étale algebra in the unitary modular tensor category $\mathrm{Rep}(\mathcal{A}_U)$. By (2.10), $\mathrm{FPdim}\,\mathrm{Rep}(\mathcal{A}_U) = \mu_{\mathcal{A}_U} = d(\Theta_x)^2$, so that Θ_x is Lagrangian. By part 1 of Proposition 2.2, $\mathrm{Rep}(\mathcal{A}_U)$ is braided equivalent to $Z(\mathcal{F}_x)$, and (2.7) gives $\mathrm{FPdim}(\mathcal{F}_x) = \frac{9(1+\delta)^2}{3(1+\delta)} = 3(1+\delta)$. The

category $\mathcal{F}_x$ of Θ_x-modules is equivalent to the category of α-induced sectors of the inclusion $\mathcal{A}_U(I) \subset \mathcal{A}_H(I)$, which is a unitary fusion category [BKLR15, Section 4.6 p. 54], see also [Bis17a, Proposition 4.5 p. 21]. By Lemma 3.1 and part 2, its rank is $\dim \mathrm{Hom}(\Theta_x, \Theta_x) = 1+1+4 = 6$. By [BKLR15, Proposition 4.11 p. 49], the intermediate subfactor $\mathcal{A}_U(I) \subset \mathcal{A}_{V'}(I) \subset \mathcal{A}_H(I)$ corresponds to a projection in $\mathrm{Hom}(\Theta_x, \Theta_x)$ whose range, with the restricted multiplication, is the Q-system θ of $\mathcal{A}_U \subset \mathcal{A}_{V'}$. Hence θ is a connected étale subalgebra of Θ_x. By Lemma 3.1, under the equivalence $\mathrm{Rep}(\mathcal{A}_U) \cong Z(\mathcal{F}_x)$, the algebra Θ_x corresponds to the algebra $I(1)$, and by part 3 of Proposition 2.2, θ corresponds to a fusion subcategory $\mathcal{P}_x$ of $\mathcal{F}_x$ with $\mathrm{FPdim}(\theta)\,\mathrm{FPdim}(\mathcal{P}_x) = \mathrm{FPdim}(\mathcal{F}_x)$, that is, $\mathrm{FPdim}(\mathcal{P}_x) = 3$. A fusion category of prime Frobenius–Perron dimension is pointed [ENO05, Corollary 8.30 p. 37]. ■

Proposition 5.7 *Under the hypotheses of Theorem 5.6, there is a proper action of the hypergroup $K = \{k_0, k_1\}$ with*

$$k_1 k_1 = \frac{1}{\delta} k_0 + \Big(1 - \frac{1}{\delta}\Big) k_1$$

on the net $\mathcal{A}_{V'}$ such that $\mathcal{A}_U = \mathcal{A}_{V'}^K$. The weight of K is $1+\delta = [\mathcal{A}_{V'} : \mathcal{A}_U]$.

Proof : By part 1 of Theorem 5.6, the net $\mathcal{A}_U$ is completely rational and $[\mathcal{A}_{V'} : \mathcal{A}_U] < \infty$. By part 3 of Proposition 2.6, there is a canonical hypergroup K acting properly on $\mathcal{A}_{V'}$ with $\mathcal{A}_U = \mathcal{A}_{V'}^K$, and its weight is $[\mathcal{A}_{V'} : \mathcal{A}_U] = 1+\delta$. By part 2 of Theorem 5.6, the dual canonical endomorphism of $\mathcal{A}_U(I) \subset \mathcal{A}_{V'}(I)$ is $\theta = \pi_0 \oplus \pi_b$ with π_b irreducible and inequivalent to π_0. Hence $\langle \gamma, \gamma \rangle = \langle \theta, \theta \rangle = 2$, where γ is the canonical endomorphism, and by [Bis17b, Section 4.6 p. 25], the canonical hypergroup is $K = \{k_0, k_1\}$ with the stated multiplication, where $\delta = [\mathcal{A}_{V'} : \mathcal{A}_U] - 1$. The elements k_0, k_1 are denoted c_0, c_1 in [Bis17b]. ■

Proposition 5.8 *Under the hypotheses of Theorem 5.6, we have $\delta > 1$. Consequently, the fusion category $\mathcal{F}_x$ has exactly three invertible simple objects and three non-invertible simple objects.*

Proof : Assume that $\delta = 1$. By Proposition 5.7, there is a proper action of a hypergroup $K = \{k_0, k_1\}$ on $\mathcal{A}_{V'}$ with $\mathcal{A}_U = \mathcal{A}_{V'}^K$ and $k_1 k_1 = \delta^{-1} k_0 + (1 - \delta^{-1}) k_1 = k_0$. Thus the vacuum preserving unital completely positive maps $\sigma_I = \phi_{k_1}^I$ of $\mathcal{A}_{V'}(I)$ satisfy $\sigma_I^2 = \mathrm{id}$. By the Kadison–Schwarz inequality and the positivity of σ_I, we have $x^*x = \sigma_I^2(x^*x) \geqslant \sigma_I(\sigma_I(x)^*\sigma_I(x)) \geqslant x^*x$. Since σ_I is injective and positive, we obtain $\sigma_I(x^*x) = \sigma_I(x)^*\sigma_I(x)$ for any x, so that σ_I is multiplicative by the multiplicative domain argument, hence an automorphism. It is nontrivial since $\mathcal{A}_U \neq \mathcal{A}_{V'}$. Since σ_I preserves the vacuum state and Ω is cyclic for $\mathcal{A}_{V'}(I)$, the formula $u_I x \Omega = \sigma_I(x)\Omega$ defines a unitary u_I with $u_I x u_I^* = \sigma_I(x)$ and $u_I \Omega = \Omega$. The maps σ_I are compatible [Bis17b, Proposition 4.21 p. 23], so that $u_I = u_J$ if $I \subset J$. Since any two intervals contain subintervals which are contained in a common interval, $g \stackrel{\mathrm{def}}{=} u_I$ does not depend on I, and g is an automorphism of the net $\mathcal{A}_{V'}$ in the sense of [CKLW18, (41) p. 21]. By [CKLW18, Theorem 6.9 p. 53], g is a unitary automorphism of V' of order two. By [CKLW18, Proposition 7.6 and Theorem 7.5 p. 58], we obtain $U = (V')^g$, so that U_1 is the fixed point subalgebra of the automorphism g of order at most two of $(V')_1 \cong \mathfrak{e}_6 \oplus \mathfrak{sl}_3$. The automorphism g preserves the two simple ideals $\mathfrak{e}_6$ and $\mathfrak{sl}_3$ of $(V')_1$, which are not isomorphic, and it is nontrivial on $(V')_1$ since V' is generated by $(V')_1$. The fixed point subalgebra of an involution of $\mathfrak{e}_6$ is isomorphic to $\mathfrak{sp}_8$, $\mathfrak{sl}_6 \oplus \mathfrak{sl}_2$, $\mathfrak{so}_{10} \oplus \mathbb{C}$ or $\mathfrak{f}_4$, of dimension 36, 38, 46 or 52, and that of an involution of $\mathfrak{sl}_3$ is isomorphic to $\mathfrak{so}_3$ or $\mathfrak{gl}_2$, of dimension 3 or 4 [Kac90, Chapter 8], [Hel78, Chapter X]. Taking into account the case where g is trivial on one of the two ideals, of dimensions 78 and 8, we obtain $\dim U_1 \geqslant \min(36+3, 36+8, 78+3) = 39$, which contradicts $\dim U_1 = 6$ [ArhI, Theorem

7.1]. A similar argument, with the rank of the fixed point subalgebra, appears in [Hua26b, pp. 7–8].

The invertible objects of $\mathcal{F}_x$ form a group containing the group of order three of $\mathcal{P}_x$. Since $\mathcal{F}_x$ has rank six, the order of this group is three or six by Lagrange's theorem. If this order were six, then $\mathcal{F}_x$ would be pointed of dimension $6 = 3(1+\delta)$, so that $\delta = 1$. ■

Corollary 5.9 *Assume* ($\mathrm{H_J}$). *Then Theorem 5.6 holds for $U = S$, with the unitary structure of [ArhI, Theorem 10.10]. In particular, $\mathcal{A}_S$ is completely rational, and* $\mathrm{Rep}(\mathcal{A}_S)$ *is braided equivalent to the Drinfeld center of a unitary fusion category of rank six with a pointed fusion subcategory of dimension three.*

Proof : By [ArhI, Theorem 10.24], ($\mathrm{H_J}$) implies that S is a Haagerup algebra. It contains the conformal vector of V' and is a unitary subalgebra for H_χ by [ArhI, Theorem 10.10]. ■

Remark 5.10 The proof of Theorem 5.6 uses the transport modules $\overline{M} \boxtimes_t M$ of Proposition 5.2 for the simple summands M of H_Λ, together with [Ten24, Theorem 4.19]. Corollary 5.5 uses the proposition for a different purpose. Under ($\mathrm{H_J}$), the fusion rules of $Z(\mathcal{H}_3)$ show that $M_b \boxtimes M_b$ contains every simple module, so the invariant scalar product on this product also unitarizes the six modules M_{d_l}. This use of the fusion rules is separate from the index argument.

For the algebra of [Hua26b], the complete rationality of the net and the unitarity of all simple modules are announced in [Hua26b, Theorem 1 p. 1 and Section 6 pp. 8–9]. In the outline of [Hua26b, Section 6 p. 8], the finiteness of the index of the subnet generated by the bounded insertions is obtained from [Ten24] and the regularity of the algebra, without reference to a specific statement, and the unitarity of the simple modules is derived afterwards from the twelve sectors of the resulting completely rational net, through [HeT25, Theorem 9.5 and Remark 9.8]. The finite index statements [Ten24, Theorem B and Corollary C p. 6, Theorem 4.20 p. 55 and Corollaries 4.21 and 4.24 pp. 55–56] do not list a unitarity hypothesis, but their proofs use [Ten24, Corollary 4.5 p. 44], which requires a unitary structure on $\overline{M} \boxtimes M$, see also [Ten24, Remark 3.18 p. 37]. For $M = M_b$, this module contains every simple module, including the six modules M_{d_l}. Proposition 5.2 supplies the transport module for algebras with bounded localized vertex operators as soon as π^M exists. Hence this step of the outline can be completed as in the proofs of Corollary 5.5 and Theorem 5.6, provided that the algebra of [Hua26b] is realized as a unitary subalgebra of V' containing its conformal vector, as in [ArhIII, Remark 7.14]. The determination of the fusion rules and of the associator in [Hua26b, Section 5 pp. 6–8] only uses the pseudo-unitarity of the category of modules over the extension algebra, and it does not depend on this issue. The application of the classification of C^*-generalized Haagerup categories [Izu18, Theorem 5.9 p. 36] in [Hua26b, Section 6 p. 9] uses the unitarity of this category, which is deduced there from the complete unitarity of the algebra. In Section 6, we apply this classification to the category $\mathcal{F}_x$ of the net, which is unitary by construction, see Remark 6.12.

The value $\delta = d^2$ is obtained in Theorem 6.11.

6 A recognition theorem for the Haagerup categories

In this section we prove Theorem 6.1, which is Theorem 1.4 of the introduction, and we deduce from it that each category $\mathcal{F}_x$ of Theorem 5.6 is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$ (Theorem 6.11). The strategy for the fusion rules, which combines the induction to the Drinfeld center, Frobenius–Schur indicators and lower bounds for sums of roots of unity, is the one outlined by Huang

[Hua26b, Sections 5 and 6 pp. 7–9], and we compare our proofs with this outline in Remark 6.12.

Throughout this section, $\mathcal{F}$ is a unitary fusion category. We equip it with the spherical structure for which the dimension $d(X)$ of every simple object X is its Frobenius–Perron dimension [ENO05, Proposition 8.23 p. 34], and we put $\dim \mathcal{F} \stackrel{\text{def}}{=} \sum_{X \in \mathrm{Irr}(\mathcal{F})} d(X)^2$, where $\mathrm{Irr}(\mathcal{F})$ is the set of isomorphism classes of simple objects. We denote by $F\colon Z(\mathcal{F}) \to \mathcal{F}$ the forgetful functor and by $I\colon \mathcal{F} \to Z(\mathcal{F})$ its adjoint. Since both categories are semisimple, we have $\dim \mathrm{Hom}(Z, I(V)) = \dim \mathrm{Hom}(I(V), Z) = \dim \mathrm{Hom}(F(Z), V)$ for any objects Z of $Z(\mathcal{F})$ and V of $\mathcal{F}$. We denote by θ_Z the twist of a simple object Z of $Z(\mathcal{F})$, which is a root of unity [NgS07a, p. 9 and Theorem 5.5 p. 13], and by $K(\mathcal{F})$ the Grothendieck ring of $\mathcal{F}$. For an object V of $\mathcal{F}$ and $n \geqslant 1$, the Frobenius–Schur indicator $\nu_n(V)$ is the trace of the linear map $E_V^{(n)}$ of $\mathrm{Hom}(1, V^{\otimes n})$ defined in [NgS07b, p. 8]. We have $(E_V^{(n)})^n = \mathrm{id}$ [NgS07b, Theorem 5.1 p. 12]. In particular, $\nu_1(V) = \dim \mathrm{Hom}(1, V)$, and if $\mathrm{Hom}(1, V^{\otimes n})$ is one-dimensional, then $\nu_n(V)$ is an n-th root of unity. By [NgS07a, Theorem 4.1 p. 10], we have

$$\nu_n(V) = \frac{1}{\dim \mathcal{F}} \sum_{Z \in \mathrm{Irr}(Z(\mathcal{F}))} \theta_Z^n d(Z) \dim \mathrm{Hom}(F(Z), V). \tag{6.1}$$

We equip $Z(\mathcal{F})$ with the braiding of [NgS07a, Theorem 4.1 p. 10]. For the reverse braiding, the twists are replaced by their complex conjugates [NgS07a, Remark 4.3 p. 10]. Since the right-hand sides of (6.3)–(6.5) below are real, the only effect of this replacement on Lemma 6.4 and Proposition 6.9 is that $\nu_3(\alpha)$ is replaced by $\overline{\nu_3(\alpha)}$ in the identities $\zeta^3 = \theta_{Z_i}^3 = \nu_3(\alpha)$.

Theorem 6.1 *Let $\mathcal{F}$ be a unitary fusion category with six simple objects up to isomorphism, three of which are invertible, and with noncommutative Grothendieck ring.*

1. *There exist an invertible object α of order three and a simple object ρ such that the simple objects of $\mathcal{F}$ are 1, α, α^2, ρ, $\alpha\rho$, $\alpha^2\rho$ up to isomorphism, the objects $\alpha^j\rho$ are self-dual, and*
$$\rho\alpha \cong \alpha^2\rho, \quad \rho^2 \cong 1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho.$$
In particular, $\mathcal{F}$ has the fusion rules (2.2) of $\mathcal{H}_2$ and $\mathcal{H}_3$, and $d(\rho) = d$.

2. *If the fusion subcategory of $\mathcal{F}$ generated by α is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator, then $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$.*

Conversely, $\mathcal{H}_2$ and $\mathcal{H}_3$ satisfy the hypotheses of part 2.

6.1 The fusion rules

Lemma 6.2 *Let $\mathcal{F}$ be as in Theorem 6.1. Then there exist an invertible object α of order three, a simple object ρ and an integer $m \geqslant 1$ such that the simple objects of $\mathcal{F}$ are 1, α, α^2, ρ, $\alpha\rho$, $\alpha^2\rho$, up to isomorphism, and*

$$\rho\alpha \cong \alpha^2\rho, \quad (\alpha^j\rho)^* \cong \alpha^j\rho \ (j \in \{0,1,2\}), \quad \rho^2 \cong 1 \oplus m(\rho \oplus \alpha\rho \oplus \alpha^2\rho).$$

Consequently, $d(\rho) = \frac{3m+\sqrt{9m^2+4}}{2}$ and $d(\rho)^2 = 1 + 3md(\rho)$.

Proof : The invertible simple objects form a group of order three, generated by some α. Let $\mathcal{X}$ be the set of the isomorphism classes of the three non-invertible simple objects. The group acts on $\mathcal{X}$ by left and by right multiplication.

The algebra $K(\mathcal{F}) \otimes \mathbb{C}$ is semisimple. Indeed, consider the basis $\mathrm{Irr}(\mathcal{F})$, the involution $X \mapsto X^*$ extended antilinearly, and the linear form τ which gives the coefficient of 1. We have $\tau(X^*Y) = \delta_{X,Y}$ for simple X, Y, so that $\tau(x^*x) > 0$ for $x \neq 0$. Hence $\langle x, y\rangle \stackrel{\text{def}}{=} \tau(y^*x)$ defines a scalar product on $K(\mathcal{F}) \otimes \mathbb{C}$, for which left multiplication is a faithful $*$-representation. Thus $K(\mathcal{F}) \otimes \mathbb{C}$ is a finite-dimensional C^*-algebra, hence semisimple. Since $K(\mathcal{F}) \otimes \mathbb{C}$ has dimension six and is noncommutative, it is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$, which has two characters.

Assume that $\alpha X \cong X$ for every $X \in \mathcal{X}$. For $X \in \mathcal{X}$, we have $X^* \in \mathcal{X}$, so that $(X\alpha)^* \cong \alpha^2 X^* \cong X^*$ and $X\alpha \cong X$. Hence, for $X, Y \in \mathcal{X}$, the objects XY and αXY are isomorphic, and the invertible summands of XY form a multiple of $1 \oplus \alpha \oplus \alpha^2$. For a primitive cube root of unity ω, the linear map $\chi_\omega \colon K(\mathcal{F}) \to \mathbb{C}$ with $\chi_\omega(\alpha^j) = \omega^j$ and $\chi_\omega(X) = 0$ for $X \in \mathcal{X}$ is therefore a character. The two characters χ_ω, χ_{ω^2} and the Frobenius–Perron character are three distinct characters, a contradiction. Since the orbits of a group of order three have one or three elements, the left action of the group on $\mathcal{X}$ is free, and $\mathcal{X} = \{\rho, \alpha\rho, \alpha^2\rho\}$ for any $\rho \in \mathcal{X}$. If $\rho\alpha \cong \rho$, then $\alpha^2\rho^* \cong (\rho\alpha)^* \cong \rho^*$, which contradicts the freeness. Hence the right action is free as well.

Write $\rho\alpha \cong \alpha^s\rho$. We have $s \neq 0$, since the right action is free. If $s = 1$, then α commutes with every simple object, and $(\alpha^i\rho)(\alpha^j\rho) \cong \alpha^{i+j}\rho^2 \cong (\alpha^j\rho)(\alpha^i\rho)$, so that $K(\mathcal{F})$ is commutative, a contradiction. Hence $s = 2$, and $\rho\alpha^i \cong \alpha^{2i}\rho$ for any i. Write $\rho^* \cong \alpha^k\rho$. Then $(\alpha^j\rho)^* \cong \rho^*\alpha^{-j} \cong \alpha^k\rho\alpha^{-j} \cong \alpha^{k+j}\rho$. Applying this twice, $\alpha^j\rho \cong (\alpha^j\rho)^{**} \cong \alpha^{2k+j}\rho$, so that $k = 0$ by freeness, and every non-invertible simple object is self-dual.

Since ρ is self-dual, $\dim \mathrm{Hom}(\alpha^i, \rho^2) = \dim \mathrm{Hom}(\alpha^i\rho, \rho) = \delta_{i,0}$, so that $\rho^2 \cong 1 \oplus \bigoplus_j n_j\alpha^j\rho$ for some integers $n_j \geqslant 0$. Moreover $\rho^2\alpha \cong \rho\alpha^2\rho \cong \alpha\rho^2$. The coefficients of $\alpha^k\rho$ in $\alpha\rho^2$ and in $\rho^2\alpha$ are n_{k-1} and n_{k-2}, so that $n_0 = n_1 = n_2 \stackrel{\text{def}}{=} m$. If $m = 0$, then ρ is invertible, a contradiction. Taking Frobenius–Perron dimensions gives $d(\rho)^2 = 1 + 3md(\rho)$. ∎

By Lemma 6.2, $\mathcal{F}$ is a quadratic category with $(\mathbb{Z}_3, -1, m)$ in the sense of [Izu18, Definition 2.1 p. 6]. We put $d_m \stackrel{\text{def}}{=} d(\rho)$ and $\Sigma_\rho \stackrel{\text{def}}{=} \rho \oplus \alpha\rho \oplus \alpha^2\rho$. Then $d_m - d_m^{-1} = 3m$, $d_m + d_m^{-1} = \sqrt{9m^2+4}$ and

$$\dim \mathcal{F} = 3 + 3d_m^2 = 3(2 + 3md_m). \tag{6.2}$$

For $m = 1$, we recover $d_1 = d$ and the fusion rules (2.2).

6.2 The Drinfeld center

Lemma 6.3 *Let $\mathcal{F}$ be as in Theorem 6.1, with the notation of Lemma 6.2. The simple objects of $Z(\mathcal{F})$ are, up to isomorphism, of the following three kinds.*

1. *The summands of $I(1) \cong 1 \oplus B \oplus 2X$, where B and X are simple and non-isomorphic, $F(B) \cong 1 \oplus m\Sigma_\rho$, $F(X) \cong 2 \cdot 1 \oplus m\Sigma_\rho$, $d(B) = d_m^2$, $d(X) = 1 + d_m^2$ and $\theta_B = \theta_X = 1$.*
2. *The summands of $I(\alpha) \cong I(\alpha^2) \cong Z_1 \oplus Z_2 \oplus Z_3$, where the Z_i are simple and pairwise non-isomorphic, and $F(Z_i) \cong \alpha \oplus \alpha^2 \oplus k_i\Sigma_\rho$ with integers $k_i \geqslant 0$ such that $k_1 + k_2 + k_3 = 3m$.*
3. *Simple objects Y_j, $j \in J$, with $F(Y_j) \cong n_j\Sigma_\rho$ for some integers $n_j \geqslant 1$.*

Proof : Let Z be a simple object of $Z(\mathcal{F})$. The half-braiding gives $\alpha F(Z) \cong F(Z)\alpha$, so that $F(Z)$ is invariant under conjugation by α. By Lemma 6.2, $\alpha(\alpha^j\rho)\alpha^{-1} \cong \alpha^{j+1}\rho\alpha^2 \cong \alpha^{j+2}\rho$, so that conjugation by α permutes ρ, $\alpha\rho$, $\alpha^2\rho$ cyclically. Hence the multiplicities of ρ, $\alpha\rho$ and $\alpha^2\rho$ in $F(Z)$ are equal. Consequently, Z is a summand of $I(1)$, of $I(\alpha)$ or of $I(\alpha^2)$, or $F(Z)$ is a multiple of Σ_ρ.

By [ENO05, Proposition 5.4 p. 21], we have $F(I(V)) \cong \bigoplus_Y Y \otimes V \otimes Y^*$, the sum running over $\mathrm{Irr}(\mathcal{F})$. For $V = 1$, Lemma 6.2 gives $F(I(1)) \cong 3 \cdot 1 \oplus 3\rho^2 \cong 6 \cdot 1 \oplus 3m\Sigma_\rho$. For $V = \alpha$ and $Y = \alpha^j\rho$, we have $Y^* \cong \rho\alpha^{-j}$ and $Y\alpha Y^* \cong \alpha^j\rho\alpha\rho\alpha^{-j} \cong \alpha^{j+2}\rho^2\alpha^{-j} \cong \alpha^2\rho^2$, since $\rho^2\alpha \cong \alpha\rho^2$ by the proof of Lemma 6.2. Hence $F(I(\alpha)) \cong 3\alpha \oplus 3\alpha^2 \oplus 3m\Sigma_\rho$, and the same computation gives $F(I(\alpha^2)) \cong 3\alpha \oplus 3\alpha^2 \oplus 3m\Sigma_\rho$. By adjunction, we obtain

$$\dim \mathrm{End}(I(\alpha)) = \dim \mathrm{End}(I(\alpha^2)) = \dim \mathrm{Hom}(I(\alpha), I(\alpha^2)) = 3,$$
$$\dim \mathrm{Hom}(I(1), I(\alpha)) = \dim \mathrm{Hom}(I(1), I(\alpha^2)) = 0.$$

The first equalities show that $I(\alpha)$ and $I(\alpha^2)$ are both direct sums of three pairwise non-isomorphic simple objects, and that the three summands Z_1, Z_2, Z_3 of $I(\alpha)$ are summands of $I(\alpha^2)$, so that $I(\alpha^2) \cong I(\alpha)$. The second ones show that $F(Z_i)$ does not contain 1, and that $F(Z)$ contains neither α nor α^2 for any summand Z of $I(1)$. Since $\dim \mathrm{Hom}(F(Z_i), \alpha) = \dim \mathrm{Hom}(F(Z_i), \alpha^2) = 1$, we get $F(Z_i) \cong \alpha \oplus \alpha^2 \oplus k_i\Sigma_\rho$, and $\sum_i k_i = 3m$ by comparison with $F(I(\alpha))$.

By [Ost15, Theorem 2.13 p. 6], the simple summands of $I(1)$ are in bijection with the irreducible representations E of $K(\mathcal{F})$, the summand A_E attached to E has multiplicity $\dim E$ and dimension $\frac{\dim \mathcal{F}}{f_E}$, where f_E is the formal codegree of E. The unit object of $Z(\mathcal{F})$ has multiplicity one in $I(1)$, so that it is A_E for a one-dimensional E with $f_E = \dim \mathcal{F}$. A character χ of $K(\mathcal{F})$ satisfies $\chi(\alpha)\chi(\rho) = \chi(\rho\alpha) = \chi(\alpha)^2\chi(\rho)$ and $\chi(\rho)^2 = 1 + m\chi(\rho)(1 + \chi(\alpha) + \chi(\alpha)^2)$. The second relation gives $\chi(\rho) \neq 0$, so that $\chi(\alpha) = 1$ and $\chi(\rho) \in \{d_m, -d_m^{-1}\}$. The formal codegrees of these two characters are $\sum_Y \chi(Y)\chi(Y^*) = 3+3\chi(\rho)^2$, that is, $\dim \mathcal{F}$ and $3+3d_m^{-2}$. The corresponding summands of $I(1)$ are the unit object and a simple object B of dimension $\frac{\dim \mathcal{F}}{3+3d_m^{-2}} = d_m^2$ by (6.2). The summand X attached to the two-dimensional representation has multiplicity two, and $d(I(1)) = \dim \mathcal{F} = 1 + d_m^2 + 2d(X)$ gives $d(X) = 1 + d_m^2$. By adjunction, 1 has multiplicity one in $F(B)$ and two in $F(X)$. Writing $F(B) \cong 1 \oplus s\Sigma_\rho$ and $F(X) \cong 2 \cdot 1 \oplus s'\Sigma_\rho$, the dimensions give $1 + 3sd_m = d_m^2$ and $2 + 3s'd_m = 1 + d_m^2$, so that $s = s' = m$ by Lemma 6.2. Finally $\theta_{I(1)} = \mathrm{id}_{I(1)}$ by [Ost15, Theorem 2.5 p. 4]. ∎

Lemma 6.4 *In the situation of Lemma 6.3, put* $\Delta \overset{\text{def}}{=} 9m^2+4$ *and* $\varepsilon \overset{\text{def}}{=} \nu_2(\rho)$. *Then* $\varepsilon \in \{1, -1\}$ *and* $k_1 = k_2 = k_3 = m$. *The twists of* Z_1, Z_2, Z_3 *are* ζ, $\zeta\omega$, $\zeta\omega^2$ *for some root of unity* ζ *with* $\zeta^3 = \nu_3(\alpha)$ *and a primitive cube root of unity* ω. *Moreover*

$$\sum_{j \in J} n_j^2 = 4m^2 + 2, \tag{6.3}$$

$$\sum_{j \in J} n_j^2 \theta_{Y_j} = -\frac{m^2 + m\sqrt{\Delta}}{2}, \tag{6.4}$$

$$\sum_{j \in J} n_j^2 \theta_{Y_j}^2 = \frac{-m^2 + (2\varepsilon - m)\sqrt{\Delta}}{2}. \tag{6.5}$$

Proof : Since ρ is simple and self-dual, $\mathrm{Hom}(1, \rho^{\otimes 2})$ is one-dimensional, so that $\varepsilon \in \{1, -1\}$. Similarly, $\mathrm{Hom}(1, \alpha^{\otimes 3})$ is one-dimensional, so that $|\nu_3(\alpha)| = 1$. Moreover $\nu_1(\alpha) = \nu_1(\rho) = 0$. Put $\ell_i \overset{\text{def}}{=} d(Z_i) = 2 + 3k_i d_m > 0$, so that $\sum_i \ell_i = \dim \mathcal{F}$ by (6.2). By (6.1) and Lemma 6.3, we have

$$\nu_n(\alpha) = \frac{1}{\dim \mathcal{F}} \sum_{i=1}^{3} \theta_{Z_i}^n \ell_i.$$

For $n=3$, this gives $\sum_i \theta_{Z_i}^3 \ell_i = \nu_3(\alpha)\sum_i \ell_i$ with $|\nu_3(\alpha)|=1$. Since the ℓ_i are positive and the θ_{Z_i} have modulus one, we obtain $\theta_{Z_1}^3 = \theta_{Z_2}^3 = \theta_{Z_3}^3 = \nu_3(\alpha)$. Write $\theta_{Z_i} = \zeta u_i$ with $\zeta = \theta_{Z_1}$ and $u_i^3 = 1$. For $n=1$, we obtain $\sum_i u_i \ell_i = 0$. If two of the u_i coincide, then $\sum_i u_i \ell_i$ is a combination with positive coefficients of two cube roots of unity whose quotient is not real, or a positive multiple of a cube root of unity, which is nonzero. Hence the u_i are the three distinct cube roots of unity, and $1+\omega+\omega^2=0$ and the linear independence of 1 and ω over $\mathbb{R}$ give $\ell_1=\ell_2=\ell_3$, that is, $k_i=m$ for all i. In particular $d(Z_i) = 2+3md_m = 1+d_m^2$, and $\sum_i \theta_{Z_i}^n = 0$ for $n \in \{1,2\}$.

Since $d(Z) = d(F(Z))$ is the Frobenius–Perron dimension of Z for every simple object Z of $Z(\mathcal{F})$, we have $\sum_Z d(Z)^2 = (\dim \mathcal{F})^2 = 9(1+d_m^2)^2$ by [ENO05, Proposition 8.12 p. 32], and $d(Y_j) = 3n_j d_m$. By Lemma 6.3, we obtain

$$9d_m^2 \sum_j n_j^2 = 9(1+d_m^2)^2 - 1 - d_m^4 - (1+d_m^2)^2 - 3(1+d_m^2)^2 = 4d_m^4 + 10d_m^2 + 4,$$

and $\sum_j n_j^2 = \frac{4(d_m^2+d_m^{-2})+10}{9} = 4m^2+2$, since $d_m^2 + d_m^{-2} = (d_m - d_m^{-1})^2 + 2 = 9m^2+2$.

By (6.1) for $V=\rho$ and $n \in \{1,2\}$, using $\nu_1(\rho)=0$, $\nu_2(\rho)=\varepsilon$, Lemma 6.3 and $\sum_i \theta_{Z_i}^n = 0$, we get

$$md_m^2 + m(1+d_m^2) + 3d_m \sum_j n_j^2 \theta_{Y_j} = 0, \quad md_m^2 + m(1+d_m^2) + 3d_m \sum_j n_j^2 \theta_{Y_j}^2 = \varepsilon \dim \mathcal{F}.$$

Since $1+2d_m^2 = 3+6md_m$ and $d_m^{-1} = d_m - 3m$, the first equality gives $\sum_j n_j^2 \theta_{Y_j} = -m(d_m^{-1} + 2m) = m^2 - md_m$, which is (6.4) since $d_m = \frac{3m+\sqrt{\Delta}}{2}$. By (6.2), the second one gives $\sum_j n_j^2 \theta_{Y_j}^2 = \varepsilon(d_m + d_m^{-1}) + m^2 - md_m = \varepsilon\sqrt{\Delta} + m^2 - md_m$, which is (6.5). ∎

The identities (6.3), (6.4) and (6.5) are those of [Hua26b, (25) p. 7]. For $m=1$ and the Haagerup categories, the objects Y_j are $d_1, \dots, d_6$, with $n_j = 1$ and twists $\mathrm{e}^{\frac{2\pi \mathrm{i} 6l^2}{13}}$ by (2.4) and (2.5), since $d(Y_j) = 3n_j d$. Since $13 \equiv 1 \bmod 4$, and since $12 \equiv -1$ is a square modulo 13 whereas 6 is not, the quadratic Gauss sums give $\sum_{l=1}^6 \mathrm{e}^{\frac{2\pi \mathrm{i} 6l^2}{13}} = \frac{-1-\sqrt{13}}{2}$ and $\sum_{l=1}^6 \mathrm{e}^{\frac{2\pi \mathrm{i} 12l^2}{13}} = \frac{-1+\sqrt{13}}{2}$, which are (6.4) and (6.5) for $m=1$ and $\varepsilon=1$.

6.3 Sums of roots of unity

For $n \geqslant 1$, we denote by $r(n)$ the sum of the primitive n-th roots of unity, that is, the Möbius function of n. Thus $r(n) \in \{0,1,-1\}$, $r(n)=0$ if 4 divides n, $r(1)=1$, $r(2)=-1$, and $r(kl) = r(k)r(l)$ if k and l are coprime. We denote by φ the Euler function.

Let $D>1$ be a squarefree integer. For a root of unity ζ, choose $L \geqslant 1$ such that $\zeta^L = 1$ and $\sqrt{D} \in \mathbb{Q}(\mathrm{e}^{\frac{2\pi \mathrm{i}}{L}})$, put $G_L \overset{\text{def}}{=} \mathrm{Gal}(\mathbb{Q}(\mathrm{e}^{\frac{2\pi \mathrm{i}}{L}})/\mathbb{Q}(\sqrt{D}))$ and

$$\langle \zeta \rangle_D \overset{\text{def}}{=} \frac{1}{|G_L|} \sum_{\sigma \in G_L} \sigma(\zeta) = A_D(\zeta) + B_D(\zeta)\sqrt{D}, \quad A_D(\zeta), B_D(\zeta) \in \mathbb{Q}.$$

This does not depend on L, since the restriction maps $G_{L'} \to G_L$ for $L \mid L'$ are surjective with fibers of the same cardinality. If $\zeta_1, \dots, \zeta_N$ are roots of unity with $\sum_q \zeta_q = a + b\sqrt{D}$ for rational numbers a, b, then averaging over G_L for a common L gives

$$\sum_{q=1}^N A_D(\zeta_q) = a, \quad \sum_{q=1}^N B_D(\zeta_q) = b. \tag{6.6}$$

Lemma 6.5 *Let ζ be a root of unity of order n.*

1. *If 40 does not divide n, then $\langle\zeta\rangle_{10} = \frac{r(n)}{\varphi(n)}$.*
2. *If $n = 40k$ with $\gcd(k, 40) > 1$, then $\langle\zeta\rangle_{10} = 0$.*
3. *If $n = 40k$ with $\gcd(k, 40) = 1$, then $A_{10}(\zeta) = 0$ and $|B_{10}(\zeta)| = \frac{|r(k)|}{8\varphi(k)}$.*

Consequently, $|A_{10}(\zeta)| + 8|B_{10}(\zeta)| \leqslant 1$, with equality if and only if $\zeta \in \{1, -1\}$ or $n = 40$. If $\zeta_1, \dots, \zeta_N$ are roots of unity with $\sum_q \zeta_q = a+b\sqrt{10}$ for rational numbers a, b, then $N \geqslant |a|+8|b|$.

Proof : The field $\mathbb{Q}(\sqrt{10})$ has discriminant 40, so that $\sqrt{10} \in \mathbb{Q}(\zeta)$ if and only if 40 divides n. If $40 \nmid n$, then $\mathbb{Q}(\zeta) \cap \mathbb{Q}(\sqrt{10}) = \mathbb{Q}$, the group G_L maps onto $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, and $\langle\zeta\rangle_{10}$ is the average of the primitive n-th roots of unity, which is $\frac{r(n)}{\varphi(n)}$.

Assume that $n = 40k$, and take $L = n$. We identify $\mathrm{Gal}(\mathbb{Q}(\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}})/\mathbb{Q})$ with $(\mathbb{Z}/n\mathbb{Z})^\times$. The subgroup G_n has index two, and we denote by χ the character of $(\mathbb{Z}/n\mathbb{Z})^\times$ with kernel G_n. It factors through $(\mathbb{Z}/40\mathbb{Z})^\times$, since the kernel of the map $(\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/40\mathbb{Z})^\times$ fixes $\mathbb{Q}(\mathrm{e}^{\frac{2\pi\mathrm{i}}{40}})$, which contains $\sqrt{10}$. Put $\tau_n \stackrel{\mathrm{def}}{=} \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} \chi(u)\mathrm{e}^{\frac{2\pi\mathrm{i}u}{n}}$. Writing $\zeta = \mathrm{e}^{\frac{2\pi\mathrm{i}v}{n}}$ with $v \in (\mathbb{Z}/n\mathbb{Z})^\times$, we obtain

$$\langle\zeta\rangle_{10} = \frac{2}{\varphi(n)} \sum_{u\in G_n} \mathrm{e}^{\frac{2\pi\mathrm{i}uv}{n}} = \frac{1}{\varphi(n)} \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} (1+\chi(u))\mathrm{e}^{\frac{2\pi\mathrm{i}uv}{n}} = \frac{r(n)+\chi(v)\tau_n}{\varphi(n)} = \frac{\chi(v)\tau_n}{\varphi(n)},$$

since $r(n) = 0$. For $\sigma \notin G_n$ we have $\sigma(\tau_n) = -\tau_n$, so that $\tau_n \in \mathbb{Q}\sqrt{10}$ and $A_{10}(\zeta) = 0$.

If $\gcd(k, 40) > 1$, choose a prime p dividing k and 40. Then p divides $\frac{n}{p}$ and 40 divides $\frac{n}{p}$. Hence $u \mapsto u + \frac{n}{p}$ permutes $(\mathbb{Z}/n\mathbb{Z})^\times$ and preserves χ, and it multiplies $\mathrm{e}^{\frac{2\pi\mathrm{i}u}{n}}$ by $\mathrm{e}^{\frac{2\pi\mathrm{i}}{p}} \neq 1$, so that $\tau_n = 0$. This proves part 2.

Assume that $\gcd(k, 40) = 1$. By the Chinese remainder theorem, $\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}} = \mathrm{e}^{\frac{2\pi\mathrm{i}x}{40}}\mathrm{e}^{\frac{2\pi\mathrm{i}y}{k}}$ with $x \in (\mathbb{Z}/40\mathbb{Z})^\times$ and $y \in (\mathbb{Z}/k\mathbb{Z})^\times$, and $(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/40\mathbb{Z})^\times \times (\mathbb{Z}/k\mathbb{Z})^\times$. Hence

$$\tau_n = \Big(\sum_{u\in(\mathbb{Z}/40\mathbb{Z})^\times} \chi(u)\mathrm{e}^{\frac{2\pi\mathrm{i}xu}{40}}\Big)\Big(\sum_{u\in(\mathbb{Z}/k\mathbb{Z})^\times} \mathrm{e}^{\frac{2\pi\mathrm{i}yu}{k}}\Big) = \chi(x)\tau_{40}r(k).$$

Let s_+ and s_- be the sums of the primitive 40-th roots of unity in the two cosets of G_{40}, so that $\tau_{40} = \pm(s_+ - s_-)$. They are fixed by G_{40} and exchanged by the other elements of the Galois group. By [Lar14, proof of Theorem 6.2, Case 2, p. 18], applied with $c = 10$, we have $\{s_+, s_-\} = \{\sqrt{10}, -\sqrt{10}\}$. Consequently, $|\tau_{40}| = 2\sqrt{10}$, and $|B_{10}(\zeta)|\sqrt{10} = \frac{2\sqrt{10}|r(k)|}{\varphi(40)\varphi(k)} = \frac{2\sqrt{10}|r(k)|}{16\varphi(k)}$, which is part 3.

By parts 1, 2 and 3, $|A_{10}(\zeta)|+8|B_{10}(\zeta)|$ is $\frac{|r(n)|}{\varphi(n)}$, 0 or $\frac{|r(k)|}{\varphi(k)}$ in the three cases. This is at most 1, with equality if and only if $n \in \{1, 2\}$ in the first case and $k = 1$ in the third case, since k is then odd. The last assertion follows from (6.6), since $|a|+8|b| \leqslant \sum_q(|A_{10}(\zeta_q)|+8|B_{10}(\zeta_q)|) \leqslant N$. ■

For a general squarefree integer $D > 1$ and rational numbers a, b with $2a, 2b \in \mathbb{Z}$, a sum of N roots of unity equal to $a + b\sqrt{D}$ satisfies

$$N \geqslant |b|\varphi(2D). \tag{6.7}$$

This follows from [Lar14, Theorem 1.3 p. 2 and Theorem 6.2 p. 18], applied to the sum of $2N$ roots of unity obtained by repeating each root twice, which is $2a + 2b\sqrt{D}$ with $2a, 2b \in \mathbb{Z}$.

6.4 The multiplicity

Proposition 6.6 *In Lemma 6.2, we have $m = 1$.*

Proof : Write $\Delta = 9m^2+4 = t^2D$ with $t \geqslant 1$ and D squarefree. Since $(3m)^2 < \Delta < (3m+1)^2$, we have $D > 1$. Since $\Delta \equiv 1 \bmod 3$, the integer t is prime to 3 and $D \equiv 1 \bmod 3$. An odd prime p dividing D divides $(3m)^2+4$ and does not divide $3m$, so that -1 is a square modulo p and $p \equiv 1 \bmod 4$. Hence every prime factor of D is 2 or congruent to 1 modulo 4, and $D \notin \{2,5\}$ since $D \equiv 1 \bmod 3$. Write $D = 2^e \prod_i p_i$ with distinct odd primes p_i and $e \in \{0,1\}$. Then

$$\frac{\varphi(2D)}{\sqrt{D}} = 2^{\frac{e}{2}} \prod_i \frac{p_i - 1}{\sqrt{p_i}},$$

with $\frac{p-1}{\sqrt{p}} \geqslant \frac{4}{\sqrt{5}} > 1$ for $p \geqslant 5$ and $\frac{p-1}{\sqrt{p}} \geqslant \frac{12}{\sqrt{13}} > 3$ for $p \geqslant 13$. If some p_i is at least 13, the product is at least $\frac{12}{\sqrt{13}} > 3$, since the other factors are at least 1. Otherwise every p_i equals 5, so that $D \in \{2,5,10\}$, and $D = 10$. Hence $\frac{\varphi(2D)}{\sqrt{D}} > 3$ unless $D = 10$.

Consider the family of $N = 4m^2+2$ roots of unity in which each twist θ_{Y_j} of Lemma 6.4 is repeated n_j^2 times. By (6.3) and (6.4), its sum is $a + b\sqrt{D}$ with $a = -\frac{m^2}{2}$ and $b = -\frac{mt}{2}$. Note that $t\sqrt{D} = \sqrt{\Delta} > 3m$.

Assume that $D \neq 10$. By (6.7), we have $4m^2+2 \geqslant \frac{mt}{2}\varphi(2D) = \frac{m}{2}t\sqrt{D}\frac{\varphi(2D)}{\sqrt{D}} > \frac{9m^2}{2}$, so that $m^2 < 4$ and $m = 1$.

Assume that $D = 10$. By Lemma 6.5, we have $4m^2+2 \geqslant \frac{m^2}{2} + 4mt > \frac{m^2}{2} + \frac{12m^2}{\sqrt{10}}$, so that $m^2 < 7$. Since $D = 13$ for $m = 1$, we get $m = 2$ and $t = 2$. Then $N = 18 = |a| + 8|b|$. By (6.6), we have $\sum_q(-A_{10}(\zeta_q) - 8B_{10}(\zeta_q)) = 18$, and each term is at most 1 by Lemma 6.5. Hence $-A_{10}(\zeta_q) - 8B_{10}(\zeta_q) = 1$ for every root ζ_q of the family. Since this quantity is -1 for $\zeta_q = 1$, Lemma 6.5 shows that each ζ_q is equal to -1 or is a primitive 40-th root of unity, and that $A_{10}(\zeta_q) = 0$ in the second case. Since $\sum_q A_{10}(\zeta_q) = a = -2$ by (6.6), the root -1 occurs exactly twice in the family. Hence the family of the squares ζ_q^2 consists of 1, twice, and of 16 primitive 20-th roots of unity, and $A_{10}(\zeta_q^2) = \frac{r(20)}{\varphi(20)} = 0$ for the latter by part 1 of Lemma 6.5. By (6.5), the sum of this family is $\frac{-4+(2\varepsilon-2)\sqrt{40}}{2} = -2 + 2(\varepsilon-1)\sqrt{10}$. Comparing the rational parts by (6.6), we obtain $-2 = \sum_q A_{10}(\zeta_q^2) = 2$, a contradiction. This excludes $m = 2$. ∎

6.5 Pointed parts and proof of the recognition theorem

Lemma 6.7 *Let $\mathcal{P}$ be a fusion category with three simple objects, all invertible, so that $\mathcal{P}$ is equivalent to $\mathrm{Vec}^{\varpi}_{\mathbb{Z}_3}$ for some 3-cocycle ϖ.*

1. *If a left $\mathcal{P}$-module category $\mathcal{M}$ contains a simple object M with $g \otimes M \cong M$ for every invertible object g of $\mathcal{P}$, then ϖ is a coboundary.*

2. *If $\mathcal{P}$ has two inequivalent indecomposable left module categories, then ϖ is a coboundary.*

Proof : 1. The full subcategory of $\mathcal{M}$ whose objects are the multiples of M is stable under the action of $\mathcal{P}$, and it is equivalent to Vec. A structure of $\mathcal{P}$-module category on Vec is the same thing as a fiber functor $\mathcal{P} \to \mathrm{Vec}$ [Ost03, Example (ii) pp. 6–7]. The monoidal structure of a fiber functor on $\mathrm{Vec}^{\varpi}_{\mathbb{Z}_3}$ consists of nonzero scalars $j(g,h)$, after a choice of basis vectors of the lines attached to the simple objects, and its compatibility with the associators says that ϖ is the coboundary of j.

2. Let $\mathcal{M}$ be an indecomposable $\mathcal{P}$-module category. The invertible objects permute the isomorphism classes of simple objects of $\mathcal{M}$. The full subcategory whose objects are the direct sums of simple objects in one orbit is a module subcategory and a direct summand of $\mathcal{M}$, so that the action is transitive, with orbits of cardinality one or three. If a simple object is fixed, then ϖ is a coboundary by part 1. Otherwise the action is free. Let M be a simple object of $\mathcal{M}$, and let $\underline{\mathrm{Hom}}(M,M)$ be its internal Hom, characterized by $\mathrm{Hom}_{\mathcal{P}}(Y,\underline{\mathrm{Hom}}(M,M)) \cong \mathrm{Hom}_{\mathcal{M}}(Y\otimes M, M)$ [Ost03, Definition 11 p. 9]. For $Y = g$ invertible, this space is nonzero if and only if $g \cong 1$, so that $\underline{\mathrm{Hom}}(M,M) \cong 1$ as an object. Its algebra structure is then isomorphic to the trivial algebra 1, and by [Ost03, Theorem 1 and its proof pp. 10–11], $\mathcal{M}$ is equivalent to the category of modules over 1, that is, to the regular module category $\mathcal{P}$. Hence, if ϖ is not a coboundary, every indecomposable $\mathcal{P}$-module category is equivalent to $\mathcal{P}$. ∎

We now prove Theorem 6.1.

Proof : Part 1 follows from Lemma 6.2 and Proposition 6.6, with $d(\rho) = d_1 = d$.

We next prove the converse statement. The categories $\mathcal{H}_2$ and $\mathcal{H}_3$ are unitary fusion categories with the fusion rules (2.2) [GrS12, p. 14, Theorem 1.4 p. 4 and Corollary 3.24 p. 24], the category $\mathcal{H}_3$ being the category of bimodules over a Q-system in $\mathcal{H}_2$ [GrS12, Lemma 3.14 p. 21 and Definition 3.19 p. 23]. They have six simple objects, three of which are invertible, and their Grothendieck rings are isomorphic and noncommutative since $\xi\alpha \cong \alpha^2\xi \ncong \alpha\xi$ in $\mathcal{H}_2$. The pointed part of $\mathcal{H}_2$ has trivial associator by [GrS12, Lemma 2.21 p. 15]. For $\mathcal{H}_3$, let $A \overset{\text{def}}{=} 1\oplus\alpha\oplus\alpha^2$ with this algebra structure, so that $\mathcal{H}_3$ is the category of A-A-bimodules in $\mathcal{H}_2$. The category $\mathcal{M}$ of left A-modules in $\mathcal{H}_2$ is a left $\mathcal{H}_3$-module category for the relative tensor product $\otimes_A$ and a right $\mathcal{H}_2$-module category, and the two actions commute. Since the Q-system A is separable, the category $\mathcal{M}$ is semisimple [DMNO13, Proposition 2.7 p. 5]. For objects Y, Y' of $\mathcal{H}_2$, we have $\mathrm{Hom}_A(A\otimes Y, A\otimes Y') \cong \mathrm{Hom}(Y, A\otimes Y')$, and every simple left A-module is a quotient, hence a summand, of some $A\otimes Y$ with Y simple [Ost03, Lemma 4 p. 9]. By (2.2), we deduce that the simple objects of $\mathcal{M}$ are $A \cong A\otimes\alpha \cong A\otimes\alpha^2$ and $A\otimes\xi \cong A\otimes\alpha\xi \cong A\otimes\alpha^2\xi$, up to isomorphism. Moreover $(A\otimes\xi)\otimes\xi \cong A\oplus 3(A\otimes\xi)$ is not simple. Hence A is, up to isomorphism, the only simple object M of $\mathcal{M}$ such that $M\otimes\xi$ is simple. For an invertible object g of $\mathcal{H}_3$, the functor $g\otimes_A -$ is an autoequivalence of $\mathcal{M}$ commuting with the right action of $\mathcal{H}_2$, so that $(g\otimes_A A)\otimes\xi \cong g\otimes_A (A\otimes\xi)$ is simple and $g\otimes_A A \cong A$. By part 1 of Lemma 6.7, the pointed part of $\mathcal{H}_3$ has trivial associator.

We finally prove part 2. By part 1, $\mathcal{F}$ is a quadratic category with $(\mathbb{Z}_3, -1, 1)$ [Izu18, Definition 2.1 p. 6]. Its invariant $c_{0,3}(\mathcal{F}) \in H^3(\mathbb{Z}_3, \mathbb{C}^\times)$ is the class of the associator of its pointed part [Izu18, p. 9], which vanishes by assumption. The invariant $c_{1,2}(\mathcal{F})$ belongs to $H^1(\mathbb{Z}_2, H^2(\mathbb{Z}_3, \mathbb{C}^\times))$ [Izu18, pp. 10–11], which is zero since $H^2(\mathbb{Z}_3, \mathbb{C}^\times) = 0$. Hence $\mathcal{F}$ is a generalized Haagerup category with group $\mathbb{Z}_3$ [Izu18, Definition 2.7 p. 12] which is a C^*-fusion category. For $G = \mathbb{Z}_3$, the group $\Gamma = (H^2(G,\mathbb{T})\times G/2G) \rtimes \mathrm{Aut}(G)$ of [Izu18, Theorem 5.9 p. 36] is $\mathrm{Aut}(\mathbb{Z}_3)$, and since G is odd one may take the signs $\varepsilon_h(g)$ of [Izu18] equal to 1 [Izu18, p. 56]. Hence the number of Γ-orbits of gauge equivalence classes of solutions in [Izu18, Theorem 5.9 p. 36] is at most the number of solutions up to group automorphism in [Izu18, Theorem 9.1 p. 57], and there are at most two C^*-generalized Haagerup categories with group $\mathbb{Z}_3$ up to equivalence. By the converse statement, $\mathcal{H}_2$ and $\mathcal{H}_3$ are two such categories, and they are not equivalent by part 1 of Proposition 2.1. Hence $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$. ∎

6.6 The associator of the pointed part and the twists

Lemma 6.8 *Let $\mathcal{P}$ be a fusion category with three simple objects, all invertible, equipped with the spherical structure for which all simple objects have dimension 1, and let α be a non-trivial*

simple object. Then $\nu_3(\alpha) = 1$ if the associator of $\mathcal{P}$ is trivial, and $\nu_3(\alpha)$ is a primitive cube root of unity otherwise.

Proof : The category $\mathcal{P}$ is pseudo-unitary, since $\dim \mathcal{P} = \mathrm{FPdim}\, \mathcal{P} = 3$. Hence its spherical structure with positive dimensions is unique [ENO05, Proposition 8.23 p. 34]. By [NgS07b, Corollary 4.4 p. 11], Frobenius–Schur indicators are invariant under tensor equivalences which preserve the pivotal structures, and every tensor equivalence between such pointed categories preserves the spherical structures with positive dimensions. Hence we may assume that $\mathcal{P}$ is the category $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, \omega_t)$ of [NgS07a, p. 30] with the pivotal structure of dimension 1 described there, where $t \in \{0, 1, 2\}$ and the class of ω_t is trivial if and only if $t = 0$, and that α is the object V_1 or V_2. By [NgS07a, p. 31], we have $\nu_3(V_1) = \mathrm{e}^{\frac{2\pi i t}{3}}$. Since $V_2 \cong V_1^*$, we also have $\nu_3(V_2) = \nu_3(V_1)$ [NgS07a, Remark 4.2 p. 10]. Hence $\nu_3(\alpha) = \mathrm{e}^{\frac{2\pi i t}{3}}$, which is equal to 1 if and only if $t = 0$. ■

Proposition 6.9 *Let $\mathcal{F}$ be as in Theorem 6.1, and let Z_1, Z_2, Z_3 be the simple summands of $I(\alpha)$ (Lemma 6.3). Then $\theta_{Z_i}^3 = \nu_3(\alpha)$ for $i \in \{1, 2, 3\}$.*

1. *If the fusion subcategory generated by α has trivial associator, then $\theta_{Z_i}^3 = 1$, and $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$.*
2. *Otherwise the twists θ_{Z_i} are primitive ninth roots of unity. In particular, $Z(\mathcal{F})$ is not braided equivalent to $Z(\mathcal{H}_3)$.*

Consequently, $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$ if and only if no simple object of $Z(\mathcal{F})$ has a twist of order nine.

Proof : The equality $\theta_{Z_i}^3 = \nu_3(\alpha)$ is proved in Lemma 6.4. The indicator $\nu_3(\alpha)$ is computed in the fusion subcategory $\mathcal{P}$ generated by α, since $\mathrm{Hom}(1, \alpha^{\otimes 3})$ and the map $E_\alpha^{(3)}$ only involve objects of $\mathcal{P}$ [NgS07a, p. 31], and the spherical structure of $\mathcal{F}$ restricts to the spherical structure of $\mathcal{P}$ with dimensions 1. Part 1 then follows from Lemma 6.8 and part 2 of Theorem 6.1. In the situation of part 2, $\nu_3(\alpha)$ is a primitive cube root of unity by Lemma 6.8, so that $\theta_{Z_i}^9 = 1 \neq \theta_{Z_i}^3$. A braided equivalence between pseudo-unitary modular tensor categories preserves the spherical structures with positive dimensions [ENO05, Proposition 8.23 p. 34], hence the twists. By (2.5), the twists of $Z(\mathcal{H}_3)$ have order 1, 3 or 13, which proves part 2. The last assertion follows from parts 1 and 2, since $Z(\mathcal{H}_2)$ and $Z(\mathcal{H}_3)$ are braided equivalent to $Z(\mathcal{H}_3)$ (Section 2.1). ■

6.7 Application to the conformal nets

Lemma 6.10 *Under the hypotheses of Theorem 5.6, let $x \in \{a, c_0\}$. Then the pointed fusion subcategory $\mathcal{P}_x$ of $\mathcal{F}_x$ is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator.*

Proof : By part 4 of Theorem 5.6, part 1 of Proposition 2.2 and Lemma 3.1, there is a braided equivalence $\mathrm{Rep}(\mathcal{A}_U) \cong Z(\mathcal{F}_x)$ under which Θ_x corresponds to $I(1)$. By the proof of part 4 of Theorem 5.6, the connected étale subalgebra $\theta = \pi_0 \oplus \pi_b$ of Θ_x corresponds to the étale subalgebra A_x of $I(1)$ attached to $\mathcal{P}_x$ by [DMNO13, Theorem 4.10 p. 18]. By part 5 of Proposition 2.2, the category of local modules $Z(\mathcal{F}_x)_{A_x}^0$ is braided equivalent to $Z(\mathcal{P}_x)$. On the other hand, the category of local θ-modules $\mathrm{Rep}(\mathcal{A}_U)_\theta^0$ is braided equivalent to $\mathrm{Rep}(\mathcal{A}_{V'})$ [BKL15, Proposition 6.4 p. 33], see also [Müg10, Theorem 3.1 p. 2]. Hence there is a braided equivalence $\Psi_x \colon \mathrm{Rep}(\mathcal{A}_{V'}) \to Z(\mathcal{P}_x)$. If the braiding conventions of [BKL15, Proposition 6.4 p. 33] and of part 5 of Proposition 2.2 differ, then Ψ_x is a braided equivalence onto $Z(\mathcal{P}_x)^{\mathrm{rev}}$.

This does not affect the argument below, since an algebra which is commutative for a braiding is commutative for the reverse braiding, so that $Z(\mathcal{P}_x)$ and $Z(\mathcal{P}_x)^{\mathrm{rev}}$ have the same Lagrangian algebras.

The net $\mathcal{A}_{V'}$ is a covariant subnet of $\mathcal{A}_{H_{\Lambda_a}}$ and of $\mathcal{A}_{H_{\Lambda_{c_0}}}$ by Theorem 4.5. Let $y \in \{a, c_0\}$. The inclusion $\mathcal{A}_{V'} \subset \mathcal{A}_{H_{\Lambda_y}}$ is irreducible by Theorem 4.5, and of finite index by part 2 of Proposition 4.3. By part 3 of Proposition 2.5, its dual canonical endomorphism Θ'_y, which is the restriction of the vacuum representation of $\mathcal{A}_{H_{\Lambda_y}}$ to $\mathcal{A}_{V'}$, is a connected étale algebra in $\mathrm{Rep}(\mathcal{A}_{V'})$. By part 3 of Proposition 2.5 and part 2 of Proposition 4.3, we have $d(\Theta'_y) = [\mathcal{A}_{H_{\Lambda_y}} : \mathcal{A}_{V'}] = 3$ and $\mu_{\mathcal{A}_{V'}} = 9 = d(\Theta'_y)^2$, so that Θ'_y is Lagrangian by (2.10). The restriction of Θ'_y to $\mathcal{A}_U$ is the restriction of the vacuum representation of $\mathcal{A}_{H_{\Lambda_y}}$, that is, $\pi_0 \oplus \pi_b \oplus 2\pi_y$ by part 2 of Theorem 5.6. Since π_a and π_{c_0} are inequivalent, the algebras Θ'_a and Θ'_{c_0} are not isomorphic. Hence $\Psi_x(\Theta'_a)$ and $\Psi_x(\Theta'_{c_0})$ are two non-isomorphic Lagrangian algebras in $Z(\mathcal{P}_x)$. By part 2 of Proposition 2.2, which gives a bijection between the isomorphism classes of Lagrangian algebras of $Z(\mathcal{P}_x)$ and the equivalence classes of indecomposable $\mathcal{P}_x$-module categories, the category $\mathcal{P}_x$ has two inequivalent indecomposable module categories. We conclude with part 2 of Lemma 6.7. ■

Theorem 6.11 *Let $U \subset V'$ be as in Theorem 5.6, and let $x \in \{a, c_0\}$. Then the fusion category $\mathcal{F}_x$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$, and $\delta = d^2$. Consequently, $\mathrm{Rep}(\mathcal{A}_U)$ is equivalent to $Z(\mathcal{H}_3)$ as a unitary modular tensor category. In particular, $(\mathrm{H_J})$ implies (N_0).*

Proof : By part 4 of Theorem 5.6 and Proposition 5.8, the category $\mathcal{F}_x$ is a unitary fusion category with six simple objects, three of which are invertible. By part 2 of Theorem 5.6, we have $\Theta_x \cong \pi_0 \oplus \pi_b \oplus 2\pi_x$ with pairwise inequivalent irreducible summands, so that $\mathrm{Hom}(\Theta_x, \Theta_x) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$. Since Θ_x corresponds to $I(1)$ (Lemma 3.1), and since $K(\mathcal{F}_x) \otimes \mathbb{C}$ is isomorphic to $\mathrm{End}(I(1))$ for the spherical category $\mathcal{F}_x$ [Ost15, Corollary 2.16 p. 7], see also [BiJ22, Corollary 3.4 p. 16], the Grothendieck ring of $\mathcal{F}_x$ is noncommutative. The fusion subcategory $\mathcal{P}_x$ is pointed of dimension three, so that its three simple objects are invertible, and $\mathcal{F}_x$ has three invertible simple objects by Proposition 5.8. Hence $\mathcal{P}_x$ is the fusion subcategory generated by the invertible objects, and it has trivial associator by Lemma 6.10. By Theorem 6.1, $\mathcal{F}_x$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$. Comparing $\mathrm{FPdim}(\mathcal{F}_x) = 3(1 + \delta)$ with (2.3), we obtain $\delta = d^2$. By part 4 of Theorem 5.6 and Section 2.1, $\mathrm{Rep}(\mathcal{A}_U)$ is braided equivalent to $Z(\mathcal{F}_x)$, hence to $Z(\mathcal{H}_3)$, and such an equivalence is naturally isomorphic to a unitary braided equivalence [Reu23, Theorem 5 p. 2]. Under $(\mathrm{H_J})$, these results apply to $U = S$ by Corollary 5.9. ■

Remark 6.12 For the algebra of [Hua26b], the fusion rules of part 1 of Theorem 6.1, the triviality of the associator of the pointed part and the identification with $\mathcal{H}_2$ or $\mathcal{H}_3$ are stated in [Hua26b, Sections 5 and 6 pp. 7–9], with an outline of proof, and the proofs of this section follow that outline. The outline leaves the following points implicit or only sketched.

1. The free actions of the invertible objects (Lemma 6.2), which are stated in [Hua26b, p. 7], and the self-duality of the non-invertible simple objects, which is implicit in [Hua26b, (24) p. 7].

2. The structure of $Z(\mathcal{F})$ (Lemma 6.3) and the equalities $k_1 = k_2 = k_3 = m$, which are obtained from $\nu_1(\alpha)$ and $\nu_3(\alpha)$ (Lemma 6.4). The identities [Hua26b, (25) p. 7], attributed there to the first two Frobenius–Schur indicators, follow from them.

3. The restrictions on the squarefree part D of $9m^2 + 4$ in the proof of Proposition 6.6. For $D \neq 10$, the second inequality displayed in [Hua26b, p. 7] amounts to $\varphi(2D) \geqslant \frac{12\sqrt{D}}{\sqrt{13}}$. It

holds for the values of D that can occur, but not for every $D \neq 10$, for instance not for $D \in \{2, 5, 7\}$. The value $D = 7$ is excluded since the odd prime factors of D are congruent to 1 modulo 4, and the values $D \in \{2, 5\}$ are excluded by the congruence $D \equiv 1 \bmod 3$.

4. The bound of Lemma 6.5 for $D = 10$, which is asserted in [Hua26b, p. 7], and the analysis of the case $m = 2$, which is sketched in [Hua26b, pp. 7–8].

5. The triviality of the associator of the pointed part. It is derived in [Hua26b, p. 7] from the transversality of the two isotropic lines, by induction from the second one, and it is used in [Hua26b, pp. 8–9]. Lemma 6.10 uses the same input, the second lattice extension, through the module categories of $\mathcal{P}_x$.

6. The converse statement of Theorem 6.1, which is needed to deduce from [Izu18, Theorems 5.9 and 9.1 pp. 36 and 57] that the category is $\mathcal{H}_2$ or $\mathcal{H}_3$.

The use of [Hua26b] in Section 5 is different, see Remark 5.10.

7 Consequences for the net of S, complete unitarity and (N)

The proofs in this section take ($\mathrm{H_J}$) as a hypothesis. For the explicit S it is supplied by [ArhIII, Theorem 6.1], and there is no circularity. The proof of [ArhIII, Theorem 6.1] uses no statement of the present article which assumes ($\mathrm{H_J}$). It uses the results of Sections 2, 3 and 6 which do not concern Haagerup algebras, Section 4 and Proposition 5.2, and it repeats, with (I1) and (F1)–(F3) of [ArhIII, Theorems 2.1 and 2.2] in place of ($\mathrm{H_J}$), the arguments of Sections 5.2 and 6.7, of the proof of Lemma 7.1, of the categorical part of the proof of Proposition 7.12 and of the proof of Theorem 7.17. The proofs of Lemma 7.1, of parts 1–3 of Theorem 7.2, of Corollaries 7.4 and 7.6, of Proposition 7.5 and of Remark 7.8 use ($\mathrm{H_J}$) only through Corollary 5.9 and the assertion ($\mathrm{N_0}$) below, except for the last assertion of Lemma 7.1, which involves $\mathrm{Rep}(S)$. We use the notation of Section 5 for $U = S$. In particular, $\mathcal{A}_S$ is completely rational by Corollary 5.9. We consider the two assertions

($\mathrm{N_0}$) the unitary modular tensor category $\mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to $Z(\mathcal{H}_3)$,

(N) there exists a braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_S) \to \mathrm{Rep}(S)$ with $\Phi(\pi_0) \cong S$ and $\Phi(\pi_M) \cong M$ for $M \in \{M_b, M_a, M_{c_0}\}$.

Under ($\mathrm{H_J}$), the category $\mathrm{Rep}(S)$ is braided equivalent to $Z(\mathcal{H}_3)$, so that (N) implies ($\mathrm{N_0}$). By Theorem 6.11, ($\mathrm{N_0}$) holds under ($\mathrm{H_J}$). We prove (N) under ($\mathrm{H_J}$) in Theorem 7.17, at the end of this section. Its proof uses Theorem 4.5, Lemma 4.7, Propositions 7.12 and 7.16 and the results of Gui [Gui26, Gui25], but neither Theorem 6.11 nor any statement of this section which assumes (N) or the complete unitarity of S. Since (N) implies ($\mathrm{N_0}$) under ($\mathrm{H_J}$), Theorem 7.17 gives a second, computer-assisted, proof of ($\mathrm{N_0}$), independent of Section 6.

7.1 The representation category and the holomorphic extensions

Lemma 7.1 *Assume* ($\mathrm{H_J}$)*, and let* $\Psi\colon \mathrm{Rep}(\mathcal{A}_S) \to Z(\mathcal{H}_3)$ *be a braided equivalence, which exists by Theorem 6.11. Then* $\Psi(\pi_b) \cong b$*,* $\{\Psi(\pi_a), \Psi(\pi_{c_0})\} = \{a, c_0\}$ *up to isomorphism, and* $\delta = d^2 = 3d + 1$*. Moreover,* Ψ *is unique up to natural isomorphism and naturally isomorphic to a unitary braided equivalence. Consequently,* (N) *holds if and only if* $\Psi(\pi_a) \cong a$*.*

Proof : The uniqueness follows from Section 2.1, and the unitarity from [Reu23, Theorem 5 p. 2]. In the unitary category $\mathrm{Rep}(\mathcal{A}_S)$, the Frobenius–Perron dimension of an irreducible sector is its statistical dimension, and $\mathrm{FPdim}\,\mathrm{Rep}(\mathcal{A}_S) = \mu_{\mathcal{A}_S}$ by (2.10). By the existence of Ψ, (2.6) and part 3 of Theorem 5.6, we obtain $9(1+\delta)^2 = 9(3d+2)^2$, that is, $\delta = 3d+1 = d^2$. The simple object $\Psi(\pi_b)$ has dimension $3d+1$, so that $\Psi(\pi_b) \cong b$ by (2.4). For $x \in \{a, c_0\}$, the object $\Psi(\pi_x)$ is a simple summand of dimension $3d+2$ of the Lagrangian algebra $\Psi(\Theta_x)$, hence it belongs to $\{a, c_0\}$ by Lemma 3.2. The objects $\Psi(\pi_a)$ and $\Psi(\pi_{c_0})$ are not isomorphic since π_a and π_{c_0} are inequivalent by Theorem 5.6. Finally, under ($\mathrm{H_J}$), the braided equivalence $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$ is unique up to natural isomorphism, and it maps M_z to z. A braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_S) \to \mathrm{Rep}(S)$ is therefore isomorphic to the composition of Ψ with the inverse of this equivalence, and $\Phi(\pi_b) \cong M_b$ by the first part. Hence (N) amounts to $\Psi(\pi_a) \cong a$. ∎

Theorem 7.2 *Assume* ($\mathrm{H_J}$).

1. $\mathrm{Rep}(\mathcal{A}_S)$ *is equivalent to* $Z(\mathcal{H}_3)$ *as a unitary modular tensor category. We have* $d(\pi_b) = 3d+1$, $d(\pi_a) = d(\pi_{c_0}) = 3d+2$, $[\mathcal{A}_{V'} : \mathcal{A}_S] = 3d+2$, $[\mathcal{A}_{E_8} : \mathcal{A}_S] = 3(3d+2)$ *for both lattice extensions, and* $\mu_{\mathcal{A}_S} = 9(3d+2)^2$.

2. *Up to isomorphism of extensions,* $\mathcal{A}_S$ *has exactly three irreducible local extensions* $\mathcal{B}$ *with* $\mu_{\mathcal{B}} = 1$. *Their dual canonical endomorphisms* Θ_1, Θ_2, Θ_3 *satisfy* $\mathrm{Rep}(\mathcal{A}_S)_{\Theta_i} \cong \mathcal{H}_i$, $\Theta_1 \cong \pi_0 \oplus \pi_b \oplus \pi_a \oplus \pi_{c_0}$ *and*

$$\{\Theta_2, \Theta_3\} = \{\Theta_a, \Theta_{c_0}\}, \quad \Theta_x \cong \pi_0 \oplus \pi_b \oplus 2\pi_x.$$

   The extensions attached to Θ_a *and* Θ_{c_0} *are the nets* $\mathcal{A}_{H_{\Lambda_a}}$ *and* $\mathcal{A}_{H_{\Lambda_{c_0}}}$ *of the lattice extensions, both isomorphic to* $\mathcal{A}_{E_8}$.

3. *The only intermediate nets between* $\mathcal{A}_S$ *and the extension* $\mathcal{B}_1$ *attached to* Θ_1 *are* $\mathcal{A}_S$ *and* $\mathcal{B}_1$. *For* $x \in \{a, c_0\}$, *the only intermediate nets between* $\mathcal{A}_S$ *and* $\mathcal{A}_{H_{\Lambda_x}}$ *are* $\mathcal{A}_S$, $\mathcal{A}_{V'}$ *and* $\mathcal{A}_{H_{\Lambda_x}}$.

4. *Assume* (N). *Then* $\Theta_2 = \Theta_a$ *and* $\Theta_3 = \Theta_{c_0}$, *that is, the net* $\mathcal{A}_{H_{\Lambda_a}}$ *carries* $\mathcal{H}_2$ *and the net* $\mathcal{A}_{H_{\Lambda_{c_0}}}$ *carries* $\mathcal{H}_3$, *as the vertex operator algebras* H_{Λ_a} *and* $H_{\Lambda_{c_0}}$ *do.*

Proof : 1. This follows from Lemma 7.1 and part 3 of Theorem 5.6.

2. Let $\mathcal{B}$ be an irreducible local extension of $\mathcal{A}_S$, that is, $\mathcal{A}_S(I)' \cap \mathcal{B}(I) = \mathbb{C}$. Then $\mathcal{A}_S(S^1)' \cap \mathcal{B}(I) = \mathbb{C}$, so that $[\mathcal{B} : \mathcal{A}_S] < \infty$ by part 2 of Proposition 2.5, and $\mathcal{B}$ is completely rational with $\mu_{\mathcal{B}} = \mu_{\mathcal{A}_S}/d(\Theta_{\mathcal{B}})^2$ by part 1 of the same proposition, where $\Theta_{\mathcal{B}}$ is the dual canonical endomorphism. Hence $\mu_{\mathcal{B}} = 1$ if and only if the connected étale algebra $\Theta_{\mathcal{B}}$ is Lagrangian in $\mathrm{Rep}(\mathcal{A}_S)$. Irreducible local extensions of $\mathcal{A}_S$ up to isomorphism correspond to commutative Q-systems Θ with $\dim \mathrm{Hom}(\mathrm{id}, \Theta) = 1$ up to unitary equivalence [LoR95, Theorem 4.9 p. 25], [BKLR15, Section 5.2.1 p. 77], see also [Lon03, Lemma 15 p. 14]. By [CGGH23, Theorem 3.2 p. 11 and Theorem 3.9 p. 15], the latter correspond to connected étale algebras up to isomorphism. By Lemma 7.1 and Propositions 3.3 and 3.6, there are exactly three Lagrangian algebras in $\mathrm{Rep}(\mathcal{A}_S)$, with the stated categories of modules and objects. Their objects are pairwise non-isomorphic. By part 2 of Theorem 5.6, $\Theta_x \cong \pi_0 \oplus \pi_b \oplus 2\pi_x$ is the dual canonical endomorphism of $\mathcal{A}_S \subset \mathcal{A}_{H_{\Lambda_x}}$, and by part 4 of the same theorem it is Lagrangian. The nets $\mathcal{A}_{H_{\Lambda_x}}$ are isomorphic to $\mathcal{A}_{E_8}$ by Theorem 4.5.

3. By [BKLR15, Proposition 4.11 p. 49], an intermediate net gives, at a fixed interval, a projection in $\mathrm{Hom}(\Theta_i, \Theta_i)$ whose range is a connected étale subalgebra of Θ_i, and distinct

intermediate nets give distinct subalgebras. As in the proof of part 3 of Theorem 3.9, using Lemma 3.1, the connected étale subalgebras of Θ_i correspond to the fusion subcategories of $\mathcal{H}_i$, which are given by part 4 of Proposition 2.1. Equivalently, one can use the Galois correspondence of part 1 of Proposition 2.6. For Θ_x, the subalgebra of Frobenius–Perron dimension $3d+2$ is the dual canonical endomorphism $\pi_0 \oplus \pi_b$ of the intermediate net $\mathcal{A}_{V'}$.

4. By Lemma 7.1, the braided equivalence Ψ maps π_a to a and π_{c_0} to c_0, so that it maps Θ_a to a Lagrangian algebra isomorphic, as an object, to $0 \oplus b \oplus 2a$, that is, to L_2 by Proposition 3.6. Similarly $\Psi(\Theta_{c_0}) \cong L_3$. ∎

Remark 7.3 Part 2 alone shows that the two lattice nets carry the categories $\mathcal{H}_2$ and $\mathcal{H}_3$ in some order. Under (N), the order is the same as for the vertex operator algebras, by Proposition 3.6, and (N) holds under ($\mathrm{H_J}$) by Theorem 7.17, see Corollary 7.19.

7.2 The Haagerup subfactor and the generalized orbifold

Corollary 7.4 *Assume* ($\mathrm{H_J}$). *Then the Haagerup subfactor arises from the completely rational net* $\mathcal{A}_S$ *in the sense of [Bis17a, Definition 4.1 p. 20], that is, the hyperfinite type* III_1 *subfactor with the standard invariant of the Haagerup subfactor is of the form described there.*

Proof : The Haagerup subfactor $N \subset M$ has finite index and finite depth, and its quantum double $D(N \subset M)$ is the Drinfeld center of either of its even parts $\mathcal{H}_1$ and $\mathcal{H}_2$, which is braided equivalent to $Z(\mathcal{H}_3)$. By part 1 of Theorem 7.2, $\mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to $D(N \subset M)$, and the conclusion follows from part 2 of Proposition 2.6. ∎

Proposition 7.5 *Assume* ($\mathrm{H_J}$). *Then there is a proper action of the hypergroup* $K = \{k_0, k_1\}$ *with*

$$k_1 k_1 = \frac{1}{\delta} k_0 + \Big(1 - \frac{1}{\delta}\Big) k_1$$

on the net $\mathcal{A}_{V'}$ *such that* $\mathcal{A}_S = \mathcal{A}_{V'}^K$. *The weight of* K *is* $1 + \delta = [\mathcal{A}_{V'} : \mathcal{A}_S]$.

Proof : This is Proposition 5.7 for $U = S$, which applies by Corollary 5.9. ∎

Corollary 7.6 *Assume* ($\mathrm{H_J}$). *Then the hypergroup of Proposition 7.5 has parameter* $w = \delta = d^2 = \frac{11+3\sqrt{13}}{2}$. *With* $\tilde{k}_1 \stackrel{\mathrm{def}}{=} d k_1 = w^{1/2} k_1$, *we have* $\tilde{k}_1 \tilde{k}_1 = k_0 + 3\tilde{k}_1$, *and* $\mathrm{Rep}(\mathcal{A}_{V'}^K) = \mathrm{Rep}(\mathcal{A}_S)$ *is braided equivalent to the quantum double of the Haagerup subfactor. The net* $\mathcal{A}_{V'}$ *is the conformal net* $\mathcal{A}_{E_6,1} \otimes \mathcal{A}_{SU(3),1}$, *denoted* $E_{6,1} \times SU(3)_1$ *in [Bis17b, Table 1 p. 25], of the lattice* $A_2 \times E_6$. *Hence, under these hypotheses,* $\mathcal{A}_S$ *solves [Bis17b, Problem 4.27 p. 27].*

Proof : By Lemma 7.1, $\delta = d^2 = 3d + 1 = \frac{11+3\sqrt{13}}{2}$. Then

$$\tilde{k}_1 \tilde{k}_1 = d^2 k_1 k_1 = k_0 + (d^2 - 1) k_1 = k_0 + \frac{d^2 - 1}{d} \tilde{k}_1 = k_0 + 3\tilde{k}_1,$$

since $d^2 - 1 = 3d$. The other assertions follow from Proposition 7.5, Corollary 7.4, part 1 of Theorem 7.2 and part 2 of Proposition 4.3. ∎

Remark 7.7 The vertex algebraic counterpart of Problem 4.27 of [Bis17b] is formulated in [GaR26, Claim 4.4 and (4.54) p. 86]. The existence of the hypergroup action is a general fact [Bis17b, Theorem 4.22 p. 24]. The new input of Corollary 7.6 is the identification of $\mathrm{Rep}(\mathcal{A}_S)$ in Theorem 6.11. The value of w is consistent with [Bis17b, Corollary 4.20 p. 23 and Lemma

4.26 p. 27]. Indeed, by Theorem 6.11, the canonical hypergroup of the inclusion $\mathcal{A}_S \subset \mathcal{A}_{E_8}$ is attached to a fusion category with the fusion rules (2.2) [Bis17b, Theorem 5.16 p. 31], the intermediate net $\mathcal{A}_{V'}$ is the fixed point net of its maximal subgroup $\mathbb{Z}_3$, and [Bis17b, Lemma 4.26 p. 27] with $n = 3$ gives $w = \frac{2+9+3\sqrt{13}}{2}$.

Remark 7.8 Bischoff deduced from the modular invariant of [EvG11, (2.4) p. 20] that the unitary fusion category attached to the inclusion $\mathcal{A}_S \subset \mathcal{A}_{V'}$ should be a near-group category of type $\mathbb{Z}_3^2 + 9$ [Bis17b, Remark 5.14 and Proposition 5.15 p. 31]. Under ($\mathrm{H_J}$), this holds for the category $\mathrm{Rep}(\mathcal{A}_S)_\theta$ of modules over the dual canonical endomorphism $\theta = \pi_0 \oplus \pi_b$. Indeed, by part 1 of Theorem 7.2, we may compute in $Z(\mathcal{H}_3)$ with the algebra $A = 0 \oplus b$. For simple objects z and z', the free module adjunction gives $\mathrm{Hom}_A(z \otimes A, z' \otimes A) \cong \mathrm{Hom}(z, z' \otimes A)$, and the fusion rules of $Z(\mathcal{H}_3)$, computed from the S-matrix of Section 2.2 by the Verlinde formula,

$$b \otimes b \cong \bigoplus_z z, \quad y \otimes b \cong y \oplus \bigoplus_{z \neq 0} z, \quad d_l \otimes b \cong \bigoplus_{z \notin \{0, d_l\}} z,$$

for $y \in \{a, c_0, c_1, c_2\}$ and $1 \leqslant l \leqslant 6$, give the following. The free modules $d_l \otimes A$ are simple and pairwise isomorphic, to an object ρ. The module $b \otimes A$ is isomorphic to $A \oplus \rho$. For $y \in \{a, c_0, c_1, c_2\}$, the module $y \otimes A$ is isomorphic to $\rho \oplus g_y \oplus g'_y$, where the eight simple modules g_y, g'_y are pairwise non-isomorphic and not isomorphic to A or ρ. Since every simple A-module is a summand of a free module, the category $\mathrm{Rep}(\mathcal{A}_S)_\theta$ has ten simple objects. Since the free module functor $z \mapsto z \otimes A$ is a tensor functor, it preserves Frobenius–Perron dimensions. Hence ρ has dimension $3d$ and $\mathrm{FPdim}(g_y) + \mathrm{FPdim}(g'_y) = 2$. Since the Frobenius–Perron dimension of a simple object is at least one [EGNO15, Proposition 3.3.4 p. 53], and a simple object of Frobenius–Perron dimension one is invertible [EGNO15, Corollary 3.3.10 p. 54], the nine objects A, g_y, g'_y are invertible. They form a group of order nine. By (2.7), the category $\mathrm{Rep}(\mathcal{A}_S)^0_\theta$ of local modules has Frobenius–Perron dimension nine. Since $\mathrm{FPdim}(\rho)^2 = 9d^2 > 9$, it cannot contain ρ, so its simple objects are these nine invertible objects. By [BKL15, Proposition 6.4 p. 33] and the proof of Lemma 6.10, it is braided equivalent to $\mathrm{Rep}(\mathcal{A}_{V'})$ and to $Z(\mathcal{P}_x) \cong Z(\mathrm{Vec}_{\mathbb{Z}_3})$, whose group of invertible objects is $\mathbb{Z}_3 \times \mathbb{Z}_3$. For an invertible object g, the object $\rho \otimes g$ is simple of dimension $3d$, so that $\rho \otimes g \cong \rho$, and similarly $\rho^* \cong \rho$. Hence $\mathrm{Hom}(\rho \otimes \rho, g) \cong \mathrm{Hom}(\rho, g \otimes \rho)$ is one-dimensional, and comparing dimensions with $9d^2 = 9 + 9 \cdot 3d$ gives $\rho \otimes \rho \cong \bigoplus_g g \oplus 9\rho$. Thus $\mathrm{Rep}(\mathcal{A}_S)_\theta$ is a near-group category of type $\mathbb{Z}_3^2 + 9$, of Frobenius–Perron dimension $9(3d + 2)$.

Remark 7.9 By [Bis17b, Theorem 4.22 p. 24] and the computation of [Bis17b, Section 4.6 p. 25], the action of k_1 is implemented on $\mathcal{H}_{V'} = \mathcal{H}_S \oplus \mathcal{H}_{N_b}$ by the operator $T = P_S - \delta^{-1} P_{N_b}$, where P_S and P_{N_b} are the orthogonal projections. In other words, the stochastic map ϕ_{k_1} on $\mathcal{A}_{V'}(I)$ is determined by $\phi_{k_1}(x)\Omega = Tx\Omega$. The restriction of T to $V' = S \oplus N_b$ is the linear map $\mathrm{id}_S \oplus (-\delta^{-1})\mathrm{id}_{N_b}$, which is not an automorphism of V'. Compare with the hypergroup symmetries of [GaR26, Section 4.3 pp. 83–88].

7.3 The third holomorphic extension

We say that S is completely unitary if it is completely unitary in the sense of [CGGH23, Definition 4.2 p. 18], that is, every S-module is unitarizable and the invariant sesquilinear forms on the tensor products of S-modules of [Gui19] are positive definite.

Proposition 7.10 *Assume* ($\mathrm{H_J}$) *and that S is completely unitary. Let B_1 be the holomorphic extension of S attached to L_1 by Theorem 3.9.*

1. *The vertex operator algebra B_1 has a unique unitary structure extending the unitary structure of S given by H_χ. With this structure, it is strongly local and its conformal net $\mathcal{A}_{B_1}$ is isomorphic to $\mathcal{A}_{E_8}$.*

2. *The net $\mathcal{A}_{B_1}$ is an irreducible local extension of $\mathcal{A}_S$, and its vacuum representation restricts to $\mathcal{A}_S$ as $\pi_0 \oplus \pi_b \oplus \pi_a \oplus \pi_{c_0}$. In particular, $\mathcal{A}_{H_{\Lambda_a}}$, $\mathcal{A}_{H_{\Lambda_{c_0}}}$ and $\mathcal{A}_{B_1}$ are three pairwise non-isomorphic extensions of $\mathcal{A}_S$ with μ-index one, each isomorphic to $\mathcal{A}_{E_8}$ as a conformal net.*

3. *The net $\mathcal{A}_{B_1}$ is the extension $\mathcal{B}_1$ of part 2 of Theorem 7.2.*

Proof : 1. By [CGGH23, Theorem 4.7 p. 19], B_1 is a unitary extension of S, with a unique unitary structure extending the given one on S. By part 1 of Theorem 3.9, $B_1 \cong V_{E_8}$. The argument of part 1 of Proposition 4.3 applies to any unitary structure on a vertex operator algebra isomorphic to V_{E_8}, so that B_1 is strongly local, has bounded localized vertex operators and $\mathcal{A}_{B_1} \cong \mathcal{A}_{E_8}$.

2. By Lemma 4.4, the coset of $\mathcal{A}_S$ in $\mathcal{A}_{B_1}$ is trivial and $\mathcal{A}_S(I)' \cap \mathcal{A}_{B_1}(I) = \mathbb{C}$. As in the proof of Theorem 5.6, the index is finite. Since $L_1 \cong 0 \oplus b \oplus a \oplus c_0$ is multiplicity free, the isotypic components of the S-module B_1 are simple modules isomorphic to S, M_b, M_a, M_{c_0}. They are pairwise orthogonal, since the orthogonal projection onto the closure of a simple submodule maps B_1 into this closure and commutes with the modes of S, as in the proof of part 1 of Lemma 4.7. By part 3 of Lemma 4.7, applied to $V_1 = B_1$ and $V_2 \in \{H_{\Lambda_a}, H_{\Lambda_{c_0}}\}$, the corresponding representations are π_0, π_b, π_a, π_{c_0}. The three extensions are pairwise non-isomorphic since their dual canonical endomorphisms are not equivalent.

3. By part 2 of Theorem 7.2, Θ_1 is the unique Lagrangian Q-system isomorphic to $\pi_0 \oplus \pi_b \oplus \pi_a \oplus \pi_{c_0}$. ∎

Remark 7.11 Under complete unitarity of S, the comparison of [CGGH23, Theorem 4.10 p. 21] between vertex algebra extensions of S and net extensions of $\mathcal{A}_S$ requires an equivalence $\mathrm{Rep}^u(S) \cong \mathrm{Rep}^f(\mathcal{A}_S)$ of unitary modular tensor categories. If S is completely unitary, such an equivalence follows from (N_0), (H_J) and [Reu23, Theorem 5 p. 2], and it does not involve the correspondence $\pi_M \mapsto M$ of (N). Proposition 7.10 only uses the unitarity of the extension B_1 and Lemma 4.7. Both the complete unitarity of S and this equivalence hold under (H_J) (Theorem 7.17 and Corollary 7.19).

7.4 A diagonal holomorphic extension of $S \otimes S$

The following statement only uses (H_J). It realizes S as its own commutant in $V_{E_8 \oplus E_8}$.

Proposition 7.12 *Assume* (H_J).

1. *There is a braided equivalence $F\colon \mathrm{Rep}(S)^{\mathrm{rev}} \to \mathrm{Rep}(S)$, and it is unique up to natural isomorphism. The induced involution π of the set of simple objects satisfies $\pi(0) = 0$, $\pi(b) = b$, $\pi(a) = a$, $\pi(c_0) = c_0$, $\pi(c_1) = c_2$, $\pi(d_1) = d_5$, $\pi(d_2) = d_3$ and $\pi(d_4) = d_6$.*

2. *The $S \otimes S$-module $W \overset{\mathrm{def}}{=} \bigoplus_z M_z \otimes M_{\pi(z)}$ carries the structure of a simple vertex operator algebra of CFT type extending $S \otimes S$, with the same conformal vector. It is holomorphic, it is isomorphic to $V_{E_8 \oplus E_8}$, and $\mathrm{Com}_W(S \otimes 1) = 1 \otimes S$, $\mathrm{Com}_W(1 \otimes S) = S \otimes 1$.*

Proof : 1. For a fusion category $\mathcal{C}$, let $\mathcal{C}^{\otimes \mathrm{op}}$ be the category $\mathcal{C}$ with the reversed tensor product. A half-braiding γ of an object X in $\mathcal{C}^{\otimes \mathrm{op}}$ gives the half-braiding γ^{-1} of X in $\mathcal{C}$, and the functor

$(X,\gamma) \mapsto (X,\gamma^{-1})$, with the monoidal structure given by the braiding of $Z(\mathcal{C})$, is a braided equivalence from $Z(\mathcal{C}^{\otimes\mathrm{op}})$ onto $Z(\mathcal{C})^{\mathrm{rev}}$. If $\mathcal{C}$ is a unitary fusion category, the functor $f \mapsto f^*$ is a linear monoidal equivalence from the complex conjugate category $\overline{\mathcal{C}}$, which has the same objects and the complex conjugate spaces of morphisms, to the opposite category $\mathcal{C}^{\mathrm{op}}$, and the duality functor $X \mapsto X^\vee$ is a monoidal equivalence from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}^{\otimes\mathrm{op}}$. Hence $\mathcal{C}^{\otimes\mathrm{op}} \cong \overline{\mathcal{C}}$ and $Z(\mathcal{C})^{\mathrm{rev}} \cong Z(\overline{\mathcal{C}})$. Apply this to the even parts $\mathcal{H}_1$ and $\mathcal{H}_2$ of the Haagerup subfactor. The categories $\overline{\mathcal{H}_1}$ and $\overline{\mathcal{H}_2}$ are the even parts of the complex conjugate subfactor, whose planar algebra is the complex conjugate of the Haagerup planar algebra, with the same index and the same principal graphs. By [JMS14, Theorem 4.1 p. 21], it is the Haagerup planar algebra or its dual, whose even parts are $\mathcal{H}_1$ and $\mathcal{H}_2$, of ranks four and six. Comparing ranks, we obtain $\overline{\mathcal{H}_1} \cong \mathcal{H}_1$ and $\overline{\mathcal{H}_2} \cong \mathcal{H}_2$. Hence $Z(\mathcal{H}_3)^{\mathrm{rev}} \cong Z(\mathcal{H}_1)^{\mathrm{rev}} \cong Z(\overline{\mathcal{H}_1}) \cong Z(\mathcal{H}_1) \cong Z(\mathcal{H}_3)$ by Section 2.1, and ($\mathrm{H_J}$) gives F. Its uniqueness follows from the triviality of the braided autoequivalences of $Z(\mathcal{H}_3)$ (Section 2.1). Since F is also a braided equivalence $\mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$, the composite $F \circ F$ is a braided autoequivalence of $\mathrm{Rep}(S)$. For the same reason, it is isomorphic to the identity functor, so that π is an involution. The equivalence F preserves Frobenius–Perron dimensions, and it maps the canonical twist of $\mathrm{Rep}(S)^{\mathrm{rev}}$, which is θ^{-1}, to the canonical twist θ of $\mathrm{Rep}(S)$ [ENO05, Proposition 8.23 p. 34]. By (2.4) and (2.5), we obtain $\pi(0) = 0$, $\pi(b) = b$, $\pi(\{a, c_0\}) = \{a, c_0\}$ and $\pi(c_1) = c_2$. Moreover $\pi(d_l) = d_{l'}$ with $6l'^2 \equiv -6l^2 \bmod 13$, that is, $l' \equiv \pm 5l \bmod 13$, since $5^2 \equiv -1 \bmod 13$. Finally, under $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$, the Lagrangian algebra $L_2 \cong 0 \oplus b \oplus 2a$ of Corollary 3.8 is also a connected étale algebra of $\mathrm{Rep}(S)^{\mathrm{rev}}$, since an algebra is commutative for a braiding if and only if it is commutative for the reverse braiding. The two tensor structures on the category of modules over a connected étale algebra defined by a braiding and by its reverse are opposite to each other [DMNO13, p. 9], so that $(\mathrm{Rep}(S)^{\mathrm{rev}})_{L_2}$ is equivalent to $\mathcal{H}_2^{\otimes\mathrm{op}} \cong \overline{\mathcal{H}_2} \cong \mathcal{H}_2$. Hence $F(L_2)$ is a Lagrangian algebra of $\mathrm{Rep}(S)$ whose category of modules is equivalent to $\mathcal{H}_2$. By Proposition 3.3, $F(L_2) \cong L_2$, so that $0 \oplus b \oplus 2\pi(a) \cong 0 \oplus b \oplus 2a$, and $\pi(a) = a$, $\pi(c_0) = c_0$.

2. Every simple S-module is self-dual (Section 2.2), and F inverts the twists. By [CKM22, Main Theorem 3 pp. 4–5 and Theorem 5.10 p. 51], applied to the braid-reversed equivalence F, the space W carries the structure of a simple conformal vertex algebra extending $S \otimes S$. It is $\mathbb{Z}$-graded since $\theta_z \theta_{\pi(z)} = 1$, and of CFT type since S is positive. Under the braided equivalence $\mathrm{Rep}(S \otimes S) \cong \mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S)$ [CKM22, Theorem 5.5 p. 48], W corresponds to the haploid commutative algebra $\bigoplus_z M_z \boxtimes M_{\pi(z)}$ with trivial twist [HKL15, Theorem 3.2 p. 10], which is a connected étale algebra by [CKM22, Theorem 5.12 p. 52 and its proof] and [DMNO13, Proposition 2.7 p. 5]. Its Frobenius–Perron dimension $\sum_z d(M_z) d(M_{\pi(z)}) = \mathrm{FPdim}\,\mathrm{Rep}(S)$ is the square root of $\mathrm{FPdim}\,\mathrm{Rep}(S \otimes S)$. It is therefore Lagrangian, so that $\mathrm{Rep}(W) \cong \mathrm{Vec}$ by [HKL15, Theorem 3.4 p. 16] and [DMNO13, Remark 4.3 p. 16], and W is strongly rational by [CKM22, Corollary 5.13 p. 53]. By [DoM04, Theorem 2 p. 2], W is isomorphic to $V_{E_8 \oplus E_8}$ or to $V_{D_{16}^+}$. The weight-one space W_1 is the sum of $(S \otimes S)_1 = (\mathfrak{g}' \oplus \mathfrak{g}'') \oplus (\mathfrak{g}' \oplus \mathfrak{g}'')$ and of the spaces $R_z \otimes R_{\pi(z)}$ with $h_z + h_{\pi(z)} = 1$, that is, $z \in \{c_1, c_2, d_1, d_5, d_4, d_6\}$, where R_z is the lowest weight space of M_z. Write $\mathfrak{g}'_1 = \mathfrak{g}' \otimes 1$ and $\mathfrak{g}'_2 = 1 \otimes \mathfrak{g}'$. By [ArhI, Theorem 8.5 and Table 2], the $\mathfrak{g}'$-weights occurring in these spaces R_z are at most 2, so that W_1 contains no irreducible $\mathfrak{g}'_k$-module of dimension 4, and the subspace of $(\mathfrak{g}'_1 \oplus \mathfrak{g}'_2)$-invariant vectors of W_1 has dimension $6 + 2 \cdot 18 + 2 \cdot 15 + 2 \cdot 3 = 78$. Assume that $W \cong V_{D_{16}^+}$, so that $W_1 \cong \mathfrak{so}(E)$ with $\dim E = 32$ and $W_1 \cong \Lambda^2 E$. The affine subalgebra generated by $\mathfrak{g}'_k$ has level 3, so that the Dynkin index of $\mathfrak{g}'_k \subset \mathfrak{so}(E)$ is 3. Since the vector representation of $\mathfrak{so}(E)$ has Dynkin index 2, the restriction of E to $\mathfrak{g}'_k$ has Dynkin index 6. Hence $E|_{\mathfrak{g}'_k}$ is $6V(1) \oplus 20V(0)$ or $V(2) \oplus 2V(1) \oplus 25V(0)$, where $V(n)$ is the irreducible module of highest weight n. In the second case $\Lambda^2 E \supset V(2) \otimes V(1) \supset V(3)$, which is excluded. Thus $E|_{\mathfrak{g}'_1 \oplus \mathfrak{g}'_2} = pV(1,1) \oplus (6-2p)(V(1,0) \oplus V(0,1)) \oplus (8+4p)V(0,0)$

for some $p \in \{0,1,2,3\}$. The multiplicity spaces of $V(1,1)$ and $V(0,0)$ carry nondegenerate symmetric forms and those of $V(1,0)$ and $V(0,1)$ nondegenerate alternating forms, so that the centralizer of $\mathfrak{g}'_1 \oplus \mathfrak{g}'_2$ in $\mathfrak{so}(E)$ is $\mathfrak{so}_p \oplus \mathfrak{so}_{8+4p} \oplus \mathfrak{sp}_{6-2p} \oplus \mathfrak{sp}_{6-2p}$, of dimension 70, 86, 127 or 193. This contradicts the value 78, so that $W \cong V_{E_8 \oplus E_8}$. A vector of W annihilated by the nonnegative modes of $S \otimes 1$ is annihilated by $L^S(0) \otimes 1$. Since $h_z > 0$ for $z \neq 0$, it lies in $S_0 \otimes S = \mathbf{1} \otimes S$. This gives the commutants. ∎

Remark 7.13 The vertex operator algebra $W \cong V_{E_8 \oplus E_8}$ is generated by $W_1 \cong \mathfrak{e}_8 \oplus \mathfrak{e}_8$, so that its unitary structures are determined by compact real forms of W_1, and $S \otimes S$ is generated by its weight-one and weight-two spaces. By Corollary 5.5, every module $M_z \otimes M_{\pi(z)}$ is unitarizable. If S is completely unitary, then so is $S \otimes S$ [Gui22, Proposition 3.31 p. 48], and W is a unitary extension of $S \otimes S$ [CGGH23, Theorem 4.7 p. 19]. Without complete unitarity, it suffices to find the PCT operator of a unitary structure on W which preserves $(S \otimes S)_1$ and $\omega_{C \otimes 1}$, see the proof of Proposition 7.16. Such an operator is found in Section 7.5.

7.5 Complete unitarity of S and the comparison (N)

We solve the existence problem of the preceding remark, and we deduce the complete unitarity of S and (N) from the results of Gui on cosets [Gui26, Gui25]. We assume ($\mathrm{H_J}$) and keep the notation of Proposition 7.12. Let A be the vertex subalgebra of S generated by $S_1 = \mathfrak{g}' \oplus \mathfrak{g}''$, which is isomorphic to $L_{\mathfrak{sl}_2}(3,0) \otimes L_{\mathfrak{sl}_2}(13,0)$ [ArhI, Corollary 9.2], and let $C = \mathrm{Com}_S(A)$. The vector $\omega_C = \omega - \omega_A$, where ω_A is the Sugawara vector of A, is the conformal vector of C, of central charge $8 - \frac{9}{5} - \frac{13}{5} = \frac{18}{5}$. We write $\omega_{S \otimes 1} = \omega \otimes \mathbf{1}$, $\omega_{A \otimes 1} = \omega_A \otimes \mathbf{1}$, $\omega_{C \otimes 1} = \omega_C \otimes \mathbf{1}$, and similarly $\omega_{1 \otimes S}$, $\omega_{1 \otimes A}$, $\omega_{1 \otimes C}$, and we denote by $L_n^{S \otimes 1}$, $L_n^{A \otimes 1}, \dots$ the modes of the corresponding Virasoro fields. We put $\omega_{A \otimes A} = \omega_{A \otimes 1} + \omega_{1 \otimes A}$ and $\omega_{C \otimes C} = \omega_{C \otimes 1} + \omega_{1 \otimes C}$, so that $\omega_W = \omega_{S \otimes 1} + \omega_{1 \otimes S} = \omega_{A \otimes A} + \omega_{C \otimes C}$ is the conformal vector of W. We denote by $\mathrm{Rep}^u(S)$ the category of unitary S-modules [Gui26, Section 3.4 p. 44] and $\mathrm{Rep}^f(\mathcal{A}_S) = \mathrm{Rep}(\mathcal{A}_S)$ is as in Section 2.6.

Let $\Gamma = E_8 \oplus E_8 \subset \mathbb{R}^8 \oplus \mathbb{R}^8$, where E_8 is the lattice of the vectors of $\mathbb{R}^8$ whose coordinates are all in $\mathbb{Z}$ or all in $\frac{1}{2} + \mathbb{Z}$ and have an even sum, with the simple roots $\alpha_1, \dots, \alpha_8$ and the fundamental weights $\varpi_1, \dots, \varpi_8$ of [Bou68, Planche VII]. Let ε_8 be a bimultiplicative cocycle on E_8 such that $\varepsilon_8(\alpha,\beta)\varepsilon_8(\beta,\alpha)^{-1} = (-1)^{\langle \alpha, \beta \rangle}$ and $\varepsilon_8(\alpha,\alpha) = (-1)^{\langle \alpha, \alpha \rangle / 2}$, and let $V_\Gamma = M(1) \otimes \mathbb{C}_\varepsilon[\Gamma]$ be the lattice vertex operator algebra built with the cocycle $\varepsilon(\alpha,\beta) = \varepsilon_8(\alpha^{(1)},\beta^{(1)})\varepsilon_8(\alpha^{(2)},\beta^{(2)})$, where $\alpha^{(1)}$ and $\alpha^{(2)}$ are the components of $\alpha \in \Gamma$. The cocycle ε satisfies the same two identities on Γ, and it is invariant under the exchange of the two summands of Γ. Let ϕ_1 be the antilinear involution of V_Γ given by the formula of Section 4.1. For a character $\kappa \colon \Gamma \to \mathbb{R}_{>0}$, written $\kappa(\lambda) = \mathrm{e}^{\langle h_\kappa, \lambda \rangle}$, we put $\phi_\kappa = \phi_1 \circ \exp(h_\kappa(0))$, so that $\phi_\kappa(\mathrm{e}^\lambda) = \kappa(\lambda)\mathrm{e}^{-\lambda}$. By [DoL14, Theorem 4.12 p. 24], ϕ_1 is the PCT operator of a structure of unitary vertex operator algebra on V_Γ, as in part 1 of Lemma 4.2. By Proposition 7.12, there is an isomorphism of vertex operator algebras $W \to V_\Gamma$. We fix one, and we identify W with V_Γ. The space W_1 is the Lie algebra $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ for the bracket $[x,y] = x_{(0)}y$, with the invariant form defined by $x_{(1)}y = \langle x, y \rangle \mathbf{1}$, for which the roots have norm two. The Dynkin index of a homomorphism $\psi \colon \mathfrak{sl}_2 \to \mathfrak{e}_8$ is $\frac{1}{2}\langle \psi(h), \psi(h) \rangle$, where h is the standard Cartan element of $\mathfrak{sl}_2$. The level of the affine vertex subalgebra of W generated by the image of a homomorphism $\mathfrak{sl}_2 \to \mathfrak{e}_8 \oplus \mathfrak{e}_8$ is the sum of the Dynkin indices of its two components.

An explicit subalgebra. In each copy of $\mathfrak{e}_8$, the vectors $h(-1)\mathbf{1}$, $h \in \mathbb{C}^8$, and e^α, for the roots $\alpha = \pm e_i \pm e_j$ of $D_8 \subset E_8$, span a Lie subalgebra isomorphic to $\mathfrak{so}(16)$. We identify it with the orthogonal Lie algebra of $X = \mathbb{C}^{16}$, with basis $f_1^\pm, \dots, f_8^\pm$ and the symmetric form

B such that $B(f_i^+, f_j^-)$ is 1 for $i = j$ and 0 otherwise, and $B(f_i^\pm, f_j^\pm) = 0$, in such a way that $e_i(-1)\mathbf{1}$ corresponds to $f_i^+ \wedge f_i^-$ and $\mathrm{e}^{se_i+te_j}$ to $\pm f_i^s \wedge f_j^t$ for $s, t \in \{\pm\}$ and $i < j$, where $(u \wedge v)y = B(v, y)u - B(u, y)v$. Consider the orthogonal decomposition $X = X_3 \oplus X_4 \oplus X_9$ with $X_3 = \langle f_1^+, f_1^-, f_2^+ + f_2^- \rangle$, $X_4 = \langle f_3^\pm, f_4^\pm \rangle$ and $X_9 = \langle f_2^+ - f_2^-, f_5^\pm, \dots, f_8^\pm \rangle$, and fix a linear isomorphism $X_9 \cong \mathbb{C}^3 \otimes \mathbb{C}^3$ which multiplies the symmetric forms by a nonzero constant, where $\mathbb{C}^3$ carries the invariant form of the adjoint representation of $\mathfrak{sl}_2$. Let $\mathfrak{a} = \mathfrak{so}(X_3)$, let $\mathfrak{b}^\pm$ be the root subalgebras of $e_3 \pm e_4$, so that $\mathfrak{so}(X_4) = \mathfrak{b}^+ \oplus \mathfrak{b}^-$, and let $\mathfrak{c}_1, \mathfrak{c}_2 \subset \mathfrak{so}(X_9)$ be the images of $\mathfrak{sl}_2$ acting on the first and on the second factor of $\mathbb{C}^3 \otimes \mathbb{C}^3$. These are commuting subalgebras isomorphic to $\mathfrak{sl}_2$, of Dynkin indices 2, 1, 1, 6 and 6 in $\mathfrak{e}_8$. Let $\mathfrak{d}$ be the diagonal of $\mathfrak{b}^- \oplus \mathfrak{c}_2$ with respect to $\mathfrak{sl}_2$-triples of the two summands, of index 7. We define $\mathfrak{s}_1, \dots, \mathfrak{s}_4 \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8$ as the diagonal subalgebras, with respect to $\mathfrak{sl}_2$-triples,

$$\mathfrak{s}_1 \subset \mathfrak{a} \oplus \mathfrak{b}^+, \quad \mathfrak{s}_2 \subset \mathfrak{c}_1 \oplus \mathfrak{d}, \quad \mathfrak{s}_3 \subset \mathfrak{b}^+ \oplus \mathfrak{a}, \quad \mathfrak{s}_4 \subset \mathfrak{d} \oplus \mathfrak{c}_1,$$

the first summand lying in the first copy and the second summand in the second copy. Their Dynkin indices in the two copies are $(2, 1)$, $(6, 7)$, $(1, 2)$ and $(7, 6)$, so that they generate affine subalgebras of levels 3, 13, 3 and 13. We take for ε_8 the bimultiplicative map with $\varepsilon_8(\alpha_i, \alpha_j) = -1$ if $i = j$, or if $i < j$ and $\{i, j\}$ is one of the edges $\{1, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{4, 5\}$, $\{5, 6\}$, $\{6, 7\}$, $\{7, 8\}$ of the Dynkin diagram, and $\varepsilon_8(\alpha_i, \alpha_j) = 1$ otherwise. For $\lambda, \mu \in E_8$ and $x, y \in \mathbb{C}^8$, we write $\mathrm{e}^{\lambda \oplus \mu}$ and $(x \oplus y)(-1)\mathbf{1}$ for the corresponding vectors of V_Γ. Put

$$E_1 = -\mathrm{e}^{(e_1-e_2)\oplus 0} - \mathrm{e}^{(e_1+e_2)\oplus 0} - \mathrm{e}^{0\oplus(e_3+e_4)}, \quad H_1 = (2e_1 \oplus (e_3 + e_4))(-1)\mathbf{1},$$

$$\begin{aligned} E_2 = {} & \mathrm{e}^{(e_5-e_2)\oplus 0} + \mathrm{e}^{(e_2+e_5)\oplus 0} + 2\mathrm{e}^{(e_7-e_6)\oplus 0} - 2\mathrm{e}^{(e_6+e_8)\oplus 0} + \mathrm{e}^{0\oplus(e_3-e_4)} \\ & - \mathrm{e}^{0\oplus(e_6-e_2)} - \mathrm{e}^{0\oplus(e_2+e_6)} - 2\mathrm{e}^{0\oplus(e_7-e_5)} + \mathrm{e}^{0\oplus(e_5-e_8)}, \end{aligned}$$

$$H_2 = ((2e_5 + 2e_7 + 2e_8) \oplus (e_3 - e_4 + 2e_6 + 2e_7 - 2e_8))(-1)\mathbf{1},$$

let E_3, H_3, E_4, H_4 be the images of E_1, H_1, E_2, H_2 under the exchange of the two summands of Γ, and put $F_k \overset{\text{def}}{=} -\phi_\kappa(E_k)$, with κ as in Proposition 7.16. A direct computation in the root system shows that (E_k, H_k, F_k) are $\mathfrak{sl}_2$-triples spanning $\mathfrak{s}_1, \dots, \mathfrak{s}_4$ for a suitable choice of the signs and of the isomorphism $X_9 \cong \mathbb{C}^3 \otimes \mathbb{C}^3$.

Lemma 7.14 *Assume* ($\mathrm{H_J}$). *After composing the isomorphism $W \to V_\Gamma$ with an automorphism of V_Γ, the subalgebras $\mathfrak{g}' \otimes \mathbf{1}$, $\mathfrak{g}'' \otimes \mathbf{1}$, $\mathbf{1} \otimes \mathfrak{g}'$ and $\mathbf{1} \otimes \mathfrak{g}''$ of W_1 are $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ and $\mathfrak{s}_4$.*

Proof : For $x \in W_1$, the zero mode $x_{(0)}$ is a derivation of W which preserves the finite-dimensional spaces W_n, so that $\exp(x_{(0)})$ is an automorphism of W. These automorphisms act on W_1 by the inner automorphisms of the two copies of $\mathfrak{e}_8$, independently. Since ε is invariant under the exchange of the two summands of Γ, this exchange is also an automorphism of V_Γ.

Subalgebras of small index. Let $\psi\colon \mathfrak{sl}_2 \to \mathfrak{e}_8$ be a homomorphism of Dynkin index $j \in \{1, 2, 3\}$. Up to an inner automorphism, $\psi(h)$ lies in the Cartan subalgebra $\{y(-1)\mathbf{1} : y \in \mathbb{C}^8\}$ and in the closed dominant chamber defined by $\alpha_1, \dots, \alpha_8$, so that $\psi(h) = \sum_i a_i \varpi_i$, where the fundamental weights are identified with coweights and $a_i = \alpha_i(\psi(h))$ are nonnegative integers. The entries $\langle \varpi_i, \varpi_k \rangle$ of the inverse Cartan matrix of E_8 are integers at least 2, with diagonal entries 4, 8, 14, 30, 20, 12, 6, 2 [Bou68, Planche VII]. Hence $\langle \psi(h), \psi(h) \rangle \geqslant 2(\sum_i a_i)^2$, and $2j = \langle \psi(h), \psi(h) \rangle \leqslant 6$ gives $\sum_i a_i = 1$. Thus $\psi(h) = \varpi_8$, ϖ_1 or ϖ_7 for $j = 1$, 2 or 3. By Mal'cev's theorem [CoM93, Theorem 3.4.12], ψ is determined up to inner automorphisms by $\psi(h)$. For $j = 3$, the highest root θ satisfies $\theta(\varpi_7) = 3$, so that $\mathfrak{e}_8$ contains an irreducible $\psi(\mathfrak{sl}_2)$-module of dimension 4. For $j = 2$, ψ is conjugate to the inclusion of $\mathfrak{a}$, which has Dynkin index 2 since X restricts to $\mathfrak{a}$ as the sum of the adjoint representation and of 13 trivial ones.

Under $\mathfrak{a} \oplus \mathfrak{so}(X_3^\perp)$, the decomposition $\mathfrak{e}_8 = \mathfrak{so}(16) \oplus \Delta^+$, where Δ^+ is a half-spin representation, gives

$$\mathfrak{e}_8 = \Lambda^2 X_3 \oplus (X_3 \otimes X_3^\perp) \oplus \Lambda^2 X_3^\perp \oplus (\Delta_3 \otimes \Delta_{13}),$$

where Δ_3 and Δ_{13} are the spin representations of dimensions 2 and 64. Hence the centralizer of $\mathfrak{a}$ is $\mathfrak{so}(X_3^\perp) \cong \mathfrak{so}(13)$.

The two factors of level three. By [ArhI, Theorem 8.5 and Table 2], the $\mathfrak{g}'$-weights of the lowest weight spaces R_z occurring in W_1 are at most 2, see the proof of Proposition 7.12. Hence W_1 has no irreducible $\mathfrak{g}' \otimes \mathbf{1}$-submodule or $\mathbf{1} \otimes \mathfrak{g}'$-submodule of dimension 4, and the two components of each of these subalgebras have Dynkin indices $(2,1)$ or $(1,2)$. Using the exchange of the two copies, we may assume that $\mathfrak{g}' \otimes \mathbf{1}$ has index 2 in the first copy. Assume that $\mathbf{1} \otimes \mathfrak{g}'$ also has index 2 in the first copy. Then its component there lies in the centralizer $\mathfrak{so}(13)$ of the component of $\mathfrak{g}' \otimes \mathbf{1}$, and $\mathbb{C}^{13}$ restricts to it as $V(2) \oplus 10V(0)$ or $4V(1) \oplus 5V(0)$. By the decomposition displayed above, the vectors of the first copy which are invariant under $\mathfrak{g}' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}'$ are the invariant vectors of $\Lambda^2 \mathbb{C}^{13}$. Since $\Lambda^2(V(2) \oplus 10V(0)) \cong 11V(2) \oplus 45V(0)$ and $\Lambda^2(4V(1) \oplus 5V(0)) \cong 6V(2) \oplus 20V(1) \oplus 20V(0)$, they form a space of dimension 45 or 20. In the second copy, the centralizer of the first subalgebra of index one is $\mathfrak{e}_7$, the second one has index one in $\mathfrak{e}_7$, and their joint centralizer is $\mathfrak{so}(12)$, of dimension 66. Hence the subspace of W_1 of vectors invariant under $\mathfrak{g}' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}'$ has dimension 111 or 86, whereas it has dimension 78 by the proof of Proposition 7.12. Therefore $\mathbf{1} \otimes \mathfrak{g}'$ has index 2 in the second copy.

Reduction to orthogonal representations. Conjugating in each copy, we may assume that the component of $\mathfrak{g}' \otimes \mathbf{1}$ in the first copy and that of $\mathbf{1} \otimes \mathfrak{g}'$ in the second copy are equal to $\mathfrak{a}$. The components in the first copy of the three other factors commute with $\mathfrak{a}$, so that they define an orthogonal representation σ_1 of $\mathfrak{sl}_2^{\oplus 3}$ on $X_3^\perp \cong \mathbb{C}^{13}$, and similarly σ_2 in the second copy. Two isomorphic orthogonal representations of a semisimple Lie algebra are conjugate under the orthogonal group, by the isotypic decomposition and the uniqueness up to equivalence of nondegenerate symmetric or alternating forms on the multiplicity spaces. Since 13 is odd, they are conjugate under $SO(13,\mathbb{C})$. The automorphisms $\exp(x_{(0)})$, $x \in \mathfrak{so}(X_3^\perp)$, fix $\mathfrak{a}$ pointwise and induce on $X_3^\perp$ all the elements of $SO(13,\mathbb{C})$. Hence the embedding of $(S \otimes S)_1$ is determined, up to automorphisms of V_Γ, by the isomorphism classes of σ_1 and σ_2. The Dynkin indices of the factors and the character of W_1 as a module over $(S \otimes S)_1 \cong \mathfrak{sl}_2^{\oplus 4}$ are functions of these classes, by the decomposition displayed above. The Dynkin index in $\mathfrak{e}_8$ of a subalgebra of $\mathfrak{so}(X_3^\perp)$ is half the Dynkin index of its representation on $X_3^\perp$, and the Dynkin index of $V(n)$ is $\frac{1}{6}n(n+1)(n+2)$. Since the indices of the factors in each copy are at most 13, only highest weights at most 4 occur, and there are 640 isomorphism classes of such orthogonal representations of $\mathfrak{sl}_2^{\oplus 3}$ on $\mathbb{C}^{13}$. An exhaustive computer enumeration shows that exactly one pair (σ_1, σ_2) gives the character of $W_1 = (S \otimes S)_1 \oplus \bigoplus_z R_z \otimes R_{\pi(z)}$ computed from [ArhI, Table 2], namely

$$\sigma_1 \cong V(2,0,2) \oplus V(0,1,1), \quad \sigma_2 \cong V(0,2,2) \oplus V(1,1,0),$$

where σ_1 is a representation of $\mathfrak{g}'' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}' \oplus \mathbf{1} \otimes \mathfrak{g}''$ and σ_2 a representation of $\mathfrak{g}' \otimes \mathbf{1} \oplus \mathfrak{g}'' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}''$. This is the pair defined by $\mathfrak{s}_1, \ldots, \mathfrak{s}_4$. This gives the assertion. ∎

From now on, we assume that the identification $W = V_\Gamma$ satisfies the conclusion of Lemma 7.14. Let $\mathcal{G} = \mathrm{Com}_W(A \otimes A) \cap W_2$.

Lemma 7.15 *Assume* (H$_\mathrm{J}$). *The space $\mathcal{G}$ has dimension* 16, *and $\omega_{C\otimes 1}$ is one of the four vectors $u_0, \ldots, u_3 \in \mathcal{G}$ defined in the proof, whose coordinates in a rational basis of $\mathcal{G}$ lie in $\mathbb{Q}(\sqrt{-6})$.*

Proof : The vector $\omega_{C\otimes 1}$ commutes with $A \otimes \mathbf{1}$ by definition of C, and with $\mathbf{1} \otimes A$, so that it belongs to $\mathcal{G}$. The space $\mathcal{G}$ is the space of vectors of W_2 annihilated by the modes $x_{(0)}$, $x_{(1)}$,

$x_{(2)}$ of the twelve vectors E_k, H_k, F_k. Since the vectors H_k lie in the Cartan subalgebra, $\mathcal{G}$ is contained in the subspace W_2^0 of vectors of weight zero for $H_1, \dots, H_4$. No root of Γ is orthogonal to $H_1, \dots, H_4$, so that W_2^0 is spanned by the 152 vectors $h(-1)h'(-1)\mathbf{1}$ and $h(-2)\mathbf{1}$, for h, h' in a basis of $\mathbb{C}^{16}$, and by the 492 vectors e^{λ} with $\langle \lambda, \lambda \rangle = 4$ and $\langle \lambda, H_k \rangle = 0$ for $k = 1, \dots, 4$. An exact computation of the common kernel of these 36 modes on W_2^0, of dimension 644, gives $\dim \mathcal{G} = 16$. The vertex subalgebra $\mathrm{Com}_W(A \otimes A)$ has a one-dimensional weight-zero space, and its weight-one space is zero because W_1 contains no nonzero $(S \otimes S)_1$-invariant vector by [ArhI, Table 2]. Hence the product $u_{(1)}v$ is a commutative product on $\mathcal{G}$, and $u_{(3)}v$ is an invariant bilinear form. The vector $u = \omega_{C \otimes 1}$ satisfies $u_{(1)}u = 2u$ and $u_{(3)}u = \frac{9}{5}\mathbf{1}$, half the central charge of C. Moreover, $u_{(1)}$ commutes with $(S \otimes S)_1$ and acts on W_1 as $L_0^{S \otimes 1} - L_0^{A \otimes 1}$. It vanishes on $(S \otimes S)_1$. On the component of $R_z \otimes R_{\pi(z)}$ on which $\mathfrak{g}' \otimes \mathbf{1} \oplus \mathfrak{g}'' \otimes \mathbf{1}$ acts through $V(m) \otimes V(n)$, it is the scalar $h_z - \frac{m(m+2)}{20} - \frac{n(n+2)}{60}$, since the vectors of R_z are primary for A. The values $h_{c_1} = \frac{1}{3}$, $h_{c_2} = \frac{2}{3}$, $h_{d_1} = \frac{6}{13}$, $h_{d_4} = \frac{5}{13}$, $h_{d_5} = \frac{7}{13}$ and $h_{d_6} = \frac{8}{13}$ are given by [ArhI, (4.1)], and the decompositions of R_z by [ArhI, Table 2]. On each of the twenty isotypic components of W_1 for $(S \otimes S)_1$, this prescribes the eigenvalues of $u_{(1)}$ on the corresponding multiplicity space, one for each of the spaces $(S \otimes S)_1$ and $R_z \otimes R_{\pi(z)}$ which contain the component. We impose the trace of $u_{(1)}$ on every multiplicity space, the second elementary symmetric function of its eigenvalues on the ten multiplicity spaces of the components which occur in two of the spaces $R_z \otimes R_{\pi(z)}$, and the equations $u_{(1)}u = 2u$ and $u_{(3)}u = \frac{9}{5}\mathbf{1}$. These conditions are polynomial equations of degree at most two in the coordinates of u in a rational basis of $\mathcal{G}$, with rational coefficients. Let $g_0, \dots, g_{15}$ be the ordered rational basis of $\mathcal{G}$ given in the file `omega_certificate.json` of the supplementary data, and write $u = \sum_{j=0}^{15} x_j g_j$. The twenty trace equations have rank eight. Their solution space is parametrized by $t_0, \dots, t_7$ through

$$(x_0, \dots, x_7) = \left(\frac{7}{195}, \frac{1}{13}, \frac{5}{39}, t_0, -\frac{t_1}{2} - \frac{7}{195}, t_1, \frac{1}{65}, 0 \right),$$
$$(x_8, \dots, x_{15}) = \left(t_2, t_3, \frac{2}{65}, t_4, t_5, t_6, t_7, \frac{14}{195} \right).$$

After this substitution, the remaining conditions give twenty-five nonzero quadratic equations. Their ideal is equivalently given by

$$t_0 = -\frac{1}{39}, \quad t_1 = -\frac{8}{195}, \quad t_3 = t_5 = 0, \quad t_7 = -26 t_2 t_4, \quad t_6 = \frac{t_7}{2},$$
$$t_2^2 = -\frac{1}{1014}, \quad t_4^2 = -\frac{2}{507}.$$

This equality of ideals is checked by comparing reduced Gröbner bases over $\mathbb{Q}$, see the data availability statement. Hence, if $J \subset \mathbb{Q}[t_0, \dots, t_7]$ denotes the ideal of the twenty-five equations, then $\mathbb{Q}[t_0, \dots, t_7]/J$ is isomorphic to

$$\mathbb{Q}[t_2, t_4]/(1014 t_2^2 + 1, 507 t_4^2 + 2).$$

Both quadratic polynomials are separable, so this algebra has dimension four over $\mathbb{Q}$ and is reduced. Its four geometric points, which define $u_0, \dots, u_3$, are obtained by choosing independently $a, d \in \{-1, 1\}$ and setting

$$t_2 = \frac{a\sqrt{-6}}{78}, \quad t_4 = \frac{d\sqrt{-6}}{39}, \quad t_6 = \frac{ad}{39}, \quad t_7 = \frac{2ad}{39},$$

with the other coordinates as above. These are necessary constraints on $\omega_{C \otimes 1}$. Proposition 7.16 checks that the PCT operator fixes all four candidates. ∎

Proposition 7.16 *Assume* (H$_\text{J}$)*, and identify W with V_Γ as in Lemma 7.14. Let $\kappa(\lambda) = 2^{-\langle\lambda,\varpi\rangle}$, where $\varpi = (\varpi_8, \varpi_8)$ and $\varpi_8 = e_7 + e_8$, so that $\langle\lambda, \varpi\rangle$ is the sum of the coefficients of α_8 in the two components of λ. Then ϕ_κ is the PCT operator of a structure of unitary vertex operator algebra on W, and $S \otimes \mathbf{1}$, $\mathbf{1} \otimes S$ and $S \otimes S$ are unitary subalgebras for this structure.*

Proof : Let $g = \exp(-\frac{1}{2}h_\kappa(0))$, an automorphism of V_Γ acting by $\kappa(\lambda)^{-1/2}$ on $M(1) \otimes \mathrm{e}^\lambda$. Then $g\phi_1 g^{-1} = \phi_\kappa$, so that ϕ_κ is the PCT operator of the normalized invariant scalar product $(g^{-1}u|g^{-1}v)$, where $(\cdot|\cdot)$ is the scalar product attached to ϕ_1, and g^{-1} is a unitary isomorphism between the two structures.

By definition of F_k, we have $\phi_\kappa(E_k) = -F_k$ for $k = 1, \dots, 4$. Since ϕ_κ is an involution, $\phi_\kappa(F_k) = -E_k$ and $\phi_\kappa(H_k) = [\phi_\kappa(E_k), \phi_\kappa(F_k)] = [F_k, E_k] = -H_k$. Hence ϕ_κ preserves $S_1 \otimes \mathbf{1}$ and $\mathbf{1} \otimes S_1$, and therefore the affine subalgebras $A \otimes \mathbf{1}$ and $\mathbf{1} \otimes A$ and their Sugawara vectors, whose coefficients are real. It fixes ω_W, hence $\omega_{C\otimes C}$, and a second exact computation gives $\phi_\kappa(u_k) = u_k$ for $k = 0, \dots, 3$. By Lemma 7.15, ϕ_κ fixes $\omega_{C\otimes 1}$ and $\omega_{1\otimes C} = \omega_{C\otimes C} - \omega_{C\otimes 1}$.

We claim that $S \otimes \mathbf{1}$ is the set of vectors $v \in W$ annihilated by the modes $x_{(n)}$, $n \geqslant 0$, of all $x \in (\mathbf{1} \otimes S_1) \cup \{\omega_{1\otimes C}\}$. These vectors are annihilated by the nonnegative modes of the vertex subalgebra generated by this set [CKLW18, Example 5.27 p. 44], which contains the Sugawara vector $\omega_{1\otimes A}$. Hence $L_0^{1\otimes A} v = 0$ and $L_0^{1\otimes C} v = 0$, so that $L_0^{1\otimes S} v = 0$. On $M_z \otimes M_{\pi(z)}$ with $z \neq 0$, the spectrum of $L_0^{1\otimes S}$ is contained in $h_{\pi(z)} + \mathbb{N}$, with $h_{\pi(z)} > 0$. Hence $v \in S \otimes S_0 = S \otimes \mathbf{1}$. Conversely, $S \otimes \mathbf{1}$ commutes with $\mathbf{1} \otimes S$. Similarly, $\mathbf{1} \otimes S$ is the commutant of $(S_1 \otimes \mathbf{1}) \cup \{\omega_{C\otimes 1}\}$. The antilinear automorphism ϕ_κ maps the commutant of a set onto the commutant of its image. Hence ϕ_κ preserves $S \otimes \mathbf{1}$, $\mathbf{1} \otimes S$ and $S \otimes S$. These subalgebras are also stable under L_1, since $L_1 = L_1^{S\otimes 1} + L_1^{1\otimes S}$, and they are unitary subalgebras by [CKLW18, Proposition 5.23 p. 43]. ■

Theorem 7.17 *Assume* (H$_\text{J}$)*. Then S, with the unitary structure of [ArhI, Theorem 10.10], satisfies condition B of [Gui26, Section 3.4 p. 44] and Condition I+ of [Gui25, Definition 6.17 p. 70]. In particular, the following hold.*

1. *S is completely unitary, and $\mathrm{Rep}^u(S)$ is a unitary modular tensor category.*
2. *Every unitary S-module is strongly integrable, and the functor $\Pi\colon \mathrm{Rep}^u(S) \to \mathrm{Rep}^f(\mathcal{A}_S)$ of [Gui26, Section 3.4 p. 45] is an equivalence of braided C^*-tensor categories. Here the target carries the tensor product and braiding of [Gui26]. The comparison with the DHR convention used in the preceding sections is given in the proof of part 3.*
3. *For every S-submodule M of a lattice extension H_Λ, with the restriction of the scalar product H_χ of part 3 of Lemma 4.2, $\Pi(M)$ is the representation π_M of Lemma 4.7 on $\mathcal{H}_M$. The assertion* (N) *holds.*

Proof : Equip W with the unitary structure of Proposition 7.16, and let $V = \mathbf{1} \otimes S$. By Proposition 7.12, its coset is $V^c = \mathrm{Com}_W(V) = S \otimes \mathbf{1}$, and $V^{cc} = V$. By [CKLW18, Propositions 5.29 and 5.31 p. 45], $\omega_W = \nu + \nu'$ with $\nu \in V$ and $\nu' \in V^c$ the conformal vectors of the unitary vertex operator algebras V and V^c. Then $\nu - \omega_{1\otimes S} = \omega_{S\otimes 1} - \nu'$ lies in $V \cap V^c \cap W_2 = \{0\}$, so that V and V^c are unitary vertex operator algebras isomorphic to S, with the conformal vector of S. In particular, V is a unitary sub-VOA of W in the sense of [Gui26, Section 3.6.1 p. 49], with commutant V^c. Both are regular by (H$_\text{J}$) and [ABD04, Theorem 4.5]. The lattice vertex operator algebra V_Γ satisfies condition B [Gui26, Theorem 3.46 p. 57]. This property is preserved by unitary isomorphisms, and two structures of unitary vertex operator algebra on a simple vertex operator algebra are unitarily isomorphic [CKLW18, Proposition 5.19 p. 40 and p. 41], so that W satisfies condition B. By [Gui26, Theorem 3.34 p. 51], V^c satisfies condition

B. By [CKLW18, Proposition 5.19 p. 40 and p. 41] again, the isomorphism $S \to V^c$, composed with an automorphism of S, is a unitary isomorphism from S, with the unitary structure of [ArhI, Theorem 10.10], onto V^c, so that S satisfies condition B. The lattice vertex operator algebra V_Γ satisfies Condition I+ [Gui25, Example 6.19(b) p. 71], and so does W by the same transport. Since V and V^c are isomorphic to S, they are C_2-cofinite and rational under $(\mathrm{H_J})$. Hence V^c satisfies Condition I+ by [Gui25, Theorem 6.18, part 3, p. 70], and so does S, through the unitary isomorphism $S \to V^c$ above.

1. This is [Gui26, Theorem 3.15 p. 45].

2. By [Gui26, Theorems 3.16 and 3.17 p. 45], every unitary S-module is strongly integrable and Π is a fully faithful $*$-functor. By [Gui26, Corollary 3.23 p. 48], it is an isomorphism onto a braided C^*-tensor subcategory of $\mathrm{Rep}(\mathcal{A}_S)$. The functor Π is the functor of [Gui25, Definition 4.22 and Theorem 4.24 p. 43], called the CWX functor there, since both send M to the representation characterized by [Gui26, (3.8) p. 41], that is, by [Gui25, (4.6) p. 43]. The category $\mathrm{Rep}^f(\mathcal{A}_S)$ of [Gui25, p. 68] consists of the dualizable representations, which are the representations with finite statistical dimension. Since S satisfies Condition I+, this functor is essentially surjective onto $\mathrm{Rep}^f(\mathcal{A}_S)$ by [Gui25, Theorem 6.14 p. 69 and Definition 6.17 p. 70]. Hence Π is a braided equivalence, with the tensor product and braiding specified in the statement.

3. Since S is a unitary subalgebra of (H_Λ, H_χ) by Theorem 4.5, the scalar product H_χ makes M a unitary S-module. By part 1 of Lemma 4.7 and its proof, π_M is a representation of $\mathcal{A}_S$ on $\mathcal{H}_M$ which maps the closed operator $Y(v,f)|_{\mathcal{H}_S}$, f supported in I, to the closed operator $Y_M(v,f)$, of which M is a core. By the energy bounds, $Y_M(v,f)$ is also the closure of the smeared field of M on the smooth vectors. This is the defining property of the representation $\Pi(M)$, which is unique since $\mathcal{A}_S(I)$ is generated by these operators [Gui26, (3.8) p. 41], see also [Gui25, Definition 4.22 p. 43]. Hence $\Pi(M) = \pi_M$. The braided C^*-tensor category of [Gui26] is defined by Connes fusion [Gui26, p. 10]. By [Gui21, Theorem 6.6 p. 77 and (6.2) p. 74], it is equivalent to the category $\mathrm{DHR}_{I_0}(\mathcal{A}_S)$ of transportable endomorphisms localized in an interval I_0, with the tensor product $\rho \odot \sigma = \sigma \circ \rho$ and the braiding $\widetilde{c}$ of [Gui21, (6.1) p. 73], through a $*$-functor G such that $\pi_0 \circ G(\mathcal{H}_i, \xi)$ is unitarily equivalent to π_i for every object $(\mathcal{H}_i, \xi)$. For the tensor product $\rho \otimes \sigma = \rho \circ \sigma$, put $b_{\rho,\sigma} = \widetilde{c}_{\sigma,\rho}$. The identity functor, with tensor constraint $J_{\rho,\sigma} = b_{\rho,\sigma} \colon \rho \circ \sigma \to \sigma \circ \rho$, is a braided monoidal equivalence from $(\mathrm{DHR}_{I_0}(\mathcal{A}_S), \odot, \widetilde{c})$ to $(\mathrm{DHR}_{I_0}(\mathcal{A}_S), \otimes, b)$. Indeed, by the hexagon identities its coherence reduces to the Yang–Baxter equation, and $\widetilde{c}_{\rho,\sigma} J_{\rho,\sigma} = J_{\sigma,\rho} b_{\rho,\sigma}$. The two orientations in the definition of the DHR braiding give b and $b^{\mathrm{rev}}_{\rho,\sigma} = b^{-1}_{\sigma,\rho}$. The forgetful functor $\mathrm{Rep}^u(S) \to \mathrm{Rep}(S)$ is a braided equivalence, since every S-module is unitarizable and its tensor product and braiding are those of $\mathrm{Rep}(S)$ [Gui19, Theorem 7.9 p. 53], [CGGH23, Remark 4.3 p. 18]. Combining these equivalences with a quasi-inverse of Π gives a braided equivalence from $(\mathrm{DHR}_{I_0}(\mathcal{A}_S), \otimes, b)$ to $\mathrm{Rep}(S)$ which maps the class of π_M to that of M. The same functor is a braided equivalence from $(\mathrm{DHR}_{I_0}(\mathcal{A}_S), \otimes, b^{\mathrm{rev}})$ to $\mathrm{Rep}(S)^{\mathrm{rev}}$, and its composition with the functor F of Proposition 7.12, which fixes the classes of S, M_b, M_a and M_{c_0}, is a braided equivalence into $\mathrm{Rep}(S)$ with the same property. Hence (N) holds for either braiding.

■

Remark 7.18 The proof of Theorem 7.17 does not use Section 5.1. The unitarizability of the simple S-modules also follows directly from Proposition 7.16. For $0 \neq v \in M_{\pi(z)}$, the restriction of the scalar product of W to $M_z \otimes \mathbb{C}v$ is an invariant scalar product for the unitary structure induced on $S \otimes \mathbf{1}$, and a unitary isomorphism from S onto $S \otimes \mathbf{1}$ transports these scalar products to all simple S-modules, possibly permuting their isomorphism classes.

The computations of Lemmas 7.14 and 7.15 and of Proposition 7.16 are exact. For the subalgebras $\mathfrak{s}_k$ fixed above, they take place in W_1 and in the space W_2^0 of Lemma 7.15, of dimensions 496 and 644. A second implementation, with other choices of the subalgebras $\mathfrak{s}_k$ and of κ, gives $\dim \mathcal{G} = 16$, four solutions of the stronger system which prescribes the characteristic polynomial of $u_{(1)}$ on every multiplicity space, with the same Gram matrix (u_k, u_l), and a positive character κ for which ϕ_κ fixes them. The second implementation also shows that the sign characters of Γ which centralize $(S \otimes S)_1$ permute $u_0, \dots, u_3$ transitively, so that the data used in Lemma 7.15 cannot distinguish them.

Corollary 7.19 *Assume* ($\mathrm{H_J}$).

1. *The conclusions of part 4 of Theorem 7.2 hold. In particular, the net $\mathcal{A}_{H_{\Lambda_a}}$ carries $\mathcal{H}_2$, the net $\mathcal{A}_{H_{\Lambda_{c_0}}}$ carries $\mathcal{H}_3$, and $\mathcal{F}_a \cong \mathcal{H}_2$, $\mathcal{F}_{c_0} \cong \mathcal{H}_3$ for the categories of Theorem 5.6.*
2. *The conclusions of Proposition 7.10 hold.*
3. *The unitary modular tensor categories $\mathrm{Rep}^u(S)$ and $\mathrm{Rep}^f(\mathcal{A}_S)$ are equivalent, so that the correspondence of [CGGH23, Theorem 4.10 p. 21] between vertex operator algebra extensions of S and irreducible local extensions of $\mathcal{A}_S$ applies.*

Proof : 1. This follows from Theorem 7.17 and part 4 of Theorem 7.2. Since $\mathcal{F}_x = \mathrm{Rep}(\mathcal{A}_S)_{\Theta_x}$ by part 4 of Theorem 5.6, the last assertion follows from part 2 of Theorem 7.2.

2. By part 1 of Theorem 7.17, S is completely unitary.

3. This follows from parts 1 and 2 of Theorem 7.17, since S is strongly local by Theorem 4.5 and $\mathcal{A}_S$ is completely rational by Corollary 5.9. ∎

8 The algebra of case (I) and the twin algebra

Theorem 4.5 applies to any unitary conformal subalgebra of V', and Corollary 5.5 and Theorems 5.6 and 6.11 apply to any such subalgebra which is a Haagerup algebra. We record what is known for the Haagerup algebras of case (I).

Riesen's algebra. Let $\psi_{\mathrm{R}}(U^{\mathrm{R}}) \subset V'$ be the image of the algebra of [Rie26a] described in [ArhI, Proposition 10.36]. Its currents have levels $(1, 39)$ [ArhI, Proposition 10.36]. It contains the Sugawara vector ω_A of its currents and the vector $\frac{1}{2}(\omega - \omega_A)$, the unit of the Griess algebra of [ArhI, Lemma 10.1], which lies in the span of $\psi_{\mathrm{R}}(e)$ and $\psi_{\mathrm{R}}(f)$ [ArhI, Proposition 10.3 and Proposition 10.36], hence the conformal vector ω of V'. It is a Haagerup algebra in case (I) by [ArhIII, Theorem 7.13(1)], which supplies ($\mathrm{H_R}$). If its generators are fixed by a positive rescaling of the standard antilinear involution of V', as asserted in [Rie26a, Theorem 1 p. 1 and p. 4], see also the working manuscript [Rie26b, Sections 2–3 pp. 16–19], then $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ is a unitary subalgebra of V' [CKLW18, Proposition 5.23 p. 43], and Theorem 4.5 applies to it. With this compatible unitary structure, Corollary 5.5 and Theorems 5.6 and 6.11 apply as well, and the statements of Section 7 which do not depend on Sections 7.4 and 7.5 hold with ($\mathrm{H_J}$) replaced by ($\mathrm{H_R}$) and S replaced by U^{R}. The proofs of Sections 7.4 and 7.5 use the currents of levels $(3, 13)$ of an algebra of case (II). They apply to the twin $(U^{\mathrm{R}})^\natural$ by the case (II) argument of Theorem 8.6. The identification with Huang's algebra and the complete unitarity of U^{R} are established in [ArhIII, Theorem 7.13(1)–(2)]. With the induced unitary structure on the specified embedding, Proposition 7.10 applies as well.

The twin of the explicit algebra. By [ArhIII, Theorem 6.1], S is a Haagerup algebra. Its twin $S^\natural$ is the algebra of case (I) constructed in [ArhI, Theorems 12.3 and 12.4(1)]. That construction gives an embedding into V' whose image contains $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ and agrees with it through weight four. The isomorphism of the full vertex operator algebras $S^\natural \cong U^{\mathrm{R}} \cong U^{\mathrm{H}}$, where U^{H} is Huang's algebra, follows from the common fixed point construction in [ArhIII, Theorem 7.9 and Theorem 7.13(1)].

Complete unitarity of the twin algebras. Following [Gui22, Definition 1.8 p. 13], a strongly rational unitary vertex operator algebra V is completely unitary if every V-module is unitarizable and if the invariant sesquilinear forms of [Gui19, Section 6.2 p. 36] on the spaces of intertwining operators are positive definite. We use the geometric description of this positivity given by Gui [Gui24]. Let W_1, W_2 be unitary V-modules, let $A\colon W \to W'$ denote the antiunitary map $w \mapsto \langle \cdot | w \rangle$, and let θ be the PCT operator of V. For positive trinions P and Q and a sewing parameter $0 < q < r_{P,Q}$, where $r_{P,Q} > 0$ is the bound denoted $\delta\rho$ in [Gui24, Section 7.1 p. 43], put

$$K_q^V(w_1, w_2 | w_1', w_2') \stackrel{\mathrm{def}}{=} \sum_{n \geqslant 0} q^n \sum_{e \in E_n} \omega_{W_1', P}(A w_1' \otimes w_1 \otimes e) \omega_{W_2, Q}(w_2 \otimes A w_2' \otimes \theta e) \tag{8.1}$$

for $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$, where E_n is an orthonormal basis of V_n and $\omega_{W,P}$ is the basic conformal block of [Gui24, Theorem 6.11 p. 40]. This is the sewn conformal block $\omega_{W_1',P} \#_q \omega_{W_2,Q}$ evaluated at $A w_1' \otimes w_1 \otimes w_2 \otimes A w_2'$, since the sewing contracts V_n against its dual through the identification $A\theta$ of V with its contragredient module [Gui24, Definition 3.4 p. 23 and (6.1e) p. 37]. The fusion product $W_1 \boxtimes W_2$ is geometrically positive if $\sum_{k,l} K_q^V(w_k, \tilde{w}_k | w_l, \tilde{w}_l) \geqslant 0$ for all finite families (w_k) in W_1 and $(\tilde{w}_k)$ in W_2. This does not depend on P, Q and q [Gui24, Lemma 7.2 p. 44]. If V is of CFT type and strongly rational and if $W_1 \boxtimes W_2$ is unitarizable, then $W_1 \boxtimes W_2$ is geometrically positive if and only if the form of [Gui19] on it is positive [Gui24, Corollary 8.4 p. 53]. Consequently, a strongly rational unitary vertex operator algebra all of whose modules are unitarizable is completely unitary if and only if $W_1 \boxtimes W_2$ is geometrically positive for all irreducible unitary V-modules W_1 and W_2, see also [Gui24, p. 47].

Lemma 8.1 *Let U be a Haagerup algebra equipped with a structure of unitary vertex operator algebra, with PCT operator θ. Then U_1 has a Cartan subalgebra $\mathfrak{h}$ with $\theta(\mathfrak{h}) = \mathfrak{h}$. Put $\mathfrak{h}_{\mathbb{R}} \stackrel{\mathrm{def}}{=} \{h \in \mathfrak{h} : \theta h = -h\}$. Then $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus \mathrm{i}\mathfrak{h}_{\mathbb{R}}$ as real vector spaces. On the algebraic finite-energy subspace of any ordinary unitary U-module, the operators $h_{(0)}$, $h \in \mathfrak{h}_{\mathbb{R}}$, are symmetric and essentially self-adjoint. Their self-adjoint closures strongly commute.*

Proof : Since U is unitary, it is self-contragredient and of CFT type, so that $L_1 U_1 = 0$ [Li94]. For $x \in U_1$, the formula [CKLW18, (95) p. 43] for the adjoint of a mode gives $x_{(n)}^+ = -(\theta x)_{(-n)}$ on U, and the same formula holds on every unitary U-module by the definition of unitary modules. The map θ restricts to an antilinear involutive automorphism of the Lie algebra $U_1 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ [ArhI, Theorem 8.5]. Its fixed point set $\mathfrak{k}$ is a real form of U_1, and for $x \in \mathfrak{k}$ the operator $x_{(0)}$ is skew-symmetric on the algebraic finite-energy subspace. Its restriction to U_1 is skew-adjoint. Hence $\mathrm{ad}(\mathfrak{k})$ consists of skew-adjoint operators for the scalar product of U_1, and since the adjoint representation of the semisimple Lie algebra U_1 is faithful, $\mathfrak{k}$ is a compact real form. A maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ gives the Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathrm{i}\mathfrak{t}$, which is θ-stable, and $\mathfrak{h}_{\mathbb{R}} = \mathrm{i}\mathfrak{t}$. For $h \in \mathfrak{h}_{\mathbb{R}}$, we have $h_{(0)}^+ = -(\theta h)_{(0)} = h_{(0)}$. Each $h_{(0)}$ preserves the finite-dimensional eigenspaces of L_0. On each such space, the operators $h_{(0)}$, $h \in \mathfrak{h}_{\mathbb{R}}$, are

commuting Hermitian matrices and therefore admit a common orthonormal eigenbasis. Taking the union of these bases gives a common orthonormal eigenbasis of the Hilbert space completion. The operators on the algebraic finite-energy subspace are consequently essentially self-adjoint, and their closures are diagonal in this common basis. In particular, their spectral projections commute. ■

All Cartan subalgebras of U_1 are conjugate under inner automorphisms $\exp(x_{(0)})$ of U, $x \in U_1$, and the Heisenberg commutant and the associated lattice do not depend on this choice up to isomorphism [ArhI, Section 12]. We therefore apply the results of [ArhI, Section 12] to a θ-stable Cartan subalgebra.

Lemma 8.2 *Let U be a Haagerup algebra equipped with a structure of unitary vertex operator algebra, let $\mathfrak{h} \subset U_1$ be a θ-stable Cartan subalgebra and let $\Omega = \mathrm{Com}_U(M(\mathfrak{h}))$, with the unitary structure induced by U. If U is completely unitary, then Ω is completely unitary.*

Proof : Unitary subalgebras. The Heisenberg subalgebra $M(\mathfrak{h})$ is generated by the θ-stable space $\mathfrak{h}$ of quasi-primary vectors, hence it is a unitary subalgebra of U [CKLW18, Example 5.26 p. 44]. By [CKLW18, Example 5.27 p. 44], Ω and $V_L \overset{\text{def}}{=} \mathrm{Com}_U(\Omega)$ are unitary subalgebras, and Ω and V_L form a dual pair, L being the associated lattice [ArhI, Proposition 12.2]. By [Ten19a, Proposition 2.21 p. 16], the span B of the vectors $a_{(-1)}b$, $a \in \Omega$, $b \in V_L$, is a unitary conformal subalgebra of U, unitarily isomorphic to $\Omega \otimes V_L$. We write $B = \Omega \otimes V_L$.

Weights. The algebra Ω is strongly rational by [ArhI, Proposition 12.2]. By the same proposition, $U = \bigoplus_{\alpha \in L'/L} U^\alpha$ with $U^\alpha = \Omega^{\tau(\alpha)} \otimes V_{\alpha+L}$ and $U^0 = B$, and the $\mathfrak{h}$-weights of U^α lie in $\alpha + L$. Let X be a unitary U-module. By Lemma 8.1, the finite-dimensional eigenspaces of L_0 in X have common orthonormal eigenbases for the operators $h_{(0)}$, $h \in \mathfrak{h}_\mathbb{R}$. Thus X is the algebraic direct sum of its mutually orthogonal $\mathfrak{h}$-weight spaces, and its Hilbert space completion is their Hilbert direct sum. For $v \in U^\alpha$, the operator $Y_X(v,z)$ shifts the weights by elements of $\alpha + L$, and the Virasoro modes of U, which are those of B, preserve the weights and each U^α.

Strong unitarity of Ω. Let μ be an irreducible Ω-module. By the proof of [ArhI, Proposition 12.2, part 2], there are $\beta \in L'/L$ and a U-module X containing $\mu \otimes V_{\beta+L}$ as a B-submodule, and we may take X irreducible. Since U is completely unitary, X admits a unitary structure. Fix a non-zero $m \in V_{\beta+L}$ and put $\langle u|u'\rangle_\mu = \langle u \otimes m|u' \otimes m\rangle_X$. For $a \in \Omega$, the modes of $a \otimes \mathbf{1}$ act on the first factor, $\theta(a \otimes \mathbf{1}) = \theta a \otimes \mathbf{1}$, and $L_n(a \otimes \mathbf{1}) = L_n^\Omega a \otimes \mathbf{1}$ for $n \geqslant 0$. Hence $\langle \cdot|\cdot\rangle_\mu$ is an invariant scalar product on μ. Thus every Ω-module is unitarizable, and so is every B-module, by [Gui22, Proposition 3.31 p. 48] and its proof, since V_L is completely unitary [Gui22, Corollary 3.32 p. 49].

Slices. Let μ_1, μ_2 be irreducible Ω-modules, and choose irreducible unitary U-modules X_i containing $Y_i = \mu_i \otimes V_{\beta_i+L}$ as above. We give Y_i the scalar product of X_i. Two invariant scalar products on the irreducible B-module Y_i differ by a positive endomorphism of Y_i, which is a positive scalar by Schur's lemma. Hence the scalar product of Y_i is a positive multiple of the tensor product of the scalar product of μ_i and of an invariant scalar product of V_{β_i+L}. The map A of X_1 sends Y_1 onto the B-submodule of X_1' formed by the linear forms $\langle \cdot|w\rangle$, $w \in Y_1$, which vanish on the orthogonal complement of Y_1, a B-submodule since the adjoints of the modes of B are modes of B, so that $A(Y_1)$ is unitarily isomorphic to Y_1', with weights in $-\beta_1 + L$. Let $w_1, w_1' \in Y_1$ and $w_2, w_2' \in Y_2$, and choose the bases E_n in (8.1) adapted to the orthogonal decomposition $U_n = \bigoplus_\alpha U_n^\alpha$. By [Gui24, (6.11) and (6.12) p. 40], $\omega_{X_1',P}(Aw_1' \otimes w_1 \otimes e)$ is of the form $\langle Y_{X_1'}(\tilde{e}, \mathrm{i})T_1 Aw_1'|AT_2 w_1\rangle$, where $\tilde{e} = 2^{L_0}\mathrm{e}^{-\mathrm{i}L_1}T_3 e$ and T_1, T_2, T_3 are the operators $U_0(\cdot)$ of [Gui24, (6.12) p. 40]. They are built from the modes L_n, $n \geqslant 0$, which commute with the operators $h_{(0)}$, $h \in \mathfrak{h}$, and preserve each U^α. Hence $\tilde{e} \in U^\alpha$ if $e \in U^\alpha$, and T_1, T_2 preserve

the $\mathfrak{h}$-weights. For $\alpha \neq 0$, the two vectors have weights in the disjoint cosets $-\beta_1 + \alpha + L$ and $-\beta_1 + L$, so that this term vanishes. For $\alpha = 0$, we have $e \in B$, $\theta e \in B$, and the two factors are the basic conformal blocks of the vertex operator algebra B for the unitary modules Y_1' and Y_2. Hence

$$K_q^U(w_1, w_2|w_1', w_2') = K_q^B(w_1, w_2|w_1', w_2'). \tag{8.2}$$

Since U is completely unitary, $X_1 \boxtimes X_2$ is unitarizable and its form is positive, so that it is geometrically positive [Gui24, Corollary 8.4 p. 53]. By (8.2), $Y_1 \boxtimes Y_2$ is geometrically positive for B, up to the positive factors coming from the normalization of the scalar products, which multiply K_q^B by a positive constant.

From B to Ω. On elementary tensors, the formula [Gui24, (6.11) p. 40] and the decompositions $L_n = L_n^\Omega \otimes 1 + 1 \otimes L_n^{V_L}$ and $Y(u \otimes m, z) = Y(u, z) \otimes Y(m, z)$ show that the basic conformal blocks of B are the products of those of Ω and of V_L, and the contraction over $B_n = \bigoplus_{a+b=n} \Omega_a \otimes (V_L)_b$ factorizes. The three sewing series converge absolutely for $0 < q < r_{P,Q}$ [Gui24, Section 7.1 p. 43], which justifies regrouping the sum. Hence $K_q^B = K_q^\Omega \cdot K_q^{V_L}$ on elementary tensors. Fix non-zero vectors $m_i \in V_{\beta_i + L}$, and put $w_k = u_k \otimes m_1$, $\tilde{w}_k = \tilde{u}_k \otimes m_2$ for finite families (u_k) in μ_1 and $(\tilde{u}_k)$ in μ_2. Then

$$\sum_{k,l} K_q^B(w_k, \tilde{w}_k|w_l, \tilde{w}_l) = K_q^{V_L}(m_1, m_2|m_1, m_2) \sum_{k,l} K_q^\Omega(u_k, \tilde{u}_k|u_l, \tilde{u}_l) \geqslant 0.$$

The series defining $K_q^{V_L}(m_1, m_2|m_1, m_2)$ converges absolutely for $0 \leqslant q < r_{P,Q}$, and its value at $q = 0$ is $\omega_{N_1', P}(Am_1 \otimes m_1 \otimes \mathbf{1}) \omega_{N_2, Q}(m_2 \otimes Am_2 \otimes \mathbf{1})$, with $N_i = V_{\beta_i + L}$, which is positive by [Gui24, Proposition 6.13 p. 42]. For q small enough, $K_q^{V_L}(m_1, m_2|m_1, m_2) > 0$, and $\mu_1 \boxtimes \mu_2$ is geometrically positive for Ω. Since $\mu_1 \boxtimes \mu_2$ is unitarizable, its form is positive by [Gui24, Corollary 8.4 p. 53]. This holds for all irreducible μ_1, μ_2, so that Ω is completely unitary. ■

Theorem 8.3 *Let U be a Haagerup algebra and $U^\natural$ its twin [ArhI, Theorem 12.3].*

1. *If U is completely unitary for some structure of unitary vertex operator algebra, then $U^\natural$ carries, for each structure of unitary vertex operator algebra on $V_{L^\natural}$, one and only one structure of unitary vertex operator algebra extending the tensor product structure of $\Omega_U \otimes V_{L^\natural}$, where Ω_U has the structure of Lemma 8.2, and $U^\natural$ is completely unitary for this structure.*

2. *The following are equivalent: U is completely unitary, $U^\natural$ is completely unitary, and Ω_U is completely unitary, each for some, equivalently every, structure of unitary vertex operator algebra.*

Proof : Complete unitarity is preserved by unitary isomorphisms of unitary vertex operator algebras, by transport of structure. Since two structures of unitary vertex operator algebra on a simple vertex operator algebra are unitarily isomorphic [CKLW18, Proposition 5.19 p. 40 and p. 41], see also [CGGH23, Remark 4.4 p. 18], complete unitarity of a simple unitary vertex operator algebra does not depend on its unitary structure.

1. Choose a θ-stable Cartan subalgebra of U_1 (Lemma 8.1). By Lemma 8.2, Ω_U is completely unitary, and so is the lattice algebra $V_{L^\natural}$ [Gui22, Corollary 3.32 p. 49]. By [Gui22, Proposition 3.31 p. 48], the regular algebra $\Omega_U \otimes V_{L^\natural}$ is completely unitary. By [ArhI, Theorem 12.3, part 1], $U^\natural$ is a simple vertex operator algebra of CFT type extending $\Omega_U \otimes V_{L^\natural}$ with the same conformal vector. The assertion follows from [CGGH23, Theorem 4.7 p. 19].

2. If U is completely unitary, so is $U^\natural$ by part 1. If $U^\natural$ is completely unitary, so is its Heisenberg commutant by Lemma 8.2, and it is isomorphic to Ω_U [ArhI, Theorem 12.3(1)]. If Ω_U is completely unitary, so is the regular algebra $\Omega_U \otimes V_L$ [Gui22, Proposition 3.31 p. 48 and Corollary 3.32 p. 49]. Since U is a simple extension of $\Omega_U \otimes V_L$ of CFT type with the same conformal vector [ArhI, Proposition 12.2, part 1], U is completely unitary by [CGGH23, Theorem 4.7 p. 19]. ■

Remark 8.4 Theorem 8.3 shows that complete unitarity is a property of the pair $\{U, U^\natural\}$, which forms a single hyperbolic genus [ArhI, Theorem 12.3, part 4], and of their common Heisenberg commutant, and its proof uses no computer calculation beyond those of [ArhI, Theorem 8.5 and Section 12]. Under $(\mathrm{H_J})$, S is completely unitary by Theorem 7.17, so that $S^\natural$ and Ω_S are completely unitary by part 2 of Theorem 8.3. Every Haagerup algebra with currents of levels $(1, 39)$ is then isomorphic to $S^\natural$ [ArhI, Theorem 12.4, part 1], hence completely unitary. This applies to the algebra of [Hua26b], whose currents have levels $(1, 39)$ [Hua26b, Abstract p. 1 and Section 3 p. 3], as soon as it is a Haagerup algebra, and to U^{R} under $(\mathrm{H_R})$, whose currents have levels $(1, 39)$ [ArhI, Proposition 10.36]. These two conclusions hold without $(\mathrm{H_J})$ by Theorem 8.6. Conversely, if the algebra of [Hua26b] is a Haagerup algebra and is completely unitary, part 2 of Theorem 8.3 gives the complete unitarity of S under $(\mathrm{H_J})$ without the computations of Section 7.5.

Corollary 8.5 *Assume* $(\mathrm{H_J})$.

1. *The vertex operator algebras Ω_S and $S^\natural$ satisfy, for every structure of unitary vertex operator algebra, in particular for the structure induced by S and for that of part 1 of Theorem 8.3, Condition I+ of [Gui25, Definition 6.17 p. 70]. In particular, they are strongly local, their conformal nets are completely rational, and their CWX functors [Gui25, Theorem 4.24 p. 43], equipped with the Wassermann tensorators, are equivalences of braided C^*-tensor categories onto the categories of representations of finite statistical dimension.*

2. *Let $U \subset V'$ be a Haagerup algebra which contains the conformal vector of V' and is a unitary subalgebra for a normalized invariant scalar product $\{\cdot|\cdot\}$ on V', and equip each H_Λ with the scalar product $\{\cdot|\cdot\}_\Lambda$ of part 2 of Lemma 4.2. Then U satisfies Condition I+ for the induced unitary structure. For each U-submodule M of H_Λ, with the restriction of $\{\cdot|\cdot\}_\Lambda$, the representation of $\mathcal{A}_U$ attached to M by [Gui25, Definition 4.22 p. 43] is the representation π_M of Lemma 4.7 on $\mathcal{H}_M$. Consequently, with the notation of Section 5.2, there is a braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_U) \to \mathrm{Rep}(U)$ with $\Phi(\pi_0) \cong U$ and $\Phi(\pi_x) \cong M_x$ for $x \in \{b, a, c_0\}$. We call this assertion the analogue of* (N) *for U.*

In particular, under $(\mathrm{H_J})$ *and* $(\mathrm{H_R})$*, part 2 applies to $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ as soon as it is a unitary subalgebra of V' for some normalized invariant scalar product, as asserted in [Rie26a, Theorem 1 p. 1 and p. 4].*

Proof : 1. By Theorem 7.17, S satisfies Condition I+. Choose the Cartan subalgebra of Lemma 8.1 for the prescribed unitary structure on S. The associated lattice algebra V_L is a unitary subalgebra of S, its commutant is Ω_S, Ω_S is strongly rational by [ArhI, Proposition 12.2], and V_L is regular by [Gui22, Corollary 3.32 p. 49]. Consequently, [Gui25, Theorem 6.18(3) p. 70] implies that Ω_S satisfies Condition I+. The even lattice algebra $V_{L^\natural}$ satisfies Condition I+ by [Gui25, Example 6.19(b) p. 71]. Thus $\Omega_S \otimes V_{L^\natural}$ satisfies Condition I+ by [Gui25, Theorem 6.18(1) p. 70]. The twin $S^\natural$ is a simple extension of this algebra with the same conformal vector by [ArhI, Theorem 12.3]. Since Condition I includes complete unitarity [Gui25, Definition 4.15 p. 41], the algebra $\Omega_S \otimes V_{L^\natural}$ is completely unitary, and the unique unitary structure supplied by

[CGGH23, Theorem 4.7 p. 19] makes $S^\natural$ a unitary extension of it. Applying [Gui25, Theorem 6.18(2) p. 70], we obtain Condition I+ for $S^\natural$. The assertions about the nets and the functors follow from [Gui25, (0.3) p. 8, Remark 4.16 p. 41, Theorem 6.14 pp. 68–69 and Definition 6.17 p. 70].

2. Under ($\mathrm{H_J}$), every Haagerup algebra is isomorphic to S or $S^\natural$ by [ArhI, Theorem 12.4]. Condition I+ is preserved under unitary isomorphism, since such an isomorphism transports the vertex operators, their energy bounds, the local algebras and the CWX functor. Two unitary structures on a simple vertex operator algebra are unitarily isomorphic by [CKLW18, Proposition 5.19 pp. 40–41]. Hence U satisfies Condition I+ for its prescribed induced structure.

By Lemma 4.2, H_Λ can be equipped with a unitary structure extending that of V', so that U is a unitary conformal subalgebra of H_Λ. For a U-submodule $M \subset H_\Lambda$, Lemma 4.7 identifies the closed smeared field in the representation π_M with the closure of the smeared module field on M. This is the defining property (4.6) of the representation of [Gui25, Definition 4.22 p. 43]. The uniqueness in this definition therefore identifies the two representations on $\mathcal{H}_M$. Taking the inverse of the CWX braided equivalence and forgetting the unitary structures gives the required equivalence with $\mathrm{Rep}(U)$. As in the proof of Theorem 7.17, the passage to DHR endomorphisms uses [Gui21, Theorem 6.6 p. 77]. If the angular orientation of Section 7 gives the reversed braiding, this is handled by the label-preserving equivalence of part 1 of Proposition 7.12, transported through $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$. This equivalence fixes the classes 0, b, a and c_0, so the stated identifications of modules are preserved. ∎

Theorem 8.6 *Let U be a Haagerup algebra.*

1. *The algebra U admits a structure of unitary vertex operator algebra and satisfies Condition I+ of [Gui25, Definition 6.17 p. 70] for every such structure. In particular, U is completely unitary.*

2. *Suppose that $U \subset V'$ contains the conformal vector of V' and is a unitary subalgebra for a normalized invariant scalar product on V'. Then the analogue of* (N) *holds for this embedding: there is a braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_U) \to \mathrm{Rep}(U)$ with $\Phi(\pi_0) \cong U$ and $\Phi(\pi_x) \cong M_x$ for $x \in \{b, a, c_0\}$, where the representations π_x are those of Lemma 4.7 in the two lattice extensions of V'.*

Proof : Case (II). Let X be a Haagerup algebra in case (II). The construction in the proof of Proposition 7.12 uses only the braided equivalence $\mathrm{Rep}(X) \cong Z(\mathcal{H}_3)$, positivity, strong rationality, and the levels and module data of [ArhI, (4.1), Theorem 8.5 and Table 2]. It therefore gives a holomorphic extension

$$W_X = \bigoplus_z M_z \otimes M_{\pi(z)} \cong V_{E_8 \oplus E_8}$$

of $X \otimes X$, with $X \otimes \mathbf{1}$ and $\mathbf{1} \otimes X$ as mutual commutants. No unitary structure on X is needed for this construction.

The affine subalgebra generated by X_1 and its commutant have the properties of [ArhI, Corollary 9.2]. Hence the proofs of Lemmas 7.14 and 7.15 apply to W_X: they use only these properties and the module data above, and their computations take place in V_Γ. The PCT operator constructed in Proposition 7.16 preserves both copies of X_1 and fixes all four candidates for the conformal vector of the affine commutant. The commutant argument in its proof therefore makes $X \otimes \mathbf{1}$ and $\mathbf{1} \otimes X$ unitary subalgebras of W_X. Their induced conformal vectors are the prescribed ones, as in the first paragraph of the proof of Theorem 7.17. This gives a unitary structure on X. The lattice algebra W_X satisfies Condition I+ by [Gui25, Example

6.19(b) p. 71] and transport of its unitary structure. Since the two mutual commutants are strongly rational, [Gui25, Theorem 6.18(3) p. 70] gives Condition I+ for X.

Case (I). If U is in case (I), then $X = U^{\natural}$ is in case (II) and $X^{\natural} \cong U$ by [ArhI, Theorem 12.3, parts 1 and 3]. Give X the unitary structure just constructed and choose a stable Cartan subalgebra as in Lemma 8.1. Its associated lattice algebra V_L is a unitary subalgebra, as in the proof of Lemma 8.2, and is regular by [Gui22, Corollary 3.32 p. 49]. Its commutant Ω_X is strongly rational by [ArhI, Proposition 12.2]. Thus [Gui25, Theorem 6.18(3) p. 70] gives Condition I+ for Ω_X. The algebra $\Omega_X \otimes V_{L^\natural}$ satisfies Condition I+ by [Gui25, Theorem 6.18(1) and Example 6.19(b) pp. 70–71]. In particular, it is completely unitary, since Condition I includes complete unitarity [Gui25, Definition 4.15 p. 41]. The simple extension $X^{\natural}$ of this algebra is therefore a unitary extension by [CGGH23, Theorem 4.7 p. 19], and it satisfies Condition I+ by [Gui25, Theorem 6.18(2) p. 70]. This proves the existence assertion for U. In both cases, any two unitary structures on the simple algebra U are unitarily isomorphic by [CKLW18, Proposition 5.19 pp. 40–41]. Transport of vertex operators, energy bounds and the CWX functor preserves Condition I+, so it holds for every unitary structure on U. This proves part 1.

The prescribed embedding. Under the hypotheses of part 2, equip U with the induced unitary structure and extend the scalar product of V' to each H_Λ by Lemma 4.2. Part 1 gives Condition I+ for this structure on U. Its net is consequently completely rational and its CWX functor is a braided equivalence onto the representations of finite statistical dimension [Gui25, (0.3) p. 8, Remark 4.16 p. 41, Theorem 6.14 pp. 68–69 and Definition 6.17 p. 70]. For every U-submodule $M \subset H_\Lambda$, Lemma 4.7 identifies the closed smeared fields in π_M with those defining the representation of [Gui25, Definition 4.22 p. 43], so the two representations coincide. Taking the inverse equivalence and forgetting the unitary structures gives the stated correspondence. The passage to the DHR convention is the one in part 3 of the proof of Theorem 7.17, using [Gui21, Theorem 6.6 p. 77 and (6.2) p. 74]. For the opposite angular orientation, the categorical argument of part 1 of Proposition 7.12, transported through $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$, gives a braid-reversing equivalence fixing the classes $0, b, a, c_0$. Thus the correspondence is unchanged, and part 2 follows. ∎

For the case (I) algebra, part 2 applies to the unitary conformal embedding of Huang's algebra described in [ArhIII, Remark 7.14]. It applies to $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ for a normalized invariant scalar product on V' inducing a unitary conformal embedding, as asserted in [Rie26a, Theorem 1 p. 1 and p. 4]. For that embedding, the analogue of part 4 of Theorem 7.2 and the conclusions of Proposition 7.10 follow as well.

AI statement. The author acknowledges the use of AI tools for language polishing, LaTeX editing, exploratory mathematical discussions and feedback, for drafting parts of the arguments, and for writing and running the computer programs of Section 7.5 and the certificate `omega_certificate.json`. The author takes full responsibility for the content of the manuscript.

Competing interests The author declares that he has no competing interests.

Data availability No data sets were generated during this study, apart from the exact outputs of the computer programs of Section 7.5. The proofs of Section 6 do not use computer calculations. The identities of Lemma 6.4 and the arithmetic of Lemma 6.5 and Proposition 6.6 were checked with short Python scripts, which are included in the archive described below.

Apart from these checks and from the computations of Section 7.5, the paper uses no computation beyond those of [ArhI]. The programs for Section 7.5, in two implementations, their common input tables `t12_table.json` and `t12_W1.json`, which encode [ArhI, Table 2], and their exact outputs, including the cocycle ε_8, the $\mathfrak{sl}_2$-triples of $\mathfrak{s}_1, \dots, \mathfrak{s}_4$, the vectors $u_0, \dots, u_3$ and the character κ, are collected in the supplementary archive `Calculs-HU-v08`. The file `README-Calculs-HU-v08.txt` of this archive gives the order of execution and lists the consistency checks which are not used in the proofs and require additional data from [ArhI]. The files `omega_certificate.json` and `verify_omega_certificate.py` of the same archive give a portable certificate for the affine coordinates, ideal equality and PCT checks used in Lemma 7.15 and Proposition 7.16. The verifier requires only Python and SymPy. It takes the mode-derived product and action tensors as input and does not recompute the full commutant kernel or these tensors from vertex operators. The additional programs in `verification-independante` reconstruct the commutant kernel, the product and bilinear form on $\mathcal{G}$, and the traces and traces of products of the multiplicity-space actions directly from lattice current modes, without importing the original vertex operator algebra programs. These trace invariants determine all the linear and quadratic constraints used in Lemma 7.15. The accompanying README states the inputs, dependencies and exact rank checks. These calculations are exact algebraic computations, not a formal verification of the vertex operator algebra implementation.

Bibliography

[ABD04] T. Abe, G. Buhl and C. Dong. Rationality, regularity, and C_2-cofiniteness. Trans. Amer. Math. Soc. 356 (2004), no. 8, 3391–3402. 30, 31, 53

[AHKZ26] J. Albert, Y. Honda, J. Kaidi and Y. Zheng. Haagerup symmetry in $(E_8)_1$? Phys. Rev. Lett. 136 (2026), 091603. Page and equation references are to arXiv:2512.08225v2. 3, 4, 18, 20

[ArhI] C. Arhancet. Haagerup vertex operator algebras at central charge eight. Preprint, 2026. arXiv:2611.01618. 2, 3, 4, 8, 10, 11, 12, 13, 17, 18, 19, 20, 21, 22, 23, 29, 30, 33, 48, 49, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62

[ArhIII] C. Arhancet. The Haagerup center at central charge eight via conformal nets. Preprint, 2026. 1, 3, 4, 6, 7, 8, 13, 33, 43, 55, 56, 61

[AsH99] M. Asaeda and U. Haagerup. Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$. Comm. Math. Phys. 202 (1999), no. 1, 1–63. 2

[Bis16] M. Bischoff. A remark on CFT realization of quantum doubles of subfactors: case index < 4. Lett. Math. Phys. 106 (2016), no. 3, 341–363. 2

[Bis17a] M. Bischoff. The relation between subfactors arising from conformal nets and the realization of quantum doubles. In Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday, 15–24, Proc. Centre Math. Appl. Austral. Nat. Univ. 46, Austral. Nat. Univ., Canberra, 2017. Page references are to this published version of arXiv:1511.08931. 2, 6, 15, 32, 45

[Bis17b] M. Bischoff. Generalized orbifold construction for conformal nets. Rev. Math. Phys. 29 (2017), no. 1, 1750002. Page references are to arXiv:1608.00253. 6, 15, 32, 45, 46

[BiJ22] M. Bischoff and C. Jones. Computing fusion rules for spherical G-extensions of fusion categories. Selecta Math. (N.S.) 28 (2022), no. 2, Paper No. 26. Page references are to arXiv:1909.02816v2. 42

[BKL15] M. Bischoff, Y. Kawahigashi and R. Longo. Characterization of 2D rational local conformal nets and its boundary conditions: the maximal case. Doc. Math. 20 (2015), 1137–1184. Page references are to arXiv:1410.8848v2. 41, 46

[BKLR15] M. Bischoff, Y. Kawahigashi, R. Longo and K.-H. Rehren. Tensor categories and endomorphisms of von Neumann algebras. With applications to quantum field theory.

SpringerBriefs in Mathematical Physics 3, Springer, Cham, 2015. Page references are to arXiv:1407.4793v3. 14, 15, 32, 44

[BKLR16] M. Bischoff, Y. Kawahigashi, R. Longo and K.-H. Rehren. Phase boundaries in algebraic conformal QFT. Comm. Math. Phys. 342 (2016), no. 1, 1–45.

[BoS25] L. E. Bottini and S. Schäfer-Nameki. Construction of a gapless phase with Haagerup symmetry. Phys. Rev. Lett. 134 (2025), 191602. arXiv version: arXiv:2410.19040v4. 3

[Bou68] N. Bourbaki. Groupes et algèbres de Lie, Chapitres IV, V et VI. Hermann, Paris, 1968. 49, 50

[Bro20] A. Brothier. On Jones' connections between subfactors, conformal field theory, Thompson's groups and knots. Celebratio Mathematica, Vaughan F. R. Jones volume, Mathematical Sciences Publishers, 2020. Page references are to arXiv:1912.07140v2. 2

[CGGH23] S. Carpi, T. Gaudio, L. Giorgetti and R. Hillier. Haploid algebras in C^*-tensor categories and the Schellekens list. Comm. Math. Phys. 402 (2023), no. 1, 169–212. Page references are to arXiv:2211.12790v3. 15, 16, 44, 46, 47, 49, 54, 55, 58, 59, 60, 61

[CKLW18] S. Carpi, Y. Kawahigashi, R. Longo and M. Weiner. From vertex operator algebras to conformal nets and back. Mem. Amer. Math. Soc. 254 (2018), no. 1213. Page and equation references are to arXiv:1503.01260v4. 13, 14, 21, 22, 23, 24, 25, 31, 32, 53, 54, 55, 56, 57, 58, 60, 61

[CKM22] T. Creutzig, S. Kanade and R. McRae. Gluing vertex algebras. Adv. Math. 396 (2022), Paper No. 108174. Page references are to arXiv:1906.00119v2. 19, 48

[CKM24] T. Creutzig, S. Kanade and R. McRae. Tensor categories for vertex operator superalgebra extensions. Mem. Amer. Math. Soc. 295 (2024), no. 1472. Page references are to arXiv:1705.05017v2. 17

[CoM93] D. H. Collingwood and W. M. McGovern. Nilpotent orbits in semisimple Lie algebras. Van Nostrand Reinhold, New York, 1993. 50

[DLM00] C. Dong, H. Li and G. Mason. Modular-invariance of trace functions in orbifold theory and generalized Moonshine. Comm. Math. Phys. 214 (2000), no. 1, 1–56. Page references are to arXiv:q-alg/9703016. 26

[DMNO13] A. Davydov, M. Müger, D. Nikshych and V. Ostrik. The Witt group of non-degenerate braided fusion categories. J. Reine Angew. Math. 677 (2013), 135–177. Page references are to arXiv:1009.2117v2. 10, 11, 12, 15, 16, 17, 18, 19, 40, 41, 48

[Don93] C. Dong. Vertex algebras associated with even lattices. J. Algebra 161 (1993), no. 1, 245–265. 29

[DJX13] C. Dong, X. Jiao and F. Xu. Quantum dimensions and quantum Galois theory. Trans. Amer. Math. Soc. 365 (2013), no. 12, 6441–6469. Page references are to arXiv:1201.2738v1. 12

[DoL14] C. Dong and X. Lin. Unitary vertex operator algebras. J. Algebra 397 (2014), 252–277. Page references are to arXiv:1308.2361v1. 21, 49

[DoM04] C. Dong and G. Mason. Holomorphic vertex operator algebras of small central charge. Pacific J. Math. 213 (2004), no. 2, 253–266. Page references are to arXiv:math/0203005. 19, 48

[DoX06] C. Dong and F. Xu. Conformal nets associated with lattices and their orbifolds. Adv. Math. 206 (2006), no. 1, 279–306. Page references are to arXiv:math/0411499v2. 22, 23

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik. Tensor categories. Mathematical Surveys and Monographs 205, American Mathematical Society, Providence, RI, 2015. 16, 46

[ENO05] P. Etingof, D. Nikshych and V. Ostrik. On fusion categories. Ann. of Math. (2) 162 (2005), no. 2, 581–642. Page references are to arXiv:math/0203060v11. 16, 18, 32, 34, 36, 37, 41, 48

[ENO10] P. Etingof, D. Nikshych and V. Ostrik. Fusion categories and homotopy theory. Quantum Topol. 1 (2010), no. 3, 209–273. Page references are to arXiv:0909.3140v2. 9

[ENO11] P. Etingof, D. Nikshych and V. Ostrik. Weakly group-theoretical and solvable fusion categories. Adv. Math. 226 (2011), no. 1, 176–205. Page references are to arXiv:0809.3031v2. 9

[EvG11] D. E. Evans and T. Gannon. The exoticness and realisability of twisted Haagerup–Izumi modular data. Comm. Math. Phys. 307 (2011), no. 2, 463–512. Page and equation references are to arXiv:1006.1326v1. 2, 3, 4, 8, 10, 15, 16, 17, 18, 30, 46

[FHL93] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky. On axiomatic approaches to vertex operator algebras and modules. Mem. Amer. Math. Soc. 104 (1993), no. 494. 12

[FLM88] I. Frenkel, J. Lepowsky and A. Meurman. Vertex operator algebras and the Monster. Pure and Applied Mathematics 134, Academic Press, Boston, 1988. 20, 21, 22

[GaR26] T. Gannon and B. C. Rayhaun. Hypergroup symmetry in relative quantum field theories and chiral algebras. Preprint, arXiv:2606.05279v1. Page references are to this version. 20, 45, 46

[GSY26] T. Gannon, A. Schopieray and H. Yadav. On Haagerup–Izumi fusion categories. Preprint, arXiv:2609.25185.

[GrS12] P. Grossman and N. Snyder. Quantum subgroups of the Haagerup fusion categories. Comm. Math. Phys. 311 (2012), no. 3, 617–643. Page references are to arXiv:1102.2631v2. 2, 4, 5, 7, 9, 10, 16, 18, 40

[Gui19] B. Gui. Unitarity of the modular tensor categories associated to unitary vertex operator algebras, II. Comm. Math. Phys. 372 (2019), no. 3, 893–950. Page references are to arXiv:1712.04931v6. 46, 54, 56

[Gui21] B. Gui. Categorical extensions of conformal nets. Comm. Math. Phys. 383 (2021), no. 2, 763–839. Page references are to arXiv:1812.04470v8. 54, 60, 61

[Gui22] B. Gui. Q-systems and extensions of completely unitary vertex operator algebras. Int. Math. Res. Not. IMRN 2022, no. 10, 7550–7614. Page references are to arXiv:1908.03484v3. 27, 49, 56, 57, 58, 59, 61

[Gui24] B. Gui. Geometric positivity of the fusion products of unitary vertex operator algebra modules. Comm. Math. Phys. 405 (2024), Paper No. 72. Page references are to arXiv:2306.11856v2. 56, 57, 58

[Gui25] B. Gui. Comparison of extensions of unitary vertex operator algebras and conformal nets. Preprint, arXiv:2505.03235v3. 5, 7, 43, 49, 53, 54, 59, 60, 61

[Gui26] B. Gui. Unbounded field operators in categorical extensions of conformal nets. Invent. Math., published online 2026. Page references are to arXiv:2001.03095v5. 5, 6, 7, 43, 49, 53, 54

[Hel78] S. Helgason. Differential geometry, Lie groups, and symmetric spaces. Pure and Applied Mathematics 80. Academic Press, New York–London, 1978. 32

[HeT25] A. G. Henriques and J. E. Tener. Every conformal net has an associated unitary VOA. Preprint, arXiv:2507.20735v3. 33

[Hua26a] T.-C. Huang. Cyclic Haagerup–Izumi fusion categories at every odd order. Preprint, arXiv:2609.15986.

[Hua26b] T.-C. Huang. Haagerup conformal field theory at central charge eight. Preprint, arXiv:2610.14441v2. The results are announced, with an outline of proof. 2, 3, 4, 5, 6, 7, 15, 20, 33, 34, 37, 42, 43, 59

[HL+22] T.-C. Huang, Y.-H. Lin, K. Ohmori, Y. Tachikawa and M. Tezuka. Numerical evidence for a Haagerup conformal field theory. Phys. Rev. Lett. 128 (2022), 231603. arXiv:2110.03008. 2

[Hua05] Y.-Z. Huang. Differential equations and intertwining operators. Commun. Contemp. Math. 7 (2005), no. 3, 375–400. 27

[HLZ12] Y.-Z. Huang, J. Lepowsky and L. Zhang, *Logarithmic tensor category theory, II: Logarithmic formal calculus and properties of logarithmic intertwining operators.* arXiv:1012.4196v2 (2012). 27

[Hua08] Y.-Z. Huang. Rigidity and modularity of vertex tensor categories. Commun. Contemp. Math. 10 (2008), suppl. 1, 871–911. 11, 12, 27

[HKL15] Y.-Z. Huang, A. Kirillov Jr. and J. Lepowsky. Braided tensor categories and extensions of vertex operator algebras. Comm. Math. Phys. 337 (2015), no. 3, 1143–1159. Page references are to arXiv:1406.3420v1. 12, 17, 19, 29, 48

[Izu01] M. Izumi. The structure of sectors associated with Longo–Rehren inclusions. II. Examples. Rev. Math. Phys. 13 (2001), no. 5, 603–674. 10

[Izu18] M. Izumi. The classification of 3^n subfactors and related fusion categories. Quantum Topol. 9 (2018), no. 3, 473–562. Page references are to arXiv:1609.07604v2. 5, 10, 33, 35, 40, 43

[Jia26] X. Jiao. An explicit lattice construction for the Haagerup center. Preprint, October 4, 2026, 54 pages, https://www.researchgate.net/publication/415561299_An_explicit_lattice_construction_for_the_Haagerup_center. 2, 3, 4, 13

[Jon18] V. F. R. Jones. A no-go theorem for the continuum limit of a periodic quantum spin chain. Comm. Math. Phys. 357 (2018), no. 1, 295–317.

[JMS14] V. F. R. Jones, S. Morrison and N. Snyder. The classification of subfactors of index at most 5. Bull. Amer. Math. Soc. (N.S.) 51 (2014), no. 2, 277–327. Page references are to arXiv:1304.6141. 2, 48

[Kac90] V. G. Kac. Infinite-dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990. 32

[Kaw18] Y. Kawahigashi. Conformal field theory, vertex operator algebras and operator algebras. Proceedings of the International Congress of Mathematicians, Rio de Janeiro 2018. Vol. III, 2597–2616, World Sci. Publ., Hackensack, NJ, 2018. Page references are to arXiv:1711.11349v1. 5

[KaL04] Y. Kawahigashi and R. Longo. Classification of local conformal nets. Case $c < 1$. Ann. of Math. (2) 160 (2004), no. 2, 493–522. Page references are to arXiv:math-ph/0201015v3. 14, 31

[KLM01] Y. Kawahigashi, R. Longo and M. Müger. Multi-interval subfactors and modularity of representations in conformal field theory. Comm. Math. Phys. 219 (2001), no. 3, 631–669. Page references are to arXiv:math/9903104v2. 2, 14

[KiO02] A. Kirillov Jr. and V. Ostrik. On a q-analogue of the McKay correspondence and the ADE classification of $\widehat{\mathfrak{sl}}_2$ conformal field theories. Adv. Math. 171 (2002), no. 2, 183–227. Page references are to arXiv:math/0101219v3. 12

[Kös04] S. Köster. Local nature of coset models. Rev. Math. Phys. 16 (2004), no. 3, 353–382. 23

[Lar14] H. K. Larson. Pseudo-unitary non-self-dual fusion categories of rank 4. J. Algebra 415 (2014), 184–213. Page references are to arXiv:1401.1879v2. 5, 38

[Li94] H. Li. Symmetric invariant bilinear forms on vertex operator algebras. J. Pure Appl. Algebra 96 (1994), no. 3, 279–297. 12, 25, 56

[Lon03] R. Longo. Conformal subnets and intermediate subfactors. Comm. Math. Phys. 237 (2003), no. 1-2, 7–30. Page references are to arXiv:math/0102196v2. 5, 14, 44

[LoR95] R. Longo and K.-H. Rehren. Nets of subfactors. Rev. Math. Phys. 7 (1995), no. 4, 567–597. Page references are to arXiv:hep-th/9411077v2. 15, 44

[LoX04] R. Longo and F. Xu. Topological sectors and a dichotomy in conformal field theory. Comm. Math. Phys. 251 (2004), no. 2, 321–364. 14

[MTW18] V. Morinelli, Y. Tanimoto and M. Weiner. Conformal covariance and the split property. Comm. Math. Phys. 357 (2018), no. 1, 379–406. 14

[Müg10] M. Müger. On superselection theory of quantum fields in low dimensions. In XVIth International Congress on Mathematical Physics, 496–503. World Sci. Publ., Hackensack, NJ, 2010. Page references are to arXiv:0909.2537v1. 41

[NgS07a] S.-H. Ng and P. Schauenburg. Frobenius–Schur indicators and exponents of spherical categories. Adv. Math. 211 (2007), no. 1, 34–71. Page references are to arXiv:math/0601012v3. 5, 34, 41

[NgS07b] S.-H. Ng and P. Schauenburg. Higher Frobenius–Schur indicators for pivotal categories. In Hopf algebras and generalizations, 63–90. Contemp. Math. 441. Amer. Math. Soc., Providence, RI, 2007. Page references are to arXiv:math/0503167v2. 5, 34, 41

[Ost03] V. Ostrik. Module categories, weak Hopf algebras and modular invariants. Transform. Groups 8 (2003), no. 2, 177–206. Page references are to arXiv:math/0111139v1. 39, 40

[Ost15] V. Ostrik. Pivotal fusion categories of rank 3. Mosc. Math. J. 15 (2015), no. 2, 373–396. Page references are to arXiv:1309.4822v1. 4, 18, 36, 42

[Reu23] D. Reutter. Uniqueness of unitary structure for unitarizable fusion categories. Comm. Math. Phys. 397 (2023), no. 1, 37–52. Page references are to arXiv:1906.09710v2. 15, 42, 44, 47

[Rie26a] A. Riesen. The Haagerup VOA at $c = 8$. Research announcement, arXiv:2610.13933v1 (October 9, 2026). 2, 3, 4, 6, 7, 13, 55, 59, 61

[Rie26b] A. Riesen. The Haagerup VOA at central charge eight. Working manuscript `haagerup_readable_proof.pdf` in the package `Haagerup_VOA_POST.zip`, release `snapshot-2026-10-09`, https://github.com/Doctorfm98/Haagerup_VOA/releases/tag/snapshot-2026-10-09. Section, page and statement numbers refer to this release. 55

[Ten19a] J. E. Tener. Geometric realization of algebraic conformal field theories. Adv. Math. 349 (2019), 488–563. Page references are to arXiv:1611.01176v2. 57

[Ten19b] J. E. Tener. Representation theory in chiral conformal field theory: from fields to observables. Selecta Math. (N.S.) 25 (2019), no. 5, Paper No. 76. Page references are to arXiv:1810.08168v2. 5, 22, 25, 26, 30, 31

[Ten24] J. E. Tener. Fusion and positivity in chiral conformal field theory. Geom. Funct. Anal. 34 (2024), no. 4, 1226–1296. Page references are to arXiv:1910.08257v3. 5, 14, 26, 27, 28, 29, 31, 33

[V+22] R. Vanhove, L. Lootens, M. Van Damme, R. Wolf, T. J. Osborne, J. Haegeman and F. Verstraete. Critical lattice model for a Haagerup conformal field theory. Phys. Rev. Lett. 128 (2022), 231602. arXiv:2110.03532. 2

Cédric Arhancet
6 rue Didier Daurat, 81000 Albi, France
URL: https://sites.google.com/site/cedricarhancet
cedric.arhancet@protonmail.com
ORCID: 0000-0002-5179-6972
https://github.com/cedricarhancet