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# Haagerup symmetries of the $E_8$ theory and conformal nets at central charge eight

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**Abstract**

A Haagerup algebra is a positive strongly rational vertex operator algebra $U$ of central charge eight with $\mathrm{Rep}(U)$ braided equivalent to the Drinfeld center $Z(\mathcal{H}_3)$ of the Haagerup fusion category. Such an algebra has been proposed by Jiao, with a computer-assisted proof, and announced by Riesen and by Huang. We show that $U$ has exactly three holomorphic extensions, each isomorphic to $V_{E_8}$, and that the categories of modules over the corresponding Lagrangian algebras are the three Haagerup fusion categories of Grossman and Snyder. The extensions attached to $\mathcal{H}_2$ and $\mathcal{H}_3$ are the two lattice extensions of $V_{E_6\oplus A_2}$. Along a strategy outlined by Huang, we prove a recognition theorem for $\mathcal{H}_2$ and $\mathcal{H}_3$ among unitary fusion categories of rank six with three invertible objects, a noncommutative fusion ring and a pointed part with trivial associator. For regular unitary vertex operator algebras with Tener's bounded localized vertex operators, we prove that the fusion product of two unitary modules is unitarizable when one of them integrates to the conformal net. The explicit unitary algebra $S \subset V_{E_6\oplus A_2}$ of our previous work, isomorphic to Jiao's algebra, is strongly local. Granting that Jiao's algebra is a Haagerup algebra, we deduce that the conformal net $\mathcal{A}_S$ is completely rational with $\mathrm{Rep}(\mathcal{A}_S) \cong Z(\mathcal{H}_3)$, that the Haagerup subfactor arises from $\mathcal{A}_S$, and that $\mathcal{A}_S$ solves a problem of Bischoff on generalized orbifolds. Under the same hypothesis, using a unitary structure on $V_{E_8\oplus E_8}$ found by exact computer calculations, we prove that $S$ is completely unitary and that $\mathrm{Rep}(\mathcal{A}_S) \cong \mathrm{Rep}(S)$ through the correspondence given by smeared vertex operators. The diagonal construction and the passage to the twin algebra also show that every Haagerup algebra is completely unitary and satisfies Condition I+.

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# 1 Introduction

From a completely rational conformal net one obtains finite index subfactors, and the representation category of the net is a unitary modular tensor category [KLM01]. Whether every finite depth subfactor is related in this way to a chiral conformal field theory is a question raised by Jones, see [Bro20, Section 1.3 p. 2]. Bischoff gave a precise formulation [Bis17a, Definition 4.1 p. 20]: a subfactor $N \subset M$ arises from a completely rational net $\mathcal{A}$ if it is of the form $\beta(\mathcal{B}_a(I)) \subset \mathcal{B}_b(I)$ for two irreducible relatively local extensions $\mathcal{B}_a, \mathcal{B}_b \supset \mathcal{A}$ and a sector $\beta$ contained in $\iota_b \rho \bar{\iota}_a$ for some $\rho \in \mathrm{Rep}(\mathcal{A})$, where $\iota_a$ and $\iota_b$ are the inclusions $\mathcal{A}(I) \subset \mathcal{B}_a(I)$ and $\mathcal{A}(I) \subset \mathcal{B}_b(I)$. He proved that $N \subset M$ arises from $\mathcal{A}$ as soon as $\mathrm{Rep}(\mathcal{A})$ is braided equivalent to the quantum double $D(N \subset M)$ [Bis17a, Proposition 4.7 p. 22], and he formulated, following [EvG11], the problem of realizing such quantum doubles by conformal nets [Bis17a, Problem 1.3 p. 16]. For all subfactors of index less than four this problem was solved in [Bis16]. The Haagerup subfactor of index $\frac{5+\sqrt{13}}{2}$ [AsH99], which has the smallest index above four among finite depth subfactors [JMS14, Section 4.2.1 p. 22], is the first case of index above four. Its quantum double is the Drinfeld center $Z(\mathcal{H}_3)$ of any of the three Morita equivalent Haagerup fusion categories $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ of Grossman and Snyder [GrS12].

Evans and Gannon predicted that $Z(\mathcal{H}_3)$ is realized by a chiral theory of central charge eight [EvG11], and numerical work suggests a Haagerup-symmetric theory with $c \approx 2$ [HL+22, V+22]. At central charge eight, explicit vertex operator algebras inside $V_{E_6 \oplus A_2}$ have recently been proposed by Jiao [Jia26], Riesen [Rie26a] and Huang [Hua26b]. Jiao gives a computer-assisted proof that his algebra realizes $Z(\mathcal{H}_3)$, in a preprint dated 4 October 2026, and Riesen and Huang announce the same property for their algebras, with outlines of proof. In [Arh26] we

showed that every Haagerup algebra lies in the chain $V_{E_6\oplus A_2} \subset V_{E_8}$, determined its currents, and proved that there are exactly two Haagerup algebras up to isomorphism, granting one of the existence results of [Jia26, Rie26a]. The two algebras are distinguished by the levels of their currents, $(1, 39)$ in case (I) and $(3, 13)$ in case (II), and they are exchanged by a twin construction [Arh26, Theorems 12.3 and 12.4 pp. 65 and 67]. The algebras of [Rie26a, Hua26b] belong to case (I) and the algebra of [Jia26] to case (II). Albert, Honda, Kaidi and Zheng suggested that the chiral $(\mathfrak{e}_8)_1$ theory carries the symmetries $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, and listed the three Lagrangian algebras of $Z(\mathcal{H}_3)$ through the tube algebra [AHKZ26, Appendix A pp. 6–7], see also [BoS25]. Huang announced, with an outline of proof, that his algebra is strongly rational and completely unitary with $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$ as unitary ribbon categories, that its conformal net is completely rational, and that $\mathcal{H}_3$ acts by topological defects on a holomorphic extension isomorphic to $V_{E_8}$ [Hua26b, Theorem 1 p. 1 and Sections 6 and 7 pp. 8–10]. For the lattice extension, his argument gives $\mathcal{H}_2$ or $\mathcal{H}_3$ [Hua26b, Section 6 p. 9 and Section 7 p. 10]. Jiao observes that his construction does not distinguish $\mathcal{H}_2$ from $\mathcal{H}_3$ and that the construction of a conformal net is a separate problem [Jia26, Section 1.2 p. 3].

In this paper we treat both points. On the categorical side, we determine the three Lagrangian algebras of $Z(\mathcal{H}_3)$ and the Haagerup category attached to each of them, without vertex algebras, and the three holomorphic extensions of a Haagerup algebra. On the operator algebraic side, we construct the conformal net of the explicit algebra $S$ of [Arh26], and we prove its complete rationality and identify its representation category under the hypothesis $(\mathrm{H_J})$, stated below, that the algebra of Jiao is a Haagerup algebra. For this purpose, we prove that, in Tener's framework of bounded localized vertex operators, the fusion product of two unitary modules is unitarizable when one of them integrates to the conformal net. Under $(\mathrm{H_J})$, this gives the unitarizability of all simple $S$-modules. The identification of the representation category uses a recognition theorem for the Haagerup categories, and we obtain the consequences of [Bis17a, Bis17b]. Finally, under $(\mathrm{H_J})$, we realize $S \otimes S$ as a unitary subalgebra of an explicit unitary structure of $V_{E_8\oplus E_8}$, found by exact computer calculations, in which $S \otimes 1$ and $1 \otimes S$ are mutual commutants, and we deduce from the results of Gui on cosets that $S$ is completely unitary and that the representation categories of $S$ and of its conformal net are equivalent through the correspondence $M \mapsto \pi_M$ of Theorem 1.6.

Throughout, $d = \frac{3+\sqrt{13}}{2}$, so that $d^2 = 3d+1$. We use the labels $0, b, a, c_0, c_1, c_2, d_1, \dots, d_6$ of [EvG11] for the simple objects of $Z(\mathcal{H}_3)$, whose quantum dimensions are $1$, $3d+1$, $3d+2$ (four times) and $3d$ (six times). For a Haagerup algebra $U$ we denote by $M_z$ the simple $U$-module attached to $z$, and we put $V' = V_{E_6\oplus A_2}$. By [Arh26, Proposition 4.1 p. 11], $U$ is contained in $V'$ and $V' = U \oplus M_b$ as a $U$-module.

**Theorem 1.1** *Let $U$ be a Haagerup algebra.*

1. *Up to isomorphism of extensions, $U$ has exactly three holomorphic extensions $B_1$, $B_2$, $B_3$, that is, simple vertex operator algebras $B$ containing $U$ with the same conformal vector and with $\dim B_0 = 1$, every $B$-module of which is a direct sum of copies of $B$. Each $B_i$ is isomorphic to $V_{E_8}$. The corresponding Lagrangian algebras $L_i$ of $\mathrm{Rep}(U)$ satisfy $\mathrm{Rep}(U)_{L_i} \cong \mathcal{H}_i$, and*

$$L_1 \cong 0 \oplus b \oplus a \oplus c_0, \quad L_2 \cong 0 \oplus b \oplus 2a, \quad L_3 \cong 0 \oplus b \oplus 2c_0$$

   *as objects, in the labels of [EvG11].*

2. *The extensions $B_2$ and $B_3$ are the two lattice vertex operator algebras $V' \oplus V'_w \oplus V'_{-w}$ attached to the two isotropic lines $\{0, \pm w\}$ of the discriminant group $\mathbb{Z}_3 \times \mathbb{Z}_3$ of $E_6 \oplus A_2$.*

*Here $V'_w$ denotes the simple $V'$-module attached to $w$, and $B_2$ is the extension for which the $U$-modules $V'_{\pm w}$ are isomorphic to $M_a$.*

3. *The subspace $U \oplus M_b$ of $B_1$ is not a vertex subalgebra, and $U$ and $B_1$ are the only vertex subalgebras of $B_1$ containing $U$. For $i \in \{2,3\}$, the simple vertex subalgebras of $B_i$ containing $U$ are $U$, $V'$ and $B_i$.*

This is Theorem 3.9 with Propositions 3.3 and 3.6. The Lagrangian algebras attached to the two even parts $\mathcal{H}_2$ and $\mathcal{H}_1$ of the Haagerup subfactor were computed by Evans and Gannon in the language of Longo–Rehren inclusions [EvG11, pp. 18–19], the second one by a Frobenius reciprocity argument of which Lemma 3.1 is a categorical version. The three Lagrangian algebras, with the categories $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ attached to them, are listed in [AHKZ26, (A8) p. 7 and pp. 3–4], in labels 1, $\pi_1$, $\pi_2$, $\sigma_1$ coming from the tube algebra of $\mathcal{H}_3$. We prove this identification in the labels of [EvG11], which are defined through the half-braidings on the even part $\mathcal{H}_2$ of the Haagerup subfactor, by means of the classification of module categories [GrS12] and of a formula of Ostrik for the dimensions of the spaces of morphisms between Lagrangian algebras [Ost15, Remark 2.19 pp. 7–8]. This proof does not use vertex algebras (Corollary 3.8), and it gives $\pi_2 = c_0$ and $\sigma_1 = a$ (Remark 3.7). The realization of these Lagrangian algebras by vertex operator algebras uses the restriction of the lattice modules computed in [Arh26, Lemma 4.3 p. 13]. If the algebra of [Hua26b] is a Haagerup algebra, part 2 shows that the two alternatives left open for its lattice extension in [Hua26b, Section 7 p. 10] are realized by the two isotropic lines, and that the extension attached in [Hua26b, Section 7 pp. 9–10] to the algebra $I_{\mathcal{H}_3}(1)$ is the lattice extension of type $c_0$. This corrects [Arh26, Remark 2.1 p. 8], where the algebra $0 \oplus b \oplus 2a$ is attributed to $\mathcal{H}_3$ (Remark 3.7). For the explicit algebra $S$ in the explicit model of $V'$, we do not know which of the two concrete isotropic lines is the one of type $a$. Part 3 shows that there is no vertex subalgebra strictly between $U$ and the extension $B_1$ attached to $\mathcal{H}_1$.

We now turn to conformal nets. Let $S \subset V'$ be the vertex algebra generated by the weight-one and weight-two spaces of [Arh26, Theorem 1.3 p. 4]. It is unitary for an explicit unitary structure of $V'$ [Arh26, Theorem 10.10 p. 33], and it is isomorphic to the algebra of Jiao [Arh26, Theorem 10.24 p. 42]. We write ($\mathrm{H_J}$) for the assertion that Jiao's algebra is a Haagerup algebra [Arh26, p. 5], as stated in [Jia26], and ($\mathrm{H_U}$) for the assertion that every simple $S$-module admits a scalar product which is invariant with respect to the PCT operator of $S$ induced by the scalar product of $V'$. Under ($\mathrm{H_J}$), all the simple $S$-modules except the six modules $M_{d_l}$ are submodules of unitary $V'$-modules [Arh26, Proposition 4.1 p. 11, Lemma 4.3 p. 13 and Lemma 11.8 p. 60], so that ($\mathrm{H_U}$) concerns these six modules. We prove ($\mathrm{H_U}$) under ($\mathrm{H_J}$) (Theorem 1.3). We write ($\mathrm{H_R}$) for the analogue of ($\mathrm{H_J}$) for the algebra of [Rie26a]. Let $\mathcal{A}_{E_8}$ be the conformal net of $V_{E_8}$, that is, the loop group net of $E_8$ at level one. For an $S$-submodule $M$ of a lattice extension of $V'$, we denote by $\pi_M$ the representation of the conformal net $\mathcal{A}_S$ of $S$ on the closure of $M$ (Lemma 4.7), and we put $\pi_0 = \pi_S$, $\pi_b = \pi_{M_b}$, $\pi_a = \pi_{M_a}$, $\pi_{c_0} = \pi_{M_{c_0}}$, where $M_a$ and $M_{c_0}$ are taken in the two lattice extensions.

**Theorem 1.2** *1. The unitary vertex operator algebra $S$ is strongly local. Its conformal net $\mathcal{A}_S$ is a covariant subnet of the conformal net $\mathcal{A}_{V'}$ of $V'$, which is itself a covariant subnet of index three of the net $\mathcal{A}_{H_\Lambda} \cong \mathcal{A}_{E_8}$ of each of the two lattice extensions $H_\Lambda = V' \oplus V'_w \oplus V'_{-w}$ attached to the isotropic lines $\Lambda = \{0, \pm w\}$ of the discriminant group of $E_6 \oplus A_2$. These inclusions are irreducible, that is, $\mathcal{A}_S(I)' \cap \mathcal{A}_{E_8}(I) = \mathbb{C}$ and $\mathcal{A}_{V'}(I)' \cap \mathcal{A}_{E_8}(I) = \mathbb{C}$ for any interval $I$.*

2. *Assume ($\mathrm{H_J}$). Then $\mathcal{A}_S$ is completely rational. The vacuum representation of $\mathcal{A}_{E_8}$ restricts to $\mathcal{A}_S$ as $\pi_0 \oplus \pi_b \oplus 2\pi_x$, where $x = a$ for one lattice extension and $x = c_0$ for the*

*other, and $\pi_0$, $\pi_b$, $\pi_a$, $\pi_{c_0}$ are pairwise inequivalent irreducible representations with finite statistical dimensions $1$, $\delta$, $1+\delta$, $1+\delta$ for some $\delta > 1$. The indices are $[\mathcal{A}_{V'} : \mathcal{A}_S] = 1+\delta$ and $[\mathcal{A}_{E_8} : \mathcal{A}_S] = 3(1+\delta)$. Finally, $\mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to the Drinfeld center of a unitary fusion category of rank six with a pointed fusion subcategory of dimension three.*

This is Theorems 4.5 and 5.6, Proposition 5.8 and Corollary 5.9. Theorem 4.5 applies to any unitary conformal subalgebra of $V'$, and Theorem 5.6 to any such subalgebra which is a Haagerup algebra. Part 1 rests on the computer-assisted unitarity theorem [Arh26, Theorem 10.10 p. 33]. For part 2, we use that $V_{E_8}$ has bounded localized vertex operators and that its net is isomorphic to the net of [CKLW18] [Ten19b, Theorem 7.4 p. 71 and Corollary 7.7 p. 72]. By a theorem of Tener [Ten24, Corollary 4.24 p. 56], the regularity of $S$ then implies that the inclusion of the nets of [Ten19b] has finite index, so that complete rationality follows from [Lon03, Theorem 24 p. 20], and the comparison [Ten19b, Lemma 7.6 p. 72] identifies the net of $S$ of [Ten19b] with the net $\mathcal{A}_S$ of [CKLW18]. The proof of [Ten24, Corollary 4.24] goes through [Ten24, Theorem 4.4 p. 44], which requires unitary structures on the fusion products $M \boxtimes \overline{M}$ of the modules involved. Here these are $M_b \boxtimes M_b$ and $M_x \boxtimes M_x$, which contain every simple $S$-module by the fusion rules of $Z(\mathcal{H}_3)$. We obtain these unitary structures from the following general result.

**Theorem 1.3** *Let $V$ be a simple regular unitary vertex operator algebra with bounded localized vertex operators, and let $M$ and $N$ be ordinary unitary $V$-modules such that $N$ integrates to a representation $\pi^N$ of the conformal net of $V$ in the sense of [Ten24, Definition 2.18 p. 20]. Then the fusion product $M \boxtimes N$ admits a unitary structure for which it is a transport module for $M$ and $N$. Consequently, under* ($\mathrm{H_J}$)*, every simple $S$-module is unitarizable, that is,* ($\mathrm{H_U}$) *holds.*

This is Proposition 5.2 and Corollary 5.5. Tener proved that, when $\pi^N$ exists, the transport form is positive semidefinite near the unit circle [Ten24, Proposition 4.2 p. 43], and that, if $M \boxtimes N$ admits a unitary structure, it is a transport module as soon as the transport form is positive semidefinite at one point $(z, z)$ of $\mathcal{R}$ [Ten24, Proposition 3.16 p. 36], which gives [Ten24, Theorem 4.4 and Corollary 4.5 p. 44] under this hypothesis. He noted that the unitarizability of $M \boxtimes N$ is open in general [Ten24, Remark 3.18 p. 37]. Our proof replaces the density argument in a Hilbert space completion of $M \boxtimes N$ by an argument which does not use any Hilbert space completion of $M \boxtimes N$. We use a nondegenerate invariant Hermitian form on $M \boxtimes N$, the invariance of the relevant spaces under polynomials in $L_0$, and a moment argument (Lemma 5.1). In particular, the unitarity of $M \boxtimes \overline{M}$ used in the proofs of the finite index theorems [Ten24, Corollaries 4.21 and 4.24 pp. 55–56] holds in their setting (Remark 5.3), and Remark 5.10 explains how this bears on the outline of [Hua26b, Section 6 p. 8].

The equivalence of $\mathrm{Rep}(\mathcal{A}_S)$ with $\mathrm{Rep}(S)$ is an instance of the general conjecture relating unitary vertex operator algebras and conformal nets, see [Kaw18, Conjecture 7.6 p. 14] and [Gui26, Gui25, Ten24]. We prove it for $S$ under ($\mathrm{H_J}$) in Theorem 1.6 below. Denote by ($\mathrm{N_0}$) the statement that $\mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to $Z(\mathcal{H}_3)$, and by (N) the assertion that there is a braided equivalence $\mathrm{Rep}(\mathcal{A}_S) \to \mathrm{Rep}(S)$ which maps $\pi_M$ to $M$ for $M \in \{M_b, M_a, M_{c_0}\}$. Under ($\mathrm{H_J}$), (N) implies ($\mathrm{N_0}$). We first prove ($\mathrm{N_0}$) under ($\mathrm{H_J}$), through the following recognition theorem, which does not use vertex algebras.

**Theorem 1.4** *Let $\mathcal{F}$ be a unitary fusion category with six simple objects up to isomorphism, three of which are invertible, and with noncommutative Grothendieck ring. Then $\mathcal{F}$ has the fusion rules of $\mathcal{H}_2$ and $\mathcal{H}_3$. If moreover the fusion subcategory generated by the invertible objects is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator, then $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$.*

This is Theorem 6.1. We follow the strategy outlined by Huang [Hua26b, Sections 5 and 6 pp. 7–9] and give complete proofs (Remark 6.12). The hypotheses force $\rho^2 \cong 1 \oplus m(\rho \oplus \alpha\rho \oplus \alpha^2\rho)$ for a non-invertible simple object $\rho$, an invertible object $\alpha$ of order three and some $m \geqslant 1$ (Lemma 6.2). The Frobenius–Schur indicators $\nu_1$, $\nu_2$, $\nu_3$ [NgS07a, NgS07b] give three identities for the twists of $Z(\mathcal{F})$ (Lemma 6.4), and lower bounds for sums of roots of unity [Lar14], completed by an average over the Galois group of $\mathbb{Q}(\sqrt{10})$ when the squarefree part of $9m^2+4$ is 10 (Lemma 6.5), give $m = 1$ (Proposition 6.6). The classification of generalized Haagerup categories by Izumi [Izu18] then gives $\mathcal{H}_2$ or $\mathcal{H}_3$. The assumption on the associator is detected by the Drinfeld center. The third Frobenius–Schur indicator of $\alpha$ is equal to 1 if and only if this associator is trivial (Lemma 6.8), and $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or $\mathcal{H}_3$ if and only if no simple object of $Z(\mathcal{F})$ has a twist of order nine (Proposition 6.9). This partly answers the question of [GrS12, p. 5] on the fusion categories with the Grothendieck ring of $\mathcal{H}_2$. For the rank six categories of part 2 of Theorem 1.2, denoted $\mathcal{F}_x$ in part 4 of Theorem 5.6, one for each lattice extension, the triviality of the associator follows from the existence of two lattice extensions of $V'$ (Lemma 6.10), an input also used in [Hua26b, p. 7], and Theorem 1.4 gives ($\mathrm{N}_0$) (Theorem 6.11).

**Theorem 1.5** *Assume* ($\mathrm{H}_\mathrm{J}$).

1. $\mathrm{Rep}(\mathcal{A}_S)$ *is equivalent to* $Z(\mathcal{H}_3)$ *as a unitary modular tensor category, in particular* ($\mathrm{N}_0$) *holds. Every braided equivalence maps* $\pi_b$ *to* $b$*, and* $\delta = d^2$*.*

2. $\mathcal{A}_S$ *has exactly three irreducible local extensions with trivial representation category, up to isomorphism of extensions. Two of them are the nets* $\mathcal{A}_{H_\Lambda}$ *of the two lattice extensions, both isomorphic to* $\mathcal{A}_{E_8}$*, and the categories of modules over their dual canonical endomorphisms are* $\mathcal{H}_2$ *and* $\mathcal{H}_3$*. For the third one, this category is* $\mathcal{H}_1$*, and the only intermediate nets are* $\mathcal{A}_S$ *and the extension itself. For each lattice extension, the only intermediate nets are* $\mathcal{A}_S$*,* $\mathcal{A}_{V'}$ *and the extension itself. The net of the lattice extension of type a carries* $\mathcal{H}_2$*, as the vertex algebra does.*

3. *The Haagerup subfactor arises from* $\mathcal{A}_S$ *in the sense of [Bis17a, Definition 4.1 p. 20].*

4. $\mathcal{A}_S$ *is the generalized orbifold of* $\mathcal{A}_{V'}$ *by the hypergroup* $K = \{k_0, k_1\}$ *with* $k_1 k_1 = d^{-2}k_0 + (1-d^{-2})k_1$*. Hence, under these hypotheses,* $\mathcal{A}_S$ *solves [Bis17b, Problem 4.27 p. 27].*

This is Theorem 7.2 with Theorem 6.11 and Corollaries 7.4, 7.6 and 7.19. The last assertion of part 2 uses Theorem 1.6 below. Part 1 follows from Theorem 1.4 and the dimensions computed in Theorem 1.2 (Theorem 6.11 and Lemma 7.1). Part 4 rests on the fact, which does not use Theorem 1.4, that $\mathcal{A}_S$ is a generalized orbifold of $\mathcal{A}_{V'}$ by a two-element hypergroup with parameter $\delta$ (Proposition 7.5). By Theorem 1.6, the three extensions of part 2 are all isomorphic to $\mathcal{A}_{E_8}$, the third one being the net of the holomorphic extension of $S$ attached to $\mathcal{H}_1$ (Proposition 7.10 and Corollary 7.19). The statements of Theorem 1.5 hold for the algebra of Riesen in case (I), granting ($\mathrm{H}_\mathrm{R}$) and its unitarity inside $V'$ asserted in [Rie26a, Theorem 1 p. 1 and p. 4] (Section 8). For the last assertion of part 2 and for the isomorphism of the third extension of its net with $\mathcal{A}_{E_8}$, the proof goes through the computer-assisted construction of Sections 7.4 and 7.5 applied to the twin algebra (Theorem 8.6).

**Theorem 1.6** *Assume* ($\mathrm{H}_\mathrm{J}$).

1. *The algebra* $S \otimes S$ *has a holomorphic extension isomorphic to* $V_{E_8 \oplus E_8}$ *in which* $S \otimes 1$ *and* $1 \otimes S$ *are mutual commutants, and* $V_{E_8 \oplus E_8}$ *has a structure of unitary vertex operator algebra for which* $S \otimes 1$ *and* $1 \otimes S$ *are unitary subalgebras.*

2. *The algebra $S$ is completely unitary, every unitary $S$-module is strongly integrable, and the functor of [Gui26, Section 3.4 p. 45] is an equivalence of braided $C^*$-tensor categories from the category $\mathrm{Rep}^u(S)$ of unitary $S$-modules to the category $\mathrm{Rep}^f(\mathcal{A}_S)$ of representations of $\mathcal{A}_S$ with finite statistical dimension, which maps $M$ to $\pi_M$ for every $S$-submodule $M$ of a lattice extension. In particular,* (N) *holds.*

3. *The twin algebra of $S$ in case (I) and the common Heisenberg commutant of the two algebras are completely unitary.*

This is Propositions 7.12 and 7.16, Theorems 7.17 and 8.3 and Remark 8.4. Part 1 rests on exact computer calculations in the weight-one and weight-two spaces of $V_{E_8\oplus E_8}$, which reduce, in the first implementation, to spaces of dimensions 496 and 644, carried out with two separate implementations (Remark 7.18). We determine the embedding of the weight-one space of $S\otimes S$ up to automorphisms, the conformal vector of the coset of the affine subalgebra of $S\otimes 1$ up to four choices, and a positive rescaling of the standard antilinear involution of the lattice algebra which fixes these four vectors. Since $S\otimes 1$ and $1\otimes S$ are commutants of explicit sets of vectors, they are then stable under this involution. Part 2 follows from the theorems of Gui on the strongly rational cosets of lattice vertex operator algebras [Gui26, Theorems 3.34 and 3.46 pp. 51 and 57], [Gui25, Theorem 6.18 p. 70]. It gives a second proof of ($\mathrm{H_U}$), independent of Theorem 1.3 but computer-assisted. Part 3 follows from part 2 and the twin construction (Theorem 8.3). In addition, Corollary 8.5 transfers Condition I+ of [Gui25, Definition 6.17 p. 70] from $S$ to the twin algebra $S^\natural$ and to the common Heisenberg commutant $\Omega_S$ of $S$ and $S^\natural$, and it proves the analogue of (N) for every Haagerup algebra realized as a unitary conformal subalgebra of $V'$. The proofs of Propositions 7.12 and 7.16, of Lemmas 7.14 and 7.15 and of parts 1 and 2 of Theorem 7.17 use no property of $S$ beyond those of an arbitrary Haagerup algebra in case (II). Consequently, every Haagerup algebra is completely unitary, and the analogue of (N) holds for the algebra $U^{\mathrm{R}}$ of [Rie26a] under ($\mathrm{H_R}$) and the unitarity asserted in [Rie26a], without ($\mathrm{H_J}$) (Theorem 8.6). In particular, if the algebra of [Hua26b] is a Haagerup algebra, it is completely unitary, as announced in [Hua26b, Theorem 1 p. 1].

Table 1 summarizes the dependence of the statements on the hypotheses.

**Structure of the paper.** Section 2 collects the facts on Haagerup categories, Lagrangian algebras, vertex operator algebras and conformal nets that we use. Section 3 proves Theorem 1.1. Section 4 constructs the net of $S$. Section 5 proves the unitarity of fusion products, the unitarizability of the simple modules and the complete rationality. Section 6 proves Theorem 1.4 and deduces ($\mathrm{N_0}$). Section 7 derives the consequences of ($\mathrm{N_0}$), constructs the unitary structure on $V_{E_8\oplus E_8}$ and proves the complete unitarity of $S$ and (N). Section 8 discusses case (I) and the twin algebra.

# 2 Preliminaries

## 2.1 The Haagerup fusion categories

We follow the conventions of Grossman and Snyder [GrS12]. The two even parts of the Haagerup subfactor are unitary fusion categories $\mathcal{H}_1$ and $\mathcal{H}_2$ with four and six simple objects respectively [GrS12, p. 14].

The simple objects of $\mathcal{H}_1$ are $1$, $\nu$, $\eta$, $\mu$, with Frobenius–Perron dimensions $1$, $d+1$, $d$, $d-1$. They are self-dual and the fusion rules are given in [GrS12, Table 2 p. 15]. In particular

$$\mu\otimes\mu\cong 1\oplus\nu,\quad \eta\otimes\eta\cong 1\oplus\nu\oplus\eta\oplus\mu,\quad \nu\otimes\nu\cong 1\oplus 2\nu\oplus 2\eta\oplus\mu. \tag{2.1}$$

| Statement | Status |
|---|---|
| Three Lagrangian algebras of $Z(\mathcal{H}_3)$, their objects in the labels of [EvG11] and their categories of modules (Propositions 3.3 and 3.6, Corollary 3.8) | proved, no hypothesis |
| Holomorphic extensions of a Haagerup algebra, realization of $L_2$ and $L_3$ by the lattice extensions (Theorem 3.9, Propositions 3.5 and 3.6) | proved for any Haagerup algebra, such algebras being known to exist only under ($\mathrm{H_J}$) or ($\mathrm{H_R}$) |
| Strong locality of $S$, inclusions of nets (Theorem 4.5) | proved, using the computer-assisted [Arh26, Theorem 10.10] |
| Recognition theorem for $\mathcal{H}_2$ and $\mathcal{H}_3$, twists of order nine (Theorem 6.1, Proposition 6.9) | proved, no hypothesis |
| Unitarity of fusion products for algebras with bounded localized vertex operators (Proposition 5.2) | proved, no hypothesis |
| Unitarizability of all simple $S$-modules, that is, ($\mathrm{H_U}$) (Corollary 5.5) | assuming ($\mathrm{H_J}$) |
| Complete rationality of $\mathcal{A}_S$, indices, decompositions, $\delta > 1$, generalized orbifold by a two-element hypergroup (Theorem 5.6, Corollary 5.9, Propositions 5.7 and 5.8) | assuming ($\mathrm{H_J}$) |
| $\mathcal{F}_x \cong \mathcal{H}_2$ or $\mathcal{H}_3$, ($\mathrm{N_0}$), unitary modular equivalence, $\delta = d^2$, three holomorphic extensions of $\mathcal{A}_S$, intermediate nets, Bischoff's consequences, near-group category $\mathbb{Z}_3^2 + 9$ (Theorem 6.11, Section 7 and Remark 7.8) | assuming ($\mathrm{H_J}$) |
| The net of the lattice extension of type $a$ carries $\mathcal{H}_2$, the three holomorphic extensions of $\mathcal{A}_S$ are isomorphic to $\mathcal{A}_{E_8}$ (Theorem 7.2, Proposition 7.10, Corollary 7.19) | assuming ($\mathrm{H_J}$), computer-assisted through Theorem 7.17 |
| Diagonal extension $V_{E_8 \oplus E_8} \supset S \otimes S$ (Proposition 7.12), with a unitary structure for which $S \otimes 1$ and $1 \otimes S$ are unitary subalgebras (Proposition 7.16) | assuming ($\mathrm{H_J}$), computer-assisted for the unitary structure |
| Complete unitarity of $S$, equivalence $\mathrm{Rep}^u(S) \cong \mathrm{Rep}^f(\mathcal{A}_S)$ through $M \mapsto \pi_M$, (N) (Theorem 7.17) | assuming ($\mathrm{H_J}$), computer-assisted, using [Gui26] |
| Equivalence of the complete unitarity of a Haagerup algebra, of its twin algebra and of their Heisenberg commutant (part 2 of Theorem 8.3) | proved, no hypothesis, no computer calculation beyond those of [Arh26, Theorem 8.5 and Section 12] |
| Complete unitarity of $S^\natural$ and $\Omega_S$, Condition I+, analogue of (N) for the Haagerup subalgebras of $V'$ (Remark 8.4, Corollary 8.5) | assuming ($\mathrm{H_J}$), through Theorem 7.17 |
| The same, without ($\mathrm{H_J}$), for every Haagerup algebra (Theorem 8.6) | computer-assisted |
| Strong locality, unitarizability of the simple modules, complete rationality, ($\mathrm{N_0}$), $\delta = d^2$, the three local extensions with $\mu$-index one and the intermediate nets, Bischoff's consequences, for the algebra $U^{\mathrm{R}}$ of case (I) (Section 8) | assuming ($\mathrm{H_R}$) and the unitarity asserted in [Rie26a] |
| Complete unitarity of $U^{\mathrm{R}}$, analogues of the last assertion of part 2 of Theorem 1.5 and of Proposition 7.10 (Theorem 8.6) | assuming ($\mathrm{H_R}$), computer-assisted, and for the analogues the unitarity asserted in [Rie26a] |
| Which concrete isotropic line is of type $a$ for $S$ | open |

Table 1: Dependence of the main statements on the hypotheses.

The simple objects of $\mathcal{H}_2$ are 1, $\alpha$, $\alpha^2$, $\xi$, $\alpha\xi$, $\alpha^2\xi$, with Frobenius–Perron dimensions 1, 1, 1, $d$, $d$, $d$ and fusion rules

$$\alpha^3 \cong 1, \quad \alpha\xi \cong \xi\alpha^2, \quad \xi^2 \cong 1 \oplus \xi \oplus \alpha\xi \oplus \alpha^2\xi. \tag{2.2}$$

The category $\mathcal{H}_3$ is defined as the category of $(1+\alpha+\alpha^2)$-$(1+\alpha+\alpha^2)$-bimodules in $\mathcal{H}_2$ [GrS12, Definition 3.19 p. 23]. It has six simple objects, three of dimension 1 and three of dimension $d$, and the same fusion ring as $\mathcal{H}_2$ [GrS12, Theorem 1.4 p. 4]. The three categories have the same global dimension

$$\dim \mathcal{H}_i = 3 + 3d^2 = 3(3d+2), \quad i \in \{1,2,3\}. \tag{2.3}$$

We will use the following facts.

**Proposition 2.1** *1. The Morita equivalence class of $\mathcal{H}_1$ consists of exactly the three fusion categories $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, which are pairwise inequivalent.*

*2. For any $i \in \{1,2,3\}$, the category $\mathcal{H}_i$ has exactly three indecomposable module categories up to equivalence, and their dual categories are $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$.*

*3. The category $\mathcal{H}_1$ has rank four, and $\mathcal{H}_2$, $\mathcal{H}_3$ have rank six.*

*4. The only fusion subcategories of $\mathcal{H}_1$ are* Vec *and $\mathcal{H}_1$. For $i \in \{2,3\}$, the only fusion subcategories of $\mathcal{H}_i$ are* Vec*, the pointed subcategory $\mathcal{P}_i$ generated by the invertible objects, and $\mathcal{H}_i$. Moreover $\mathcal{P}_i$ has dimension three.*

*Proof* : 1. This is [GrS12, Theorem 1.4 p. 4 and Corollary 3.22 p. 24].

2. For $\mathcal{H}_2$, the three module categories are given by [GrS12, Corollary 3.16 p. 22], and their duals are described in [GrS12, Lemmas 3.17 and 3.18 p. 22, Definition 3.19 p. 23 and Corollary 3.22 p. 24]. For $\mathcal{H}_1$ and $\mathcal{H}_3$ this is [GrS12, Theorem 3.25 p. 24].

3. This was recalled above.

4. A fusion subcategory is determined by the set of its simple objects, which is closed under fusion and duality. By (2.1), any fusion subcategory of $\mathcal{H}_1$ containing $\mu$ contains $\nu$, any fusion subcategory containing $\nu$ or $\eta$ contains $\eta$, $\mu$ and $\nu$. Hence a nontrivial fusion subcategory of $\mathcal{H}_1$ is $\mathcal{H}_1$. By (2.2), each object $\alpha^k\xi$ is self-dual and $\alpha^k\xi \otimes \alpha^k\xi \cong \alpha^{3k}\xi^2 \cong \xi^2$. Hence a fusion subcategory of $\mathcal{H}_2$ containing one of the objects $\alpha^k\xi$ contains the three objects of dimension $d$. It then contains $\xi \otimes \alpha\xi$, which contains $\alpha^2$ by [GrS12, Table 1 p. 15], so that it is $\mathcal{H}_2$. The same argument applies to $\mathcal{H}_3$, which has the same fusion ring. The remaining subcategories are Vec and $\mathcal{P}_i$, and $\dim \mathcal{P}_i = 3$. ■

Since the categories $\mathcal{H}_i$ are Morita equivalent, their Drinfeld centers $Z(\mathcal{H}_1)$, $Z(\mathcal{H}_2)$ and $Z(\mathcal{H}_3)$ are braided equivalent [ENO11, Theorem 3.1 p. 9], and we denote any of them by $Z(\mathcal{H}_3)$. By [GrS12, Theorem 1.5 p. 5], there is exactly one Morita equivalence between any two of the categories $\mathcal{H}_i$. Hence the Brauer–Picard group of $\mathcal{H}_3$ is trivial, so that every braided autoequivalence of $Z(\mathcal{H}_3)$ is isomorphic to the identity functor [ENO10, Theorem 1.1 p. 3]. Consequently, any two braided equivalences between a braided fusion category and $Z(\mathcal{H}_3)$ are monoidally naturally isomorphic, and the labels of the simple objects of $Z(\mathcal{H}_3)$ used below are intrinsic.

## 2.2 The Drinfeld center

We use the modular data of $Z(\mathcal{H}_3)$ in the form of [EvG11, equations (2.5a) and (2.5b) p. 20], as recalled in [Arh26, Section 2.1 p. 7]. The simple objects are labelled

$$0,\ b,\ a,\ c_0,\ c_1,\ c_2,\ d_1,\dots,d_6,$$

with 0 the unit. Their quantum dimensions are

$$1,\quad 3d+1,\quad 3d+2,\quad 3d+2,\quad 3d+2,\quad 3d+2,\quad 3d,\dots,3d, \tag{2.4}$$

and their twists are

$$1,\ 1,\ 1,\ 1,\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}},\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}},\ \mathrm{e}^{\frac{2\pi\mathrm{i}6l^2}{13}}\quad (1\leqslant l\leqslant 6). \tag{2.5}$$

The charge conjugation $S^2$ is trivial, since the $S$-matrix of [EvG11, (2.5a)–(2.5b) p. 20] is real, symmetric and unitary, so that every simple object is self-dual, see also [Arh26, p. 7]. Since $Z(\mathcal{H}_3)$ is unitary, the quantum dimensions coincide with the Frobenius–Perron dimensions, and

$$\operatorname{FPdim} Z(\mathcal{H}_3) = (\dim \mathcal{H}_3)^2 = 9(3d+2)^2. \tag{2.6}$$

The twists $\mathrm{e}^{\frac{2\pi\mathrm{i}6l^2}{13}}$ are different from 1, since 13 does not divide $6l^2$ for $1\leqslant l\leqslant 6$. Hence the simple objects with trivial twist are exactly 0, $b$, $a$ and $c_0$.

The modular data do not distinguish $a$ from $c_0$. In [EvG11], the labels are defined through the half-braidings of [Izu01, Section 8] on Izumi's model $\{1,\alpha,\alpha^2,\xi,\alpha\xi,\alpha^2\xi\}$ of the six-object even part of the Haagerup subfactor [EvG11, p. 17], [Izu18, pp. 56–57]. This even part is $\mathcal{H}_2$, which is defined in [GrS12, p. 14] as the even part with six simple objects of the Haagerup subfactor. With $\mu=\xi\oplus\alpha\xi\oplus\alpha^2\xi$, the forgetful functor $F\colon Z(\mathcal{H}_2)\to\mathcal{H}_2$ maps $b$ to $1\oplus\mu$, $a$ to $2\cdot 1\oplus\mu$, the objects $c_j$ to $\alpha\oplus\alpha^2\oplus\mu$ and the objects $d_l$ to $\mu$ [EvG11, p. 18]. By Section 2.1, these labels are intrinsic. We use them throughout.

## 2.3 Étale and Lagrangian algebras

Let $\mathcal{C}$ be a non-degenerate braided fusion category. An étale algebra in $\mathcal{C}$ is a commutative separable algebra, and it is connected if $\dim \operatorname{Hom}_{\mathcal{C}}(1,A)=1$. For a connected étale algebra $A$, the category $\mathcal{C}_A$ of right $A$-modules is a fusion category and the full subcategory $\mathcal{C}_A^0$ of local (dyslectic) modules is a non-degenerate braided fusion category [DMNO13, Corollary 3.30 p. 15], with

$$\operatorname{FPdim}(\mathcal{C}_A)=\frac{\operatorname{FPdim}(\mathcal{C})}{\operatorname{FPdim}(A)},\quad \operatorname{FPdim}(\mathcal{C}_A^0)=\frac{\operatorname{FPdim}(\mathcal{C})}{\operatorname{FPdim}(A)^2}, \tag{2.7}$$

see [DMNO13, Lemma 3.11 p. 10 and Corollary 3.32 p. 15]. A connected étale algebra $L$ is called Lagrangian if $\operatorname{FPdim}(L)^2=\operatorname{FPdim}(\mathcal{C})$ [DMNO13, Definition 4.6 p. 17], which is equivalent to $\mathcal{C}_L^0\cong\mathrm{Vec}$ [DMNO13, Remark 4.3 p. 16].

**Proposition 2.2** *Let $\mathcal{C}$ be a non-degenerate braided fusion category.*

1. *Let $L$ be a Lagrangian algebra in $\mathcal{C}$. The free module functor $F_L\colon\mathcal{C}\to\mathcal{C}_L$, $X\mapsto X\otimes L$, lifts to a braided equivalence $\mathcal{C}\to Z(\mathcal{C}_L)$, under which $F_L$ corresponds to the forgetful functor $Z(\mathcal{C}_L)\to\mathcal{C}_L$.*

2. *Assume that $\mathcal{C} = Z(\mathcal{D})$ for a fusion category $\mathcal{D}$. Then the isomorphism classes of Lagrangian algebras of $\mathcal{C}$ are in bijection with the equivalence classes of indecomposable $\mathcal{D}$-module categories, and for the Lagrangian algebra $L$ corresponding to $\mathcal{M}$ the fusion category $\mathcal{C}_L$ is equivalent to the dual category $\mathcal{D}^*_{\mathcal{M}}$.*

3. *Let $\mathcal{D}$ be a fusion category and $I\colon \mathcal{D} \to Z(\mathcal{D})$ the right adjoint of the forgetful functor. The étale subalgebras of the Lagrangian algebra $I(1)$ are in order-reversing bijection with the fusion subcategories of $\mathcal{D}$, and if $B \subset I(1)$ corresponds to $\mathcal{B} \subset \mathcal{D}$ then* $\mathrm{FPdim}(B)\,\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{D})$.

4. *Let $A$ be a connected étale algebra in $\mathcal{C}$. The commutative algebras $B$ in $\mathcal{C}$ equipped with a unital algebra morphism $A \to B$, which is automatically injective [DMNO13, p. 12], correspond to the commutative algebras in $\mathcal{C}_A^0$. Under this correspondence, connected étale algebras correspond to connected étale algebras.*

5. *Let $\mathcal{D}$ be a fusion category, let $\mathcal{B} \subset \mathcal{D}$ be a fusion subcategory and let $A_{\mathcal{B}} \subset I(1)$ be the étale subalgebra attached to $\mathcal{B}$ in part 3. Then the category $Z(\mathcal{D})^0_{A_{\mathcal{B}}}$ of local $A_{\mathcal{B}}$-modules is braided equivalent to $Z(\mathcal{B})$.*

*Proof* : 1. This is [DMNO13, Corollary 4.1 p. 16], with the central structure [DMNO13, (17) p. 10].

2. This is [DMNO13, Proposition 4.8 and Remark 4.9 p. 17], with the 2-equivalence of [DMNO13, Proposition 4.4 p. 16].

3. This is [DMNO13, Theorem 4.10 p. 18].

4. This is [DMNO13, Section 3.6 and Proposition 3.16 p. 12].

5. Put $\mathcal{C} = Z(\mathcal{D})$ and $A = A_{\mathcal{B}}$. In the proof of [DMNO13, Theorem 4.10 p. 18], the category $\mathcal{C}_A$ is identified with the relative center $Z_{\mathcal{B}}(\mathcal{D})$, whose objects are the objects of $\mathcal{D}$ equipped with a half-braiding with respect to the objects of $\mathcal{B}$, in such a way that the free module functor $\mathcal{C} \to \mathcal{C}_A$, with its central structure [DMNO13, (17) p. 10], corresponds to the forgetful functor $Z(\mathcal{D}) \to Z_{\mathcal{B}}(\mathcal{D})$ with the central structure given by the half-braidings. Since $\mathcal{C}$ is non-degenerate, there is a braided equivalence $Z(\mathcal{C}_A) \cong \mathcal{C} \boxtimes (\mathcal{C}_A^0)^{\mathrm{rev}}$ under which $\mathcal{C} \boxtimes 1$ corresponds to the image of the central structure of the free module functor [DMNO13, Corollary 3.30 and Remark 3.31 p. 15]. On the other hand, $Z_{\mathcal{B}}(\mathcal{D})$ is the dual of $\mathcal{D} \boxtimes \mathcal{B}^{\mathrm{op}}$ with respect to the module category $\mathcal{D}$ [DMNO13, p. 18], so that Schauenburg's equivalence [DMNO13, (12) p. 7] gives a braided equivalence $Z(Z_{\mathcal{B}}(\mathcal{D})) \cong Z(\mathcal{D}) \boxtimes Z(\mathcal{B})^{\mathrm{rev}}$, under which the central structure of the forgetful functor $Z(\mathcal{D}) \to Z_{\mathcal{B}}(\mathcal{D})$ corresponds to $Z(\mathcal{D}) \boxtimes 1$ [DMNO13, pp. 18–19]. The centralizer of $\mathcal{C} \boxtimes 1$ in $\mathcal{C} \boxtimes \mathcal{E}$ is $1 \boxtimes \mathcal{E}$ for any braided fusion category $\mathcal{E}$, since $\mathcal{C}$ is non-degenerate. Comparing the centralizers of $\mathcal{C} \boxtimes 1$ in the two descriptions of $Z(\mathcal{C}_A)$, we obtain a braided equivalence $(\mathcal{C}_A^0)^{\mathrm{rev}} \cong Z(\mathcal{B})^{\mathrm{rev}}$, hence $\mathcal{C}_A^0 \cong Z(\mathcal{B})$. This is the claim proved in [DMNO13, proof of Theorem 4.10 p. 18]. ■

## 2.4 Haagerup algebras and their extensions

A vertex operator algebra is strongly rational if it is simple, of CFT type, self-contragredient, rational and $C_2$-cofinite. It is positive if every irreducible module not isomorphic to the vertex operator algebra itself has positive lowest conformal weight. Following [Arh26], a Haagerup algebra is a positive strongly rational vertex operator algebra $U$ of central charge eight together with a braided equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$. The modular tensor category $\mathrm{Rep}(U)$ is given by [Hua08]. Since $Z(\mathcal{H}_3)$ has no nontrivial invertible object, such an equivalence preserves the

ribbon structures [Arh26, p. 8]. By Section 2.1, it is unique up to natural isomorphism, so that the simple $U$-modules $M_z$ below are well defined.

Let $U$ be a strongly rational vertex operator algebra. An extension of $U$ is a vertex operator algebra $B$ containing $U$ as a vertex subalgebra with the same conformal vector and with $\dim B_0 = 1$. Two extensions are isomorphic if there is an isomorphism of vertex operator algebras between them which restricts to the identity of $U$. An extension $B$ is holomorphic if it is simple and if every $B$-module is a direct sum of copies of $B$, that is, $\mathrm{Rep}(B) \cong \mathrm{Vec}$. By [HKL15, Theorem 3.2 p. 10], extensions of $U$ correspond to haploid commutative associative algebras with trivial twist in $\mathrm{Rep}(U)$, and by [HKL15, Theorem 3.4 p. 16] the category of modules of an extension $B$ is isomorphic to the category of local modules of the corresponding algebra. If the algebra is moreover rigid in the sense of [KiO02, Definition 1.11 p. 10], then its category of modules is semisimple [KiO02, Theorem 3.3 p. 17], so that it is separable [DMNO13, Proposition 2.7 p. 5], hence étale.

**Lemma 2.3** *Let $U$ be a positive strongly rational vertex operator algebra and $B$ a simple extension of $U$ with $B \cong B'$. Let $W$ be a vertex subalgebra of $B$ containing $U$. Then $W$ is simple if and only if the invariant bilinear form of $B$ restricts to a nondegenerate form on $W$. In this case, the algebra in $\mathrm{Rep}(U)$ corresponding to $W$ is a connected étale subalgebra of the algebra corresponding to $B$.*

*Proof* : Since $B$ is simple and isomorphic to $B'$, it carries a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$, unique up to a scalar [FHL93], [Li94]. Let $R = \{w \in W : (w, W) = 0\}$ be the radical of its restriction to $W$. The vertex subalgebra $W$ contains the conformal vector, hence it is stable under $L_1$. The invariance of the form expresses the adjoint of a mode $u_{(n)}$, for $u \in W$, as a finite combination of modes of the vectors $L_1^j u \in W$. Hence $R$ is an ideal of $W$. Since $(\mathbf{1}, \mathbf{1}) \neq 0$, we have $R \neq W$. Consequently, if $W$ is simple then $R = 0$.

Conversely, assume that $R = 0$, and let $J \subset W$ be a proper ideal. The space $J$ does not contain $\mathbf{1}$, and since $\dim W_0 = 1$ and $J$ is graded, we have $J \cap W_0 = 0$. For $j \in J$ and $w \in W$, the invariance of the form gives $(j, w) = (j_{(-1)}\mathbf{1}, w) = (\mathbf{1}, j^\dagger w)$, where $j^\dagger$ is a finite linear combination of modes of the vectors $L_1^k j$. These vectors lie in $J$, since $J$ is an ideal containing $j$ and the conformal vector belongs to $W$. Hence $j^\dagger w \in J$. Since $J \cap W_0 = 0$, $J$ is graded and $(\mathbf{1}, W_n) = 0$ for $n \neq 0$, we obtain $(j, w) = 0$. Thus $J \subset R = 0$, and $W$ is simple.

Finally, assume that $W$ is simple. By [HKL15, Theorem 3.2 p. 10], $W$ corresponds to a haploid commutative algebra $A_W$ with trivial twist, which is a subalgebra of the algebra $A_B$ attached to $B$. The multiplication of $A_W$ corresponds to the vertex operator of $W$ [HKL15, proof of Theorem 3.2 p. 10], so that the subobjects of $A_W$ which are $A_W$-submodules correspond to the ideals of $W$. Since $W$ is simple, $A_W$ is simple as an $A_W$-module. The categorical dimension of $A_W$ is a sum of quantum dimensions of simple $U$-modules, which are positive for positive strongly rational $U$ [DJX13, Lemma 4.2 p. 13], [Hua08], so that it is nonzero. Since $\theta_{A_W} = \mathrm{id}$, [KiO02, Lemma 1.20 p. 15] shows that $A_W$ is rigid in the sense of [KiO02, Definition 1.11 p. 10], its category of modules is semisimple [KiO02, Theorem 3.3 p. 17], and $A_W$ is separable [DMNO13, Proposition 2.7 p. 5], hence étale. ∎

We recall the facts from [Arh26] that we use. Let $U$ be a Haagerup algebra.

1. By [Arh26, Proposition 4.1 p. 11], the subspace $V' = U \oplus M_b$ of the holomorphic extension of $U$ attached to an algebra isomorphic to $0 \oplus b \oplus 2a$ is a vertex operator algebra isomorphic to $V_{E_6 \oplus A_2}$, and $U$ is a conformal subalgebra of $V'$. We fix such an isomorphism. The discriminant group of $E_6 \oplus A_2$ is $\mathbb{Z}_3 \times \mathbb{Z}_3$, with quadratic form $q(i,j) = \frac{2i^2+j^2}{3}$ modulo one. We denote by $V'_w$, $w \in \mathbb{Z}_3 \times \mathbb{Z}_3$, the simple $V'$-modules.

2. Let $A_0$ be the algebra in $\mathrm{Rep}(U)$ attached to $V'$. By [Arh26, Lemma 4.3 p. 13] and its proof, for $w \neq 0$ the restriction of $V'_w$ to $U$ is simple. This restriction is isomorphic to $M_{c_1}$ if $w \in \{(0,1),(0,2)\}$, to $M_{c_2}$ if $w \in \{(1,0),(2,0)\}$, and the four remaining classes, which are the nonzero elements of the two isotropic lines, restrict to $M_a$ for the two classes of one line and to $M_{c_0}$ for the two classes of the other line. Indeed, the proof of [Arh26, Lemma 4.3 p. 13] shows that exactly two of these four classes restrict to $M_{c_0}$, and the set of such classes is stable under $w \mapsto -w$ since $V'_{-w}$ is the contragredient of $V'_w$ and $c_0$ is self-dual. In the coordinates chosen in [Arh26, Section 4.2 p. 12], for which the holomorphic extension of [Arh26, Proposition 4.1 p. 11], attached to an algebra isomorphic to $0 \oplus b \oplus 2a$, corresponds to the line generated by $(1,1)$, the classes restricting to $M_a$ are $(1,1)$ and $(2,2)$. For the explicit model of $V'$ of [Arh26, Section 10.1 p. 27], the corresponding line is not known.

We denote by $\Lambda_a$ and $\Lambda_{c_0}$ the isotropic lines whose nonzero classes restrict to $M_a$ and $M_{c_0}$ respectively, and we put

$$H_a \overset{\text{def}}{=} \bigoplus_{w \in \Lambda_a} V'_w, \quad H_{c_0} \overset{\text{def}}{=} \bigoplus_{w \in \Lambda_{c_0}} V'_w. \tag{2.8}$$

Each isotropic line $\Lambda$ defines an even unimodular overlattice of $E_6 \oplus A_2$, which is isometric to $E_8$, and $\bigoplus_{w \in \Lambda} V'_w$ is the corresponding lattice vertex operator algebra, see Lemma 4.1 and the paragraph following it. Hence $H_a \cong H_{c_0} \cong V_{E_8}$, and as $U$-modules

$$H_a = U \oplus M_b \oplus M_a \oplus M_a, \quad H_{c_0} = U \oplus M_b \oplus M_{c_0} \oplus M_{c_0}. \tag{2.9}$$

For the explicit algebras we use the following external hypotheses, introduced in [Arh26, p. 5]. We write ($\mathrm{H_J}$) for the assertion that the explicit conformal algebra of [Jia26] is a Haagerup algebra, and ($\mathrm{H_R}$) for the same assertion for the algebra $U^{\mathrm{R}}$ of [Rie26a]. Positivity is part of these hypotheses. By [Arh26, Theorem 10.24 p. 42], Jiao's algebra is isomorphic to the algebra $S$, so that ($\mathrm{H_J}$) implies that $S$ is a Haagerup algebra.

## 2.5 Unitary vertex operator algebras and strong locality

We use the framework of [CKLW18]. A unitary vertex operator algebra is a vertex operator algebra $V$ with a normalized invariant scalar product, whose PCT operator $\theta$ is an antilinear involutive automorphism of $V$ [CKLW18, Section 5]. A vertex subalgebra $W$ of a unitary vertex operator algebra $V$ is a unitary subalgebra if and only if $\theta W \subset W$ and $L_1 W \subset W$ [CKLW18, Proposition 5.23 p. 43]. A unitary vertex operator algebra with $V_0 = \mathbb{C}\mathbf{1}$ is simple [CKLW18, Proposition 5.3 p. 34].

A simple unitary vertex operator algebra is strongly local if its vertex operators satisfy energy bounds and the von Neumann algebras generated by the smeared vertex operators localized in disjoint intervals commute. A strongly local vertex operator algebra $V$ defines an irreducible conformal net $\mathcal{A}_V$ on the Hilbert space completion $\mathcal{H}_V$ of $V$, by

$$\mathcal{A}_V(I) = W^*\big(\{Y(a,f) : a \in V,\ f \in C^\infty(S^1),\ \operatorname{supp} f \subset I\}\big),$$

the von Neumann algebra generated by the closures $Y(a,f)$ of the smeared vertex operators $Y_0(a,f)$ with domain $V$ [CKLW18, (106) p. 49 and (114) p. 50], see [CKLW18, Theorem 6.8 p. 53]. We will use the following results.

**Proposition 2.4** *Let $V$ be a strongly local simple unitary vertex operator algebra.*

1. *If $\{\cdot|\cdot\}$ is another normalized invariant scalar product on $V$, then $(V, \{\cdot|\cdot\})$ is strongly local, and the two conformal nets are isomorphic [CKLW18, Theorem 6.8 p. 53].*

2. *A unitary subalgebra $W$ of $V$ is strongly local, and $\mathcal{A}_W$ embeds as a covariant subnet of $\mathcal{A}_V$ [CKLW18, Theorem 7.1 p. 55]. The map $W \mapsto \mathcal{A}_W$ is a bijection between unitary subalgebras of $V$ and Möbius covariant subnets of $\mathcal{A}_V$ [CKLW18, Theorem 7.5 p. 58].*

3. *If $G$ is a closed subgroup of the group of unitary automorphisms of $V$, then $\mathcal{A}_{V^G} = \mathcal{A}_V^G$ [CKLW18, Proposition 7.6 p. 58].*

4. *For a unitary subalgebra $W$ of $V$, the coset subnet $\mathcal{A}_W^c$, given by $\mathcal{A}_W^c(I) = \mathcal{A}_W(S^1)' \cap \mathcal{A}_V(I)$, is the net of the coset $W^c$ [CKLW18, Proposition 7.8 p. 58].*

5. *A tensor product $V^\alpha \otimes V^\beta$ of strongly local vertex operator algebras is strongly local, and $\mathcal{A}_{V^\alpha \otimes V^\beta} = \mathcal{A}_{V^\alpha} \otimes \mathcal{A}_{V^\beta}$ [CKLW18, Corollary 8.2 p. 62]. For a complex simple Lie algebra $\mathfrak{g}$ and a positive integer $k$, the simple affine vertex operator algebra $V_{\mathfrak{g},k}$ is strongly local and its net is the loop group net $\mathcal{A}_{G_k}$ [CKLW18, Example 8.7 p. 63].*

## 2.6 Conformal nets

Throughout, $\mathrm{Rep}(\mathcal{A})$ denotes the category of finite direct sums of irreducible representations of a completely rational net. Equivalently, it is the category of representations of finite statistical dimension, also denoted by $\mathrm{Rep}^f(\mathcal{A})$ below. This convention excludes infinite direct sums when we refer to a fusion category.

We refer to [KLM01, BKLR15] for conformal nets, their representations and Q-systems. A conformal net $\mathcal{A}$ is completely rational if it is split, strongly additive and has finite $\mu$-index. In this case every representation on a separable Hilbert space is a direct sum of irreducible representations with finite statistical dimension [KLM01, Corollary 39 p. 27], there are finitely many irreducible sectors, $\mathrm{Rep}(\mathcal{A})$ is a unitary modular tensor category, and

$$\mu_{\mathcal{A}} = \sum_{\pi \in \mathrm{Irr}(\mathrm{Rep}(\mathcal{A}))} d(\pi)^2 \tag{2.10}$$

by [KLM01, Theorem 33 p. 24]. By [LoX04, MTW18], complete rationality is equivalent to rationality in the sense of [Ten24, Section 4.4 p. 54], namely finitely many irreducible representations, all with finite index.

For an inclusion $\mathcal{A} \subset \mathcal{B}$ of conformal nets, the index $[\mathcal{B} : \mathcal{A}]$ is the Jones index of $\mathcal{A}(I) \subset \mathcal{B}(I)$, which does not depend on $I$. We will use the following facts.

**Proposition 2.5** *Let $\mathcal{A} \subset \mathcal{B}$ be a conformal subnet.*

1. *If $[\mathcal{B} : \mathcal{A}] < \infty$, then $\mathcal{A}$ is completely rational if and only if $\mathcal{B}$ is [Lon03, Theorem 24 p. 20]. In this case $\mu_{\mathcal{A}} = [\mathcal{B} : \mathcal{A}]^2 \mu_{\mathcal{B}}$ [KLM01, Proposition 24 p. 19].*

2. *If $\mathcal{A}$ is completely rational and $\mathcal{B}$ is an irreducible local extension of $\mathcal{A}$, in the sense that $\mathcal{A}(S^1)' \cap \mathcal{B}(I) = \mathbb{C}$ [KaL04, p. 9], then $[\mathcal{B} : \mathcal{A}] < \infty$ [KaL04, Proposition 2.3 p. 12]. The hypotheses of [KaL04, Proposition 2.3] on the sectors of $\mathcal{A}$ hold by [KLM01, Theorem 33 p. 24 and Corollary 39 p. 27].*

3. *If $\mathcal{A}$ is completely rational and $[\mathcal{B} : \mathcal{A}] < \infty$, the restriction of the vacuum representation of $\mathcal{B}$ to $\mathcal{A}$ is the dual canonical endomorphism $\theta$ of the inclusion, which carries the structure of a commutative* Q*-system in* $\mathrm{Rep}(\mathcal{A})$*, and $[\mathcal{B} : \mathcal{A}] = d(\theta)$ [LoR95, Theorem 4.9*

*p. 25]. Conversely, a commutative* Q*-system* $\Theta$ *in* $\mathrm{Rep}(\mathcal{A})$ *with* $\dim \mathrm{Hom}(\mathrm{id}, \Theta) = 1$ *defines an irreducible local extension of* $\mathcal{A}$*, unique up to isomorphism of extensions [LoR95, Theorem 4.9 p. 25], [BKLR15, Section 5.2.1 p. 77]. Intermediate extensions correspond to intermediate* Q*-systems [BKLR15, Proposition 4.11 p. 49].*

A commutative Q-system $\Theta$ with $\dim \mathrm{Hom}(\mathrm{id}, \Theta) = 1$ is a connected étale algebra in the braided fusion category $\mathrm{Rep}(\mathcal{A})$. Conversely, by [CGGH23, Theorem 3.2 p. 11], a haploid algebra in a $C^*$-tensor category is equivalent to a Q-system if and only if it is rigid. Since a connected étale algebra is rigid [DMNO13, Remark 3.4 p. 7], a connected étale algebra in $\mathrm{Rep}(\mathcal{A})$ is equivalent to a commutative Q-system. A braided equivalence between unitary braided fusion categories is monoidally naturally isomorphic to a unitary braided equivalence [Reu23, Theorem 5 p. 2].

We recall the results of Bischoff that we use. A conformal net $\mathcal{A}$ is called holomorphic if it is completely rational with $\mu_{\mathcal{A}} = 1$, that is, if $\mathrm{Rep}(\mathcal{A}) \cong \mathrm{Vec}$.

**Proposition 2.6** *1. Let* $\mathcal{A}$ *be a completely rational net. Then* $\mathrm{Rep}(\mathcal{A})$ *is braided equivalent to* $Z(\mathcal{F})$ *for some unitary fusion category* $\mathcal{F}$ *if and only if* $\mathcal{A}$ *has a holomorphic finite index extension. In this case,* $\mathcal{F}$ *gives a holomorphic extension* $\mathcal{B}_{\mathcal{F}}$ *of* $\mathcal{A}$*, and there is a Galois correspondence between full fusion subcategories of* $\mathcal{F}$ *and intermediate nets* $\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}_{\mathcal{F}}$ *[Bis17a, Proposition 4.5 p. 21].*

*2. Let* $N \subset M$ *be a finite index, finite depth subfactor. If there is a completely rational net* $\mathcal{A}$ *such that* $\mathrm{Rep}(\mathcal{A})$ *is braided equivalent to* $D(N \subset M)$*, then* $N \subset M$ *arises from* $\mathcal{A}$ *in the sense of [Bis17a, Definition 4.1 p. 20], see [Bis17a, Proposition 4.7 p. 22].*

*3. Let* $\mathcal{A} \subset \mathcal{B}$ *be a finite index subnet. There is a canonical hypergroup* $K$ *acting properly on* $\mathcal{B}$ *by vacuum preserving unital completely positive maps such that* $\mathcal{A} = \mathcal{B}^K$*, and* $[\mathcal{B} : \mathcal{A}]$ *is the weight of* $K$ *[Bis17b, Theorem 1.3 p. 3 and Theorem 4.22 p. 24].*

# 3 Lagrangian algebras and holomorphic extensions

## 3.1 A rank formula

The following standard fact is the main tool of this section. In the language of Longo–Rehren inclusions, it is the Frobenius reciprocity argument used in [EvG11, p. 19] for the four-object even part of the Haagerup subfactor, and it is also used in [Hua26b, (23) p. 7]. We include a proof.

**Lemma 3.1** *Let* $\mathcal{C}$ *be a non-degenerate braided fusion category and let* $L$ *be a Lagrangian algebra in* $\mathcal{C}$*. Then* $\dim \mathrm{Hom}_{\mathcal{C}}(L, L)$ *is equal to the number of isomorphism classes of simple objects of the fusion category* $\mathcal{D} = \mathcal{C}_L$*. Moreover, the braided equivalence* $\Phi\colon \mathcal{C} \to Z(\mathcal{D})$ *of part 1 of Proposition 2.2 maps the algebra* $L$ *to an algebra isomorphic to* $I(1)$*, where* $I$ *is the right adjoint of the forgetful functor* $Z(\mathcal{D}) \to \mathcal{D}$*.*

*Proof* : Put $\mathcal{D} = \mathcal{C}_L$. By part 1 of Proposition 2.2, there is a braided equivalence $\Phi\colon \mathcal{C} \to Z(\mathcal{D})$ such that $F \circ \Phi$ is isomorphic to the free module functor $F_L\colon X \mapsto X \otimes L$, where $F\colon Z(\mathcal{D}) \to \mathcal{D}$ is the forgetful functor. The right adjoint of $F_L$ is the functor $\mathcal{D} \to \mathcal{C}$ which forgets the module structure, and it maps the unit object $L$ of $\mathcal{D}$ to $L$. Since $\Phi$ is an equivalence, the right adjoint of $F \circ \Phi$ is $\Phi^{-1} \circ I$, where $I$ is the right adjoint of $F$. By uniqueness of adjoints, we obtain an isomorphism $\Phi(L) \cong I(1)$. It is an isomorphism of algebras, since the algebra structures of $L$

and $I(1)$ are induced by the lax monoidal structures of the right adjoints of the tensor functors $F_L$ and $F$, and $F \circ \Phi \cong F_L$ as tensor functors [DMNO13, Lemma 3.9 p. 10]. Consequently, we have

$$\mathrm{Hom}_{\mathcal{C}}(L, L) \cong \mathrm{Hom}_{Z(\mathcal{D})}(I(1), I(1)) \cong \mathrm{Hom}_{\mathcal{D}}(F(I(1)), 1).$$

By [ENO05, Proposition 5.4 p. 21], [EGNO15, Proposition 9.2.2 p. 278], we have $F(I(1)) \cong \bigoplus_X X \otimes X^*$, where $X$ runs through a set of representatives of the isomorphism classes of simple objects of $\mathcal{D}$. Since $\dim \mathrm{Hom}_{\mathcal{D}}(X \otimes X^*, 1) = \dim \mathrm{Hom}_{\mathcal{D}}(X, X) = 1$ for any simple object $X$, the conclusion follows. This is [ENO05, Proposition 5.6 p. 21], and the identification of $L$ with $I(1)$ is also a consequence of [DMNO13, Proposition 4.4 p. 16]. ∎

## 3.2 The three Lagrangian algebras of the Haagerup double

**Lemma 3.2** *Let $L$ be a Lagrangian algebra in $Z(\mathcal{H}_3)$. Then $L$ is isomorphic, as an object, to $0 \oplus b \oplus X$ with $X \in \{a \oplus a,\ c_0 \oplus c_0,\ a \oplus c_0\}$.*

*Proof* : The pairing $L \otimes L \to L \to 1$ is nondegenerate [DMNO13, Remark 3.4 p. 7], so that $L$ is a rigid haploid algebra in the unitary braided fusion category $Z(\mathcal{H}_3)$. By [CGGH23, Theorem 3.2 p. 11], $L$ is equivalent to a standard commutative Q-system, which has trivial twist by [CGGH23, Proposition 2.22 p. 9]. Hence $L$ is a direct sum of simple objects with trivial twist. By Section 2.2, we can write $L \cong 0 \oplus n_b b \oplus n_a a \oplus n_{c_0} c_0$ with nonnegative integers $n_b$, $n_a$, $n_{c_0}$, where the multiplicity of 0 is one since $L$ is connected. By (2.6), the Lagrangian condition reads

$$1 + n_b(3d+1) + (n_a + n_{c_0})(3d+2) = 3(3d+2) = 9d + 6.$$

Since $d$ is irrational, we can compare the coefficients of 1 and $d$ and obtain $1+n_b+2(n_a+n_{c_0}) = 6$ and $n_b + n_a + n_{c_0} = 3$. Hence $n_a + n_{c_0} = 2$ and $n_b = 1$. ∎

**Proposition 3.3** *The braided fusion category $Z(\mathcal{H}_3)$ has exactly three Lagrangian algebras up to isomorphism, $L_1$, $L_2$ and $L_3$, characterized by $Z(\mathcal{H}_3)_{L_i} \cong \mathcal{H}_i$. As objects, $L_1 \cong 0 \oplus b \oplus a \oplus c_0$, and $L_2$, $L_3$ belong to $\{0 \oplus b \oplus 2a,\ 0 \oplus b \oplus 2c_0\}$.*

*Proof* : By part 2 of Proposition 2.2 applied to $\mathcal{D} = \mathcal{H}_3$, the Lagrangian algebras of $Z(\mathcal{H}_3)$ are in bijection with the indecomposable $\mathcal{H}_3$-module categories, and the category of modules over the Lagrangian algebra attached to $\mathcal{M}$ is the dual category $(\mathcal{H}_3)^*_{\mathcal{M}}$. By part 2 of Proposition 2.1, there are three such module categories and their duals are $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. Since these three categories are pairwise inequivalent, the three Lagrangian algebras are pairwise non-isomorphic, and we can label them so that $Z(\mathcal{H}_3)_{L_i} \cong \mathcal{H}_i$.

By Lemma 3.1 and part 3 of Proposition 2.1, we have

$$\dim \mathrm{Hom}(L_1, L_1) = 4, \quad \dim \mathrm{Hom}(L_2, L_2) = \dim \mathrm{Hom}(L_3, L_3) = 6. \tag{3.1}$$

For the three objects of Lemma 3.2, the dimensions of the endomorphism spaces are $1+1+4 = 6$ for $0 \oplus b \oplus 2a$ and $0 \oplus b \oplus 2c_0$, and $1+1+1+1 = 4$ for $0 \oplus b \oplus a \oplus c_0$, since 0, $b$, $a$, $c_0$ are pairwise non-isomorphic simple objects. This proves the assertion. ∎

**Remark 3.4** The objects of $L_1$ and $L_2$ appear in [EvG11, pp. 18–19] as the dual canonical endomorphisms of the Longo–Rehren inclusions of the two even parts of the Haagerup subfactor, together with the three monomial modular invariants $|\chi_0 + \chi_b + 2\chi_a|^2$, $|\chi_0 + \chi_b + \chi_a + \chi_{c_0}|^2$ and $|\chi_0 + \chi_b + 2\chi_{c_0}|^2$ [EvG11, Proposition 4 p. 32]. Proposition 3.6 below attaches the third invariant to the category $\mathcal{H}_3$, which was found later [GrS12].

We now realize the Lagrangian algebras of objects $0 \oplus b \oplus 2a$ and $0 \oplus b \oplus 2c_0$ by means of a Haagerup algebra.

**Proposition 3.5** *Let $U$ be a Haagerup algebra, and let $L_{\Lambda_a}$ and $L_{\Lambda_{c_0}}$ be the algebras in* $\mathrm{Rep}(U)$ *attached to the extensions $H_a$ and $H_{c_0}$ of (2.8).*

1. *$L_{\Lambda_a}$ and $L_{\Lambda_{c_0}}$ are Lagrangian algebras, isomorphic as objects to $0 \oplus b \oplus 2a$ and $0 \oplus b \oplus 2c_0$ respectively.*
2. *Under the braided equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$, we have $\{L_{\Lambda_a}, L_{\Lambda_{c_0}}\} = \{L_2, L_3\}$ and the remaining Lagrangian algebra $L_1$ is isomorphic to $0 \oplus b \oplus a \oplus c_0$ as an object.*
3. *The algebra $A_0$ attached to $V' = U \oplus M_b$ is a connected étale algebra, and $L_{\Lambda_a}$, $L_{\Lambda_{c_0}}$ are the only Lagrangian algebras of $\mathrm{Rep}(U)$ containing $A_0$. In particular $L_1$ does not contain $A_0$ as a subalgebra.*

*Proof* : 1. The vertex operator algebra $H_a$ is a simple extension of $U$ isomorphic to $V_{E_8}$, hence holomorphic and self-contragredient. By [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16] and Lemma 2.3 applied to $W = B = H_a$, the algebra $L_{\Lambda_a}$ is a connected étale algebra in $\mathrm{Rep}(U)$ whose category of local modules is equivalent to $\mathrm{Rep}(H_a) \cong \mathrm{Vec}$. By [DMNO13, Remark 4.3 p. 16], it is Lagrangian. Its object is read off from (2.9). The same argument applies to $H_{c_0}$.

2. Since $a$ and $c_0$ are non-isomorphic, the objects of $L_{\Lambda_a}$ and $L_{\Lambda_{c_0}}$ are non-isomorphic, so these two Lagrangian algebras are distinct. By Proposition 3.3, they have endomorphism spaces of dimension six, hence they are $L_2$ and $L_3$ in some order, and $L_1$ is the remaining one, with the stated object.

3. The algebra $A_0$ corresponds to the simple vertex subalgebra $V' \subset H_a$, so it is a connected étale subalgebra of $L_{\Lambda_a}$ by Lemma 2.3. By [HKL15, Theorem 3.4 p. 16] and [CKM24, Theorem 3.65 p. 85], the braided category $\mathrm{Rep}(U)^0_{A_0}$ is braided equivalent to $\mathrm{Rep}(V')$, which is the pointed modular category of the discriminant form $(\mathbb{Z}_3 \times \mathbb{Z}_3, q)$. By part 4 of Proposition 2.2, the connected étale algebras $L$ of $\mathrm{Rep}(U)$ containing $A_0$ correspond to the connected étale algebras $B$ of $\mathrm{Rep}(U)^0_{A_0} \cong \mathrm{Rep}(V')$. If $W$ is the corresponding extension of $U$ and of $V'$, then [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16], applied to $U \subset W$ and to $V' \subset W$, give isomorphisms of categories $\mathrm{Rep}(U)^0_L \cong \mathrm{Rep}(W) \cong \mathrm{Rep}(V')^0_B$. Hence $L$ is Lagrangian if and only if $B$ is, by [DMNO13, Remark 4.3 p. 16], and the Lagrangian algebras of $\mathrm{Rep}(U)$ containing $A_0$ correspond to the holomorphic extensions of $V'$, by [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16] and Lemma 2.3. A holomorphic extension of the lattice vertex operator algebra $V'$ is a direct sum of simple currents $V'_w$ with $w$ in a subgroup $\Lambda$ of $\mathbb{Z}_3 \times \mathbb{Z}_3$ of order three on which $q$ vanishes, since its Frobenius–Perron dimension is three. Indeed, the simple summands of a connected étale algebra in the pointed category $\mathrm{Rep}(V')$ have trivial twist, appear with multiplicity at most one and form a subgroup, by the stabilizer argument of the proof of [Arh26, Proposition 4.1 p. 11]. Since $q$ vanishes on $\Lambda$, the restriction of the pointed braided category $\mathrm{Rep}(V')$ to $\Lambda$ is equivalent to $\mathrm{Rep}(\mathbb{Z}_3)$ as a symmetric category, with trivial associator and braiding, so that an algebra structure on $\bigoplus_{w \in \Lambda} V'_w$ is given by a normalized 2-cochain with values in $\mathbb{C}^\times$ and trivial coboundary, the components of the multiplication being nonzero since the pairing of a connected étale algebra is nondegenerate [DMNO13, Remark 3.4 p. 7]. Hence this algebra structure is unique up to isomorphism since $H^2(\mathbb{Z}_3, \mathbb{C}^\times) = 0$. The form $q(i,j) = \frac{2i^2+j^2}{3}$ vanishes exactly on the two lines generated by $(1,1)$ and $(1,2)$. Hence there are exactly two Lagrangian algebras of $\mathrm{Rep}(V')$, giving $H_a$ and $H_{c_0}$. The last statement follows from part 2. ∎

**Proposition 3.6** *In the labels of [EvG11] recalled in Section 2.2, we have $L_2 \cong 0 \oplus b \oplus 2a$ and $L_3 \cong 0 \oplus b \oplus 2c_0$ as objects. If $U$ is a Haagerup algebra, then the extension $H_a$ of $U$ carries $\mathcal{H}_2$ while $H_{c_0}$ carries $\mathcal{H}_3$, that is, $L_{\Lambda_a} \cong L_2$ and $L_{\Lambda_{c_0}} \cong L_3$.*

*Proof* : By Lemma 3.1 applied to $\mathcal{D} = \mathcal{H}_2$, the Lagrangian algebra $L_2$ corresponds to $I(1)$ under $Z(\mathcal{H}_3) \cong Z(\mathcal{H}_2)$. By adjunction, the multiplicity of a simple object $z$ of $Z(\mathcal{H}_2)$ in $I(1)$ is $\dim \mathrm{Hom}_{\mathcal{H}_2}(F(z), 1)$, where $F$ is the forgetful functor. By Section 2.2, this multiplicity is 1 for $z \in \{0, b\}$, 2 for $z = a$ and 0 for the other simple objects. Hence $L_2 \cong 0 \oplus b \oplus 2a$. This is the computation of [EvG11, p. 18].

We now compute $L_3$. Let $\Phi\colon Z(\mathcal{H}_3) \to Z(\mathcal{H}_2)$ be a braided equivalence with $\Phi(L_2) \cong I(1)$, given by Lemma 3.1. For $i \in \{1, 2, 3\}$, the algebra $\Phi(L_i)$ is a Lagrangian algebra of $Z(\mathcal{H}_2)$ with category of modules equivalent to $\mathcal{H}_i$. By part 2 of Proposition 2.2, it is the Lagrangian algebra attached in [DMNO13, Section 4.2 p. 17] to an indecomposable $\mathcal{H}_2$-module category $\mathcal{N}_i$, and $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ are pairwise inequivalent, since the algebras $\Phi(L_i)$ are pairwise non-isomorphic, and they are, up to equivalence, all the indecomposable $\mathcal{H}_2$-module categories (part 2 of Proposition 2.1). Here $\mathcal{N}_2$ is the regular module category, since $\Phi(L_2) \cong I(1)$ and $I(1)$ is the Lagrangian algebra attached to $\mathcal{H}_2$ as a module category over itself [DMNO13, Remark 4.9 p. 17]. We equip $\mathcal{H}_2$ with its spherical structure with positive dimensions [ENO05, Proposition 8.23 p. 34]. The induced dimensions of the simple objects of $Z(\mathcal{H}_2)$ are positive, so that the twists of the Lagrangian algebras $\Phi(L_i)$ are trivial [Ost15, Remark 2.19 p. 8]. Hence, by [Ost15, Remark 2.19 pp. 7–8], the number of simple objects of the category $\mathrm{Fun}_{\mathcal{H}_2}(\mathcal{N}_2, \mathcal{N}_i)$ of module functors is $\dim \mathrm{Hom}(\Phi(L_2), \Phi(L_i)) = \dim \mathrm{Hom}(L_2, L_i)$. Since a module functor from the regular module category is determined by the image of the unit object, $\mathrm{Fun}_{\mathcal{H}_2}(\mathcal{N}_2, \mathcal{N}_i)$ is equivalent to $\mathcal{N}_i$, and $\dim \mathrm{Hom}(L_2, L_i)$ is the rank of $\mathcal{N}_i$. By Proposition 3.3 and the first paragraph, $\dim \mathrm{Hom}(L_2, L_1) = \dim \mathrm{Hom}(0 \oplus b \oplus 2a, 0 \oplus b \oplus a \oplus c_0) = 4$, so that $\mathcal{N}_1$ has rank four.

Let $A = 1 \oplus \alpha \oplus \alpha^2$ with the structure of Q-system in $\mathcal{H}_2$ used to define $\mathcal{H}_3$ [GrS12, Lemma 3.14 p. 21 and Definition 3.19 p. 23], and let $\mathcal{M}$ be the category of right $A$-modules in $\mathcal{H}_2$, which is a semisimple left $\mathcal{H}_2$-module category since $A$ is separable [DMNO13, Proposition 2.7 p. 5]. For objects $Y$, $Y'$ of $\mathcal{H}_2$, we have $\mathrm{Hom}_A(Y \otimes A, Y' \otimes A) \cong \mathrm{Hom}(Y, Y' \otimes A)$, and since $A$ is a Q-system, every simple right $A$-module is a summand of some $Y \otimes A$ with $Y$ simple. Since $\xi\alpha \cong \alpha^2\xi$ and $\xi\alpha^2 \cong \alpha\xi$ by (2.2), we have $\alpha^j\xi \otimes A \cong \alpha^j\xi \oplus \alpha^{j+1}\xi \oplus \alpha^{j+2}\xi$ for $j \in \{0, 1, 2\}$. Hence $\dim \mathrm{Hom}_A(Y \otimes A, Y' \otimes A)$ is equal to 1 if $Y$ and $Y'$ belong both to $\{1, \alpha, \alpha^2\}$ or both to $\{\xi, \alpha\xi, \alpha^2\xi\}$, and to 0 otherwise. We obtain that the simple objects of $\mathcal{M}$ are $A \cong \alpha \otimes A \cong \alpha^2 \otimes A$ and $\xi \otimes A \cong \alpha\xi \otimes A \cong \alpha^2\xi \otimes A$, up to isomorphism. Since every simple object of $\mathcal{M}$ is a summand of $Y \otimes A$ for some $Y$, the module category $\mathcal{M}$ is generated by $A$, hence indecomposable. It has rank two, so that it is equivalent neither to $\mathcal{N}_2$, of rank six, nor to $\mathcal{N}_1$, of rank four. Hence $\mathcal{M} \cong \mathcal{N}_3$ and $\dim \mathrm{Hom}(L_2, L_3) = 2$. Since $\dim \mathrm{Hom}(0 \oplus b \oplus 2a, 0 \oplus b \oplus 2a) = 6$, Proposition 3.3 gives $L_3 \cong 0 \oplus b \oplus 2c_0$.

Finally, let $U$ be a Haagerup algebra. By part 1 of Proposition 3.5, $L_{\Lambda_a}$ and $L_{\Lambda_{c_0}}$ are Lagrangian algebras isomorphic to $0 \oplus b \oplus 2a$ and $0 \oplus b \oplus 2c_0$ as objects. By the above and Proposition 3.3, $L_2$ and $L_3$ are the only Lagrangian algebras with these objects, so that $L_{\Lambda_a} \cong L_2$ and $L_{\Lambda_{c_0}} \cong L_3$. ∎

**Remark 3.7** In [Arh26, Remark 2.1 p. 8], the algebra $0 \oplus b \oplus 2a$ was attributed to $\mathcal{H}_3$ after [EvG11, Section 2.2]. By Proposition 3.6, it is attached to $\mathcal{H}_2$, since the system $\Delta$ of [EvG11, p. 17] is $\mathcal{H}_2$ (Section 2.2), and the labels $a$ and $c_0$ should be exchanged in the first and third sentences of that remark, while its second sentence should read $I_{\mathcal{H}_3}(1) \cong 0 \oplus b \oplus 2c_0$ (Proposition 3.6), the algebra $0 \oplus b \oplus 2a$ being $I_{\mathcal{H}_2}(1)$ [EvG11, p. 18]. That remark reproduced the dictionary of [Hua26b, p. 2], which matches the labels 0, $b$, $a$, $c_0$ of [Hua26b, Section 7 pp. 9–10], intended as those of [EvG11], with the labels 1, $\pi_1$, $\pi_2$, $\sigma_1$ of [AHKZ26]. In [AHKZ26, (A8) p. 7], the Lagrangian algebra attached to $\mathcal{H}_3$ is $1 \oplus \pi_1 \oplus 2\pi_2$, and this $\mathcal{H}_3$ is the category of [GrS12]. Indeed, gauging an algebra $1 \oplus \rho \oplus \alpha\rho$ in it gives $\mathcal{H}_1$ [AHKZ26, p. 4], and among the categories $\mathcal{H}_2$ and

$\mathcal{H}_3$, only $\mathcal{H}_3$ is the principal even part of a subfactor of index $4+\sqrt{13}$ [GrS12, Theorems 1.3 p. 3 and 1.4 p. 4]. Hence, by Proposition 3.6 and the triviality of the braided autoequivalences of $Z(\mathcal{H}_3)$, the label $\pi_2$ of [AHKZ26] is the label $c_0$ of [EvG11], and $\sigma_1$ is $a$. The exchange of $a$ and $c_0$ preserves the modular data (Section 2.2), and the results of [Arh26] are not affected. Indeed, [Arh26, Proposition 4.1 p. 11] only uses a Lagrangian algebra isomorphic to $0\oplus b\oplus 2a$, which exists by Proposition 3.6, and its subalgebra attached by [DMNO13, Theorem 4.10 p. 18] to the fusion subcategory generated by the invertible objects, whose associator is not used in [Arh26, p. 12].

**Corollary 3.8** *The three Lagrangian algebras of $Z(\mathcal{H}_3)$ are isomorphic, as objects, to*

$$0\oplus b\oplus a\oplus c_0,\quad 0\oplus b\oplus 2a,\quad 0\oplus b\oplus 2c_0,$$

*with categories of modules $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ respectively.*

*Proof* : This is Proposition 3.3 with Proposition 3.6. ∎

## 3.3 Holomorphic extensions of a Haagerup algebra

**Theorem 3.9** *Let $U$ be a Haagerup algebra.*

1. *Up to isomorphism of extensions, $U$ has exactly three holomorphic extensions $B_1$, $B_2$, $B_3$. The algebra $L_i$ in $\mathrm{Rep}(U)$ attached to $B_i$ is the Lagrangian algebra of Proposition 3.3, so that $\mathrm{Rep}(U)_{L_i}\cong\mathcal{H}_i$. Each $B_i$ is a strongly rational holomorphic vertex operator algebra isomorphic to $V_{E_8}$.*
2. *$B_2=H_a$ and $B_3=H_{c_0}$ as extensions of $U$.*
3. *The subspace $U\oplus M_b$ of $B_1$ is not a vertex subalgebra. The only vertex subalgebras of $B_1$ containing $U$ are $U$ and $B_1$. For $i\in\{2,3\}$, the only simple vertex subalgebras of $B_i$ containing $U$ are $U$, $V'$ and $B_i$.*

*Proof* : 1. Let $B$ be a holomorphic extension of $U$. Since $\mathrm{Rep}(B)$ is trivial, the contragredient module $B'$ is isomorphic to $B$. By [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16] and Lemma 2.3, the algebra $L_B$ attached to $B$ is a connected étale algebra with $\mathrm{Rep}(U)^0_{L_B}\cong\mathrm{Rep}(B)\cong$ Vec, hence a Lagrangian algebra. Conversely, a Lagrangian algebra $L$ of $\mathrm{Rep}(U)$ is a haploid commutative algebra with trivial twist (Lemma 3.2 and its proof), so it defines an extension $B_L$ of $U$ with $\mathrm{Rep}(B_L)\cong\mathrm{Rep}(U)^0_L\cong$ Vec by [HKL15, Theorems 3.2 and 3.4 pp. 10 and 16]. Since $L$ is connected, $L$ is a simple object of $\mathrm{Rep}(U)_L$ [DMNO13, p. 10], and the ideals of $B_L$ correspond to the subobjects of $L$ in $\mathrm{Rep}(U)_L$ [HKL15, proof of Theorem 3.2 p. 10]. Hence $B_L$ is simple. These two constructions are inverse to each other up to isomorphism, by [HKL15, Theorem 3.2 p. 10]. By Proposition 3.3, there are exactly three Lagrangian algebras, which proves the first two assertions. The nonvacuum summands of $L_i$ are $b$, $a$ and $c_0$, whose lowest conformal weights are equal to one [Arh26, (4.1) p. 11]. Hence $B_i$ is of CFT type. It is self-contragredient since $\mathrm{Rep}(B_i)\cong$ Vec, and it is $C_2$-cofinite and rational by the argument of the proof of [CKM22, Corollary 5.13 p. 53], since its category of modules is semisimple, see also [Arh26, proof of Proposition 4.1 p. 11]. A strongly rational holomorphic vertex operator algebra of central charge eight is isomorphic to $V_{E_8}$ [DoM04, Theorem 1 p. 2].

2. This is Proposition 3.6.

3. Let $W$ be a simple vertex subalgebra of $B_i$ containing $U$. By Lemma 2.3, the corresponding algebra $A_W$ is a connected étale subalgebra of $L_i$. By Lemma 3.1, under the braided

equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_i)$ given by part 1 of Proposition 2.2, the algebra $L_i$ corresponds to $I_{\mathcal{H}_i}(1)$. By part 3 of Proposition 2.2, the connected étale subalgebras of $L_i$ correspond to the fusion subcategories of $\mathcal{H}_i$, with $\mathrm{FPdim}(A_W)\,\mathrm{FPdim}(\mathcal{B}) = \dim \mathcal{H}_i$ if $A_W$ corresponds to $\mathcal{B}$. By part 4 of Proposition 2.1, the connected étale subalgebras of $L_1$ are the unit object 0 and $L_1$, and for $i \in \{2,3\}$ the connected étale subalgebras of $L_i$ are 0, $L_i$ and a unique subalgebra of Frobenius–Perron dimension $\frac{3(3d+2)}{3} = 3d+2$, corresponding to $\mathcal{P}_i$. For $i \in \{2,3\}$, the extension $B_i$ contains $V'$, which is simple, so this subalgebra is $A_0$. This proves the statements on simple vertex subalgebras.

Let now $W$ be any vertex subalgebra of $B_1$ containing $U$, and let $(\cdot,\cdot)$ be the invariant bilinear form of $B_1$. It is $U$-invariant, so it pairs the isotypic component of type $z$ with that of type $z^*$, and its restriction to each isotypic component of a self-dual type is nondegenerate. Since $L_1 \cong 0 \oplus b \oplus a \oplus c_0$ is multiplicity free, $W$ is a sum of isotypic components of $B_1$, all of self-dual types. Hence the restriction of $(\cdot,\cdot)$ to $W$ is nondegenerate, $W$ is simple by Lemma 2.3, and the description of the connected étale subalgebras of $L_1$ obtained above gives $W \in \{U, B_1\}$. In particular, $U \oplus M_b$ is not a vertex subalgebra of $B_1$. ∎

**Remark 3.10** By part 3 of Theorem 3.9, there is no vertex subalgebra strictly between $U$ and $B_1 \cong V_{E_8}$. The weight-one space of $B_1$ is $U_1 \oplus R_b \oplus R_a \oplus R_{c_0}$, where $R_z$ is the lowest weight space of $M_z$, and its dimension is $(6+13\gamma) + (80-13\gamma) + 81 + 81 = 248 = \dim \mathfrak{e}_8$ by [Arh26, (4.1) p. 11], where $\gamma \in \{0,1\}$ is the parameter of loc. cit.

**Remark 3.11** Let $L$ be a Lagrangian algebra in $\mathrm{Rep}(U)$ with holomorphic extension $B$. In the language of symmetry topological field theories, the fusion category $\mathrm{Rep}(U)_L \cong \mathcal{H}_i$ of $L$-modules is interpreted as the category of the topological defect lines of $B$ which preserve $U$, see [AHKZ26, GaR26, Hua26b]. Theorem 3.9 thus gives a vertex algebraic version of the suggestion of [AHKZ26] that the chiral $(\mathfrak{e}_8)_1$ theory carries the three symmetries $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, under the assumption that a Haagerup algebra exists. An operator algebraic version is given in Section 7, under ($\mathrm{H_J}$).

# 4 The conformal net of $S$

Throughout this section, $L = E_6 \oplus A_2$ denotes the root lattice and $V' = V_L = M(1) \otimes \mathbb{C}_\varepsilon[L]$ is the lattice vertex operator algebra of [Arh26, Section 10], built as in [FLM88, Chapter 8] with a bimultiplicative cocycle $\varepsilon\colon L \times L \to \{\pm 1\}$ satisfying $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)^{-1} = (-1)^{\langle\alpha,\beta\rangle}$ and $\varepsilon(\alpha,\alpha) = (-1)^{\langle\alpha,\alpha\rangle/2}$ for any $\alpha,\beta \in L$. The second property holds for the cocycle of [Arh26, Section 10.1], which satisfies it on the simple roots, since $\alpha \mapsto \varepsilon(\alpha,\alpha)(-1)^{\langle\alpha,\alpha\rangle/2}$ is a group homomorphism $L \to \{\pm 1\}$ by the first property. We denote by $L^\circ$ the dual lattice, so that $L^\circ/L \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ with the quadratic form $q$ of Section 2.4. No hypothesis on Haagerup algebras is made in this section.

## 4.1 The lattice extensions and their unitary structures

Let $\Lambda \subset L^\circ/L$ be one of the two isotropic lines and let $L_\Lambda \subset L^\circ$ be its preimage. Since $q$ vanishes on $\Lambda$, the lattice $L_\Lambda$ is even, and it is unimodular since $[L_\Lambda : L] = 3$ and $|L^\circ/L| = 9$. Hence $L_\Lambda$ is isometric to $E_8$.

**Lemma 4.1** *1. The cocycle $\varepsilon$ extends uniquely to a bimultiplicative map $\varepsilon_\Lambda\colon L_\Lambda \times L_\Lambda \to \{\pm 1\}$. We have $\varepsilon_\Lambda(\alpha,\beta)\varepsilon_\Lambda(\beta,\alpha)^{-1} = (-1)^{\langle\alpha,\beta\rangle}$ and $\varepsilon_\Lambda(\alpha,\alpha) = (-1)^{\langle\alpha,\alpha\rangle/2}$ for any $\alpha,\beta \in L_\Lambda$.*

2. *The lattice vertex operator algebra*

$$H_\Lambda \overset{\text{def}}{=} M(1)\otimes\mathbb{C}_{\varepsilon_\Lambda}[L_\Lambda] = \bigoplus_{w\in\Lambda} V'_w, \quad V'_w \overset{\text{def}}{=} M(1)\otimes\mathbb{C}_{\varepsilon_\Lambda}[L+\lambda_w],$$

*where $\lambda_w \in L^\circ$ is a representative of $w$, contains $V' = V'_0$ as a vertex subalgebra with the same conformal vector, and $H_\Lambda$ is isomorphic to $V_{E_8}$.*

*Proof* : 1. A bimultiplicative map $M\times M\to\{\pm1\}$ on a lattice $M$ is the same thing as a bilinear form on the $\mathbb{F}_2$-vector space $M/2M$. The inclusion $L\subset L_\Lambda$ induces an isomorphism $L/2L\to L_\Lambda/2L_\Lambda$. Indeed, both spaces have dimension eight, and if $x\in L$ satisfies $x=2y$ with $y\in L_\Lambda$, then $3y\in L$ and $2y\in L$, hence $y\in L$. This gives the existence and uniqueness of $\varepsilon_\Lambda$. The maps $(\alpha,\beta)\mapsto\varepsilon_\Lambda(\alpha,\beta)\varepsilon_\Lambda(\beta,\alpha)^{-1}$ and $(\alpha,\beta)\mapsto(-1)^{\langle\alpha,\beta\rangle}$ are bimultiplicative on $L_\Lambda$, since $L_\Lambda$ is integral, and they coincide on $L$. By the same isomorphism, they coincide on $L_\Lambda$. Finally, the map $s(\alpha)=\varepsilon_\Lambda(\alpha,\alpha)(-1)^{\langle\alpha,\alpha\rangle/2}$ satisfies $s(\alpha+\beta)=s(\alpha)s(\beta)$ by the commutator identity, so it is a group homomorphism $L_\Lambda\to\{\pm1\}$. It is trivial on $L$, and $[L_\Lambda:L]=3$ is odd, so that $s$ is trivial.

2. By part 1, the construction of lattice vertex operator algebras [FLM88, Chapter 8] applies to $(L_\Lambda,\varepsilon_\Lambda)$, and the subspace $M(1)\otimes\mathbb{C}_\varepsilon[L]$ is a vertex subalgebra equal to $V'$ with the same conformal vector. Up to isomorphism, the lattice vertex operator algebra does not depend on the choice of the cocycle, and $L_\Lambda\cong E_8$, so that $H_\Lambda\cong V_{E_8}$. ■

The subspace $V'_w$ of $H_\Lambda$ realizes the simple $V'$-module attached to $w$. An extension of $V'$ whose underlying $V'$-module is $\bigoplus_{w\in\Lambda}V'_w$ is unique up to isomorphism of extensions, by the argument of the proof of part 3 of Proposition 3.5, since $H^2(\mathbb{Z}_3,\mathbb{C}^\times)=0$.

Hence, in the situation of Section 2.4, the vertex operator algebras $H_a$ and $H_{c_0}$ of (2.8) are isomorphic, as extensions of $V'$, to $H_{\Lambda_a}$ and $H_{\Lambda_{c_0}}$. For a character $\chi\colon L\to\mathbb{R}_{>0}$, we write $\chi(\lambda)=\mathrm{e}^{\langle h_\chi,\lambda\rangle}$ with $h_\chi$ in the real span of $L$, and we extend $\chi$ to $L^\circ$ by the same formula. Let $\phi_1$ be the antilinear map of $H_\Lambda$ defined by [Arh26, (10.3) p. 33] for the trivial character, that is

$$\phi_1\big(ch_1(-n_1)\cdots h_k(-n_k)\mathrm{e}^\lambda\big) = (-1)^k\overline{c}h_1(-n_1)\cdots h_k(-n_k)\mathrm{e}^{-\lambda}, \quad c\in\mathbb{C},\ \lambda\in L_\Lambda,$$

where $h_1,\dots,h_k$ belong to the real span of $L$.

**Lemma 4.2** *1. The map $\phi_1$ is an antilinear involutive automorphism of $H_\Lambda$. If $\beta$ denotes the normalized invariant bilinear form of $H_\Lambda$, then $H_1(u,v)\overset{\text{def}}{=}\beta(u,\phi_1(v))$ is a normalized invariant scalar product on $H_\Lambda$.*

2. *Let $\{\cdot|\cdot\}$ be a normalized invariant scalar product on $V'$. There exists $x$ in $(V')_1$ such that $\tilde{h}=\exp(x(0))$ is an automorphism of $H_\Lambda$ preserving $V'$, $\{u|v\}=H_1(\tilde{h}u,\tilde{h}v)$ for any $u,v\in V'$ and $H_1(u,\tilde{h}u)>0$ for any nonzero $u\in V'$. Consequently, $\{u|v\}_\Lambda\overset{\text{def}}{=}H_1(\tilde{h}u,\tilde{h}v)$ is a normalized invariant scalar product on $H_\Lambda$ which extends $\{\cdot|\cdot\}$, and $V'$ is a unitary subalgebra of $(H_\Lambda,\{\cdot|\cdot\}_\Lambda)$. The scalar product $\{\cdot|\cdot\}_\Lambda$ does not depend on the choice of $x$.*

3. *For the unitary structure $\phi_\chi$ of [Arh26, Theorem 10.10 p. 33], one can take $x=\frac{1}{2}h_\chi(-1)\mathbf{1}$ in part 2. The corresponding PCT operator of $H_\Lambda$ is $\phi_\chi=\phi_1\circ\exp(h_\chi(0))$.*

*Proof* : 1. The cocycle $\varepsilon_\Lambda$ is bimultiplicative, so that $\varepsilon_\Lambda(-\alpha,-\beta)=\varepsilon_\Lambda(\alpha,\beta)$, and $\varepsilon_\Lambda(\alpha,\alpha)=(-1)^{\langle\alpha,\alpha\rangle/2}$ by Lemma 4.1. Hence $\mathrm{e}^\alpha\mathrm{e}^{-\alpha}=\varepsilon_\Lambda(\alpha,-\alpha)\mathrm{e}^0=(-1)^{\langle\alpha,\alpha\rangle/2}\mathrm{e}^0$ for any $\alpha\in L_\Lambda$, which is the relation used in [DoL14, Lemma 4.11 p. 23], and $\phi_1$ is the antilinear involution for which lattice vertex operator algebras are unitary [DoL14, Theorem 4.12 p. 24], [CKLW18, Example 5.9 p. 35], as in the proof of [Arh26, Theorem 10.10 p. 33].

2. By part 1, the restriction of $H_1$ to $V'$ is a normalized invariant scalar product on $V'$. Since $V'$ is simple, there exists by [CKLW18, Proposition 5.19 p. 40] an automorphism $h$ of $V'$ with $\{u|v\} = H_1(hu, hv)$ for any $u, v \in V'$ and $H_1(u, hu) > 0$ for any nonzero $u \in V'$. In particular $h$ preserves each $(V')_n$ and is diagonalizable with positive eigenvalues. Let $D$ be the linear map acting by $\log \lambda$ on the eigenspace of $h$ for the eigenvalue $\lambda$. Since $h(u_{(n)}v) = \lambda\mu u_{(n)}v$ if $hu = \lambda u$ and $hv = \mu v$, the map $D$ is a derivation of $V'$ preserving the grading. Its restriction to $(V')_1$ is a derivation of the semisimple Lie algebra $((V')_1, [u, v] = u_{(0)}v) \cong \mathfrak{e}_6 \oplus \mathfrak{sl}_3$, so that there is $x \in (V')_1$ with $D(u) = x_{(0)}u$ for $u \in (V')_1$. The derivations $D$ and $x_{(0)} = x(0)$ of $V'$ coincide on $(V')_1$ and vanish on the vacuum. The space $(V')_1$ contains the vectors $h(-1)\mathbf{1}$ and $\mathrm{e}^{\alpha}$ for the roots $\alpha$ of $L$, the lattice $L$ is spanned by its roots, and $\mathrm{e}^{\alpha+\beta}$ is a nonzero multiple of $\mathrm{e}^{\alpha}_{(-\langle\alpha,\beta\rangle-1)}\mathrm{e}^{\beta}$ [FLM88, Chapter 8]. Hence $(V')_1$ generates $V'$, and $D = x(0)$. Now $x(0)$ is a derivation of $H_\Lambda$ which preserves the finite-dimensional homogeneous spaces, so that $\tilde{h} = \exp(x(0))$ is an automorphism of $H_\Lambda$ which extends $h = \exp(D)$. The transport of a normalized invariant scalar product by an automorphism is a normalized invariant scalar product, and the PCT operator of $\{\cdot|\cdot\}_\Lambda$ is $\tilde{h}^{-1}\phi_1\tilde{h}$, which preserves $V'$. Since $L_1V' \subset V'$, the last assertion follows from [CKLW18, Proposition 5.23 p. 43]. Two choices of $x$ give automorphisms of $H_\Lambda$ which coincide on $V'$, since a positive square root is unique. They differ by an automorphism of $H_\Lambda$ fixing $V'$ pointwise, which acts on each simple $V'$-module $V'_w$, $w \in \Lambda$, by a scalar, hence by a power of the automorphism $g_\Lambda$ of Proposition 4.3. This automorphism is unitary for $H_1$ and commutes with $\tilde{h}$, so that $\{\cdot|\cdot\}_\Lambda$ does not depend on the choice of $x$.

3. With $g = \exp(h_\chi(0))$, which acts by $\chi(\lambda)$ on $M(1) \otimes \mathrm{e}^{\lambda}$, we have $\phi_\chi = \phi_1 \circ g = g^{-1/2} \circ \phi_1 \circ g^{1/2}$ on $V'$ by the proof of [Arh26, Theorem 10.10 p. 33], and the same computation holds on $H_\Lambda$. We still denote by $H_\chi(u, v) = H_1(g^{1/2}u, g^{1/2}v)$ the corresponding scalar product on $H_\Lambda$, which extends the scalar product $H_\chi$ of $V'$ of [Arh26, Theorem 10.10 p. 33]. ■

**Proposition 4.3** *Let $\Lambda$ be an isotropic line and let $\{\cdot|\cdot\}$ be a normalized invariant scalar product on $H_\Lambda$.*

1. *The unitary vertex operator algebra $(H_\Lambda, \{\cdot|\cdot\})$ is strongly local and has bounded localized vertex operators in the sense of [Ten19b, Definition 4.12 p. 42]. The net of [Ten19b] and the net of [CKLW18] of $(H_\Lambda, \{\cdot|\cdot\})$ are isomorphic, and they are isomorphic to the loop group net $\mathcal{A}_{E_8}$ of $E_8$ at level one. This net is completely rational and $\mu_{\mathcal{A}_{E_8}} = 1$.*

2. *Assume that $\{\cdot|\cdot\}$ is one of the scalar products of part 2 of Lemma 4.2. Let $\beta_\Lambda \in L^\circ$ with $\langle\beta_\Lambda, \lambda_w\rangle \in \frac{1}{3} + \mathbb{Z}$ for a generator $w$ of $\Lambda$, and put $g_\Lambda \stackrel{\text{def}}{=} \mathrm{e}^{2\pi\mathrm{i}\beta_\Lambda(0)}$. Then $g_\Lambda$ is a unitary automorphism of order three of $H_\Lambda$ with fixed point algebra $V'$, and $\mathcal{A}_{V'} = \mathcal{A}_{H_\Lambda}^{\langle g_\Lambda\rangle}$. The net $\mathcal{A}_{V'}$ is completely rational, $\mu_{\mathcal{A}_{V'}} = 9$, $[\mathcal{A}_{H_\Lambda} : \mathcal{A}_{V'}] = 3$ and $\mathcal{A}_{V'}(I)' \cap \mathcal{A}_{H_\Lambda}(I) = \mathbb{C}$ for any interval $I$. Moreover $\mathcal{A}_{V'}$ is isomorphic to $\mathcal{A}_{E_6,1} \otimes \mathcal{A}_{SU(3),1}$.*

*Proof* : 1. The lattice vertex operator algebra of $E_8$ coincides with $V(E_8, 1)$ [Ten19b, Proof of Theorem 7.4 p. 71]. With its unitary structure, it has bounded localized vertex operators [Ten19b, Theorem 7.4 p. 71], it is strongly local with net $\mathcal{A}_{E_8}$ [CKLW18, Example 8.7 p. 63], and the nets of [Ten19b] and [CKLW18] are isomorphic [Ten19b, Corollary 7.7 p. 72]. Let $\psi\colon H_\Lambda \to V(E_8, 1)$ be an isomorphism of vertex operator algebras. The pull back by $\psi$ of the scalar product of $V(E_8, 1)$ is a normalized invariant scalar product on $H_\Lambda$. By [CKLW18, Proposition 5.19 p. 40], it is of the form $\{hu|hv\}$ for some automorphism $h$ of $H_\Lambda$. Then $\psi \circ h^{-1}$ is a unitary isomorphism from $(H_\Lambda, \{\cdot|\cdot\})$ onto $V(E_8, 1)$, and all the properties are transported. Here bounded localized vertex operators refer to the system of generalized annuli fixed in [Ten19b, Section 7]. A unitary isomorphism of vertex operator algebras preserves

the conformal vectors, hence intertwines the representations of the Virasoro algebra and the operators attached to the annuli, so that it transports these properties and the nets. Finally, by [Ten19b, Proposition 7.5 p. 72], the net of [Ten19b] of $V(E_8, 1)$ is isomorphic to the lattice net of $E_8$ of Dong and Xu, since $E_8$ is the code lattice of the extended Hamming code [Ten19b, Example 7.2 p. 70]. By [DoX06, Corollary 3.19 p. 14], this net is completely rational with $\mu$-index $|E_8^\circ/E_8| = 1$.

2. The map $g_\Lambda$ acts on $M(1) \otimes \mathrm{e}^\lambda$ by the scalar $\mathrm{e}^{2\pi\mathrm{i}\langle\beta_\Lambda,\lambda\rangle}$, which is 1 on $V'$ and a primitive cube root of unity on $V'_w$ for $w \neq 0$. Such an element $\beta_\Lambda$ exists since the pairing $L^\circ/L_\Lambda \times L_\Lambda/L \to \mathbb{Q}/\mathbb{Z}$ induced by $\langle\cdot,\cdot\rangle$ is perfect. Hence $g_\Lambda$ is an automorphism of order three with fixed point algebra $V'$. The sectors $M(1) \otimes \mathrm{e}^\lambda$ are pairwise orthogonal for $H_1$, since $\beta$ pairs the sector of $\lambda$ with the sector of $-\lambda$ and $\phi_1$ exchanges them. So $g_\Lambda$ is unitary for $H_1$. The automorphism $\tilde{h} = \exp(x(0))$, $x \in (V')_1$, of Lemma 4.2 commutes with $g_\Lambda$, since $g_\Lambda x(0) g_\Lambda^{-1} = (g_\Lambda x)(0) = x(0)$. Hence $g_\Lambda$ is also unitary for $\{\cdot|\cdot\}$. By [CKLW18, Theorem 6.9 p. 53 and Proposition 7.6 p. 58], $g_\Lambda$ defines a vacuum preserving automorphism of the net $\mathcal{A}_{H_\Lambda}$ and $\mathcal{A}_{V'} = \mathcal{A}_{H_\Lambda}^{\langle g_\Lambda\rangle}$. The group $\langle g_\Lambda\rangle$ acts faithfully and properly in the sense of [DoX06, Definition 2.1 p. 5]. Indeed, if $g \in \langle g_\Lambda\rangle$ acts trivially on $\mathcal{A}_{H_\Lambda}(I)$ for some interval $I$, then $gx\Omega = x\Omega$ for any $x \in \mathcal{A}_{H_\Lambda}(I)$, so that $g = 1$ since $\Omega$ is cyclic for $\mathcal{A}_{H_\Lambda}(I)$. By part 1 and [DoX06, Theorem 2.4 p. 6], the net $\mathcal{A}_{V'}$ is completely rational with $\mu_{\mathcal{A}_{V'}} = 3^2 \mu_{\mathcal{A}_{H_\Lambda}} = 9$.

The vertex operator algebra $V'$ contains the conformal vector of $H_\Lambda$, so that $\mathcal{A}_{V'}(I)' \cap \mathcal{A}_{H_\Lambda}(I) = \mathbb{C}$ by Lemma 4.4 below. In particular $\mathcal{A}_{V'}(I) \subset \mathcal{A}_{H_\Lambda}(I)$ is an irreducible subfactor. By part 2 of Proposition 2.5, its index is finite, and by part 1 of the same proposition, $[\mathcal{A}_{H_\Lambda} : \mathcal{A}_{V'}]^2 = \mu_{\mathcal{A}_{V'}}/\mu_{\mathcal{A}_{H_\Lambda}} = 9$. Finally $V' \cong V_{E_6} \otimes V_{A_2}$, and the Frenkel–Kac construction gives $V_{E_6} \cong V_{\mathfrak{e}_6,1}$ and $V_{A_2} \cong V_{\mathfrak{sl}_3,1}$. Transporting the scalar product along this isomorphism as in part 1, the last assertion follows from parts 1 and 5 of Proposition 2.4. ∎

## 4.2 Strong locality of $S$

**Lemma 4.4** *Let $V$ be a strongly local simple unitary vertex operator algebra and let $W$ be a unitary subalgebra of $V$ containing the conformal vector of $V$. Then for any interval $I$ we have*

$$\mathcal{A}_W(I)' \cap \mathcal{A}_V(I) = \mathcal{A}_W(S^1)' \cap \mathcal{A}_V(I) = \mathbb{C},$$

*where $\mathcal{A}_W(S^1)$ is the von Neumann algebra generated by the algebras $\mathcal{A}_W(J)$.*

*Proof* : The first equality holds for any covariant subnet of an irreducible conformal net [CKLW18, Example 3.7 p. 24], [Kös04]. By [CKLW18, Proposition 7.8 p. 58], the coset subnet $\mathcal{A}_W(S^1)' \cap \mathcal{A}_V(I)$ is $\mathcal{A}_{W^c}(I)$, where $W^c$ is the coset of $W$ in $V$. By [CKLW18, Proposition 5.31 p. 45], the conformal vector $\omega$ of $V$ decomposes as $\omega = \omega^W + \omega^{W^c}$. Here $\omega^W = \omega$ since $\omega \in W$, so that $\omega^{W^c} = 0$ and $W^c = \mathbb{C}\Omega$ by [CKLW18, Remark 5.30 p. 45]. Hence $\mathcal{A}_{W^c}(I) = \mathbb{C}$. ∎

**Theorem 4.5** *Let $W \subset V'$ be a vertex subalgebra containing the conformal vector of $V'$ which is a unitary subalgebra for a normalized invariant scalar product $\{\cdot|\cdot\}$ on $V'$. Let $\Lambda$ be an isotropic line and equip $H_\Lambda$ with the scalar product $\{\cdot|\cdot\}_\Lambda$ of Lemma 4.2.*

1. *$W$ is strongly local, and $\mathcal{A}_W \subset \mathcal{A}_{V'} \subset \mathcal{A}_{H_\Lambda}$ are covariant subnets, where $\mathcal{A}_{H_\Lambda} \cong \mathcal{A}_{E_8}$ and $[\mathcal{A}_{H_\Lambda} : \mathcal{A}_{V'}] = 3$.*
2. *The coset subnets of $\mathcal{A}_W$ in $\mathcal{A}_{V'}$ and in $\mathcal{A}_{H_\Lambda}$ are trivial, and the inclusions are irreducible: $\mathcal{A}_W(I)' \cap \mathcal{A}_{H_\Lambda}(I) = \mathbb{C}$ and $\mathcal{A}_{V'}(I)' \cap \mathcal{A}_{H_\Lambda}(I) = \mathbb{C}$ for any interval $I$.*

*In particular, all these statements hold for the algebra $S$ of [Arh26, Theorem 10.10 p. 33], with the scalar product $H_\chi$.*

*Proof* : By Lemma 4.2, the vertex operator algebra $V'$ is a unitary subalgebra of $(H_\Lambda, \{\cdot|\cdot\}_\Lambda)$. Since $W$ contains the conformal vector, $L_1 W \subset W$, and the PCT operator of $H_\Lambda$ restricts to that of $V'$, which preserves $W$. Hence $W$ is a unitary subalgebra of $H_\Lambda$ by [CKLW18, Proposition 5.23 p. 43]. By Proposition 4.3, $H_\Lambda$ is strongly local, and by part 2 of Proposition 2.4, the unitary subalgebras $W$ and $V'$ are strongly local with nets embedded as covariant subnets. The remaining assertions follow from Proposition 4.3 and Lemma 4.4. For $S$, the hypothesis holds by [Arh26, Theorem 10.10 p. 33], and the scalar product $\{\cdot|\cdot\}_\Lambda$ is $H_\chi$ by part 3 of Lemma 4.2. ■

**Remark 4.6** By part 1 of Proposition 2.4, the net $\mathcal{A}_W$ does not depend, up to isomorphism, on the normalized invariant scalar product on $W$.

## 4.3 Submodules and representations

Let $V$ be a strongly local simple unitary vertex operator algebra and $W$ a unitary subalgebra containing the conformal vector of $V$. Let $e_W$ be the orthogonal projection onto the closure $\mathcal{H}_W$ of $W$. By [CKLW18, Theorem 7.1 p. 55], the conformal net of $W$ is $I \mapsto \mathcal{B}_W(I)e_W$, where $\mathcal{B}_W(I) = \mathcal{A}_V(I) \cap \{e_W\}'$. Let $\mathcal{C}_W(I)$ be the von Neumann algebra generated by the closed operators $Y(w,f)$, $w \in W$, $\operatorname{supp} f \subset I$. These operators are affiliated with $\mathcal{A}_V(I)$ and commute with $e_W$, and the restriction of $Y(w,f)$ to $\mathcal{H}_W$, with domain $D(Y(w,f)) \cap \mathcal{H}_W$, is the smeared vertex operator of $W$ [CKLW18, proof of Theorem 7.1 p. 55]. Hence $\mathcal{C}_W(I) \subset \mathcal{B}_W(I)$. Since the reduction by $e_W$ of the von Neumann algebra generated by closed operators commuting with $e_W$ is the von Neumann algebra generated by their restrictions to $\mathcal{H}_W$, the algebra $\mathcal{C}_W(I)e_W$ is generated by the smeared vertex operators of $W$, so that $\mathcal{C}_W(I)e_W = \mathcal{B}_W(I)e_W$ [CKLW18, Theorem 7.1 p. 55]. Since $\Omega$ is separating for $\mathcal{A}_V(I)$, the map $x \mapsto xe_W$ is injective on $\mathcal{B}_W(I)$, and we obtain $\mathcal{C}_W(I) = \mathcal{B}_W(I)$. We denote this algebra by $\mathcal{A}_W(I)$. The restriction map $r_I \colon \mathcal{A}_W(I) \to \mathcal{A}_W(I)|_{\mathcal{H}_W}$ is an isomorphism of von Neumann algebras onto the local algebra of the net of $W$. Let $M \subset V$ be a $W$-submodule. Since $\omega \in W$, the subspace $M$ is graded by the eigenvalues of $L_0$. We denote by $\mathcal{H}_M$ its closure in $\mathcal{H}_V$.

**Lemma 4.7** *1. We have $V = M \oplus (M^\perp \cap V)$ as $W$-modules. For any interval $I$, the subspace $\mathcal{H}_M$ is invariant under $\mathcal{A}_W(I)$, and, for any $w \in W$ and any $f \in C^\infty(S^1)$, the restriction $Y_M(w,f) \stackrel{\text{def}}{=} Y(w,f)|_{\mathcal{H}_M}$, with domain $D(Y(w,f)) \cap \mathcal{H}_M$, is a closed operator on $\mathcal{H}_M$, and $M$ is a core for it. The formula $\pi_M(x) \stackrel{\text{def}}{=} r_I^{-1}(x)|_{\mathcal{H}_M}$, for $x \in \mathcal{A}_W(I)|_{\mathcal{H}_W}$, defines a representation $\pi_M$ of the conformal net of $W$ on $\mathcal{H}_M$.*

*2. If $M$ is a simple $W$-module, then $\pi_M$ is irreducible.*

*3. Let $V_1$ and $V_2$ be strongly local simple unitary vertex operator algebras which both contain $W$ as a unitary subalgebra with the same conformal vector, the scalar products of $V_1$ and $V_2$ restricting to the same scalar product on $W$. Let $M_1 \subset V_1$ and $M_2 \subset V_2$ be simple $W$-submodules. Then the representations $\pi_{M_1}$ and $\pi_{M_2}$ are unitarily equivalent if and only if the $W$-modules $M_1$ and $M_2$ are isomorphic.*

*Proof* : 1. Let $P$ be the orthogonal projection onto $\mathcal{H}_M$. Since the homogeneous spaces are finite-dimensional and orthogonal, $P$ maps $V$ onto $M$ and $(1-P)V = M^\perp \cap V$ is the sum of the orthogonal complements of the spaces $M(n) \stackrel{\text{def}}{=} M \cap V_n$ in $V_n$. Let $w \in W$ be homogeneous.

By [CKLW18, (95) p. 43], the adjoint of a mode $w_n$ on $V$ is a finite linear combination of modes of the vectors $L_1^j\theta w$, which lie in $W$. Hence the modes of $w$ preserve $M$ and $M^\perp \cap V$. Consequently, the operator $Y_0(w,f)$ with domain $V$ satisfies $PY_0(w,f)b = Y_0(w,f)Pb$ for any $b \in V$. Since $Y(w,f)$ is the closure of $Y_0(w,f)$ [CKLW18, p. 49] and $PV \subset V$, we obtain $PY(w,f) \subset Y(w,f)P$. The same holds for the adjoint, so that $P$ commutes with the von Neumann algebra generated by the operators $Y(w,f)$, $\operatorname{supp} f \subset I$, that is, with $\mathcal{A}_W(I)$. If $\xi \in \mathcal{H}_M$ belongs to the domain of $Y(w,f)$, and if $b_k \in V$ satisfies $b_k \to \xi$ and $Y(w,f)b_k \to Y(w,f)\xi$, then $Pb_k \in M$, $Pb_k \to \xi$ and $Y(w,f)Pb_k = PY(w,f)b_k \to Y(w,f)\xi$. Hence $M$ is a core for $Y_M(w,f)$. The compatibility of the maps $\pi_M$ with the inclusions of intervals holds by construction.

Since the normal isomorphism $r_I$ maps the operator $Y(w,f)$ affiliated with $\mathcal{A}_W(I)$ to its restriction to $\mathcal{H}_W$, the representation $\pi_M$ maps the smeared vertex operator $Y(w,f)|_{\mathcal{H}_W}$ of $W$ to $Y_M(w,f)$.

2. We first show that $A \subset L_0|_{\mathcal{H}_M}$ for the operator $A$ defined below. Choose real functions $f_1, f_2 \in C^\infty(S^1)$ with support in proper intervals and $f_1 + f_2 = 1$. The vector $\omega$ is quasi-primary and fixed by $\theta$, so that $Y(\omega, f_j)$ is symmetric by [CKLW18, (113) p. 50]. Hence the operator $A = Y_M(\omega,f_1) + Y_M(\omega,f_2)$ with domain $D(Y_M(\omega,f_1)) \cap D(Y_M(\omega,f_2))$ is symmetric. On $M$, it coincides with $Y_0(\omega,1) = L_0$. The restriction of $L_0$ to $\mathcal{H}_M$ is self-adjoint and $M$ is a core for it. Thus the closure of $A$ is a symmetric extension of a self-adjoint operator, so it is equal to it, and $A \subset L_0|_{\mathcal{H}_M}$.

Let $T \in \pi_M(\mathcal{A}_W)'$ be a projection. By part 1, $TY_M(w,f) \subset Y_M(w,f)T$ whenever $\operatorname{supp} f$ is contained in a proper interval. For $\xi \in M$, we obtain $T\xi \in D(A)$ and $AT\xi = TA\xi = TL_0\xi$. Hence $L_0T\xi = TL_0\xi$, and $T$ maps each eigenspace $M(n)$ of $L_0|_{\mathcal{H}_M}$ into itself. Now let $w \in W$ be homogeneous and $\xi \in M(k)$. The vector $Y_0(w,f)\xi = \sum_n \hat{f}_n w_n \xi$ has component $\hat{f}_n w_n \xi$ in $M(k-n)$. Comparing the components of $TY_M(w,f)\xi = Y_M(w,f)T\xi$, we obtain $\hat{f}_n T w_n \xi = \hat{f}_n w_n T\xi$. For any $n$ there exists $f$ supported in a proper interval with $\hat{f}_n \neq 0$, for instance $f(z) = z^n g(z)$ with $g \geqslant 0$ nonzero and supported in a proper interval. Hence $T|_M$ is an endomorphism of the simple $W$-module $M$, so it is a scalar, and $T \in \{0,1\}$.

3. The restrictions to $W$ of the normalized invariant bilinear forms of $V_1$ and $V_2$ are normalized invariant bilinear forms on $W$, hence they coincide [Li94]. Since the scalar products also coincide on $W$, the PCT operators of $V_1$ and $V_2$ restrict to the same antilinear map $\theta$ of $W$, and the adjoint formula [CKLW18, (95) p. 43] is the same in $V_1$ and $V_2$. If $\phi\colon M_1 \to M_2$ is an isomorphism of $W$-modules, then $(u,v) \mapsto (\phi u|\phi v)$ is an invariant scalar product on $M_1$. The invariant sesquilinear forms on $M_1$ correspond to the module maps from the simple module $\overline{M_1}$, with conjugate scalars and the action of $w$ given by that of $\theta w$, to the simple contragredient module $M_1'$. Hence they form a space of dimension at most one. Hence $\phi$ is a positive multiple of a unitary map $u\colon M_1 \to M_2$. It preserves the grading and extends to a unitary $u\colon \mathcal{H}_{M_1} \to \mathcal{H}_{M_2}$. Since $M_k$ is a core for $Y_{M_k}(w,f)$, we obtain $uY_{M_1}(w,f)u^* = Y_{M_2}(w,f)$ for any $w \in W$ and any $f$ with support in $I$. By part 1, the normal representations $x \mapsto u\pi_{M_1}(x)u^*$ and $\pi_{M_2}$ of $\mathcal{A}_W(I)|_{\mathcal{H}_W}$ map the closed operator $Y(w,f)|_{\mathcal{H}_W}$ to the same closed operator. Hence they coincide on the partial isometries and on the spectral projections of the moduli of the polar decompositions of these operators, which generate $\mathcal{A}_W(I)|_{\mathcal{H}_W}$. Thus $u$ intertwines $\pi_{M_1}$ and $\pi_{M_2}$.

Conversely, let $T\colon \mathcal{H}_{M_1} \to \mathcal{H}_{M_2}$ be a unitary intertwiner. The operator $\left(\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right)$ commutes with $(\pi_{M_1} \oplus \pi_{M_2})(\mathcal{A}_W(I))$, with which the closed operator $Y_{M_1}(w,f) \oplus Y_{M_2}(w,f)$ is affiliated by part 1. Hence $TY_{M_1}(w,f) \subset Y_{M_2}(w,f)T$. The argument of part 2, applied to the operators $A_k = Y_{M_k}(\omega,f_1) + Y_{M_k}(\omega,f_2) \subset L_0|_{\mathcal{H}_{M_k}}$, shows that $T$ maps $M_1(n)$ into $M_2(n)$ and that $T|_{M_1}\colon M_1 \to M_2$ is a nonzero morphism of $W$-modules. Since $M_1$ and $M_2$ are simple, it is an isomorphism. ∎

**Remark 4.8** In the framework of bounded localized vertex operators, the statements of Lemma 4.7 are counterparts of [Ten19b, Proposition 5.4 p. 50, Proposition 5.5 p. 51 and Theorem 5.6 p. 52].

# 5 Unitarity of fusion products and complete rationality

In Section 5.1, $V$ is a general simple regular unitary vertex operator algebra. From Section 5.2 on, $U \subset V'$ is a Haagerup algebra which is a vertex subalgebra of $V'$ containing the conformal vector of $V'$. The main example is $U = S$ under $(\mathrm{H_J})$.

## 5.1 Positivity of transport forms and unitarity of fusion products

Let $V$ be a simple regular unitary vertex operator algebra and let $M$, $N$ be unitary $V$-modules. Throughout this subsection, modules are ordinary grading-restricted modules. By regularity they are finite direct sums of irreducible ordinary modules. Their conformal weights are rational [DLM00, Theorem 11.3 p. 48], in particular real, and their $L_0$-eigenspaces are finite-dimensional. All forms and module homomorphisms are initially defined on the algebraic finite-energy subspaces. Tener defined the transport form $\langle\cdot,\cdot\rangle_{z,w}$ on $M \otimes N$, for $(z,w)$ in the domain $\mathcal{R}$ of [Ten24, (3.5) p. 24], by means of the annihilation operator $\mathcal{Y}_M^-$ of $M$ and of the vertex operator of $N$ [Ten24, Definition 3.1 p. 25]. A transport module for $M$ and $N$ is a unitary $V$-module $K$ with a dominant intertwining operator $\mathcal{Y}$ of type $\binom{K}{MN}$ belonging to the space $\mathcal{I}_{\mathrm{Hilb}}$ of [Ten24, Definition 2.8 p. 12], such that $\langle a_1 \otimes b_1, a_2 \otimes b_2\rangle_{z,w} = \langle \mathcal{Y}(a_1,z)b_1, \mathcal{Y}(a_2,w)b_2\rangle$ on $\mathcal{R}$ [Ten24, Definition 3.3 p. 26]. Tener proved that the fusion product $M \boxtimes N$ is a transport module as soon as the transport form is positive semidefinite at one point $(z,z)$ of $\mathcal{R}$, under the hypothesis that $M \boxtimes N$ admits a unitary structure [Ten24, Proposition 3.16 p. 36], and he noted that the unitarizability of $M \boxtimes N$ is open in general [Ten24, Remark 3.18 p. 37]. We show that this hypothesis holds in the setting of bounded localized vertex operators when the representation $\pi^N$ exists. We denote by $P = M \boxtimes N$ the fusion product, by $\mathcal{Y}_\boxtimes$ its universal intertwining operator, by $P(s)$ the eigenspace of $L_0$ for the eigenvalue $s$ and by $P_s$ the projection onto $P(s)$ defined on the algebraic completion of $P$.

**Lemma 5.1** *Let $\Sigma$ be a countable subset of $[s_0,\infty)$ for some $s_0 \in \mathbb{R}$, let $\varepsilon > 0$ and let $(c_s)_{s\in\Sigma}$ be complex numbers with $\sum_s |c_s|\mathrm{e}^{\varepsilon s} < \infty$. If $\sum_s |p(s)|^2 c_s \geqslant 0$ for every polynomial $p \in \mathbb{C}[X]$, then $c_s \geqslant 0$ for every $s \in \Sigma$.*

*Proof* : Let $\nu = \sum_s |c_s|\delta_s$, a finite positive measure on $[s_0,\infty)$. Let $f \in \mathrm{L}^2(\nu)$ be orthogonal to all polynomials. For $\operatorname{Im}\zeta > -\frac{\varepsilon}{2}$ and $s \geqslant s_0$, we have $|\mathrm{e}^{\mathrm{i}\zeta s}| \leqslant C_\zeta \mathrm{e}^{\frac{\varepsilon}{2}s}$ for a constant $C_\zeta$ which is locally bounded in $\zeta$, and $\int |f|\mathrm{e}^{\frac{\varepsilon}{2}s}\,\mathrm{d}\nu \leqslant \|f\|_2(\int \mathrm{e}^{\varepsilon s}\,\mathrm{d}\nu)^{\frac{1}{2}} < \infty$. Hence the function $F(\zeta) = \int f(s)\mathrm{e}^{\mathrm{i}\zeta s}\,\mathrm{d}\nu(s)$ is holomorphic on the half-plane $\operatorname{Im}\zeta > -\frac{\varepsilon}{2}$, and all its derivatives at 0 vanish. Thus $F = 0$, the Fourier transform of the finite complex measure $f\nu$ vanishes, and $f = 0$ in $\mathrm{L}^2(\nu)$. So the polynomials are dense in $\mathrm{L}^2(\nu)$. Fix $t \in \Sigma$ and choose polynomials $p_n$ with $\|p_n - 1_{\{t\}}\|_2 \to 0$. Using $\big||\alpha|^2 - |\beta|^2\big| \leqslant |\alpha-\beta|(|\alpha|+|\beta|)$, we obtain

$$\Big|\sum_s |p_n(s)|^2 c_s - c_t\Big| \leqslant \int \big||p_n|^2 - 1_{\{t\}}\big|\,\mathrm{d}\nu \leqslant \|p_n - 1_{\{t\}}\|_2\big(\|p_n\|_2 + \|1_{\{t\}}\|_2\big) \xrightarrow[n\to\infty]{} 0.$$

Since the sums on the left are nonnegative, $c_t \geqslant 0$. ∎

**Proposition 5.2** *Let $V$ be a simple regular unitary vertex operator algebra with bounded localized vertex operators, and let $M$ and $N$ be ordinary unitary $V$-modules such that the representation $\pi^N$ of [Ten24, Definition 2.18 p. 20] exists. Then $M \boxtimes N$ admits an invariant scalar product for which $(M \boxtimes N, \mathcal{Y}_\boxtimes)$ is a transport module for $M$ and $N$.*

*Proof : A nondegenerate invariant Hermitian form.* The scalar products of the unitary modules $M$ and $N$ give isomorphisms $\overline{M} \cong M'$ and $\overline{N} \cong N'$, where, for a $V$-module $X$, the complex conjugate module $\overline{X}$ is the complex conjugate vector space of $X$ with the action $Y_{\overline{X}}(v,x)\overline{a} \stackrel{\text{def}}{=} \overline{Y_X(\theta v, x)a}$, $v \in V$, $a \in X$, the conjugation being applied to the coefficients and $\theta$ being the PCT operator of $V$, see [Gui22, (1.24) p. 10]. A sesquilinear form $B$ on $X$ is invariant if $B(Y_X(v,x)a,b) = B(a, Y_X(\mathrm{e}^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1})b)$ for any $v \in V$ and any $a, b \in X$. Conjugation of intertwining operators as in [Gui22, (1.25) p. 11], which only uses the complex conjugate modules and not their scalar products, gives for every $V$-module $K$ an antilinear bijection from the intertwining operators of type $\binom{K}{MN}$ onto those of type $\binom{\overline{K}}{\overline{M}\,\overline{N}}$. Hence the conjugate of $\mathcal{Y}_\boxtimes$ is a universal intertwining operator of type $\binom{\overline{P}}{\overline{M}\,\overline{N}}$, and $\overline{P} \cong \overline{M} \boxtimes \overline{N}$. Since $\mathrm{Rep}(V)$ is a rigid braided tensor category in which the contragredient module is the dual object [Hua08], we have $M' \boxtimes N' \cong (M \boxtimes N)'$. Hence $\overline{P} \cong P'$, that is, the finite-dimensional space $\mathcal{F} \cong \mathrm{Hom}_V(\overline{P}, P')$ of invariant sesquilinear forms on $P$ contains a nondegenerate form. The map $B \mapsto B^*$, $B^*(x,y) = \overline{B(y,x)}$, is an antilinear involution of $\mathcal{F}$, whose fixed points, the Hermitian forms, constitute a real form $\mathcal{F}_\mathbb{R}$ of $\mathcal{F}$. Let $B \in \mathcal{F}$. Since $L_0$ is symmetric for $B$ and the weights of $P$ are real, we have $B(P(s), P(t)) = 0$ for $s \neq t$. By invariance, the left radical $\{p \in P : B(p,P) = 0\}$ of $B$ is a submodule of $P$, and a nonzero submodule of $P$ contains a simple submodule, whose lowest weight belongs to the finite set $S_0$ of lowest weights of simple summands of $P$. Hence $B$ is nondegenerate if and only if its restriction to $P(s) \times P(s)$ is nondegenerate for any $s \in S_0$. In bases of the spaces $P(s)$, this is the non-vanishing of the product over $s \in S_0$ of the determinants of the matrices of these restrictions, which is a holomorphic polynomial function on $\mathcal{F}$. This function is not identically zero, hence not identically zero on $\mathcal{F}_\mathbb{R}$, since $\mathcal{F}_\mathbb{R}$ is a real form of $\mathcal{F}$. Thus there is a nondegenerate invariant Hermitian form $Q_0$ on $P$. The operator $L_0$ is symmetric for $Q_0$, so that the spaces $P(s)$ are mutually orthogonal and $Q_0$ is nondegenerate on each of them.

*The transport endomorphism.* By associativity of intertwining operators and the universal property of $P$, there is an intertwining operator $\mathcal{Z}$ of type $\binom{N}{\overline{M}\;P}$ such that

$$Y^N(\mathcal{Y}^-_M(c, \overline{w}^{-1} - z)a, z)b = \mathcal{Z}(c, \overline{w}^{-1})\mathcal{Y}_\boxtimes(a,z)b$$

as matrix coefficients in the associativity domain, with the branches used in [Ten24, Proposition 3.12 p. 34]. Indeed, associativity first gives a finite sum of products through an intermediate module $K$. The first intertwining operator factors through $\mathcal{Y}_\boxtimes$ by a module homomorphism $P \to K$, which we compose with the second intertwining operator.

Using the nondegenerate invariant Hermitian form $Q_0$ on $P$, which is linear in the first variable, and the scalar products of $M$ and $N$, we define the adjoint $\mathcal{Z}^\dagger$ by [Gui22, (1.27) p. 11], where the adjoint of each mode, which maps a space $N(t)$ into a space $P(t')$, is taken with respect to the scalar product of $N(t)$ and to the restriction of $Q_0$ to $P(t')$. It exists since $Q_0$ is nondegenerate on $P(t')$. As in [Gui22, (1.26) p. 11], $\mathcal{Z}^\dagger$ is the contragredient intertwining operator of $\mathcal{Z}$ composed with the isomorphisms $\overline{P} \cong P'$ and $\overline{N} \cong N'$ given by $Q_0$ and by the scalar product of $N$. Hence $\mathcal{Z}^\dagger$ is an intertwining operator of type $\binom{P}{M\;N}$, and $\mathcal{Z}^{\dagger\dagger} = \mathcal{Z}$. By the universal property there is $\Gamma \in \mathrm{End}_V(P)$ such that $\mathcal{Z}^\dagger = \Gamma\mathcal{Y}_\boxtimes$. Let $\Gamma^\sharp$ be its adjoint for $Q_0$, determined by $Q_0(\Gamma x, y) = Q_0(x, \Gamma^\sharp y)$ for $x, y \in P$. This adjoint exists on each homogeneous subspace, and invariance shows that it is a $V$-module endomorphism. Put

$\Lambda = \Gamma^\sharp$. Let $(z,w) \in \mathcal{R}$ and let $c \in \overline{M}$ be the vector attached to $a_2$ in [Ten24, (3.6) p. 25]. For $s \in \mathbb{R}$ and $\xi \in P(s)$, the definition of $\mathcal{Z}^\dagger$, the identity $\mathcal{Z}^\dagger = \Gamma \mathcal{Y}_\boxtimes$ and the orthogonality of the spaces $P(t)$ for $Q_0$ give, as in the proof of [Ten24, Proposition 3.12 p. 34],

$$\langle \mathcal{Z}(c, \overline{w}^{-1})\xi, b_2 \rangle_N = Q_0\big(\xi, \Gamma P_s \mathcal{Y}_\boxtimes(a_2, w) b_2\big) = Q_0\big(\Lambda\xi, P_s \mathcal{Y}_\boxtimes(a_2, w) b_2\big).$$

Taking $\xi = P_s \mathcal{Y}_\boxtimes(a_1, z) b_1$ and summing over $s$, we obtain

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{z,w} = \sum_s Q_0\big(\Lambda P_s \mathcal{Y}_\boxtimes(a_1, z) b_1, P_s \mathcal{Y}_\boxtimes(a_2, w) b_2\big). \tag{5.1}$$

Here $(z,w) \in \mathcal{R}$, and the series converges absolutely by convergence of products of intertwining operators [Hua05]. We show that $\Lambda$ is invertible, by the argument of [Ten24, Lemma 3.15 p. 35] written without orthonormal bases of $P$. In the braided tensor category $\mathrm{Rep}(V)$, identity (5.1) and the associativity of intertwining operators [Hua05], evaluated at real points $2^{-\frac{1}{2}} < z = w < 1$ where all logarithms are real, give

$$u \circ (\mathrm{id}_{\overline{M}} \boxtimes \Lambda) = \ell_N \circ (y \boxtimes \mathrm{id}_N) \circ \alpha^{-1},$$

where $y \in \mathrm{Hom}_V(\overline{M} \boxtimes M, V)$ and $u \in \mathrm{Hom}_V(\overline{M} \boxtimes P, N)$ correspond to $\mathcal{Y}_M^-$ and to the adjoint of $\mathcal{Y}_\boxtimes$ with respect to $Q_0$ and to the scalar products of $M$ and $N$, $\alpha \colon (\overline{M} \boxtimes M) \boxtimes N \to \overline{M} \boxtimes P$ is the associativity isomorphism and $\ell_N$ is the unit isomorphism. The module $\overline{M} \cong M'$ is a dual object of $M$, with evaluation $\mathrm{ev}_M \colon \overline{M} \boxtimes M \to V$ and coevaluation $\mathrm{coev}_M \colon V \to M \boxtimes \overline{M}$ [Hua08]. Let $M = \bigoplus_i M_i$ be an orthogonal decomposition into simple submodules, and take $\mathrm{ev}_M = \bigoplus_i \mathrm{ev}_{M_i}$. The operator $\mathcal{Y}_M^-$ vanishes on $\overline{M_i} \otimes M_j$ for $i \neq j$. Indeed, $\mathcal{Y}_M^- = (\mathcal{Y}_M^+)^\dagger$ [Ten24, Definition 2.7 p. 12], the creation operator satisfies $\mathcal{Y}_M^+(a,x)b = \mathrm{e}^{xL_{-1}} Y^M(b,-x)a \in M_i\{x\}$ for any $a \in M_i$ and any $b \in V$, and $M_i$ is orthogonal to $M_j$. The restriction of $\mathcal{Y}_M^-$ to $\overline{M_i} \otimes M_i$ is the annihilation operator of $M_i$, which is non-zero. Since $\mathrm{Hom}_V(\overline{M_i} \boxtimes M_i, V)$ is one-dimensional, we get $y = \mathrm{ev}_M \circ (D \boxtimes \mathrm{id}_M)$ for an automorphism $D$ of $\overline{M}$ acting on each $\overline{M_i}$ by a non-zero scalar. Put

$$X = (\mathrm{id}_M \boxtimes u) \circ \alpha' \circ (\mathrm{coev}_M \boxtimes \mathrm{id}_P) \circ \ell_P^{-1} \colon P \to M \boxtimes N = P,$$

where $\alpha' \colon (M \boxtimes \overline{M}) \boxtimes P \to M \boxtimes (\overline{M} \boxtimes P)$ is the associativity isomorphism. By naturality of $\alpha'$ and of $\ell_P$, the displayed identity, the zig-zag identity for $\mathrm{ev}_M$ and $\mathrm{coev}_M$ and the coherence of the associativity isomorphisms, we obtain $X \circ \Lambda = D' \boxtimes \mathrm{id}_N$, where $\mathrm{coev}_M = \bigoplus_i \mathrm{coev}_{M_i}$ is the coevaluation associated with $\mathrm{ev}_M$, and $D' \overset{\mathrm{def}}{=} \bigoplus_i d_i \mathrm{id}_{M_i}$ if $D$ acts on $\overline{M_i}$ by the scalar $d_i$, so that $(\mathrm{id}_M \boxtimes D) \circ \mathrm{coev}_M = (D' \boxtimes \mathrm{id}_{\overline{M}}) \circ \mathrm{coev}_M$. Hence $\Lambda$ has a left inverse. Since $\Lambda$ preserves the finite-dimensional spaces $P(s)$, it is invertible. Consequently, the invariant sesquilinear form $Q \overset{\mathrm{def}}{=} Q_0(\Lambda \cdot, \cdot)$ is nondegenerate, and the spaces $P(s)$ are mutually orthogonal for $Q$.

*Positivity of the transport form.* By [Ten24, Proposition 3.11 p. 33], $\mathcal{Y}_M^-$ belongs to the space $\mathcal{I}_{\mathrm{Hilb}}$ of [Ten24, Definition 2.8 p. 12]. By [Ten24, Proposition 4.2 p. 43], whose hypotheses do not involve $P$, there is $r < 1$ such that the form $\langle \cdot, \cdot \rangle_{z,z}$ on $M \otimes N$ is positive semidefinite whenever $(z,z) \in \mathcal{R}$ and $|z| > r$. Fix a real number $z$ with $\max(r, 2^{-\frac{1}{2}}) < z < 1$, so that $(z,z) \in \mathcal{R}$. For $x \in M \otimes N$, put $\xi = \mathcal{Y}_\boxtimes(x,z)$, $\xi_s = P_s \xi$ and $c_s(x) = Q(\xi_s, \xi_s)$. By (5.1), $\sum_s c_s(x) = \langle x, x \rangle_{z,z} \geqslant 0$.

*Invariance under $L_0$.* For an intertwining operator $\mathcal{Y}$, we have $[L_0, \mathcal{Y}(a,x)] = x \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{Y}(a,x) + \mathcal{Y}(L_0 a, x)$ and $\frac{\mathrm{d}}{\mathrm{d}x} \mathcal{Y}(a,x) = \mathcal{Y}(L_{-1} a, x)$. Hence

$$L_0 \mathcal{Y}_\boxtimes(a,z)b = \mathcal{Y}_\boxtimes(L_0 a + z L_{-1} a, z)b + \mathcal{Y}_\boxtimes(a,z) L_0 b,$$

where $L_0$ acts componentwise on the algebraic completion $\prod_s P(s)$ of $P$, which contains the vectors $\mathcal{Y}_\boxtimes(x,z)$, $x \in M \otimes N$, defined by linearity. With the operator $T_z \overset{\mathrm{def}}{=} (L_0 + z L_{-1}) \otimes \mathrm{id}_N +$

$\mathrm{id}_M \otimes L_0$ of $M \otimes N$, we obtain $p(L_0)\mathcal{Y}_{\boxtimes}(x,z) = \mathcal{Y}_{\boxtimes}(p(T_z)x,z)$ for any polynomial $p$ and any $x \in M \otimes N$. Put $x_p \overset{\text{def}}{=} p(T_z)x$. Since $P_s p(L_0)\xi = p(s)\xi_s$, we get $\sum_s |p(s)|^2 c_s(x) = \langle x_p, x_p\rangle_{z,z} \geqslant 0$.

*Exponential decay.* Let $1 < \lambda < z^{-1}$. If $x = \sum_k a_k \otimes b_k$ with homogeneous $a_k$ and $b_k$ of weights $h_k$ and $h'_k$, put $x' = \sum_k \lambda^{h_k+h'_k} a_k \otimes b_k$. Then $P_s\mathcal{Y}_{\boxtimes}(x',\lambda z) = \lambda^s \xi_s$, since $\lambda^{L_0}\mathcal{Y}_{\boxtimes}(a,z)b = \mathcal{Y}_{\boxtimes}(\lambda^{L_0}a,\lambda z)\lambda^{L_0}b$. The absolute convergence of (5.1) at the point $(\lambda z, \lambda z) \in \mathcal{R}$ gives $\sum_s \lambda^{2s}|c_s(x)| < \infty$. The weights of $P$ are bounded below.

*Conclusion.* By Lemma 5.1, $c_s(x) \geqslant 0$ for all $s$ and all $x$. For homogeneous $a \in M$ and $b \in N$ of weights $h$ and $h'$, we have $P_s\mathcal{Y}_{\boxtimes}(a,z)b = z^{-n-1}a_{(n)}b$ with $n = h + h' - s - 1$. By the commutator formula, the coefficients $a_{(n)}b$ span a submodule of $P$, which is equal to $P$ by uniqueness in the universal property of $(P, \mathcal{Y}_{\boxtimes})$ and semisimplicity. That is, $\mathcal{Y}_{\boxtimes}$ is dominant. Hence every $v \in P(s)$ can be written $v = \sum_k (a_k)_{(n_k)}b_k = P_s\mathcal{Y}_{\boxtimes}(x,z)$ with $x = \sum_k z^{n_k+1}a_k \otimes b_k$, and $Q(v,v) = c_s(x) \geqslant 0$. By polarization, $Q$ is a positive semidefinite Hermitian form. It is invariant and nondegenerate, hence positive definite, and $P$ is a unitary $V$-module for $Q$. Finally, (5.1) reads $\langle x, y\rangle_{z,w} = Q(\mathcal{Y}_{\boxtimes}(x,z), \mathcal{Y}_{\boxtimes}(y,w))$ on $\mathcal{R}$, and $\mathcal{Y}_{\boxtimes}$ is dominant and lies in $\mathcal{I}_{\mathrm{Hilb}}$ by [Ten24, Proposition 3.11 p. 33], so that $(P, \mathcal{Y}_{\boxtimes})$ is a transport module. ■

**Remark 5.3** In the proof of [Ten24, Proposition 3.16 p. 36], the positivity of $\Lambda$ is deduced from that of the transport form through the density of the vectors $\mathcal{Y}_{\boxtimes}(a,z)b$ in a Hilbert space completion of $M \boxtimes N$ [Ten24, Lemma 3.6 p. 26], which presupposes a unitary structure on $M \boxtimes N$. In the proof above, this density is replaced by the invariance of the spaces $\{\mathcal{Y}_{\boxtimes}(x,z) : x \in M \otimes N\}$ under polynomials in $L_0$ and by Lemma 5.1, and the scalar product of $M \boxtimes N$ by a nondegenerate invariant Hermitian form. The proof of the invertibility of $\Lambda$ is that of [Ten24, Lemma 3.15 p. 35], with the form $Q_0$ in place of a scalar product on $M \boxtimes N$. Consequently, for a simple regular unitary vertex operator algebra with bounded localized vertex operators, the hypothesis that $M \boxtimes \overline{M}$ admits a unitary structure, used in the proof of [Ten24, Theorem 4.20 p. 55] through [Ten24, Theorem 4.4 and Corollary 4.5 p. 44], holds as soon as $\pi^{\overline{M}}$ exists, as in the setting of [Ten24, Theorem 4.20 p. 55].

## 5.2 Unitarity of the modules and complete rationality

**Lemma 5.4** *The $U$-module $V'$ decomposes as $V' = U \oplus M$ with $M \cong M_b$. Consequently, for each isotropic line $\Lambda$, the vertex operator algebra $H_\Lambda$ of Lemma 4.1 is a holomorphic extension of $U$ containing $V' = U \oplus M_b$, and the facts recalled in Section 2.4 apply to $U \subset V' \subset H_\Lambda$. In particular, the isotropic lines $\Lambda_a$ and $\Lambda_{c_0}$ are defined, and for $w \in \Lambda_x \setminus \{0\}$, $x \in \{a, c_0\}$, the $U$-module $V'_w$ is simple and isomorphic to $M_x$.*

*Proof* : The vertex operator algebra $V'$ is a simple extension of $U$, and $V' \cong (V')'$. By Lemma 2.3 applied to $W = B = V'$ and by [HKL15, Theorem 3.4 p. 16], the algebra $A$ in $\mathrm{Rep}(U)$ attached to $V'$ is a connected étale algebra, and the category $\mathrm{Rep}(U)^0_A$ is isomorphic to $\mathrm{Rep}(V')$, which has nine simple objects [Don93]. Hence $\mathrm{FPdim}\,\mathrm{Rep}(U)^0_A \geqslant 9$, and by (2.7) and (2.6), we have $\mathrm{FPdim}(A)^2 \leqslant \frac{9(3d+2)^2}{9}$, that is, $\mathrm{FPdim}(A) \leqslant 3d+2$. As in the proof of Lemma 3.2, $A$ is a direct sum of simple objects with trivial twist containing 0 once. By (2.4), the only such objects of Frobenius–Perron dimension at most $3d+2$ are 0 and $0 \oplus b$. Since $V' \neq U$, we obtain $A \cong 0 \oplus b$. The other assertions follow, since $H_\Lambda \cong V_{E_8}$ is holomorphic and contains $V'$ by Lemma 4.1. ■

Assume now that $U$ is a unitary subalgebra of $V'$ for a normalized invariant scalar product, and equip each $H_\Lambda$ with the scalar product of part 2 of Lemma 4.2. Let $N_b$ be the orthogonal

complement of $U$ in $V'$, which is a $U$-submodule isomorphic to $M_b$ by Lemma 5.4 and part 1 of Lemma 4.7. With the notation of Lemma 4.7, we put

$$\pi_0 \overset{\text{def}}{=} \pi_U, \quad \pi_b \overset{\text{def}}{=} \pi_{N_b}, \quad \pi_x \overset{\text{def}}{=} \pi_{V'_w} \text{ for } w \in \Lambda_x \setminus \{0\},\ x \in \{a, c_0\},$$

where $V'_w$ is viewed as a $U$-submodule of $H_{\Lambda_x}$. By part 3 of Lemma 4.7, $\pi_x$ does not depend on the choice of $w$ up to unitary equivalence, and $\pi_0$ is the vacuum representation of $\mathcal{A}_U$.

**Corollary 5.5** *Let $U \subset V'$ be a Haagerup algebra containing the conformal vector of $V'$ which is a unitary subalgebra for a normalized invariant scalar product on $V'$. Then every simple $U$-module admits a scalar product which is invariant with respect to the PCT operator of $U$ induced by this scalar product. In particular, under* ($\mathrm{H_J}$)*, every simple $S$-module admits a scalar product which is invariant for the PCT operator of $S$ induced by $H_\chi$, which is the assertion* ($\mathrm{H_U}$) *of the introduction.*

*Proof* : Fix an isotropic line $\Lambda$ and put $H = H_\Lambda$, with the scalar product of part 2 of Lemma 4.2. By part 1 of Proposition 4.3, $H$ has bounded localized vertex operators. As in the proof of Theorem 4.5, $U$ is a unitary subalgebra of $H$ containing its conformal vector, so that $U$ has bounded localized vertex operators, for the same system of generalized annuli [Ten19b, Proposition 4.16 p. 43]. The $U$-module $N_b$ is a unitary $U$-submodule of $H$ isomorphic to $M_b$, and the representation $\pi^{N_b}$ exists by [Ten19b, Theorem 5.6 p. 52], applied to the $H$-module $H$, for which $\pi^H$ is the vacuum representation. The strongly rational algebra $U$ is regular [ABD04, Theorem 4.5]. By Proposition 5.2, $N_b \boxtimes N_b$ admits an invariant scalar product. In the normalized $S$-matrix of [EvG11, (2.5a)–(2.5b) p. 20], put $x = \frac{13-3\sqrt{13}}{26}$ and $y = \frac{3}{\sqrt{13}}$. Then $x(1-x) = \frac{1}{13}$ and $1 - 2x = y$. Outside the columns 0 and $b$, the squares of the entries in the rows 0 and $b$ agree. Since all simple objects are self-dual, $S^2 = 1$, and the Verlinde formula gives

$$N_{bb}^k = \delta_{0k} + \frac{1-2x}{3}\left(\frac{S_{k0}}{x} - \frac{S_{kb}}{1-x}\right).$$

For $k = 0$ the added term vanishes. For $k \in \{b, a, c_0, c_1, c_2\}$ it is $\frac{(1-2x)^2}{9x(1-x)} = 1$, and for $k = d_l$ it is $\frac{(1-2x)y}{9x(1-x)} = 1$. Thus $b \otimes b$ is the direct sum of the twelve simple objects, each with multiplicity one. Hence every simple $U$-module is isomorphic to a submodule of $N_b \boxtimes N_b$, and the restriction of the scalar product to this submodule is invariant. The last assertion follows, since under ($\mathrm{H_J}$) the algebra $S$ is a Haagerup algebra which contains the conformal vector of $V'$ and is a unitary subalgebra of $V'$ [Arh26, Theorem 10.10 p. 33 and Theorem 10.24 p. 42]. ■

**Theorem 5.6** *Let $U \subset V'$ be a Haagerup algebra containing the conformal vector of $V'$ which is a unitary subalgebra for a normalized invariant scalar product on $V'$.*

1. *The conformal net $\mathcal{A}_U$ is completely rational. For $x \in \{a, c_0\}$, the index $[\mathcal{A}_{H_{\Lambda_x}} : \mathcal{A}_U]$ is finite.*

2. *The representations $\pi_0$, $\pi_b$, $\pi_a$, $\pi_{c_0}$ are irreducible and pairwise inequivalent. The vacuum representation of $\mathcal{A}_{V'}$ restricts to $\mathcal{A}_U$ as $\pi_0 \oplus \pi_b$, and the vacuum representation of $\mathcal{A}_{H_{\Lambda_x}}$ restricts to $\mathcal{A}_U$ as $\pi_0 \oplus \pi_b \oplus 2\pi_x$.*

3. *Put $\delta \overset{\text{def}}{=} d(\pi_b)$. Then $\delta \geqslant 1$, $[\mathcal{A}_{V'} : \mathcal{A}_U] = 1 + \delta$, $d(\pi_a) = d(\pi_{c_0}) = 1 + \delta$, $[\mathcal{A}_{H_{\Lambda_x}} : \mathcal{A}_U] = 3(1+\delta)$ and $\mu_{\mathcal{A}_U} = 9(1+\delta)^2$.*

4. *For $x \in \{a, c_0\}$, the dual canonical endomorphism $\Theta_x$ of $\mathcal{A}_U \subset \mathcal{A}_{H_{\Lambda_x}}$ is a Lagrangian algebra in* $\mathrm{Rep}(\mathcal{A}_U)$*, and the fusion category $\mathcal{F}_x \overset{\text{def}}{=} \mathrm{Rep}(\mathcal{A}_U)_{\Theta_x}$ is a unitary fusion category of rank six and Frobenius–Perron dimension $3(1+\delta)$ such that* $\mathrm{Rep}(\mathcal{A}_U)$ *is braided equivalent to $Z(\mathcal{F}_x)$. The category $\mathcal{F}_x$ contains a pointed fusion subcategory of dimension three, corresponding to the intermediate net $\mathcal{A}_{V'}$.*

*Proof* : 1. Fix $x$ and put $H = H_{\Lambda_x}$, with the scalar product of Lemma 4.2. By Proposition 4.3, $H$ is strongly local, has bounded localized vertex operators, and its nets in the sense of [Ten19b] and [CKLW18] are isomorphic. By the proof of Theorem 4.5, $U$ is a unitary subalgebra of $H$ containing its conformal vector. The strongly rational vertex operator algebra $U$ is regular [ABD04, Theorem 4.5]. The proof of [Ten24, Corollary 4.24 p. 56] reduces, through [Ten24, Corollary 4.21 p. 55], to [Ten24, Theorem 4.20 p. 55] for the simple $U$-submodules $M$ of $H$, that is, for $U$, $N_b$ and $V'_{\pm w}$. The proof of this theorem uses [Ten24, Corollary 4.5 p. 44], hence a unitary structure on $M \boxtimes \overline{M}$, see Remark 5.3. In the proof of [Ten24, Corollary 4.21 p. 55], the representations $\pi^M$ and $\pi^{\overline{M}}$ of $\mathcal{A}_U^{\mathrm{T}}$ exist, so that Proposition 5.2, applied to $M$ and $N = \overline{M}$, gives such a structure. By [Ten24, Corollary 4.24 p. 56], the net $\mathcal{A}_U^{\mathrm{T}}$ of [Ten19b] attached to $U \subset H$ has finite index in the net $\mathcal{A}_H^{\mathrm{T}}$, which is completely rational by part 1 of Proposition 4.3. By part 1 of Proposition 2.5, $\mathcal{A}_U^{\mathrm{T}}$ is completely rational. By [Ten19b, Lemma 7.6 p. 72], $\mathcal{A}_U^{\mathrm{T}}$ is isomorphic to the net $\mathcal{A}_U$ of [CKLW18], which is therefore completely rational.

Now consider the inclusion $\mathcal{A}_U \subset \mathcal{A}_H$ of nets of [CKLW18]. By Theorem 4.5, its coset is trivial and $\mathcal{A}_U(I)' \cap \mathcal{A}_H(I) = \mathbb{C}$. In particular, $\mathcal{A}_H$ is an irreducible local extension of $\mathcal{A}_U$ in the sense of [KaL04, p. 9]. Since $\mathcal{A}_U$ is completely rational, the index $[\mathcal{A}_H : \mathcal{A}_U]$ is finite by part 2 of Proposition 2.5. The same argument applies to the inclusion $\mathcal{A}_U \subset \mathcal{A}_{V'}$.

2. As a $U$-module, $H = U \oplus N_b \oplus V'_w \oplus V'_{-w}$ for $w \in \Lambda_x \setminus \{0\}$, and the summands are pairwise orthogonal. Indeed, the lattice sectors are orthogonal for $H_1$, the automorphism $\tilde{h} = \exp(y(0))$, $y \in (V')_1$, of Lemma 4.2 preserves each $V'_w$, and $N_b$ is the orthogonal complement of $U$ in $V'$. Hence $\mathcal{H}_H = \mathcal{H}_U \oplus \mathcal{H}_{N_b} \oplus \mathcal{H}_{V'_w} \oplus \mathcal{H}_{V'_{-w}}$, and by part 1 of Lemma 4.7, the vacuum representation of $\mathcal{A}_H$ restricts to $\pi_0 \oplus \pi_b \oplus \pi_{V'_w} \oplus \pi_{V'_{-w}}$. By Lemma 5.4, the summands are simple $U$-modules, isomorphic to $U$, $M_b$, $M_x$, $M_x$. These are pairwise non-isomorphic except for the last two. By parts 2 and 3 of Lemma 4.7, we obtain the decomposition $\pi_0 \oplus \pi_b \oplus 2\pi_x$ into irreducible representations, with $\pi_0$, $\pi_b$, $\pi_x$ pairwise inequivalent. Applying part 3 of Lemma 4.7 to $V_1 = H_{\Lambda_a}$, $V_2 = H_{\Lambda_{c_0}}$ and to the non-isomorphic $U$-modules $M_a$ and $M_{c_0}$, we see that $\pi_a$ and $\pi_{c_0}$ are inequivalent. The same argument applied to $\mathcal{H}_{V'} = \mathcal{H}_U \oplus \mathcal{H}_{N_b}$ gives the restriction of the vacuum representation of $\mathcal{A}_{V'}$.

3. By part 1 and part 3 of Proposition 2.5, the dual canonical endomorphism of $\mathcal{A}_U \subset \mathcal{A}_{V'}$ is $\theta = \pi_0 \oplus \pi_b$, so that $[\mathcal{A}_{V'} : \mathcal{A}_U] = d(\theta) = 1 + \delta$. Here $\delta \geqslant 1$, as for any statistical dimension. By part 1 of Proposition 2.5 applied to the two finite index inclusions $\mathcal{A}_U \subset \mathcal{A}_{V'}$ and $\mathcal{A}_U \subset \mathcal{A}_H$, and by Proposition 4.3, we have

$$\mu_{\mathcal{A}_U} = [\mathcal{A}_{V'} : \mathcal{A}_U]^2 \mu_{\mathcal{A}_{V'}} = 9(1+\delta)^2 \quad \text{and} \quad \mu_{\mathcal{A}_U} = [\mathcal{A}_H : \mathcal{A}_U]^2 \mu_{\mathcal{A}_H} = [\mathcal{A}_H : \mathcal{A}_U]^2.$$

Hence $1 + \delta + 2d(\pi_x) = d(\Theta_x) = [\mathcal{A}_H : \mathcal{A}_U] = 3(1+\delta)$, so that $d(\pi_x) = 1 + \delta$.

4. By part 3 of Proposition 2.5 and part 1, $\Theta_x$ is a commutative Q-system with

$$\dim \mathrm{Hom}(\mathrm{id}, \Theta_x) = \dim \mathcal{A}_U(I)' \cap \mathcal{A}_H(I) = 1,$$

hence a connected étale algebra in the unitary modular tensor category $\mathrm{Rep}(\mathcal{A}_U)$. By (2.10), $\mathrm{FPdim}\,\mathrm{Rep}(\mathcal{A}_U) = \mu_{\mathcal{A}_U} = d(\Theta_x)^2$, so that $\Theta_x$ is Lagrangian. By part 1 of Proposition 2.2, $\mathrm{Rep}(\mathcal{A}_U)$ is braided equivalent to $Z(\mathcal{F}_x)$, and (2.7) gives $\mathrm{FPdim}(\mathcal{F}_x) = \frac{9(1+\delta)^2}{3(1+\delta)} = 3(1+\delta)$. The

category $\mathcal{F}_x$ of $\Theta_x$-modules is equivalent to the category of $\alpha$-induced sectors of the inclusion $\mathcal{A}_U(I) \subset \mathcal{A}_H(I)$, which is a unitary fusion category [BKLR15, Section 4.6 p. 54], see also [Bis17a, Proposition 4.5 p. 21]. By Lemma 3.1 and part 2, its rank is $\dim \mathrm{Hom}(\Theta_x, \Theta_x) = 1+1+4 = 6$. By [BKLR15, Proposition 4.11 p. 49], the intermediate subfactor $\mathcal{A}_U(I) \subset \mathcal{A}_{V'}(I) \subset \mathcal{A}_H(I)$ corresponds to a projection in $\mathrm{Hom}(\Theta_x, \Theta_x)$ whose range, with the restricted multiplication, is the Q-system $\theta$ of $\mathcal{A}_U \subset \mathcal{A}_{V'}$. Hence $\theta$ is a connected étale subalgebra of $\Theta_x$. By Lemma 3.1, under the equivalence $\mathrm{Rep}(\mathcal{A}_U) \cong Z(\mathcal{F}_x)$, the algebra $\Theta_x$ corresponds to the algebra $I(1)$, and by part 3 of Proposition 2.2, $\theta$ corresponds to a fusion subcategory $\mathcal{P}_x$ of $\mathcal{F}_x$ with $\mathrm{FPdim}(\theta)\,\mathrm{FPdim}(\mathcal{P}_x) = \mathrm{FPdim}(\mathcal{F}_x)$, that is, $\mathrm{FPdim}(\mathcal{P}_x) = 3$. A fusion category of prime Frobenius–Perron dimension is pointed [ENO05, Corollary 8.30 p. 37]. ∎

**Proposition 5.7** *Under the hypotheses of Theorem 5.6, there is a proper action of the hypergroup $K = \{k_0, k_1\}$ with*

$$k_1 k_1 = \frac{1}{\delta} k_0 + \Big(1 - \frac{1}{\delta}\Big) k_1$$

*on the net $\mathcal{A}_{V'}$ such that $\mathcal{A}_U = \mathcal{A}_{V'}^K$. The weight of $K$ is $1+\delta = [\mathcal{A}_{V'} : \mathcal{A}_U]$.*

*Proof* : By part 1 of Theorem 5.6, the net $\mathcal{A}_U$ is completely rational and $[\mathcal{A}_{V'} : \mathcal{A}_U] < \infty$. By part 3 of Proposition 2.6, there is a canonical hypergroup $K$ acting properly on $\mathcal{A}_{V'}$ with $\mathcal{A}_U = \mathcal{A}_{V'}^K$, and its weight is $[\mathcal{A}_{V'} : \mathcal{A}_U] = 1+\delta$. By part 2 of Theorem 5.6, the dual canonical endomorphism of $\mathcal{A}_U(I) \subset \mathcal{A}_{V'}(I)$ is $\theta = \pi_0 \oplus \pi_b$ with $\pi_b$ irreducible and inequivalent to $\pi_0$. Hence $\langle \gamma, \gamma \rangle = \langle \theta, \theta \rangle = 2$, where $\gamma$ is the canonical endomorphism, and by [Bis17b, Section 4.6 p. 25], the canonical hypergroup is $K = \{k_0, k_1\}$ with the stated multiplication, where $\delta = [\mathcal{A}_{V'} : \mathcal{A}_U] - 1$. The elements $k_0$, $k_1$ are denoted $c_0$, $c_1$ in [Bis17b]. ∎

**Proposition 5.8** *Under the hypotheses of Theorem 5.6, we have $\delta > 1$. Consequently, the fusion category $\mathcal{F}_x$ has exactly three invertible simple objects and three non-invertible simple objects.*

*Proof* : Assume that $\delta = 1$. By Proposition 5.7, there is a proper action of a hypergroup $K = \{k_0, k_1\}$ on $\mathcal{A}_{V'}$ with $\mathcal{A}_U = \mathcal{A}_{V'}^K$ and $k_1 k_1 = \delta^{-1} k_0 + (1 - \delta^{-1}) k_1 = k_0$. Thus the vacuum preserving unital completely positive maps $\sigma_I = \phi_{k_1}^I$ of $\mathcal{A}_{V'}(I)$ satisfy $\sigma_I^2 = \mathrm{id}$. By the Kadison–Schwarz inequality and the positivity of $\sigma_I$, we have $x^*x = \sigma_I^2(x^*x) \geqslant \sigma_I(\sigma_I(x)^*\sigma_I(x)) \geqslant x^*x$. Since $\sigma_I$ is injective and positive, we obtain $\sigma_I(x^*x) = \sigma_I(x)^*\sigma_I(x)$ for any $x$, so that $\sigma_I$ is multiplicative by the multiplicative domain argument, hence an automorphism. It is nontrivial since $\mathcal{A}_U \neq \mathcal{A}_{V'}$. Since $\sigma_I$ preserves the vacuum state and $\Omega$ is cyclic for $\mathcal{A}_{V'}(I)$, the formula $u_I x \Omega = \sigma_I(x)\Omega$ defines a unitary $u_I$ with $u_I x u_I^* = \sigma_I(x)$ and $u_I \Omega = \Omega$. The maps $\sigma_I$ are compatible [Bis17b, Proposition 4.21 p. 23], so that $u_I = u_J$ if $I \subset J$. Since any two intervals contain subintervals which are contained in a common interval, $g \stackrel{\mathrm{def}}{=} u_I$ does not depend on $I$, and $g$ is an automorphism of the net $\mathcal{A}_{V'}$ in the sense of [CKLW18, (41) p. 21]. By [CKLW18, Theorem 6.9 p. 53], $g$ is a unitary automorphism of $V'$ of order two. By [CKLW18, Proposition 7.6 and Theorem 7.5 p. 58], we obtain $U = (V')^g$, so that $U_1$ is the fixed point subalgebra of the automorphism $g$ of order at most two of $(V')_1 \cong \mathfrak{e}_6 \oplus \mathfrak{sl}_3$. The automorphism $g$ preserves the two simple ideals $\mathfrak{e}_6$ and $\mathfrak{sl}_3$ of $(V')_1$, which are not isomorphic, and it is nontrivial on $(V')_1$ since $V'$ is generated by $(V')_1$. The fixed point subalgebra of an involution of $\mathfrak{e}_6$ is isomorphic to $\mathfrak{sp}_8$, $\mathfrak{sl}_6 \oplus \mathfrak{sl}_2$, $\mathfrak{so}_{10} \oplus \mathbb{C}$ or $\mathfrak{f}_4$, of dimension 36, 38, 46 or 52, and that of an involution of $\mathfrak{sl}_3$ is isomorphic to $\mathfrak{so}_3$ or $\mathfrak{gl}_2$, of dimension 3 or 4 [Kac90, Chapter 8], [Hel78, Chapter X]. Taking into account the case where $g$ is trivial on one of the two ideals, of dimensions 78 and 8, we obtain $\dim U_1 \geqslant \min(36+3, 36+8, 78+3) = 39$, which contradicts $\dim U_1 \in \{6, 19\}$

[Arh26, (4.1) p. 11]. A similar argument, with the rank of the fixed point subalgebra, appears in [Hua26b, pp. 7–8].

The invertible objects of $\mathcal{F}_x$ form a group containing the group of order three of $\mathcal{P}_x$. Since $\mathcal{F}_x$ has rank six, the order of this group is three or six by Lagrange's theorem. If this order were six, then $\mathcal{F}_x$ would be pointed of dimension $6 = 3(1+\delta)$, so that $\delta = 1$. ■

**Corollary 5.9** *Assume* ($\mathrm{H_J}$). *Then Theorem 5.6 holds for $U = S$, with the unitary structure of [Arh26, Theorem 10.10 p. 33]. In particular, $\mathcal{A}_S$ is completely rational, and* $\mathrm{Rep}(\mathcal{A}_S)$ *is braided equivalent to the Drinfeld center of a unitary fusion category of rank six with a pointed fusion subcategory of dimension three.*

*Proof* : By [Arh26, Theorem 10.24 p. 42], ($\mathrm{H_J}$) implies that $S$ is a Haagerup algebra. It contains the conformal vector of $V'$ and is a unitary subalgebra for $H_\chi$ by [Arh26, Theorem 10.10 p. 33]. ■

**Remark 5.10** The unitarizability of the fusion products $M \boxtimes \overline{M}$ enters the proof of Theorem 5.6 only through [Ten24, Corollary 4.5 p. 44], that is, through [Ten24, Theorem 4.4 p. 44 and Proposition 3.16 p. 36]. By the fusion rules of $Z(\mathcal{H}_3)$, the object $b \otimes b$ contains every simple object of $Z(\mathcal{H}_3)$, so that, for the application of [Ten24, Theorem 4.4 p. 44], the unitarizability of the modules $M_b$, $M_a$, $M_{c_0}$ occurring in $H_\Lambda$ does not suffice. Under ($\mathrm{H_J}$), all the simple $S$-modules except the $M_{d_l}$ are submodules of unitary $V'$-modules [Arh26, Proposition 4.1 p. 11, Lemma 4.3 p. 13 and Lemma 11.8 p. 60]. In the proof of Theorem 5.6, the unitary structures on the modules $M \boxtimes \overline{M}$ are given directly by Proposition 5.2, applied to $M$ and $N = \overline{M}$, and the six modules $M_{d_l}$ are unitarizable by Corollary 5.5.

For the algebra of [Hua26b], the complete rationality of the net and the unitarity of all simple modules are announced in [Hua26b, Theorem 1 p. 1 and Section 6 pp. 8–9]. In the outline of [Hua26b, Section 6 p. 8], the finiteness of the index of the subnet generated by the bounded insertions is obtained from [Ten24] and the regularity of the algebra, without reference to a specific statement, and the unitarity of the simple modules is derived afterwards from the twelve sectors of the resulting completely rational net, through [HeT25, Theorem 9.5 and Remark 9.8]. The finite index statements [Ten24, Theorem B and Corollary C p. 6, Theorem 4.20 p. 55 and Corollaries 4.21 and 4.24 pp. 55–56] do not list a unitarity hypothesis, but their proofs use [Ten24, Corollary 4.5 p. 44], which requires a unitary structure on $M \boxtimes \overline{M}$, see also [Ten24, Remark 3.18 p. 37]. For $M = M_b$, the module $M \boxtimes \overline{M}$ contains every simple module, including the six modules $M_{d_l}$. By Proposition 5.2 and Remark 5.3, this requirement is met for algebras with bounded localized vertex operators, as soon as $\pi^{\overline{M}}$ exists. Hence this step of the outline can be completed as in the proofs of Corollary 5.5 and Theorem 5.6, provided that the algebra of [Hua26b] is realized as a unitary subalgebra of $V'$ containing its conformal vector, for instance as in Section 8. The determination of the fusion rules and of the associator in [Hua26b, Section 5 pp. 6–8] only uses the pseudo-unitarity of the category of modules over the extension algebra, and it does not depend on this issue. The application of the classification of $C^*$-generalized Haagerup categories [Izu18, Theorem 5.9 p. 36] in [Hua26b, Section 6 p. 9] uses the unitarity of this category, which is deduced there from the complete unitarity of the algebra. In Section 6, we apply this classification to the category $\mathcal{F}_x$ of the net, which is unitary by construction, see Remark 6.12.

The value $\delta = d^2$ is obtained in Theorem 6.11.

# 6 A recognition theorem for the Haagerup categories

In this section we prove Theorem 6.1, which is Theorem 1.4 of the introduction, and we deduce from it that each category $\mathcal{F}_x$ of Theorem 5.6 is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$ (Theorem 6.11). The strategy for the fusion rules, which combines the induction to the Drinfeld center, Frobenius–Schur indicators and lower bounds for sums of roots of unity, is the one outlined by Huang [Hua26b, Sections 5 and 6 pp. 7–9], and we compare our proofs with this outline in Remark 6.12.

Throughout this section, $\mathcal{F}$ is a unitary fusion category. We equip it with the spherical structure for which the dimension $d(X)$ of every simple object $X$ is its Frobenius–Perron dimension [ENO05, Proposition 8.23 p. 34], and we put $\dim \mathcal{F} \overset{\text{def}}{=} \sum_{X \in \mathrm{Irr}(\mathcal{F})} d(X)^2$, where $\mathrm{Irr}(\mathcal{F})$ is the set of isomorphism classes of simple objects. We denote by $F\colon Z(\mathcal{F}) \to \mathcal{F}$ the forgetful functor and by $I\colon \mathcal{F} \to Z(\mathcal{F})$ its adjoint. Since both categories are semisimple, we have $\dim \mathrm{Hom}(Z, I(V)) = \dim \mathrm{Hom}(I(V), Z) = \dim \mathrm{Hom}(F(Z), V)$ for any objects $Z$ of $Z(\mathcal{F})$ and $V$ of $\mathcal{F}$. We denote by $\theta_Z$ the twist of a simple object $Z$ of $Z(\mathcal{F})$, which is a root of unity [NgS07a, p. 9 and Theorem 5.5 p. 13], and by $K(\mathcal{F})$ the Grothendieck ring of $\mathcal{F}$. For an object $V$ of $\mathcal{F}$ and $n \geqslant 1$, the Frobenius–Schur indicator $\nu_n(V)$ is the trace of the linear map $E_V^{(n)}$ of $\mathrm{Hom}(1, V^{\otimes n})$ defined in [NgS07b, p. 8]. We have $(E_V^{(n)})^n = \mathrm{id}$ [NgS07b, Theorem 5.1 p. 12]. In particular, $\nu_1(V) = \dim \mathrm{Hom}(1, V)$, and if $\mathrm{Hom}(1, V^{\otimes n})$ is one-dimensional, then $\nu_n(V)$ is an $n$-th root of unity. By [NgS07a, Theorem 4.1 p. 10], we have

$$\nu_n(V) = \frac{1}{\dim \mathcal{F}} \sum_{Z \in \mathrm{Irr}(Z(\mathcal{F}))} \theta_Z^n d(Z) \dim \mathrm{Hom}(F(Z), V). \tag{6.1}$$

We equip $Z(\mathcal{F})$ with the braiding of [NgS07a, Theorem 4.1 p. 10]. For the reverse braiding, the twists are replaced by their complex conjugates [NgS07a, Remark 4.3 p. 10]. Since the right-hand sides of (6.3)–(6.5) below are real, the only effect of this replacement on Lemma 6.4 and Proposition 6.9 is that $\nu_3(\alpha)$ is replaced by $\overline{\nu_3(\alpha)}$ in the identities $\zeta^3 = \theta_{Z_i}^3 = \nu_3(\alpha)$.

**Theorem 6.1** *Let $\mathcal{F}$ be a unitary fusion category with six simple objects up to isomorphism, three of which are invertible, and with noncommutative Grothendieck ring.*

1. *There exist an invertible object $\alpha$ of order three and a simple object $\rho$ such that the simple objects of $\mathcal{F}$ are $1$, $\alpha$, $\alpha^2$, $\rho$, $\alpha\rho$, $\alpha^2\rho$ up to isomorphism, the objects $\alpha^j\rho$ are self-dual, and*
$$\rho\alpha \cong \alpha^2\rho, \quad \rho^2 \cong 1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho.$$
*In particular, $\mathcal{F}$ has the fusion rules (2.2) of $\mathcal{H}_2$ and $\mathcal{H}_3$, and $d(\rho) = d$.*
2. *If the fusion subcategory of $\mathcal{F}$ generated by $\alpha$ is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator, then $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$.*

*Conversely, $\mathcal{H}_2$ and $\mathcal{H}_3$ satisfy the hypotheses of part 2.*

## 6.1 The fusion rules

**Lemma 6.2** *Let $\mathcal{F}$ be as in Theorem 6.1. Then there exist an invertible object $\alpha$ of order three, a simple object $\rho$ and an integer $m \geqslant 1$ such that the simple objects of $\mathcal{F}$ are $1$, $\alpha$, $\alpha^2$, $\rho$, $\alpha\rho$, $\alpha^2\rho$, up to isomorphism, and*

$$\rho\alpha \cong \alpha^2\rho, \quad (\alpha^j\rho)^* \cong \alpha^j\rho \ (j \in \{0,1,2\}), \quad \rho^2 \cong 1 \oplus m(\rho \oplus \alpha\rho \oplus \alpha^2\rho).$$

*Consequently, $d(\rho) = \frac{3m+\sqrt{9m^2+4}}{2}$ and $d(\rho)^2 = 1 + 3md(\rho)$.*

*Proof* : The invertible simple objects form a group of order three, generated by some $\alpha$. Let $\mathcal{X}$ be the set of the isomorphism classes of the three non-invertible simple objects. The group acts on $\mathcal{X}$ by left and by right multiplication.

The algebra $K(\mathcal{F}) \otimes \mathbb{C}$ is semisimple. Indeed, consider the basis $\mathrm{Irr}(\mathcal{F})$, the involution $X \mapsto X^*$ extended antilinearly, and the linear form $\tau$ which gives the coefficient of 1. We have $\tau(X^*Y) = \delta_{X,Y}$ for simple $X$, $Y$, so that $\tau(x^*x) > 0$ for $x \neq 0$. Hence $\langle x, y\rangle \stackrel{\text{def}}{=} \tau(y^*x)$ defines a scalar product on $K(\mathcal{F}) \otimes \mathbb{C}$, for which left multiplication is a faithful $*$-representation. Thus $K(\mathcal{F}) \otimes \mathbb{C}$ is a finite-dimensional $C^*$-algebra, hence semisimple. Since $K(\mathcal{F}) \otimes \mathbb{C}$ has dimension six and is noncommutative, it is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$, which has two characters.

Assume that $\alpha X \cong X$ for every $X \in \mathcal{X}$. For $X \in \mathcal{X}$, we have $X^* \in \mathcal{X}$, so that $(X\alpha)^* \cong \alpha^2 X^* \cong X^*$ and $X\alpha \cong X$. Hence, for $X, Y \in \mathcal{X}$, the objects $XY$ and $\alpha XY$ are isomorphic, and the invertible summands of $XY$ form a multiple of $1 \oplus \alpha \oplus \alpha^2$. For a primitive cube root of unity $\omega$, the linear map $\chi_\omega \colon K(\mathcal{F}) \to \mathbb{C}$ with $\chi_\omega(\alpha^j) = \omega^j$ and $\chi_\omega(X) = 0$ for $X \in \mathcal{X}$ is therefore a character. The two characters $\chi_\omega$, $\chi_{\omega^2}$ and the Frobenius–Perron character are three distinct characters, a contradiction. Since the orbits of a group of order three have one or three elements, the left action of the group on $\mathcal{X}$ is free, and $\mathcal{X} = \{\rho, \alpha\rho, \alpha^2\rho\}$ for any $\rho \in \mathcal{X}$. If $\rho\alpha \cong \rho$, then $\alpha^2\rho^* \cong (\rho\alpha)^* \cong \rho^*$, which contradicts the freeness. Hence the right action is free as well.

Write $\rho\alpha \cong \alpha^s\rho$. We have $s \neq 0$, since the right action is free. If $s = 1$, then $\alpha$ commutes with every simple object, and $(\alpha^i\rho)(\alpha^j\rho) \cong \alpha^{i+j}\rho^2 \cong (\alpha^j\rho)(\alpha^i\rho)$, so that $K(\mathcal{F})$ is commutative, a contradiction. Hence $s = 2$, and $\rho\alpha^i \cong \alpha^{2i}\rho$ for any $i$. Write $\rho^* \cong \alpha^k\rho$. Then $(\alpha^j\rho)^* \cong \rho^*\alpha^{-j} \cong \alpha^k\rho\alpha^{-j} \cong \alpha^{k+j}\rho$. Applying this twice, $\alpha^j\rho \cong (\alpha^j\rho)^{**} \cong \alpha^{2k+j}\rho$, so that $k = 0$ by freeness, and every non-invertible simple object is self-dual.

Since $\rho$ is self-dual, $\dim \mathrm{Hom}(\alpha^i, \rho^2) = \dim \mathrm{Hom}(\alpha^i\rho, \rho) = \delta_{i,0}$, so that $\rho^2 \cong 1 \oplus \bigoplus_j n_j\alpha^j\rho$ for some integers $n_j \geqslant 0$. Moreover $\rho^2\alpha \cong \rho\alpha^2\rho \cong \alpha\rho^2$. The coefficients of $\alpha^k\rho$ in $\alpha\rho^2$ and in $\rho^2\alpha$ are $n_{k-1}$ and $n_{k-2}$, so that $n_0 = n_1 = n_2 \stackrel{\text{def}}{=} m$. If $m = 0$, then $\rho$ is invertible, a contradiction. Taking Frobenius–Perron dimensions gives $d(\rho)^2 = 1 + 3md(\rho)$. ∎

By Lemma 6.2, $\mathcal{F}$ is a quadratic category with $(\mathbb{Z}_3, -1, m)$ in the sense of [Izu18, Definition 2.1 p. 6]. We put $d_m \stackrel{\text{def}}{=} d(\rho)$ and $\Sigma_\rho \stackrel{\text{def}}{=} \rho \oplus \alpha\rho \oplus \alpha^2\rho$. Then $d_m - d_m^{-1} = 3m$, $d_m + d_m^{-1} = \sqrt{9m^2+4}$ and

$$\dim \mathcal{F} = 3 + 3d_m^2 = 3(2 + 3md_m). \tag{6.2}$$

For $m = 1$, we recover $d_1 = d$ and the fusion rules (2.2).

## 6.2 The Drinfeld center

**Lemma 6.3** *Let $\mathcal{F}$ be as in Theorem 6.1, with the notation of Lemma 6.2. The simple objects of $Z(\mathcal{F})$ are, up to isomorphism, of the following three kinds.*

1. *The summands of $I(1) \cong 1 \oplus B \oplus 2X$, where $B$ and $X$ are simple and non-isomorphic, $F(B) \cong 1 \oplus m\Sigma_\rho$, $F(X) \cong 2 \cdot 1 \oplus m\Sigma_\rho$, $d(B) = d_m^2$, $d(X) = 1 + d_m^2$ and $\theta_B = \theta_X = 1$.*
2. *The summands of $I(\alpha) \cong I(\alpha^2) \cong Z_1 \oplus Z_2 \oplus Z_3$, where the $Z_i$ are simple and pairwise non-isomorphic, and $F(Z_i) \cong \alpha \oplus \alpha^2 \oplus k_i\Sigma_\rho$ with integers $k_i \geqslant 0$ such that $k_1 + k_2 + k_3 = 3m$.*
3. *Simple objects $Y_j$, $j \in J$, with $F(Y_j) \cong n_j\Sigma_\rho$ for some integers $n_j \geqslant 1$.*

*Proof* : Let $Z$ be a simple object of $Z(\mathcal{F})$. The half-braiding gives $\alpha F(Z) \cong F(Z)\alpha$, so that $F(Z)$ is invariant under conjugation by $\alpha$. By Lemma 6.2, $\alpha(\alpha^j\rho)\alpha^{-1} \cong \alpha^{j+1}\rho\alpha^2 \cong \alpha^{j+2}\rho$, so

that conjugation by $\alpha$ permutes $\rho$, $\alpha\rho$, $\alpha^2\rho$ cyclically. Hence the multiplicities of $\rho$, $\alpha\rho$ and $\alpha^2\rho$ in $F(Z)$ are equal. Consequently, $Z$ is a summand of $I(1)$, of $I(\alpha)$ or of $I(\alpha^2)$, or $F(Z)$ is a multiple of $\Sigma_\rho$.

By [ENO05, Proposition 5.4 p. 21], we have $F(I(V)) \cong \bigoplus_Y Y \otimes V \otimes Y^*$, the sum running over $\mathrm{Irr}(\mathcal{F})$. For $V = 1$, Lemma 6.2 gives $F(I(1)) \cong 3 \cdot 1 \oplus 3\rho^2 \cong 6 \cdot 1 \oplus 3m\Sigma_\rho$. For $V = \alpha$ and $Y = \alpha^j\rho$, we have $Y^* \cong \rho\alpha^{-j}$ and $Y\alpha Y^* \cong \alpha^j\rho\alpha\rho\alpha^{-j} \cong \alpha^{j+2}\rho^2\alpha^{-j} \cong \alpha^2\rho^2$, since $\rho^2\alpha \cong \alpha\rho^2$ by the proof of Lemma 6.2. Hence $F(I(\alpha)) \cong 3\alpha \oplus 3\alpha^2 \oplus 3m\Sigma_\rho$, and the same computation gives $F(I(\alpha^2)) \cong 3\alpha \oplus 3\alpha^2 \oplus 3m\Sigma_\rho$. By adjunction, we obtain

$$\dim \mathrm{End}(I(\alpha)) = \dim \mathrm{End}(I(\alpha^2)) = \dim \mathrm{Hom}(I(\alpha), I(\alpha^2)) = 3,$$
$$\dim \mathrm{Hom}(I(1), I(\alpha)) = \dim \mathrm{Hom}(I(1), I(\alpha^2)) = 0.$$

The first equalities show that $I(\alpha)$ and $I(\alpha^2)$ are both direct sums of three pairwise non-isomorphic simple objects, and that the three summands $Z_1$, $Z_2$, $Z_3$ of $I(\alpha)$ are summands of $I(\alpha^2)$, so that $I(\alpha^2) \cong I(\alpha)$. The second ones show that $F(Z_i)$ does not contain 1, and that $F(Z)$ contains neither $\alpha$ nor $\alpha^2$ for any summand $Z$ of $I(1)$. Since $\dim \mathrm{Hom}(F(Z_i), \alpha) = \dim \mathrm{Hom}(F(Z_i), \alpha^2) = 1$, we get $F(Z_i) \cong \alpha \oplus \alpha^2 \oplus k_i\Sigma_\rho$, and $\sum_i k_i = 3m$ by comparison with $F(I(\alpha))$.

By [Ost15, Theorem 2.13 p. 6], the simple summands of $I(1)$ are in bijection with the irreducible representations $E$ of $K(\mathcal{F})$, the summand $A_E$ attached to $E$ has multiplicity $\dim E$ and dimension $\frac{\dim \mathcal{F}}{f_E}$, where $f_E$ is the formal codegree of $E$. The unit object of $Z(\mathcal{F})$ has multiplicity one in $I(1)$, so that it is $A_E$ for a one-dimensional $E$ with $f_E = \dim \mathcal{F}$. A character $\chi$ of $K(\mathcal{F})$ satisfies $\chi(\alpha)\chi(\rho) = \chi(\rho\alpha) = \chi(\alpha)^2\chi(\rho)$ and $\chi(\rho)^2 = 1 + m\chi(\rho)(1 + \chi(\alpha) + \chi(\alpha)^2)$. The second relation gives $\chi(\rho) \neq 0$, so that $\chi(\alpha) = 1$ and $\chi(\rho) \in \{d_m, -d_m^{-1}\}$. The formal codegrees of these two characters are $\sum_Y \chi(Y)\chi(Y^*) = 3 + 3\chi(\rho)^2$, that is, $\dim \mathcal{F}$ and $3 + 3d_m^{-2}$. The corresponding summands of $I(1)$ are the unit object and a simple object $B$ of dimension $\frac{\dim \mathcal{F}}{3+3d_m^{-2}} = d_m^2$ by (6.2). The summand $X$ attached to the two-dimensional representation has multiplicity two, and $d(I(1)) = \dim \mathcal{F} = 1 + d_m^2 + 2d(X)$ gives $d(X) = 1 + d_m^2$. By adjunction, 1 has multiplicity one in $F(B)$ and two in $F(X)$. Writing $F(B) \cong 1 \oplus s\Sigma_\rho$ and $F(X) \cong 2 \cdot 1 \oplus s'\Sigma_\rho$, the dimensions give $1 + 3sd_m = d_m^2$ and $2 + 3s'd_m = 1 + d_m^2$, so that $s = s' = m$ by Lemma 6.2. Finally $\theta_{I(1)} = \mathrm{id}_{I(1)}$ by [Ost15, Theorem 2.5 p. 4]. ∎

**Lemma 6.4** *In the situation of Lemma 6.3, put* $\Delta \overset{\text{def}}{=} 9m^2+4$ *and* $\varepsilon \overset{\text{def}}{=} \nu_2(\rho)$. *Then* $\varepsilon \in \{1, -1\}$ *and* $k_1 = k_2 = k_3 = m$. *The twists of* $Z_1$, $Z_2$, $Z_3$ *are* $\zeta$, $\zeta\omega$, $\zeta\omega^2$ *for some root of unity* $\zeta$ *with* $\zeta^3 = \nu_3(\alpha)$ *and a primitive cube root of unity* $\omega$. *Moreover*

$$\sum_{j\in J} n_j^2 = 4m^2 + 2, \tag{6.3}$$

$$\sum_{j\in J} n_j^2\theta_{Y_j} = -\frac{m^2 + m\sqrt{\Delta}}{2}, \tag{6.4}$$

$$\sum_{j\in J} n_j^2\theta_{Y_j}^2 = \frac{-m^2 + (2\varepsilon - m)\sqrt{\Delta}}{2}. \tag{6.5}$$

*Proof* : Since $\rho$ is simple and self-dual, $\mathrm{Hom}(1, \rho^{\otimes 2})$ is one-dimensional, so that $\varepsilon \in \{1, -1\}$. Similarly, $\mathrm{Hom}(1, \alpha^{\otimes 3})$ is one-dimensional, so that $|\nu_3(\alpha)| = 1$. Moreover $\nu_1(\alpha) = \nu_1(\rho) = 0$. Put $\ell_i \overset{\text{def}}{=} d(Z_i) = 2 + 3k_id_m > 0$, so that $\sum_i \ell_i = \dim \mathcal{F}$ by (6.2). By (6.1) and Lemma 6.3, we

have

$$\nu_n(\alpha) = \frac{1}{\dim \mathcal{F}} \sum_{i=1}^{3} \theta_{Z_i}^n \ell_i.$$

For $n = 3$, this gives $\sum_i \theta_{Z_i}^3 \ell_i = \nu_3(\alpha) \sum_i \ell_i$ with $|\nu_3(\alpha)| = 1$. Since the $\ell_i$ are positive and the $\theta_{Z_i}$ have modulus one, we obtain $\theta_{Z_1}^3 = \theta_{Z_2}^3 = \theta_{Z_3}^3 = \nu_3(\alpha)$. Write $\theta_{Z_i} = \zeta u_i$ with $\zeta = \theta_{Z_1}$ and $u_i^3 = 1$. For $n = 1$, we obtain $\sum_i u_i \ell_i = 0$. If two of the $u_i$ coincide, then $\sum_i u_i \ell_i$ is a combination with positive coefficients of two cube roots of unity whose quotient is not real, or a positive multiple of a cube root of unity, which is nonzero. Hence the $u_i$ are the three distinct cube roots of unity, and $1 + \omega + \omega^2 = 0$ and the linear independence of 1 and $\omega$ over $\mathbb{R}$ give $\ell_1 = \ell_2 = \ell_3$, that is, $k_i = m$ for all $i$. In particular $d(Z_i) = 2 + 3md_m = 1 + d_m^2$, and $\sum_i \theta_{Z_i}^n = 0$ for $n \in \{1, 2\}$.

Since $d(Z) = d(F(Z))$ is the Frobenius–Perron dimension of $Z$ for every simple object $Z$ of $Z(\mathcal{F})$, we have $\sum_Z d(Z)^2 = (\dim \mathcal{F})^2 = 9(1 + d_m^2)^2$ by [ENO05, Proposition 8.12 p. 32], and $d(Y_j) = 3n_j d_m$. By Lemma 6.3, we obtain

$$9d_m^2 \sum_j n_j^2 = 9(1 + d_m^2)^2 - 1 - d_m^4 - (1 + d_m^2)^2 - 3(1 + d_m^2)^2 = 4d_m^4 + 10d_m^2 + 4,$$

and $\sum_j n_j^2 = \frac{4(d_m^2 + d_m^{-2}) + 10}{9} = 4m^2 + 2$, since $d_m^2 + d_m^{-2} = (d_m - d_m^{-1})^2 + 2 = 9m^2 + 2$.

By (6.1) for $V = \rho$ and $n \in \{1, 2\}$, using $\nu_1(\rho) = 0$, $\nu_2(\rho) = \varepsilon$, Lemma 6.3 and $\sum_i \theta_{Z_i}^n = 0$, we get

$$md_m^2 + m(1 + d_m^2) + 3d_m \sum_j n_j^2 \theta_{Y_j} = 0, \quad md_m^2 + m(1 + d_m^2) + 3d_m \sum_j n_j^2 \theta_{Y_j}^2 = \varepsilon \dim \mathcal{F}.$$

Since $1 + 2d_m^2 = 3 + 6md_m$ and $d_m^{-1} = d_m - 3m$, the first equality gives $\sum_j n_j^2 \theta_{Y_j} = -m(d_m^{-1} + 2m) = m^2 - md_m$, which is (6.4) since $d_m = \frac{3m + \sqrt{\Delta}}{2}$. By (6.2), the second one gives $\sum_j n_j^2 \theta_{Y_j}^2 = \varepsilon(d_m + d_m^{-1}) + m^2 - md_m = \varepsilon\sqrt{\Delta} + m^2 - md_m$, which is (6.5). ∎

The identities (6.3), (6.4) and (6.5) are those of [Hua26b, (25) p. 7]. For $m = 1$ and the Haagerup categories, the objects $Y_j$ are $d_1, \dots, d_6$, with $n_j = 1$ and twists $\mathrm{e}^{\frac{2\pi \mathrm{i} 6l^2}{13}}$ by (2.4) and (2.5), since $d(Y_j) = 3n_j d$. Since $13 \equiv 1 \bmod 4$, and since $12 \equiv -1$ is a square modulo 13 whereas 6 is not, the quadratic Gauss sums give $\sum_{l=1}^{6} \mathrm{e}^{\frac{2\pi \mathrm{i} 6l^2}{13}} = \frac{-1 - \sqrt{13}}{2}$ and $\sum_{l=1}^{6} \mathrm{e}^{\frac{2\pi \mathrm{i} 12l^2}{13}} = \frac{-1 + \sqrt{13}}{2}$, which are (6.4) and (6.5) for $m = 1$ and $\varepsilon = 1$.

## 6.3 Sums of roots of unity

For $n \geqslant 1$, we denote by $r(n)$ the sum of the primitive $n$-th roots of unity, that is, the Möbius function of $n$. Thus $r(n) \in \{0, 1, -1\}$, $r(n) = 0$ if 4 divides $n$, $r(1) = 1$, $r(2) = -1$, and $r(kl) = r(k)r(l)$ if $k$ and $l$ are coprime. We denote by $\varphi$ the Euler function.

Let $D > 1$ be a squarefree integer. For a root of unity $\zeta$, choose $L \geqslant 1$ such that $\zeta^L = 1$ and $\sqrt{D} \in \mathbb{Q}(\mathrm{e}^{\frac{2\pi \mathrm{i}}{L}})$, put $G_L \overset{\text{def}}{=} \mathrm{Gal}(\mathbb{Q}(\mathrm{e}^{\frac{2\pi \mathrm{i}}{L}})/\mathbb{Q}(\sqrt{D}))$ and

$$\langle \zeta \rangle_D \overset{\text{def}}{=} \frac{1}{|G_L|} \sum_{\sigma \in G_L} \sigma(\zeta) = A_D(\zeta) + B_D(\zeta)\sqrt{D}, \quad A_D(\zeta), B_D(\zeta) \in \mathbb{Q}.$$

This does not depend on $L$, since the restriction maps $G_{L'} \to G_L$ for $L \mid L'$ are surjective with fibers of the same cardinality. If $\zeta_1, \dots, \zeta_N$ are roots of unity with $\sum_q \zeta_q = a + b\sqrt{D}$ for rational

numbers $a$, $b$, then averaging over $G_L$ for a common $L$ gives

$$\sum_{q=1}^{N} A_D(\zeta_q) = a, \quad \sum_{q=1}^{N} B_D(\zeta_q) = b. \tag{6.6}$$

**Lemma 6.5** *Let $\zeta$ be a root of unity of order $n$.*

1. *If 40 does not divide $n$, then $\langle\zeta\rangle_{10} = \frac{r(n)}{\varphi(n)}$.*
2. *If $n = 40k$ with $\gcd(k, 40) > 1$, then $\langle\zeta\rangle_{10} = 0$.*
3. *If $n = 40k$ with $\gcd(k, 40) = 1$, then $A_{10}(\zeta) = 0$ and $|B_{10}(\zeta)| = \frac{|r(k)|}{8\varphi(k)}$.*

*Consequently, $|A_{10}(\zeta)| + 8|B_{10}(\zeta)| \leqslant 1$, with equality if and only if $\zeta \in \{1, -1\}$ or $n = 40$. If $\zeta_1, \dots, \zeta_N$ are roots of unity with $\sum_q \zeta_q = a + b\sqrt{10}$ for rational numbers $a$, $b$, then $N \geqslant |a| + 8|b|$.*

*Proof* : The field $\mathbb{Q}(\sqrt{10})$ has discriminant 40, so that $\sqrt{10} \in \mathbb{Q}(\zeta)$ if and only if 40 divides $n$. If $40 \nmid n$, then $\mathbb{Q}(\zeta) \cap \mathbb{Q}(\sqrt{10}) = \mathbb{Q}$, the group $G_L$ maps onto $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, and $\langle\zeta\rangle_{10}$ is the average of the primitive $n$-th roots of unity, which is $\frac{r(n)}{\varphi(n)}$.

Assume that $n = 40k$, and take $L = n$. We identify $\mathrm{Gal}(\mathbb{Q}(\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}})/\mathbb{Q})$ with $(\mathbb{Z}/n\mathbb{Z})^\times$. The subgroup $G_n$ has index two, and we denote by $\chi$ the character of $(\mathbb{Z}/n\mathbb{Z})^\times$ with kernel $G_n$. It factors through $(\mathbb{Z}/40\mathbb{Z})^\times$, since the kernel of the map $(\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/40\mathbb{Z})^\times$ fixes $\mathbb{Q}(\mathrm{e}^{\frac{2\pi\mathrm{i}}{40}})$, which contains $\sqrt{10}$. Put $\tau_n \stackrel{\mathrm{def}}{=} \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} \chi(u)\mathrm{e}^{\frac{2\pi\mathrm{i}u}{n}}$. Writing $\zeta = \mathrm{e}^{\frac{2\pi\mathrm{i}v}{n}}$ with $v \in (\mathbb{Z}/n\mathbb{Z})^\times$, we obtain

$$\langle\zeta\rangle_{10} = \frac{2}{\varphi(n)} \sum_{u\in G_n} \mathrm{e}^{\frac{2\pi\mathrm{i}uv}{n}} = \frac{1}{\varphi(n)} \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} (1 + \chi(u))\mathrm{e}^{\frac{2\pi\mathrm{i}uv}{n}} = \frac{r(n) + \chi(v)\tau_n}{\varphi(n)} = \frac{\chi(v)\tau_n}{\varphi(n)},$$

since $r(n) = 0$. For $\sigma \notin G_n$ we have $\sigma(\tau_n) = -\tau_n$, so that $\tau_n \in \mathbb{Q}\sqrt{10}$ and $A_{10}(\zeta) = 0$.

If $\gcd(k, 40) > 1$, choose a prime $p$ dividing $k$ and 40. Then $p$ divides $\frac{n}{p}$ and 40 divides $\frac{n}{p}$. Hence $u \mapsto u + \frac{n}{p}$ permutes $(\mathbb{Z}/n\mathbb{Z})^\times$ and preserves $\chi$, and it multiplies $\mathrm{e}^{\frac{2\pi\mathrm{i}u}{n}}$ by $\mathrm{e}^{\frac{2\pi\mathrm{i}}{p}} \neq 1$, so that $\tau_n = 0$. This proves part 2.

Assume that $\gcd(k, 40) = 1$. By the Chinese remainder theorem, $\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}} = \mathrm{e}^{\frac{2\pi\mathrm{i}x}{40}}\mathrm{e}^{\frac{2\pi\mathrm{i}y}{k}}$ with $x \in (\mathbb{Z}/40\mathbb{Z})^\times$ and $y \in (\mathbb{Z}/k\mathbb{Z})^\times$, and $(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/40\mathbb{Z})^\times \times (\mathbb{Z}/k\mathbb{Z})^\times$. Hence

$$\tau_n = \Big(\sum_{u\in(\mathbb{Z}/40\mathbb{Z})^\times} \chi(u)\mathrm{e}^{\frac{2\pi\mathrm{i}xu}{40}}\Big)\Big(\sum_{u\in(\mathbb{Z}/k\mathbb{Z})^\times} \mathrm{e}^{\frac{2\pi\mathrm{i}yu}{k}}\Big) = \chi(x)\tau_{40}r(k).$$

Let $s_+$ and $s_-$ be the sums of the primitive 40-th roots of unity in the two cosets of $G_{40}$, so that $\tau_{40} = \pm(s_+ - s_-)$. They are fixed by $G_{40}$ and exchanged by the other elements of the Galois group. By [Lar14, proof of Theorem 6.2, Case 2, p. 18], applied with $c = 10$, we have $\{s_+, s_-\} = \{\sqrt{10}, -\sqrt{10}\}$. Consequently, $|\tau_{40}| = 2\sqrt{10}$, and $|B_{10}(\zeta)|\sqrt{10} = \frac{2\sqrt{10}|r(k)|}{\varphi(40)\varphi(k)} = \frac{2\sqrt{10}|r(k)|}{16\varphi(k)}$, which is part 3.

By parts 1, 2 and 3, $|A_{10}(\zeta)| + 8|B_{10}(\zeta)|$ is $\frac{|r(n)|}{\varphi(n)}$, 0 or $\frac{|r(k)|}{\varphi(k)}$ in the three cases. This is at most 1, with equality if and only if $n \in \{1, 2\}$ in the first case and $k = 1$ in the third case, since $k$ is then odd. The last assertion follows from (6.6), since $|a| + 8|b| \leqslant \sum_q (|A_{10}(\zeta_q)| + 8|B_{10}(\zeta_q)|) \leqslant N$. ∎

For a general squarefree integer $D > 1$ and rational numbers $a$, $b$ with $2a, 2b \in \mathbb{Z}$, a sum of $N$ roots of unity equal to $a + b\sqrt{D}$ satisfies

$$N \geqslant |b|\varphi(2D). \tag{6.7}$$

This follows from [Lar14, Theorem 1.3 p. 2 and Theorem 6.2 p. 18], applied to the sum of $2N$ roots of unity obtained by repeating each root twice, which is $2a + 2b\sqrt{D}$ with $2a, 2b \in \mathbb{Z}$.

## 6.4 The multiplicity

**Proposition 6.6** *In Lemma 6.2, we have $m = 1$.*

*Proof* : Write $\Delta = 9m^2 + 4 = t^2 D$ with $t \geqslant 1$ and $D$ squarefree. Since $(3m)^2 < \Delta < (3m+1)^2$, we have $D > 1$. Since $\Delta \equiv 1 \bmod 3$, the integer $t$ is prime to 3 and $D \equiv 1 \bmod 3$. An odd prime $p$ dividing $D$ divides $(3m)^2 + 4$ and does not divide $3m$, so that $-1$ is a square modulo $p$ and $p \equiv 1 \bmod 4$. Hence every prime factor of $D$ is 2 or congruent to 1 modulo 4, and $D \notin \{2, 5\}$ since $D \equiv 1 \bmod 3$. Write $D = 2^e \prod_i p_i$ with distinct odd primes $p_i$ and $e \in \{0, 1\}$. Then

$$\frac{\varphi(2D)}{\sqrt{D}} = 2^{\frac{e}{2}} \prod_i \frac{p_i - 1}{\sqrt{p_i}},$$

with $\frac{p-1}{\sqrt{p}} \geqslant \frac{4}{\sqrt{5}} > 1$ for $p \geqslant 5$ and $\frac{p-1}{\sqrt{p}} \geqslant \frac{12}{\sqrt{13}} > 3$ for $p \geqslant 13$. If some $p_i$ is at least 13, the product is at least $\frac{12}{\sqrt{13}} > 3$, since the other factors are at least 1. Otherwise every $p_i$ equals 5, so that $D \in \{2, 5, 10\}$, and $D = 10$. Hence $\frac{\varphi(2D)}{\sqrt{D}} > 3$ unless $D = 10$.

Consider the family of $N = 4m^2 + 2$ roots of unity in which each twist $\theta_{Y_j}$ of Lemma 6.4 is repeated $n_j^2$ times. By (6.3) and (6.4), its sum is $a + b\sqrt{D}$ with $a = -\frac{m^2}{2}$ and $b = -\frac{mt}{2}$. Note that $t\sqrt{D} = \sqrt{\Delta} > 3m$.

Assume that $D \neq 10$. By (6.7), we have $4m^2 + 2 \geqslant \frac{mt}{2}\varphi(2D) = \frac{m}{2} t\sqrt{D} \frac{\varphi(2D)}{\sqrt{D}} > \frac{9m^2}{2}$, so that $m^2 < 4$ and $m = 1$.

Assume that $D = 10$. By Lemma 6.5, we have $4m^2 + 2 \geqslant \frac{m^2}{2} + 4mt > \frac{m^2}{2} + \frac{12m^2}{\sqrt{10}}$, so that $m^2 < 7$. Since $D = 13$ for $m = 1$, we get $m = 2$ and $t = 2$. Then $N = 18 = |a| + 8|b|$. By (6.6), we have $\sum_q (-A_{10}(\zeta_q) - 8B_{10}(\zeta_q)) = 18$, and each term is at most 1 by Lemma 6.5. Hence $-A_{10}(\zeta_q) - 8B_{10}(\zeta_q) = 1$ for every root $\zeta_q$ of the family. Since this quantity is $-1$ for $\zeta_q = 1$, Lemma 6.5 shows that each $\zeta_q$ is equal to $-1$ or is a primitive 40-th root of unity, and that $A_{10}(\zeta_q) = 0$ in the second case. Since $\sum_q A_{10}(\zeta_q) = a = -2$ by (6.6), the root $-1$ occurs exactly twice in the family. Hence the family of the squares $\zeta_q^2$ consists of 1, twice, and of 16 primitive 20-th roots of unity, and $A_{10}(\zeta_q^2) = \frac{r(20)}{\varphi(20)} = 0$ for the latter by part 1 of Lemma 6.5. By (6.5), the sum of this family is $\frac{-4+(2\varepsilon-2)\sqrt{40}}{2} = -2 + 2(\varepsilon - 1)\sqrt{10}$. Comparing the rational parts by (6.6), we obtain $-2 = \sum_q A_{10}(\zeta_q^2) = 2$, a contradiction. This excludes $m = 2$. ∎

## 6.5 Pointed parts and proof of the recognition theorem

**Lemma 6.7** *Let $\mathcal{P}$ be a fusion category with three simple objects, all invertible, so that $\mathcal{P}$ is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}^{\varpi}$ for some 3-cocycle $\varpi$.*

1. *If a left $\mathcal{P}$-module category $\mathcal{M}$ contains a simple object $M$ with $g \otimes M \cong M$ for every invertible object $g$ of $\mathcal{P}$, then $\varpi$ is a coboundary.*

*2. If $\mathcal{P}$ has two inequivalent indecomposable left module categories, then $\varpi$ is a coboundary.*

*Proof* : 1. The full subcategory of $\mathcal{M}$ whose objects are the multiples of $M$ is stable under the action of $\mathcal{P}$, and it is equivalent to Vec. A structure of $\mathcal{P}$-module category on Vec is the same thing as a fiber functor $\mathcal{P} \to \mathrm{Vec}$ [Ost03, Example (ii) pp. 6–7]. The monoidal structure of a fiber functor on $\mathrm{Vec}_{\mathbb{Z}_3}^{\varpi}$ consists of nonzero scalars $j(g,h)$, after a choice of basis vectors of the lines attached to the simple objects, and its compatibility with the associators says that $\varpi$ is the coboundary of $j$.

2. Let $\mathcal{M}$ be an indecomposable $\mathcal{P}$-module category. The invertible objects permute the isomorphism classes of simple objects of $\mathcal{M}$. The full subcategory whose objects are the direct sums of simple objects in one orbit is a module subcategory and a direct summand of $\mathcal{M}$, so that the action is transitive, with orbits of cardinality one or three. If a simple object is fixed, then $\varpi$ is a coboundary by part 1. Otherwise the action is free. Let $M$ be a simple object of $\mathcal{M}$, and let $\underline{\mathrm{Hom}}(M,M)$ be its internal Hom, characterized by $\mathrm{Hom}_{\mathcal{P}}(Y, \underline{\mathrm{Hom}}(M,M)) \cong \mathrm{Hom}_{\mathcal{M}}(Y \otimes M, M)$ [Ost03, Definition 11 p. 9]. For $Y = g$ invertible, this space is nonzero if and only if $g \cong 1$, so that $\underline{\mathrm{Hom}}(M,M) \cong 1$ as an object. Its algebra structure is then isomorphic to the trivial algebra 1, and by [Ost03, Theorem 1 and its proof pp. 10–11], $\mathcal{M}$ is equivalent to the category of modules over 1, that is, to the regular module category $\mathcal{P}$. Hence, if $\varpi$ is not a coboundary, every indecomposable $\mathcal{P}$-module category is equivalent to $\mathcal{P}$. ∎

We now prove Theorem 6.1.

*Proof* : Part 1 follows from Lemma 6.2 and Proposition 6.6, with $d(\rho) = d_1 = d$.

We next prove the converse statement. The categories $\mathcal{H}_2$ and $\mathcal{H}_3$ are unitary fusion categories with the fusion rules (2.2) [GrS12, p. 14, Theorem 1.4 p. 4 and Corollary 3.24 p. 24], the category $\mathcal{H}_3$ being the category of bimodules over a Q-system in $\mathcal{H}_2$ [GrS12, Lemma 3.14 p. 21 and Definition 3.19 p. 23]. They have six simple objects, three of which are invertible, and their Grothendieck rings are isomorphic and noncommutative since $\xi\alpha \cong \alpha^2\xi \ncong \alpha\xi$ in $\mathcal{H}_2$. The pointed part of $\mathcal{H}_2$ has trivial associator by [GrS12, Lemma 2.21 p. 15]. For $\mathcal{H}_3$, let $A \overset{\text{def}}{=} 1 \oplus \alpha \oplus \alpha^2$ with this algebra structure, so that $\mathcal{H}_3$ is the category of $A$-$A$-bimodules in $\mathcal{H}_2$. The category $\mathcal{M}$ of left $A$-modules in $\mathcal{H}_2$ is a left $\mathcal{H}_3$-module category for the relative tensor product $\otimes_A$ and a right $\mathcal{H}_2$-module category, and the two actions commute. Since the Q-system $A$ is separable, the category $\mathcal{M}$ is semisimple [DMNO13, Proposition 2.7 p. 5]. For objects $Y$, $Y'$ of $\mathcal{H}_2$, we have $\mathrm{Hom}_A(A \otimes Y, A \otimes Y') \cong \mathrm{Hom}(Y, A \otimes Y')$, and every simple left $A$-module is a quotient, hence a summand, of some $A \otimes Y$ with $Y$ simple [Ost03, Lemma 4 p. 9]. By (2.2), we deduce that the simple objects of $\mathcal{M}$ are $A \cong A \otimes \alpha \cong A \otimes \alpha^2$ and $A \otimes \xi \cong A \otimes \alpha\xi \cong A \otimes \alpha^2\xi$, up to isomorphism. Moreover $(A \otimes \xi) \otimes \xi \cong A \oplus 3(A \otimes \xi)$ is not simple. Hence $A$ is, up to isomorphism, the only simple object $M$ of $\mathcal{M}$ such that $M \otimes \xi$ is simple. For an invertible object $g$ of $\mathcal{H}_3$, the functor $g \otimes_A -$ is an autoequivalence of $\mathcal{M}$ commuting with the right action of $\mathcal{H}_2$, so that $(g \otimes_A A) \otimes \xi \cong g \otimes_A (A \otimes \xi)$ is simple and $g \otimes_A A \cong A$. By part 1 of Lemma 6.7, the pointed part of $\mathcal{H}_3$ has trivial associator.

We finally prove part 2. By part 1, $\mathcal{F}$ is a quadratic category with $(\mathbb{Z}_3, -1, 1)$ [Izu18, Definition 2.1 p. 6]. Its invariant $c_{0,3}(\mathcal{F}) \in H^3(\mathbb{Z}_3, \mathbb{C}^\times)$ is the class of the associator of its pointed part [Izu18, p. 9], which vanishes by assumption. The invariant $c_{1,2}(\mathcal{F})$ belongs to $H^1(\mathbb{Z}_2, H^2(\mathbb{Z}_3, \mathbb{C}^\times))$ [Izu18, pp. 10–11], which is zero since $H^2(\mathbb{Z}_3, \mathbb{C}^\times) = 0$. Hence $\mathcal{F}$ is a generalized Haagerup category with group $\mathbb{Z}_3$ [Izu18, Definition 2.7 p. 12] which is a $C^*$-fusion category. For $G = \mathbb{Z}_3$, the group $\Gamma = (H^2(G, \mathbb{T}) \times G/2G) \rtimes \mathrm{Aut}(G)$ of [Izu18, Theorem 5.9 p. 36] is $\mathrm{Aut}(\mathbb{Z}_3)$, and since $G$ is odd one may take the signs $\varepsilon_h(g)$ of [Izu18] equal to 1 [Izu18, p. 56]. Hence the number of $\Gamma$-orbits of gauge equivalence classes of solutions in [Izu18, Theorem 5.9 p. 36] is at most the number of solutions up to group automorphism in [Izu18, Theorem 9.1 p. 57], and there are at most two $C^*$-generalized Haagerup categories with group $\mathbb{Z}_3$ up to

equivalence. By the converse statement, $\mathcal{H}_2$ and $\mathcal{H}_3$ are two such categories, and they are not equivalent by part 1 of Proposition 2.1. Hence $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$. ■

## 6.6 The associator of the pointed part and the twists

**Lemma 6.8** *Let $\mathcal{P}$ be a fusion category with three simple objects, all invertible, equipped with the spherical structure for which all simple objects have dimension 1, and let $\alpha$ be a non-trivial simple object. Then $\nu_3(\alpha) = 1$ if the associator of $\mathcal{P}$ is trivial, and $\nu_3(\alpha)$ is a primitive cube root of unity otherwise.*

*Proof* : The category $\mathcal{P}$ is pseudo-unitary, since $\dim \mathcal{P} = \mathrm{FPdim}\, \mathcal{P} = 3$. Hence its spherical structure with positive dimensions is unique [ENO05, Proposition 8.23 p. 34]. By [NgS07b, Corollary 4.4 p. 11], Frobenius–Schur indicators are invariant under tensor equivalences which preserve the pivotal structures, and every tensor equivalence between such pointed categories preserves the spherical structures with positive dimensions. Hence we may assume that $\mathcal{P}$ is the category $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, \omega_t)$ of [NgS07a, p. 30] with the pivotal structure of dimension 1 described there, where $t \in \{0, 1, 2\}$ and the class of $\omega_t$ is trivial if and only if $t = 0$, and that $\alpha$ is the object $V_1$ or $V_2$. By [NgS07a, p. 31], we have $\nu_3(V_1) = \mathrm{e}^{\frac{2\pi i t}{3}}$. Since $V_2 \cong V_1^*$, we also have $\nu_3(V_2) = \nu_3(V_1)$ [NgS07a, Remark 4.2 p. 10]. Hence $\nu_3(\alpha) = \mathrm{e}^{\frac{2\pi i t}{3}}$, which is equal to 1 if and only if $t = 0$. ■

**Proposition 6.9** *Let $\mathcal{F}$ be as in Theorem 6.1, and let $Z_1$, $Z_2$, $Z_3$ be the simple summands of $I(\alpha)$ (Lemma 6.3). Then $\theta_{Z_i}^3 = \nu_3(\alpha)$ for $i \in \{1, 2, 3\}$.*

1. *If the fusion subcategory generated by $\alpha$ has trivial associator, then $\theta_{Z_i}^3 = 1$, and $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$.*
2. *Otherwise the twists $\theta_{Z_i}$ are primitive ninth roots of unity. In particular, $Z(\mathcal{F})$ is not braided equivalent to $Z(\mathcal{H}_3)$.*

*Consequently, $\mathcal{F}$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$ if and only if no simple object of $Z(\mathcal{F})$ has a twist of order nine.*

*Proof* : The equality $\theta_{Z_i}^3 = \nu_3(\alpha)$ is proved in Lemma 6.4. The indicator $\nu_3(\alpha)$ is computed in the fusion subcategory $\mathcal{P}$ generated by $\alpha$, since $\mathrm{Hom}(1, \alpha^{\otimes 3})$ and the map $E_\alpha^{(3)}$ only involve objects of $\mathcal{P}$ [NgS07a, p. 31], and the spherical structure of $\mathcal{F}$ restricts to the spherical structure of $\mathcal{P}$ with dimensions 1. Part 1 then follows from Lemma 6.8 and part 2 of Theorem 6.1. In the situation of part 2, $\nu_3(\alpha)$ is a primitive cube root of unity by Lemma 6.8, so that $\theta_{Z_i}^9 = 1 \neq \theta_{Z_i}^3$. A braided equivalence between pseudo-unitary modular tensor categories preserves the spherical structures with positive dimensions [ENO05, Proposition 8.23 p. 34], hence the twists. By (2.5), the twists of $Z(\mathcal{H}_3)$ have order 1, 3 or 13, which proves part 2. The last assertion follows from parts 1 and 2, since $Z(\mathcal{H}_2)$ and $Z(\mathcal{H}_3)$ are braided equivalent to $Z(\mathcal{H}_3)$ (Section 2.1). ■

## 6.7 Application to the conformal nets

**Lemma 6.10** *Under the hypotheses of Theorem 5.6, let $x \in \{a, c_0\}$. Then the pointed fusion subcategory $\mathcal{P}_x$ of $\mathcal{F}_x$ is equivalent to $\mathrm{Vec}_{\mathbb{Z}_3}$ with trivial associator.*

*Proof* : By part 4 of Theorem 5.6, part 1 of Proposition 2.2 and Lemma 3.1, there is a braided equivalence $\mathrm{Rep}(\mathcal{A}_U) \cong Z(\mathcal{F}_x)$ under which $\Theta_x$ corresponds to $I(1)$. By the proof of

part 4 of Theorem 5.6, the connected étale subalgebra $\theta = \pi_0 \oplus \pi_b$ of $\Theta_x$ corresponds to the étale subalgebra $A_x$ of $I(1)$ attached to $\mathcal{P}_x$ by [DMNO13, Theorem 4.10 p. 18]. By part 5 of Proposition 2.2, the category of local modules $Z(\mathcal{F}_x)^0_{A_x}$ is braided equivalent to $Z(\mathcal{P}_x)$. On the other hand, the category of local $\theta$-modules $\mathrm{Rep}(\mathcal{A}_U)^0_\theta$ is braided equivalent to $\mathrm{Rep}(\mathcal{A}_{V'})$ [BKL15, Proposition 6.4 p. 33], see also [Müg10, Theorem 3.1 p. 2]. Hence there is a braided equivalence $\Psi_x \colon \mathrm{Rep}(\mathcal{A}_{V'}) \to Z(\mathcal{P}_x)$. If the braiding conventions of [BKL15, Proposition 6.4 p. 33] and of part 5 of Proposition 2.2 differ, then $\Psi_x$ is a braided equivalence onto $Z(\mathcal{P}_x)^{\mathrm{rev}}$. This does not affect the argument below, since an algebra which is commutative for a braiding is commutative for the reverse braiding, so that $Z(\mathcal{P}_x)$ and $Z(\mathcal{P}_x)^{\mathrm{rev}}$ have the same Lagrangian algebras.

The net $\mathcal{A}_{V'}$ is a covariant subnet of $\mathcal{A}_{H_{\Lambda_a}}$ and of $\mathcal{A}_{H_{\Lambda_{c_0}}}$ by Theorem 4.5. Let $y \in \{a, c_0\}$. The inclusion $\mathcal{A}_{V'} \subset \mathcal{A}_{H_{\Lambda_y}}$ is irreducible by Theorem 4.5, and of finite index by part 2 of Proposition 4.3. By part 3 of Proposition 2.5, its dual canonical endomorphism $\Theta'_y$, which is the restriction of the vacuum representation of $\mathcal{A}_{H_{\Lambda_y}}$ to $\mathcal{A}_{V'}$, is a connected étale algebra in $\mathrm{Rep}(\mathcal{A}_{V'})$. By part 3 of Proposition 2.5 and part 2 of Proposition 4.3, we have $d(\Theta'_y) = [\mathcal{A}_{H_{\Lambda_y}} : \mathcal{A}_{V'}] = 3$ and $\mu_{\mathcal{A}_{V'}} = 9 = d(\Theta'_y)^2$, so that $\Theta'_y$ is Lagrangian by (2.10). The restriction of $\Theta'_y$ to $\mathcal{A}_U$ is the restriction of the vacuum representation of $\mathcal{A}_{H_{\Lambda_y}}$, that is, $\pi_0 \oplus \pi_b \oplus 2\pi_y$ by part 2 of Theorem 5.6. Since $\pi_a$ and $\pi_{c_0}$ are inequivalent, the algebras $\Theta'_a$ and $\Theta'_{c_0}$ are not isomorphic. Hence $\Psi_x(\Theta'_a)$ and $\Psi_x(\Theta'_{c_0})$ are two non-isomorphic Lagrangian algebras in $Z(\mathcal{P}_x)$. By part 2 of Proposition 2.2, which gives a bijection between the isomorphism classes of Lagrangian algebras of $Z(\mathcal{P}_x)$ and the equivalence classes of indecomposable $\mathcal{P}_x$-module categories, the category $\mathcal{P}_x$ has two inequivalent indecomposable module categories. We conclude with part 2 of Lemma 6.7. ∎

**Theorem 6.11** *Let $U \subset V'$ be as in Theorem 5.6, and let $x \in \{a, c_0\}$. Then the fusion category $\mathcal{F}_x$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$, and $\delta = d^2$. Consequently,* $\mathrm{Rep}(\mathcal{A}_U)$ *is equivalent to $Z(\mathcal{H}_3)$ as a unitary modular tensor category. In particular,* ($\mathrm{H_J}$) *implies* ($\mathrm{N_0}$).

*Proof* : By part 4 of Theorem 5.6 and Proposition 5.8, the category $\mathcal{F}_x$ is a unitary fusion category with six simple objects, three of which are invertible. By part 2 of Theorem 5.6, we have $\Theta_x \cong \pi_0 \oplus \pi_b \oplus 2\pi_x$ with pairwise inequivalent irreducible summands, so that $\mathrm{Hom}(\Theta_x, \Theta_x) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$. Since $\Theta_x$ corresponds to $I(1)$ (Lemma 3.1), and since $K(\mathcal{F}_x) \otimes \mathbb{C}$ is isomorphic to $\mathrm{End}(I(1))$ for the spherical category $\mathcal{F}_x$ [Ost15, Corollary 2.16 p. 7], see also [BiJ22, Corollary 3.4 p. 16], the Grothendieck ring of $\mathcal{F}_x$ is noncommutative. The fusion subcategory $\mathcal{P}_x$ is pointed of dimension three, so that its three simple objects are invertible, and $\mathcal{F}_x$ has three invertible simple objects by Proposition 5.8. Hence $\mathcal{P}_x$ is the fusion subcategory generated by the invertible objects, and it has trivial associator by Lemma 6.10. By Theorem 6.1, $\mathcal{F}_x$ is equivalent to $\mathcal{H}_2$ or to $\mathcal{H}_3$. Comparing $\mathrm{FPdim}(\mathcal{F}_x) = 3(1 + \delta)$ with (2.3), we obtain $\delta = d^2$. By part 4 of Theorem 5.6 and Section 2.1, $\mathrm{Rep}(\mathcal{A}_U)$ is braided equivalent to $Z(\mathcal{F}_x)$, hence to $Z(\mathcal{H}_3)$, and such an equivalence is naturally isomorphic to a unitary braided equivalence [Reu23, Theorem 5 p. 2]. Under ($\mathrm{H_J}$), these results apply to $U = S$ by Corollary 5.9. ∎

**Remark 6.12** For the algebra of [Hua26b], the fusion rules of part 1 of Theorem 6.1, the triviality of the associator of the pointed part and the identification with $\mathcal{H}_2$ or $\mathcal{H}_3$ are stated in [Hua26b, Sections 5 and 6 pp. 7–9], with an outline of proof, and the proofs of this section follow that outline. The outline leaves the following points implicit or only sketched.

1. The free actions of the invertible objects (Lemma 6.2), which are stated in [Hua26b, p. 7], and the self-duality of the non-invertible simple objects, which is implicit in [Hua26b, (24) p. 7].

2. The structure of $Z(\mathcal{F})$ (Lemma 6.3) and the equalities $k_1 = k_2 = k_3 = m$, which are obtained from $\nu_1(\alpha)$ and $\nu_3(\alpha)$ (Lemma 6.4). The identities [Hua26b, (25) p. 7], attributed there to the first two Frobenius–Schur indicators, follow from them.

3. The restrictions on the squarefree part $D$ of $9m^2 + 4$ in the proof of Proposition 6.6. For $D \neq 10$, the second inequality displayed in [Hua26b, p. 7] amounts to $\varphi(2D) \geqslant \frac{12\sqrt{D}}{\sqrt{13}}$. It holds for the values of $D$ that can occur, but not for every $D \neq 10$, for instance not for $D \in \{2, 5, 7\}$. The value $D = 7$ is excluded since the odd prime factors of $D$ are congruent to 1 modulo 4, and the values $D \in \{2, 5\}$ are excluded by the congruence $D \equiv 1 \bmod 3$.

4. The bound of Lemma 6.5 for $D = 10$, which is asserted in [Hua26b, p. 7], and the analysis of the case $m = 2$, which is sketched in [Hua26b, pp. 7–8].

5. The triviality of the associator of the pointed part. It is derived in [Hua26b, p. 7] from the transversality of the two isotropic lines, by induction from the second one, and it is used in [Hua26b, pp. 8–9]. Lemma 6.10 uses the same input, the second lattice extension, through the module categories of $\mathcal{P}_x$.

6. The converse statement of Theorem 6.1, which is needed to deduce from [Izu18, Theorems 5.9 and 9.1 pp. 36 and 57] that the category is $\mathcal{H}_2$ or $\mathcal{H}_3$.

The use of [Hua26b] in Section 5 is different, see Remark 5.10.

# 7 Consequences for the net of $S$, complete unitarity and (N)

In this section we assume ($\mathrm{H_J}$), and we use the notation of Section 5 for $U = S$. In particular, $\mathcal{A}_S$ is completely rational by Corollary 5.9. We consider the two assertions

($\mathrm{N_0}$) the unitary modular tensor category $\mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to $Z(\mathcal{H}_3)$,

(N) there exists a braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_S) \to \mathrm{Rep}(S)$ with $\Phi(\pi_0) \cong S$ and $\Phi(\pi_M) \cong M$ for $M \in \{M_b, M_a, M_{c_0}\}$.

Under ($\mathrm{H_J}$), the category $\mathrm{Rep}(S)$ is braided equivalent to $Z(\mathcal{H}_3)$, so that (N) implies ($\mathrm{N_0}$). By Theorem 6.11, ($\mathrm{N_0}$) holds under ($\mathrm{H_J}$). We prove (N) under ($\mathrm{H_J}$) in Theorem 7.17, at the end of this section. Its proof uses Theorem 4.5, Lemma 4.7, Propositions 7.12 and 7.16 and the results of Gui [Gui26, Gui25], but neither Theorem 6.11 nor any statement of this section which assumes (N) or the complete unitarity of $S$. Since (N) implies ($\mathrm{N_0}$) under ($\mathrm{H_J}$), Theorem 7.17 gives a second, computer-assisted, proof of ($\mathrm{N_0}$), independent of Section 6.

## 7.1 The representation category and the holomorphic extensions

**Lemma 7.1** *Assume* ($\mathrm{H_J}$)*, and let* $\Psi\colon \mathrm{Rep}(\mathcal{A}_S) \to Z(\mathcal{H}_3)$ *be a braided equivalence, which exists by Theorem 6.11. Then* $\Psi(\pi_b) \cong b$, $\{\Psi(\pi_a), \Psi(\pi_{c_0})\} = \{a, c_0\}$ *up to isomorphism, and* $\delta = d^2 = 3d + 1$*. Moreover,* $\Psi$ *is unique up to natural isomorphism and naturally isomorphic to a unitary braided equivalence. Consequently,* (N) *holds if and only if* $\Psi(\pi_a) \cong a$*.*

*Proof* : The uniqueness follows from Section 2.1, and the unitarity from [Reu23, Theorem 5 p. 2]. In the unitary category $\mathrm{Rep}(\mathcal{A}_S)$, the Frobenius–Perron dimension of an irreducible sector is its statistical dimension, and $\mathrm{FPdim}\,\mathrm{Rep}(\mathcal{A}_S) = \mu_{\mathcal{A}_S}$ by (2.10). By the existence of

$\Psi$, (2.6) and part 3 of Theorem 5.6, we obtain $9(1+\delta)^2 = 9(3d+2)^2$, that is, $\delta = 3d+1 = d^2$. The simple object $\Psi(\pi_b)$ has dimension $3d+1$, so that $\Psi(\pi_b) \cong b$ by (2.4). For $x \in \{a, c_0\}$, the object $\Psi(\pi_x)$ is a simple summand of dimension $3d+2$ of the Lagrangian algebra $\Psi(\Theta_x)$, hence it belongs to $\{a, c_0\}$ by Lemma 3.2. The objects $\Psi(\pi_a)$ and $\Psi(\pi_{c_0})$ are not isomorphic since $\pi_a$ and $\pi_{c_0}$ are inequivalent by Theorem 5.6. Finally, under ($\mathrm{H_J}$), the braided equivalence $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$ is unique up to natural isomorphism, and it maps $M_z$ to $z$. A braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_S) \to \mathrm{Rep}(S)$ is therefore isomorphic to the composition of $\Psi$ with the inverse of this equivalence, and $\Phi(\pi_b) \cong M_b$ by the first part. Hence (N) amounts to $\Psi(\pi_a) \cong a$. ∎

**Theorem 7.2** *Assume* ($\mathrm{H_J}$).

1. $\mathrm{Rep}(\mathcal{A}_S)$ *is equivalent to* $Z(\mathcal{H}_3)$ *as a unitary modular tensor category. We have* $d(\pi_b) = 3d+1$, $d(\pi_a) = d(\pi_{c_0}) = 3d+2$, $[\mathcal{A}_{V'} : \mathcal{A}_S] = 3d+2$, $[\mathcal{A}_{E_8} : \mathcal{A}_S] = 3(3d+2)$ *for both lattice extensions, and* $\mu_{\mathcal{A}_S} = 9(3d+2)^2$.

2. *Up to isomorphism of extensions,* $\mathcal{A}_S$ *has exactly three irreducible local extensions* $\mathcal{B}$ *with* $\mu_{\mathcal{B}} = 1$. *Their dual canonical endomorphisms* $\Theta_1$, $\Theta_2$, $\Theta_3$ *satisfy* $\mathrm{Rep}(\mathcal{A}_S)_{\Theta_i} \cong \mathcal{H}_i$, $\Theta_1 \cong \pi_0 \oplus \pi_b \oplus \pi_a \oplus \pi_{c_0}$ *and*

$$\{\Theta_2, \Theta_3\} = \{\Theta_a, \Theta_{c_0}\}, \quad \Theta_x \cong \pi_0 \oplus \pi_b \oplus 2\pi_x.$$

   *The extensions attached to* $\Theta_a$ *and* $\Theta_{c_0}$ *are the nets* $\mathcal{A}_{H_{\Lambda_a}}$ *and* $\mathcal{A}_{H_{\Lambda_{c_0}}}$ *of the lattice extensions, both isomorphic to* $\mathcal{A}_{E_8}$.

3. *The only intermediate nets between* $\mathcal{A}_S$ *and the extension* $\mathcal{B}_1$ *attached to* $\Theta_1$ *are* $\mathcal{A}_S$ *and* $\mathcal{B}_1$. *For* $x \in \{a, c_0\}$, *the only intermediate nets between* $\mathcal{A}_S$ *and* $\mathcal{A}_{H_{\Lambda_x}}$ *are* $\mathcal{A}_S$, $\mathcal{A}_{V'}$ *and* $\mathcal{A}_{H_{\Lambda_x}}$.

4. *Assume* (N). *Then* $\Theta_2 = \Theta_a$ *and* $\Theta_3 = \Theta_{c_0}$, *that is, the net* $\mathcal{A}_{H_{\Lambda_a}}$ *carries* $\mathcal{H}_2$ *and the net* $\mathcal{A}_{H_{\Lambda_{c_0}}}$ *carries* $\mathcal{H}_3$, *as the vertex operator algebras* $H_{\Lambda_a}$ *and* $H_{\Lambda_{c_0}}$ *do.*

*Proof* : 1. This follows from Lemma 7.1 and part 3 of Theorem 5.6.

2. Let $\mathcal{B}$ be an irreducible local extension of $\mathcal{A}_S$, that is, $\mathcal{A}_S(I)' \cap \mathcal{B}(I) = \mathbb{C}$. Then $\mathcal{A}_S(S^1)' \cap \mathcal{B}(I) = \mathbb{C}$, so that $[\mathcal{B} : \mathcal{A}_S] < \infty$ by part 2 of Proposition 2.5, and $\mathcal{B}$ is completely rational with $\mu_{\mathcal{B}} = \mu_{\mathcal{A}_S}/d(\Theta_{\mathcal{B}})^2$ by part 1 of the same proposition, where $\Theta_{\mathcal{B}}$ is the dual canonical endomorphism. Hence $\mu_{\mathcal{B}} = 1$ if and only if the connected étale algebra $\Theta_{\mathcal{B}}$ is Lagrangian in $\mathrm{Rep}(\mathcal{A}_S)$. Irreducible local extensions of $\mathcal{A}_S$ up to isomorphism correspond to commutative Q-systems $\Theta$ with $\dim \mathrm{Hom}(\mathrm{id}, \Theta) = 1$ up to unitary equivalence [LoR95, Theorem 4.9 p. 25], [BKLR15, Section 5.2.1 p. 77], see also [Lon03, Lemma 15 p. 14]. By [CGGH23, Theorem 3.2 p. 11 and Theorem 3.9 p. 15], the latter correspond to connected étale algebras up to isomorphism. By Lemma 7.1 and Propositions 3.3, 3.5 and 3.6, there are exactly three Lagrangian algebras in $\mathrm{Rep}(\mathcal{A}_S)$, with the stated categories of modules and objects. Their objects are pairwise non-isomorphic. By part 2 of Theorem 5.6, $\Theta_x \cong \pi_0 \oplus \pi_b \oplus 2\pi_x$ is the dual canonical endomorphism of $\mathcal{A}_S \subset \mathcal{A}_{H_{\Lambda_x}}$, and by part 4 of the same theorem it is Lagrangian. The nets $\mathcal{A}_{H_{\Lambda_x}}$ are isomorphic to $\mathcal{A}_{E_8}$ by Theorem 4.5.

3. By [BKLR15, Proposition 4.11 p. 49], an intermediate net gives, at a fixed interval, a projection in $\mathrm{Hom}(\Theta_i, \Theta_i)$ whose range is a connected étale subalgebra of $\Theta_i$, and distinct intermediate nets give distinct subalgebras. As in the proof of part 3 of Theorem 3.9, using Lemma 3.1, the connected étale subalgebras of $\Theta_i$ correspond to the fusion subcategories of $\mathcal{H}_i$, which are given by part 4 of Proposition 2.1. Equivalently, one can use the Galois correspondence

of part 1 of Proposition 2.6. For $\Theta_x$, the subalgebra of Frobenius–Perron dimension $3d+2$ is the dual canonical endomorphism $\pi_0 \oplus \pi_b$ of the intermediate net $\mathcal{A}_{V'}$.

4. By Lemma 7.1, the braided equivalence $\Psi$ maps $\pi_a$ to $a$ and $\pi_{c_0}$ to $c_0$, so that it maps $\Theta_a$ to a Lagrangian algebra isomorphic, as an object, to $0 \oplus b \oplus 2a$, that is, to $L_2$ by Proposition 3.6. Similarly $\Psi(\Theta_{c_0}) \cong L_3$. ■

**Remark 7.3** Part 2 alone shows that the two lattice nets carry the categories $\mathcal{H}_2$ and $\mathcal{H}_3$ in some order. Under (N), the order is the same as for the vertex operator algebras, by Proposition 3.6, and (N) holds under ($\mathrm{H_J}$) by Theorem 7.17, see Corollary 7.19.

## 7.2 The Haagerup subfactor and the generalized orbifold

**Corollary 7.4** *Assume* ($\mathrm{H_J}$). *Then the Haagerup subfactor arises from the completely rational net $\mathcal{A}_S$ in the sense of [Bis17a, Definition 4.1 p. 20], that is, the hyperfinite type* $\mathrm{III}_1$ *subfactor with the standard invariant of the Haagerup subfactor is of the form described there.*

*Proof* : The Haagerup subfactor $N \subset M$ has finite index and finite depth, and its quantum double $D(N \subset M)$ is the Drinfeld center of either of its even parts $\mathcal{H}_1$ and $\mathcal{H}_2$, which is braided equivalent to $Z(\mathcal{H}_3)$. By part 1 of Theorem 7.2, $\mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to $D(N \subset M)$, and the conclusion follows from part 2 of Proposition 2.6. ■

**Proposition 7.5** *Assume* ($\mathrm{H_J}$). *Then there is a proper action of the hypergroup $K = \{k_0, k_1\}$ with*

$$k_1 k_1 = \frac{1}{\delta} k_0 + \Big(1 - \frac{1}{\delta}\Big) k_1$$

*on the net $\mathcal{A}_{V'}$ such that $\mathcal{A}_S = \mathcal{A}_{V'}^K$. The weight of $K$ is $1 + \delta = [\mathcal{A}_{V'} : \mathcal{A}_S]$.*

*Proof* : This is Proposition 5.7 for $U = S$, which applies by Corollary 5.9. ■

**Corollary 7.6** *Assume* ($\mathrm{H_J}$). *Then the hypergroup of Proposition 7.5 has parameter $w = \delta = d^2 = \frac{11+3\sqrt{13}}{2}$. With $\tilde{k}_1 \stackrel{\mathrm{def}}{=} d k_1 = w^{1/2} k_1$, we have $\tilde{k}_1 \tilde{k}_1 = k_0 + 3\tilde{k}_1$, and $\mathrm{Rep}(\mathcal{A}_{V'}^K) = \mathrm{Rep}(\mathcal{A}_S)$ is braided equivalent to the quantum double of the Haagerup subfactor. The net $\mathcal{A}_{V'}$ is the conformal net $\mathcal{A}_{E_6,1} \otimes \mathcal{A}_{SU(3),1}$, denoted $E_{6,1} \times SU(3)_1$ in [Bis17b, Table 1 p. 25], of the lattice $A_2 \times E_6$. Hence, under these hypotheses, $\mathcal{A}_S$ solves [Bis17b, Problem 4.27 p. 27].*

*Proof* : By Lemma 7.1, $\delta = d^2 = 3d + 1 = \frac{11+3\sqrt{13}}{2}$. Then

$$\tilde{k}_1 \tilde{k}_1 = d^2 k_1 k_1 = k_0 + (d^2 - 1) k_1 = k_0 + \frac{d^2 - 1}{d} \tilde{k}_1 = k_0 + 3\tilde{k}_1,$$

since $d^2 - 1 = 3d$. The other assertions follow from Proposition 7.5, Corollary 7.4, part 1 of Theorem 7.2 and part 2 of Proposition 4.3. ■

**Remark 7.7** The vertex algebraic counterpart of Problem 4.27 of [Bis17b] is formulated in [GaR26, Claim 4.4 and (4.54) p. 86]. The existence of the hypergroup action is a general fact [Bis17b, Theorem 4.22 p. 24]. The new input of Corollary 7.6 is the identification of $\mathrm{Rep}(\mathcal{A}_S)$ in Theorem 6.11. The value of $w$ is consistent with [Bis17b, Corollary 4.20 p. 23 and Lemma 4.26 p. 27]. Indeed, by Theorem 6.11, the canonical hypergroup of the inclusion $\mathcal{A}_S \subset \mathcal{A}_{E_8}$ is attached to a fusion category with the fusion rules (2.2) [Bis17b, Theorem 5.16 p. 31], the intermediate net $\mathcal{A}_{V'}$ is the fixed point net of its maximal subgroup $\mathbb{Z}_3$, and [Bis17b, Lemma 4.26 p. 27] with $n = 3$ gives $w = \frac{2+9+3\sqrt{13}}{2}$.

**Remark 7.8** Bischoff deduced from the modular invariant of [EvG11, (2.4) p. 20] that the unitary fusion category attached to the inclusion $\mathcal{A}_S \subset \mathcal{A}_{V'}$ should be a near-group category of type $\mathbb{Z}_3^2 + 9$ [Bis17b, Remark 5.14 and Proposition 5.15 p. 31]. Under ($\mathrm{H_J}$), this holds for the category $\mathrm{Rep}(\mathcal{A}_S)_\theta$ of modules over the dual canonical endomorphism $\theta = \pi_0 \oplus \pi_b$. Indeed, by part 1 of Theorem 7.2, we may compute in $Z(\mathcal{H}_3)$ with the algebra $A = 0 \oplus b$. For simple objects $z$ and $z'$, the free module adjunction gives $\mathrm{Hom}_A(z \otimes A, z' \otimes A) \cong \mathrm{Hom}(z, z' \otimes A)$, and the fusion rules of $Z(\mathcal{H}_3)$, computed from the $S$-matrix of Section 2.2 by the Verlinde formula,

$$b \otimes b \cong \bigoplus_z z, \quad y \otimes b \cong y \oplus \bigoplus_{z \neq 0} z, \quad d_l \otimes b \cong \bigoplus_{z \notin \{0, d_l\}} z,$$

for $y \in \{a, c_0, c_1, c_2\}$ and $1 \leqslant l \leqslant 6$, give the following. The free modules $d_l \otimes A$ are simple and pairwise isomorphic, to an object $\rho$. The module $b \otimes A$ is isomorphic to $A \oplus \rho$. For $y \in \{a, c_0, c_1, c_2\}$, the module $y \otimes A$ is isomorphic to $\rho \oplus g_y \oplus g'_y$, where the eight simple modules $g_y$, $g'_y$ are pairwise non-isomorphic and not isomorphic to $A$ or $\rho$. Since every simple $A$-module is a summand of a free module, the category $\mathrm{Rep}(\mathcal{A}_S)_\theta$ has ten simple objects. Since the free module functor $z \mapsto z \otimes A$ is a tensor functor, it preserves Frobenius–Perron dimensions. Hence $\rho$ has dimension $3d$ and $\mathrm{FPdim}(g_y) + \mathrm{FPdim}(g'_y) = 2$. Since the Frobenius–Perron dimension of a simple object is at least one [EGNO15, Proposition 3.3.4 p. 53], and a simple object of Frobenius–Perron dimension one is invertible [EGNO15, Corollary 3.3.10 p. 54], the nine objects $A$, $g_y$, $g'_y$ are invertible. They form a group of order nine. By (2.7), the category $\mathrm{Rep}(\mathcal{A}_S)_\theta^0$ of local modules has Frobenius–Perron dimension nine. Since $\mathrm{FPdim}(\rho)^2 = 9d^2 > 9$, it cannot contain $\rho$, so its simple objects are these nine invertible objects. By [BKL15, Proposition 6.4 p. 33] and the proof of Lemma 6.10, it is braided equivalent to $\mathrm{Rep}(\mathcal{A}_{V'})$ and to $Z(\mathcal{P}_x) \cong Z(\mathrm{Vec}_{\mathbb{Z}_3})$, whose group of invertible objects is $\mathbb{Z}_3 \times \mathbb{Z}_3$. For an invertible object $g$, the object $\rho \otimes g$ is simple of dimension $3d$, so that $\rho \otimes g \cong \rho$, and similarly $\rho^* \cong \rho$. Hence $\mathrm{Hom}(\rho \otimes \rho, g) \cong \mathrm{Hom}(\rho, g \otimes \rho)$ is one-dimensional, and comparing dimensions with $9d^2 = 9 + 9 \cdot 3d$ gives $\rho \otimes \rho \cong \bigoplus_g g \oplus 9\rho$. Thus $\mathrm{Rep}(\mathcal{A}_S)_\theta$ is a near-group category of type $\mathbb{Z}_3^2 + 9$, of Frobenius–Perron dimension $9(3d + 2)$.

**Remark 7.9** By [Bis17b, Theorem 4.22 p. 24] and the computation of [Bis17b, Section 4.6 p. 25], the action of $k_1$ is implemented on $\mathcal{H}_{V'} = \mathcal{H}_S \oplus \mathcal{H}_{N_b}$ by the operator $T = P_S - \delta^{-1} P_{N_b}$, where $P_S$ and $P_{N_b}$ are the orthogonal projections. In other words, the stochastic map $\phi_{k_1}$ on $\mathcal{A}_{V'}(I)$ is determined by $\phi_{k_1}(x)\Omega = Tx\Omega$. The restriction of $T$ to $V' = S \oplus N_b$ is the linear map $\mathrm{id}_S \oplus (-\delta^{-1})\mathrm{id}_{N_b}$, which is not an automorphism of $V'$. Compare with the hypergroup symmetries of [GaR26, Section 4.3 pp. 83–88].

## 7.3 The third holomorphic extension

We say that $S$ is completely unitary if it is completely unitary in the sense of [CGGH23, Definition 4.2 p. 18], that is, every $S$-module is unitarizable and the invariant sesquilinear forms on the tensor products of $S$-modules of [Gui19] are positive definite.

**Proposition 7.10** *Assume* ($\mathrm{H_J}$) *and that $S$ is completely unitary. Let $B_1$ be the holomorphic extension of $S$ attached to $L_1$ by Theorem 3.9.*

1. *The vertex operator algebra $B_1$ has a unique unitary structure extending the unitary structure of $S$ given by $H_\chi$. With this structure, it is strongly local and its conformal net $\mathcal{A}_{B_1}$ is isomorphic to $\mathcal{A}_{E_8}$.*

2. *The net $\mathcal{A}_{B_1}$ is an irreducible local extension of $\mathcal{A}_S$, and its vacuum representation restricts to $\mathcal{A}_S$ as $\pi_0 \oplus \pi_b \oplus \pi_a \oplus \pi_{c_0}$. In particular, $\mathcal{A}_{H_{\Lambda_a}}$, $\mathcal{A}_{H_{\Lambda_{c_0}}}$ and $\mathcal{A}_{B_1}$ are three*

*pairwise non-isomorphic extensions of $\mathcal{A}_S$ with $\mu$-index one, each isomorphic to $\mathcal{A}_{E_8}$ as a conformal net.*

3. *The net $\mathcal{A}_{B_1}$ is the extension $\mathcal{B}_1$ of part 2 of Theorem 7.2.*

*Proof* : 1. By [CGGH23, Theorem 4.7 p. 19], $B_1$ is a unitary extension of $S$, with a unique unitary structure extending the given one on $S$. By part 1 of Theorem 3.9, $B_1 \cong V_{E_8}$. The argument of part 1 of Proposition 4.3 applies to any unitary structure on a vertex operator algebra isomorphic to $V_{E_8}$, so that $B_1$ is strongly local, has bounded localized vertex operators and $\mathcal{A}_{B_1} \cong \mathcal{A}_{E_8}$.

2. By Lemma 4.4, the coset of $\mathcal{A}_S$ in $\mathcal{A}_{B_1}$ is trivial and $\mathcal{A}_S(I)' \cap \mathcal{A}_{B_1}(I) = \mathbb{C}$. As in the proof of Theorem 5.6, the index is finite. Since $L_1 \cong 0 \oplus b \oplus a \oplus c_0$ is multiplicity free, the isotypic components of the $S$-module $B_1$ are simple modules isomorphic to $S$, $M_b$, $M_a$, $M_{c_0}$. They are pairwise orthogonal, since the orthogonal projection onto the closure of a simple submodule maps $B_1$ into this closure and commutes with the modes of $S$, as in the proof of part 1 of Lemma 4.7. By part 3 of Lemma 4.7, applied to $V_1 = B_1$ and $V_2 \in \{H_{\Lambda_a}, H_{\Lambda_{c_0}}\}$, the corresponding representations are $\pi_0$, $\pi_b$, $\pi_a$, $\pi_{c_0}$. The three extensions are pairwise non-isomorphic since their dual canonical endomorphisms are not equivalent.

3. By part 2 of Theorem 7.2, $\Theta_1$ is the unique Lagrangian Q-system isomorphic to $\pi_0 \oplus \pi_b \oplus \pi_a \oplus \pi_{c_0}$. ∎

**Remark 7.11** Under complete unitarity of $S$, the comparison of [CGGH23, Theorem 4.10 p. 21] between vertex algebra extensions of $S$ and net extensions of $\mathcal{A}_S$ requires an equivalence $\mathrm{Rep}^u(S) \cong \mathrm{Rep}^f(\mathcal{A}_S)$ of unitary modular tensor categories. If $S$ is completely unitary, such an equivalence follows from ($\mathrm{N}_0$), ($\mathrm{H}_\mathrm{J}$) and [Reu23, Theorem 5 p. 2], and it does not involve the correspondence $\pi_M \mapsto M$ of (N). Proposition 7.10 only uses the unitarity of the extension $B_1$ and Lemma 4.7. Both the complete unitarity of $S$ and this equivalence hold under ($\mathrm{H}_\mathrm{J}$) (Theorem 7.17 and Corollary 7.19).

## 7.4 A diagonal holomorphic extension of $S \otimes S$

The following statement only uses ($\mathrm{H}_\mathrm{J}$). It realizes $S$ as its own commutant in $V_{E_8 \oplus E_8}$.

**Proposition 7.12** *Assume* ($\mathrm{H}_\mathrm{J}$).

1. *There is a braided equivalence $F\colon \mathrm{Rep}(S)^{\mathrm{rev}} \to \mathrm{Rep}(S)$, and it is unique up to natural isomorphism. The induced involution $\pi$ of the set of simple objects satisfies $\pi(0) = 0$, $\pi(b) = b$, $\pi(a) = a$, $\pi(c_0) = c_0$, $\pi(c_1) = c_2$, $\pi(d_1) = d_5$, $\pi(d_2) = d_3$ and $\pi(d_4) = d_6$.*

2. *The $S \otimes S$-module $W \stackrel{\mathrm{def}}{=} \bigoplus_z M_z \otimes M_{\pi(z)}$ carries the structure of a simple vertex operator algebra of CFT type extending $S \otimes S$, with the same conformal vector. It is holomorphic, it is isomorphic to $V_{E_8 \oplus E_8}$, and $\mathrm{Com}_W(S \otimes 1) = 1 \otimes S$, $\mathrm{Com}_W(1 \otimes S) = S \otimes 1$.*

*Proof* : 1. For a fusion category $\mathcal{C}$, let $\mathcal{C}^{\otimes \mathrm{op}}$ be the category $\mathcal{C}$ with the reversed tensor product. A half-braiding $\gamma$ of an object $X$ in $\mathcal{C}^{\otimes \mathrm{op}}$ gives the half-braiding $\gamma^{-1}$ of $X$ in $\mathcal{C}$, and the functor $(X, \gamma) \mapsto (X, \gamma^{-1})$, with the monoidal structure given by the braiding of $Z(\mathcal{C})$, is a braided equivalence from $Z(\mathcal{C}^{\otimes \mathrm{op}})$ onto $Z(\mathcal{C})^{\mathrm{rev}}$. If $\mathcal{C}$ is a unitary fusion category, the functor $f \mapsto f^*$ is a linear monoidal equivalence from the complex conjugate category $\overline{\mathcal{C}}$, which has the same objects and the complex conjugate spaces of morphisms, to the opposite category $\mathcal{C}^{\mathrm{op}}$, and the duality functor $X \mapsto X^\vee$ is a monoidal equivalence from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}^{\otimes \mathrm{op}}$. Hence $\mathcal{C}^{\otimes \mathrm{op}} \cong \overline{\mathcal{C}}$ and $Z(\mathcal{C})^{\mathrm{rev}} \cong Z(\overline{\mathcal{C}})$. Apply this to the even parts $\mathcal{H}_1$ and $\mathcal{H}_2$ of the Haagerup subfactor. The

categories $\overline{\mathcal{H}_1}$ and $\overline{\mathcal{H}_2}$ are the even parts of the complex conjugate subfactor, whose planar algebra is the complex conjugate of the Haagerup planar algebra, with the same index and the same principal graphs. By [JMS14, Theorem 4.1 p. 21], it is the Haagerup planar algebra or its dual, whose even parts are $\mathcal{H}_1$ and $\mathcal{H}_2$, of ranks four and six. Comparing ranks, we obtain $\overline{\mathcal{H}_1} \cong \mathcal{H}_1$ and $\overline{\mathcal{H}_2} \cong \mathcal{H}_2$. Hence $Z(\mathcal{H}_3)^{\mathrm{rev}} \cong Z(\mathcal{H}_1)^{\mathrm{rev}} \cong Z(\overline{\mathcal{H}_1}) \cong Z(\mathcal{H}_1) \cong Z(\mathcal{H}_3)$ by Section 2.1, and ($\mathrm{H_J}$) gives $F$. Its uniqueness follows from the triviality of the braided autoequivalences of $Z(\mathcal{H}_3)$ (Section 2.1). Since $F$ is also a braided equivalence $\mathrm{Rep}(S) \to \mathrm{Rep}(S)^{\mathrm{rev}}$, the composite $F \circ F$ is a braided autoequivalence of $\mathrm{Rep}(S)$. For the same reason, it is isomorphic to the identity functor, so that $\pi$ is an involution. The equivalence $F$ preserves Frobenius–Perron dimensions, and it maps the canonical twist of $\mathrm{Rep}(S)^{\mathrm{rev}}$, which is $\theta^{-1}$, to the canonical twist $\theta$ of $\mathrm{Rep}(S)$ [ENO05, Proposition 8.23 p. 34]. By (2.4) and (2.5), we obtain $\pi(0) = 0$, $\pi(b) = b$, $\pi(\{a, c_0\}) = \{a, c_0\}$ and $\pi(c_1) = c_2$. Moreover $\pi(d_l) = d_{l'}$ with $6l'^2 \equiv -6l^2 \bmod 13$, that is, $l' \equiv \pm 5l \bmod 13$, since $5^2 \equiv -1 \bmod 13$. Finally, under $\mathrm{Rep}(S) \cong Z(\mathcal{H}_3)$, the Lagrangian algebra $L_2 \cong 0 \oplus b \oplus 2a$ of Corollary 3.8 is also a connected étale algebra of $\mathrm{Rep}(S)^{\mathrm{rev}}$, since an algebra is commutative for a braiding if and only if it is commutative for the reverse braiding. The two tensor structures on the category of modules over a connected étale algebra defined by a braiding and by its reverse are opposite to each other [DMNO13, p. 9], so that $(\mathrm{Rep}(S)^{\mathrm{rev}})_{L_2}$ is equivalent to $\mathcal{H}_2^{\otimes \mathrm{op}} \cong \overline{\mathcal{H}_2} \cong \mathcal{H}_2$. Hence $F(L_2)$ is a Lagrangian algebra of $\mathrm{Rep}(S)$ whose category of modules is equivalent to $\mathcal{H}_2$. By Proposition 3.3, $F(L_2) \cong L_2$, so that $0 \oplus b \oplus 2\pi(a) \cong 0 \oplus b \oplus 2a$, and $\pi(a) = a$, $\pi(c_0) = c_0$.

2. Every simple $S$-module is self-dual (Section 2.2), and $F$ inverts the twists. By [CKM22, Main Theorem 3 pp. 4–5 and Theorem 5.10 p. 51], applied to the braid-reversed equivalence $F$, the space $W$ carries the structure of a simple conformal vertex algebra extending $S \otimes S$. It is $\mathbb{Z}$-graded since $\theta_z \theta_{\pi(z)} = 1$, and of CFT type since $S$ is positive. Under the braided equivalence $\mathrm{Rep}(S \otimes S) \cong \mathrm{Rep}(S) \boxtimes \mathrm{Rep}(S)$ [CKM22, Theorem 5.5 p. 48], $W$ corresponds to the haploid commutative algebra $\bigoplus_z M_z \boxtimes M_{\pi(z)}$ with trivial twist [HKL15, Theorem 3.2 p. 10], which is a connected étale algebra by [CKM22, Theorem 5.12 p. 52 and its proof] and [DMNO13, Proposition 2.7 p. 5]. Its Frobenius–Perron dimension $\sum_z d(M_z) d(M_{\pi(z)}) = \mathrm{FPdim}\, \mathrm{Rep}(S)$ is the square root of $\mathrm{FPdim}\, \mathrm{Rep}(S \otimes S)$. It is therefore Lagrangian, so that $\mathrm{Rep}(W) \cong \mathrm{Vec}$ by [HKL15, Theorem 3.4 p. 16] and [DMNO13, Remark 4.3 p. 16], and $W$ is strongly rational by [CKM22, Corollary 5.13 p. 53]. By [DoM04, Theorem 2 p. 2], $W$ is isomorphic to $V_{E_8 \oplus E_8}$ or to $V_{D_{16}^+}$. The weight-one space $W_1$ is the sum of $(S \otimes S)_1 = (\mathfrak{g}' \oplus \mathfrak{g}'') \oplus (\mathfrak{g}' \oplus \mathfrak{g}'')$ and of the spaces $R_z \otimes R_{\pi(z)}$ with $h_z + h_{\pi(z)} = 1$, that is, $z \in \{c_1, c_2, d_1, d_5, d_4, d_6\}$, where $R_z$ is the lowest weight space of $M_z$. Write $\mathfrak{g}'_1 = \mathfrak{g}' \otimes 1$ and $\mathfrak{g}'_2 = 1 \otimes \mathfrak{g}'$. By [Arh26, Theorem 8.5 p. 23 and Table 2 p. 24], the $\mathfrak{g}'$-weights occurring in these spaces $R_z$ are at most 2, so that $W_1$ contains no irreducible $\mathfrak{g}'_k$-module of dimension 4, and the subspace of $(\mathfrak{g}'_1 \oplus \mathfrak{g}'_2)$-invariant vectors of $W_1$ has dimension $6 + 2 \cdot 18 + 2 \cdot 15 + 2 \cdot 3 = 78$. Assume that $W \cong V_{D_{16}^+}$, so that $W_1 \cong \mathfrak{so}(E)$ with $\dim E = 32$ and $W_1 \cong \Lambda^2 E$. The affine subalgebra generated by $\mathfrak{g}'_k$ has level 3, so that the Dynkin index of $\mathfrak{g}'_k \subset \mathfrak{so}(E)$ is 3. Since the vector representation of $\mathfrak{so}(E)$ has Dynkin index 2, the restriction of $E$ to $\mathfrak{g}'_k$ has Dynkin index 6. Hence $E|_{\mathfrak{g}'_k}$ is $6V(1) \oplus 20V(0)$ or $V(2) \oplus 2V(1) \oplus 25V(0)$, where $V(n)$ is the irreducible module of highest weight $n$. In the second case $\Lambda^2 E \supset V(2) \otimes V(1) \supset V(3)$, which is excluded. Thus $E|_{\mathfrak{g}'_1 \oplus \mathfrak{g}'_2} = pV(1,1) \oplus (6-2p)(V(1,0) \oplus V(0,1)) \oplus (8+4p)V(0,0)$ for some $p \in \{0, 1, 2, 3\}$. The multiplicity spaces of $V(1,1)$ and $V(0,0)$ carry nondegenerate symmetric forms and those of $V(1,0)$ and $V(0,1)$ nondegenerate alternating forms, so that the centralizer of $\mathfrak{g}'_1 \oplus \mathfrak{g}'_2$ in $\mathfrak{so}(E)$ is $\mathfrak{so}_p \oplus \mathfrak{so}_{8+4p} \oplus \mathfrak{sp}_{6-2p} \oplus \mathfrak{sp}_{6-2p}$, of dimension 70, 86, 127 or 193. This contradicts the value 78, so that $W \cong V_{E_8 \oplus E_8}$. A vector of $W$ annihilated by the nonnegative modes of $S \otimes 1$ is annihilated by $L^S(0) \otimes 1$. Since $h_z > 0$ for $z \neq 0$, it lies in $S_0 \otimes S = \mathbf{1} \otimes S$. This gives the commutants. ∎

**Remark 7.13** The vertex operator algebra $W \cong V_{E_8 \oplus E_8}$ is generated by $W_1 \cong \mathfrak{e}_8 \oplus \mathfrak{e}_8$, so that its unitary structures are determined by compact real forms of $W_1$, and $S \otimes S$ is generated by its weight-one and weight-two spaces. By Corollary 5.5, every module $M_z \otimes M_{\pi(z)}$ is unitarizable. If $S$ is completely unitary, then so is $S \otimes S$ [Gui22, Proposition 3.31 p. 48], and $W$ is a unitary extension of $S \otimes S$ [CGGH23, Theorem 4.7 p. 19]. Without complete unitarity, it suffices to find the PCT operator of a unitary structure on $W$ which preserves $(S \otimes S)_1$ and $\omega_{C\otimes 1}$, see the proof of Proposition 7.16. Such an operator is found in Section 7.5.

## 7.5 Complete unitarity of $S$ and the comparison (N)

We solve the existence problem of the preceding remark, and we deduce the complete unitarity of $S$ and (N) from the results of Gui on cosets [Gui26, Gui25]. We assume ($\mathrm{H_J}$) and keep the notation of Proposition 7.12. Let $A$ be the vertex subalgebra of $S$ generated by $S_1 = \mathfrak{g}' \oplus \mathfrak{g}''$, which is isomorphic to $L_{\mathfrak{sl}_2}(3,0) \otimes L_{\mathfrak{sl}_2}(13,0)$ [Arh26, Corollary 9.2 p. 26], and let $C = \mathrm{Com}_S(A)$. The vector $\omega_C = \omega - \omega_A$, where $\omega_A$ is the Sugawara vector of $A$, is the conformal vector of $C$, of central charge $8 - \frac{9}{5} - \frac{13}{5} = \frac{18}{5}$. We write $\omega_{S\otimes 1} = \omega \otimes \mathbf{1}$, $\omega_{A\otimes 1} = \omega_A \otimes \mathbf{1}$, $\omega_{C\otimes 1} = \omega_C \otimes \mathbf{1}$, and similarly $\omega_{1\otimes S}$, $\omega_{1\otimes A}$, $\omega_{1\otimes C}$, and we denote by $L_n^{S\otimes 1}$, $L_n^{A\otimes 1}, \dots$ the modes of the corresponding Virasoro fields. We put $\omega_{A\otimes A} = \omega_{A\otimes 1} + \omega_{1\otimes A}$ and $\omega_{C\otimes C} = \omega_{C\otimes 1} + \omega_{1\otimes C}$, so that $\omega_W = \omega_{S\otimes 1} + \omega_{1\otimes S} = \omega_{A\otimes A} + \omega_{C\otimes C}$ is the conformal vector of $W$. We denote by $\mathrm{Rep}^u(S)$ the category of unitary $S$-modules [Gui26, Section 3.4 p. 44] and $\mathrm{Rep}^f(\mathcal{A}_S) = \mathrm{Rep}(\mathcal{A}_S)$ is as in Section 2.6.

Let $\Gamma = E_8 \oplus E_8 \subset \mathbb{R}^8 \oplus \mathbb{R}^8$, where $E_8$ is the lattice of the vectors of $\mathbb{R}^8$ whose coordinates are all in $\mathbb{Z}$ or all in $\frac{1}{2}+\mathbb{Z}$ and have an even sum, with the simple roots $\alpha_1, \dots, \alpha_8$ and the fundamental weights $\varpi_1, \dots, \varpi_8$ of [Bou68, Planche VII]. Let $\varepsilon_8$ be a bimultiplicative cocycle on $E_8$ such that $\varepsilon_8(\alpha,\beta)\varepsilon_8(\beta,\alpha)^{-1} = (-1)^{\langle\alpha,\beta\rangle}$ and $\varepsilon_8(\alpha,\alpha) = (-1)^{\langle\alpha,\alpha\rangle/2}$, and let $V_\Gamma = M(1) \otimes \mathbb{C}_\varepsilon[\Gamma]$ be the lattice vertex operator algebra built with the cocycle $\varepsilon(\alpha,\beta) = \varepsilon_8(\alpha^{(1)},\beta^{(1)})\varepsilon_8(\alpha^{(2)},\beta^{(2)})$, where $\alpha^{(1)}$ and $\alpha^{(2)}$ are the components of $\alpha \in \Gamma$. The cocycle $\varepsilon$ satisfies the same two identities on $\Gamma$, and it is invariant under the exchange of the two summands of $\Gamma$. Let $\phi_1$ be the antilinear involution of $V_\Gamma$ given by the formula of Section 4.1. For a character $\kappa\colon \Gamma \to \mathbb{R}_{>0}$, written $\kappa(\lambda) = \mathrm{e}^{\langle h_\kappa, \lambda\rangle}$, we put $\phi_\kappa = \phi_1 \circ \exp(h_\kappa(0))$, so that $\phi_\kappa(\mathrm{e}^\lambda) = \kappa(\lambda)\mathrm{e}^{-\lambda}$. By [DoL14, Theorem 4.12 p. 24], $\phi_1$ is the PCT operator of a structure of unitary vertex operator algebra on $V_\Gamma$, as in part 1 of Lemma 4.2. By Proposition 7.12, there is an isomorphism of vertex operator algebras $W \to V_\Gamma$. We fix one, and we identify $W$ with $V_\Gamma$. The space $W_1$ is the Lie algebra $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ for the bracket $[x,y] = x_{(0)}y$, with the invariant form defined by $x_{(1)}y = \langle x,y\rangle\mathbf{1}$, for which the roots have norm two. The Dynkin index of a homomorphism $\psi\colon \mathfrak{sl}_2 \to \mathfrak{e}_8$ is $\frac{1}{2}\langle\psi(h),\psi(h)\rangle$, where $h$ is the standard Cartan element of $\mathfrak{sl}_2$. The level of the affine vertex subalgebra of $W$ generated by the image of a homomorphism $\mathfrak{sl}_2 \to \mathfrak{e}_8 \oplus \mathfrak{e}_8$ is the sum of the Dynkin indices of its two components.

**An explicit subalgebra.** In each copy of $\mathfrak{e}_8$, the vectors $h(-1)\mathbf{1}$, $h \in \mathbb{C}^8$, and $\mathrm{e}^\alpha$, for the roots $\alpha = \pm e_i \pm e_j$ of $D_8 \subset E_8$, span a Lie subalgebra isomorphic to $\mathfrak{so}(16)$. We identify it with the orthogonal Lie algebra of $X = \mathbb{C}^{16}$, with basis $f_1^\pm, \dots, f_8^\pm$ and the symmetric form $B$ such that $B(f_i^+, f_j^-)$ is 1 for $i = j$ and 0 otherwise, and $B(f_i^\pm, f_j^\pm) = 0$, in such a way that $e_i(-1)\mathbf{1}$ corresponds to $f_i^+ \wedge f_i^-$ and $\mathrm{e}^{se_i+te_j}$ to $\pm f_i^s \wedge f_j^t$ for $s,t \in \{\pm\}$ and $i < j$, where $(u \wedge v)y = B(v,y)u - B(u,y)v$. Consider the orthogonal decomposition $X = X_3 \oplus X_4 \oplus X_9$ with $X_3 = \langle f_1^+, f_1^-, f_2^+ + f_2^-\rangle$, $X_4 = \langle f_3^\pm, f_4^\pm\rangle$ and $X_9 = \langle f_2^+ - f_2^-, f_5^\pm, \dots, f_8^\pm\rangle$, and fix a linear isomorphism $X_9 \cong \mathbb{C}^3 \otimes \mathbb{C}^3$ which multiplies the symmetric forms by a nonzero constant, where $\mathbb{C}^3$ carries the invariant form of the adjoint representation of $\mathfrak{sl}_2$. Let $\mathfrak{a} = \mathfrak{so}(X_3)$, let $\mathfrak{b}^\pm$ be the root subalgebras of $e_3 \pm e_4$, so that $\mathfrak{so}(X_4) = \mathfrak{b}^+ \oplus \mathfrak{b}^-$, and let $\mathfrak{c}_1, \mathfrak{c}_2 \subset \mathfrak{so}(X_9)$ be the images of

$\mathfrak{sl}_2$ acting on the first and on the second factor of $\mathbb{C}^3 \otimes \mathbb{C}^3$. These are commuting subalgebras isomorphic to $\mathfrak{sl}_2$, of Dynkin indices 2, 1, 1, 6 and 6 in $\mathfrak{e}_8$. Let $\mathfrak{d}$ be the diagonal of $\mathfrak{b}^- \oplus \mathfrak{c}_2$ with respect to $\mathfrak{sl}_2$-triples of the two summands, of index 7. We define $\mathfrak{s}_1, \dots, \mathfrak{s}_4 \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8$ as the diagonal subalgebras, with respect to $\mathfrak{sl}_2$-triples,

$$\mathfrak{s}_1 \subset \mathfrak{a} \oplus \mathfrak{b}^+, \quad \mathfrak{s}_2 \subset \mathfrak{c}_1 \oplus \mathfrak{d}, \quad \mathfrak{s}_3 \subset \mathfrak{b}^+ \oplus \mathfrak{a}, \quad \mathfrak{s}_4 \subset \mathfrak{d} \oplus \mathfrak{c}_1,$$

the first summand lying in the first copy and the second summand in the second copy. Their Dynkin indices in the two copies are $(2,1)$, $(6,7)$, $(1,2)$ and $(7,6)$, so that they generate affine subalgebras of levels 3, 13, 3 and 13. We take for $\varepsilon_8$ the bimultiplicative map with $\varepsilon_8(\alpha_i, \alpha_j) = -1$ if $i = j$, or if $i < j$ and $\{i,j\}$ is one of the edges $\{1,3\}$, $\{2,4\}$, $\{3,4\}$, $\{4,5\}$, $\{5,6\}$, $\{6,7\}$, $\{7,8\}$ of the Dynkin diagram, and $\varepsilon_8(\alpha_i, \alpha_j) = 1$ otherwise. For $\lambda, \mu \in E_8$ and $x, y \in \mathbb{C}^8$, we write $\mathrm{e}^{\lambda \oplus \mu}$ and $(x \oplus y)(-1)\mathbf{1}$ for the corresponding vectors of $V_\Gamma$. Put

$$E_1 = -\mathrm{e}^{(e_1 - e_2) \oplus 0} - \mathrm{e}^{(e_1 + e_2) \oplus 0} - \mathrm{e}^{0 \oplus (e_3 + e_4)}, \quad H_1 = (2e_1 \oplus (e_3 + e_4))(-1)\mathbf{1},$$

$$\begin{aligned} E_2 = {} & \mathrm{e}^{(e_5 - e_2) \oplus 0} + \mathrm{e}^{(e_2 + e_5) \oplus 0} + 2\mathrm{e}^{(e_7 - e_6) \oplus 0} - 2\mathrm{e}^{(e_6 + e_8) \oplus 0} + \mathrm{e}^{0 \oplus (e_3 - e_4)} \\ & - \mathrm{e}^{0 \oplus (e_6 - e_2)} - \mathrm{e}^{0 \oplus (e_2 + e_6)} - 2\mathrm{e}^{0 \oplus (e_7 - e_5)} + \mathrm{e}^{0 \oplus (e_5 - e_8)}, \end{aligned}$$

$$H_2 = ((2e_5 + 2e_7 + 2e_8) \oplus (e_3 - e_4 + 2e_6 + 2e_7 - 2e_8))(-1)\mathbf{1},$$

let $E_3$, $H_3$, $E_4$, $H_4$ be the images of $E_1$, $H_1$, $E_2$, $H_2$ under the exchange of the two summands of $\Gamma$, and put $F_k \overset{\text{def}}{=} -\phi_\kappa(E_k)$, with $\kappa$ as in Proposition 7.16. A direct computation in the root system shows that $(E_k, H_k, F_k)$ are $\mathfrak{sl}_2$-triples spanning $\mathfrak{s}_1, \dots, \mathfrak{s}_4$ for a suitable choice of the signs and of the isomorphism $X_9 \cong \mathbb{C}^3 \otimes \mathbb{C}^3$.

**Lemma 7.14** *Assume* $(\mathrm{H_J})$. *After composing the isomorphism* $W \to V_\Gamma$ *with an automorphism of* $V_\Gamma$, *the subalgebras* $\mathfrak{g}' \otimes \mathbf{1}$, $\mathfrak{g}'' \otimes \mathbf{1}$, $\mathbf{1} \otimes \mathfrak{g}'$ *and* $\mathbf{1} \otimes \mathfrak{g}''$ *of* $W_1$ *are* $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ *and* $\mathfrak{s}_4$.

*Proof* : For $x \in W_1$, the zero mode $x_{(0)}$ is a derivation of $W$ which preserves the finite-dimensional spaces $W_n$, so that $\exp(x_{(0)})$ is an automorphism of $W$. These automorphisms act on $W_1$ by the inner automorphisms of the two copies of $\mathfrak{e}_8$, independently. Since $\varepsilon$ is invariant under the exchange of the two summands of $\Gamma$, this exchange is also an automorphism of $V_\Gamma$.

*Subalgebras of small index.* Let $\psi \colon \mathfrak{sl}_2 \to \mathfrak{e}_8$ be a homomorphism of Dynkin index $j \in \{1,2,3\}$. Up to an inner automorphism, $\psi(h)$ lies in the Cartan subalgebra $\{y(-1)\mathbf{1} : y \in \mathbb{C}^8\}$ and in the closed dominant chamber defined by $\alpha_1, \dots, \alpha_8$, so that $\psi(h) = \sum_i a_i \varpi_i$, where the fundamental weights are identified with coweights and $a_i = \alpha_i(\psi(h))$ are nonnegative integers. The entries $\langle \varpi_i, \varpi_k \rangle$ of the inverse Cartan matrix of $E_8$ are integers at least 2, with diagonal entries 4, 8, 14, 30, 20, 12, 6, 2 [Bou68, Planche VII]. Hence $\langle \psi(h), \psi(h) \rangle \geqslant 2(\sum_i a_i)^2$, and $2j = \langle \psi(h), \psi(h) \rangle \leqslant 6$ gives $\sum_i a_i = 1$. Thus $\psi(h) = \varpi_8$, $\varpi_1$ or $\varpi_7$ for $j = 1$, 2 or 3. By Mal'cev's theorem [CoM93, Theorem 3.4.12], $\psi$ is determined up to inner automorphisms by $\psi(h)$. For $j = 3$, the highest root $\theta$ satisfies $\theta(\varpi_7) = 3$, so that $\mathfrak{e}_8$ contains an irreducible $\psi(\mathfrak{sl}_2)$-module of dimension 4. For $j = 2$, $\psi$ is conjugate to the inclusion of $\mathfrak{a}$, which has Dynkin index 2 since $X$ restricts to $\mathfrak{a}$ as the sum of the adjoint representation and of 13 trivial ones. Under $\mathfrak{a} \oplus \mathfrak{so}(X_3^\perp)$, the decomposition $\mathfrak{e}_8 = \mathfrak{so}(16) \oplus \Delta^+$, where $\Delta^+$ is a half-spin representation, gives

$$\mathfrak{e}_8 = \Lambda^2 X_3 \oplus (X_3 \otimes X_3^\perp) \oplus \Lambda^2 X_3^\perp \oplus (\Delta_3 \otimes \Delta_{13}),$$

where $\Delta_3$ and $\Delta_{13}$ are the spin representations of dimensions 2 and 64. Hence the centralizer of $\mathfrak{a}$ is $\mathfrak{so}(X_3^\perp) \cong \mathfrak{so}(13)$.

*The two factors of level three.* By [Arh26, Theorem 8.5 p. 23 and Table 2 p. 24], the $\mathfrak{g}'$-weights of the lowest weight spaces $R_z$ occurring in $W_1$ are at most 2, see the proof of Proposition

7.12. Hence $W_1$ has no irreducible $\mathfrak{g}' \otimes \mathbf{1}$-submodule or $\mathbf{1} \otimes \mathfrak{g}'$-submodule of dimension 4, and the two components of each of these subalgebras have Dynkin indices $(2,1)$ or $(1,2)$. Using the exchange of the two copies, we may assume that $\mathfrak{g}' \otimes \mathbf{1}$ has index 2 in the first copy. Assume that $\mathbf{1} \otimes \mathfrak{g}'$ also has index 2 in the first copy. Then its component there lies in the centralizer $\mathfrak{so}(13)$ of the component of $\mathfrak{g}' \otimes \mathbf{1}$, and $\mathbb{C}^{13}$ restricts to it as $V(2) \oplus 10V(0)$ or $4V(1) \oplus 5V(0)$. By the decomposition displayed above, the vectors of the first copy which are invariant under $\mathfrak{g}' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}'$ are the invariant vectors of $\Lambda^2\mathbb{C}^{13}$. Since $\Lambda^2(V(2) \oplus 10V(0)) \cong 11V(2) \oplus 45V(0)$ and $\Lambda^2(4V(1) \oplus 5V(0)) \cong 6V(2) \oplus 20V(1) \oplus 20V(0)$, they form a space of dimension 45 or 20. In the second copy, the centralizer of the first subalgebra of index one is $\mathfrak{e}_7$, the second one has index one in $\mathfrak{e}_7$, and their joint centralizer is $\mathfrak{so}(12)$, of dimension 66. Hence the subspace of $W_1$ of vectors invariant under $\mathfrak{g}' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}'$ has dimension 111 or 86, whereas it has dimension 78 by the proof of Proposition 7.12. Therefore $\mathbf{1} \otimes \mathfrak{g}'$ has index 2 in the second copy.

*Reduction to orthogonal representations.* Conjugating in each copy, we may assume that the component of $\mathfrak{g}' \otimes \mathbf{1}$ in the first copy and that of $\mathbf{1} \otimes \mathfrak{g}'$ in the second copy are equal to $\mathfrak{a}$. The components in the first copy of the three other factors commute with $\mathfrak{a}$, so that they define an orthogonal representation $\sigma_1$ of $\mathfrak{sl}_2^{\oplus 3}$ on $X_3^\perp \cong \mathbb{C}^{13}$, and similarly $\sigma_2$ in the second copy. Two isomorphic orthogonal representations of a semisimple Lie algebra are conjugate under the orthogonal group, by the isotypic decomposition and the uniqueness up to equivalence of nondegenerate symmetric or alternating forms on the multiplicity spaces. Since 13 is odd, they are conjugate under $SO(13,\mathbb{C})$. The automorphisms $\exp(x_{(0)})$, $x \in \mathfrak{so}(X_3^\perp)$, fix $\mathfrak{a}$ pointwise and induce on $X_3^\perp$ all the elements of $SO(13,\mathbb{C})$. Hence the embedding of $(S \otimes S)_1$ is determined, up to automorphisms of $V_\Gamma$, by the isomorphism classes of $\sigma_1$ and $\sigma_2$. The Dynkin indices of the factors and the character of $W_1$ as a module over $(S \otimes S)_1 \cong \mathfrak{sl}_2^{\oplus 4}$ are functions of these classes, by the decomposition displayed above. The Dynkin index in $\mathfrak{e}_8$ of a subalgebra of $\mathfrak{so}(X_3^\perp)$ is half the Dynkin index of its representation on $X_3^\perp$, and the Dynkin index of $V(n)$ is $\frac{1}{6}n(n+1)(n+2)$. Since the indices of the factors in each copy are at most 13, only highest weights at most 4 occur, and there are 640 isomorphism classes of such orthogonal representations of $\mathfrak{sl}_2^{\oplus 3}$ on $\mathbb{C}^{13}$. An exhaustive computer enumeration shows that exactly one pair $(\sigma_1, \sigma_2)$ gives the character of $W_1 = (S \otimes S)_1 \oplus \bigoplus_z R_z \otimes R_{\pi(z)}$ computed from [Arh26, Table 2 p. 24], namely

$$\sigma_1 \cong V(2,0,2) \oplus V(0,1,1), \quad \sigma_2 \cong V(0,2,2) \oplus V(1,1,0),$$

where $\sigma_1$ is a representation of $\mathfrak{g}'' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}' \oplus \mathbf{1} \otimes \mathfrak{g}''$ and $\sigma_2$ a representation of $\mathfrak{g}' \otimes \mathbf{1} \oplus \mathfrak{g}'' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}''$. This is the pair defined by $\mathfrak{s}_1, \dots, \mathfrak{s}_4$. This gives the assertion. ∎

From now on, we assume that the identification $W = V_\Gamma$ satisfies the conclusion of Lemma 7.14. Let $\mathcal{G} = \mathrm{Com}_W(A \otimes A) \cap W_2$.

**Lemma 7.15** *Assume* (H$_\mathrm{J}$). *The space $\mathcal{G}$ has dimension* 16, *and $\omega_{C\otimes 1}$ is one of the four vectors $u_0, \dots, u_3 \in \mathcal{G}$ defined in the proof, whose coordinates in a rational basis of $\mathcal{G}$ lie in $\mathbb{Q}(\sqrt{-6})$.*

*Proof* : The vector $\omega_{C\otimes 1}$ commutes with $A \otimes \mathbf{1}$ by definition of $C$, and with $\mathbf{1} \otimes A$, so that it belongs to $\mathcal{G}$. The space $\mathcal{G}$ is the space of vectors of $W_2$ annihilated by the modes $x_{(0)}$, $x_{(1)}$, $x_{(2)}$ of the twelve vectors $E_k$, $H_k$, $F_k$. Since the vectors $H_k$ lie in the Cartan subalgebra, $\mathcal{G}$ is contained in the subspace $W_2^0$ of vectors of weight zero for $H_1, \dots, H_4$. No root of $\Gamma$ is orthogonal to $H_1, \dots, H_4$, so that $W_2^0$ is spanned by the 152 vectors $h(-1)h'(-1)\mathbf{1}$ and $h(-2)\mathbf{1}$, for $h, h'$ in a basis of $\mathbb{C}^{16}$, and by the 492 vectors $\mathrm{e}^\lambda$ with $\langle \lambda, \lambda \rangle = 4$ and $\langle \lambda, H_k \rangle = 0$ for $k = 1, \dots, 4$. An exact computation of the common kernel of these 36 modes on $W_2^0$, of dimension 644, gives $\dim \mathcal{G} = 16$. The vertex subalgebra $\mathrm{Com}_W(A \otimes A)$ has a one-dimensional weight-zero space, and its weight-one space is zero because $W_1$ contains no nonzero $(S \otimes S)_1$-invariant vector by [Arh26, Table 2 p. 24]. Hence the product $u_{(1)}v$ is a commutative product on $\mathcal{G}$, and $u_{(3)}v$ is

an invariant bilinear form. The vector $u = \omega_{C\otimes 1}$ satisfies $u_{(1)}u = 2u$ and $u_{(3)}u = \frac{9}{5}\mathbf{1}$, half the central charge of $C$. Moreover, $u_{(1)}$ commutes with $(S\otimes S)_1$ and acts on $W_1$ as $L_0^{S\otimes 1} - L_0^{A\otimes 1}$. It vanishes on $(S\otimes S)_1$. On the component of $R_z \otimes R_{\pi(z)}$ on which $\mathfrak{g}'\otimes\mathbf{1}\oplus\mathfrak{g}''\otimes\mathbf{1}$ acts through $V(m)\otimes V(n)$, it is the scalar $h_z - \frac{m(m+2)}{20} - \frac{n(n+2)}{60}$, since the vectors of $R_z$ are primary for $A$. The values $h_{c_1} = \frac{1}{3}$, $h_{c_2} = \frac{2}{3}$, $h_{d_1} = \frac{6}{13}$, $h_{d_4} = \frac{5}{13}$, $h_{d_5} = \frac{7}{13}$ and $h_{d_6} = \frac{8}{13}$ are given by [Arh26, (4.1) p. 11], and the decompositions of $R_z$ by [Arh26, Table 2 p. 24]. On each of the twenty isotypic components of $W_1$ for $(S\otimes S)_1$, this prescribes the eigenvalues of $u_{(1)}$ on the corresponding multiplicity space, one for each of the spaces $(S\otimes S)_1$ and $R_z\otimes R_{\pi(z)}$ which contain the component. We impose the trace of $u_{(1)}$ on every multiplicity space, the second elementary symmetric function of its eigenvalues on the ten multiplicity spaces of the components which occur in two of the spaces $R_z\otimes R_{\pi(z)}$, and the equations $u_{(1)}u = 2u$ and $u_{(3)}u = \frac{9}{5}\mathbf{1}$. These conditions are polynomial equations of degree at most two in the coordinates of $u$ in a rational basis of $\mathcal{G}$, with rational coefficients. Let $g_0,\dots,g_{15}$ be the ordered rational basis of $\mathcal{G}$ given in the file `omega_certificate.json` of the supplementary data, and write $u = \sum_{j=0}^{15} x_j g_j$. The twenty trace equations have rank eight. Their solution space is parametrized by $t_0,\dots,t_7$ through

$$(x_0,\dots,x_7) = \left(\frac{7}{195}, \frac{1}{13}, \frac{5}{39}, t_0, -\frac{t_1}{2} - \frac{7}{195}, t_1, \frac{1}{65}, 0\right),$$
$$(x_8,\dots,x_{15}) = \left(t_2, t_3, \frac{2}{65}, t_4, t_5, t_6, t_7, \frac{14}{195}\right).$$

After this substitution, the remaining conditions give twenty-five nonzero quadratic equations. Their ideal is equivalently given by

$$t_0 = -\frac{1}{39}, \quad t_1 = -\frac{8}{195}, \quad t_3 = t_5 = 0, \quad t_7 = -26t_2t_4, \quad t_6 = \frac{t_7}{2},$$
$$t_2^2 = -\frac{1}{1014}, \quad t_4^2 = -\frac{2}{507}.$$

This equality of ideals is checked by comparing reduced Gröbner bases over $\mathbb{Q}$, see the data availability statement. Hence, if $J\subset\mathbb{Q}[t_0,\dots,t_7]$ denotes the ideal of the twenty-five equations, then $\mathbb{Q}[t_0,\dots,t_7]/J$ is isomorphic to

$$\mathbb{Q}[t_2,t_4]/(1014t_2^2+1, 507t_4^2+2).$$

Both quadratic polynomials are separable, so this algebra has dimension four over $\mathbb{Q}$ and is reduced. Its four geometric points, which define $u_0,\dots,u_3$, are obtained by choosing independently $a,d\in\{-1,1\}$ and setting

$$t_2 = \frac{a\sqrt{-6}}{78}, \quad t_4 = \frac{d\sqrt{-6}}{39}, \quad t_6 = \frac{ad}{39}, \quad t_7 = \frac{2ad}{39},$$

with the other coordinates as above. These are necessary constraints on $\omega_{C\otimes 1}$. We do not need to determine which candidate is $\omega_{C\otimes 1}$, since Proposition 7.16 checks that the PCT operator fixes all four candidates. ∎

**Proposition 7.16** *Assume* (H$_J$)*, and identify $W$ with $V_\Gamma$ as in Lemma 7.14. Let $\kappa(\lambda) = 2^{-\langle\lambda,\varpi\rangle}$, where $\varpi = (\varpi_8,\varpi_8)$ and $\varpi_8 = e_7+e_8$, so that $\langle\lambda,\varpi\rangle$ is the sum of the coefficients of $\alpha_8$ in the two components of $\lambda$. Then $\phi_\kappa$ is the PCT operator of a structure of unitary vertex operator algebra on $W$, and $S\otimes\mathbf{1}$, $\mathbf{1}\otimes S$ and $S\otimes S$ are unitary subalgebras for this structure.*

*Proof* : Let $g = \exp(-\frac{1}{2}h_\kappa(0))$, an automorphism of $V_\Gamma$ acting by $\kappa(\lambda)^{-1/2}$ on $M(1)\otimes \mathrm{e}^\lambda$. Then $g\phi_1 g^{-1} = \phi_\kappa$, so that $\phi_\kappa$ is the PCT operator of the normalized invariant scalar product $(g^{-1}u|g^{-1}v)$, where $(\cdot|\cdot)$ is the scalar product attached to $\phi_1$, and $g^{-1}$ is a unitary isomorphism between the two structures.

By definition of $F_k$, we have $\phi_\kappa(E_k) = -F_k$ for $k = 1,\dots,4$. Since $\phi_\kappa$ is an involution, $\phi_\kappa(F_k) = -E_k$ and $\phi_\kappa(H_k) = [\phi_\kappa(E_k), \phi_\kappa(F_k)] = [F_k, E_k] = -H_k$. Hence $\phi_\kappa$ preserves $S_1\otimes\mathbf{1}$ and $\mathbf{1}\otimes S_1$, and therefore the affine subalgebras $A\otimes\mathbf{1}$ and $\mathbf{1}\otimes A$ and their Sugawara vectors, whose coefficients are real. It fixes $\omega_W$, hence $\omega_{C\otimes C}$, and a second exact computation gives $\phi_\kappa(u_k) = u_k$ for $k = 0,\dots,3$. By Lemma 7.15, $\phi_\kappa$ fixes $\omega_{C\otimes 1}$ and $\omega_{1\otimes C} = \omega_{C\otimes C} - \omega_{C\otimes 1}$.

We claim that $S\otimes\mathbf{1}$ is the set of vectors $v\in W$ annihilated by the modes $x_{(n)}$, $n\geqslant 0$, of all $x\in(\mathbf{1}\otimes S_1)\cup\{\omega_{1\otimes C}\}$. These vectors are annihilated by the nonnegative modes of the vertex subalgebra generated by this set [CKLW18, Example 5.27 p. 44], which contains the Sugawara vector $\omega_{1\otimes A}$. Hence $L_0^{1\otimes A}v = 0$ and $L_0^{1\otimes C}v = 0$, so that $L_0^{1\otimes S}v = 0$. On $M_z\otimes M_{\pi(z)}$ with $z\neq 0$, the spectrum of $L_0^{1\otimes S}$ is contained in $h_{\pi(z)}+\mathbb{N}$, with $h_{\pi(z)} > 0$. Hence $v\in S\otimes S_0 = S\otimes\mathbf{1}$. Conversely, $S\otimes\mathbf{1}$ commutes with $\mathbf{1}\otimes S$. Similarly, $\mathbf{1}\otimes S$ is the commutant of $(S_1\otimes\mathbf{1})\cup\{\omega_{C\otimes 1}\}$. The antilinear automorphism $\phi_\kappa$ maps the commutant of a set onto the commutant of its image. Hence $\phi_\kappa$ preserves $S\otimes\mathbf{1}$, $\mathbf{1}\otimes S$ and $S\otimes S$. These subalgebras are also stable under $L_1$, since $L_1 = L_1^{S\otimes 1} + L_1^{1\otimes S}$, and they are unitary subalgebras by [CKLW18, Proposition 5.23 p. 43]. ■

**Theorem 7.17** *Assume* ($\mathrm{H_J}$). *Then $S$, with the unitary structure of [Arh26, Theorem 10.10 p. 33], satisfies condition B of [Gui26, Section 3.4 p. 44] and Condition I+ of [Gui25, Definition 6.17 p. 70]. In particular, the following hold.*

1. *$S$ is completely unitary, and $\mathrm{Rep}^u(S)$ is a unitary modular tensor category.*
2. *Every unitary $S$-module is strongly integrable, and the functor $\Pi\colon \mathrm{Rep}^u(S)\to\mathrm{Rep}^f(\mathcal{A}_S)$ of [Gui26, Section 3.4 p. 45] is an equivalence of braided $C^*$-tensor categories. Here the target carries the tensor product and braiding of [Gui26]. The comparison with the DHR convention used in the preceding sections is given in the proof of part 3.*
3. *For every $S$-submodule $M$ of a lattice extension $H_\Lambda$, with the restriction of the scalar product $H_\chi$ of part 3 of Lemma 4.2, $\Pi(M)$ is the representation $\pi_M$ of Lemma 4.7 on $\mathcal{H}_M$. The assertion* (N) *holds.*

*Proof* : Equip $W$ with the unitary structure of Proposition 7.16, and let $V = \mathbf{1}\otimes S$. By Proposition 7.12, its coset is $V^c = \mathrm{Com}_W(V) = S\otimes\mathbf{1}$, and $V^{cc} = V$. By [CKLW18, Propositions 5.29 and 5.31 p. 45], $\omega_W = \nu+\nu'$ with $\nu\in V$ and $\nu'\in V^c$ the conformal vectors of the unitary vertex operator algebras $V$ and $V^c$. Then $\nu - \omega_{1\otimes S} = \omega_{S\otimes 1} - \nu'$ lies in $V\cap V^c\cap W_2 = \{0\}$, so that $V$ and $V^c$ are unitary vertex operator algebras isomorphic to $S$, with the conformal vector of $S$. In particular, $V$ is a unitary sub-VOA of $W$ in the sense of [Gui26, Section 3.6.1 p. 49], with commutant $V^c$. Both are regular by ($\mathrm{H_J}$) and [ABD04, Theorem 4.5]. The lattice vertex operator algebra $V_\Gamma$ satisfies condition B [Gui26, Theorem 3.46 p. 57]. This property is preserved by unitary isomorphisms, and two structures of unitary vertex operator algebra on a simple vertex operator algebra are unitarily isomorphic [CKLW18, Proposition 5.19 p. 40 and p. 41], so that $W$ satisfies condition B. By [Gui26, Theorem 3.34 p. 51], $V^c$ satisfies condition B. By [CKLW18, Proposition 5.19 p. 40 and p. 41] again, the isomorphism $S\to V^c$, composed with an automorphism of $S$, is a unitary isomorphism from $S$, with the unitary structure of [Arh26, Theorem 10.10 p. 33], onto $V^c$, so that $S$ satisfies condition B. The lattice vertex operator algebra $V_\Gamma$ satisfies Condition I+ [Gui25, Example 6.19(b) p. 71], and so does $W$ by the same transport. Since $V$ and $V^c$ are isomorphic to $S$, they are $C_2$-cofinite and rational under ($\mathrm{H_J}$).

Hence $V^c$ satisfies Condition I+ by [Gui25, Theorem 6.18, part 3, p. 70], and so does $S$, through the unitary isomorphism $S \to V^c$ above.

1. This is [Gui26, Theorem 3.15 p. 45].

2. By [Gui26, Theorems 3.16 and 3.17 p. 45], every unitary $S$-module is strongly integrable and $\Pi$ is a fully faithful $*$-functor. By [Gui26, Corollary 3.23 p. 48], it is an isomorphism onto a braided $C^*$-tensor subcategory of $\mathrm{Rep}(\mathcal{A}_S)$. The functor $\Pi$ is the CWX functor of [Gui25, Theorem 4.24 p. 43], since both send $M$ to the representation characterized by [Gui26, (3.8) p. 41], that is, by [Gui25, (4.6) p. 43]. The category $\mathrm{Rep}^f(\mathcal{A}_S)$ of [Gui25, p. 68] consists of the dualizable representations, which are the representations with finite statistical dimension. Since $S$ satisfies Condition I+, this functor is essentially surjective onto $\mathrm{Rep}^f(\mathcal{A}_S)$ by [Gui25, Theorem 6.14 p. 69 and Definition 6.17 p. 70]. Hence $\Pi$ is a braided equivalence, with the tensor product and braiding specified in the statement.

3. Since $S$ is a unitary subalgebra of $(H_\Lambda, H_\chi)$ by Theorem 4.5, the scalar product $H_\chi$ makes $M$ a unitary $S$-module. By part 1 of Lemma 4.7 and its proof, $\pi_M$ is a representation of $\mathcal{A}_S$ on $\mathcal{H}_M$ which maps the closed operator $Y(v,f)|_{\mathcal{H}_S}$, $f$ supported in $I$, to the closed operator $Y_M(v,f)$, of which $M$ is a core. By the energy bounds, $Y_M(v,f)$ is also the closure of the smeared field of $M$ on the smooth vectors. This is the defining property of the representation $\Pi(M)$, which is unique since $\mathcal{A}_S(I)$ is generated by these operators [Gui26, (3.8) p. 41], see also [Gui25, Definition 4.22 p. 43]. Hence $\Pi(M) = \pi_M$. The braided $C^*$-tensor category of [Gui26] is defined by Connes fusion [Gui26, p. 10]. By [Gui21, Theorem 6.6 p. 77 and (6.2) p. 74], it is equivalent to the category $\mathrm{DHR}_{I_0}(\mathcal{A}_S)$ of transportable endomorphisms localized in an interval $I_0$, with the tensor product $\rho \odot \sigma = \sigma \circ \rho$ and the braiding $\widetilde{c}$ of [Gui21, (6.1) p. 73], through a $*$-functor $G$ such that $\pi_0 \circ G(\mathcal{H}_i, \xi)$ is unitarily equivalent to $\pi_i$ for every object $(\mathcal{H}_i, \xi)$. For the tensor product $\rho \otimes \sigma = \rho \circ \sigma$, put $b_{\rho,\sigma} = \widetilde{c}_{\sigma,\rho}$. The identity functor, with tensor constraint $J_{\rho,\sigma} = b_{\rho,\sigma} \colon \rho \circ \sigma \to \sigma \circ \rho$, is a braided monoidal equivalence from $(\mathrm{DHR}_{I_0}(\mathcal{A}_S), \odot, \widetilde{c})$ to $(\mathrm{DHR}_{I_0}(\mathcal{A}_S), \otimes, b)$. Indeed, by the hexagon identities its coherence reduces to the Yang–Baxter equation, and $\widetilde{c}_{\rho,\sigma} J_{\rho,\sigma} = J_{\sigma,\rho} b_{\rho,\sigma}$. The two orientations in the definition of the DHR braiding give $b$ and $b^{\mathrm{rev}}_{\rho,\sigma} = b^{-1}_{\sigma,\rho}$. The forgetful functor $\mathrm{Rep}^u(S) \to \mathrm{Rep}(S)$ is a braided equivalence, since every $S$-module is unitarizable and its tensor product and braiding are those of $\mathrm{Rep}(S)$ [Gui19, Theorem 7.9 p. 53], [CGGH23, Remark 4.3 p. 18]. Combining these equivalences with a quasi-inverse of $\Pi$ gives a braided equivalence from $(\mathrm{DHR}_{I_0}(\mathcal{A}_S), \otimes, b)$ to $\mathrm{Rep}(S)$ which maps the class of $\pi_M$ to that of $M$. The same functor is a braided equivalence from $(\mathrm{DHR}_{I_0}(\mathcal{A}_S), \otimes, b^{\mathrm{rev}})$ to $\mathrm{Rep}(S)^{\mathrm{rev}}$, and its composition with the functor $F$ of Proposition 7.12, which fixes the classes of $S$, $M_b$, $M_a$ and $M_{c_0}$, is a braided equivalence into $\mathrm{Rep}(S)$ with the same property. Hence (N) holds for either braiding.

■

**Remark 7.18** The proof of Theorem 7.17 does not use Section 5.1. The unitarizability of the simple $S$-modules also follows directly from Proposition 7.16. For $0 \neq v \in M_{\pi(z)}$, the restriction of the scalar product of $W$ to $M_z \otimes \mathbb{C}v$ is an invariant scalar product for the unitary structure induced on $S \otimes \mathbf{1}$, and a unitary isomorphism from $S$ onto $S \otimes \mathbf{1}$ transports these scalar products to all simple $S$-modules, possibly permuting their isomorphism classes.

The computations of Lemmas 7.14 and 7.15 and of Proposition 7.16 are exact. For the subalgebras $\mathfrak{s}_k$ fixed above, they take place in $W_1$ and in the space $W_2^0$ of Lemma 7.15, of dimensions 496 and 644. A second implementation, with other choices of the subalgebras $\mathfrak{s}_k$ and of $\kappa$, gives $\dim \mathcal{G} = 16$, four solutions of the stronger system which prescribes the characteristic polynomial of $u_{(1)}$ on every multiplicity space, with the same Gram matrix $(u_k, u_l)$, and a positive character $\kappa$ for which $\phi_\kappa$ fixes them. The second implementation also shows that the

sign characters of $\Gamma$ which centralize $(S \otimes S)_1$ permute $u_0, \ldots, u_3$ transitively, so that the data used in Lemma 7.15 cannot distinguish them.

**Corollary 7.19** *Assume* ($\mathrm{H_J}$).

1. *The conclusions of part 4 of Theorem 7.2 hold. In particular, the net $\mathcal{A}_{H_{\Lambda_a}}$ carries $\mathcal{H}_2$, the net $\mathcal{A}_{H_{\Lambda_{c_0}}}$ carries $\mathcal{H}_3$, and $\mathcal{F}_a \cong \mathcal{H}_2$, $\mathcal{F}_{c_0} \cong \mathcal{H}_3$ for the categories of Theorem 5.6.*
2. *The conclusions of Proposition 7.10 hold.*
3. *The unitary modular tensor categories $\mathrm{Rep}^u(S)$ and $\mathrm{Rep}^f(\mathcal{A}_S)$ are equivalent, so that the correspondence of [CGGH23, Theorem 4.10 p. 21] between vertex operator algebra extensions of $S$ and irreducible local extensions of $\mathcal{A}_S$ applies.*

*Proof* : 1. This follows from Theorem 7.17 and part 4 of Theorem 7.2. Since $\mathcal{F}_x = \mathrm{Rep}(\mathcal{A}_S)_{\Theta_x}$ by part 4 of Theorem 5.6, the last assertion follows from part 2 of Theorem 7.2.

2. By part 1 of Theorem 7.17, $S$ is completely unitary.

3. This follows from parts 1 and 2 of Theorem 7.17, since $S$ is strongly local by Theorem 4.5 and $\mathcal{A}_S$ is completely rational by Corollary 5.9. ∎

# 8 The algebra of case (I) and the twin algebra

Theorem 4.5 applies to any unitary conformal subalgebra of $V'$, and Corollary 5.5 and Theorems 5.6 and 6.11 apply to any such subalgebra which is a Haagerup algebra. We record what is known for the Haagerup algebras of case (I).

**Riesen's algebra.** Let $\psi_{\mathrm{R}}(U^{\mathrm{R}}) \subset V'$ be the image of the algebra of [Rie26a] described in [Arh26, Proposition 10.36 p. 49]. Its currents have levels $(1, 39)$ [Arh26, Proposition 10.36 p. 49]. It contains the Sugawara vector $\omega_A$ of its currents and the vector $\frac{1}{2}(\omega - \omega_A)$, the unit of the Griess algebra of [Arh26, Lemma 10.1 p. 28], which lies in the span of $\psi_{\mathrm{R}}(e)$ and $\psi_{\mathrm{R}}(f)$ [Arh26, Proposition 10.3 p. 29 and Proposition 10.36 p. 49], hence the conformal vector $\omega$ of $V'$. Under ($\mathrm{H_R}$), it is a Haagerup algebra in case (I). If its generators are fixed by a positive rescaling of the standard antilinear involution of $V'$, as asserted in [Rie26a, Theorem 1 p. 1 and p. 4], see also the working manuscript [Rie26b, Sections 2–3 pp. 16–19], then $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ is a unitary subalgebra of $V'$ [CKLW18, Proposition 5.23 p. 43], and Theorem 4.5 applies to it. Under ($\mathrm{H_R}$) and this unitarity, Corollary 5.5 and Theorems 5.6 and 6.11 apply as well, and the statements of Section 7 which do not depend on Sections 7.4 and 7.5 hold with ($\mathrm{H_J}$) replaced by ($\mathrm{H_R}$) and $S$ replaced by $U^{\mathrm{R}}$. The proofs of Sections 7.4 and 7.5 use the currents of levels $(3, 13)$ of an algebra of case (II). Under ($\mathrm{H_R}$) alone, they apply to the twin $(U^{\mathrm{R}})^\natural$, which is a Haagerup algebra in case (II) [Arh26, Theorem 12.4, part 2, p. 67], see Theorem 8.6. If the algebra announced in [Hua26b] is a Haagerup algebra, it is isomorphic to $U^{\mathrm{R}}$ under ($\mathrm{H_R}$) [Arh26, Theorem 10.38 p. 50]. If $U^{\mathrm{R}}$ is completely unitary, for instance under ($\mathrm{H_R}$) by Theorem 8.6, Proposition 7.10 applies to $U^{\mathrm{R}}$ as well.

**The twin algebra.** Assume ($\mathrm{H_J}$). By [Arh26, Theorem 12.4 p. 67], there is a second Haagerup algebra $S^\natural$, in case (I), which embeds into $V'$ with image containing $\psi_{\mathrm{R}}(U^{\mathrm{R}})$, with equality in weights at most four. If ($\mathrm{H_R}$) holds as well, then $S^\natural \cong U^{\mathrm{R}}$ [Arh26, Theorem 12.4 p. 67].

**Complete unitarity of the twin algebras.** Following [Gui22, Definition 1.8 p. 13], a strongly rational unitary vertex operator algebra $V$ is completely unitary if every $V$-module is unitarizable and if the invariant sesquilinear forms of [Gui19, Section 6.2 p. 36] on the spaces of intertwining operators are positive definite. We use the geometric description of this positivity given by Gui [Gui24]. Let $W_1$, $W_2$ be unitary $V$-modules, let $A\colon W \to W'$ denote the antiunitary map $w \mapsto \langle \cdot | w \rangle$, and let $\theta$ be the PCT operator of $V$. For positive trinions $P$ and $Q$ and a sewing parameter $0 < q < r_{P,Q}$, where $r_{P,Q} > 0$ is the bound denoted $\delta\rho$ in [Gui24, Section 7.1 p. 43], put

$$K_q^V(w_1, w_2 | w_1', w_2') \overset{\text{def}}{=} \sum_{n \geqslant 0} q^n \sum_{e \in E_n} \omega_{W_1', P}(A w_1' \otimes w_1 \otimes e) \omega_{W_2, Q}(w_2 \otimes A w_2' \otimes \theta e) \tag{8.1}$$

for $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$, where $E_n$ is an orthonormal basis of $V_n$ and $\omega_{W,P}$ is the basic conformal block of [Gui24, Theorem 6.11 p. 40]. This is the sewn conformal block $\omega_{W_1',P} \#_q \omega_{W_2,Q}$ evaluated at $A w_1' \otimes w_1 \otimes w_2 \otimes A w_2'$, since the sewing contracts $V_n$ against its dual through the identification $A\theta$ of $V$ with its contragredient module [Gui24, Definition 3.4 p. 23 and (6.1e) p. 37]. The fusion product $W_1 \boxtimes W_2$ is geometrically positive if $\sum_{k,l} K_q^V(w_k, \tilde{w}_k | w_l, \tilde{w}_l) \geqslant 0$ for all finite families $(w_k)$ in $W_1$ and $(\tilde{w}_k)$ in $W_2$. This does not depend on $P$, $Q$ and $q$ [Gui24, Lemma 7.2 p. 44]. If $V$ is of CFT type and strongly rational and if $W_1 \boxtimes W_2$ is unitarizable, then $W_1 \boxtimes W_2$ is geometrically positive if and only if the form of [Gui19] on it is positive [Gui24, Corollary 8.4 p. 53]. Consequently, a strongly rational unitary vertex operator algebra all of whose modules are unitarizable is completely unitary if and only if $W_1 \boxtimes W_2$ is geometrically positive for all irreducible unitary $V$-modules $W_1$ and $W_2$, see also [Gui24, p. 47].

**Lemma 8.1** *Let $U$ be a Haagerup algebra equipped with a structure of unitary vertex operator algebra, with PCT operator $\theta$. Then $U_1$ has a Cartan subalgebra $\mathfrak{h}$ with $\theta(\mathfrak{h}) = \mathfrak{h}$. Put $\mathfrak{h}_{\mathbb{R}} \overset{\text{def}}{=} \{h \in \mathfrak{h} : \theta h = -h\}$. Then $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus \mathrm{i}\mathfrak{h}_{\mathbb{R}}$ as real vector spaces. On the algebraic finite-energy subspace of any ordinary unitary $U$-module, the operators $h_{(0)}$, $h \in \mathfrak{h}_{\mathbb{R}}$, are symmetric and essentially self-adjoint. Their self-adjoint closures strongly commute.*

*Proof* : Since $U$ is unitary, it is self-contragredient and of CFT type, so that $L_1 U_1 = 0$ [Li94]. For $x \in U_1$, the formula [CKLW18, (95) p. 43] for the adjoint of a mode gives $x^+_{(n)} = -(\theta x)_{(-n)}$ on $U$, and the same formula holds on every unitary $U$-module by the definition of unitary modules. The map $\theta$ restricts to an antilinear involutive automorphism of the Lie algebra $U_1 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ [Arh26, Theorem 8.5 p. 23]. Its fixed point set $\mathfrak{k}$ is a real form of $U_1$, and for $x \in \mathfrak{k}$ the operator $x_{(0)}$ is skew-symmetric on the algebraic finite-energy subspace. Its restriction to $U_1$ is skew-adjoint. Hence $\mathrm{ad}(\mathfrak{k})$ consists of skew-adjoint operators for the scalar product of $U_1$, and since the adjoint representation of the semisimple Lie algebra $U_1$ is faithful, $\mathfrak{k}$ is a compact real form. A maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ gives the Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathrm{i}\mathfrak{t}$, which is $\theta$-stable, and $\mathfrak{h}_{\mathbb{R}} = \mathrm{i}\mathfrak{t}$. For $h \in \mathfrak{h}_{\mathbb{R}}$, we have $h^+_{(0)} = -(\theta h)_{(0)} = h_{(0)}$. Each $h_{(0)}$ preserves the finite-dimensional eigenspaces of $L_0$. On each such space, the operators $h_{(0)}$, $h \in \mathfrak{h}_{\mathbb{R}}$, are commuting Hermitian matrices and therefore admit a common orthonormal eigenbasis. Taking the union of these bases gives a common orthonormal eigenbasis of the Hilbert space completion. The operators on the algebraic finite-energy subspace are consequently essentially self-adjoint, and their closures are diagonal in this common basis. In particular, their spectral projections commute. ∎

All Cartan subalgebras of $U_1$ are conjugate under inner automorphisms $\exp(x_{(0)})$ of $U$, $x \in U_1$, and the Heisenberg commutant and the associated lattice do not depend on this choice

up to isomorphism [Arh26, p. 61]. We therefore apply the results of [Arh26, Section 12 pp. 61–68] to a $\theta$-stable Cartan subalgebra.

**Lemma 8.2** *Let $U$ be a Haagerup algebra equipped with a structure of unitary vertex operator algebra, let $\mathfrak{h} \subset U_1$ be a $\theta$-stable Cartan subalgebra and let $\Omega = \mathrm{Com}_U(M(\mathfrak{h}))$, with the unitary structure induced by $U$. If $U$ is completely unitary, then $\Omega$ is completely unitary.*

*Proof : Unitary subalgebras.* The Heisenberg subalgebra $M(\mathfrak{h})$ is generated by the $\theta$-stable space $\mathfrak{h}$ of quasi-primary vectors, hence it is a unitary subalgebra of $U$ [CKLW18, Example 5.26 p. 44]. By [CKLW18, Example 5.27 p. 44], $\Omega$ and $V_L \overset{\text{def}}{=} \mathrm{Com}_U(\Omega)$ are unitary subalgebras, and $\Omega$ and $V_L$ form a dual pair, $L$ being the associated lattice [Arh26, Proposition 12.2 p. 63]. By [Ten19a, Proposition 2.21 p. 16], the span $B$ of the vectors $a_{(-1)}b$, $a \in \Omega$, $b \in V_L$, is a unitary conformal subalgebra of $U$, unitarily isomorphic to $\Omega \otimes V_L$. We write $B = \Omega \otimes V_L$.

*Weights.* The algebra $\Omega$ is strongly rational by [Arh26, Proposition 12.2 p. 63]. By the same proposition, $U = \bigoplus_{\alpha \in L'/L} U^\alpha$ with $U^\alpha = \Omega^{\tau(\alpha)} \otimes V_{\alpha+L}$ and $U^0 = B$, and the $\mathfrak{h}$-weights of $U^\alpha$ lie in $\alpha + L$. Let $X$ be a unitary $U$-module. By Lemma 8.1, the finite-dimensional eigenspaces of $L_0$ in $X$ have common orthonormal eigenbases for the operators $h_{(0)}$, $h \in \mathfrak{h}_\mathbb{R}$. Thus $X$ is the algebraic direct sum of its mutually orthogonal $\mathfrak{h}$-weight spaces, and its Hilbert space completion is their Hilbert direct sum. For $v \in U^\alpha$, the operator $Y_X(v,z)$ shifts the weights by elements of $\alpha + L$, and the Virasoro modes of $U$, which are those of $B$, preserve the weights and each $U^\alpha$.

*Strong unitarity of $\Omega$.* Let $\mu$ be an irreducible $\Omega$-module. By the proof of [Arh26, Proposition 12.2, part 2, pp. 63–64], there are $\beta \in L'/L$ and a $U$-module $X$ containing $\mu \otimes V_{\beta+L}$ as a $B$-submodule, and we may take $X$ irreducible. Since $U$ is completely unitary, $X$ admits a unitary structure. Fix a non-zero $m \in V_{\beta+L}$ and put $\langle u|u'\rangle_\mu = \langle u \otimes m | u' \otimes m\rangle_X$. For $a \in \Omega$, the modes of $a \otimes \mathbf{1}$ act on the first factor, $\theta(a \otimes \mathbf{1}) = \theta a \otimes \mathbf{1}$, and $L_n(a \otimes \mathbf{1}) = L_n^\Omega a \otimes \mathbf{1}$ for $n \geqslant 0$. Hence $\langle \cdot | \cdot \rangle_\mu$ is an invariant scalar product on $\mu$. Thus every $\Omega$-module is unitarizable, and so is every $B$-module, by [Gui22, Proposition 3.31 p. 48] and its proof, since $V_L$ is completely unitary [Gui22, Corollary 3.32 p. 49].

*Slices.* Let $\mu_1$, $\mu_2$ be irreducible $\Omega$-modules, and choose irreducible unitary $U$-modules $X_i$ containing $Y_i = \mu_i \otimes V_{\beta_i+L}$ as above. We give $Y_i$ the scalar product of $X_i$. Two invariant scalar products on the irreducible $B$-module $Y_i$ differ by a positive endomorphism of $Y_i$, which is a positive scalar by Schur's lemma. Hence the scalar product of $Y_i$ is a positive multiple of the tensor product of the scalar product of $\mu_i$ and of an invariant scalar product of $V_{\beta_i+L}$. The map $A$ of $X_1$ sends $Y_1$ onto the $B$-submodule of $X_1'$ formed by the linear forms $\langle \cdot | w \rangle$, $w \in Y_1$, which vanish on the orthogonal complement of $Y_1$, a $B$-submodule since the adjoints of the modes of $B$ are modes of $B$, so that $A(Y_1)$ is unitarily isomorphic to $Y_1'$, with weights in $-\beta_1 + L$. Let $w_1, w_1' \in Y_1$ and $w_2, w_2' \in Y_2$, and choose the bases $E_n$ in (8.1) adapted to the orthogonal decomposition $U_n = \bigoplus_\alpha U_n^\alpha$. By [Gui24, (6.11) and (6.12) p. 40], $\omega_{X_1', P}(Aw_1' \otimes w_1 \otimes e)$ is of the form $\langle Y_{X_1'}(\tilde{e}, \mathrm{i}) T_1 A w_1' | A T_2 w_1 \rangle$, where $\tilde{e} = 2^{L_0} \mathrm{e}^{-\mathrm{i}L_1} T_3 e$ and $T_1$, $T_2$, $T_3$ are the operators $U_0(\cdot)$ of [Gui24, (6.12) p. 40]. They are built from the modes $L_n$, $n \geqslant 0$, which commute with the operators $h_{(0)}$, $h \in \mathfrak{h}$, and preserve each $U^\alpha$. Hence $\tilde{e} \in U^\alpha$ if $e \in U^\alpha$, and $T_1$, $T_2$ preserve the $\mathfrak{h}$-weights. For $\alpha \neq 0$, the two vectors have weights in the disjoint cosets $-\beta_1 + \alpha + L$ and $-\beta_1 + L$, so that this term vanishes. For $\alpha = 0$, we have $e \in B$, $\theta e \in B$, and the two factors are the basic conformal blocks of the vertex operator algebra $B$ for the unitary modules $Y_1'$ and $Y_2$. Hence

$$K_q^U(w_1, w_2 | w_1', w_2') = K_q^B(w_1, w_2 | w_1', w_2'). \tag{8.2}$$

Since $U$ is completely unitary, $X_1 \boxtimes X_2$ is unitarizable and its form is positive, so that it is geometrically positive [Gui24, Corollary 8.4 p. 53]. By (8.2), $Y_1 \boxtimes Y_2$ is geometrically positive

for $B$, up to the positive factors coming from the normalization of the scalar products, which multiply $K_q^B$ by a positive constant.

*From $B$ to $\Omega$.* On elementary tensors, the formula [Gui24, (6.11) p. 40] and the decompositions $L_n = L_n^\Omega \otimes 1 + 1 \otimes L_n^{V_L}$ and $Y(u \otimes m, z) = Y(u, z) \otimes Y(m, z)$ show that the basic conformal blocks of $B$ are the products of those of $\Omega$ and of $V_L$, and the contraction over $B_n = \bigoplus_{a+b=n} \Omega_a \otimes (V_L)_b$ factorizes. The three sewing series converge absolutely for $0 < q < r_{P,Q}$ [Gui24, Section 7.1 p. 43], which justifies regrouping the sum. Hence $K_q^B = K_q^\Omega \cdot K_q^{V_L}$ on elementary tensors. Fix non-zero vectors $m_i \in V_{\beta_i+L}$, and put $w_k = u_k \otimes m_1$, $\tilde{w}_k = \tilde{u}_k \otimes m_2$ for finite families $(u_k)$ in $\mu_1$ and $(\tilde{u}_k)$ in $\mu_2$. Then

$$\sum_{k,l} K_q^B(w_k, \tilde{w}_k | w_l, \tilde{w}_l) = K_q^{V_L}(m_1, m_2 | m_1, m_2) \sum_{k,l} K_q^\Omega(u_k, \tilde{u}_k | u_l, \tilde{u}_l) \geqslant 0.$$

The series defining $K_q^{V_L}(m_1, m_2 | m_1, m_2)$ converges absolutely for $0 \leqslant q < r_{P,Q}$, and its value at $q = 0$ is $\omega_{N_1',P}(Am_1 \otimes m_1 \otimes \mathbf{1}) \omega_{N_2,Q}(m_2 \otimes Am_2 \otimes \mathbf{1})$, with $N_i = V_{\beta_i+L}$, which is positive by [Gui24, Proposition 6.13 p. 42]. For $q$ small enough, $K_q^{V_L}(m_1, m_2 | m_1, m_2) > 0$, and $\mu_1 \boxtimes \mu_2$ is geometrically positive for $\Omega$. Since $\mu_1 \boxtimes \mu_2$ is unitarizable, its form is positive by [Gui24, Corollary 8.4 p. 53]. This holds for all irreducible $\mu_1$, $\mu_2$, so that $\Omega$ is completely unitary. ■

**Theorem 8.3** *Let $U$ be a Haagerup algebra and $U^\natural$ its twin [Arh26, Theorem 12.3 p. 65].*

1. *If $U$ is completely unitary for some structure of unitary vertex operator algebra, then $U^\natural$ carries, for each structure of unitary vertex operator algebra on $V_{L^\natural}$, one and only one structure of unitary vertex operator algebra extending the tensor product structure of $\Omega_U \otimes V_{L^\natural}$, where $\Omega_U$ has the structure of Lemma 8.2, and $U^\natural$ is completely unitary for this structure.*

2. *The following are equivalent: $U$ is completely unitary, $U^\natural$ is completely unitary, and $\Omega_U$ is completely unitary, each for some, equivalently every, structure of unitary vertex operator algebra.*

*Proof* : Complete unitarity is preserved by unitary isomorphisms of unitary vertex operator algebras, by transport of structure. Since two structures of unitary vertex operator algebra on a simple vertex operator algebra are unitarily isomorphic [CKLW18, Proposition 5.19 p. 40 and p. 41], see also [CGGH23, Remark 4.4 p. 18], complete unitarity of a simple unitary vertex operator algebra does not depend on its unitary structure.

1. Choose a $\theta$-stable Cartan subalgebra of $U_1$ (Lemma 8.1). By Lemma 8.2, $\Omega_U$ is completely unitary, and so is the lattice algebra $V_{L^\natural}$ [Gui22, Corollary 3.32 p. 49]. By [Gui22, Proposition 3.31 p. 48], the regular algebra $\Omega_U \otimes V_{L^\natural}$ is completely unitary. By [Arh26, Theorem 12.3, part 1, p. 65], $U^\natural$ is a simple vertex operator algebra of CFT type extending $\Omega_U \otimes V_{L^\natural}$ with the same conformal vector. The assertion follows from [CGGH23, Theorem 4.7 p. 19].

2. If $U$ is completely unitary, so is $U^\natural$ by part 1. If $U^\natural$ is completely unitary, so is its Heisenberg commutant by Lemma 8.2, and it is isomorphic to $\Omega_U$ [Arh26, p. 61 and Theorem 12.3, part 1, p. 65]. If $\Omega_U$ is completely unitary, so is the regular algebra $\Omega_U \otimes V_L$ [Gui22, Proposition 3.31 p. 48 and Corollary 3.32 p. 49]. Since $U$ is a simple extension of $\Omega_U \otimes V_L$ of CFT type with the same conformal vector [Arh26, Proposition 12.2, part 1, p. 63], $U$ is completely unitary by [CGGH23, Theorem 4.7 p. 19]. ■

**Remark 8.4** Theorem 8.3 shows that complete unitarity is a property of the pair $\{U, U^\natural\}$, which forms a single hyperbolic genus [Arh26, Theorem 12.3, part 4, p. 65], and of their common

Heisenberg commutant, and its proof uses no computer calculation beyond those of [Arh26, Theorem 8.5 and Section 12]. Under ($\mathrm{H_J}$), $S$ is completely unitary by Theorem 7.17, so that $S^\natural$ and $\Omega_S$ are completely unitary by part 2 of Theorem 8.3. Every Haagerup algebra with currents of levels $(1, 39)$ is then isomorphic to $S^\natural$ [Arh26, Theorem 12.4, part 1, p. 67], hence completely unitary. This applies to the algebra of [Hua26b], whose currents have levels $(1, 39)$ [Hua26b, Abstract p. 1 and Section 3 p. 3], as soon as it is a Haagerup algebra, and to $U^{\mathrm{R}}$ under ($\mathrm{H_R}$), whose currents have levels $(1, 39)$ [Arh26, Proposition 10.36 p. 49]. These two conclusions hold without ($\mathrm{H_J}$) by Theorem 8.6. Conversely, if the algebra of [Hua26b] is a Haagerup algebra and is completely unitary, part 2 of Theorem 8.3 gives the complete unitarity of $S$ under ($\mathrm{H_J}$) without the computations of Section 7.5.

**Corollary 8.5** *Assume* ($\mathrm{H_J}$).

1. *The vertex operator algebras $\Omega_S$ and $S^\natural$ satisfy, for every structure of unitary vertex operator algebra, in particular for the structure induced by $S$ and for that of part 1 of Theorem 8.3, Condition I+ of [Gui25, Definition 6.17 p. 70]. In particular, they are strongly local, their conformal nets are completely rational, and their CWX functors, equipped with the Wassermann tensorators, are equivalences of braided $C^*$-tensor categories onto the categories of representations of finite statistical dimension.*

2. *Let $U \subset V'$ be a Haagerup algebra which contains the conformal vector of $V'$ and is a unitary subalgebra for a normalized invariant scalar product $\{\cdot|\cdot\}$ on $V'$, and equip each $H_\Lambda$ with the scalar product $\{\cdot|\cdot\}_\Lambda$ of part 2 of Lemma 4.2. Then $U$ satisfies Condition I+ for the induced unitary structure. For each $U$-submodule $M$ of $H_\Lambda$, with the restriction of $\{\cdot|\cdot\}_\Lambda$, the CWX representation of $M$ is the representation $\pi_M$ of Lemma 4.7 on $\mathcal{H}_M$. Consequently, with the notation of Section 5.2, there is a braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_U) \to \mathrm{Rep}(U)$ with $\Phi(\pi_0) \cong U$ and $\Phi(\pi_x) \cong M_x$ for $x \in \{b, a, c_0\}$. We call this assertion the analogue of* (N) *for $U$.*

*In particular, under* ($\mathrm{H_J}$) *and* ($\mathrm{H_R}$)*, part 2 applies to $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ as soon as it is a unitary subalgebra of $V'$ for some normalized invariant scalar product, as asserted in [Rie26a, Theorem 1 p. 1 and p. 4].*

*Proof* : 1. By Theorem 7.17, $S$ satisfies Condition I+. Choose the Cartan subalgebra of Lemma 8.1 for the prescribed unitary structure on $S$. The associated lattice algebra $V_L$ is a unitary subalgebra of $S$, its commutant is $\Omega_S$, $\Omega_S$ is strongly rational by [Arh26, Proposition 12.2 p. 63], and $V_L$ is regular by [Gui22, Corollary 3.32 p. 49]. Consequently, [Gui25, Theorem 6.18(3) p. 70] implies that $\Omega_S$ satisfies Condition I+. The even lattice algebra $V_{L^\natural}$ satisfies Condition I+ by [Gui25, Example 6.19(b) p. 71]. Thus $\Omega_S \otimes V_{L^\natural}$ satisfies Condition I+ by [Gui25, Theorem 6.18(1) p. 70]. The twin $S^\natural$ is a simple extension of this algebra with the same conformal vector by [Arh26, Theorem 12.3 p. 65]. Since Condition I includes complete unitarity [Gui25, Definition 4.15 p. 41], the algebra $\Omega_S \otimes V_{L^\natural}$ is completely unitary, and the unique unitary structure supplied by [CGGH23, Theorem 4.7 p. 19] makes $S^\natural$ a unitary extension of it. Applying [Gui25, Theorem 6.18(2) p. 70], we obtain Condition I+ for $S^\natural$. The assertions about the nets and the functors follow from [Gui25, (0.3) p. 8, Remark 4.16 p. 41, Theorem 6.14 pp. 68–69 and Definition 6.17 p. 70].

2. Under ($\mathrm{H_J}$), every Haagerup algebra is isomorphic to $S$ or $S^\natural$ by [Arh26, Theorem 12.4 p. 67]. Condition I+ is preserved under unitary isomorphism, since such an isomorphism transports the vertex operators, their energy bounds, the local algebras and the CWX functor. Two unitary structures on a simple vertex operator algebra are unitarily isomorphic by [CKLW18, Proposition 5.19 pp. 40–41]. Hence $U$ satisfies Condition I+ for its prescribed induced structure.

By Lemma 4.2, $H_\Lambda$ can be equipped with a unitary structure extending that of $V'$, so that $U$ is a unitary conformal subalgebra of $H_\Lambda$. For a $U$-submodule $M \subset H_\Lambda$, Lemma 4.7 identifies the closed smeared field in the representation $\pi_M$ with the closure of the smeared module field on $M$. This is the defining property of the strongly integrated CWX representation [Gui25, Definition 4.22 p. 43]. The uniqueness in this definition therefore identifies the two representations on $\mathcal{H}_M$. Taking the inverse of the CWX braided equivalence and forgetting the unitary structures gives the required equivalence with $\mathrm{Rep}(U)$. As in the proof of Theorem 7.17, the passage to DHR endomorphisms uses [Gui21, Theorem 6.6 p. 77]. If the angular orientation of Section 7 gives the reversed braiding, this is handled by the label-preserving equivalence of part 1 of Proposition 7.12, transported through $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$. This equivalence fixes the classes 0, $b$, $a$ and $c_0$, so the stated identifications of modules are preserved. ∎

**Theorem 8.6** *Let $U$ be a Haagerup algebra.*

1. *The algebra $U$ admits a structure of unitary vertex operator algebra and satisfies Condition I+ of [Gui25, Definition 6.17 p. 70] for every such structure. In particular, $U$ is completely unitary.*

2. *Suppose that $U \subset V'$ contains the conformal vector of $V'$ and is a unitary subalgebra for a normalized invariant scalar product on $V'$. Then the analogue of* (N) *holds for this embedding: there is a braided equivalence $\Phi\colon \mathrm{Rep}(\mathcal{A}_U) \to \mathrm{Rep}(U)$ with $\Phi(\pi_0) \cong U$ and $\Phi(\pi_x) \cong M_x$ for $x \in \{b, a, c_0\}$, where the representations $\pi_x$ are those of Lemma 4.7 in the two lattice extensions of $V'$.*

*Proof* : *Case (II).* Let $X$ be a Haagerup algebra in case (II). The construction in the proof of Proposition 7.12 uses only the braided equivalence $\mathrm{Rep}(X) \cong Z(\mathcal{H}_3)$, positivity, strong rationality, and the levels and module data of [Arh26, (4.1) p. 11, Theorem 8.5 p. 23 and Table 2 p. 24]. It therefore gives a holomorphic extension

$$W_X = \bigoplus_z M_z \otimes M_{\pi(z)} \cong V_{E_8 \oplus E_8}$$

of $X \otimes X$, with $X \otimes \mathbf{1}$ and $\mathbf{1} \otimes X$ as mutual commutants. No unitary structure on $X$ is needed for this construction.

The affine subalgebra generated by $X_1$ and its commutant have the properties of [Arh26, Corollary 9.2 p. 26]. Hence the proofs of Lemmas 7.14 and 7.15 apply to $W_X$: they use only these properties and the module data above, and their computations take place in $V_\Gamma$. The PCT operator constructed in Proposition 7.16 preserves both copies of $X_1$ and fixes all four candidates for the conformal vector of the affine commutant. The commutant argument in its proof therefore makes $X \otimes \mathbf{1}$ and $\mathbf{1} \otimes X$ unitary subalgebras of $W_X$. Their induced conformal vectors are the prescribed ones, as in the first paragraph of the proof of Theorem 7.17. This gives a unitary structure on $X$. The lattice algebra $W_X$ satisfies Condition I+ by [Gui25, Example 6.19(b) p. 71] and transport of its unitary structure. Since the two mutual commutants are strongly rational, [Gui25, Theorem 6.18(3) p. 70] gives Condition I+ for $X$.

*Case (I).* If $U$ is in case (I), then $X = U^\natural$ is in case (II) and $X^\natural \cong U$ by [Arh26, Theorem 12.3, parts 1 and 3, p. 65]. Give $X$ the unitary structure just constructed and choose a stable Cartan subalgebra as in Lemma 8.1. Its associated lattice algebra $V_L$ is a unitary subalgebra, as in the proof of Lemma 8.2, and is regular by [Gui22, Corollary 3.32 p. 49]. Its commutant $\Omega_X$ is strongly rational by [Arh26, Proposition 12.2 p. 63]. Thus [Gui25, Theorem 6.18(3) p. 70] gives Condition I+ for $\Omega_X$. The algebra $\Omega_X \otimes V_{L^\natural}$ satisfies Condition I+ by [Gui25, Theorem 6.18(1) and Example 6.19(b) pp. 70–71]. In particular, it is completely unitary, since

Condition I includes complete unitarity [Gui25, Definition 4.15 p. 41]. The simple extension $X^\natural$ of this algebra is therefore a unitary extension by [CGGH23, Theorem 4.7 p. 19], and it satisfies Condition I+ by [Gui25, Theorem 6.18(2) p. 70]. This proves the existence assertion for $U$. In both cases, any two unitary structures on the simple algebra $U$ are unitarily isomorphic by [CKLW18, Proposition 5.19 pp. 40–41]. Transport of vertex operators, energy bounds and the CWX functor preserves Condition I+, so it holds for every unitary structure on $U$. This proves part 1.

*The prescribed embedding.* Under the hypotheses of part 2, equip $U$ with the induced unitary structure and extend the scalar product of $V'$ to each $H_\Lambda$ by Lemma 4.2. Part 1 gives Condition I+ for this structure on $U$. Its net is consequently completely rational and its CWX functor is a braided equivalence onto the representations of finite statistical dimension [Gui25, (0.3) p. 8, Remark 4.16 p. 41, Theorem 6.14 pp. 68–69 and Definition 6.17 p. 70]. For every $U$-submodule $M \subset H_\Lambda$, Lemma 4.7 identifies the closed smeared fields in $\pi_M$ with those defining the CWX representation, so the two representations coincide. Taking the inverse equivalence and forgetting the unitary structures gives the stated correspondence. The passage to the DHR convention is the one in part 3 of the proof of Theorem 7.17, using [Gui21, Theorem 6.6 p. 77 and (6.2) p. 74]. For the opposite angular orientation, the categorical argument of part 1 of Proposition 7.12, transported through $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$, gives a braid-reversing equivalence fixing the classes $0, b, a, c_0$. Thus the correspondence is unchanged, and part 2 follows. ∎

Under $(\mathrm{H_R})$, the algebra $U^{\mathrm{R}}$ is a Haagerup algebra, so it is completely unitary by Theorem 8.6. If moreover $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ is a unitary subalgebra of $V'$ for some normalized invariant scalar product, as asserted in [Rie26a, Theorem 1 p. 1 and p. 4], then the analogue of (N), the analogue of part 4 of Theorem 7.2 and the conclusions of Proposition 7.10 hold for $\psi_{\mathrm{R}}(U^{\mathrm{R}})$. Similarly, the algebra of [Hua26b] is completely unitary as soon as it is a Haagerup algebra. The proofs in Sections 2–7 do not use Theorem 8.6.

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**Competing interests** The author declares that he has no competing interests.

**Data availability** No data sets were generated during this study, apart from the exact outputs of the computer programs of Section 7.5. The proofs of Section 6 do not use computer calculations. The identities of Lemma 6.4 and the arithmetic of Lemma 6.5 and Proposition 6.6 were checked with short Python scripts, which are included in the archive described below. Apart from these checks and from the computations of Section 7.5, the paper uses no computation beyond those of [Arh26]. The programs for Section 7.5, in two implementations, their common input tables `t12_table.json` and `t12_W1.json`, which encode [Arh26, Table 2 p. 24], and their exact outputs, including the cocycle $\varepsilon_8$, the $\mathfrak{sl}_2$-triples of $\mathfrak{s}_1, \dots, \mathfrak{s}_4$, the vectors $u_0, \dots, u_3$ and the character $\kappa$, are deposited in the archive `Calculs-HU-v08`, to be made available as ancillary files of the arXiv submission. The file `README-Calculs-HU-v08.txt` of this archive gives the order of execution and lists the consistency checks which are not used in the proofs and require additional data from [Arh26]. The files `omega_certificate.json` and `verify_omega_certificate.py` of the same archive give a portable certificate for the affine coordinates, ideal equality and PCT checks used in Lemma 7.15 and Proposition 7.16. The verifier

requires only Python and SymPy. It takes the mode-derived product and action tensors as input and does not recompute the full commutant kernel or these tensors from vertex operators. The additional programs in `verification-independante` reconstruct the commutant kernel, the product and bilinear form on $\mathcal{G}$, and the traces and traces of products of the multiplicity-space actions directly from lattice current modes, without importing the original vertex operator algebra programs. These trace invariants determine all the linear and quadratic constraints used in Lemma 7.15. The accompanying README states the inputs, dependencies and exact rank checks. These calculations are exact algebraic computations, not a formal verification of the vertex operator algebra implementation.

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