hexagon:2611.00033v1

On Bounds for Reciprocal Sums and Residues in Collatz Orbits, and Their Consequences

Behzad Nikzad* Karl Frinkle† Fatemeh Bagherzadeh‡

September 22, 2026 (draft) October 26, 2026 (revised)

We prove uniform bounds for reciprocal sums and residue products along Collatz paths. Let $\mathcal{F}$ be a finite nonrepeating shortcut path whose input states are greater than 1, let $\mathcal{R}_{\mathcal{F}}$ be the sum of the reciprocals of its odd inputs, and let $\rho_{\mathcal{F}}$ be its finite-path residue. Then

$$\mathcal{R}_{\mathcal{F}} < 1, \qquad \rho_{\mathcal{F}} < \frac{4}{3}.$$

For the first-arrival trajectory of a terminating positive integer n, these inequalities give

$$\mathcal{R}(n) < 1, \qquad \rho(n) = \frac{2^{E(n)}}{3^{O(n)}n} < \frac{4}{3},$$

where $E(n)$ and $O(n)$ are the division and odd-operation counts. More generally, when each odd value greater than 1 is counted once, the corresponding reciprocal-sum and residue bounds hold for every forward orbit, without a termination assumption.

The proof combines verified finite height-delay records and residue bounds, canonical-block reciprocal estimates, and an exact finite count organized by gap brackets. Gap brackets, closely related to parity vectors, record the divisions by two between successive odd inputs and organize the finite counting argument; together with a related coordinate representation, they also yield exact formulas for reconstructing finite trajectories.

After taking base-2 logarithms, the bound $\rho_{\mathcal{F}} < 4/3$ leaves an uncertainty of less than $1/2$ in the relation between the even- and odd-operation counts. Since these counts are integers, either count is determined exactly by the other count and the path endpoints. We obtain nearest-integer restrictions on the changes in E and O between adjacent integers, apart from explicit small exceptions, and prove that adjacent terminating integers of equal height have identical operation counts. Thus every equal-height streak lies in a single operation-count class.

We further introduce the charge $\lambda = E - 2O$ and associated height and charge slice functions to study the persistence and eventual break of equal-height streaks. We distinguish class records, which are the least starting integers attaining specified values, from delay records, which attain new extrema relative to all smaller starting integers. The resulting record theory shows that λ-delay records form a subsequence of the odd-count delay records, which form a subsequence of the common division-count and height-delay list. On the class side, the nonpositive charge-class records lie among the odd-count class records, which lie among the division-count class records, which in turn lie among the height-class records. The corresponding hierarchies also hold for first-arrival paths to a common endpoint z when $z = 1$ or when $z > 2$ is nonperiodic under the shortcut map.

*behzad.nikzad@tempered.ai
†kfrinkle@se.edu
‡nasrin.bagherzadeh@tempered.ai

Keywords. Collatz map; $3x + 1$ problem; reciprocal sums; residue products; gap brackets; equal-height streaks; class records; delay records.

Introduction

The Collatz problem asks whether repeated iteration of the map

$$C(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2 \end{cases}$$

reaches 1 for every positive integer n. Whenever the trajectory of n reaches 1, we say that it *terminates*. For such an n, let $E(n)$ denote the number of divisions by two and let $O(n)$ denote the number of odd operations $x \mapsto 3x + 1$ before the first arrival at 1. The *height* is the total number of ordinary Collatz operations,

$$H(n) = E(n) + O(n) = \min\{k \geq 0 : C^k(n) = 1\}.$$

The problem, its history, and the limitations of current general methods are surveyed by Lagarias [7, 9]; see also Terras's foundational work on stopping times [20] and Tao's almost-all result [19].

Our central concern is the multiplicative correction that relates the operation counts to the endpoints of a Collatz path. We work primarily with the shortcut map, which sends an even input n to $n/2$ and combines an odd operation with the division by two that immediately follows it into the single step $(3n + 1)/2$.

For a finite shortcut path $\mathcal{F} = (a, \ldots, z)$, every shortcut step represents one division by two, while each odd input also represents one operation $x \mapsto 3x + 1$. Accordingly, let $E_{\mathcal{F}}$ be the number of shortcut steps, let $O_{\mathcal{F}}$ be the number of odd inputs, and put

$$H_{\mathcal{F}} = E_{\mathcal{F}} + O_{\mathcal{F}}.$$

The finite-path residue and odd-input reciprocal sum are

$$\rho_{\mathcal{F}} = \frac{2^{E_{\mathcal{F}}} z}{3^{O_{\mathcal{F}}} a}, \qquad \mathcal{R}_{\mathcal{F}} = \sum_{\substack{x \text{ an odd input} \\ \text{of } \mathcal{F}}} \frac{1}{x}.$$

The residue also has the product representation

$$\rho_{\mathcal{F}} = \prod_{\substack{x \text{ an odd input} \\ \text{of } \mathcal{F}}} \left(1 + \frac{1}{3x}\right),$$

so it measures the accumulated correction contributed by the odd inputs. On the complete first-arrival path of a terminating integer n, these quantities specialize to

$$\rho(n) = \frac{2^{E(n)}}{3^{O(n)} n} \qquad \text{and} \qquad \mathcal{R}(n) = \sum_{x} \frac{1}{x},$$

where the latter sum is over the odd inputs before the first arrival at 1. Roosendaal calls $\rho(n)$ the residue and conjectures that $\rho(993)$ is its global maximum [14].

The central result of this paper is that every finite nonrepeating shortcut path whose input states are greater than 1 satisfies

$$\mathcal{R}_{\mathcal{F}} < 1, \qquad \rho_{\mathcal{F}} < \frac{4}{3}.$$

No restriction is imposed on the endpoint z or on the subsequent orbit. In particular, the theorem is finite-path in nature and does not assume that the starting integer terminates. Passing to successively longer initial portions of an arbitrary forward orbit gives

$$\mathcal{R}^{\circ}(n) < 1, \qquad \rho^{\circ}(n) < \frac{4}{3},$$

when each distinct odd value greater than 1 is counted once. These once-counted bounds therefore include nonterminating orbits and cycles. On a complete first-arrival path they give $\mathcal{R}(n) < 1$ and $\rho(n) < 4/3$ for every terminating positive integer n.

The proof combines analytic estimates with explicitly stated finite inputs. The two record bounds used in Section 2 are (34) and (35): the height-delay bound through

$$D = 280\,190\,771\,772\,317\,584\,95$$

reported by Roosendaal [14], and the residue record $\rho(993)$ below 2^{32}, recorded in OEIS A127789 [11]. These finite records are taken as inputs. Above the checked range, canonical blocks give uniform reciprocal estimates, and Appendix 1 sharpens them by an exact finite count organized by gap brackets. A gap bracket records the divisions by two between successive odd inputs. Besides organizing the count, gap and coordinate brackets give exact affine formulas that reconstruct a finite trajectory from its division data and endpoint. Thus the same encoding has both an algebraic role in describing trajectories and a combinatorial role in proving the residue bound. Related finite affine remainders and the use of delay-record data are studied by Rozier and Terracol [15], whose first-descent and paradoxical-sequence results are distinct from the finite-path estimate proved here.

The four-thirds threshold has exact consequences because the operation counts are integers. For a finite path $\mathcal{F} = (a, \ldots, z)$, the residue identity and the main bound give

$$O_{\mathcal{F}} \log_2 3 + \log_2(a/z) \le E_{\mathcal{F}} < O_{\mathcal{F}} \log_2 3 + \log_2(a/z) + \log_2(4/3).$$

The interval has length $\log_2(4/3) < 1/2$, so its integer member is uniquely determined. Consequently, for fixed endpoints, either operation count determines the other exactly, without any assumption about the orbit after z. On complete first-arrival paths, any one of $H(n)$, $E(n)$, and $O(n)$, together with n, determines the other two. These results supply the residue hypothesis in Luo's conditional recovery formulas [10] and prove Ruggiero's proposed height ceiling formula [16], with a nearest-integer refinement.

The same estimate compares different paths. For two finite paths from starting values a and b to a common endpoint z, the ratio of their residues confines the differences of their even and odd counts to a logarithmic strip of width $2\log_2(4/3) < 1$. This gives an exact nearest-integer formula for one count difference in terms of the other and the ratio a/b. In particular, adjacent finite paths of equal height that end at a common point have identical operation counts. For terminating integers, every equal-height streak is therefore a consecutive block in one operation-count class. More generally, if $a < b$, $H(a) = H(b)$, and $d = O(a) - O(b)$, then

$$\frac{3}{4}\,6^d < \frac{b}{a} < \frac{4}{3}\,6^d,$$

so the odd-count difference controls the possible separation of equal-height starting values.

Consecutive integers of equal height have been studied from several directions. Garner used finite trajectory patterns to obtain sufficient conditions for consecutive trajectories to coalesce and conjectured that his patterns were necessary [5]; his convention counts shortcut steps rather than ordinary operations, as discussed by LaDue [6]. Elia and Tucker constructed infinitely many counterexamples to the necessity direction [1]. Gao studied consecutive integers having both a common height and a common total stopping time [4], and

Lagarias's annotated bibliography attributes the corresponding infinite-recurrence assertion to Penning [8, entry 68]. LaDue obtained a recursive criterion for certain adjacent even–odd pairs whose trajectories coalesce [6], while Frinkle et al. developed high-performance methods for finding long equal-height streaks beginning at integers of the form $2^k + 1$ [3].

To study these streaks structurally, we introduce the charge

$$\lambda = E - 2O.$$

The residue bound makes H, E, O, and λ tightly linked: for example, adjacent height equality implies adjacent charge equality, and the converse fails only at three explicitly listed small pairs. For integers

$$A + r, \qquad A = 2^{\alpha}3^{\beta}b, \qquad \gcd(b, 6) = 1,$$

exact formulas relate the first α shortcut steps of $A + r$ to the corresponding steps of r. They then express the height and charge of $A + r$ in terms of this finite orbit data and the height and charge of the resulting boundary integer. The associated height and charge slices G and Q, together with the residual corrections δ_G and δ_Q, give exact criteria for the persistence and eventual break of equal-height runs. We call a maximal block of consecutive positive integers having the same height an *equal-height streak*; the first integer after the block is its *streak break*. The projected G- and Q-plateaus coincide apart from one stated exception, but an actual height break depends on the residual correction as well as the projected slice.

The charge also completes a hierarchy of record phenomena. A class record for a statistic $X \in \{H, E, O, \lambda\}$ is the least starting integer attaining a specified value of X, whereas a delay record attains a new extreme relative to all smaller starting integers. For H, E, and O, delay records attain new greater values; for λ, they attain new lower values. Writing CR_X and DR_X for the corresponding sets of starting integers, and $\mathrm{CR}_{\lambda}^{\leq 0}$ for the nonpositive charge-class records, we prove the strict global hierarchies

$$\mathrm{DR}_{\lambda} \subsetneq \mathrm{DR}_O \subsetneq \mathrm{DR}_E = \mathrm{DR}_H, \qquad \mathrm{CR}_{\lambda}^{\leq 0} \subsetneq \mathrm{CR}_O \subsetneq \mathrm{CR}_E \subsetneq \mathrm{CR}_H.$$

Thus the height-delay records are exactly the division-count delay records, every λ-delay record is an odd-count delay record, and every realized nonpositive charge-class record is an odd-count class record and hence also a division-count and height-class record. The two hierarchies intersect exactly as follows:

$$\mathrm{DR}_O = \mathrm{CR}_O \cap \mathrm{DR}_H,$$

and

$$\mathrm{DR}_{\lambda} = \mathrm{CR}_{\lambda}^{\leq 0} \cap \mathrm{DR}_O = \mathrm{CR}_{\lambda}^{\leq 0} \cap \mathrm{DR}_E = \mathrm{CR}_{\lambda}^{\leq 0} \cap \mathrm{DR}_H.$$

The class-record terminology appears in the computational work of Scollo [17, 18], while Roosendaal also makes the distinction between class and delay records [14].

These relations are not confined to complete trajectories ending at 1. For $z = 1$, or for a nonperiodic $z > 2$, we consider all sources whose first-arrival shortcut paths end at z and define relative values of H, E, O, and λ. The same class and delay constructions then give

$$\mathrm{DR}_{\lambda}(z) \subseteq \mathrm{DR}_O(z) \subseteq \mathrm{DR}_E(z) = \mathrm{DR}_H(z), \qquad \mathrm{CR}_{\lambda}^{\leq 0}(z) \subseteq \mathrm{CR}_O(z) \subseteq \mathrm{CR}_E(z) \subseteq \mathrm{CR}_H(z),$$

together with the corresponding intersection identities. The containments are written non-strictly because the integers distinguishing two global record sets need not reach a specified tail. When $z = 1$, these relative containments become the strict global class- and delay-record hierarchies stated above. More generally, a global record that reaches a terminating admissible tail remains a record in the corresponding same-tail problem, although the converse need not hold.

The paper is organized as follows. Section 1 develops finite path counts, gap and coordinate brackets, and the affine reconstruction formula. Section 2 introduces residues, reciprocal sums, canonical blocks, and once-counted orbit quantities, and proves the uniform four-thirds and reciprocal-sum bounds; the exact high-range count is supplied in Appendix 1. Section 3 derives the common-endpoint comparison, recovery, synchronization, and nearest-integer results, introduces class and delay records, and proves the equivalence of the height- and division-count delay properties. Section 4 introduces the charge, operation-count cells, transport formulas, and the G- and Q-slices. Section 5 uses these functions and their residual corrections to analyze plateaus, streaks, and exact break criteria. Section 6 establishes the global class and delay hierarchies, their exact intersections, and their extensions above a common tail. Finally, Section 7 summarizes the structural consequences, and Section 8 records the remaining computational and analytic questions. For convenient reference, Appendix 2 collects the principal identities, bounds, recovery formulas, operation-count orderings, and record relations proved throughout the paper.

1 Maps, Counts, and Gap Brackets

In addition to the ordinary map C, we can define two other maps that combine even and odd steps in different ways.

Definition 1.1. For a positive integer n, the *shortcut map* $T(n)$ is defined as

$$T(n) = \begin{cases} n/2, & n \text{ even}, \\ (3n+1)/2, & n \text{ odd}. \end{cases} \tag{1}$$

Since every odd step in C is followed by an even step, the shortcut map simply combines the two steps into a single operation.

The notation $\mathrm{ord}_2(m)$ denotes the exponent of 2 in the prime factorization of a positive integer m. For example, $\mathrm{ord}_2(70) = 1$ and $\mathrm{ord}_2(160) = 5$.

Definition 1.2. For a positive integer n, the *accelerated odd map* $U(n)$ is defined as

$$U(n) = \begin{cases} \dfrac{n}{2^{\mathrm{ord}_2(n)}}, & n \text{ even}, \\ \dfrac{3n+1}{2^{\mathrm{ord}_2(3n+1)}}, & n \text{ odd}. \end{cases} \tag{2}$$

Given an odd value n, $U(n)$ is the next odd value in the ordinary trajectory of n. Given an even value n, $U(n)$ is the first odd value in the ordinary trajectory of n. So repeated application of U to n produces the odd numbers in the ordinary trajectory of n – which we will refer to as the *odd iterates* of n.

Finite Paths and Count Quantities.

Definition 1.3. A *finite shortcut path* is a list

$$\mathcal{F} = (v_0, v_1, \ldots, v_N), \qquad v_{t+1} = T(v_t) \quad (0 \le t < N).$$

Its *input states* are $v_0, \ldots, v_{N-1}$, and its *endpoint* is v_N. The endpoint is not counted as an input. Define

$$E_{\mathcal{F}} := N, \qquad O_{\mathcal{F}} := \#\{0 \le t < N : v_t \text{ is odd}\}, \qquad H_{\mathcal{F}} := E_{\mathcal{F}} + O_{\mathcal{F}}.$$

Every shortcut step contains one division by two, and an odd input also contributes one operation $x \mapsto 3x+1$. Thus $H_{\mathcal{F}}$ counts the ordinary Collatz operations represented by the path. We call it the *finite-path height*.

The path is *nonrepeating* if its input states are distinct. The endpoint is not subject to this condition and may coincide with an earlier input state. Thus a path ending at the first repetition of a forward orbit is nonrepeating in this sense. Neither the endpoint nor any subsequent state is required to be 1.

For $0 \le j \le N$, write $\mathcal{F}_{\le j} = (v_0, \dots, v_j)$. We refer to truncating $\mathcal{F}$ after its first j shortcut steps as making a *cut at* j; the path $\mathcal{F}_{\le j}$ is the portion of $\mathcal{F}$ through that cut. Its states $v_0, \dots, v_j$ are iterates of T and are not required to be odd. In particular, a cut may have an even endpoint v_j. By contrast, the map U passes directly to successive odd states.

Definition 1.4. If the trajectory of n reaches 1, we say it *terminates* and its *first-arrival trajectory* is the finite portion of the trajectory beginning at n and ending at the first occurrence of 1. No subsequent steps in the cycle $1 \to 4 \to 2 \to 1$ are included. For $n = 1$, the first-arrival trajectory consists only of the initial state n, and all operation counts are zero.

Write $\mathcal{T}$ for the set of positive integers whose trajectories terminate. For $n \in \mathcal{T}$, write $\mathcal{F}(n)$ for its first-arrival shortcut path. Let $E(n)$ be the number of $x \mapsto x/2$ steps in the ordinary first-arrival trajectory, and let $O(n)$ be the number of operations $x \mapsto 3x + 1$. Then the Collatz height of the integer n is given by the formula

$$H(n) = E(n) + O(n).$$

The complete counts $E(n), O(n), H(n)$ are used only when n terminates. Finite-path counts and finite-step identities do not require termination. Later formulas involving H, λ, G, or Q are asserted only when their displayed values are defined.

Example 1.5. We present a series of calculations that can be easily reproduced by hand and that will help acquaint the reader with notation introduced later in the paper.

The ordinary trajectory of $n = 23$ is:

$$23 \to 70 \to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$$

It contains eleven divisions by two and four odd operations. Hence

$$(E, O, H)(23) = (11, 4, 15).$$

If we compare the ordinary trajectory with the trajectory consisting only of its odd iterates, we obtain

$$23 \underset{70}{\longrightarrow} 35 \underset{106}{\longrightarrow} 53 \underset{160\to80\to40\to20\to10}{\longrightarrow\!\!\!\!\longrightarrow} 5 \underset{16\to8\to4\to2}{\longrightarrow\!\!\!\!\longrightarrow} 1.$$

Passing from the ordinary trajectory to the sequence of odd iterates suppresses the intervening even states. We record the number of divisions by two performed in each suppressed portion as a *gap*. We also include an initial gap $\mathrm{ord}_2(n)$, which counts the divisions by two needed to reach the first odd state; this initial gap is 0 when n is already odd. Each subsequent gap counts the divisions by two occurring between two consecutive odd states. Since 23 is odd, its initial gap is 0, and its gaps are

$$\big(\mathrm{ord}_2(23), \mathrm{ord}_2(70), \mathrm{ord}_2(106), \mathrm{ord}_2(160), \mathrm{ord}_2(16)\big) = (0, 1, 1, 5, 4).$$

Here the initial value $\mathrm{ord}_2(23) = 0$ records the fact that 23 is already odd. Thus

$$0 + 1 + 1 + 5 + 4 = 11 = E(23),$$

while the number of odd operations is $O(23) = 4$.

For comparison, consider the even starting value 276. Its ordinary trajectory is

$$276 \to 138 \to \mathbf{69} \to 208 \to 104 \to 52 \to 26 \to \mathbf{13} \to 40 \to 20 \to 10 \to \mathbf{5} \to 16 \to 8 \to 4 \to 2 \to \mathbf{1},$$

where the odd states are shown in bold. After the initial even block, these odd iterates are

$$69 \to 13 \to 5 \to 1.$$

The original and odd-only trajectories can be compared as follows:

$$\underbrace{276 \to 138 \to 69}_{\text{two divisions; initial gap } =2} \xrightarrow[208\to104\to52\to26]{} 13 \xrightarrow[40\to20\to10]{} 5 \xrightarrow[16\to8\to4\to2]{} 1.$$

Consequently, the gaps are

$$\big(\mathrm{ord}_2(276), \mathrm{ord}_2(208), \mathrm{ord}_2(40), \mathrm{ord}_2(16)\big) = (2, 4, 3, 4).$$

Their total is

$$2 + 4 + 3 + 4 = 13 = E(276),$$

and the number of odd operations is $O(276) = 3$.

The gap data can be recorded either as the individual numbers of divisions by two or equivalently by their cumulative sums. For 23, these are $(0, 1, 1, 5, 4)$ and $(0, 1, 2, 7, 11)$, respectively. For 276, they are $(2, 4, 3, 4)$ and $(2, 6, 9, 13)$. The motivation for two representations will become clear soon.

Example 1.5 suggests two distinct ways to write the same information: we can list the gaps, or we can list their cumulative sums. We first define these brackets without assuming that their entries come from a trajectory.

Definition 1.6. Suppose $k \geq 0$, and that $g_0, \ldots, g_k$ are integers with $g_0 \geq 0$ and $g_i \geq 1$ for $1 \leq i \leq k$, and z is a positive integer, which may be even. We call

$$[g_0, g_1, \ldots, g_k \mid z] =: [\mathbf{g}]_z$$

a *gap bracket*, with

$$\mathbf{g} = (g_0, \ldots, g_k),$$

and call z its *tail*.

Definition 1.7. Given $[\mathbf{g}]_z = [g_0, g_1, \ldots, g_k \mid z]$, define

$$\ell_i(\mathbf{g}) := \sum_{j=0}^{i} g_j, \quad 0 \leq i \leq k,$$

and write

$$\boldsymbol{\ell}(\mathbf{g}) := \big(\ell_0(\mathbf{g}), \ldots, \ell_k(\mathbf{g})\big).$$

The corresponding *coordinate bracket* is

$$\langle \boldsymbol{\ell}(\mathbf{g}) \rangle_z := \langle \ell_0(\mathbf{g}), \ell_1(\mathbf{g}), \ldots, \ell_k(\mathbf{g}) \mid z \rangle.$$

We use square brackets for gaps and angle brackets for coordinates. When the gap bracket is fixed, we write $\ell_i = \ell_i(\mathbf{g})$ and $\boldsymbol{\ell} = \boldsymbol{\ell}(\mathbf{g})$. The gaps are recovered by

$$g_0 = \ell_0, \qquad g_i = \ell_i - \ell_{i-1} \quad 1 \leq i \leq k.$$

Thus we pass from gaps to coordinates by taking cumulative sums and return by taking consecutive differences. The tail does not change.

Example 1.8. The gap bracket

$$[0, 1, 1, 5, 4 \mid 1]$$

has coordinate bracket

$$\langle 0, 1, 2, 7, 11 \mid 1 \rangle.$$

Definition 1.9. Given a gap bracket $[\mathbf{g}]_z = [g_0, g_1, \ldots, g_k \mid z]$ and its corresponding coordinate bracket, where k is the last index of their entries, define their *evaluation* by

$$\begin{aligned} \mathrm{ev}\big([\mathbf{g}]_z\big) &= \mathrm{ev}\big(\langle \boldsymbol{\ell}(\mathbf{g}) \rangle_z\big) \\ &:= \frac{2^{\ell_k} z}{3^k} - \sum_{i=0}^{k-1} \frac{2^{\ell_i}}{3^{i+1}}. \end{aligned} \tag{3}$$

For $k = 0$, the sum is empty, and we define

$$\mathrm{ev}\big([g_0]_z\big) = \mathrm{ev}\big(\langle \ell_0 \rangle_z\big) := 2^{g_0} z = 2^{\ell_0} z.$$

The sum subtracted in the evaluation can be written over the common denominator 3^k:

$$\sum_{i=0}^{k-1} \frac{2^{\ell_i}}{3^{i+1}} = \frac{1}{3^k} \sum_{i=0}^{k-1} 3^{k-1-i} 2^{\ell_i}.$$

This motivates associating with each bracket the integer

$$J\big([\mathbf{g}]_z\big) = J\big(\langle \boldsymbol{\ell}(\mathbf{g}) \rangle_z\big) := \sum_{i=0}^{k-1} 3^{k-1-i} 2^{\ell_i}. \tag{4}$$

For $k = 0$, this is the empty sum $J([g_0]_z) = J(\langle \ell_0 \rangle_z) = 0$. With this notation, the evaluation takes the affine form

$$\mathrm{ev}\big([\mathbf{g}]_z\big) = \mathrm{ev}\big(\langle \boldsymbol{\ell}(\mathbf{g}) \rangle_z\big) = \frac{2^{\ell_k} z - J([\mathbf{g}]_z)}{3^k}. \tag{5}$$

The evaluation need not be an integer; J depends on the entries of the bracket, but not on its tail.

So far, gap brackets and their evaluations have been defined algebraically. The powers of 2 and 3 in the evaluation reflect the Collatz map, but no condition has yet been imposed that a bracket actually occur along a Collatz trajectory. We now associate brackets with finite shortcut paths: the gap entries record the divisions by two, and the tail is the state reached at the chosen cut.

Definition 1.10. Let $\mathcal{F} = (v_0, \ldots, v_N)$ be a finite shortcut path and let $0 \le j \le N$. Suppose its odd inputs before the cut at j are at positions

$$0 \le p_1 < \cdots < p_k < j.$$

If $k > 0$, define

$$g_0 = p_1, \qquad g_i = p_{i+1} - p_i \quad \text{for } 1 \le i < k, \qquad g_k = j - p_k.$$

We call the entries $g_0, g_1, \ldots, g_k$ the *gaps* of $\mathcal{F}$ at the cut after j shortcut steps. The entry g_0 is the *initial gap*: it counts the shortcut steps preceding the first odd input and may equal 0. For $1 \le i < k$, the entry g_i counts the shortcut steps from the odd input at position p_i to the next odd input at position p_{i+1}. The final gap g_k counts the steps from the last odd input before the cut to the endpoint v_j. It may be only part of the full gap leading to the next odd state when v_j is even.

The cumulative coordinates of Definition 1.7 then satisfy

$$\ell_i = \sum_{r=0}^{i} g_r = p_{i+1} \quad \text{for } 0 \le i < k, \qquad \ell_k = j.$$

Thus the first k coordinates are precisely the shortcut-step positions of the odd inputs, while the final coordinate is the position of the cut. The *gap bracket after j shortcut steps* is

$$\mathrm{Gap}_j(\mathcal{F}) := [g_0, g_1, \ldots, g_k \mid v_j]. \tag{6}$$

If $k = 0$, there are no odd inputs before the cut. In this case, define the single gap $g_0 = j$ and put

$$\mathrm{Gap}_j(\mathcal{F}) = [j \mid v_j].$$

In either case, the gaps sum to j, while $k = O_{\mathcal{F}_{\le j}}$ counts the odd inputs. These gaps count the divisions by two represented by the shortcut steps: the shortcut step from an odd input includes the first division following the operation $x \mapsto 3x + 1$. Consequently, whenever the transition from an odd state x to the next odd state is completed by the cut, the corresponding gap is the full exponent $\mathrm{ord}_2(3x + 1)$ appearing in the accelerated odd map $U(x)$. If the endpoint v_j is even, however, the final gap records only the divisions performed before the cut and need not extend to the next odd state.

We write

$$\mathrm{Gap}(\mathcal{F}) := \mathrm{Gap}_N(\mathcal{F})$$

for the bracket of the entire finite path. With this notation,

$$\mathrm{Gap}_j(\mathcal{F}) = \mathrm{Gap}(\mathcal{F}_{\le j}).$$

For a starting integer n, define

$$\mathrm{Gap}_j(n) := \mathrm{Gap}\big((n, T(n), \ldots, T^j(n))\big).$$

Here, as above, the subscript j is the number of shortcut steps, not the number of odd inputs or gaps. In particular,

$$\mathrm{Gap}_0(\mathcal{F}) = [0 \mid v_0], \qquad \mathrm{Gap}_0(n) = [0 \mid n].$$

The notation $\mathrm{Gap}_j(g)$ is used in the same way when the starting integer is denoted by g; the bold symbol $\mathbf{g}$ denotes a gap list. No termination assumption is needed for any of these cuts.

Example 1.11. Consider $n = 14$. We first write $14 = 2 \cdot 7$, so the initial exponent is $e = \mathrm{ord}_2(14) = 1$ and the first odd state is $x_1 = 7$. The successive odd states are

$$7,\ 11,\ 17,\ 13,\ 5,\ 1,$$

since

$$3(7) + 1 = 2 \cdot 11, \qquad 3(11) + 1 = 2 \cdot 17, \qquad 3(17) + 1 = 2^2 \cdot 13,$$
$$3(13) + 1 = 2^3 \cdot 5, \qquad 3(5) + 1 = 2^4 \cdot 1.$$

The corresponding shortcut path begins

$$14 \xrightarrow{T} 7 \xrightarrow{T} 11 \xrightarrow{T} 17 \xrightarrow{T} 26 \xrightarrow{T} 13.$$

Thus the cut after four shortcut steps ends at the even value 26 and gives

$$\mathrm{Gap}_4(14) = [1, 1, 1, 1 \mid 26].$$

The final entry 1 records only the first of the two divisions following the odd input 17. One shortcut step later, the second division reaches the next odd state 13, and the final gap is complete:

$$\mathrm{Gap}_5(14) = [1, 1, 1, 2 \mid 13].$$

The first entry 1 records the initial factor of 2 removed from 14, while the following entries $1, 1, 2$ record the powers of 2 removed in the first three odd transitions.

Continuing to the first arrival at 1 gives

$$\mathrm{Gap}_{12}(14) = [1, 1, 1, 2, 3, 4 \mid 1].$$

Definition 1.12. Given $\mathrm{Gap}_j(\mathcal{F}) = [g_0, \ldots, g_k \mid v_j]$, its *coordinate bracket after j shortcut steps* is

$$\mathrm{CBrack}_j(\mathcal{F}) := \langle \ell_0, \ell_1, \ldots, \ell_k \mid v_j \rangle = \langle \boldsymbol{\ell}(\mathbf{g}) \rangle_{v_j}.$$

Its last coordinate is $\ell_k = j$. We use $\mathrm{CBrack}_j(n)$ for the path starting at n, with the same shortcut-step index. Thus both indexed brackets describe exactly the same cut.

Example 1.13. Continuing with $n = 14$, recall that $14 = 2 \cdot 7$ and that its successive odd states are

$$7,\ 11,\ 17,\ 13,\ 5,\ 1.$$

For $j = 5$, the first five shortcut steps have exactly $k = 3$ odd inputs, namely 7, 11, and 17; the odd value 13 is the endpoint of the cut and is therefore not counted as an input. The corresponding gap bracket is

$$\mathrm{Gap}_5(14) = [1, 1, 1, 2 \mid 13].$$

Taking cumulative sums of the gap entries gives

$$1,\ 2,\ 3,\ 5,$$

so

$$\mathrm{CBrack}_5(14) = \langle 1, 2, 3, 5 \mid 13 \rangle.$$

Thus the coordinate 5 records that a total of five factors of 2 have been removed through the first three odd transitions, including the initial factor removed from 14.

For $j = 12$, the first-arrival shortcut path reaches 1. Its gap bracket has $k = 5$ odd inputs and is

$$\mathrm{Gap}_{12}(14) = [1, 1, 1, 2, 3, 4 \mid 1].$$

Its cumulative sums are

$$1,\ 2,\ 3,\ 5,\ 8,\ 12,$$

and hence

$$\mathrm{CBrack}_{12}(14) = \langle 1, 2, 3, 5, 8, 12 \mid 1 \rangle.$$

In particular, the final coordinate 12 is the total number of factors of 2 removed before the trajectory first reaches 1. The operation counts can therefore be read directly from this first-arrival bracket. Its gaps sum to 12, so $E(14) = 12$. The bracket has six gap entries and hence $k = 5$ odd inputs, so

$$O(14) = 5, \qquad H(14) = E(14) + O(14) = 17.$$

Likewise, at the earlier cut $j = 5$, the bracket $\mathrm{Gap}_5(14) = [1, 1, 1, 2 \mid 13]$ gives the finite-path counts

$$E_{\mathcal{F}(14)_{\le 5}} = 5, \qquad O_{\mathcal{F}(14)_{\le 5}} = 3, \qquad H_{\mathcal{F}(14)_{\le 5}} = 8.$$

Thus the final coordinate gives the division count, while one less than the number of gap entries gives the odd-input count.

It should be noted that these definitions do not require n to reach 1. If a finite bracket is extended beyond the first arrival at 1, the cycle $1 \to 4 \to 2 \to 1$ appends gaps of 2; such extensions are not included in the complete first-arrival brackets defined below Table 1.

Table 1: Gap brackets of 14 and 23 at the same shortcut-step cuts. The endpoint may be even. The first arrivals at 1 occur at $j = 12$ and $j = 11$, respectively; each later complete passage through the shortcut cycle $1 \to 2 \to 1$ adds two to j and appends a gap of 2.

j	$\mathrm{Gap}_j(14)$	$\mathrm{Gap}_j(23)$
0	[0 | 14]	[0 | 23]
1	[1 | 7]	[0, 1 | 35]
2	[1, 1 | 11]	[0, 1, 1 | 53]
3	[1, 1, 1 | 17]	[0, 1, 1, 1 | 80]
4	[1, 1, 1, 1 | 26]	[0, 1, 1, 2 | 40]
5	[1, 1, 1, 2 | 13]	[0, 1, 1, 3 | 20]
6	[1, 1, 1, 2, 1 | 20]	[0, 1, 1, 4 | 10]
7	[1, 1, 1, 2, 2 | 10]	[0, 1, 1, 5 | 5]
8	[1, 1, 1, 2, 3 | 5]	[0, 1, 1, 5, 1 | 8]
9	[1, 1, 1, 2, 3, 1 | 8]	[0, 1, 1, 5, 2 | 4]
10	[1, 1, 1, 2, 3, 2 | 4]	[0, 1, 1, 5, 3 | 2]
11	[1, 1, 1, 2, 3, 3 | 2]	[0, 1, 1, 5, 4 | 1]
12	[1, 1, 1, 2, 3, 4 | 1]	[0, 1, 1, 5, 4, 1 | 2]
13	[1, 1, 1, 2, 3, 4, 1 | 2]	[0, 1, 1, 5, 4, 2 | 1]
14	[1, 1, 1, 2, 3, 4, 2 | 1]	[0, 1, 1, 5, 4, 2, 1 | 2]

Example 1.14. Table 1 shows that once a trajectory reaches 1, each further odd-to-odd passage through the cycle $1 \to 4 \to 2 \to 1$ removes two factors of 2, since $3(1) + 1 = 4 = 2^2 \cdot 1$. Thus each such passage increases the shortcut-step index j by 2, appends a gap of 2, and leaves the tail equal to 1.

For example, the first-arrival gap bracket of 23 is

$$\mathrm{Gap}_{11}(23) = [0, 1, 1, 5, 4 \mid 1].$$

Extending it through one additional passage around the cycle gives

$$\mathrm{Gap}_{13}(23) = [0, 1, 1, 5, 4, 2 \mid 1].$$

We can also see directly from the evaluation formula (3) why this longer bracket still evaluates to 23. The leading term of the original evaluation satisfies

$$\frac{2^{11}}{3^4} = \frac{2^{11}}{3^4}\left(\frac{2^2 - 1}{3}\right) = \frac{2^{13}}{3^5} - \frac{2^{11}}{3^5}.$$

Consequently,

$$\begin{aligned} 23 &= \frac{2^{11}}{3^4} - \frac{2^7}{3^4} - \frac{2^2}{3^3} - \frac{2^1}{3^2} - \frac{2^0}{3} \\ &= \frac{2^{13}}{3^5} - \frac{2^{11}}{3^5} - \frac{2^7}{3^4} - \frac{2^2}{3^3} - \frac{2^1}{3^2} - \frac{2^0}{3}. \end{aligned}$$

Thus appending the trailing gap 2 produces a longer bracket with the same evaluation. The same argument applies whenever a trailing gap of 2 is appended to a bracket with tail 1. Such extensions are not included in the first-arrival counts E, O, and H.

Theorem 1.15 (Affine Gap Formula)**.** *Let $\mathcal{F} = (n, v_1, \ldots, v_N)$ be a finite shortcut path, let $0 \le j \le N$, and put $k = O_{\mathcal{F}_{\le j}}$. Write $\mathrm{Gap}_j(\mathcal{F}) = [\mathbf{g}]_z$ and $\mathrm{CBrack}_j(\mathcal{F}) = \langle \boldsymbol{\ell}(\mathbf{g}) \rangle_z$, where $z = v_j$ and $\ell_k = j$. Then both brackets evaluate to n, and*

$$n = \frac{2^j z - J([\mathbf{g}]_z)}{3^k} = \frac{2^{\ell_k} z - J([\mathbf{g}]_z)}{3^k}. \tag{7}$$

Proof. For $k = 0$, we have

$$\mathrm{Gap}_j(\mathcal{F}) = [j \mid z], \qquad g_0 = \ell_0 = j,$$

so

$$n = 2^{g_0} z = 2^{\ell_0} z.$$

Since $J([\mathbf{g}]_z) = 0$ in this case, the result follows.

Now assume $k \ge 1$. Let x_i be the odd input at position p_i, and set $x_{k+1} = z$. The last of these values may be even. From the divisions recorded in each gap,

$$x_i = \frac{2^{g_i} x_{i+1} - 1}{3}, \qquad 1 \le i \le k.$$

Starting from $n = 2^{g_0} x_1$, the first substitution gives

$$n = 2^{g_0} \left(\frac{2^{g_1} x_2 - 1}{3} \right) = \frac{2^{g_0 + g_1} x_2}{3} - \frac{2^{g_0}}{3}.$$

Since

$$\ell_0 = g_0 \qquad \text{and} \qquad \ell_1 = g_0 + g_1,$$

this becomes

$$n = \frac{2^{\ell_1} x_2}{3} - \frac{2^{\ell_0}}{3}.$$

Continuing this substitution through $x_{k+1} = z$ accumulates the successive gaps into the coordinates $\ell_i = g_0 + g_1 + \cdots + g_i$ and gives

$$n = \frac{2^{\ell_k} z}{3^k} - \sum_{i=0}^{k-1} \frac{2^{\ell_i}}{3^{i+1}}.$$

By Definition 1.9, the right-hand side is the evaluation of both associated brackets. Hence

$$n = \mathrm{ev}([\mathbf{g}]_z) = \mathrm{ev}(\langle \boldsymbol{\ell}(\mathbf{g}) \rangle_z).$$

Finally, by the definition of J,

$$J([\mathbf{g}]_z) = \sum_{i=0}^{k-1} 3^{k-1-i} 2^{\ell_i},$$

so

$$\sum_{i=0}^{k-1} \frac{2^{\ell_i}}{3^{i+1}} = \frac{J([\mathbf{g}]_z)}{3^k}.$$

Therefore

$$n = \frac{2^{\ell_k} z - J([\mathbf{g}]_z)}{3^k},$$

which proves (7). □

Example 1.16. Consider again the first-arrival trajectory of 23. Its gap bracket at first arrival at 1 is

$$\mathrm{Gap}_{11}(23) = [0, 1, 1, 5, 4 \mid 1] =: [\mathbf{g}]_1.$$

Taking cumulative sums of the gaps gives

$$0,\ 1,\ 2,\ 7,\ 11,$$

so the corresponding coordinate bracket is

$$\mathrm{CBrack}_{11}(23) = \langle 0, 1, 2, 7, 11 \mid 1 \rangle = \langle \boldsymbol{\ell}(\mathbf{g}) \rangle_1.$$

Here $j = 11$ and $k = 4$, the tail is $z = 1$, and the coordinates are

$$\ell_0 = 0, \qquad \ell_1 = 1, \qquad \ell_2 = 2, \qquad \ell_3 = 7, \qquad \ell_4 = 11.$$

By the definition of J,

$$J([\mathbf{g}]_1) = \sum_{i=0}^{3} 3^{3-i} 2^{\ell_i}.$$

Substituting the four coordinates $\ell_0, \ldots, \ell_3$ gives

$$\begin{aligned} J([\mathbf{g}]_1) &= 3^3 2^0 + 3^2 2^1 + 3^1 2^2 + 3^0 2^7 \\ &= 27 + 18 + 12 + 128 \\ &= 185. \end{aligned}$$

Notice that the final coordinate $\ell_4 = 11$ does not occur in the sum defining J. Instead, it appears in the leading term of the affine gap formula. Since the tail is $z = 1$, Theorem 1.15 gives

$$23 = \frac{2^{11} \cdot 1 - 185}{3^4} = \frac{2048 - 185}{81} = \frac{1863}{81} = 23.$$

Thus the gap bracket $[0, 1, 1, 5, 4 \mid 1]$ encodes the same trajectory information as the coordinate bracket $\langle 0, 1, 2, 7, 11 \mid 1 \rangle$. The value $J = 185$ collects the contributions of the nonfinal coordinates into a single term in the formula; together with $j = 11$, $k = 4$, and $z = 1$, it recovers the starting integer 23. The value of J alone need not determine the starting integer: the same value may occur for brackets associated with distinct starting integers.

Theorem 1.15 applies to any finite trajectory segment, with an arbitrary positive tail z, including even endpoints. We now specialize to a terminating trajectory and stop at its first arrival at 1. In this case the tail is $z = 1$, the number of odd transitions is $O(n)$, and the final coordinate records the total number of divisions by 2.

Definition 1.17. Assume that n reaches 1, and put $E = E(n)$ and $O = O(n)$. The *complete gap bracket* and *complete coordinate bracket* of n are

$$\mathrm{Gap}_E(n) = [g_0, \ldots, g_O \mid 1] =: [\mathbf{g}]_1,$$
$$\mathrm{CBrack}_E(n) = \langle \ell_0, \ldots, \ell_O \mid 1 \rangle = \langle \boldsymbol{\ell}(\mathbf{g}) \rangle_1.$$

These brackets end at the first arrival at 1, without extra trailing gaps of 2. In particular, the last coordinate is $\ell_O = E$. If $O = 0$, the complete bracket is $[E \mid 1]$.

For a complete bracket, the Affine Gap Formula takes a particularly simple form. Since $z = 1$, $k = O$, and $\ell_O = E$, Theorem 1.15 gives

$$n = \frac{2^E - J([\mathbf{g}]_1)}{3^O}.$$

We therefore write

$$J(n) := J\big([\mathbf{g}]_1\big) = J\big(\langle \boldsymbol{\ell}(\mathbf{g}) \rangle_1\big),$$

so that

$$2^E = 3^O n + J(n), \tag{8}$$

or equivalently,

$$J(n) = 2^{E(n)} - 3^{O(n)} n. \tag{9}$$

When $O = 0$, we have $n = 2^E$ and hence $J(n) = 0$.

If we consider the underlying affine relation of (8) with the gap entries fixed, J is constant, giving it the form $y = 3^k x + J$, where $x = n$ and $y = 2^{\ell_k} z$. We therefore call J the *intercept* associated with the bracket. For a complete bracket, $z = 1$, hence $y = 2^{\ell_k}$.

To illustrate, Example 1.16 gives $E(23) = 11$, $O(23) = 4$, and $J(23) = 185$. Thus

$$2^{11} = 3^4(23) + 185,$$

which is exactly the complete-bracket form of the Affine Gap Formula for 23.

The complete-bracket identity also gives a natural normalization of the intercept correction. Dividing (8) by $3^{O(n)} n$ gives

$$\frac{2^{E(n)}}{3^{O(n)} n} = 1 + \frac{J(n)}{3^{O(n)} n}.$$

The quantity on the left is the residue defined in the next section. Thus we can interpret the residue as a measure of the intercept's correction relative to the main term $3^{O(n)} n$.

2 The Residue and its Upper Bounds

Definition 2.1. Define the *residue* $\rho(n)$ of a terminating positive integer n by

$$\rho(n) = \frac{2^{E(n)}}{3^{O(n)} n}. \tag{10}$$

Roosendaal denotes ρ by $\mathrm{Res}(n)$ [14]. The residue function can be expressed in a variety of ways. For instance, using the fact that $E(n) = H(n) - O(n)$, we also have:

$$\rho(n) = \frac{2^{H(n)}}{6^{O(n)} n}. \tag{11}$$

By the definition of J in (8), ρ can also be expressed as

$$\rho(n) = 1 + \frac{J(n)}{3^{O(n)}n}. \tag{12}$$

Let $e_0 = \mathrm{ord}_2(n)$. If $O(n) > 0$, enumerate the odd inputs before first arrival at 1 as $x_1, \ldots, x_{O(n)}$, with $x_1 = n/2^{e_0}$, and put $x_{O(n)+1} = 1$. For $1 \le i \le O(n)$, set $g_i = \mathrm{ord}_2(3x_i + 1)$. Then

$$x_{i+1} = \frac{3x_i + 1}{2^{g_i}}, \qquad E(n) = e_0 + \sum_{i=1}^{O(n)} g_i.$$

Multiplication and cancellation give

$$\prod_{i=1}^{O(n)} \left(1 + \frac{1}{3x_i}\right) = \frac{2^{E(n)}}{3^{O(n)}n} = \rho(n). \tag{13}$$

When $O(n) = 0$, the starting integer is a power of two and the empty product is 1. In particular, $\rho(n) \ge 1$, with equality if and only if n is a power of two. This product representation also appears in [10, Lemma 2.2].

Theorem 2.2. *If n terminates, then*

$$2^{E(n)} \ge 3^{O(n)}n. \tag{14}$$

Equality holds if and only if n is a power of two.

Proof. By (13), $\rho(n) \ge 1$, with equality if and only if n is a power of two. Since (10) gives

$$2^{E(n)} = 3^{O(n)}n\rho(n),$$

the stated inequality and equality condition follow. □

For the following two definitions, let $n > 1$ be terminating, and let

$$x_1, \ldots, x_{O(n)}$$

denote the odd input states of its first-arrival shortcut path $\mathcal{F}(n)$. Equivalently, these are the odd iterates encountered before the first arrival at 1. The terminal value 1 is the endpoint of the path and is not included among its inputs.

Definition 2.3. The *Reciprocal function* $\mathcal{R}(n)$ is the sum of the reciprocals of the odd inputs of n:

$$\mathcal{R}(n) = \sum_{i=1}^{O(n)} \frac{1}{x_i}. \tag{15}$$

Definition 2.4. The *Correction Product* $\Pi(n)$ is defined over the odd inputs of n as follows:

$$\Pi(n) = \prod_{i=1}^{O(n)} \frac{2x_i + 1}{2x_i - 1}. \tag{16}$$

We can extend the definitions of $\mathcal{R}$ and Π to $n = 1$ by using the standard conventions of

$$\mathcal{R}(1) = \sum_{\varnothing} = 0, \qquad \Pi(1) = \prod_{\varnothing} = 1.$$

For a terminating odd integer $x > 1$, the residue product gives:

$$\rho(x) = \left(1 + \frac{1}{3x}\right)\rho(U(x)). \tag{17}$$

2.1 Finite Paths and Finite-Path Residues

The product representation (13) has a finite-path version that does not require termination. This finite-path residue and its product identity appear in [10, Definition 5.2 and Lemma 5.3]. We use the finite shortcut path $\mathcal{F} = (v_0, \dots, v_N)$ and the counts $E_{\mathcal{F}}, O_{\mathcal{F}}, H_{\mathcal{F}}$ from Definition 1.3. The finite-path identities below allow arbitrary positive input states. When a result requires nonrepetition or excludes 1 as an input, we state this explicitly.

Definition 2.5. The *finite-path reciprocal sum* of $\mathcal{F}$ is

$$\mathcal{R}_{\mathcal{F}} := \sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t},$$

where the empty sum is understood to be 0.

The *finite-path residue* of $\mathcal{F}$ is similarly defined as

$$\rho_{\mathcal{F}} := \prod_{\substack{0 \le t < N \\ v_t \text{ odd}}} \left(1 + \frac{1}{3v_t}\right),$$

where the empty product is understood to be 1.

The *finite-path correction product* is

$$\Pi_{\mathcal{F}} := \prod_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{2v_t + 1}{2v_t - 1},$$

with empty product 1. Each input occurrence is counted in these finite-path quantities. No input at the endpoint is included.

Proposition 2.6. *For every finite shortcut path $\mathcal{F} = (v_0, v_1, \dots, v_N)$, we have*

$$\rho_{\mathcal{F}} = \frac{2^{E_{\mathcal{F}}} v_N}{3^{O_{\mathcal{F}}} v_0}. \tag{18}$$

Proof. For each $0 \le t < N$, the shortcut map gives

$$2v_{t+1} = \begin{cases} v_t, & v_t \text{ even}, \\ 3v_t + 1, & v_t \text{ odd}. \end{cases}$$

Hence

$$\frac{2v_{t+1}}{v_t} = \begin{cases} 1, & v_t \text{ even}, \\ 3\left(1 + \dfrac{1}{3v_t}\right), & v_t \text{ odd}. \end{cases}$$

Multiplying over $0 \le t < N$ gives

$$\frac{2^N v_N}{v_0} = 3^{O_{\mathcal{F}}} \prod_{\substack{0 \le t < N \\ v_t \text{ odd}}} \left(1 + \frac{1}{3v_t}\right) = 3^{O_{\mathcal{F}}} \rho_{\mathcal{F}}.$$

Since $E_{\mathcal{F}} = N$, rearranging gives

$$\rho_{\mathcal{F}} = \frac{2^{E_{\mathcal{F}}} v_N}{3^{O_{\mathcal{F}}} v_0}.$$

□

If $\mathcal{F}$ is the complete first-arrival shortcut trajectory of a terminating integer n, then $v_0 = n$, $v_N = 1$, $E_{\mathcal{F}} = E(n)$, and $O_{\mathcal{F}} = O(n)$. Therefore

$$\mathcal{R}_{\mathcal{F}} = \mathcal{R}(n), \qquad \rho_{\mathcal{F}} = \rho(n), \qquad \Pi_{\mathcal{F}} = \Pi(n).$$

The same product identity gives $\rho_{\mathcal{F}} \geq 1$ and hence

$$2^{E_{\mathcal{F}}} v_N \geq 3^{O_{\mathcal{F}}} v_0,$$

with equality exactly when $O_{\mathcal{F}} = 0$. This is the finite-path version of Theorem 2.2, with no termination assumption.

Thus the finite-path residue extends the original residue without requiring the endpoint v_N to be 1 or the starting integer v_0 to terminate. In particular, $\rho_{\mathcal{F}}$ depends only on the finite path $\mathcal{F}$ and makes no assertion about the subsequent behavior of v_N.

Canonical Blocks and Dyadic Shells. The finite-path residue is a product over the odd inputs of the path, so bounding it requires control of both the sizes of those inputs and their number. We organize the odd inputs $x > 1$ according to the unique *dyadic shell* $(2^j, 2^{j+1})$ that contains them, where $j = \lfloor \log_2 x \rfloor$. As j ranges over the positive integers, these shells partition the odd integers greater than 1. Writing an odd input in its shell as $x = 2^j + \bar{x}$ for $0 < \bar{x} < 2^j$, the next result compares the first j shortcut steps of x and $\bar{x}$. The resulting restrictions on their endpoints will bound the number of odd inputs from a fixed shell whose j-step segments lie within a nonrepeating finite path.

Theorem 2.7. *Let $x > 1$ be odd. Write*

$$j = \lfloor \log_2 x \rfloor, \qquad x = 2^j + \bar{x}, \qquad 0 < \bar{x} < 2^j.$$

For $0 \leq t \leq j$, define

$$s_t := \#\{0 \leq r < t : T^r(x) \text{ is odd}\}.$$

Then $T^r(x)$ and $T^r(\bar{x})$ have the same parity for every $0 \leq r < j$, so s_t also counts the odd inputs among the first t shortcut steps of $\bar{x}$. Moreover,

$$T^t(x) = 2^{j-t} 3^{s_t} + T^t(\bar{x}) \qquad 0 \leq t \leq j. \tag{19}$$

Proof. At step zero,

$$T^0(x) - T^0(\bar{x}) = x - \bar{x} = 2^j 3^0,$$

so the formula holds with $s_0 = 0$.

Suppose that it holds at some step $t < j$. Then

$$T^t(x) - T^t(\bar{x}) = 2^{j-t} 3^{s_t},$$

which is even because $j - t \geq 1$. Therefore $T^t(x)$ and $T^t(\bar{x})$ have the same parity.

If both current values are even, the next shortcut step halves each of them, and $s_{t+1} = s_t$. Therefore

$$\begin{aligned} T^{t+1}(x) - T^{t+1}(\bar{x}) &= \frac{T^t(x)}{2} - \frac{T^t(\bar{x})}{2} \\ &= \frac{1}{2}\left(T^t(x) - T^t(\bar{x})\right) \\ &= \frac{1}{2}\left(2^{j-t} 3^{s_t}\right) \end{aligned}$$

$$= 2^{j-(t+1)}3^{s_{t+1}}.$$

If both current values are odd, the odd branch of the shortcut map is applied to each, and $s_{t+1} = s_t + 1$. Hence

$$\begin{aligned} T^{t+1}(x) - T^{t+1}(\bar{x}) &= \frac{3T^t(x)+1}{2} - \frac{3T^t(\bar{x})+1}{2} \\ &= \frac{3}{2}\left(T^t(x) - T^t(\bar{x})\right) \\ &= \frac{3}{2}\left(2^{j-t}3^{s_t}\right) \\ &= 2^{j-(t+1)}3^{s_t+1} \\ &= 2^{j-(t+1)}3^{s_{t+1}}. \end{aligned}$$

Induction proves both the parity agreement through the first j shortcut inputs and equation (19). □

Definition 2.8. Let $x_0 > 1$ be odd and set $j = \lfloor \log_2 x_0 \rfloor$ so that $2^j < x_0 < 2^{j+1}$. We call the j-step shortcut trajectory segment $(x_0, T(x_0), ..., T^j(x_0))$ the *canonical block* of x_0. Its final state $T^j(x_0)$ is called the *canonical endpoint* of x_0. The interval $(2^j, 2^{j+1})$ is called the *dyadic shell of index* j, or simply *shell* j.

Example 2.9. To illustrate Definition 2.8 and the relation in Theorem 2.7, take $x = 13$. Since $2^3 < 13 < 2^4$, we have $j = 3$, so 13 lies in dyadic shell 3. Its canonical block is

$$(13, T(13), T^2(13), T^3(13)) = (13, 20, 10, 5),$$

and its canonical endpoint is $T^3(13) = 5$. Writing $13 = 2^3 + \bar{x}$ gives $\bar{x} = 5$, whose corresponding three-step segment is

$$(5, T(5), T^2(5), T^3(5)) = (5, 8, 4, 2).$$

The two trajectories have the same parity through steps $0 \le t \le 2$. Since $T^0(x)$ is odd,

$$T^1(x) - T^1(\bar{x}) = \frac{3}{2}\big(T^0(x) - T^0(\bar{x})\big).$$

The next two inputs are even, so

$$T^2(x) - T^2(\bar{x}) = \frac{1}{2}\big(T^1(x) - T^1(\bar{x})\big), \qquad T^3(x) - T^3(\bar{x}) = \frac{1}{2}\big(T^2(x) - T^2(\bar{x})\big).$$

Table 2: Theorem 2.7 applied to $x = 13$ for $0 \le t \le 3$.

t	$3-t$	s_t	$T^t(x)$	$T^t(\bar{x})$	$2^{3-t}3^{s_t}$
0	3	0	13	5	8
1	2	1	20	8	12
2	1	1	10	4	6
3	0	1	5	2	3

The final row gives the canonical endpoint $T^3(13) = 5$; the comparison value $T^3(5) = 2$ differs from it by $5 - 2 = 3 = 2^{3-3}3^{s_3}$, as asserted by Theorem 2.7.

More generally, setting $t = j$ in Theorem 2.7 gives

$$T^j(x) = 3^s + q, \tag{20}$$

where $s = s_j$ and $q = T^j(\bar{x})$. We will shortly show that $0 < q < 3^s$, therefore yielding the bound $3^s < T^j(x) < 2 \cdot 3^s$ and restricting where an element of an orbit can lie.

Theorem 2.10. *For $0 \le t \le j$, the following inequality holds:*

$$T^t(\bar{x}) < 2^{j-t}3^{s_t}. \tag{21}$$

Proof. By definition, $0 < \bar{x} < 2^j$. Hence, when $t = 0$, we have $T^0(\bar{x}) = \bar{x} < 2^j = 2^j 3^0$, since $s_0 = 0$.

If we assume that equation (21) holds true up through step t, then at step $t + 1$ there are two cases.

Case 1: If $T^t(\bar{x})$ is even, then no new odd step is counted and $s_t = s_{t+1}$ and

$$T^{t+1}(\bar{x}) = \frac{1}{2}T^t(\bar{x}),$$

so

$$T^{t+1}(\bar{x}) < \frac{2^{j-t}3^{s_t}}{2} = 2^{j-(t+1)}3^{s_{t+1}}.$$

Case 2: If $T^t(\bar{x})$ is odd, then $s_{t+1} = s_t + 1$ and

$$T^{t+1}(\bar{x}) = \frac{3T^t(\bar{x}) + 1}{2}.$$

The induction hypothesis gives $T^t(\bar{x}) < 2^{j-t}3^{s_t}$, which we can rewrite as

$$T^t(\bar{x}) \le 2^{j-t}3^{s_t} - 1.$$

Multiplying the above inequality by 3 and then adding 1 gives

$$3T^t(\bar{x}) + 1 \le 3(2^{j-t}3^{s_t} - 1) + 1 = 2^{j-t}3^{s_t+1} - 2.$$

Dividing by 2 gives

$$T^{t+1}(\bar{x}) = \frac{3T^t(\bar{x}) + 1}{2} \le 2^{j-(t+1)}3^{s_{t+1}} - 1 < 2^{j-(t+1)}3^{s_{t+1}}.$$

□

Corollary 2.11. *With $s = s_j$ and $q = T^j(\bar{x})$ as above, $0 < q < 3^s$.*

Proof. Setting $t = j$ in Theorem 2.10 gives $0 < q = T^j(\bar{x}) < 2^{j-j}3^{s_j} = 3^s$. □

Thus the canonical endpoint $T^j(x) = 3^s + q$ lies in the interval $(3^s, 2 \cdot 3^s)$ by Corollary 2.11. It is also not divisible by 3. To see this, suppose that the odd inputs among the first j shortcut steps occur at positions $0 = p_1 < p_2 < \cdots < p_s < j$. Starting from x, each shortcut step contributes a factor $1/2$, while each odd-input step also contributes a factor 3 to all terms already present and introduces a new additive term of $1/2$. Iterating these affine transformations through the j steps gives

$$2^j T^j(x) = 3^s x + \sum_{i=1}^{s} 3^{s-i}2^{p_i}.$$

Indeed, the term introduced at the odd step in position p_i is multiplied by 3 once for each of the $s - i$ later odd inputs and, after clearing the common denominator 2^j, contributes $3^{s-i}2^{p_i}$.

Reducing this identity modulo 3, the term $3^s x$ and every term of the sum with $i < s$ vanish. Only the contribution from the final odd input remains:

$$2^j T^j(x) \equiv 2^{p_s} \not\equiv 0 \pmod 3.$$

Since 2^j is invertible modulo 3, it follows that $T^j(x) \not\equiv 0 \pmod 3$. Thus the possible canonical endpoints in $(3^s, 2 \cdot 3^s)$ are further restricted - they cannot be multiples of 3.

Definition 2.12. Let $(x, T(x), \ldots, T^j(x))$ be a canonical block. Suppose its odd inputs before the cut at j occur at the input positions $0 = p_1 < p_2 < \cdots < p_s < j$. We call the ordered tuple $(p_1, p_2, \ldots, p_s)$ the *cut signature* of the canonical block.

The following example makes explicit the combinatorial information carried by a cut signature.

Example 2.13. Suppose $j = 7$ and exactly three odd inputs occur, at positions $p_1 = 0$, $p_2 = 2$, and $p_3 = 5$. Then the first seven shortcut inputs have parities (O, E, O, E, E, O, E). Write

$$\mathbf{p} = (p_1, p_2, p_3) = (0, 2, 5)$$

for the cut signature.

The last odd input occurs at position $p_3 = 5$. Since the cut is made after seven shortcut steps, the final gap has length $j - p_3 = 7 - 5 = 2$. It consists of the shortcut steps whose inputs are at positions 5 and 6, the first of which is the odd-input step at position 5.

Because $p_1 = 0$, the initial gap is $g_0 = 0$. The three positive gap lengths following this initial gap are

$$g_1 = p_2 - p_1 = 2, \qquad g_2 = p_3 - p_2 = 3, \qquad g_3 = j - p_3 = 2.$$

Thus the full gap list at the cut is

$$(g_0, g_1, g_2, g_3) = (0, 2, 3, 2),$$

whereas the positive gap-length triple used in the composition count is

$$(g_1, g_2, g_3) = (2, 3, 2).$$

These positive gaps sum to the length of the block:

$$g_1 + g_2 + g_3 = 2 + 3 + 2 = 7 = j.$$

There are $\binom{6}{2} = 15$ positive compositions of 7 into 3 parts:

$$\{(1,1,5), (1,2,4), (1,3,3), (1,4,2), (1,5,1), (2,1,4), (2,2,3), (2,3,2),\\ (2,4,1), (3,1,3), (3,2,2), (3,3,1), (4,1,2), (4,2,1), (5,1,1)\}.$$

The corresponding candidate odd-input position tuples are recovered by reversing the cumulation process and setting $p_1 = 0$. They are listed below in lexicographic order. Note that the cut signature $(0, 2, 5)$ appears below and corresponds to the gaps $(2, 3, 2)$ given in the list above.

$$\{(0,1,2), (0,1,3), (0,1,4), (0,1,5), (0,1,6), (0,2,3), (0,2,4), (0,2,5),\\ (0,2,6), (0,3,4), (0,3,5), (0,3,6), (0,4,5), (0,4,6), (0,5,6)\}.$$

Thus these are exactly the 15 candidate position tuples satisfying

$$0 = p_1 < p_2 < p_3 < 7.$$

Every realized 7-step canonical block with three odd inputs has one of these tuples as its cut signature, but not every candidate tuple need arise from a starting integer.

More generally, the candidate odd-input position tuples satisfying

$$0 = p_1 < p_2 < \cdots < p_s < j$$

are in bijection with the positive compositions of j into s parts, of which there are

$$\binom{j-1}{s-1}.$$

Every realized cut signature is one of these candidate tuples.

The next lemma bounds the number of canonical blocks in a fixed dyadic shell having a prescribed number of odd inputs. One bound comes from the fact that each realized cut signature corresponds to at most one starting integer in the shell; the other comes from counting the possible canonical endpoints.

Lemma 2.14. *Let $\mathcal{F} = (v_0, v_1, \ldots, v_N)$ be a finite shortcut path whose input states $v_0, v_1, \ldots, v_{N-1}$ are distinct. Consider the odd input states $x = v_t > 1$ whose canonical endpoints occur strictly before the endpoint v_N; equivalently, if $j = \lfloor \log_2 x \rfloor$, then $t + j < N$.*

For integers $j \geq 1$ and $1 \leq s \leq j$, let $N_{j,s}$ be the number of these odd input states satisfying $2^j < x < 2^{j+1}$ for which exactly s of the j input states $v_t, v_{t+1}, \ldots, v_{t+j-1}$ are odd. Equivalently, the canonical block

$$(v_t, v_{t+1}, \ldots, v_{t+j})$$

contains exactly s odd inputs before its endpoint. Then

$$N_{j,s} \leq \binom{j-1}{s-1}, \qquad \sum_j N_{j,s} \leq 2 \cdot 3^{s-1}. \tag{22}$$

Proof. We first prove the left inequality in (22). Fix j and s. Suppose the odd inputs in a j-step canonical block occur at positions $0 = p_1 < p_2 < \cdots < p_s < j$. Define the corresponding gaps by $g_i = p_{i+1} - p_i$ for $1 \leq i < s$, with $g_s = j - p_s$.

Each g_i is positive, and $g_1 + \cdots + g_s = j$. The final entry $g_s = j - p_s$ records the shortcut steps from the last odd input to the cut. If the endpoint of the cut is even, this entry includes only the divisions performed before the cut and not all the divisions needed to reach the next odd state.

Conversely, a candidate odd-input position tuple is recovered from these gaps by $p_1 = 0$ and

$$p_i = g_1 + \cdots + g_{i-1} \qquad (2 \leq i \leq s).$$

Thus the candidate odd-input position tuples are in bijection with the positive compositions of j into s parts. Equivalently, one chooses the $s - 1$ dividing positions among the $j - 1$ spaces between successive steps. Since every realized cut signature is one of these candidate tuples, there are at most

$$\binom{j-1}{s-1}$$

possible realized cut signatures.

Put $\ell_0 = 0$ and $\ell_i = g_1 + \cdots + g_i$ for $1 \le i \le s$, so that $\ell_s = j$. The corresponding j-step bracket has s odd inputs, endpoint $T^j(x)$, and final coordinate $\ell_s = j$. Therefore, rearranging the affine gap formula in Theorem 1.15 gives

$$2^j T^j(x) = 3^s x + \sum_{i=0}^{s-1} 3^{s-1-i} 2^{\ell_i}.$$

Suppose two starts x and y in the shell $(2^j, 2^{j+1})$ have the same gaps. Because the gaps, and hence the coordinates $\ell_0, \ldots, \ell_{s-1}$, are the same, the sums in their affine gap formulas are equal. Subtracting the two formulas gives

$$2^j \left(T^j(x) - T^j(y)\right) = 3^s (x - y).$$

Since $\gcd(2^j, 3^s) = 1$, it follows that $2^j \mid (x - y)$. However, $x, y \in (2^j, 2^{j+1})$ implies $|x - y| < 2^j$. Therefore $x - y = 0$, and hence $x = y$. Thus each possible choice of the gaps contributes at most one starting integer in the shell. This is the required uniqueness part of the finite parity correspondence of Terras and Everett [20, 2]. Consequently,

$$N_{j,s} \le \binom{j-1}{s-1}. \tag{23}$$

We now fix s and count the possible canonical endpoints. Every canonical endpoint being counted occurs among the input states $v_0, v_1, \ldots, v_{N-1}$. By Theorem 2.7 and Corollary 2.11, it satisfies $3^s < T^j(x) < 2 \cdot 3^s$ and is not divisible by 3.

The open interval $(3^s, 2 \cdot 3^s)$ contains $3^s - 1$ integers. Of these, exactly $3^{s-1} - 1$ are divisible by 3. Therefore the number not divisible by 3 is $(3^s - 1) - (3^{s-1} - 1) = 2 \cdot 3^{s-1}$.

It remains to show that distinct starting states counted with the same value of s have distinct canonical endpoints. Suppose that $x_1 = v_{t_1}$ and $x_2 = v_{t_2}$ have the same canonical endpoint. Because the input states of $\mathcal{F}$ are distinct, $t_1 \ne t_2$; assume that $t_1 < t_2$. Both canonical endpoints occur among the distinct input states $v_0, \ldots, v_{N-1}$, so their equality implies that they occur at the same position, say t_f.

The two canonical blocks have respective lengths

$$j_1 = t_f - t_1, \qquad j_2 = t_f - t_2,$$

which need not be equal. Their input positions are

$$t_1, t_1 + 1, \ldots, t_f - 1 \quad \text{and} \quad t_2, t_2 + 1, \ldots, t_f - 1.$$

Thus the earlier block contains every odd input of the later block and also contains the odd starting state v_{t_1}. It therefore has more than s odd inputs if the later block has s, contradicting the assumption that both are counted with the same value of s. Hence distinct starting states counted in the sum over j have distinct canonical endpoints.

There are at most $2 \cdot 3^{s-1}$ available endpoints, so

$$\sum_j N_{j,s} \le 2 \cdot 3^{s-1}. \tag{24}$$

□

Example 2.15. Consider the integer 993. It has 32 odd inputs before the terminal value 1:

$$\begin{aligned} 993 &\to 745 \to 559 \to 839 \to 1259 \to 1889 \to 1417 \to 1063 \to 1595 \to 2393 \to 1795 \\ &\to 2693 \to 505 \to 379 \to 569 \to 427 \to 641 \to 481 \to 361 \to 271 \to 407 \to 611 \\ &\to 917 \to 43 \to 65 \to 49 \to 37 \to 7 \to 11 \to 17 \to 13 \to 5 \to 1. \end{aligned}$$

Table 3 displays the values of $N_{j,s}$, with s indexing the rows and j indexing the columns, and illustrates the two estimates in Lemma 2.14. Thus a row sum fixes s and sums over j, while a column fixes j.

Table 3: The values of $N_{j,s}$ for the first-arrival orbit of 993. The terminal state 1 is not counted. Every entry outside the range shown is 0.

$s \setminus j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	1	2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0
4	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
5	0	0	0	0	0	0	0	4	1	2	0	0	0	0	0
6	0	0	0	0	0	0	0	1	3	2	2	0	0	0	0
7	0	0	0	0	0	0	0	0	2	2	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

As an example, the entry in the cell $(j, s) = (8, 5)$ satisfies

$$N_{8,5} = 4 \le \binom{7}{4} = 35,$$

whereas summing the entire row $s = 5$ gives

$$\sum_j N_{j,5} = 4 + 1 + 2 = 7 \le 2 \cdot 3^4 = 162.$$

The first comparison is cellwise, while the second is rowwise. The whole-row bound is attained when $s = 1$, since

$$\sum_j N_{j,1} = 1 + 1 = 2 = 2 \cdot 3^0,$$

but it becomes considerably less sharp for the larger values of s in this example. This loss of sharpness reflects the sparsity of the table: most of the possible (j, s)-cells are unoccupied.

Theorem 2.16 (Uniform Finiteness)**.** *Let $\mathcal{F} = (v_0, v_1, \ldots, v_N)$ be a finite nonrepeating shortcut path whose input states are greater than* 1*. Then*

$$\mathcal{R}_{\mathcal{F}} < B_{\mathcal{R}} + 2,$$

where

$$B_{\mathcal{R}} = \left(\frac{2}{3}\right)^{1/3} \frac{r}{1-r}, \qquad r = \left(\frac{3}{(2\sqrt{2}-1)^2}\right)^{1/3} < 1.$$

Consequently,

$$\rho_{\mathcal{F}} < \exp\left(\frac{B_{\mathcal{R}} + 2}{3}\right).$$

If $\mathcal{F}$ is the complete first-arrival path of a terminating integer n, then every canonical block beginning at an odd input has its canonical endpoint before v_N. Consequently,

$$\mathcal{R}(n) \le B_{\mathcal{R}}, \qquad \rho(n) \le e^{B_{\mathcal{R}}/3}.$$

We will give two proofs of this theorem. The first proof uses Hölder's inequality to obtain the explicit constant $B_{\mathcal{R}}$. We also give a more visual proof of uniform boundedness which bounds $\mathcal{R}(n)$ by summing across two regions in the lattice of all values for s and j. Although it produces a weaker numerical bound, it makes explicit how the two estimates in Lemma 2.14 complement one another: the whole-row bound controls the region $s \le 3j/5$, while the binomial bound controls the region $s > 3j/5$. This row-and-column viewpoint is illustrated in Example 2.17 and is reused in Appendix 1B, particularly in the estimates for $j > 1000$.

First proof (Hölder's inequality). We divide the odd input states of $\mathcal{F}$ into two classes. An odd input $x = v_t$ is called an *interior input* if its canonical endpoint occurs before v_N, and a *boundary input* otherwise. If $j = \lfloor \log_2 x \rfloor$, these two cases are respectively

$$t + j < N \qquad \text{and} \qquad t + j \ge N.$$

We first bound the contribution from the interior inputs, and as such, let $x = v_t$ be an interior odd input. Since every input state of $\mathcal{F}$ is greater than 1, there is a unique integer $j = \lfloor \log_2 x \rfloor \ge 1$ such that $2^j < x < 2^{j+1}$. Therefore $1/x < 2^{-j}$. For fixed j and s, let $N_{j,s}$ be the number of such interior canonical blocks whose first j shortcut steps contain exactly s odd inputs, as in Lemma 2.14. Their total contribution to $\mathcal{R}_{\mathcal{F}}$ is at most $N_{j,s}2^{-j}$.

A j-step canonical block contains exactly j input positions, namely $0, 1, \ldots, j-1$. By definition, its starting state is odd, so the input at position 0 is odd and the block has at least one odd input. Since it has only j input positions altogether, $1 \le s \le j$.

For each $s \ge 1$, define

$$a_s = \sum_{j \ge s} N_{j,s} 2^{-j}.$$

If $\mathcal{R}_{\mathcal{F}}^{\text{int}}$ denotes the reciprocal contribution from the interior odd inputs, then

$$\mathcal{R}_{\mathcal{F}}^{\text{int}} \le \sum_{s \ge 1} a_s.$$

We now bound each a_s. For every $j \ge s$,

$$N_{j,s}2^{-j} = N_{j,s}^{1/3} \left(N_{j,s} 2^{-3j/2} \right)^{2/3}.$$

Applying Hölder's inequality with conjugate exponents 3 and $3/2$ gives

$$a_s \le \left(\sum_{j \ge s} N_{j,s} \right)^{1/3} \left(\sum_{j \ge s} N_{j,s} 2^{-3j/2} \right)^{2/3}.$$

We bound these two factors separately. By equation (24) in Lemma 2.14,

$$\sum_{j \ge s} N_{j,s} \le 2 \cdot 3^{s-1}.$$

Hence

$$\begin{aligned} \left(\sum_{j \ge s} N_{j,s} \right)^{1/3} &\le \left(2 \cdot 3^{s-1} \right)^{1/3} \\ &= \left(\frac{2}{3} \right)^{1/3} 3^{s/3}. \end{aligned} \tag{25}$$

For the second factor, equation (23) gives

$$N_{j,s} \le \binom{j-1}{s-1}.$$

Therefore

$$\sum_{j\ge s} N_{j,s} 2^{-3j/2} \le \sum_{j\ge s} \binom{j-1}{s-1} 2^{-3j/2}.$$

Set $z = 2^{-3/2}$, and consider the generating-function identity

$$\sum_{j\ge s} \binom{j-1}{s-1} z^j = \left(\frac{z}{1-z}\right)^s,$$

we obtain

$$\begin{aligned}\sum_{j\ge s} \binom{j-1}{s-1} 2^{-3j/2} &= \left(\frac{2^{-3/2}}{1-2^{-3/2}}\right)^s \\ &= \frac{1}{(2\sqrt{2}-1)^s}.\end{aligned}$$

It follows that

$$\left(\sum_{j\ge s} N_{j,s} 2^{-3j/2}\right)^{2/3} \le \frac{1}{(2\sqrt{2}-1)^{2s/3}}. \tag{26}$$

Combining the two estimates (25) and (26) gives

$$\begin{aligned}a_s &\le \left(\frac{2}{3}\right)^{1/3} 3^{s/3} \frac{1}{(2\sqrt{2}-1)^{2s/3}} \\ &= \left(\frac{2}{3}\right)^{1/3} \left(\frac{3}{(2\sqrt{2}-1)^2}\right)^{s/3}.\end{aligned}$$

Define

$$r = \left(\frac{3}{(2\sqrt{2}-1)^2}\right)^{1/3}.$$

Since $(2\sqrt{2}-1)^2 = 9 - 4\sqrt{2} > 3$, we have $0 < r < 1$. Numerically, $r \approx 0.96454$. Thus

$$a_s \le \left(\frac{2}{3}\right)^{1/3} r^s.$$

Summing over $s \ge 1$ and using the geometric-series identity gives

$$\begin{aligned}\mathcal{R}_{\mathcal{F}}^{\text{int}} &\le \sum_{s\ge 1} a_s \\ &\le \left(\frac{2}{3}\right)^{1/3} \sum_{s=1}^{\infty} r^s \\ &= \left(\frac{2}{3}\right)^{1/3} \frac{r}{1-r} \\ &= B_{\mathcal{R}}.\end{aligned}$$

We now bound the contribution from the boundary odd inputs. Suppose that such an input is $x = v_t$ and that $2^j < x < 2^{j+1}$. Its canonical block has length j. Since this is a boundary input, its canonical endpoint does not occur before v_N. Thus $t + j \geq N$, or equivalently $t \geq N - j$. Since $0 \leq t < N$, such a starting position must lie among

$$N - j, N - j + 1, \ldots, N - 1,$$

after omitting any negative indices when $j > N$. In either case there are at most j possible input positions at which a boundary canonical block of shell index j can begin. Every starting state in this shell satisfies $x > 2^j$, and therefore $1/x < 2^{-j}$. Hence the total reciprocal contribution from boundary canonical blocks of shell index j is less than $j2^{-j}$. If $\mathcal{R}_{\mathcal{F}}^{\text{bdry}}$ denotes the reciprocal contribution from all boundary odd inputs, then

$$\mathcal{R}_{\mathcal{F}}^{\text{bdry}} < \sum_{j \geq 1} j2^{-j}.$$

For $|z| < 1$,

$$\sum_{j \geq 1} jz^j = \frac{z}{(1 - z)^2}.$$

Setting $z = 1/2$ gives

$$\sum_{j \geq 1} j2^{-j} = \frac{1/2}{(1 - 1/2)^2} = 2.$$

Therefore

$$\mathcal{R}_{\mathcal{F}}^{\text{bdry}} < 2. \tag{27}$$

Every odd input state of $\mathcal{F}$ belongs to exactly one of the two classes. Consequently,

$$\mathcal{R}_{\mathcal{F}} = \mathcal{R}_{\mathcal{F}}^{\text{int}} + \mathcal{R}_{\mathcal{F}}^{\text{bdry}} < B_{\mathcal{R}} + 2.$$

We next convert the reciprocal bound into a residue bound. By Definition 2.5,

$$\rho_{\mathcal{F}} = \prod_{\substack{0 \leq t < N \\ v_t \text{ odd}}} \left(1 + \frac{1}{3v_t}\right).$$

Taking logarithms gives

$$\log \rho_{\mathcal{F}} = \sum_{\substack{0 \leq t < N \\ v_t \text{ odd}}} \log \left(1 + \frac{1}{3v_t}\right).$$

Since $\log(1 + u) < u$ for $(u > 0)$, we have

$$\begin{aligned} \log \rho_{\mathcal{F}} &\leq \frac{1}{3} \sum_{\substack{0 \leq t < N \\ v_t \text{ odd}}} \frac{1}{v_t} \\ &= \frac{1}{3} \mathcal{R}_{\mathcal{F}} \\ &< \frac{B_{\mathcal{R}} + 2}{3}. \end{aligned}$$

Exponentiating gives

$$\rho_{\mathcal{F}} < \exp\left(\frac{B_{\mathcal{R}} + 2}{3}\right).$$

It remains to establish the sharper bounds when $\mathcal{F}$ is the complete first-arrival path of a terminating integer n. We show that in this case every canonical block beginning at an odd input is an interior block.

Let $x > 1$ be an odd input on the first-arrival path and set $j = \lfloor \log_2 x \rfloor$. For every positive integer y, $T(y) \geq y/2$. Indeed, if y is even, then $T(y) = y/2$, while if y is odd, then

$$T(y) = \frac{3y+1}{2} > \frac{y}{2}.$$

Applying this inequality successively gives

$$T^t(x) \geq \frac{x}{2^t}.$$

Since $x > 2^j$, for every $0 \leq t \leq j$ we obtain

$$T^t(x) \geq \frac{x}{2^t} > \frac{2^j}{2^t} = 2^{j-t} \geq 1.$$

Thus $T^t(x) > 1$ for every $0 \leq t \leq j$, so none of the states in the canonical block, including its endpoint, is 1. The complete path first reaches 1 at step N. Therefore the remaining steps $N - t_0$ from $x = v_{t_0}$ to that first arrival must satisfy $N - t_0 > j$, or equivalently, $t_0 + j < N$. By definition, x is therefore an interior input. Since x was an arbitrary odd input on the first-arrival path, every canonical block beginning at an odd input is an interior block.

It follows that $\mathcal{R}_{\mathcal{F}}^{\mathrm{bdry}} = 0$ for a complete first-arrival path. Since in this case $\mathcal{R}_{\mathcal{F}} = \mathcal{R}(n)$, the interior-block estimate gives $\mathcal{R}(n) \leq B_{\mathcal{R}}$. Likewise, $\rho_{\mathcal{F}} = \rho(n)$, so

$$\log \rho(n) \leq \frac{1}{3}\mathcal{R}(n) \leq \frac{B_{\mathcal{R}}}{3}.$$

Therefore

$$\rho(n) \leq e^{B_{\mathcal{R}}/3}.$$

This proves both the finite-path bounds and the sharper bounds for terminating first-arrival paths. □

Second proof of uniform boundedness (row and column bounds). As in the first proof, divide the odd input states of $\mathcal{F}$ into interior and boundary inputs. Thus, if $x = v_t$ and $j = \lfloor \log_2 x \rfloor$, then x is an interior input when $t + j < N$, and a boundary input when $t + j \geq N$.

Consider first the interior inputs. Each such odd input $x > 1$ contributes less than 2^{-j} when $2^j < x < 2^{j+1}$. Therefore, if $\mathcal{R}_{\mathcal{F}}^{\mathrm{int}}$ denotes the reciprocal contribution from the interior odd inputs, then

$$\mathcal{R}_{\mathcal{F}}^{\mathrm{int}} \leq \sum_{j \geq 1} \sum_{s=1}^{j} N_{j,s} 2^{-j}.$$

Partition the index set of this double sum along the line $s = 3j/5$. In the region $s \leq 3j/5$, summation row by row, with s fixed, allows the use of the whole-row bound. In the region $s > 3j/5$, summation column by column, with j fixed, allows the use of the binomial bound.

The region $s \leq 3j/5$: use the row bound. In row s, these entries have

$$j \geq \left\lceil \frac{5s}{3} \right\rceil.$$

Every weight 2^{-j} in this part of the row is at most $2^{-5s/3}$. The second inequality of Lemma 2.14 gives

$$\begin{aligned}\sum_{j\ge\lceil 5s/3\rceil} N_{j,s}2^{-j} &\le 2^{-5s/3}\sum_{j\ge s} N_{j,s}\\ &\le \frac{2}{3}\left(\frac{3}{2^{5/3}}\right)^{s}.\end{aligned} \tag{28}$$

The sum in the middle includes the whole row. Including entries from the other region only increases the upper bound.

The region $s > 3j/5$: use the binomial bound. In column j, since $s > 3j/5$ these entries satisfy

$$1 < \left(\frac{2}{3}\right)^{3j/5}\left(\frac{3}{2}\right)^{s},$$

and therefore

$$\binom{j-1}{s-1} < \left(\frac{2}{3}\right)^{3j/5}\binom{j-1}{s-1}\left(\frac{3}{2}\right)^{s}.$$

Using the first inequality of Lemma 2.14, we obtain

$$\begin{aligned}2^{-j}\sum_{3j/5<s\le j} N_{j,s} &\le 2^{-j}\sum_{3j/5<s\le j}\binom{j-1}{s-1}\\ &\le 2^{-j}\left(\frac{2}{3}\right)^{3j/5}\sum_{s=1}^{j}\binom{j-1}{s-1}\left(\frac{3}{2}\right)^{s}\\ &= 2^{-j}\left(\frac{2}{3}\right)^{3j/5}\frac{3}{2}\left(1+\frac{3}{2}\right)^{j-1}\\ &= \frac{3}{5}\left(\frac{5}{4}\left(\frac{2}{3}\right)^{3/5}\right)^{j}.\end{aligned} \tag{29}$$

Here the second line extends the sum to all s after inserting the weight $(3/2)^s$, and the following line uses the binomial theorem.

Both estimates decrease geometrically. Put

$$r_1 = \frac{3}{2^{5/3}}, \qquad r_2 = \frac{5}{4}\left(\frac{2}{3}\right)^{3/5}.$$

Then

$$r_1^3 = \frac{27}{32} < 1, \qquad r_2^5 = \frac{3125}{3456} < 1.$$

Every interior entry belongs to exactly one of the two regions. Adding the first region across rows and the second region down columns gives

$$\begin{aligned}\mathcal{R}_{\mathcal{F}}^{\mathrm{int}} &< \frac{2}{3}\sum_{s\ge1} r_1^s + \frac{3}{5}\sum_{j\ge1} r_2^j\\ &= \frac{2r_1}{3(1-r_1)} + \frac{3r_2}{5(1-r_2)} < \infty.\end{aligned} \tag{30}$$

The expression on the right is independent of $\mathcal{F}$.

The boundary-input estimate is independent of the method used to bound the interior inputs. Therefore (27), established in the first proof, gives $\mathcal{R}_{\mathcal{F}}^{\mathrm{bdry}} < 2$. Consequently,

$$\begin{aligned}\mathcal{R}_{\mathcal{F}} &= \mathcal{R}_{\mathcal{F}}^{\mathrm{int}} + \mathcal{R}_{\mathcal{F}}^{\mathrm{bdry}} \\ &< \left(\frac{2r_1}{3(1-r_1)} + \frac{3r_2}{5(1-r_2)}\right) + 2 < \infty.\end{aligned}$$

Thus the reciprocal sums of all finite shortcut paths satisfying the hypotheses of the theorem are bounded by an absolute constant independent of the path.

Finally, by Definition 2.5,

$$\log \rho_{\mathcal{F}} = \sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \log\left(1 + \frac{1}{3v_t}\right) \le \frac{1}{3}\mathcal{R}_{\mathcal{F}},$$

so $\rho_{\mathcal{F}}$ also has a uniform finite upper bound.

If $\mathcal{F}$ is the complete first-arrival path of a terminating integer n, the argument in the first proof shows that every canonical block beginning at an odd input is an interior block. Hence $\mathcal{R}_{\mathcal{F}}^{\mathrm{bdry}} = 0$. Since $\mathcal{R}_{\mathcal{F}} = \mathcal{R}(n)$ and $\rho_{\mathcal{F}} = \rho(n)$, we obtain

$$\mathcal{R}(n) < \frac{2r_1}{3(1-r_1)} + \frac{3r_2}{5(1-r_2)},$$

and consequently

$$\rho(n) < \exp\left[\frac{1}{3}\left(\frac{2r_1}{3(1-r_1)} + \frac{3r_2}{5(1-r_2)}\right)\right].$$

Thus the row-and-column argument independently gives uniform finite bounds for both $\mathcal{R}(n)$ and $\rho(n)$. Empty sums, including the case of a power of two, cause no exception.

Numerically, the bounds furnished by this argument are

$$\mathcal{R}(n) < 40.941, \qquad \rho(n) < 8.45 \times 10^5.$$

These constants are intentionally coarse; the purpose of the row-and-column argument is to give an independent proof that both quantities admit uniform finite bounds. Empty sums, including the case in which n is a power of two, cause no exception. □

The sharper bounds

$$\mathcal{R}(n) \le B_{\mathcal{R}}, \qquad \rho(n) \le B_\rho := e^{B_{\mathcal{R}}/3}$$

hold for every terminating integer n. In particular, B_ρ answers Roosendaal's *Weak Residue Conjecture* [14]. We will later reduce this upper bound further.

Example 2.17. We return to the first-arrival orbit of 993 from Example 2.15. Since this is a complete first-arrival path, every canonical block beginning at an odd input is an interior block, as shown in the first proof of Theorem 2.16, thus there is no boundary contribution. The row-and-column argument of the second proof can be seen directly in the $N_{j,s}$ array.

Table 4 repeats the same values of $N_{j,s}$, with a staircase boundary separating the two regions used in the second proof. Entries to the right of the staircase satisfy $5s \le 3j$; entries below it satisfy $5s > 3j$.

The canonical block beginning at 993 has length $j = \lfloor \log_2 993 \rfloor = 9$, and six of its nine shortcut inputs are odd. It is therefore counted in the cell $N_{9,6}$. Since $N_{9,6} = 3$, the other two canonical blocks counted in this cell begin at 745 and 641. The largest occupied indices in the array are $j = 11$ and $s = 7$; as before, the display is extended to $j = 15$ and $s = 10$. Every entry outside the displayed rectangle is zero.

Put $k_s = \lceil 5s/3 \rceil$, and for each row s, let R_s denote the number of odd inputs represented by the cells to the right of the staircase:

$$R_s := \sum_{j \geq k_s} N_{j,s}.$$

Since every odd input represented in this part of the row lies in a shell of index $j \geq k_s$, its reciprocal is less than 2^{-k_s}. The two rightmost columns therefore show

$$\frac{R_s}{2^{k_s}} \qquad \text{and} \qquad \frac{2 \cdot 3^{s-1}}{2^{k_s}},$$

where

$$R_s \leq \sum_j N_{j,s} \leq 2 \cdot 3^{s-1}$$

by the whole-row bound from Lemma 2.14. Similarly, for each column j, let

$$L_j := \sum_{3j/5 < s \leq j} N_{j,s}$$

denote the number of odd inputs represented by the cells below the staircase. Every term in column j has reciprocal weight less than 2^{-j}, so the contribution from these entries is less than $L_j/2^j$. The two rows at the bottom therefore show

$$\frac{L_j}{2^j} \qquad \text{and} \qquad \frac{1}{2^j} \sum_{3j/5 < s \leq j} \binom{j-1}{s-1}.$$

Thus, for this first-arrival path, the right-hand row counts R_s and the below-staircase column counts L_j partition all of the occupied $N_{j,s}$ entries; no cell is counted twice.

In row s, the region to the right of the staircase begins at $j = k_s = \lceil 5s/3 \rceil$. For $s = 5$, one has $k_5 = 9$. Thus $N_{8,5} = 4$ belongs to the below-staircase region, while $N_{9,5} = 1$ and $N_{10,5} = 2$ lie to the right of the staircase. Hence

$$R_5 = N_{9,5} + N_{10,5} = 3,$$

whereas the whole-row count is

$$N_{8,5} + N_{9,5} + N_{10,5} = 4 + 1 + 2 = 7 \leq 162.$$

In row $s = 1$, all occupied entries lie to the right and the whole-row bound is attained:

$$R_1 = N_{2,1} + N_{3,1} = 2.$$

Assigning every entry to the right of the staircase in row s the common upper weight 2^{-k_s}, rather than retaining its individual column weight 2^{-j}, gives the coarser bound

$$\mathcal{R}(993) < \sum_{s=1}^{10} \frac{R_s}{2^{k_s}} + \sum_{j=1}^{15} \frac{L_j}{2^j} = \frac{429}{512} + \frac{13}{32} = \frac{637}{512}.$$

For example, row $s = 5$ has $k_s = 9$ and $R_5 = 3$, so its contribution is less than

$$\frac{3}{2^9} \leq \frac{162}{2^9}.$$

In column $j = 8$, the left-hand count is

$$L_8 = N_{8,5} + N_{8,6} = 5,$$

Table 4: The $N_{j,s}$ array for the first-arrival orbit of 993, with the staircase separating the regions $5s \le 3j$ and $5s > 3j$. Since every canonical block in this first-arrival path is interior, there is no boundary contribution. The first two columns on the right give the right-of-staircase row count R_s and its whole-row bound. The last two columns give the corresponding weighted count and weighted bound. The two rows at the bottom give the below-staircase column contributions $L_j/2^j$ and their binomial bounds. Powers of two are left visible to show the denominators used.

$s \setminus j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	R_s	$2\cdot3^{s-1}$	$\frac{R_s}{2^{k_s}}$	$\frac{2\cdot3^{s-1}}{2^{k_s}}$
1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	2	2	$\frac{2}{2^2}$	$\frac{2}{2^2}$
2	0	1	1	1	2	0	0	0	0	0	0	0	0	0	0	3	6	$\frac{3}{2^4}$	$\frac{6}{2^4}$
3	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	4	18	$\frac{4}{2^5}$	$\frac{18}{2^5}$
4	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	2	54	$\frac{2}{2^7}$	$\frac{54}{2^7}$
5	0	0	0	0	0	0	0	4	1	2	0	0	0	0	0	3	162	$\frac{3}{2^9}$	$\frac{162}{2^9}$
6	0	0	0	0	0	0	0	1	3	2	2	0	0	0	0	4	486	$\frac{4}{2^{10}}$	$\frac{486}{2^{10}}$
7	0	0	0	0	0	0	0	0	2	2	0	0	0	0	0	0	1458	0	$\frac{1458}{2^{12}}$
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4374	0	$\frac{4374}{2^{14}}$
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	13122	0	$\frac{13122}{2^{15}}$
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	39366	0	$\frac{39366}{2^{17}}$
below count $L_j/2^j$	0	$\frac{1}{2^2}$	$\frac{1}{2^3}$	0	0	0	0	$\frac{5}{2^8}$	$\frac{5}{2^9}$	$\frac{2}{2^{10}}$	0	0	0	0	0	$\sum_j L_j = 14$,		$\sum_j \frac{L_j}{2^j} = \frac{13}{32}$	
binomial bound $/2^j$	$\frac{1}{2^1}$	$\frac{1}{2^2}$	$\frac{3}{2^3}$	$\frac{4}{2^4}$	$\frac{5}{2^5}$	$\frac{16}{2^6}$	$\frac{22}{2^7}$	$\frac{64}{2^8}$	$\frac{93}{2^9}$	$\frac{130}{2^{10}}$	$\frac{386}{2^{11}}$	$\frac{562}{2^{12}}$	$\frac{1586}{2^{13}}$	$\frac{2380}{2^{14}}$	$\frac{3473}{2^{15}}$				

so its contribution is less than $5/2^8$, which is bounded above by $64/2^8$ using the binomial count. Once again, the right-hand row counts and below-staircase column counts are disjoint, so no cell is counted twice.

To retain the sharper bound obtained from the actual table, instead multiply every entry in column j by its own weight 2^{-j}. On the right of the staircase, the nonzero weighted row sums, in order of increasing s, give

$$\frac{3}{8} + \frac{1}{8} + \frac{27}{512} + \frac{3}{512} + \frac{1}{256} + \frac{3}{1024} = \frac{579}{1024}.$$

Below the staircase, the nonzero weighted column sums, in order of increasing j, give

$$\frac{1}{4} + \frac{1}{8} + \frac{5}{256} + \frac{5}{512} + \frac{1}{512} = \frac{13}{32}.$$

There are 18 odd inputs on the right and 14 below. Thus all 32 odd inputs of the first-arrival path are counted exactly once, with no boundary inputs remaining, and

$$\mathcal{R}(993) < \frac{579}{1024} + \frac{13}{32} = \frac{995}{1024}.$$

2.2 Once-Counted Forward-Orbit Quantities

The preceding finite-path bounds can be extended to arbitrary forward orbits without assuming termination. To do so, repeated values must be counted only once.

Definition 2.18. For a positive integer n, let

$$O(n) := \{x > 1 : x \text{ is odd and occurs in the forward orbit of } n\}$$

be the set of distinct odd values greater than 1 occurring in the forward orbit of n. Define the *once-counted reciprocal sum* by

$$\mathcal{R}^{\circ}(n) := \sum_{x \in O(n)} \frac{1}{x}, \tag{31}$$

the *once-counted residue product* by

$$\rho^{\circ}(n) := \prod_{x \in O(n)} \left(1 + \frac{1}{3x}\right), \tag{32}$$

and the *once-counted correction product* by

$$\Pi^{\circ}(n) := \prod_{x \in O(n)} \frac{2x+1}{2x-1}. \tag{33}$$

If $O(n)$ is infinite, the sum and products are understood as the limits obtained by including the distinct odd values in order of their first occurrence along the forward orbit.

Due to the lack of creativity on the part of the authors, the reader must ensure that they are aware of the difference between $O(n)$, the number of odd steps in the orbit the integer n, and $\mathcal{O}(n)$, the set of once counted integers greater than 1 i nthe forward orbit of n, as defined above.

The superscript $\circ$ distinguishes these once counted quantities from the original functions $\mathcal{R}(n)$, $\rho(n)$, and $\Pi(n)$, which were defined for terminating first-arrival trajectories. In the case of n terminating, no state can repeat before the first arrival at 1, and the terminal value 1 is excluded from both definitions. Hence

$$\mathcal{R}^{\circ}(n) = \mathcal{R}(n), \qquad \rho^{\circ}(n) = \rho(n), \qquad \Pi^{\circ}(n) = \Pi(n).$$

Corollary 2.19 (Uniform Finiteness for Forward Orbits)**.** *For every positive integer n,*

$$\mathcal{R}^{\circ}(n) \le B_{\mathcal{R}} + 2,$$

where $B_{\mathcal{R}}$ is the constant in Theorem 2.16. Consequently,

$$\rho^{\circ}(n) \le \exp\left(\frac{B_{\mathcal{R}} + 2}{3}\right)$$

and

$$\Pi^{\circ}(n) \le \exp\left(2(B_{\mathcal{R}} + 2)\right).$$

In particular, all three once-counted quantities are finite for every forward orbit.

Proof. There are three cases.

First suppose that the orbit of n reaches 1. Then its first-arrival path has distinct input states greater than 1, and the terminating specialization of Theorem 2.16 gives

$$\mathcal{R}^{\circ}(n) = \mathcal{R}(n) \le B_{\mathcal{R}}.$$

The case $n = 1$ is included by the convention $O(1) = \{\,\}$.

Next suppose that the orbit does not reach 1 but some state repeats. Let N be the first step at which $v_N = v_k$ for some $0 \le k < N$. Then $v_0, v_1, \ldots, v_{N-1}$ are distinct and greater than 1. Therefore Theorem 2.16, applied to $\mathcal{F} = (v_0, v_1, \ldots, v_N)$, gives

$$\sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t} < B_{\mathcal{R}} + 2.$$

Since T is deterministic, once $v_N = v_k$ occurs the subsequent orbit repeats the portion beginning at v_k. Hence no new states occur after step N, and therefore

$$\mathcal{R}^\circ(n) = \sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t} < B_{\mathcal{R}} + 2.$$

Finally, suppose that the orbit neither reaches 1 nor repeats a state. For each $N \ge 1$, define

$$S_N := \sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t}.$$

The finite path $\mathcal{F}_N = (v_0, v_1, \ldots, v_N)$ has distinct input states greater than 1, so Theorem 2.16 gives $S_N < B_{\mathcal{R}} + 2$. Moreover, $S_{N+1} - S_N$ is either 0 or $1/v_N$, according as v_N is even or odd. Hence (S_N) is nondecreasing. Since it is also bounded above by $B_{\mathcal{R}} + 2$, it converges to a finite limit satisfying

$$\lim_{N \to \infty} S_N \le B_{\mathcal{R}} + 2.$$

Because the orbit has no repeated states, these partial sums include every member of $O(n)$ exactly once, in order of first occurrence. Therefore, by (31),

$$\mathcal{R}^\circ(n) = \lim_{N \to \infty} S_N \le B_{\mathcal{R}} + 2.$$

It remains to derive the two product bounds. By Definition 2.18,

$$\log \rho^\circ(n) = \sum_{x \in O(n)} \log\left(1 + \frac{1}{3x}\right).$$

Since $\log(1 + u) < u$ for $u > 0$,

$$\log \rho^\circ(n) \le \frac{1}{3}\mathcal{R}^\circ(n) \le \frac{B_{\mathcal{R}} + 2}{3}.$$

Therefore

$$\rho^\circ(n) \le \exp\left(\frac{B_{\mathcal{R}} + 2}{3}\right).$$

Similarly,

$$\log \Pi^\circ(n) = \sum_{x \in O(n)} \log\left(\frac{2x+1}{2x-1}\right).$$

For every $x \in O(n)$, we have $x > 1$, and hence

$$\begin{aligned} \log\left(\frac{2x+1}{2x-1}\right) &= \log\left(1 + \frac{2}{2x-1}\right) \\ &< \frac{2}{2x-1} \\ &< \frac{2}{x}. \end{aligned}$$

It follows that $\log \Pi^\circ(n) \le 2\mathcal{R}^\circ(n) \le 2(B_{\mathcal{R}} + 2)$, and therefore

$$\Pi^\circ(n) \le \exp\left(2(B_{\mathcal{R}} + 2)\right).$$

Thus all three once-counted quantities are finite. □

2.3 The Four-Thirds and Reciprocal-Sum Bounds

We now use the finite-path reciprocal estimate proved in Appendix 1 to obtain a four-thirds bound for once-counted forward-orbit residues. The finite-path four-thirds bound then follows by viewing a finite path residue as a subproduct of the corresponding once-counted residue, while the bound for terminating integers follows as an immediate specialization. An elementary comparison between the residue product and the reciprocal sum then gives the corresponding bounds $\mathcal{R}_{\mathcal{F}} < 1$, $\mathcal{R}^{\circ}(n) < 1$, and $\mathcal{R}(n) < 1$.

Set

$$D = 28019077177231758495, \qquad M = 2^{32}.$$

We use the following two finite records:

$$H(y) \le 2456 \qquad 1 \le y \le D, \tag{34}$$

$$\rho(z) \le \rho(993) = \frac{2^{61}}{3^{32} \cdot 993} \qquad 1 \le z < M. \tag{35}$$

These are the delay-record bounds of Roosendaal [14] and the residue record through 2^{32} recorded in OEIS A127789 [11]. Heights and residues are taken up to first arrival at 1, and the two finite records are assumed. The estimate needed above D is finite-path in nature and does not require termination.

Lemma 2.20. *Let*

$$\mathcal{F} = (v_0, v_1, \dots, v_N)$$

be a finite nonrepeating shortcut path whose input states are greater than D. Then

$$\mathcal{R}_{\mathcal{F}} < \frac{9}{50}.$$

Proof. This is Theorem 1A.1 of Appendix 1. □

The counting lemma first gives a uniform reciprocal bound. Appendix 1 sharpens it above D using one block-balance identity and a finite upper count of gap cuts. The height-delay and residue bounds then control the remainder of any forward orbit that enters the checked range.

The finite affine remainder and the use of stopping-time and delay-record data are also studied by Rozier and Terracol [15]. Their results concerning stopping times and paradoxical sequences address questions different from the finite-path residue bound proved here under the stated hypotheses.

Corollary 2.21 (Once-Counted Four-Thirds Bound)**.** *For every positive integer n,*

$$\rho^{\circ}(n) < \frac{4}{3}.$$

Proof. Suppose first that the forward orbit of n enters $[1, D]$. Let q be the least nonnegative integer for which $v_q \le D$, and put $y = v_q$. Define the high-orbit reciprocal contribution by

$$\mathcal{R}_{\mathrm{high}}(n) := \sum_{\substack{0 \le t < q \\ v_t \text{ odd}}} \frac{1}{v_t}.$$

Thus $\mathcal{R}_{\mathrm{high}}(n)$ is the reciprocal sum over the odd input states occurring before the first entry into $[1, D]$.

The input states $v_0, v_1, \dots, v_{q-1}$ are distinct and greater than D; otherwise a repeated state would produce a cycle before the first entry into $[1, D]$. Consequently, $\mathcal{R}_{\mathrm{high}}(n)$ is also the contribution of these states to the once-counted sum $\mathcal{R}^{\circ}(n)$. Therefore Lemma 2.20, applied to the finite path $(v_0, v_1, \dots, v_q)$, gives

$$\mathcal{R}_{\mathrm{high}}(n) < \frac{9}{50}.$$

Since $y \le D$, the finite height bound (34) implies that y terminates and $H(y) \le 2456$. Let z be the first state at or after y that is below M. Before reaching z there are at most 2456 odd inputs, each at least M, so their reciprocal contribution is at most $2456/M$. The residue record (35) then gives

$$\rho^\circ(n) < \rho(993) \exp\left[\frac{1}{3}\left(\frac{9}{50} + \frac{2456}{M}\right)\right] < \rho(993)e^{61/1000} < \frac{4}{3}.$$

Now suppose that the forward orbit of n never enters $[1, D]$. If some state repeats, let N be the first index for which $v_N = v_k$ for some $0 \le k < N$. Then the input states $v_0, v_1, \ldots, v_{N-1}$ are distinct and greater than D. Lemma 2.20, applied to the finite path $(v_0, v_1, \ldots, v_N)$, therefore gives

$$\sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t} < \frac{9}{50}.$$

Since T is deterministic, no new states occur after the first repetition. Hence

$$\mathcal{R}^\circ(n) = \sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t} < \frac{9}{50}.$$

Now suppose that no state repeats. For each $N \ge 1$, define

$$S_N := \sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t}.$$

The finite initial path $(v_0, v_1, \ldots, v_N)$ has distinct input states greater than D, so Lemma 2.20 gives $S_N < \frac{9}{50}$. Moreover, $S_{N+1} - S_N$ is either 0 or $1/v_N$, according as v_N is even or odd. Thus (S_N) is nondecreasing and bounded above by $9/50$, so it converges and

$$\lim_{N \to \infty} S_N \le \frac{9}{50}.$$

Because the orbit has no repeated states, these partial sums include every member of $O(n)$ exactly once, in order of first occurrence. Therefore, by (31),

$$\mathcal{R}^\circ(n) = \lim_{N \to \infty} S_N \le \frac{9}{50}.$$

Thus, whether or not the orbit repeats, $\mathcal{R}^\circ(n) \le 9/50$. Therefore

$$\log \rho^\circ(n) \le \frac{1}{3}\mathcal{R}^\circ(n) \le \frac{3}{50},$$

which implies that

$$\rho^\circ(n) \le e^{3/50} < \frac{4}{3}.$$

Thus in either case, $\rho^\circ(n) < \frac{4}{3}$. □

Theorem 2.22 (Finite-Path Four-Thirds Bound)**.** *Let $\mathcal{F} = (v_0, v_1, \ldots, v_N)$ be a finite shortcut path whose input states $v_0, \ldots, v_{N-1}$ are distinct and greater than* 1*. Then*

$$\rho_{\mathcal{F}} < \frac{4}{3}.$$

No restriction is imposed on the endpoint v_N or on the subsequent orbit.

Proof. If $N = 0$, then $\mathcal{F}$ has no input states, so the product defining $\rho_{\mathcal{F}}$ is empty. Hence $\rho_{\mathcal{F}} = 1 < 4/3$.

Assume now that $N \geq 1$. The odd inputs of $\mathcal{F}$ are distinct and occur on the forward orbit of v_0. Since every input state of $\mathcal{F}$ is greater than 1, each such odd input is among the distinct odd values counted in $\rho^\circ(v_0)$. Hence $\rho_{\mathcal{F}}$ is a subproduct of $\rho^\circ(v_0)$.

Since every factor in the residue product is greater than 1, $\rho_{\mathcal{F}} \leq \rho^\circ(v_0)$. By Corollary 2.21, $\rho^\circ(v_0) < 4/3$. Consequently,

$$\rho_{\mathcal{F}} < \frac{4}{3}.$$

□

The restriction on the input states is essential. Under the shortcut map, $T(1) = 2$, so $\mathcal{F} = (1, 2)$ is a one-step finite path whose only input is the odd value 1. Consequently,

$$\rho_{\mathcal{F}} = 1 + \frac{1}{3 \cdot 1} = \frac{4}{3}.$$

Thus allowing 1 as an input would invalidate the strict conclusion of Theorem 2.22. No corresponding restriction is needed on the endpoint, because the endpoint contributes no factor to $\rho_{\mathcal{F}}$; it may equal 1 or repeat an earlier input state.

A weaker uniform bound remains available when 1 is allowed as an input. For every finite shortcut path $\mathcal{F} = (v_0, \ldots, v_N)$ with distinct positive input states,

$$\rho_{\mathcal{F}} \leq \frac{4}{3}\rho^\circ(v_0) < \frac{16}{9} < 2.$$

Indeed, distinctness allows the input 1 to occur at most once, and it contributes the factor $4/3$. Every other odd input is a distinct member of $O(v_0)$, so the product of the corresponding factors is at most $\rho^\circ(v_0)$. The strict inequality follows from Corollary 2.21.

In Luo's notation, the quantity $\mathrm{res}_3(N, N')$ is exactly the residue $\rho_{\mathcal{F}}$ of the corresponding finite path. Thus one may take $R_3 = 16/9 < 2$. This proves the $q = 3$ cases of Luo's Conjectures 5.6(a) and 5.8(a) for positive starting integers [10].

For a complete first-arrival path, however, 1 can occur only as the endpoint and never as an input. The sharper four-thirds bound therefore gives the following terminating specialization.

Theorem 2.23 (The Four-Thirds Bound)**.** *Every terminating positive integer n satisfies $\rho(n) < 4/3$.*

Proof. Let $\mathcal{F}_n$ be the complete first-arrival shortcut path of n. Its input states, if any, are distinct and greater than 1, so Theorem 2.22 gives $\rho_{\mathcal{F}_n} < 4/3$. Since the finite-path residue specializes to the ordinary residue on a complete first-arrival path, $\rho_{\mathcal{F}_n} = \rho(n)$. Therefore

$$\rho(n) < \frac{4}{3}.$$

□

Corollary 2.24 (Reciprocal Sum Below One)**.** *Let $\mathcal{F}$ be a finite nonrepeating shortcut path whose input states are greater than* 1*. Then*

$$\mathcal{R}_{\mathcal{F}} < 1.$$

Moreover, for every positive integer n,

$$\mathcal{R}^\circ(n) = \sum_{x \in O(n)} \frac{1}{x} < 1. \tag{36}$$

If n terminates, this specializes to

$$\mathcal{R}(n) < 1.$$

Proof. List the odd inputs of $\mathcal{F}$ as $x_1, \ldots, x_{O_\mathcal{F}}$. Expanding the finite-path residue product gives

$$\rho_\mathcal{F} = \prod_{i=1}^{O_\mathcal{F}} \left(1 + \frac{1}{3x_i}\right) \geq 1 + \frac{1}{3}\sum_{i=1}^{O_\mathcal{F}} \frac{1}{x_i} = 1 + \frac{\mathcal{R}_\mathcal{F}}{3}. \tag{37}$$

Theorem 2.22 therefore gives

$$1 + \frac{\mathcal{R}_\mathcal{F}}{3} \leq \rho_\mathcal{F} < \frac{4}{3}, \tag{38}$$

and hence $\mathcal{R}_\mathcal{F} < 1$.

For the once-counted reciprocal sum and residue product, list the members of $O(n)$ in order of their first occurrence as $x_1, x_2, \ldots$, with the list ending if $O(n)$ is finite. For every finite initial segment, put

$$S_K := \sum_{i=1}^{K} \frac{1}{x_i}, \qquad P_K := \prod_{i=1}^{K} \left(1 + \frac{1}{3x_i}\right).$$

Expanding the product and retaining only its constant and linear terms gives

$$P_K \geq 1 + \frac{1}{3}S_K.$$

If $O(n)$ is finite, taking K to include the entire list gives the desired inequality directly. If $O(n)$ is infinite, Definition 2.18 gives

$$S_K \longrightarrow \mathcal{R}^\circ(n), \qquad P_K \longrightarrow \rho^\circ(n),$$

and both limits are finite by Corollary 2.19. Taking the limit in the preceding inequality therefore yields

$$1 + \frac{\mathcal{R}^\circ(n)}{3} \leq \rho^\circ(n). \tag{39}$$

Corollary 2.21 now gives

$$1 + \frac{\mathcal{R}^\circ(n)}{3} \leq \rho^\circ(n) < \frac{4}{3}, \tag{40}$$

so (36) follows. When n terminates, the once-counted quantities agree with the complete first-arrival quantities, giving $\mathcal{R}(n) < 1$. □

3 Consequences of the Residue Bound $\rho < 4/3$

The preceding section establishes the bound $\rho_\mathcal{F} < 4/3$ for every finite nonrepeating shortcut path whose input states are greater than 1. Since $\log_2(4/3) < 1/2$, the resulting logarithmic correction is too small to obscure an integer operation count. We now use this fact to derive exact restrictions on the even and odd counts of individual paths and of paths having a common endpoint.

We begin by comparing two finite shortcut paths ending at the same value. Their residue ratio gives a *logarithmic strip* for the differences in their operation counts, leading to exact nearest-integer formulas, separation bounds, and synchronization results. We then recover the operation counts of a single finite path from its endpoints and one of its counts. Specializing these results to complete first-arrival paths yields the corresponding recovery formulas for terminating integers, restrictions on the changes between adjacent

starting integers, and the equivalence of the E-delay and height-delay records. We conclude by comparing the three scalar quantities ρ, $\mathcal{R}$, and Π on finite paths and once-counted forward orbits.

The ceiling and floor recovery formulas are the conditional formulas of Luo [10, Theorems 4.1 and 5.11], written here in our notation for complete and finite paths, while Ruggiero proposed the height ceiling formula [16]. Theorem 2.22 supplies the required residue bound and also yields the nearest-integer refinements.

3.1 Common-Endpoint Finite Paths

We first extend the residue-ratio viewpoint to finite shortcut paths. Let

$$\mathcal{F}_a = (a, \dots, z), \qquad \mathcal{F}_b = (b, \dots, z)$$

be two finite shortcut paths having the same endpoint z. Write

$$E_a = E_{\mathcal{F}_a}, \qquad O_a = O_{\mathcal{F}_a},$$

and

$$E_b = E_{\mathcal{F}_b}, \qquad O_b = O_{\mathcal{F}_b}.$$

Define the differences in their even- and odd-step counts by

$$e = E_a - E_b, \qquad q = O_a - O_b.$$

Definition 3.1. For two finite shortcut paths $\mathcal{F}_a$ and $\mathcal{F}_b$ from a and b, respectively, to a common endpoint z, define the *common-endpoint residue ratio* by

$$W_z(a, b) = \frac{2^e}{3^q}. \tag{41}$$

The two paths are fixed as part of this definition; $W_z(a, b)$ suppresses that dependence.

Theorem 3.2 (Common-Endpoint Residue Identity)**.** *Let $\mathcal{F}_a$ and $\mathcal{F}_b$ be finite shortcut paths from a and b, respectively, to a common endpoint z. Then*

$$W_z(a, b) = \frac{a}{b}\frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}}. \tag{42}$$

Proof. By Proposition 2.6,

$$\rho_{\mathcal{F}_a} = \frac{2^{E_a} z}{3^{O_a} a}, \qquad \rho_{\mathcal{F}_b} = \frac{2^{E_b} z}{3^{O_b} b}.$$

Since the two paths have the same endpoint, taking the ratio cancels the factor z and gives

$$\begin{aligned}
\frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}} &= \frac{2^{E_a} z}{3^{O_a} a}\frac{3^{O_b} b}{2^{E_b} z} \\
&= \frac{b}{a}\frac{2^{E_a - E_b}}{3^{O_a - O_b}} \\
&= \frac{b}{a}\frac{2^e}{3^q} \\
&= \frac{b}{a} W_z(a, b).
\end{aligned}$$

Multiplying by a/b gives

$$W_z(a, b) = \frac{a}{b}\frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}},$$

as required. □

Theorem 3.3 (Common-Endpoint Logarithmic Strip)**.** *Let $\mathcal{F}_a$ and $\mathcal{F}_b$ be finite nonrepeating shortcut paths from a and b, respectively, to a common endpoint z, and assume that the input states of both paths are greater than* 1*. With $e = E_a - E_b$ and $q = O_a - O_b$, we have*

$$\left| e - q \log_2 3 - \log_2 \frac{a}{b} \right| < \log_2 \frac{4}{3}. \tag{43}$$

Proof. By Theorem 3.2,

$$\frac{2^e}{3^q} = \frac{a}{b} \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}}.$$

Taking logarithms to base 2 gives

$$e - q \log_2 3 = \log_2 \frac{a}{b} + \log_2 \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}}.$$

Therefore

$$e - q \log_2 3 - \log_2 \frac{a}{b} = \log_2 \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}}. \tag{44}$$

By Theorem 2.22, $1 \le \rho_{\mathcal{F}_a}, \rho_{\mathcal{F}_b} < 4/3$, hence

$$\frac{3}{4} < \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}} < \frac{4}{3}.$$

Taking logarithms gives

$$-\log_2 \frac{4}{3} < \log_2 \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}} < \log_2 \frac{4}{3}.$$

Combining this with (44) yields

$$-\log_2 \frac{4}{3} < e - q \log_2 3 - \log_2 \frac{a}{b} < \log_2 \frac{4}{3},$$

which is equivalent to

$$\left| e - q \log_2 3 - \log_2 \frac{a}{b} \right| < \log_2 \frac{4}{3}.$$

□

For $t \in \mathbb{R}$, write $\operatorname{nint}(t) = \lfloor t + 1/2 \rfloor$ for the nearest integer, with half-integer ties rounded upward. However, for the applications below, there are no ties.

Corollary 3.4 (Common-Endpoint Count-Difference Formula)**.** *Under the hypotheses of Theorem 3.3,*

$$E_a - E_b = \operatorname{nint}\left((O_a - O_b) \log_2 3 + \log_2 \frac{a}{b} \right). \tag{45}$$

No nearest-integer tie can occur.

Proof. By Theorem 3.3,

$$\left| (E_a - E_b) - \left((O_a - O_b) \log_2 3 + \log_2 \frac{a}{b} \right) \right| < \log_2 \frac{4}{3} < \frac{1}{2}.$$

Since $E_a - E_b$ is an integer, it is the unique nearest integer to

$$(O_a - O_b) \log_2 3 + \log_2 \frac{a}{b}.$$

□

We next specialize the common-endpoint comparison to paths of equal finite-path height $H_{\mathcal{F}} = E_{\mathcal{F}} + O_{\mathcal{F}}$.

Theorem 3.5 (Common-Endpoint Equal-Height Separation)**.** *Let $\mathcal{F}_a$ and $\mathcal{F}_b$ be finite nonrepeating shortcut paths from a and b, respectively, to a common endpoint z, where $a < b$, and assume that the input states of both paths are greater than* 1*. If $H_{\mathcal{F}_a} = H_{\mathcal{F}_b}$, and $d = O_a - O_b$, then $d \geq 0$ and*

$$\frac{3}{4}\,6^d < \frac{b}{a} < \frac{4}{3}\,6^d. \tag{46}$$

Consequently,

$$\frac{b}{a} \notin \left[\frac{4}{3}, \frac{9}{2}\right].$$

Proof. Since $H_{\mathcal{F}_a} = E_a + O_a$ and $H_{\mathcal{F}_b} = E_b + O_b$, the assumption $H_{\mathcal{F}_a} = H_{\mathcal{F}_b}$ gives $E_a + O_a = E_b + O_b$. Therefore $E_b - E_a = O_a - O_b = d$.

By the finite-path residue identity:

$$\rho_{\mathcal{F}_a} = \frac{2^{E_a} z}{3^{O_a} a}, \qquad \rho_{\mathcal{F}_b} = \frac{2^{E_b} z}{3^{O_b} b}.$$

Taking their ratio gives

$$\begin{aligned}\frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}} &= \frac{2^{E_a} z}{3^{O_a} a} \frac{3^{O_b} b}{2^{E_b} z} \\ &= \frac{b}{a}\, 2^{E_a - E_b} 3^{O_b - O_a}.\end{aligned}$$

Since $E_b - E_a = d$ and $O_a - O_b = d$, we have $E_a - E_b = -d$ and $O_b - O_a = -d$. Hence

$$\frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}} = \frac{b}{a}\, 2^{-d} 3^{-d} = \frac{b}{a}\, 6^{-d}.$$

Solving for b/a gives

$$\frac{b}{a} = 6^d \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}}. \tag{47}$$

By Theorem 2.22, $1 \leq \rho_{\mathcal{F}_a} < \frac{4}{3}$, and $1 \leq \rho_{\mathcal{F}_b} < \frac{4}{3}$. Therefore

$$\frac{3}{4} < \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}} < \frac{4}{3}.$$

Multiplying by 6^d and using (47) gives

$$\frac{3}{4}\,6^d < \frac{b}{a} < \frac{4}{3}\,6^d.$$

It remains to show that $d \geq 0$. Suppose instead that $d \leq -1$. Then $6^d \leq 1/6$, so the upper bound gives

$$\frac{b}{a} < \frac{4}{3}\,6^d \leq \frac{4}{3} \cdot \frac{1}{6} = \frac{2}{9}.$$

But $a < b$, so $b/a > 1$, which is a contradiction. Therefore $d \geq 0$. There are now two cases. If $d = 0$, then $3/4 < b/a < 4/3$. Since $a < b$, this sharpens to $1 < b/a < 4/3$.

If $d \geq 1$, then

$$\frac{b}{a} > \frac{3}{4}\,6^d \geq \frac{3}{4} \cdot 6 = \frac{9}{2}.$$

Thus every equal-height pair satisfies either $1 < b/a < 4/3$ or $b/a > 9/2$. Consequently,

$$\frac{b}{a} \notin \left[\frac{4}{3}, \frac{9}{2}\right].$$

□

Corollary 3.6 (Adjacent Finite-Path Synchronization)**.** *Let $\mathcal{F}_a$ and $\mathcal{F}_{a+1}$ be finite nonrepeating shortcut paths from the adjacent starting integers a and $a+1$ to a common endpoint z, and assume that the input states of both paths are greater than* 1*. If $H_{\mathcal{F}_a} = H_{\mathcal{F}_{a+1}}$, then*

$$O_{\mathcal{F}_a} = O_{\mathcal{F}_{a+1}} \qquad \text{and} \qquad E_{\mathcal{F}_a} = E_{\mathcal{F}_{a+1}}.$$

Proof. Put $d = O_{\mathcal{F}_a} - O_{\mathcal{F}_{a+1}}$. By Theorem 3.5, we have $d \geq 0$. If $d \geq 1$, then the same theorem gives

$$\frac{a+1}{a} > \frac{9}{2}.$$

On the other hand, since $a \geq 1$,

$$\frac{a+1}{a} = 1 + \frac{1}{a} \leq 2,$$

which is a contradiction. Therefore $d = 0$, and hence $O_{\mathcal{F}_a} = O_{\mathcal{F}_{a+1}}$.

Finally, $H_{\mathcal{F}_a} = H_{\mathcal{F}_{a+1}}$ means $E_{\mathcal{F}_a} + O_{\mathcal{F}_a} = E_{\mathcal{F}_{a+1}} + O_{\mathcal{F}_{a+1}}$. Since the odd-operation counts are equal, it follows that $E_{\mathcal{F}_a} = E_{\mathcal{F}_{a+1}}$. □

Corollary 3.6 is the finite-path form of the synchronization phenomenon underlying equal-height streaks. When the paths are complete first-arrival paths, their common endpoint is 1, and equality of their finite-path heights is equality of the ordinary heights of their starting integers. The corollary therefore implies that adjacent terminating integers of equal height have the same even and odd operation counts. Applied successively across a consecutive equal-height block, this forces both counts to remain constant throughout the block. We state this specialization precisely in Corollary 3.12 and return to its consequences for streaks in Section 5.

3.2 Recovery of Finite-Path Operation Counts

The preceding subsection compares two finite shortcut paths ending at a common endpoint. We now turn from comparison to reconstruction and consider a single path $\mathcal{F} = (a, \ldots, z)$. The finite-path residue identity expresses its operation counts in terms of one another, the endpoint ratio a/z, and the residue correction $\rho_{\mathcal{F}}$. The bound $1 \leq \rho_{\mathcal{F}} < 4/3$ makes this correction sufficiently small to recover the missing integer counts exactly: $H_{\mathcal{F}}$ and $E_{\mathcal{F}}$ can be recovered from $O_{\mathcal{F}}$, while $O_{\mathcal{F}}$ can be recovered from either $H_{\mathcal{F}}$ or $E_{\mathcal{F}}$.

Theorem 3.7 (Finite-Path Recovery of Operation Counts)**.** *Let $\mathcal{F} = (a, \ldots, z)$ be a finite nonrepeating shortcut path whose input states are greater than* 1*. Define*

$$L_{\mathcal{F}} := \log_2\left(\frac{6^{O_{\mathcal{F}}} a}{z}\right).$$

Then

$$H_{\mathcal{F}} = \lceil L_{\mathcal{F}} \rceil = \operatorname{nint}(L_{\mathcal{F}}), \tag{48}$$

and

$$E_{\mathcal{F}} = \left\lceil \log_2\left(\frac{3^{O_{\mathcal{F}}} a}{z}\right) \right\rceil. \tag{49}$$

Moreover,

$$O_{\mathcal{F}} = \left\lfloor \frac{H_{\mathcal{F}} - \log_2(a/z)}{\log_2 6} \right\rfloor = \left\lfloor \frac{E_{\mathcal{F}} - \log_2(a/z)}{\log_2 3} \right\rfloor. \tag{50}$$

Proof. By the finite-path residue identity,

$$\rho_{\mathcal{F}} = \frac{2^{E_{\mathcal{F}}} z}{3^{O_{\mathcal{F}}} a}.$$

Taking base-2 logarithms gives

$$\log_2 \rho_{\mathcal{F}} = E_{\mathcal{F}} - O_{\mathcal{F}} \log_2 3 - \log_2 \frac{a}{z}.$$

Therefore

$$E_{\mathcal{F}} = \log_2 \left(\frac{3^{O_{\mathcal{F}}} a}{z} \right) + \log_2 \rho_{\mathcal{F}}. \tag{51}$$

By Theorem 2.22, $1 \le \rho_{\mathcal{F}} < 4/3$. Hence $0 \le \log_2 \rho_{\mathcal{F}} < \log_2(4/3) < 1$. Thus (51) expresses the integer $E_{\mathcal{F}}$ as $\log_2 \left(3^{O_{\mathcal{F}}} a/z\right)$ plus a number in $[0, 1)$. If $\log_2 \rho_{\mathcal{F}} > 0$, then

$$E_{\mathcal{F}} - 1 < \log_2 \left(\frac{3^{O_{\mathcal{F}}} a}{z} \right) < E_{\mathcal{F}},$$

and hence

$$E_{\mathcal{F}} = \left\lceil \log_2 \left(\frac{3^{O_{\mathcal{F}}} a}{z} \right) \right\rceil.$$

If $\log_2 \rho_{\mathcal{F}} = 0$, then

$$\log_2 \left(\frac{3^{O_{\mathcal{F}}} a}{z} \right) = E_{\mathcal{F}},$$

so the same ceiling formula holds. This proves (49).

Now use $H_{\mathcal{F}} = E_{\mathcal{F}} + O_{\mathcal{F}}$. Adding $O_{\mathcal{F}}$ to both sides of (51) gives

$$H_{\mathcal{F}} = O_{\mathcal{F}} + \log_2 \left(\frac{3^{O_{\mathcal{F}}} a}{z} \right) + \log_2 \rho_{\mathcal{F}}.$$

Since $O_{\mathcal{F}} = \log_2 \left(2^{O_{\mathcal{F}}}\right)$, the first two terms combine to give

$$H_{\mathcal{F}} = \log_2 \left(\frac{6^{O_{\mathcal{F}}} a}{z} \right) + \log_2 \rho_{\mathcal{F}}.$$

So, by the definition of $L_{\mathcal{F}}$,

$$H_{\mathcal{F}} = L_{\mathcal{F}} + \log_2 \rho_{\mathcal{F}}. \tag{52}$$

Again, $0 \le \log_2 \rho_{\mathcal{F}} < \log_2(4/3) < 1$, so the same argument used above gives

$$H_{\mathcal{F}} = \lceil L_{\mathcal{F}} \rceil .$$

For the nearest-integer formula, since $\log_2(4/3) < 1/2$, it follows from (52) that $0 \le H_{\mathcal{F}} - L_{\mathcal{F}} < 1/2$. Thus the integer $H_{\mathcal{F}}$ is the unique nearest integer to $L_{\mathcal{F}}$, proving (48):

$$H_{\mathcal{F}} = \operatorname{nint}(L_{\mathcal{F}}) .$$

It remains to recover $O_{\mathcal{F}}$. From (52),

$$H_{\mathcal{F}} = O_{\mathcal{F}} \log_2 6 + \log_2 \frac{a}{z} + \log_2 \rho_{\mathcal{F}}.$$

Therefore

$$\frac{H_{\mathcal{F}} - \log_2(a/z)}{\log_2 6} = O_{\mathcal{F}} + \frac{\log_2 \rho_{\mathcal{F}}}{\log_2 6}.$$

Since $0 \le \log_2 \rho_{\mathcal{F}} < \log_2(4/3) < \log_2 6$, we have

$$0 \le \frac{\log_2 \rho_{\mathcal{F}}}{\log_2 6} < 1.$$

Taking floors yields

$$O_{\mathcal{F}} = \left\lfloor \frac{H_{\mathcal{F}} - \log_2(a/z)}{\log_2 6} \right\rfloor.$$

Finally, (51) can be written as

$$E_{\mathcal{F}} = O_{\mathcal{F}} \log_2 3 + \log_2 \frac{a}{z} + \log_2 \rho_{\mathcal{F}}.$$

Rearranging terms in the above equation gives

$$\frac{E_{\mathcal{F}} - \log_2(a/z)}{\log_2 3} = O_{\mathcal{F}} + \frac{\log_2 \rho_{\mathcal{F}}}{\log_2 3}.$$

Using once again: $0 \le \log_2 \rho_{\mathcal{F}} < \log_2(4/3) < \log_2 3$, we have

$$0 \le \frac{\log_2 \rho_{\mathcal{F}}}{\log_2 3} < 1.$$

Taking floors gives

$$O_{\mathcal{F}} = \left\lfloor \frac{E_{\mathcal{F}} - \log_2(a/z)}{\log_2 3} \right\rfloor.$$

This completes the proof. □

The preceding results apply to finite shortcut paths with arbitrary endpoints. We now specialize them to complete first-arrival paths. If n terminates and $\mathcal{F}_n$ is its complete first-arrival shortcut path from n to 1, then

$$E_{\mathcal{F}_n} = E(n), \qquad O_{\mathcal{F}_n} = O(n), \qquad H_{\mathcal{F}_n} = H(n), \qquad \rho_{\mathcal{F}_n} = \rho(n).$$

In particular, the endpoint z in the finite-path formulas becomes 1. For adjacent terminating integers n and $n+1$, their complete first-arrival paths have the common endpoint 1, so the common-endpoint residue ratio introduced above specializes to the relative residue ratio $W(n)$ defined below.

3.3 Complete First-Arrival Paths

Definition 3.8. For adjacent terminating integers n and $n+1$, define the differences in their even- and odd-step counts by

$$e = \Delta E = E(n) - E(n+1), \qquad q = \Delta O = O(n) - O(n+1).$$

The *relative residue ratio* $W(n)$ is given by

$$W(n) = \frac{2^e}{3^q}. \tag{53}$$

Theorem 3.9 (W as Ratio of Successive ρ Values.)**.** *For adjacent terminating integers n and $n+1$,*

$$W(n) = \frac{n\rho(n)}{(n+1)\rho(n+1)}. \tag{54}$$

Proof. Let $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ be the complete first-arrival shortcut paths from n and $n+1$, respectively, to 1. Then

$$W_1(n, n+1) = W(n), \qquad \rho_{\mathcal{F}_n} = \rho(n), \qquad \rho_{\mathcal{F}_{n+1}} = \rho(n+1).$$

Applying Theorem 3.2 with $a = n$, $b = n+1$, and $z = 1$ gives

$$W(n) = \frac{n}{n+1}\frac{\rho(n)}{\rho(n+1)} = \frac{n\rho(n)}{(n+1)\rho(n+1)}.$$

□

Corollary 3.10 (Recovery of Operation Counts)**.** *Let n be a terminating integer, and define*

$$L(n) := \log_2\left(6^{O(n)}n\right).$$

Then

$$H(n) = \lceil L(n) \rceil = \operatorname{nint}(L(n)), \qquad E(n) = \left\lceil \log_2\left(3^{O(n)}n\right)\right\rceil. \tag{55}$$

Moreover,

$$O(n) = \left\lfloor \frac{H(n) - \log_2 n}{\log_2 6} \right\rfloor = \left\lfloor \frac{E(n) - \log_2 n}{\log_2 3} \right\rfloor. \tag{56}$$

Proof. Let $\mathcal{F}$ be the complete first-arrival shortcut path from n to 1. Then $a = n$, and $z = 1$, and, by the correspondence between the finite-path quantities and the complete terminating trajectory,

$$E_{\mathcal{F}} = E(n), \qquad O_{\mathcal{F}} = O(n), \qquad H_{\mathcal{F}} = H(n), \qquad \rho_{\mathcal{F}} = \rho(n).$$

Furthermore,

$$L_{\mathcal{F}} = \log_2\left(\frac{6^{O_{\mathcal{F}}}a}{z}\right) = \log_2\left(6^{O(n)}n\right) = L(n).$$

Applying Theorem 3.7 gives

$$H(n) = \lceil L(n) \rceil = \operatorname{nint}(L(n)), \quad \text{and} \quad E(n) = \left\lceil \log_2\left(3^{O(n)}n\right)\right\rceil.$$

This proves (55). The same theorem gives

$$O(n) = \left\lfloor \frac{H(n) - \log_2(n/1)}{\log_2 6} \right\rfloor = \left\lfloor \frac{E(n) - \log_2(n/1)}{\log_2 3} \right\rfloor.$$

Since $\log_2(n/1) = \log_2 n$, this is exactly (56). □

The ceiling formula $H(n) = \lceil L(n) \rceil$ was proposed by Ruggiero [16]. Luo [10] previously gave related formulas for determining the operation counts under additional assumptions. Here, the bound on $\rho(n)$ supplies the hypothesis needed for these formulas; the formulas themselves are not claimed as new results.

For each terminating n, the residue identity gives

$$H(n) - L(n) = \log_2 \rho(n) \geq 0.$$

Since $H(n)$ is an integer,

$$H(n) = \lceil L(n) \rceil \quad \Longleftrightarrow \quad 0 \le H(n) - L(n) < 1 \quad \Longleftrightarrow \quad \rho(n) < 2.$$

Thus, for terminating n, Ruggiero's ceiling formula is precisely equivalent to the inequality $\rho(n) < 2$. Since Theorem 2.23 gives the stronger bound $\rho(n) < 4/3$, the ceiling formula holds for every terminating positive integer.

Theorem 3.11 (Equal-Height Separation)**.** *If $x < y$ terminate and $H(x) = H(y)$, put $d = O(x) - O(y)$. Then $d \ge 0$ and*

$$\frac{3}{4}\,6^d < \frac{y}{x} < \frac{4}{3}\,6^d.$$

Consequently, no equal-height pair has a ratio in the interval $[4/3, 9/2]$.

Proof. Let $\mathcal{F}_x$ and $\mathcal{F}_y$ be the complete first-arrival shortcut paths from x and y, respectively, to 1. Then the two paths have the common endpoint $z = 1$, and

$$E_{\mathcal{F}_x} = E(x), \qquad O_{\mathcal{F}_x} = O(x), \qquad H_{\mathcal{F}_x} = H(x),$$

while

$$E_{\mathcal{F}_y} = E(y), \qquad O_{\mathcal{F}_y} = O(y), \qquad H_{\mathcal{F}_y} = H(y).$$

In particular, $H_{\mathcal{F}_x} = H_{\mathcal{F}_y}$ and $O_{\mathcal{F}_x} - O_{\mathcal{F}_y} = O(x) - O(y) = d$. Applying Theorem 3.5 with $a = x$, $b = y$, and $z = 1$ therefore gives $d \ge 0$ and

$$\frac{3}{4}\,6^d < \frac{y}{x} < \frac{4}{3}\,6^d.$$

The same theorem also gives

$$\frac{y}{x} \notin \left[\frac{4}{3}, \frac{9}{2}\right],$$

which proves the result. □

For terminating integers, Theorem 3.11 shows that the smaller member of an equal-height pair cannot have fewer odd operations than the larger member. If the two integers are adjacent, so that $y = x + 1$, then

$$\frac{y}{x} = \frac{x+1}{x} \le 2.$$

On the other hand, if $d \ge 1$, then Theorem 3.11 gives

$$\frac{y}{x} > \frac{3}{4}\,6^d \ge \frac{9}{2},$$

a contradiction. Hence $d = 0$, so the two integers have equal odd-operation counts. Since their heights are equal, their even-operation counts are equal as well.

Corollary 3.12 (Synchronization of Height Streaks)**.** *Adjacent terminating integers of equal height have equal even and odd operation counts. Every consecutive equal-height block has common* (E, O).

Proof. Let n and $n + 1$ be adjacent terminating integers with $H(n) = H(n + 1)$. Let $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ be their complete first-arrival shortcut paths to 1. Then the two paths have the common endpoint $z = 1$, and

$$E_{\mathcal{F}_n} = E(n), \qquad O_{\mathcal{F}_n} = O(n), \qquad H_{\mathcal{F}_n} = H(n),$$

while

$$E_{\mathcal{F}_{n+1}} = E(n+1), \qquad O_{\mathcal{F}_{n+1}} = O(n+1), \qquad H_{\mathcal{F}_{n+1}} = H(n+1).$$

Thus $H_{\mathcal{F}_n} = H_{\mathcal{F}_{n+1}}$, and applying Corollary 3.6 gives $O_{\mathcal{F}_n} = O_{\mathcal{F}_{n+1}}$ and $E_{\mathcal{F}_n} = E_{\mathcal{F}_{n+1}}$. Therefore $O(n) = O(n+1)$ and $E(n) = E(n+1)$.

Finally, if $n, n+1, \ldots, n+k$ is a consecutive block of terminating integers having the same height, then applying the preceding conclusion to each adjacent pair gives

$$(E(n), O(n)) = (E(n+1), O(n+1)) = \cdots = (E(n+k), O(n+k)).$$

Hence the entire block shares a common (E, O). □

Corollary 3.13. *For adjacent terminating integers n and $n+1$, the relative residue ratio $W(n)$ satisfies*

$$\frac{3n}{4(n+1)} < W(n) < \frac{4n}{3(n+1)}. \tag{57}$$

Proof. By Theorem 3.9,

$$W(n) = \frac{n}{n+1}\frac{\rho(n)}{\rho(n+1)}.$$

Since $1 \le \rho(n), \rho(n+1) < 4/3$, we have

$$\frac{3}{4} < \frac{\rho(n)}{\rho(n+1)} < \frac{4}{3}.$$

Multiplying by the positive factor $n/(n+1)$ gives (57) as required. □

Theorem 3.14 (The Adjacent Nearest-Integer Law)**.** *For every terminating adjacent pair $n, n+1$ with $n \ge 3$,*

$$\Delta E = \operatorname{nint}\big((\log_2 3)\Delta O\big). \tag{58}$$

If $q = \Delta O \ne 0$ and $e = \Delta E$, they have the same sign and

$$\frac{3}{2} \le \left|\frac{e}{q}\right| \le 2. \tag{59}$$

The upper equality can occur only for $(e, q) = (2, 1)$ or $(-2, -1)$. Lastly, if $q = 0$, then $e = 0$.

Proof. Put $e = \Delta E = E(n) - E(n+1)$ and $q = \Delta O = O(n) - O(n+1)$. Then by definition, $W(n) = 2^e/3^q$. Taking logarithms to base 2 gives $\log_2 W(n) = e - q\log_2 3$. Thus, to prove the nearest-integer formula, it is enough to show that $\left|\log_2 W(n)\right| < 1/2$, or equivalently,

$$\frac{1}{\sqrt{2}} < W(n) < \sqrt{2}. \tag{60}$$

The upper bound follows immediately from Corollary 3.13. Indeed,

$$W(n) < \frac{4n}{3(n+1)} < \frac{4}{3} < \sqrt{2}.$$

For the lower bound, Corollary 3.13 gives

$$W(n) > \frac{3n}{4(n+1)}.$$

The function $3n/(4(n+1))$ is increasing in n, so for $n \geq 17$,

$$W(n) > \frac{3n}{4(n+1)} \geq \frac{3 \cdot 17}{4 \cdot 18} = \frac{17}{24} > \frac{1}{\sqrt{2}}.$$

For the remaining integers $3 \leq n \leq 16$, direct computation of $E(n)$ and $O(n)$ gives (60). Consequently, (60) holds for every terminating adjacent pair $n, n+1$ with $n \geq 3$. Taking logarithms yields

$$\left| e - q \log_2 3 \right| < \frac{1}{2}. \tag{61}$$

Since e is an integer, it is the unique nearest integer to $q \log_2 3$. Hence $e = \operatorname{nint}\left(q \log_2 3\right)$, which proves (58).

We next prove the bounds on e/q. Suppose first that $q \neq 0$. Since $e = \operatorname{nint}\left(q \log_2 3\right)$ and $\log_2 3 > 0$, e and q have the same sign. Moreover, (61), together with the triangle inequality and reverse triangle inequality, gives

$$|q| \log_2 3 - \frac{1}{2} < |e| < |q| \log_2 3 + \frac{1}{2}.$$

For $|q| \geq 6$,

$$|q| \log_2 3 - \frac{1}{2} > \frac{3}{2}|q|,$$

while for $|q| \geq 2$,

$$|q| \log_2 3 + \frac{1}{2} < 2|q|.$$

Hence, for $|q| \geq 6$, $\frac{3}{2}|q| < |e| < 2|q|$.

It remains to consider the finitely many cases $1 \leq |q| < 6$. Since changing the sign of q changes the sign of $\operatorname{nint}(q \log_2 3)$, it suffices to list the cases $q > 0$.

Table 5: Computation of $|e/q|$ for $0 < |q| < 6$.

$\lvert q \rvert = \lvert \Delta O \rvert$	$q \log_2 3$	$e = \Delta E$	$\lvert e/q \rvert$
1	1.58496	2	2
2	3.16993	3	3/2
3	4.75489	5	5/3
4	6.33985	6	3/2
5	7.92481	8	8/5

Combining these finite cases with the bounds for $|q| \geq 6$ gives

$$\frac{3}{2} \leq \left| \frac{e}{q} \right| \leq 2,$$

which proves (59). The upper equality occurs only when $|q| = 1$, giving

$$(e, q) = (2, 1) \qquad \text{or} \qquad (e, q) = (-2, -1).$$

Finally, if $q = 0$, then (58) gives $e = \operatorname{nint}(0) = 0$. □

Theorem 3.14 shows that the changes in the even- and odd-operation counts of adjacent terminating integers are tightly coupled. Whenever $\Delta O \neq 0$, the two differences have the same sign and satisfy

$$\frac{3}{2} \leq \left|\frac{\Delta E}{\Delta O}\right| \leq 2.$$

Thus a change in the odd-operation count determines a correspondingly restricted change in the even-operation count.

Theorem 3.15 (The Sharper Residue Ratio Bound)**.** *For every terminating adjacent pair $n, n+1$,*

$$W(n) < \frac{3}{4} \quad \Longleftrightarrow \quad n \in \{1, 2, 8\},$$

and

$$W(n) = \frac{3}{4} \quad \Longleftrightarrow \quad n \in \{4, 5\}.$$

For odd $n \geq 3$, the only lower equality is at $n = 5$.

Proof. By the proof of Corollary 2.21 and the rational comparison (165), every terminating integer n satisfies $\rho(n) < \rho(993)e^{61/1000} < 1.331964$. Hence

$$\rho(n) < B, \qquad B := \frac{13333}{10000} < \frac{4}{3}.$$

By Theorem 3.9,

$$W(n) = \frac{n}{n+1}\frac{\rho(n)}{\rho(n+1)}.$$

Since $\rho(n) \geq 1$ and $\rho(n+1) < B$,

$$W(n) > \frac{1}{B}\frac{n}{n+1}.$$

Now $3B/(4-3B) = 39999$ so for $n \geq 39999$,

$$\frac{1}{B}\frac{n}{n+1} \geq \frac{3}{4}.$$

Because the preceding bound on $W(n)$ is strict, it follows that $W(n) > 3/4$ for $n \geq 39999$.

It remains only to examine the finite range below this threshold. Exact integer computation of $E(n)$ and $O(n)$ gives

$$W(n) < \frac{3}{4} \quad \Longleftrightarrow \quad n \in \{1, 2, 8\}, \qquad \text{and} \qquad W(n) = \frac{3}{4} \quad \Longleftrightarrow \quad n \in \{4, 5\}$$

throughout this finite range.

Combining the finite computation with the bound for $n \geq 39999$ proves the two stated equivalences. Since among the two equality cases 4 and 5 only 5 is odd, for odd $n \geq 3$ the only lower equality is at $n = 5$.

Finally, the upper bound $W(n) < 4/3$ holds for every terminating adjacent pair by Corollary 3.13. □

Combining Corollary 3.13 with Theorem 3.15, we obtain

$$\frac{3}{4} < W(n) < \frac{4}{3}$$

for every terminating adjacent pair $n, n+1$ with $n \notin \{1, 2, 4, 5, 8\}$.

The preceding results describe local changes in the operation counts between adjacent terminating starting integers. We now turn to a global consequence of the same residue bound: among terminating integers, E and H attain new record values at exactly the same starting integers. To state this result precisely, we must distinguish the least starting integer realizing a prescribed value from a starting integer whose value exceeds all previously attained values. For Collatz height, this distinction between class and delay records is standard in computational work on the $3x + 1$ problem; see Scollo [17, 18] and Roosendaal [14]. We adopt the distinction uniformly for the integer-valued trajectory statistics used below.

Definition 3.16. Let $f : \mathcal{T} \to \mathbb{Z}$ be a function on the terminating set $\mathcal{T}$. Define its extended class-record function by

$$c_f(k) := \min\{n \in \mathcal{T} : f(n) = k\}, \tag{62}$$

with the convention that the minimum of the empty set is $+\infty$. When k is *realized* by f, meaning that $f(n) = k$ for some $n \in \mathcal{T}$, the finite value $c_f(k)$ is called the *f-class record at value k*.

Thus $c_H(h)$, $c_E(e)$, and $c_O(o)$ are the least starting integers with the specified height, division count, or odd count. The index names the value of the class, not a position in a list of starting integers. For $X \in \{H, E, O\}$, write

$$\mathrm{CR}_X := \{c_X(k) : k \text{ is realized by } X\}$$

for the set of starting integers that are X-class records.

Definition 3.17. For $f \in \{H, E, O\}$, a terminating integer n is an *f-delay record among terminating integers* if $f(m) < f(n)$ for every $m \in \mathcal{T}$ with $m < n$.

Since $H(1) = E(1) = O(1) = 0$, set $d_f(0) = 1$. Having defined $d_f(r)$, define the next term by

$$d_f(r + 1) := \min\{n \in \mathcal{T} : n > d_f(r),\ f(n) > f(d_f(r))\}. \tag{63}$$

A next term is defined only when this set is nonempty. This recursion enumerates exactly the f-delay records among terminating integers in increasing order of their starting integers. Here r is the position in the delay-record list, not the value of f. We call $d_f(1)$ the first nontrivial record.

If every positive integer through n terminates, then an f-delay record among terminating integers is called simply an *f-delay record*. This includes the usual height-delay definition. For $X \in \{H, E, O\}$, write

$$\mathrm{DR}_X := \{d_X(r) : d_X(r) \text{ is defined}\}$$

for the corresponding set of delay-record starting integers.

For example, $c_H(2) = 4$, whereas $d_H(2) = 3$ and $H(3) = 7$. Every defined delay-record term is the class record at the value it attains:

$$d_f(r) = c_f\big(f(d_f(r))\big). \tag{64}$$

Indeed, a smaller member of the same class would contradict the strict record property. The converse need not hold: a class record need not be a delay record. For example, although $c_H(2) = 4$, the smaller integer 3 already has height $H(3) = 7 > 2$.

Theorem 3.18 (Equivalence of Height-Delay and E-Delay Records)**.** *Suppose all positive integers through n terminate. Then n is an E-delay record if and only if it is a height-delay record. Equivalently,*

$$E(n) > E(m) \text{ for every } m < n \quad \Longleftrightarrow \quad H(n) > H(m) \text{ for every } m < n.$$

Proof. Suppose first that n is a height-delay record. If some $m < n$ satisfied $E(m) \geq E(n)$, then, since $H = E + O$ and $H(n) > H(m)$, $E(n) + O(n) > E(m) + O(m)$. The assumption $E(m) \geq E(n)$ therefore forces $O(n) \geq O(m) + 1$. Applying Theorem 3.2 to the complete first-arrival shortcut paths from n and m to 1 gives

$$\frac{n}{m} = 2^{E(n)-E(m)} 3^{O(m)-O(n)} \frac{\rho(m)}{\rho(n)}.$$

Since

$$2^{E(n)-E(m)} \leq 1, \qquad 3^{O(m)-O(n)} \leq \frac{1}{3},$$

with $1 \leq \rho(n)$ and $\rho(m) < \frac{4}{3}$, we obtain

$$\frac{n}{m} < 1 \cdot \frac{1}{3} \cdot \frac{4}{3} = \frac{4}{9}.$$

This contradicts $m < n$. Hence $E(n) > E(m)$ for every $m < n$, so n is an E-delay record.

Conversely, suppose that n is an E-delay record, so that $E(n) > E(m)$ for every $m < n$. Assume, for contradiction, that some $m < n$ satisfies $H(m) \geq H(n)$, and set

$$a := E(n) - E(m) \geq 1.$$

Because $H = E + O$, the height inequality gives $E(m) + O(m) \geq E(n) + O(n)$, and therefore $O(m) - O(n) \geq a$.

Since m terminates, so does $2^a m$, and its first a shortcut steps are halving steps leading to m. Hence

$$E(2^a m) = E(m) + a = E(n).$$

Applying Theorem 3.2 to the complete first-arrival shortcut paths from m and n to 1 gives

$$\frac{m}{n} = 2^{E(m)-E(n)} 3^{O(n)-O(m)} \frac{\rho(n)}{\rho(m)}.$$

Since $E(m) - E(n) = -a$, this becomes

$$\frac{2^a m}{n} = 3^{O(n)-O(m)} \frac{\rho(n)}{\rho(m)}.$$

Using $O(n) - O(m) \leq -a$ with $1 \leq \rho(m)$ and $\rho(n) < \frac{4}{3}$ yields

$$\frac{2^a m}{n} < \frac{1}{3^a} \frac{4}{3} \leq \frac{4}{9} < 1.$$

Thus $2^a m < n$. But $E(2^a m) = E(n)$, contradicting the assumption that n is an E-delay record. Therefore $H(n) > H(m)$ for every $m < n$, so n is a height-delay record. □

The same equivalence holds for delay records among terminating integers, without assuming that every smaller positive integer terminates. Indeed, the proof compares n only with smaller terminating integers m. In the converse direction, the auxiliary integer $2^a m$ also terminates whenever m does, so it remains an admissible comparison within $\mathcal{T}$.

The Three Scalar Quantities. Having recorded this terminating-set form of the delay-record equivalence, we return to the three scalar quantities underlying the residue estimates: $\mathcal{R}$, ρ, and Π. We compare them first on a finite path and then on a whole forward orbit. The finite-path comparison is a product calculation and requires neither termination nor nonrepetition.

Theorem 3.19. *For every finite shortcut path $\mathcal{F}$,*

$$1 + \frac{\mathcal{R}_{\mathcal{F}}}{3} \le \rho_{\mathcal{F}}, \qquad 3\log\rho_{\mathcal{F}} \le \mathcal{R}_{\mathcal{F}} \le \log\Pi_{\mathcal{F}} \le 2\mathcal{R}_{\mathcal{F}}. \tag{65}$$

For every positive integer n, whether or not it terminates,

$$1 + \frac{\mathcal{R}^{\circ}(n)}{3} \le \rho^{\circ}(n), \qquad 3\log\rho^{\circ}(n) \le \mathcal{R}^{\circ}(n) \le \log\Pi^{\circ}(n) \le 2\mathcal{R}^{\circ}(n). \tag{66}$$

For a terminating n, these are also the inequalities for $\mathcal{R}(n), \rho(n), \Pi(n)$.

Proof. The first inequality in (65) follows from the product expansion (37). Although that expansion appears in the proof of Corollary 2.24, its derivation uses only the positivity of the odd input occurrences. It therefore applies to every finite shortcut path, including paths with repeated inputs or with 1 as an input. The first inequality in (66) is (39).

It remains to prove the two comparison chains. List the odd input occurrences of $\mathcal{F}$ as $x_1, \dots, x_{O_{\mathcal{F}}}$. From the definition of $\Pi_{\mathcal{F}}$,

$$\Pi_{\mathcal{F}} = \prod_i \frac{2x_i + 1}{2x_i - 1} = \prod_i \left(1 + \frac{2}{2x_i - 1}\right).$$

Using $\log(1 + t) \le t$ for $t > -1$ gives

$$\begin{aligned}\log\Pi_{\mathcal{F}} &= \sum_i \log\left(1 + \frac{2}{2x_i - 1}\right)\\ &\le \sum_i \frac{2}{2x_i - 1}\\ &\le \sum_i \frac{2}{x_i} = 2\mathcal{R}_{\mathcal{F}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\log\rho_{\mathcal{F}} &= \sum_i \log\left(1 + \frac{1}{3x_i}\right)\\ &\le \sum_i \frac{1}{3x_i} = \frac{1}{3}\mathcal{R}_{\mathcal{F}}.\end{aligned}$$

Therefore

$$3\log\rho_{\mathcal{F}} \le \mathcal{R}_{\mathcal{F}}.$$

To compare $\mathcal{R}_{\mathcal{F}}$ with $\log\Pi_{\mathcal{F}}$, use the inequality valid for $0 < t < 1$:

$$\log\frac{1 + t}{1 - t} > 2t. \tag{67}$$

Setting $t = 1/(2x_i)$ gives

$$\log\left(\frac{2x_i + 1}{2x_i - 1}\right) > \frac{1}{x_i}.$$

Thus, if $\mathcal{F}$ contains at least one odd input, summing over all odd input occurrences gives

$$\log \Pi_{\mathcal{F}} > \sum_i \frac{1}{x_i} = \mathcal{R}_{\mathcal{F}}.$$

If $\mathcal{F}$ contains no odd input, then the sum and products are empty, so

$$\rho_{\mathcal{F}} = \Pi_{\mathcal{F}} = 1, \qquad \mathcal{R}_{\mathcal{F}} = 0.$$

Consequently,

$$3 \log \rho_{\mathcal{F}} \le \mathcal{R}_{\mathcal{F}} \le \log \Pi_{\mathcal{F}} \le 2\mathcal{R}_{\mathcal{F}}$$

for every finite shortcut path $\mathcal{F}$.

For the once-counted quantities, list the members of $O(n)$ in order of first occurrence as $x_1, x_2, \ldots$, with the list ending if $O(n)$ is finite. For every finite initial segment, put

$$\mathcal{R}_K := \sum_{i=1}^{K} \frac{1}{x_i}, \qquad \rho_K := \prod_{i=1}^{K} \left(1 + \frac{1}{3x_i}\right), \qquad \Pi_K := \prod_{i=1}^{K} \frac{2x_i + 1}{2x_i - 1}.$$

The finite-path calculations above give

$$3 \log \rho_K \le \mathcal{R}_K \le \log \Pi_K \le 2\mathcal{R}_K,$$

for every K. If $O(n)$ is finite, taking K to include the entire list gives the desired inequalities directly. If $O(n)$ is infinite, Definition 2.18 gives

$$\mathcal{R}_K \longrightarrow \mathcal{R}^\circ(n), \qquad \rho_K \longrightarrow \rho^\circ(n), \qquad \Pi_K \longrightarrow \Pi^\circ(n).$$

These limits are finite by Corollary 2.19, so continuity of the logarithm allows us to pass to the limit in the preceding chain. This proves the remaining inequalities in (66).

Finally, when n terminates, the once-counted quantities agree with the complete first-arrival quantities:

$$\mathcal{R}^\circ(n) = \mathcal{R}(n), \qquad \rho^\circ(n) = \rho(n), \qquad \Pi^\circ(n) = \Pi(n).$$

Therefore (66) gives the same inequalities for $\mathcal{R}(n)$, $\rho(n)$, and $\Pi(n)$. □

These observations show that uniform boundedness of any one of ρ, $\mathcal{R}$, and Π is equivalent to uniform boundedness of all three, in each of the settings just stated. Empty sums and products are included. Corollary 2.24 gives the principal reciprocal-sum consequence of the four-thirds residue bound. For a nonterminating orbit, the high-range estimate of Lemma 2.20 and the preceding comparison inequalities yield substantially sharper bounds.

Corollary 3.20 (Sharper Bounds for Nonterminating Orbits)**.** *If the forward orbit of n does not terminate, then*

$$\mathcal{R}^\circ(n) \le \frac{9}{50} < \log 2, \qquad \rho^\circ(n) \le e^{3/50} < 2^{1/3}, \qquad \Pi^\circ(n) \le e^{9/25} < 2.$$

Proof. Recall that $D = 28\,019\,077\,177\,231\,758\,495$. The verified height bound (34) asserts that every positive integer $y \le D$ terminates, with $H(y) \le 2456$. Therefore, if the forward orbit of n entered $[1, D]$, its remaining tail would reach 1, and the orbit of n would terminate. Since the orbit is assumed not to terminate, every state in it is greater than D.

Write $v_t = T^t(n)$. Suppose first that some state repeats, and let N be the first index for which $v_N = v_k$ for some $0 \le k < N$. Then the input states $v_0, \dots, v_{N-1}$ are distinct and greater than D. Lemma 2.20, applied to the finite path $(v_0, \dots, v_N)$, gives

$$\sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t} < \frac{9}{50}.$$

Because the shortcut map is deterministic, no new states occur after the first repetition. Hence

$$\mathcal{R}^\circ(n) = \sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t} < \frac{9}{50}.$$

Now suppose that no state repeats. For $N \ge 1$, define

$$S_N := \sum_{\substack{0 \le t < N \\ v_t \text{ odd}}} \frac{1}{v_t}.$$

The finite path $(v_0, \dots, v_N)$ has distinct input states greater than D, so Lemma 2.20 gives $S_N < 9/50$. The sequence (S_N) is nondecreasing and bounded above by $9/50$. Moreover, because no state repeats, these partial sums include every member of $O(n)$ exactly once, in order of first occurrence. Therefore

$$\mathcal{R}^\circ(n) = \lim_{N \to \infty} S_N \le \frac{9}{50}.$$

Thus, whether or not the orbit repeats,

$$\mathcal{R}^\circ(n) \le \frac{9}{50}.$$

The comparison inequalities (66) now give

$$\log \rho^\circ(n) \le \frac{1}{3}\mathcal{R}^\circ(n) \le \frac{3}{50}$$

and

$$\log \Pi^\circ(n) \le 2\mathcal{R}^\circ(n) \le \frac{9}{25}.$$

Exponentiating yields

$$\rho^\circ(n) \le e^{3/50}, \qquad \Pi^\circ(n) \le e^{9/25},$$

which proves all the stated bounds. □

Theorem 3.21 (Comparison Chain)**.** *For every finite shortcut path $\mathcal{F}$,*

$$\Pi_{\mathcal{F}} < 2 \quad \Longrightarrow \quad \mathcal{R}_{\mathcal{F}} < \log 2 \quad \Longrightarrow \quad \rho_{\mathcal{F}} < 2^{1/3}.$$

The same chain holds for $\Pi^\circ(n), \mathcal{R}^\circ(n), \rho^\circ(n)$ for every positive n, and for the complete first-arrival quantities whenever n terminates.

Proof. For a finite path, Theorem 3.19 gives

$$\mathcal{R}_{\mathcal{F}} \le \log \Pi_{\mathcal{F}}, \qquad \log \rho_{\mathcal{F}} \le \frac{1}{3}\mathcal{R}_{\mathcal{F}}.$$

If $\Pi_{\mathcal{F}} < 2$, the first inequality gives $\mathcal{R}_{\mathcal{F}} < \log 2$. If $\mathcal{R}_{\mathcal{F}} < \log 2$, the second gives $\log \rho_{\mathcal{F}} < \frac{1}{3} \log 2$, and hence $\rho_{\mathcal{F}} < 2^{1/3}$. Equation (66) gives the same argument for once-counted quantities. The terminating case is their first-arrival specialization, empty input lists included. □

Neither uniform boundedness nor the bound $\rho < 4/3$ proves the reverse implications between these particular bounds. Conjecture 3.22 states the stronger bounds we seek for terminating trajectories. Corollary 3.20 already gives these bounds for the once-counted quantities of nonterminating orbits.

Conjecture 3.22 (Three Scalar Bounds)**.** For every terminating positive integer n,

$$\rho(n) \le 2^{1/3}, \qquad \mathcal{R}(n) \le \log 2, \qquad \Pi(n) \le 2.$$

Example 3.23 (The Residue at $n = 993$)**.** Direct computation gives

$$(E, O, H)(993) = (61, 32, 93),$$

and hence

$$\rho(993) = \frac{2^{61}}{3^{32} \cdot 993} = 1.253142144395068\ldots < 2^{1/3} < \frac{4}{3}.$$

Roosendaal's strong residue conjecture asserts that $\rho(993)$ is the maximum of $\rho(n)$ over terminating positive integers [14]. Since $\rho(993) < 2^{1/3}$, Roosendaal's conjecture would imply $\rho(n) < 2^{1/3}$ for every terminating positive integer n.

The threshold $2^{1/3}$ has an exact interpretation in terms of height recovery. Earlier, the threshold $\rho(n) < 2$ was shown to be equivalent to Ruggiero's ceiling formula. The following proposition shows that the sharper threshold $\rho_{\mathcal{F}} \le 2^{1/3}$ is equivalent to an exact shifted-floor formula with shift $1/3$.

Proposition 3.24 (The One-Third Height Formula)**.** *For a finite shortcut path $\mathcal{F} = (a, \ldots, z)$, the inequality $\rho_{\mathcal{F}} \le 2^{1/3}$ is equivalent to*

$$H_{\mathcal{F}} = \left\lfloor \log_2(6^{O_{\mathcal{F}}} a/z) + \frac{1}{3} \right\rfloor.$$

In particular, for a terminating integer n, $\rho(n) \le 2^{1/3}$ is equivalent to

$$H(n) = \left\lfloor \log_2(6^{O(n)} n) + \frac{1}{3} \right\rfloor.$$

Proof. Put

$$L = \log_2 \left(6^{O_{\mathcal{F}}} a/z\right).$$

By the residue identity, as in Proposition 2.6, $H_{\mathcal{F}} = L + \log_2 \rho_{\mathcal{F}}$, and hence $H_{\mathcal{F}} - L = \log_2 \rho_{\mathcal{F}}$.

Suppose first that $\rho_{\mathcal{F}} \le 2^{1/3}$. Since $\rho_{\mathcal{F}} \ge 1$, we have $0 \le \log_2 \rho_{\mathcal{F}} \le 1/3$. Therefore $0 \le H_{\mathcal{F}} - L \le 1/3$, so

$$H_{\mathcal{F}} \le L + \frac{1}{3} \le H_{\mathcal{F}} + \frac{1}{3} < H_{\mathcal{F}} + 1.$$

Since $H_{\mathcal{F}}$ is an integer, it follows that

$$H_{\mathcal{F}} = \left\lfloor L + \frac{1}{3} \right\rfloor.$$

Substituting the definition of L gives

$$H_{\mathcal{F}} = \left\lfloor \log_2 \left(6^{O_{\mathcal{F}}} a/z\right) + \frac{1}{3} \right\rfloor.$$

Conversely, suppose that

$$H_{\mathcal{F}} = \left\lfloor L + \frac{1}{3} \right\rfloor.$$

Then

$$H_{\mathcal{F}} \le L + \frac{1}{3} < H_{\mathcal{F}} + 1.$$

The left-hand inequality gives $H_{\mathcal{F}} - L \le 1/3$. Since $H_{\mathcal{F}} - L = \log_2 \rho_{\mathcal{F}}$, we obtain $\log_2 \rho_{\mathcal{F}} \le 1/3$. Exponentiating gives

$$\rho_{\mathcal{F}} \le 2^{1/3}.$$

This proves the finite-path equivalence. Taking the first-arrival path from n to 1 gives the terminating specialization. □

4 The charge λ and the functions G and Q

The preceding section showed that the residue bound recovers the even- and odd-operation counts and restricts how those counts can differ between trajectories. We now reorganize the same information using height and charge. For a finite shortcut path $\mathcal{F}$, the height

$$H_{\mathcal{F}} = E_{\mathcal{F}} + O_{\mathcal{F}}$$

records the total number of operations, while the charge

$$\lambda_{\mathcal{F}} = E_{\mathcal{F}} - 2O_{\mathcal{F}}$$

records a second combination of the same two counts. Together, $H_{\mathcal{F}}$ and $\lambda_{\mathcal{F}}$ determine $E_{\mathcal{F}}$ and $O_{\mathcal{F}}$ exactly.

We first develop the charge formulas for finite paths and then specialize them to complete first-arrival paths. We next apply these formulas to integers written as an anchor plus an offset and introduce the functions $\mathcal{G}$ and $\mathcal{Q}$. Their two-variable restrictions, G and Q, will be used to study equal-height streaks and their breaks.

4.1 Finite-Path Charge

Definition 4.1. Let $\mathcal{F} = (a, \dots, z)$ be a finite shortcut path. Define the *finite-path charge* of $\mathcal{F}$ by $\lambda_{\mathcal{F}} := E_{\mathcal{F}} - 2O_{\mathcal{F}}$.

Together with the finite-path height $H_{\mathcal{F}} = E_{\mathcal{F}} + O_{\mathcal{F}}$, the charge determines the even- and odd-operation counts:

$$E_{\mathcal{F}} = \frac{2H_{\mathcal{F}} + \lambda_{\mathcal{F}}}{3}, \qquad O_{\mathcal{F}} = \frac{H_{\mathcal{F}} - \lambda_{\mathcal{F}}}{3}, \tag{68}$$

as well as

$$E_{\mathcal{F}} = \lambda_{\mathcal{F}} + 2O_{\mathcal{F}}, \qquad H_{\mathcal{F}} = \lambda_{\mathcal{F}} + 3O_{\mathcal{F}}. \tag{69}$$

If $\mathcal{F}$ is the complete first-arrival shortcut path of a terminating integer n, then

$$E_{\mathcal{F}} = E(n), \qquad O_{\mathcal{F}} = O(n), \qquad H_{\mathcal{F}} = H(n).$$

Accordingly, define $\lambda(n) := E(n) - 2O(n)$. Then $\lambda_{\mathcal{F}} = \lambda(n)$, and (68) specializes to

$$E(n) = \frac{2H(n) + \lambda(n)}{3}, \qquad O(n) = \frac{H(n) - \lambda(n)}{3}. \tag{70}$$

The complete first-arrival quantities also satisfy the elementary one-step recursions shown in Table 6. These formulas hold for $n > 1$. At $n = 1$, the first-arrival counts satisfy $E(1) = O(1) = H(1) = \lambda(1) = 0$.

Table 6: One-step recursion formulas for $H(n)$ and $\lambda(n)$ under the shortcut map T and the ordinary Collatz map C, according to the parity of n.

	T		C	
	H	λ	H	λ
n even	$H(n/2)+1$	$\lambda(n/2)+1$	$H(n/2)+1$	$\lambda(n/2)+1$
n odd	$H((3n+1)/2)+2$	$\lambda((3n+1)/2)-1$	$H(3n+1)+1$	$\lambda(3n+1)-2$

The one-step formulas describe how height and charge change locally. The residue bound gives a complementary finite-path result: the charge and the endpoints determine the operation counts exactly.

Theorem 4.2 (Finite-Path Charge Recovery)**.** *Let $\mathcal{F} = (a, \dots, z)$ be a finite nonrepeating shortcut path whose input states are greater than* 1*, then*

$$\log_2 \rho_{\mathcal{F}} = \lambda_{\mathcal{F}} + O_{\mathcal{F}} \log_2 \frac{4}{3} - \log_2 \frac{a}{z}. \tag{71}$$

Consequently,

$$O_{\mathcal{F}} = \left\lceil \frac{\log_2(a/z) - \lambda_{\mathcal{F}}}{\log_2(4/3)} \right\rceil. \tag{72}$$

Together with (68)*, this also determines $E_{\mathcal{F}}$ and $H_{\mathcal{F}}$. Moreover,*

$$\log_2 \frac{a}{z} - O_{\mathcal{F}} \log_2 \frac{4}{3} \le \lambda_{\mathcal{F}} < \log_2 \frac{a}{z} - (O_{\mathcal{F}} - 1) \log_2 \frac{4}{3}. \tag{73}$$

Proof. By the finite-path residue identity,

$$\rho_{\mathcal{F}} = \frac{2^{E_{\mathcal{F}}} z}{3^{O_{\mathcal{F}}} a}.$$

Taking logarithms to base 2 gives

$$\log_2 \rho_{\mathcal{F}} = E_{\mathcal{F}} - O_{\mathcal{F}} \log_2 3 - \log_2 \frac{a}{z}.$$

Since $\lambda_{\mathcal{F}} = E_{\mathcal{F}} - 2O_{\mathcal{F}}$, we have $E_{\mathcal{F}} = \lambda_{\mathcal{F}} + 2O_{\mathcal{F}}$. Substitution therefore gives

$$\begin{aligned} \log_2 \rho_{\mathcal{F}} &= \lambda_{\mathcal{F}} + 2O_{\mathcal{F}} - O_{\mathcal{F}} \log_2 3 - \log_2 \frac{a}{z} \\ &= \lambda_{\mathcal{F}} + O_{\mathcal{F}} \log_2 \frac{4}{3} - \log_2 \frac{a}{z}, \end{aligned}$$

which proves (71). Next, rearranging (71) gives

$$\frac{\log_2(a/z) - \lambda_{\mathcal{F}}}{\log_2(4/3)} = O_{\mathcal{F}} - \frac{\log_2 \rho_{\mathcal{F}}}{\log_2(4/3)}.$$

By Theorem 2.22, $1 \le \rho_{\mathcal{F}} < 4/3$. Hence

$$0 \le \frac{\log_2 \rho_{\mathcal{F}}}{\log_2(4/3)} < 1,$$

and

$$O_{\mathcal{F}} - 1 < \frac{\log_2(a/z) - \lambda_{\mathcal{F}}}{\log_2(4/3)} \le O_{\mathcal{F}}.$$

Since $O_{\mathcal{F}}$ is an integer, taking ceilings gives

$$O_{\mathcal{F}} = \left\lceil \frac{\log_2(a/z) - \lambda_{\mathcal{F}}}{\log_2(4/3)} \right\rceil.$$

Finally, from (71),

$$\lambda_{\mathcal{F}} = \log_2 \frac{a}{z} - O_{\mathcal{F}} \log_2 \frac{4}{3} + \log_2 \rho_{\mathcal{F}}.$$

Using $0 \le \log_2 \rho_{\mathcal{F}} < \log_2(4/3)$ gives

$$\log_2 \frac{a}{z} - O_{\mathcal{F}} \log_2 \frac{4}{3} \le \lambda_{\mathcal{F}} < \log_2 \frac{a}{z} - (O_{\mathcal{F}} - 1) \log_2 \frac{4}{3},$$

which proves (73). □

For the complete first-arrival shortcut path of a terminating integer n, we have $a = n$, $z = 1$, and $\lambda_{\mathcal{F}} = \lambda(n)$. Thus Theorem 4.2 also gives

$$O(n) = \left\lceil \frac{\log_2 n - \lambda(n)}{\log_2(4/3)} \right\rceil,$$

with $E(n) = \lambda(n) + 2O(n)$ and $H(n) = \lambda(n) + 3O(n)$.

Corollary 4.3 (Fixed-Charge Common-Endpoint Ordering)**.** *Let $\mathcal{F}_a = (a, \dots, z)$ and $\mathcal{F}_b = (b, \dots, z)$ be finite nonrepeating shortcut paths to a common endpoint z, and assume that the input states of both paths are greater than* 1*. Suppose that the two paths have equal finite-path charge: $\lambda_{\mathcal{F}_a} = \lambda_{\mathcal{F}_b}$. Set $d := O_{\mathcal{F}_a} - O_{\mathcal{F}_b}$. Then $E_{\mathcal{F}_a} - E_{\mathcal{F}_b} = 2d$ and $H_{\mathcal{F}_a} - H_{\mathcal{F}_b} = 3d$, and*

$$\frac{a}{b} = \left(\frac{4}{3}\right)^d \frac{\rho_{\mathcal{F}_b}}{\rho_{\mathcal{F}_a}}. \tag{74}$$

Consequently,

$$\left(\frac{4}{3}\right)^{d-1} < \frac{a}{b} < \left(\frac{4}{3}\right)^{d+1}. \tag{75}$$

In particular,

$$a < b \quad \Longrightarrow \quad O_{\mathcal{F}_a} \le O_{\mathcal{F}_b}, \quad E_{\mathcal{F}_a} \le E_{\mathcal{F}_b}, \quad H_{\mathcal{F}_a} \le H_{\mathcal{F}_b},$$

while

$$a > b \quad \Longrightarrow \quad O_{\mathcal{F}_a} \ge O_{\mathcal{F}_b}, \quad E_{\mathcal{F}_a} \ge E_{\mathcal{F}_b}, \quad H_{\mathcal{F}_a} \ge H_{\mathcal{F}_b}.$$

Proof. Since the two paths have equal charge, $E_{\mathcal{F}_a} - 2O_{\mathcal{F}_a} = E_{\mathcal{F}_b} - 2O_{\mathcal{F}_b}$. Therefore $E_{\mathcal{F}_a} - E_{\mathcal{F}_b} = 2\left(O_{\mathcal{F}_a} - O_{\mathcal{F}_b}\right) = 2d$. It follows that

$$H_{\mathcal{F}_a} - H_{\mathcal{F}_b} = (E_{\mathcal{F}_a} - E_{\mathcal{F}_b}) + (O_{\mathcal{F}_a} - O_{\mathcal{F}_b}) = 3d.$$

By Theorem 3.2,

$$\frac{2^{E_{\mathcal{F}_a} - E_{\mathcal{F}_b}}}{3^{O_{\mathcal{F}_a} - O_{\mathcal{F}_b}}} = \frac{a}{b} \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}}.$$

Substituting $E_{\mathcal{F}_a} - E_{\mathcal{F}_b} = 2d$ gives

$$\left(\frac{4}{3}\right)^d = \frac{a}{b} \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}},$$

and hence

$$\frac{a}{b} = \left(\frac{4}{3}\right)^d \frac{\rho_{\mathcal{F}_b}}{\rho_{\mathcal{F}_a}}.$$

By Theorem 2.22, $1 \le \rho_{\mathcal{F}_a}, \rho_{\mathcal{F}_b} < 4/3$, and so

$$\frac{3}{4} < \frac{\rho_{\mathcal{F}_b}}{\rho_{\mathcal{F}_a}} < \frac{4}{3}.$$

Multiplying by $(4/3)^d$ yields

$$\left(\frac{4}{3}\right)^{d-1} < \frac{a}{b} < \left(\frac{4}{3}\right)^{d+1},$$

which proves (75).

Now suppose $a < b$, then $a/b < 1$. If $d \ge 1$, the lower bound in (75) gives

$$\frac{a}{b} > \left(\frac{4}{3}\right)^{d-1} \ge 1,$$

a contradiction. Hence $d \le 0$. Therefore $O_{\mathcal{F}_a} \le O_{\mathcal{F}_b}$. Since the differences in the even count and height are $2d$ and $3d$, respectively, it follows that

$$E_{\mathcal{F}_a} \le E_{\mathcal{F}_b}, \qquad H_{\mathcal{F}_a} \le H_{\mathcal{F}_b}.$$

The case $a > b$ is analogous. If $d \le -1$, then the upper bound in (75) gives

$$\frac{a}{b} < \left(\frac{4}{3}\right)^{d+1} \le 1,$$

contradicting $a/b > 1$. Hence $d \ge 0$, and the stated inequalities follow. □

4.2 Elementary Charge Bounds and Operation-Count Cells

Lemma 4.4 (Elementary Upper λ Bound)**.** *For every terminating positive integer n,*

$$\lambda(n) \le \lfloor \log_2 n \rfloor . \tag{76}$$

Proof. If $O(n) = 0$, then $n = 2^{E(n)}$, so $\lambda(n) = E(n) = \log_2 n$, and the stated inequality is an equality.

Now suppose $O(n) > 0$. If x_i ranges over the odd inputs occurring before the terminal state 1, then $3 + 1/x_i < 4$. By the product definition of the residue,

$$3^{O(n)}\rho(n) = \prod_i \left(3 + \frac{1}{x_i}\right) < 4^{O(n)}.$$

Starting with the definition of $\rho(n)$ in (10) and solving for n, along with the above inequality, we obtain

$$n > \frac{2^{E(n)}}{4^{O(n)}} = 2^{E(n)-2O(n)} = 2^{\lambda(n)}.$$

Thus $\lambda(n) < \log_2 n$. Since $\lambda(n)$ is an integer, $\lambda(n) \le \lfloor \log_2 n \rfloor$, as claimed. □

Theorem 4.5 (Elementary Lower λ Bound)**.** *For every terminating positive integer n,*

$$\lambda(n) \geq \frac{3\log_2 n + (\log_2 3 - 2)H(n)}{1 + \log_2 3}, \tag{77}$$

with strict inequality when $O(n) > 0$.

Proof. The residue identity gives $2^{E(n)} = 3^{O(n)} n\rho(n)$. Since $\rho(n) \geq 1$, $2^{E(n)} \geq 3^{O(n)} n$, and therefore $E(n) \geq O(n)\log_2 3 + \log_2 n$. Using the count inversion formulas

$$E(n) = \frac{2H(n) + \lambda(n)}{3}, \qquad O(n) = \frac{H(n) - \lambda(n)}{3},$$

we obtain

$$\frac{2H(n) + \lambda(n)}{3} \geq \frac{H(n) - \lambda(n)}{3}\log_2 3 + \log_2 n.$$

Multiplying by 3 and collecting the terms containing $\lambda(n)$ gives

$$(1 + \log_2 3)\lambda(n) \geq 3\log_2 n + (\log_2 3 - 2)H(n).$$

Since $1 + \log_2 3 > 0$, division gives

$$\lambda(n) \geq \frac{3\log_2 n + (\log_2 3 - 2)H(n)}{1 + \log_2 3}.$$

If $O(n) > 0$, then the residue product contains at least one factor and hence $\rho(n) > 1$. Thus $2^{E(n)} > 3^{O(n)} n$, so the resulting inequality for $\lambda(n)$ is strict. □

Combining Lemma 4.4 and Theorem 4.5, and using the fact that $\lambda(n)$ is an integer, gives

$$\left\lceil \frac{3\log_2 n + (\log_2 3 - 2)H(n)}{1 + \log_2 3} \right\rceil \leq \lambda(n) \leq \left\lfloor \log_2 n \right\rfloor. \tag{78}$$

When $O(n) > 0$, the lower inequality in Theorem 4.5 is strict. Hence the lower bound sharpens to

$$\lambda(n) \geq \left\lfloor \frac{3\log_2 n + (\log_2 3 - 2)H(n)}{1 + \log_2 3} \right\rfloor + 1.$$

For $n = 2^k$, one instead has $O(n) = 0$ and $H(n) = \lambda(n) = k$, with equality in the lower bound of Theorem 4.5.

Finally, since $\log_2 3 - 2 < 0$, the coefficient of $H(n)$ in the lower bound is negative. Thus an upper bound on $H(n)$ gives a lower bound on the right-hand side of (77), and hence a lower bound on $\lambda(n)$.

Corollary 4.6 (Elementary Finite-Path Charge Bounds)**.** *For every finite shortcut path $\mathcal{F} = (a, \ldots, z)$,*

$$\left\lceil \frac{3\log_2(a/z) + (\log_2 3 - 2)H_{\mathcal{F}}}{1 + \log_2 3} \right\rceil \leq \lambda_{\mathcal{F}} \leq \left\lfloor \log_2(a/z) \right\rfloor.$$

The underlying lower real inequality is strict when $O_{\mathcal{F}} > 0$. The upper real inequality $\lambda_{\mathcal{F}} \leq \log_2(a/z)$ is strict if at least one odd input is greater than 1.

Proof. The finite-path residue identity gives

$$\frac{a}{z} = \frac{2^{E_{\mathcal{F}}}}{3^{O_{\mathcal{F}}}\rho_{\mathcal{F}}}.$$

Since $\rho_{\mathcal{F}} \geq 1$, the proof of Theorem 4.5 applies with $\log_2 n$ replaced by $\log_2(a/z)$. For the upper bound, each odd input x is positive, so $3 + 1/x \leq 4$. Multiplication gives $3^{O_{\mathcal{F}}}\rho_{\mathcal{F}} \leq 4^{O_{\mathcal{F}}}$, hence $a/z \geq 2^{\lambda_{\mathcal{F}}}$. The strict cases follow from $\rho_{\mathcal{F}} > 1$ when there is an odd input and from $3 + 1/x < 4$ when $x > 1$. Taking integer bounds gives the result. □

Consequences of the Four-Thirds Threshold. We now return to the four-thirds bound obtained from the finite inputs in Subsection 2.3, namely $1 \le \rho(n) < 4/3$, or equivalently $0 \le \log_2 \rho(n) < \log_2(4/3)$. Together with the residue identities, this restricted interval gives exact information about the operation counts and their differences. We use it below to compare operation-count classes at a fixed charge, recover operation counts from charge, relate charge-class minima to height-class records, and compare the height and charge of adjacent integers.

Operation-Count Classes.

Definition 4.7. For nonnegative integers e, o, define the *operation-count class*, or *operation-count cell*, $C_{e,o}$ by

$$C_{e,o} = \{n \in \mathcal{T} : E(n) = e,\ O(n) = o\}. \tag{79}$$

We call the cell *occupied* if $C_{e,o} \neq \{\}$.

For every $n \in C_{e,o}$, the residue identity gives

$$n = \frac{2^e/3^o}{\rho(n)}, \qquad \frac{3}{4}\frac{2^e}{3^o} < n \le \frac{2^e}{3^o}.$$

We call $2^e/3^o$ the *scale* of the cell. It lets us compare entire cells before choosing their least members.

Theorem 4.8 (Ordering at a Fixed Charge)**.** *Let x and y be terminating integers with $\lambda(x) = \lambda(y)$. If $O(y) > O(x)$, then $y > x$.*

Proof. Apply Corollary 4.3 to the complete first-arrival paths of x and y, which have the common endpoint 1. At equal charge, a larger number of odd steps forces the starting integer to be larger. Hence $O(y) > O(x)$ implies $y > x$. □

Theorem 4.8 tells us that among integers having the same charge λ, having more odd steps forces the starting integer to be larger.

Example 4.9. Consider $x = 91$ and $y = 121$. Direct computation gives

$$(E, O, H, \lambda)(91) = (59, 33, 92, -7), \qquad (E, O, H, \lambda)(121) = (61, 34, 95, -7).$$

Thus the two integers have the same charge, while $O(121) - O(91) = 1$. Theorem 4.8 therefore gives $121 > 91$. Their residues satisfy

$$\rho(91) = \frac{2^{59}}{3^{33} \cdot 91}, \qquad \rho(121) = \frac{2^{61}}{3^{34} \cdot 121},$$

and hence

$$\frac{\rho(91)}{\rho(121)} = \frac{363}{364}.$$

Accordingly,

$$\frac{121}{91} = \frac{4}{3}\frac{\rho(91)}{\rho(121)} = \frac{4}{3} \cdot \frac{363}{364}.$$

In this example the exact residue ratio is therefore much closer to 1 than the general lower bound $3/4$.

Corollary 4.10 (Count Recovery From Charge)**.** *For every terminating integer n,*

$$O(n) = \left\lceil \frac{\log_2 n - \lambda(n)}{\log_2(4/3)} \right\rceil.$$

Consequently, n and $\lambda(n)$ determine all three operation counts $O(n)$, $E(n)$, and $H(n)$.

Proof. The displayed formula is the complete first-arrival specialization of Theorem 4.2. Once $O(n)$ is known, the defining identities

$$\lambda(n) = E(n) - 2O(n), \qquad H(n) = E(n) + O(n)$$

determine $E(n)$ and $H(n)$. □

Example 4.11. Continuing with the integer 91 from Example 4.9, Corollary 4.10 gives

$$O(91) = \left\lceil \frac{\log_2(91) - (-7)}{\log_2(4/3)} \right\rceil = \lceil 32.5459619\ldots \rceil = 33.$$

Now that $O(91)$ has been computed, the defining identities give

$$(E, O, H)(91) = (59, 33, 92).$$

The implication of Corollary 4.10 is that once n and its charge $\lambda(n)$ are known, the uniform four-thirds residue bound pins $O(n)$ down to a unique integer, and then $E(n)$ and $H(n)$ follow algebraically.

The preceding results describe height and charge for general finite paths. We now turn to starting integers of the form $A + r$ and separate the contribution of A from the trajectory of r. When A is divisible by 2^α, the first α shortcut steps of $A + r$ can be expressed exactly in terms of the corresponding steps of r. These initial-segment identities yield formulas for height and charge and lead to the functions introduced below.

Decomposition into an Anchor Plus an Offset. Let $A = 2^\alpha 3^\beta b$, $\gcd(b, 6) = 1$, where α, β are nonnegative integers and b is a positive integer. For a positive integer r, call A the *anchor* and r its *offset*. Let $o_t(r)$ be the number of odd inputs among the first t shortcut steps of r, and write

$$u_t(r) = T^t(r).$$

The next theorem carries the finite orbit data $u_t(r)$ and $o_t(r)$ from the offset r into exact formulas for the corresponding iterates of $A + r$. This is what we mean here by *finite transport*.

Theorem 4.12 (Exact Finite Transport)**.** *For every $0 \le t \le \alpha$,*

$$T^t(A + r) = 2^{\alpha - t} 3^{\beta + o_t(r)} b + u_t(r).$$

In particular,

$$T^\alpha(A + r) = 3^{\beta + o_\alpha(r)} b + u_\alpha(r).$$

Proof. At $t = 0$, since $o_0(r) = 0$ and $u_0(r) = r$,

$$2^\alpha 3^{\beta + o_0(r)} b + u_0(r) = 2^\alpha 3^\beta b + r = A + r,$$

so the identity holds.

Now suppose the identity holds for some $t < \alpha$. Then

$$T^t(A + r) - T^t(r) = 2^{\alpha - t} 3^{\beta + o_t(r)} b.$$

Since $t < \alpha$, this difference is even, so $T^t(A + r)$ and $T^t(r)$ have the same parity. Hence the same branch of the shortcut map is applied to both.

If the two current states are even, then

$$T^{t+1}(A + r) - T^{t+1}(r) = \frac{1}{2} \left(T^t(A + r) - T^t(r) \right).$$

Since no odd step occurs in this case, $o_{t+1}(r) = o_t(r)$, and hence

$$T^{t+1}(A + r) - T^{t+1}(r) = 2^{\alpha - (t+1)} 3^{\beta + o_{t+1}(r)} b.$$

If the two current states are odd, then $T(x) = (3x + 1)/2$ is applied to both. The $+1$ terms cancel when the two images are subtracted, giving

$$T^{t+1}(A + r) - T^{t+1}(r) = \frac{3}{2} \left(T^t(A + r) - T^t(r) \right).$$

In this case an additional odd step has occurred, so $o_{t+1}(r) = o_t(r) + 1$. Therefore

$$T^{t+1}(A + r) - T^{t+1}(r) = 2^{\alpha - (t+1)} 3^{\beta + o_{t+1}(r)} b.$$

Thus in either case,

$$T^{t+1}(A + r) = 2^{\alpha - (t+1)} 3^{\beta + o_{t+1}(r)} b + u_{t+1}(r),$$

which proves the induction step.

Setting $t = \alpha$ gives the second identity. Moreover, $T^t(A + r) \geq (A + r)/2^t > 1$ for $0 \leq t \leq \alpha$, so no first-arrival trajectory is truncated before this transport is used. □

Theorem 4.12 is entirely finite: no termination assumption is required for the anchor A, the offset r, or the integer $A + r$. The reduction of an integer of the form $2^\alpha 3^\beta b + r$ to $3^c b + u$ therefore applies independently of any complete first-arrival trajectory. We call an integer of the latter form a *boundary integer*.

Theorem 4.13 (Finite-Path Height and Charge Transport)**.** *Let $A = 2^\alpha 3^\beta b$, $\gcd(b, 6) = 1$, and set $c_r = \beta + o_\alpha(r)$, $u_r = u_\alpha(r)$. By Theorem 4.12, $T^\alpha(A + r) = 3^{c_r} b + u_r$.*

Let $\mathcal{F}_B = (3^{c_r} b + u_r, \ldots, z)$ be any finite shortcut path beginning at the boundary integer, and let $\mathcal{F}_A = (A + r, \ldots, 3^{c_r} b + u_r, \ldots, z)$ be the finite path obtained by adjoining the first α shortcut steps of $A + r$ to $\mathcal{F}_B$. Then

$$\begin{aligned} H_{\mathcal{F}_A} &= \alpha - \beta + c_r + H_{\mathcal{F}_B}, \\ \lambda_{\mathcal{F}_A} &= \alpha + 2\beta - 2c_r + \lambda_{\mathcal{F}_B}. \end{aligned} \tag{80}$$

Proof. The first α shortcut steps contain $o_\alpha(r)$ odd steps. Since every shortcut step contributes one even operation, these steps contribute $\alpha + o_\alpha(r)$ to the height and $\alpha - 2o_\alpha(r)$ to the charge. Therefore $H_{\mathcal{F}_A} = \alpha + o_\alpha(r) + H_{\mathcal{F}_B}$, and $\lambda_{\mathcal{F}_A} = \alpha - 2o_\alpha(r) + \lambda_{\mathcal{F}_B}$.

Since $c_r = \beta + o_\alpha(r)$, we have $o_\alpha(r) = c_r - \beta$. Substitution gives $H_{\mathcal{F}_A} = \alpha - \beta + c_r + H_{\mathcal{F}_B}$, and $\lambda_{\mathcal{F}_A} = \alpha + 2\beta - 2c_r + \lambda_{\mathcal{F}_B}$, as claimed. □

When the boundary integer terminates, the preceding finite-path transport formulas specialize to complete first-arrival paths. This motivates the following terminating functions.

The exact three-variable functions. For positive integers b, u and integers $c \geq 0$, whenever $3^c b + u$ terminates, define

$$\mathcal{G}(b, c, u) = c + H(3^c b + u), \qquad \mathcal{Q}(b, c, u) = \lambda(3^c b + u) - 2c. \tag{81}$$

When these functions arise from the anchor decomposition $A = 2^\alpha 3^\beta b$ with offset r, their arguments are

$$c = c_r := \beta + o_\alpha(r), \qquad u = u_r := u_\alpha(r) = T^\alpha(r).$$

Thus

$$3^{c_r} b + u_r = T^\alpha(A + r)$$

is the boundary integer supplied by Theorem 4.12. The functions themselves are defined more generally for arbitrary positive integers b and u and every integer $c \geq 0$ whenever $3^c b + u$ terminates.

Theorem 4.14 (Exact Height and λ Formulas)**.** *Set $c_r = \beta + o_\alpha(r)$ and $u_r = u_\alpha(r)$. If the boundary integer $3^{c_r} b + u_r$ terminates, then*

$$\begin{aligned} H(A + r) &= \alpha - \beta + \mathcal{G}(b, c_r, u_r), \\ \lambda(A + r) &= \alpha + 2\beta + \mathcal{Q}(b, c_r, u_r). \end{aligned} \tag{82}$$

Proof. Take $\mathcal{F}_B$ in Theorem 4.13 to be the complete first-arrival shortcut path from $3^{c_r} b + u_r$ to 1. Then $H_{\mathcal{F}_B} = H(3^{c_r} b + u_r), \qquad \lambda_{\mathcal{F}_B} = \lambda(3^{c_r} b + u_r)$, and $\mathcal{F}_A$ is the complete first-arrival shortcut path from $A + r$ to 1. Hence

$$H(A + r) = \alpha - \beta + \big(c_r + H(3^{c_r} b + u_r)\big) = \alpha - \beta + \mathcal{G}(b, c_r, u_r),$$

while

$$\lambda(A + r) = \alpha + 2\beta + \big(\lambda(3^{c_r} b + u_r) - 2c_r\big) = \alpha + 2\beta + \mathcal{Q}(b, c_r, u_r).$$

□

Before deriving structural identities for $\mathcal{G}$ and $\mathcal{Q}$, we record two elementary composition identities for the finite shortcut dynamics,

$$u_{t+s}(r) = u_s(u_t(r)), \qquad o_{t+s}(r) = o_t(r) + o_s(u_t(r)).$$

These identities hold for all nonnegative integers t and s and require no termination assumption.

The same finite-transport argument also gives, for every positive integer m,

$$u_\alpha(2^\alpha m + r) = 3^{o_\alpha(r)} m + u_\alpha(r).$$

Again, no termination assumption is required. We next explore several useful properties of $\mathcal{G}$ and its restrictions to specific level sets.

Identity 4.1. *For positive integers b, u and integers $c \geq 0$ such that $3^{c+1} b + u$ terminates,*

$$\mathcal{G}(b, c + 1, u) = \mathcal{G}(3b, c, u) + 1, \qquad \mathcal{Q}(b, c + 1, u) = \mathcal{Q}(3b, c, u) - 2.$$

Proof. We start with the definitions of $\mathcal{G}(b, c + 1, u)$ and $\mathcal{Q}(b, c + 1, u)$ and rearrange them to obtain the desired result:

$$\begin{aligned} \mathcal{G}(b, c + 1, u) &= H(3^{c+1} b + u) + c + 1 \\ &= H(3^c 3b + u) + c + 1 \\ &= \mathcal{G}(3b, c, u) + 1 \end{aligned} \qquad \begin{aligned} \mathcal{Q}(b, c + 1, u) &= \lambda(3^{c+1} b + u) - 2(c + 1) \\ &= \lambda(3^c 3b + u) - 2c - 2 \\ &= \mathcal{Q}(3b, c, u) - 2. \end{aligned}$$

□

The Two-Variable Slices.

Definition 4.15. For positive odd b and integers $K \geq 0$ such that $2^K b + 1$ terminates, define the *height* and *charge slice functions* as follows:

$$G(b, K) = H(2^K b + 1) - K, \qquad Q(b, K) = \lambda(2^K b + 1) - K. \tag{83}$$

Definition 4.16. For fixed b, a *G-plateau* is a maximal finite interval of consecutive integers $K_0, K_0+1, \ldots, K_1$ on which $G(b, K)$ is defined and constant. A *Q-plateau* is defined analogously.

We call $G(b, K)$ and $Q(b, K)$ the *height slices* and *charge slices* because they are obtained by restricting the preceding three-variable functions $\mathcal{G}$ and $\mathcal{Q}$ to the *boundary cases*

$$G(b, 2c) = \mathcal{G}(b, c, 1), \qquad G(b, 2c - 1) = \mathcal{G}(b, c, 2), \tag{84}$$

where $c \geq 0$ in the first identity and $c \geq 1$ in the second. The exactly analogous identities for Q are

$$Q(b, 2c) = \mathcal{Q}(b, c, 1), \qquad Q(b, 2c - 1) = \mathcal{Q}(b, c, 2), \tag{85}$$

with the same respective ranges for c. We refer to (84) and (85) as the *boundary-slice identities*.

For $c = 0$, the even identities follow directly from the definitions. For $c \geq 1$, begin with $2^{2c} b + 1 = 4^c b + 1$:

$$4^c b + 1 \longmapsto 4\big(3 \cdot 4^{c-1} b + 1\big) \longmapsto 2\big(3 \cdot 4^{c-1} b + 1\big) \longmapsto 3 \cdot 4^{c-1} b + 1.$$

Repeating this block c times gives $3^c b + 1$. One operation $x \mapsto 3x + 1$ followed by two halvings leaves λ unchanged. We will make use of this fact repeatedly. Thus, each block contributes 3 to the height and 0 to the charge. Hence

$$G(b, 2c) = H(2^{2c} b + 1) - 2c = H(3^c b + 1) + c = \mathcal{G}(b, c, 1),$$
$$Q(b, 2c) = \lambda(2^{2c} b + 1) - 2c = \lambda(3^c b + 1) - 2c = \mathcal{Q}(b, c, 1).$$

For the odd identities, let $c \geq 1$. After $c - 1$ applications of the same three-operation block,

$$2^{2c-1} b + 1 \longmapsto 2 \cdot 3^{c-1} b + 1.$$

One further odd operation followed by one even operation gives

$$2 \cdot 3^{c-1} b + 1 \longmapsto 2(3^c b + 2) \longmapsto 3^c b + 2.$$

Altogether these operations contribute $3c - 1$ to the height and -1 to the charge. Therefore

$$G(b, 2c - 1) = H(2^{2c-1} b + 1) - (2c - 1) = H(3^c b + 2) + c = \mathcal{G}(b, c, 2),$$
$$\begin{aligned} Q(b, 2c - 1) &= \lambda(2^{2c-1} b + 1) - (2c - 1) \\ &= \lambda(3^c b + 2) - 1 - (2c - 1) = \lambda(3^c b + 2) - 2c = \mathcal{Q}(b, c, 2). \end{aligned}$$

Identity 4.2. *For every positive odd integer b and integer $K \geq 0$ such that $2^{K+2} b + 1$ terminates—equivalently, such that $2^K (3b) + 1$ terminates—one has*

$$G(b, K + 2) = G(3b, K) + 1, \qquad Q(b, K + 2) = Q(3b, K) - 2.$$

Proof. Starting with the definitions and applying one odd Collatz operation followed by two even operations gives

$$2^{K+2}b + 1 \longmapsto 4(3 \cdot 2^K b + 1) \longmapsto 2(3 \cdot 2^K b + 1) \longmapsto 3 \cdot 2^K b + 1.$$

These three operations contribute 3 to the height and 0 to the charge. Therefore

$$\begin{aligned} G(b, K+2) &= H(2^{K+2}b + 1) - (K+2) \\ &= H(2^K 3b + 1) + 3 - (K+2) \\ &= G(3b, K) + 1 \end{aligned} \qquad \begin{aligned} Q(b, K+2) &= \lambda(2^{K+2}b + 1) - (K+2) \\ &= \lambda(2^K 3b + 1) - (K+2) \\ &= Q(3b, K) - 2. \end{aligned}$$

□

Identity 4.3. *Let $K, t \geq 0$ be integers and let b be a positive odd integer. If $2^{K+2t}b+1$ terminates—equivalently, if $2^K 3^t b + 1$ terminates—then*

$$G(b, K+2t) = t + G(3^t b, K), \qquad Q(b, K+2t) = Q(3^t b, K) - 2t.$$

Proof. The integer $2^{K+2t}b + 1$ reaches $2^K 3^t b + 1$ through t successive applications of the three-operation block used in Identity 4.2. Hence the stated termination conditions are equivalent, and all intermediate slice values are defined. Applying Identity 4.2 t times gives the two formulas. □

For $n = 2^K b + 1$, the two functions G and Q recover the complete counts without additional notation:

$$O(n) = \frac{G(b, K) - Q(b, K)}{3}, \qquad E(n) = K + \frac{2G(b, K) + Q(b, K)}{3}. \tag{86}$$

Indeed $H(n) = K + G(b, K)$ and $\lambda(n) = K + Q(b, K)$, so the formulas are just the inversion of $H = E + O$ and $\lambda = E - 2O$.

For $n = A + r$, with $A = 2^\alpha 3^\beta b$, define

$$K_r = \alpha + 2\beta - \lambda(r) \tag{87}$$

whenever r terminates. A formula involving $G(b, K_r)$ or $Q(b, K_r)$ is asserted only when $K_r \geq 0$ and $2^{K_r}b + 1$ terminates. The residuals δ_G and δ_Q introduced below additionally require $A + r$ to terminate.

The purpose of K_r is to encode the three-variable boundary data (c_r, u_r) by a single slice coordinate whenever $u_r = 1$ or $u_r = 2$. In particular, the boundary-slice identities give

$$\mathcal{G}(b, c_r, 1) = G(b, 2c_r), \qquad \mathcal{Q}(b, c_r, 1) = Q(b, 2c_r),$$

and

$$\mathcal{G}(b, c_r, 2) = G(b, 2c_r - 1), \qquad \mathcal{Q}(b, c_r, 2) = Q(b, 2c_r - 1).$$

Consequently, whenever $u_r = 1$ and $K_r = 2c_r$, or $u_r = 2$, $c_r \geq 1$, and $K_r = 2c_r - 1$, the exact formulas in equation (82) reduce to

$$\begin{aligned} H(A + r) &= \alpha - \beta + G(b, K_r), \\ \lambda(A + r) &= \alpha + 2\beta + Q(b, K_r). \end{aligned} \tag{88}$$

We next determine when these relations between K_r, c_r, and u_r occur. Using the definition of λ, we may rewrite K_r as

$$K_r = \alpha - E(r) + 2(\beta + O(r)). \tag{89}$$

For a terminating $r > 1$, the complete first-arrival shortcut path contains exactly $E(r)$ shortcut steps, and its final step is $2 \to 1$. Hence $T^{E(r)-1}(r) = 2$. The iterates $u_t(r) = T^t(r)$ are not truncated at first arrival, so thereafter they continue through the cycle $2 \leftrightarrow 1$.

Suppose

$$\alpha \geq E(r) - 1, \tag{90}$$

and write $d = \alpha - (E(r) - 1)$. Then

$$u_r = \begin{cases} 2, & d \text{ even}, \\ 1, & d \text{ odd}. \end{cases}$$

Substituting $\alpha = E(r) - 1 + d$ into equation (89) gives

$$K_r = 2(\beta + O(r)) + d - 1. \tag{91}$$

If $d = 2j$ is even, then the additional d shortcut steps around the $2 \leftrightarrow 1$ cycle contain j odd steps. Thus $c_r = \beta + O(r) + j$, and $u_r = 2$, and equation (91) gives $K_r = 2c_r - 1$. If $d = 2j + 1$ is odd, the additional d shortcut steps again contain j odd steps, so $c_r = \beta + O(r) + j$, $u_r = 1$, and $K_r = 2c_r$.

Thus, for a terminating $r > 1$, the conditions $\alpha \geq E(r) - 1$ and $K_r \geq 0$ ensure that the boundary data fall into one of the two slice cases in Equation (88). These conditions are sufficient for the two-variable formulas. The formulas can also hold outside these slice cases, as Example 4.18 shows.

The case $r = 1$ is separate because $E(1) = 0$, so the preceding argument involving the state at time $E(r) - 1$ does not apply. Nevertheless, the boundary data can be determined directly. Since $T(1) = 2$ and $T(2) = 1$, we have

$$u_\alpha(1) = \begin{cases} 1, & \alpha \text{ even}, \\ 2, & \alpha \text{ odd}, \end{cases} \qquad o_\alpha(1) = \left\lceil \frac{\alpha}{2} \right\rceil.$$

Thus

$$c_1 = \beta + \left\lceil \frac{\alpha}{2} \right\rceil.$$

Since $\lambda(1) = 0$, Equation (87) gives $K_1 = \alpha + 2\beta$.

If $\alpha = 2j$ is even, then $u_1 = 1$, $c_1 = \beta + j$, and $K_1 = 2c_1$. If $\alpha = 2j + 1$ is odd, then $u_1 = 2$, $c_1 = \beta + j + 1$, and $K_1 = 2c_1 - 1$. Hence, for every $\alpha \geq 0$, the case $r = 1$ falls into the corresponding boundary-slice case of Equation (88).

Example 4.17 (The Integer 13)**.** Write

$$13 = 12 + 1 = 2^2 \cdot 3 + 1,$$

so that $\alpha = 2$, $\beta = 1$, $b = 1$, and $r = 1$. Since α is even, the preceding $r = 1$ calculation gives

$$u_1 = 1, \qquad c_1 = \beta + \frac{\alpha}{2} = 2, \qquad K_1 = \alpha + 2\beta = 4 = 2c_1.$$

Thus the boundary integer is $3^{c_1} b + u_1 = 3^2 + 1 = 10$, and the exact three-variable functions give $\mathcal{G}(1, 2, 1) = 2 + H(10) = 8$ and $\mathcal{Q}(1, 2, 1) = \lambda(10) - 4 = -1$. Since $u_1 = 1$ and $K_1 = 2c_1$, the boundary-slice identities give $G(1, 4) = \mathcal{G}(1, 2, 1) = 8$ and $Q(1, 4) = \mathcal{Q}(1, 2, 1) = -1$. Therefore Equation (88) yields $H(13) = 2 - 1 + G(1, 4) = 9$, and $\lambda(13) = 2 + 2 + Q(1, 4) = 3$.

Example 4.18 (The Sufficient Condition is not Necessary)**.** Take $A = 10 = 2 \cdot 5$ and $r = 3$, so that $\alpha = 1$, $\beta = 0$, and $b = 5$. Since

$$(E, O, H, \lambda)(3) = (5, 2, 7, 1),$$

we have $\alpha = 1 < E(3) - 1 = 4$, so the sufficient condition in Equation (90) does not hold.

Nevertheless, the first shortcut step of the offset is $3 \to 5$, so $u_3 = 5$, $o_1(3) = 1$, and $c_3 = 1$. Thus the exact boundary integer is $3^{c_3}b + u_3 = 3 \cdot 5 + 5 = 20$. Also, $K_3 = \alpha + 2\beta - \lambda(3) = 0$.

Although $(c_3, u_3, K_3) = (1, 5, 0)$ is not one of the boundary-slice cases derived above, direct evaluation gives $\mathcal{G}(5, 1, 5) = 1 + H(20) = 8 = G(5, 0)$, and $\mathcal{Q}(5, 1, 5) = \lambda(20) - 2 = 2 = Q(5, 0)$. Consequently the two-variable formulas are still exact: $H(13) = 1 + G(5, 0) = 9$, $\lambda(13) = 1 + Q(5, 0) = 3$. Condition (90) is sufficient for the two-variable reduction, but it is not necessary.

Example 4.19 (Height and Charge Validity Can Separate)**.** Take $A = 4 = 2^2$ and $r = 7$, so that $\alpha = 2$, $\beta = 0$, and $b = 1$. Since

$$(E, O, H, \lambda)(7) = (11, 5, 16, 1),$$

we have $K_7 = \alpha + 2\beta - \lambda(7) = 1$, while $\alpha = 2 < E(7) - 1 = 10$. Thus the sufficient condition (90) does not hold.

The first two shortcut steps of the offset are $7 \to 11 \to 17$. Both steps have odd inputs, so $u_7 = 17$, $o_2(7) = 2$, and $c_7 = 2$. Hence the exact boundary integer is $3^{c_7}b + u_7 = 3^2 + 17 = 26$.

For the height function, $\mathcal{G}(1, 2, 17) = 2 + H(26) = 12$ and $G(1, 1) = 6$, so the projected value differs from the exact value by $\mathcal{G}(1, 2, 17) - G(1, 1) = 6$. For the charge function, however, $\mathcal{Q}(1, 2, 17) = \lambda(26) - 4 = 0$ and $Q(1, 1) = 0$. Thus $\mathcal{Q}(1, 2, 17) = Q(1, 1)$ even though $\mathcal{G}(1, 2, 17) \neq G(1, 1)$.

Consequently the projected formulas give $H(11) \stackrel{?}{=} 2 + G(1, 1) = 8$ and $\lambda(11) = 2 + Q(1, 1) = 2$, whereas the exact values are $H(11) = 14$ and $\lambda(11) = 2$. The height projection fails by 6, while the charge projection remains exact.

Example 4.20 (The Same K-value, Different Exact Values)**.** Take $A = 92 = 2^2 \cdot 23$, so that $\alpha = 2$, $\beta = 0$, and $b = 23$, and consider the offsets $r = 14$ and $r = 15$. The two offsets have identical complete first-arrival values of E, O, H, and λ:

$$(E, O, H, \lambda)(14) = (E, O, H, \lambda)(15) = (12, 5, 17, 2).$$

Consequently, $K_{14} = K_{15} = 2 + 0 - 2 = 0$. Their finite transport data, however, are different. For $r = 14$, $14 \to 7 \to 11$, so $o_2(14) = 1$, $c_{14} = 1$ and $u_{14} = 11$.

For $r = 15$, $15 \to 23 \to 35$, so $o_2(15) = 2$, $c_{15} = 2$, and $u_{15} = 35$. Thus the corresponding boundary integers are $3^{c_{14}}b + u_{14} = 3 \cdot 23 + 11 = 80$ and $3^{c_{15}}b + u_{15} = 3^2 \cdot 23 + 35 = 242$.

For $r = 14$, direct evaluation gives $\mathcal{G}(23, 1, 11) = 10 = G(23, 0)$ and $\mathcal{Q}(23, 1, 11) = Q(23, 0)$, so both projected formulas are exact. For $r = 15$, however, $\mathcal{G}(23, 2, 35) = 98$, and $\mathcal{G}(23, 2, 35) - G(23, 0) = 88$, while $\mathcal{Q}(23, 2, 35) - Q(23, 0) = -14$. Consequently, $H(106) = 12$ and $H(107) = 100$.

Thus two offsets can have the same complete first-arrival operation-count pair $(E(r), O(r))$, and hence the same value of K_r, while producing different exact boundary data and different errors in the two-variable projection. In particular, neither $\lambda(r)$ nor even the full first-arrival operation-count pair $(E(r), O(r))$ determines whether the projection is exact.

Exact correction of the projected formula. The preceding examples show that the two-variable projection need not agree with the exact three-variable boundary data. Whenever the quantities on the right-hand side are defined, set

$$\delta_G(r) = \mathcal{G}(b, c_r, u_r) - G(b, K_r), \qquad \delta_Q(r) = \mathcal{Q}(b, c_r, u_r) - Q(b, K_r). \tag{92}$$

Thus $\delta_G(r)$ and $\delta_Q(r)$ measure, respectively, the exact-minus-projected errors in the height and charge functions. In particular, the height projection is exact precisely when $\delta_G(r) = 0$, and the charge projection is exact precisely when $\delta_Q(r) = 0$.

Substituting Equation (92) into the exact formulas (82) gives

$$\begin{aligned} H(A+r) &= \alpha - \beta + G(b, K_r) + \delta_G(r), \\ \lambda(A+r) &= \alpha + 2\beta + Q(b, K_r) + \delta_Q(r). \end{aligned} \tag{93}$$

Hence the two-variable formulas in (88) are recovered exactly when the corresponding residual vanishes.

This decomposition separates the projected slice value from the residual correction needed to recover the exact height or charge. In particular, it provides the link between G-plateaus and the actual height streaks studied in Section 5.

5 Streak Calculations and Further Corollaries

The preceding section developed the charge bounds, operation-count cells, and the slice functions G and Q. We now compare height and charge for adjacent integers and use those comparisons to relate the plateaus of G and Q and to bound their changes between consecutive slice indices. We then combine the slice functions with the residual corrections δ_G and δ_Q to obtain exact criteria for streaks and their breaks.

Throughout this section, $H(n)$ and $\lambda(n)$ denote the complete first-arrival height and charge of a terminating integer n, as defined in the preceding sections. The discussion of streak breaks near the end of the section uses the class- and delay-record terminology of Definitions 3.16 and 3.17, particularly the height-class function c_H. The relations among the full class- and delay-record families are developed in Section 6.

Theorem 5.1 (Adjacent Height and Charge)**.** *Under the sharper bounds on $W(n)$ from Theorem 3.15, adjacent height equality implies adjacent charge equality. Conversely, adjacent charge equality implies adjacent height equality except at the three pairs $(2,3)$, $(4,5)$, and $(5,6)$.*

Proof. Suppose first that $H(n) = H(n+1)$. By Corollary 3.12, adjacent height equality implies $E(n) = E(n+1)$ and $O(n) = O(n+1)$. Therefore

$$\lambda(n) = E(n) - 2O(n) = E(n+1) - 2O(n+1) = \lambda(n+1).$$

Thus adjacent height equality always implies adjacent charge equality.

Conversely, suppose $\lambda(n) = \lambda(n+1)$, and set $e = E(n) - E(n+1)$ and $q = O(n) - O(n+1)$. Then $0 = \lambda(n) - \lambda(n+1) = e - 2q$, so $e = 2q$. Consequently,

$$W(n) = \frac{2^e}{3^q} = \left(\frac{4}{3}\right)^q.$$

For $n \notin \{1,2,4,5,8\}$, Theorem 3.15 gives $3/4 < W(n) < 4/3$. Hence

$$\frac{3}{4} < \left(\frac{4}{3}\right)^q < \frac{4}{3}.$$

Since q is an integer, this forces $q = 0$. Therefore $e = 0$ as well, giving $H(n) - H(n+1) = e + q = 0$. Thus $H(n) = H(n+1)$.

It remains only to inspect the exceptional cases $n \in \{1,2,4,5,8\}$. Direct computation gives

$$\begin{aligned} &(E,O,H,\lambda)(1) = (0,0,0,0), \quad (E,O,H,\lambda)(2) = (1,0,1,1), \quad (E,O,H,\lambda)(3) = (5,2,7,1), \\ &(E,O,H,\lambda)(4) = (2,0,2,2), \quad (E,O,H,\lambda)(5) = (4,1,5,2), \quad (E,O,H,\lambda)(6) = (6,2,8,2). \end{aligned}$$

From the above calculations, we see that $\lambda(2) = \lambda(3)$, $\lambda(4) = \lambda(5)$ and $\lambda(5) = \lambda(6)$, while the corresponding heights are unequal. Hence $(2,3)$, $(4,5)$ and $(5,6)$ are exceptions. At the remaining exceptional indices, $(1,2)$ has unequal charges, and $(8,9)$ has unequal charges since $\lambda(8) = 3$ and $\lambda(9) = 1$. There are no other exceptions. □

Plateaus and Jumps of G and Q. We now return to the functions G and Q. Theorem 5.1 relates height equality and charge equality for adjacent integers, and this relation can be translated directly into the behavior of G and Q as K increases.

Fix a positive odd integer b and $K \geq 0$ such that the relevant trajectories terminate, and put

$$g = G(b, K+1) - G(b, K), \qquad d = Q(b, K+1) - Q(b, K).$$

For $n = 2^{K+1}b + 1$, the definitions in (83) give

$$g = H(n) - H(n+1), \qquad d = \lambda(n) - \lambda(n+1). \tag{94}$$

Thus $g = 0$ means that G is constant from K to $K+1$, while $d = 0$ means the same for Q. Equation (94) therefore converts Theorem 5.1 into an equivalence between G-plateaus and Q-plateaus. Since $n = 2^{K+1}b + 1$ is odd, the only exceptional adjacent pair that can occur is $(5, 6)$, corresponding to $(b, K) = (1, 1)$. The following theorem also bounds the nonzero jumps.

Theorem 5.2 (Plateau Equivalence and Jump Bounds)**.** *Let b be a positive odd integer and let $K \geq 0$. Suppose that the slice values at K and $K+1$ are defined, and set $n = 2^{K+1}b + 1$, $g = G(b, K+1) - G(b, K)$, and $d = Q(b, K+1) - Q(b, K)$. Under the bounds of Theorem 3.15, one has $g = 0$ if and only if $d = 0$, except at $(b, K) = (1, 1)$, where $(g, d) = (-3, 0)$. Away from this exception, every nonzero jump satisfies*

$$5|d| \leq |g| \leq 8|d|, \tag{95}$$

and g and d have opposite signs. More precisely, with $c = \log_2(4/3)$ and $d \neq 0$,

$$g \in \left\{ d + 3\left\lfloor -\frac{d}{c} \right\rfloor, \quad d + 3\left\lceil -\frac{d}{c} \right\rceil \right\}.$$

Proof. Since $n + 1 = 2(2^K b + 1)$, the first shortcut step from $n + 1$ is a halving step to $2^K b + 1$. Therefore

$$H(n+1) = H(2^K b + 1) + 1, \qquad \lambda(n+1) = \lambda(2^K b + 1) + 1.$$

Using the definitions of G and Q, we consequently obtain

$$\begin{aligned} g &= H(2^{K+1}b + 1) - (K+1) - \big(H(2^K b + 1) - K\big) \\ &= H(n) - H(n+1), \end{aligned}$$

and similarly, $d = \lambda(n) - \lambda(n+1)$.

Write $e = E(n) - E(n+1)$ and $q = O(n) - O(n+1)$, Then $g = e + q$ and $d = e - 2q$. Thus

$$g = d + 3q. \tag{96}$$

Also, $\log_2 W(n) = e - q \log_2 3 = d + (2 - \log_2 3)q$. Setting $c = \log_2(4/3)$, we obtain

$$\log_2 W(n) = d + cq. \tag{97}$$

Since $n = 2^{K+1}b + 1$ is odd and $n \geq 3$, the only exceptional adjacent pair from Theorem 5.1 that can occur is $(5, 6)$. Hence $g = 0$ if and only if $d = 0$, except when $n = 5$. The equation $5 = 2^{K+1}b + 1$ with b odd and $K \geq 0$ has the unique solution $(b, K) = (1, 1)$. At this point, $g = H(5) - H(6) = -3$, and $d = \lambda(5) - \lambda(6) = 0$.

Now suppose $d \neq 0$. Then $n \neq 5$, and Theorem 3.15 gives $3/4 < W(n) < 4/3$. Using (97), this is equivalent to $|d + cq| < c$, and hence

$$\left| q - \left(-\frac{d}{c} \right) \right| < 1.$$

As q is an integer and $-d/c$ is not an integer,

$$q \in \left\{ \left\lfloor -\frac{d}{c} \right\rfloor, \left\lceil -\frac{d}{c} \right\rceil \right\}. \tag{98}$$

Since $2 < 1/c < 3$, we have $2|d| < |d|/c < 3|d|$. Equation (98) therefore gives

$$2|d| \leq |q| \leq 3|d|. \tag{99}$$

Moreover, both possible values of q have the opposite sign from d.

Using (96), we have $|g| = 3|q| - |d|$. Together with (99), this yields

$$5|d| = 3(2|d|) - |d| \leq |g| \leq 3(3|d|) - |d| = 8|d|.$$

Since $3|q| > |d|$, the sign of $g = d + 3q$ is the sign of q, and hence is opposite to the sign of d.

Finally, substituting the two possibilities for q from (98) into $g = d + 3q$ gives

$$g \in \left\{ d + 3\left\lfloor -\frac{d}{c} \right\rfloor, \quad d + 3\left\lceil -\frac{d}{c} \right\rceil \right\}.$$

□

Example 5.3 (Both Unit-Jump Bounds Occur)**.** When $d = 1$, Theorem 5.2 restricts the corresponding height jump to $g \in \{-8, -5\}$. Both possibilities occur. First, take $b = 3$ and $K = 1$. Direct computation gives

$$\big(G(3,1), Q(3,1)\big) = (15, 0), \qquad \big(G(3,2), Q(3,2)\big) = (7, 1).$$

Therefore $g = 7 - 15 = -8$ and $d = 1 - 0 = 1$.

For the other possibility, take $b = 723$ and $K = 1$. In this case,

$$\big(G(723,1), Q(723,1)\big) = (51, 3), \qquad \big(G(723,2), Q(723,2)\big) = (46, 4),$$

and hence $g = 46 - 51 = -5$ and $d = 4 - 3 = 1$. Thus a unit charge jump can produce either extremal height jump allowed by Theorem 5.2. In particular, the constants 5 and 8 in (95) are both sharp.

Proposition 5.4 (Either Slice Determines the Other)**.** *Let b be a positive odd integer and $K \geq 0$, and suppose $n = 2^K b + 1$ terminates. Put $B = \log_2(b + 2^{-K})$ and $c = \log_2(4/3)$. Then the two-variable slice values determine one another by*

$$G(b,K) = Q(b,K) + 3\left\lfloor \frac{B - Q(b,K)}{c} \right\rfloor, \tag{100}$$

and

$$Q(b,K) = G(b,K) - 3\left\lfloor \frac{G(b,K) - B}{\log_2 6} \right\rfloor. \tag{101}$$

Proof. Since $n = 2^K b + 1 = 2^K(b + 2^{-K})$, we have $\log_2 n = K + B$. Moreover, $G(b,K) - Q(b,K) = H(n) - \lambda(n) = 3O(n)$. By Corollary 4.10,

$$O(n) = \left\lfloor \frac{\log_2 n - \lambda(n)}{\log_2(4/3)} \right\rfloor.$$

Since $\log_2 n - \lambda(n) = (K + B) - (K + Q(b, K)) = B - Q(b, K)$, we obtain

$$O(n) = \left\lceil \frac{B - Q(b, K)}{c} \right\rceil.$$

Substituting this into $G(b, K) = Q(b, K) + 3O(n)$ gives (100).

For the reverse direction, the height-based recovery formula gives

$$O(n) = \left\lfloor \frac{H(n) - \log_2 n}{\log_2 6} \right\rfloor.$$

Since $H(n) - \log_2 n = (K + G(b, K)) - (K + B) = G(b, K) - B$, we obtain

$$O(n) = \left\lfloor \frac{G(b, K) - B}{\log_2 6} \right\rfloor.$$

Substituting this into $Q(b, K) = G(b, K) - 3O(n)$ gives (101). □

Conceptually, Proposition 5.4 shows that either two-variable slice determines the other. Indeed, $G(b, K) - Q(b, K) = 3\,O(2^K b + 1)$, so the difference between the two slice values is always a nonnegative multiple of 3. The ceiling formula recovers this multiple from Q, while the floor formula recovers it from G.

Proposition 5.4 also gives an exact relation between the terminating G- and Q-slices. A G-plateau alone, however, does not necessarily determine a streak of equal heights. The exact finite-transport formula is expressed through $\mathcal{G}$, while the two-variable slice G is its projection; the residual δ_G measures the difference between them. We now use this decomposition to obtain an exact criterion for a streak of complete first-arrival heights.

Exact Streak Criterion. Fix an anchor $A = 2^\alpha 3^\beta b$, and for each positive offset r set $c_r = \beta + o_\alpha(r)$ and $u_r = T^\alpha(r)$. Whenever the boundary integer $3^{c_r} b + u_r$ terminates, define

$$\Gamma_A(r) = \mathcal{G}(b, c_r, u_r). \tag{102}$$

By the exact transport formula, $H(A + r) = \alpha - \beta + \Gamma_A(r)$. Consequently, the block $\{A + 1, A + 2, \ldots, A + L\}$ has constant height if and only if

$$\Gamma_A(1) = \Gamma_A(2) = \cdots = \Gamma_A(L).$$

When the corresponding projected slice value $G(b, K_r)$ is defined, $\Gamma_A(r) = G(b, K_r) + \delta_G(r)$. Hence, for $r \geq 2$, whenever the projected slice values are defined at both $r - 1$ and r,

$$\Gamma_A(r) - \Gamma_A(r - 1) = \Delta G_r + \Delta\delta_{G,r},$$

where

$$\Delta G_r = G(b, K_r) - G(b, K_{r-1}), \qquad \Delta\delta_{G,r} = \delta_G(r) - \delta_G(r - 1).$$

Therefore the difference in successive heights can be expressed as

$$H(A + r) - H(A + r - 1) = \Delta G_r + \Delta\delta_{G,r}. \tag{103}$$

If the height is constant through $A + r - 1$, then $A + r$ is the first break exactly when

$$\Delta G_r + \Delta\delta_{G,r} \neq 0.$$

A streak break may result from a change in the projected G-value, from a change in the residual correction, or from both. Conversely, nonzero changes in these two terms may cancel, in which case no actual height change occurs.

A Dyadic Anchor Example. OEIS Sequence A277109 corresponds to the maximal streaks of integers of the same height starting at $2^N + 1$ [12], and these streaks were studied computationally in [3]. In the framework under consideration, we have the anchor $A = 2^N$ and

$$K_r = N - \lambda(r), \qquad H(2^N + r) = N + G(1, N - \lambda(r)) + \delta_G(r). \tag{104}$$

As a small example, consider $N = 8$, where it is known that the streak for $2^8 + 1$ has length 3. First we compute $\lambda(i)$ and K_i for $1 \le i \le 4$:

$$(\lambda(1), \lambda(2), \lambda(3), \lambda(4)) = (0, 1, 1, 2)$$
$$(K_1, K_2, K_3, K_4) = (8, 7, 7, 6)$$

and

$$(c_1, u_1) = (4, 1),\ (c_2, u_2) = (4, 2),\ (c_3, u_3) = (4, 2),\ (c_4, u_4) = (3, 1).$$

Although there are four offsets $1 \le r \le 4$, only three distinct evaluations are required in either G or $\mathcal{G}$. Indeed, the offsets $r = 2$ and $r = 3$ give both the same projected coordinate $K_r = 7$ and the same exact boundary pair $(c_r, u_r) = (4, 2)$. Such repetitions can substantially reduce the number of height computations required across large blocks of integers.

$$\mathcal{G}(1, 4, 1) = \mathcal{G}(1, 4, 2) = 114, \qquad \mathcal{G}(1, 3, 1) = 21$$
$$G(1, 8) = G(1, 7) = 114, \qquad G(1, 6) = 21$$

The first three integers $257, 258, 259$ therefore have height 122, while $H(260) = 29$. The computations above also give $\delta_G(r) = 0$ for $1 \le r \le 4$. Hence the exact break equation (103) gives

$$\Delta G_2 + \Delta\delta_{G,2} = \Delta G_3 + \Delta\delta_{G,3} = 0, \qquad \Delta G_4 + \Delta\delta_{G,4} = -93.$$

Thus the changes at offsets $r = 2$ and $r = 3$ preserve the common height, whereas the change at $r = 4$, from 259 to 260, does not. Hence 260 is the first break in the equal-height streak.

A Factorial Anchor Example. For $A = m!$, write $\alpha = \mathrm{ord}_2(m!)$, $\beta = \mathrm{ord}_3(m!)$, and $b = m!/(2^\alpha 3^\beta)$. The same exact formula applies with $K_r = \alpha + 2\beta - \lambda(r)$.

For the factorial anchors, we study the *anchored constant-height run* beginning at $A + 1$, in accordance with OEIS sequence A346775 [13]. An anchored run of length L consists of $A + 1, A + 2, \ldots, A + L$, with

$$H(A + 1) = H(A + 2) = \cdots = H(A + L),$$

and its first break occurs at offset $L + 1$. The anchored run need not be a maximal streak, since the same height may also occur at A, or consecutive integers preceding, and up to, A. Table 7 gives the resulting data for five factorial anchors, computed using the same approach as in the preceding dyadic example.

The preceding examples use the anchor decomposition and the slice functions to calculate heights from trajectory data. We now turn to the reverse question. Once the operation counts E and O and the intercept J are known, can the corresponding coordinate bracket be recovered? This is relevant to streaks because we now know that integers in the same operation-count class have the same values of E and O, while their intercepts change in a prescribed way as the starting integer changes.

Table 7: Anchored constant-height run data for five factorial anchors, together with the parameters α, β, and b used in the corresponding G-plateau calculations.

Anchor	α	β	b	Anchored Run Length	Common Height	First Break Offset
6!	4	2	5	3	46	4
18!	16	8	14889875	30	425	31
20!	18	8	1414538125	30	431	31
44!	41	19	$\approx 1.04 \times 10^{33}$	26	1383	27
87!	82	42	$\approx 3.98 \times 10^{87}$	1606	3247	1607

Recovering the Coordinates. Recall from (4) that the intercept of a complete coordinate bracket is determined by its coordinates. The following decoding procedure proves the converse: for fixed E and O, whenever an intercept J has such a coordinate representation, its coordinates are uniquely determined.

For integers in a fixed operation-count class $C_{e,o}$, a displacement of d in the starting integer changes the intercept by exactly $-d3^o$. Indeed, every member of the class has

$$H = e + o, \qquad \lambda = e - 2o, \qquad J(n) = 2^e - 3^o n.$$

Therefore, whenever $n, n + d \in C_{e,o}$,

$$J(n + d) = J(n) - d3^o. \tag{105}$$

In particular, if both n and $n + 1$ belong to the class, then their intercepts differ by 3^o.

Theorem 5.5 (Unique Decoder)**.** *Fix nonnegative integers E and O. Suppose an integer J admits a representation*

$$J = \sum_{i=0}^{O-1} 3^{O-1-i} 2^{\ell_i}, \qquad 0 \le \ell_0 < \cdots < \ell_{O-1} < E.$$

Then this coordinate sequence is unique. Starting from $R_0 = J$, the coordinates are recovered successively by

$$\ell_i = \mathrm{ord}_2(R_i), \qquad R_{i+1} = R_i - 3^{O-1-i} 2^{\ell_i}, \qquad 0 \le i < O.$$

After these O steps, $R_O = 0$. Append $\ell_O = E$ as the final coordinate. If $O = 0$, the sum is empty, $J = 0$, and the only coordinate is $\ell_0 = E$.

Proof. Suppose first that $O \ge 1$. Factoring 2^{ℓ_0} from J gives

$$J = 2^{\ell_0} \left(3^{O-1} + \sum_{i=1}^{O-1} 3^{O-1-i} 2^{\ell_i - \ell_0} \right).$$

Since $\ell_i > \ell_0$ for every $i \ge 1$, each exponent $\ell_i - \ell_0$ is positive. Hence every term in the sum inside the parentheses is even, whereas 3^{O-1} is odd. Therefore the quantity in parentheses is odd, and hence $\mathrm{ord}_2(J) = \ell_0$. Thus the first coordinate ℓ_0 is uniquely determined. Subtracting its contribution gives

$$R_1 = J - 3^{O-1} 2^{\ell_0} = \sum_{i=1}^{O-1} 3^{O-1-i} 2^{\ell_i}.$$

Applying the same argument to R_1 gives $\mathrm{ord}_2(R_1) = \ell_1$. Repeating this process recovers each coordinate $\ell_0, \ell_1, \ldots, \ell_{O-1}$ uniquely, and after the final subtraction we obtain $R_O = 0$. The last coordinate is then $\ell_O = E$.

If $O = 0$, the sum is empty and $J = 0$, so there is nothing to recover apart from the single coordinate $\ell_0 = E$. □

Once the coordinates are known, the gaps are recovered by the difference formula in Definition 1.7.

Example 5.6. With $E = 11$, $O = 4$, and $J = 185$, the decoding process is shown in Table 8.

Table 8: Decoding the coordinates from $E = 11$, $O = 4$, and $J = 185$.

i	R_i	$\ell_i = \mathrm{ord}_2(R_i)$	R_{i+1}
0	185	0	$185 - 3^3 \cdot 2^0 = 158$
1	158	1	$158 - 3^2 \cdot 2^1 = 140$
2	140	2	$140 - 3^1 \cdot 2^2 = 128$
3	128	7	$128 - 3^0 \cdot 2^7 = 0$

The decoder therefore recovers $(\ell_0, \ell_1, \ell_2, \ell_3) = (0, 1, 2, 7)$. Appending $\ell_4 = E = 11$ gives the complete coordinate bracket, and taking consecutive differences gives the corresponding gap bracket:

$$\langle \boldsymbol{\ell} \rangle_1 = \langle 0, 1, 2, 7, 11 \mid 1 \rangle = \mathrm{CBrack}_{11}(23),$$
$$[\mathbf{g}]_1 = [0, 1, 1, 5, 4 \mid 1] = \mathrm{Gap}_{11}(23).$$

Thus the decoder reconstructs from $(E, O, J) = (11, 4, 185)$ exactly the complete coordinate and gap brackets of 23.

When testing a proposed intercept J, the decoding fails if a remainder becomes nonpositive before all O terms have been removed, or if a recovered coordinate is not strictly larger than the preceding one or is at least E. After O steps, the final remainder must be zero.

The decoder checks only whether the proposed intercept J has the required coordinate representation for the given values of E and O. It does not by itself show that these are the complete first-arrival counts of a positive integer. After decoding, the candidate starting integer must be recovered from

$$n = \frac{2^E - J}{3^O}.$$

This value must be a positive integer, and the path encoded by the recovered bracket must reach 1 for the first time at its endpoint.

For example, take $E = 2$, $O = 1$, and $J = 1$. The decoder recovers $\ell_0 = \mathrm{ord}_2(1) = 0$ and appends $\ell_1 = E = 2$, giving the corresponding gap bracket $[0, 2 \mid 1]$. The intercept identity gives

$$n = \frac{2^2 - 1}{3} = 1.$$

This bracket encodes the finite shortcut path $1 \to 2 \to 1$, which makes one traversal of the cycle from 1 back to 1 and has $E = 2$ and $O = 1$. It is not, however, the complete first-arrival path from 1 to 1, because that path stops immediately at its starting value. Consequently,

$$E(1) = O(1) = 0.$$

Thus a successful decoding must be followed by both the positive-integer check and the first-arrival check.

Theorem 5.7 (Consecutive Intercept Block)**.** *Fix nonnegative integers e and o. Suppose*

$$J_0, J_0 - 3^o, \ldots, J_0 - (L-1)3^o$$

all pass the decoder and the first-arrival checks described above. Then the corresponding integers are L consecutive members of the operation-count class $C_{e,o}$. In particular, they all have the same values of E, O, H, and λ.

Proof. For fixed e and o, the intercept identity gives $J(n) = 2^e - 3^o n$, and hence

$$n = \frac{2^e - J(n)}{3^o}.$$

Let n_0 be the integer corresponding to J_0. For $0 \le k < L$, the intercept $J_0 - k3^o$ therefore corresponds to

$$\frac{2^e - (J_0 - k3^o)}{3^o} = \frac{2^e - J_0}{3^o} + k = n_0 + k.$$

Thus the given intercepts correspond to the L consecutive integers $n_0, n_0 + 1, \ldots, n_0 + (L - 1)$. By hypothesis, each decoded bracket gives the complete first-arrival counts $E = e$ and $O = o$. Hence all L integers belong to $C_{e,o}$, and consequently have the common values $H = e + o$ and $\lambda = e - 2o$. □

Example 5.8 (The Class $C_{78,44}$)**.** Every $n \in C_{78,44}$ satisfies $E(n) = 78$ and $O(n) = 44$. Consequently, $H(n) = 122$ and $\lambda(n) = -10$, and the corresponding intercept is $J(n) = 2^{78} - 3^{44}n$. Since an intercept is a sum of nonnegative terms, $J(n) \ge 0$. Therefore every member of $C_{78,44}$ satisfies $0 \le 2^{78} - 3^{44}n$, giving the upper bound

$$n \le \left\lfloor \frac{2^{78}}{3^{44}} \right\rfloor = 306.$$

Thus every member of $C_{78,44}$ occurs in the finite range $1 \le n \le 306$. Checking the complete first-arrival counts in this range gives

$$C_{78,44} = \{257, 258, 259, 265\}.$$

The first three points form a block of consecutive integers, and their intercepts differ successively by

$$3^{44} = 984\,770\,902\,183\,11\,232\,881.$$

The candidate corresponding to 260 fails the decoder for $C_{78,44}$. To see exactly where the failure occurs, compare the decoding of the intercepts corresponding to 257 and 260 when both are tested as candidates for $C_{78,44}$. A successful decoding must recover coordinates satisfying

$$J = \sum_{i=0}^{43} 3^{43-i} 2^{\ell_i}, \qquad 0 \le \ell_0 < \cdots < \ell_{43} < 78.$$

The values of J_{257} and J_{260} are

$$J_{257} = 2^{78} - 257 \cdot 3^{44} = 49\,145\,333\,042\,469\,206\,826\,127,$$
$$J_{260} = 2^{78} - 260 \cdot 3^{44} = 46\,191\,020\,335\,918\,373\,127\,484.$$

Since J_{257} is odd, $\mathrm{ord}_2(J_{257}) = 0$; direct calculation gives $\mathrm{ord}_2(J_{260}) = 2$.

$$R_{257,1} = J_{257} - 3^{43}2^0 = 48\,817\,076\,075\,074\,669\,748\,500,$$
$$R_{260,1} = J_{260} - 3^{43}2^2 = 44\,877\,992\,466\,340\,224\,816\,976.$$

Continuing the decoding process, Table 9 shows the first steps for 257 and 260. The decoding of 260 fails when $R_{260,8} < 0$, after the subtraction at step 7.

Continuing the table through all 44 decoder steps gives $R_{257,44} = 0$. Moreover, all 44 recovered coordinates are strictly increasing and lie below 78:

$$\begin{aligned}(\ell_0, \ell_1, \ldots, \ell_{43}) =& (0, 2, 4, 6, 9, 11, 12, 13, 14, 15, 17, 19, 20, 22, 23, 24,\\ & 26, 27, 28, 29, 31, 34, 35, 36, 38, 39, 41, 42, 43, 44,\\ & 45, 46, 49, 50, 51, 52, 56, 58, 60, 64, 67, 68, 69, 74).\end{aligned}$$

Table 9: First eight decoder steps for 257 and 260 as candidates for $C_{78,44}$. The decoding of 257 continues successfully, while that of 260 fails when $R_{260,8} < 0$.

i	$\ell_{257,i}$	$R_{257,i}$	$\ell_{260,i}$	$R_{260,i}$
0	0	49 145 333 042 469 206 826 127	2	46 191 020 335 918 373 127 484
1	2	48 817 076 075 074 669 748 500	4	44 877 992 466 340 224 816 976
2	4	48 379 400 118 548 620 311 664	6	43 127 288 640 236 027 069 632
3	6	47 795 832 176 513 887 729 216	10	40 793 016 872 097 096 739 840
4	9	47 017 741 587 134 244 285 952	11	28 343 567 442 022 801 647 616
5	11	44 942 833 348 788 528 437 248	12	20 043 934 488 639 938 252 800
6	12	42 176 289 030 994 240 638 976	14	14 510 845 853 051 362 656 256
7	13	40 331 926 152 464 715 440 128	17	7 133 394 338 933 261 860 864
8	14	39 102 350 900 111 698 640 896	–	−12 539 809 698 715 006 926 848

Thus 257 passes the decoder. Together with the first-arrival check, this confirms $257 \in C_{78,44}$. Of special interest, $265 \in C_{78,44}$, and the coordinate sequences for 257 and 265 agree from ℓ_{11} onward. Thus the two integers have different initial coordinate data but identical remaining coordinate sequences. In particular, a single operation-count class can contain more than one block of consecutive integers.

The decoder and the consecutive-intercept theorem determine when successive integers remain in a fixed operation-count class. Since an operation-count class has a fixed E, O, and H, each consecutive block in such a class is an equal-height block. We now compare this description with height-class records of the offsets. Such records provide natural candidates for streak breaks, but the actual height of $A + r$ depends on both the projected change in G and the change in the residual correction δ_G. Consequently, class-record information about r alone need not determine whether an anchored run breaks.

Class Records and the Exact Break Equation. By (64), every height-delay record is a height-class record, although a height-class record need not be a height-delay record. For example, 18 and 25 are height-delay records with $H(18) = 20$ and $H(25) = 23$, so $c_H(20) = 18$ and $c_H(23) = 25$. The intermediate height values 21 and 22 are first attained at $c_H(21) = 36$ and $c_H(22) = 72$, respectively, and neither 36 nor 72 is a height-delay record.

The class-record property alone does not decide whether an offset is a streak break. The offset $r = 27$ is the height-class record at height 111: $c_H(111) = 27$. For each of three factorial anchors A, Table 10 examines the change from $A + 26$ to $A + 27$. It decomposes the height difference $H(A + 27) - H(A + 26)$ into the projected change ΔG_{27} and the residual change $\Delta\delta_{G,27}$.

Table 10: Changes in the G-slice, residual correction, and height between offsets 26 and 27 for three factorial anchors. For 18! and 20!, ΔG is cancelled by $\Delta\delta_G$, while for 44! the change produces an actual height change.

anchor	ΔG	$\Delta\delta_G$	ΔH
18!	-31	+31	0
20!	+23	-23	0
44!	-137	0	-137

Thus the same class-record offset $r = 27$ does not break either the 18! + 1 or the 20! + 1 anchored run, because in both cases the projected change ΔG_{27} is exactly cancelled by the residual change $\Delta\delta_{G,27}$. Their

first break instead occurs at offset 31. For the 44! + 1 anchored run, however, the residual change at $r = 27$ is zero, so the nonzero projected change produces an actual height change.

These examples show that class-record status alone does not determine whether a height change occurs. By (103), the exact criterion is

$$\Delta G_r + \Delta\delta_{G,r} \neq 0.$$

Obtaining an effective criterion that computes or bounds the residual change without evaluating all relevant trajectories remains a separate problem. The formal relation between first projected plateau exits and record values is stated in Section 6, after the complete record hierarchy has been established.

6 Record Hierarchies and Common Tails

To summarize the class- and delay-record information thus far, the class-record functions c_H, c_E, and c_O are defined in Definition 3.16, and the corresponding delay-record lists d_H, d_E, and d_O are defined in definition 3.17. Theorem 3.18 establishes the equivalence of the E-delay and height-delay properties. Section 4 introduces the charge λ, the operation-count classes of Definition 4.7, and the fixed-charge ordering of Theorem 4.8. Section 5 then relates height-class records of offsets to possible streak breaks and shows, through (103), why class-record status alone does not determine whether an actual height change occurs.

We now bring these results together into a unified theory of the four class-record and delay-record families. The main mechanism is the ordering of occupied operation-count cells supplied by the bound $\rho < 4/3$: fixing one of H, E, O, or λ imposes an order on the remaining operation counts and hence on the least starting integers in the corresponding classes. We first introduce the λ-record notation and establish the global class and delay hierarchies. We then apply these relations to the first exit from a projected G-plateau and extend the theory to finite first-arrival paths above a fixed admissible tail. The global and same-tail hierarchies are summarized in (128) and (129).

6.1 Global Record Hierarchies

Definition 6.1. Specializing Definition 3.16 to $f = \lambda$, define the extended λ-class-record function by

$$c_\lambda(q) := \min\{n \in \mathcal{T} : \lambda(n) = q\}, \tag{106}$$

with the convention that the minimum of the empty set is $+\infty$. When q is realized, the finite value $c_\lambda(q)$ is called the *λ-class record at charge q*.

A terminating integer n is a *λ-delay record among terminating integers* if

$$\lambda(n) < \lambda(m) \qquad \text{for every } m \in \mathcal{T} \text{ with } m < n.$$

Since $\lambda(1) = 0$, set $d_\lambda(0) = 1$. Having defined $d_\lambda(r)$, define the next term by

$$d_\lambda(r+1) := \min\{n \in \mathcal{T} : n > d_\lambda(r),\ \lambda(n) < \lambda(d_\lambda(r))\}. \tag{107}$$

A next term is defined only when this set is nonempty. This recursion enumerates exactly the λ-delay records among terminating integers in increasing order of their starting integers. Here r is the position in the delay-record list, not a charge value.

If every positive integer through n terminates, then a λ-delay record among terminating integers is called simply a *λ-delay record*.

Write

$$\mathrm{CR}_\lambda := \{c_\lambda(q) : q \text{ is realized}\}, \qquad \mathrm{DR}_\lambda := \{d_\lambda(r) : d_\lambda(r) \text{ is defined}\},$$

and

$$\mathrm{CR}_{\lambda}^{\le 0} := \{c_\lambda(q) : q \le 0 \text{ is realized}\}.$$

To illustrate the distinction between the two indices, note that

$$c_\lambda(-1) = 251, \qquad d_\lambda(1) = 27 = c_\lambda(-12), \qquad \lambda(27) = -12.$$

Thus 251 is the least starting integer of charge -1, whereas 27 is the first nontrivial λ-delay record. Since the charge attained at that delay record is -12, the same starting integer is also the λ-class record $c_\lambda(-12)$.

For λ, delay means attaining a new lower value; for H, E, and O, it means attaining a new greater value. The restriction to nonpositive charge classes in the record hierarchies reflects the direction chosen for the λ-delay definition. The initial λ-delay record is 1, whose charge is zero. Every other λ-delay record has negative charge. The positive charge classes are nevertheless completely determined, as Lemma 4.4 gives $\lambda(n) \le \lfloor \log_2 n \rfloor$, and will be used in Theorem 6.3 to directly show that $c_\lambda(q) = 2^q$ for $q \ge 0$.

We now determine how the λ-record sets fit together with the three record families already defined. The first occurrence of each odd count provides the organizing sequence for the class hierarchy.

Lemma 6.2 (First Heights at an Odd Count)**.** *Every nonnegative odd count o occurs. Put*

$$b_o := c_O(o), \qquad e_o := E(b_o), \qquad m_o := H(b_o), \qquad q_o := \lambda(b_o) = m_o - 3o.$$

Then b_o is odd, and

$$e_o = \min\{E(n) : n \in \mathcal{T},\ O(n) = o\}, \qquad m_o = \min\{H(n) : n \in \mathcal{T},\ O(n) = o\}.$$

Moreover, for every $t \ge 0$,

$$(E, O, H)(2^t b_o) = (e_o + t, o, m_o + t).$$

Consequently, every division count at least e_o and every height at least m_o occurs at odd count o. Consecutive values satisfy

$$2 \le m_{o+1} - m_o \le 5, \qquad e_{o+1} > e_o, \qquad q_{o+1} \ge q_o - 1. \tag{108}$$

Proof. Direct calculation gives $b_0 = 1$ and $b_1 = 5$. We first show inductively that every larger odd count occurs. Suppose $o \ge 1$ is realized. Starting with any terminating integer having odd count o, remove its initial division steps. The resulting integer y is odd, terminating, and satisfies $O(y) = o$. Moreover, $y > 1$, since $O(y) = o \ge 1$. According to the residue class of y modulo 3, define

$$x = \begin{cases} \dfrac{4(4y+1)-1}{3}, & y \equiv 0 \pmod 3, \\ \dfrac{4y-1}{3}, & y \equiv 1 \pmod 3, \\ \dfrac{2y-1}{3}, & y \equiv 2 \pmod 3. \end{cases}$$

In each case, x is a positive odd integer.

First, suppose $y \equiv 0 \pmod 3$ and put $z = 4y + 1$. Then

$$z \mapsto 4(3y+1) \mapsto 2(3y+1) \mapsto 3y + 1,$$

whereas $y \mapsto 3y + 1$. Thus the paths from z and y reach the state $3y + 1$ using the same number of odd steps, but the path from z uses two additional division steps. Consequently, $O(z) = O(y)$ and $H(z) = H(y) + 2$.

If $y \equiv 1 \pmod 3$, then $x \mapsto 4y \mapsto 2y \mapsto y$, and therefore $O(x) = O(y) + 1$ and $H(x) = H(y) + 3$.

Lastly, if $y \equiv 2 \pmod 3$, then $x \mapsto 2y \mapsto y$, so the path from x reaches y after one odd step and one division step. Hence $O(x) = O(y) + 1$ and $H(x) = H(y) + 2$.

For

$$x = \frac{4z-1}{3} = \frac{4(4y+1)-1}{3},$$

we have $x \mapsto 4z \mapsto 2z \mapsto z$. This adds one odd step and two division steps, and hence

$$O(x) = O(z) + 1 = O(y) + 1, \qquad H(x) = H(z) + 3 = H(y) + 5.$$

In every residue class we therefore obtain a terminating integer with odd count $o + 1$. Since the counts 0 and 1 already occur, induction shows that every nonnegative odd count occurs.

Furthermore, if b_o were even, then $b_o/2 < b_o$ would terminate and would have the same odd count as b_o, contradicting the definition of $b_o = c_O(o)$.

Now fix an odd count o. By the residue identity and the bound $1 \le \rho(n) < 4/3$, every terminating integer with $O(n) = o$ satisfies

$$\frac{3}{4}\frac{2^{E(n)}}{3^o} < n \le \frac{2^{E(n)}}{3^o}.$$

If n and n' both have odd count o and $E(n') \ge E(n) + 1$, then

$$n' > \frac{3}{4}\frac{2^{E(n)+1}}{3^o} = \frac{3}{2}\frac{2^{E(n)}}{3^o} > \frac{2^{E(n)}}{3^o} \ge n.$$

Thus, at a fixed odd count, every occupied cell with a larger division count lies entirely above every occupied cell with a smaller division count. Since b_o is the least integer with odd count o, its division count e_o is the least division count occurring at that odd count. Because $H = E + O$, its height $m_o = e_o + o$ is likewise the least height occurring there.

For every $t \ge 0$, the first t steps in the trajectory of $2^t b_o$ are division steps leading to b_o. Therefore

$$E(2^t b_o) = e_o + t, \qquad O(2^t b_o) = o, \qquad H(2^t b_o) = m_o + t.$$

This proves that every division count at least e_o and every height at least m_o occurs at odd count o.

It remains to compare consecutive values. For $o = 0$, direct calculation gives $(e_0, m_0, q_0) = (0, 0, 0)$ and $(e_1, m_1, q_1) = (4, 5, 2)$. Thus all three assertions in (108) hold for $o = 0$, that is, for the consecutive odd counts 0 and 1.

Now suppose $o \ge 1$. Apply the three constructions above with $y = b_o$. According to the residue class of b_o modulo 3, they produce an integer with odd count $o + 1$ and height $m_o + 2$, $m_o + 3$, or $m_o + 5$. Since m_{o+1} is the least height at odd count $o + 1$, $m_{o+1} \le m_o + 5$.

For the reverse inequality, write

$$a = \operatorname{ord}_2(3b_{o+1} + 1) \ge 1, \qquad y = \frac{3b_{o+1}+1}{2^a}.$$

Deleting the first odd step and its following a division steps from the trajectory of b_{o+1} leaves the trajectory of the odd integer y, which has odd count o. Hence $H(y) = m_{o+1} - (a + 1)$. Since m_o is the least height at odd count o,

$$m_o \le H(y) = m_{o+1} - (a+1) \le m_{o+1} - 2.$$

Therefore $2 \le m_{o+1} - m_o \le 5$.

Finally, since $e_o = m_o - o$,

$$e_{o+1} - e_o = m_{o+1} - m_o - 1 \ge 1$$

so $e_{o+1} > e_o$. Similarly, since $q_o = m_o - 3o$, $q_{o+1} - q_o = m_{o+1} - m_o - 3 \ge -1$. This proves all the assertions in (108). □

Theorem 6.3 (Class-Record Hierarchy)**.** *Using the same notation as Lemma 6.2, every O-class record is an E-class record, and every E-class record is a height-class record. More precisely,*

$$c_O(o) = c_E(e_o) = c_H(m_o). \tag{109}$$

For a division count e, let

$$o = \max\{j \geq 0 : e_j \leq e\}.$$

Then

$$c_E(e) = c_H(e + o). \tag{110}$$

Moreover,

$$\mathrm{CR}_\lambda^{\leq 0} \subsetneq \mathrm{CR}_O \subsetneq \mathrm{CR}_E \subsetneq \mathrm{CR}_H. \tag{111}$$

Proof. We first establish the two cell orderings needed to identify the class minima. Suppose that $j' > j$, with $x \in C_{e_x,j'}$ and $y \in C_{e_y,j}$. The residue identity gives the general comparison

$$\frac{x}{y} = 2^{e_x - e_y} 3^{j-j'} \frac{\rho(y)}{\rho(x)}.$$

First fix the division count, so that $e_x = e_y = e$. Then

$$\frac{x}{y} = 3^{j-j'} \frac{\rho(y)}{\rho(x)} < \frac{1}{3} \cdot \frac{4}{3} = \frac{4}{9} < 1.$$

Hence, at a fixed division count, every integer in a cell with a larger odd count is smaller than every integer in a cell with a smaller odd count.

Next fix the height h. In this case, $e_x = h - j'$, and $e_y = h - j$ and therefore $e_x - e_y = j - j'$. Consequently,

$$\frac{x}{y} = 2^{j-j'} 3^{j-j'} \frac{\rho(y)}{\rho(x)} = 6^{j-j'} \frac{\rho(y)}{\rho(x)} < \frac{1}{6} \cdot \frac{4}{3} = \frac{2}{9} < 1.$$

Thus, at a fixed height, every integer in a cell with a larger odd count is again smaller than every integer in a cell with a smaller odd count. In particular, the greatest occupied odd count contains the smallest starting integers.

By Lemma 6.2, the first division counts e_o strictly increase. Since $E(b_o) = e_o$ and $O(b_o) = o$, we have $b_o \in C_{e_o,o}$, so this cell is occupied. Now let $j > o$. Then $e_j > e_o$. Because e_j is the least division count attained by any terminating integer with odd count j, no such integer can have division count e_o. Thus o is the greatest odd count represented by an occupied cell at division count e_o. The fixed-division-count ordering therefore shows that the smallest integer having division count e_o lies in $C_{e_o,o}$. Moreover, $b_o = c_O(o)$ is the least integer having odd count o, so it is in particular the least member of $C_{e_o,o}$. Hence $c_E(e_o) = b_o = c_O(o)$.

The first heights m_o also strictly increase. Consequently, no odd count greater than o is occupied at height m_o. The fixed-height ordering therefore shows, by the same argument, that $c_H(m_o) = b_o$. This proves $c_O(o) = c_E(e_o) = c_H(m_o)$, which is the entirety of (109).

Now let $e \geq 0$ be arbitrary and put $o = \max\{j \geq 0 : e_j \leq e\}$. This maximum exists because $e_0 = 0$, so the set is nonempty, and the strict increase of the integer sequence (e_j) gives $e_j \geq j$. Hence $e_j \leq e$ is possible only when $j \leq e$, so the set is finite. By Lemma 6.2, $2^{e-e_o} b_o \in C_{e,o}$, so this cell is occupied. If $j > o$, then $e_j > e$, and hence no cell with odd count j is occupied at division count e. Thus o is the greatest occupied odd count at e, and $c_E(e) = \min C_{e,o}$.

Put $h = e + o$. The same operation-count cell can be written as $C_{e,o} = C_{h-o,o}$. If $j > o$, then $e_j > e$, and therefore $m_j = e_j + j > e + o = h$. Since m_j is the least height occurring at odd count j, no odd count

greater than o is occupied at height h. Hence o is also the greatest occupied odd count at height h. The fixed-height ordering then gives

$$c_H(h) = \min C_{h-o,o} = \min C_{e,o} = c_E(e).$$

Since $h = e + o$, this proves (110).

It remains to identify the charge classes. For $q \geq 0$, Lemma 4.4 gives $n \geq 2^q$ whenever $\lambda(n) = q$, while $\lambda(2^q) = q$. So every 2^q exceeds the charge of every smaller starting integer, while no integer strictly between 2^q and 2^{q+1} can establish a new upper charge record. Hence

$$c_\lambda(q) = 2^q, \qquad q \geq 0. \tag{112}$$

Now let $q < 0$ be realized. Recall from Lemma 6.2 that, for every $j \geq 0$,

$$e_j = \min\{E(n) : n \in \mathcal{T},\ O(n) = j\}, \qquad q_j = e_j - 2j.$$

The set $\{j \geq 0 : q_j \leq q\}$ is nonempty. Indeed, choose n_0 with $\lambda(n_0) = q$ and put $j_0 = O(n_0)$. Then $E(n_0) = q + 2j_0$. Since e_{j_0} is the least division count attained at odd count j_0, we have $e_{j_0} \leq E(n_0) = q + 2j_0$, and therefore $q_{j_0} = e_{j_0} - 2j_0 \leq q$.

Let o be the least odd count satisfying $q_o \leq q$. For a fixed odd count j, any integer n with charge q and odd count j satisfies $E(n) = q + 2j$. Therefore, when $q + 2j \geq 0$, the integers having charge q and odd count j are precisely the members of $C_{q+2j,j}$; when $q + 2j < 0$, no such integers exist.

By Lemma 6.2, such an integer exists exactly when the required division count $q + 2j$ is at least e_j. Therefore

$$\{n \in \mathcal{T} : \lambda(n) = q,\ O(n) = j\} \neq \{\,\} \quad \Longleftrightarrow \quad q + 2j \geq e_j \quad \Longleftrightarrow \quad q \geq e_j - 2j = q_j.$$

Consequently, o is the least odd count attained by any integer of charge q.

We now show that the least integer of charge q lies in this first occupied cell. Choose $x \in C_{q+2o,o}$. If $c_\lambda(q)$ had odd count $j > o$, then x and $c_\lambda(q)$ would have the same charge, but $O(x) = o < j = O\big(c_\lambda(q)\big)$. Theorem 4.8 would then give $x < c_\lambda(q)$, contradicting the definition of $c_\lambda(q)$. Therefore

$$c_\lambda(q) = \min C_{q+2o,o}.$$

Since $q_0 = 0 > q$, we have $o \geq 1$. The minimality of o gives $q_{o-1} > q$. Because these quantities are integers, $q_{o-1} \geq q + 1$. Lemma 6.2 gives $q_o \geq q_{o-1} - 1 \geq q$. Together with the defining inequality $q_o \leq q$, this yields $q_o = q$. It follows that

$$C_{q+2o,o} = C_{q_o+2o,o} = C_{e_o,o}.$$

The least member of this cell is $b_o = c_O(o)$. Therefore

$$c_\lambda(q) = c_O(o) = c_E(e_o) = c_H(m_o). \tag{113}$$

For $q = 0$, equation (112) gives $c_\lambda(0) = 1 = c_O(0)$. Together with (113), this proves $\mathrm{CR}_\lambda^{\leq 0} \subseteq \mathrm{CR}_O$. Equation (109) gives $\mathrm{CR}_O \subseteq \mathrm{CR}_E$, and (110) gives $\mathrm{CR}_E \subseteq \mathrm{CR}_H$.

All three inclusions are proper. First, $5 = c_O(1)$ and $\lambda(5) = 2 > 0$, so $5 \in \mathrm{CR}_O \setminus \mathrm{CR}_\lambda^{\leq 0}$. Next, $2 = c_E(1)$, $O(2) = 0$, and $c_O(0) = 1$, so $2 \in \mathrm{CR}_E \setminus \mathrm{CR}_O$. Finally, $10 = c_H(6)$, $E(10) = 5$, and $c_E(5) = 3$, so $10 \in \mathrm{CR}_H \setminus \mathrm{CR}_E$. This proves (111). □

Example 6.4 (Three Cell Orderings)**.** The cell orderings used above can be seen in small operation-count classes. First, fix the odd count $O = 1$. Direct calculation gives

$$C_{6,1} = \{20, 21\}, \qquad C_{7,1} = \{40, 42\}, \qquad C_{8,1} = \{80, 84, 85\}.$$

Thus

$$\max C_{6,1} = 21 < 40 = \min C_{7,1}, \qquad \max C_{7,1} = 42 < 80 = \min C_{8,1}.$$

At a fixed odd count, increasing the division count therefore moves the entire cell upward.

Next, fix the division count $E = 7$. The occupied cells for increasing odd counts 0, 1, and 2 are:

$$C_{7,0} = \{128\}, \qquad C_{7,1} = \{40, 42\}, \qquad C_{7,2} = \{12, 13\}.$$

Here

$$\max C_{7,2} = 13 < 40 = \min C_{7,1}, \qquad \max C_{7,1} = 42 < 128 = \min C_{7,0}.$$

Thus, at a fixed division count, increasing the odd count moves the entire cell downward.

Finally, fix the height $H = 7$. The occupied cells at this height are

$$C_{7,0} = \{128\}, \qquad C_{6,1} = \{20, 21\}, \qquad C_{5,2} = \{3\}.$$

Once again, the cells move downward as the odd count increases: $3 < 20 < 21 < 128$. These examples illustrate that the orderings compare entire occupied cells, not merely their least members. The proof of Theorem 6.3 additionally uses the fixed-charge ordering from Theorem 4.8: among two terminating integers having the same charge, the one with the larger odd count is the larger starting integer. Example 4.9 illustrates this ordering.

The preceding theorem compares the least starting integers within fixed classes. We now pass to delay records, which compare each attained value with the values at all smaller starting integers. By (64), every H-, E-, or O-delay record is the corresponding class record at the value it attains. The same immediate argument applies to λ-delay records. One therefore expects the class-record hierarchy to induce a related delay-record hierarchy, which is indeed the case.

Theorem 6.5 (Delay-Record Hierarchy)**.** *Among terminating integers,*

$$\mathrm{DR}_\lambda \subsetneq \mathrm{DR}_O \subsetneq \mathrm{DR}_E = \mathrm{DR}_H. \tag{114}$$

The same implications hold without qualification whenever all smaller positive integers terminate.

Proof. We first rewrite the residue identity in terms of the charge. Since $E(n) = \lambda(n) + 2O(n)$, we have

$$n = \frac{2^{E(n)}}{3^{O(n)}\rho(n)} = 2^{\lambda(n)} \left(\frac{4}{3}\right)^{O(n)} \frac{1}{\rho(n)}.$$

Let n be a λ-delay record, and let $m < n$ be terminating. By the definition of a λ-delay record, $\lambda(n) < \lambda(m)$. Thus $\lambda(m) - \lambda(n)$ is a positive integer, so $\lambda(m) - \lambda(n) \geq 1$ and

$$2^{\lambda(m)-\lambda(n)} \geq 2.$$

Suppose, contrary to the desired conclusion, that $O(m) \geq O(n)$. Then

$$\left(\frac{4}{3}\right)^{O(m)-O(n)} \geq 1.$$

Moreover, the four-thirds residue bound gives $\rho(n)/\rho(m) > 3/4$. Consequently,

$$\frac{m}{n} = 2^{\lambda(m)-\lambda(n)} \left(\frac{4}{3}\right)^{O(m)-O(n)} \frac{\rho(n)}{\rho(m)} > 2 \cdot 1 \cdot \frac{3}{4} = \frac{3}{2} > 1.$$

This contradicts $m < n$. Hence $O(m) < O(n)$ for every smaller terminating m, so n is an O-delay record. Therefore $\mathrm{DR}_\lambda \subseteq \mathrm{DR}_O$.

Now let n be an O-delay record, and let $m < n$ be terminating. Then $O(m) < O(n)$, and since the odd counts are integers, $O(m) - O(n) \le -1$. Suppose that $E(m) \ge E(n)$. The three factors in the residue comparison then satisfy

$$2^{E(n)-E(m)} \le 1, \qquad 3^{O(m)-O(n)} \le \frac{1}{3}, \qquad \frac{\rho(m)}{\rho(n)} < \frac{4}{3}.$$

It follows that

$$\frac{n}{m} = 2^{E(n)-E(m)} 3^{O(m)-O(n)} \frac{\rho(m)}{\rho(n)} < 1 \cdot \frac{1}{3} \cdot \frac{4}{3} = \frac{4}{9} < 1.$$

This again contradicts $m < n$, which requires $n/m > 1$. Therefore $E(m) < E(n)$ for every smaller terminating m, and n is an E-delay record. Thus $\mathrm{DR}_O \subseteq \mathrm{DR}_E$.

Finally, Theorem 3.18, in its terminating-set form, gives $\mathrm{DR}_E = \mathrm{DR}_H$.

It remains to show that the two inclusions are proper. Direct calculation gives $O(1) = O(2) = 0$ and $O(3) = 2$. Thus 3 is an O-delay record. On the other hand, $\lambda(1) = 0$ and $\lambda(3) = 1$, so 3 does not attain a new lower charge and is not a λ-delay record. Hence $\mathrm{DR}_\lambda \subsetneq \mathrm{DR}_O$.

Similarly, $E(1) = H(1) = 0$ and $E(2) = H(2) = 1$, so 2 is both an E-delay record and a height-delay record. However, $O(2) = O(1) = 0$, so 2 is not an O-delay record. Therefore $\mathrm{DR}_O \subsetneq \mathrm{DR}_E = \mathrm{DR}_H$. This proves (114). □

Corollary 6.6 (Exact Intersections of the Two Rows)**.** *The class- and delay-record hierarchies intersect in the following exact way. An integer is an O-delay record precisely when it is both an O-class record and a height-delay record:*

$$\mathrm{DR}_O = \mathrm{CR}_O \cap \mathrm{DR}_H. \tag{115}$$

Likewise, among the nonpositive λ-class records, the λ-, O-, E-, and height-delay properties are equivalent:

$$\mathrm{DR}_\lambda = \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_O = \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_E = \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_H. \tag{116}$$

In particular, once a negative λ-class record is known, its four delay tests are equivalent.

Proof. Every O-delay record is an O-class record: a smaller member of the same O-class would contradict the strict delay-record property. Theorem 6.5 also shows that every O-delay record is a height-delay record. Therefore

$$\mathrm{DR}_O \subseteq \mathrm{CR}_O \cap \mathrm{DR}_H.$$

Conversely, suppose that $n = c_O(o) \in \mathrm{DR}_H$. In the notation of Lemma 6.2, we have $n = b_o$ and $H(n) = m_o$. Let $m < n$ be terminating. Since n is the least integer with odd count o, one cannot have $O(m) = o$.

Suppose instead that $O(m) = j > o$. Then the class minimum at odd count j satisfies $c_O(j) \le m < n$. The first heights m_j strictly increase with j, so $m_j > m_o$. Moreover, by the definitions of b_j and m_j, $H\big(c_O(j)\big) = H(b_j) = m_j > m_o = H(n)$. Thus the smaller terminating integer $c_O(j)$ has greater height than n, contradicting the assumption that n is a height-delay record.

Consequently, $O(m) < o = O(n)$ for every smaller terminating integer m. Hence n is an O-delay record, which proves

$$\mathrm{CR}_O \cap \mathrm{DR}_H \subseteq \mathrm{DR}_O.$$

Together with the first inclusion, this proves (115).

We next consider the charge records. Every λ-delay record is a λ-class record at the charge it attains. Indeed, if n is a λ-delay record and some terminating integer $m < n$ satisfied $\lambda(m) = \lambda(n)$, then this would contradict the defining inequality $\lambda(m) > \lambda(n)$ for every terminating $m < n$. Its charge is nonpositive. The initial record is $1 = c_\lambda(0)$. If $n > 1$ is a λ-delay record, comparison with the smaller starting integer 1 gives $\lambda(n) < \lambda(1) = 0$. Thus every λ-delay record belongs to $\mathrm{CR}_\lambda^{\le 0}$. Theorem 6.5 also gives $\mathrm{DR}_\lambda \subseteq \mathrm{DR}_O$. Therefore

$$\mathrm{DR}_\lambda \subseteq \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_O.$$

For the reverse inclusion, suppose first that $n = c_\lambda(q) \in \mathrm{DR}_O$ for $q < 0$. By (113), there is an odd count o such that $n = c_\lambda(q) = c_O(o)$ and $q_o = q$. In the construction leading to (113), o is the least odd count for which $q_o \le q$. Hence $q_j > q$ for every $j < o$.

Let $m < n$ be terminating, and put $j = O(m)$. Since n is an O-delay record, $j = O(m) < O(n) = o$. At odd count j, Lemma 6.2 shows that the least division count is e_j. Therefore $E(m) \ge e_j$, and consequently

$$\lambda(m) = E(m) - 2j \ge e_j - 2j = q_j.$$

Because $j < o$, we have $q_j > q$, and hence $\lambda(m) > q = \lambda(n)$. Thus every smaller terminating integer has charge strictly greater than that of n, so n is a λ-delay record.

For $q = 0$, equation (112) gives $c_\lambda(0) = 1$, and the initial starting integer 1 belongs to all four delay-record sets. We have therefore proved

$$\mathrm{DR}_\lambda = \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_O.$$

It remains to obtain the other two intersections. Theorem 6.3 gives $\mathrm{CR}_\lambda^{\le 0} \subseteq \mathrm{CR}_O$. Using (115), we therefore have

$$\begin{aligned} \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_H &= \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{CR}_O \cap \mathrm{DR}_H \\ &= \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_O \\ &= \mathrm{DR}_\lambda. \end{aligned}$$

Finally, Theorem 6.5 gives $\mathrm{DR}_E = \mathrm{DR}_H$, so

$$\mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_E = \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_H = \mathrm{DR}_\lambda.$$

This proves all the equalities in (116). □

Corollary 6.7 (Reading the Class and Delay Lists)**.** *The full height-class list, together with the E- and O-counts of each entry, determines all four class-record sets. When the height-class list is read in increasing height, the O-class record $c_O(o)$ is the first height-class record with odd count o, while the E-class record $c_E(e)$ is the last height-class record with division count e. When the resulting O-class list is read in increasing odd count, for every realized $q < 0$, the record $c_\lambda(q)$ is the first O-class record b_o for which $q_o \le q$; at this first crossing, $q_o = q$. The nonnegative λ-class records are given by* (112)*.*

For $X \in \{H, E, O\}$, a finite value $c_X(k)$ belongs to DR_X *exactly when*

$$c_X(k) < c_X(\ell) \qquad \text{for every realized } \ell > k. \tag{117}$$

For $X = \lambda$, the corresponding condition is

$$c_\lambda(k) < c_\lambda(\ell) \qquad \text{for every realized } \ell < k.$$

For every $X \in \{H, E, O, \lambda\}$ and every index r for which $d_X(r)$ is defined,

$$d_X(r) = c_X\big(X(d_X(r))\big). \tag{118}$$

Moreover,

$$d_E(r) = d_H(r) \tag{119}$$

for every index r for which these terms are defined.

Proof. We first recover the class-record sets from the height-class list. By Lemma 6.2, m_o is the least height occurring at odd count o. Equation (109) gives $c_O(o) = c_H(m_o)$. Thus no height-class record preceding $c_H(m_o)$ can have odd count o, while $c_H(m_o)$ itself does. Therefore $c_O(o)$ is the first height-class record with odd count o.

Now fix a division count e, and let $o = \max\{j \geq 0 : e_j \leq e\}$. Theorem 6.3 gives $c_E(e) = c_H(e + o)$. Suppose that a later height-class record also had division count e. Its height would have the form $h = e + j$ for some odd count $j > o$. The existence of an integer with division count e and odd count j would imply $e_j \leq e$, contradicting the maximality of o. Hence $c_E(e)$ is the last height-class record with division count e.

The O-class record recovered from the height-class list at odd count o is $b_o = c_O(o) = c_H(m_o)$, and its charge is $q_o = \lambda(b_o)$. Now let $q < 0$ be realized and put $o = \min\{j \geq 0 : q_j \leq q\}$. Thus $q_j > q$ for every $j < o$, whereas $q_o \leq q$; equivalently, o is the first position in the O-class list at which the charge falls to or below q.

Since $q_0 = 0 > q$, we have $o \geq 1$. Moreover, $q_{o-1} > q$, so $q_{o-1} \geq q + 1$ since charges are integer valued. Lemma 6.2 gives $q_o \geq q_{o-1} - 1 \geq q$. Together with $q_o \leq q$, this yields $q_o = q$. Equation (113) therefore gives $c_\lambda(q) = c_O(o)$. Consequently, the negative λ-class records are recovered by scanning the O-class list in increasing odd count and selecting the first record whose charge reaches each realized negative value. For $q \geq 0$, equation (112) gives $c_\lambda(q) = 2^q$. This proves that the height-class list and its operation counts determine all four class-record sets.

We next prove the delay criterion. Let $X \in \{H, E, O\}$ and put $n = c_X(k)$. Suppose first that $n \in \mathrm{DR}_X$. If some realized $\ell > k$ satisfied $c_X(\ell) \leq n$, there would be two possibilities. If $c_X(\ell) < n$, then $c_X(\ell)$ would be a smaller terminating integer with $X\big(c_X(\ell)\big) = \ell > k = X(n)$, contradicting the delay-record property of n. If $c_X(\ell) = n$, then $\ell = X\big(c_X(\ell)\big) = X(n) = k$, contradicting $\ell > k$. Therefore $c_X(k) < c_X(\ell)$ for every realized $\ell > k$.

Conversely, suppose that (117) holds. If some terminating $m < n$ satisfied $X(m) \geq k$, then equality $X(m) = k$ would contradict the definition of $n = c_X(k)$. Hence $\ell = X(m) > k$. The class-record property would then give $c_X(\ell) \leq m < n = c_X(k)$, contradicting (117). Thus $X(m) < X(n)$ for every smaller terminating m, and $n \in \mathrm{DR}_X$, proving (117).

The argument for λ is the same with the direction of the values reversed. Let $n = c_\lambda(k)$. If $n \in \mathrm{DR}_\lambda$ and $\ell < k$ is realized, then $c_\lambda(\ell) < n$ would give a smaller integer with charge below $\lambda(n)$, contradicting the lower-record property. Equality is again impossible because $\ell \neq k$. Hence $c_\lambda(k) < c_\lambda(\ell)$ for every realized $\ell < k$.

Conversely, suppose these inequalities hold. If a terminating $m < n$ satisfied $\lambda(m) \leq k$, then equality would contradict $n = c_\lambda(k)$. Thus $\ell = \lambda(m) < k$, and $c_\lambda(\ell) \leq m < n = c_\lambda(k)$, contradicting the assumed inequality. Therefore $\lambda(m) > \lambda(n)$ for every smaller terminating m, so $n \in \mathrm{DR}_\lambda$.

Finally, let $X \in \{H, E, O, \lambda\}$ and put $n = d_X(r)$ and $k = X(n)$. The strict delay-record property implies that no smaller terminating integer can have the same X-value as n. Thus n is the least integer in its X-class, and consequently

$$d_X(r) = n = c_X(k) = c_X\big(X(d_X(r))\big).$$

This proves (118) for all four quantities.

Theorem 6.5 gives $\mathrm{DR}_E = \mathrm{DR}_H$. By Definition 3.17, both d_E and d_H begin at index 0 with the initial record 1 and then enumerate the remaining members of this common set in increasing order. Therefore their terms agree at every defined index: $d_E(r) = d_H(r)$. This proves (119).

□

These tests compare a class record with class records at more extreme values. For $X \in \{H, E, O\}$, finding a realized $\ell > k$ with $c_X(\ell) < c_X(k)$ proves that $c_X(k)$ is not an X-delay record. For $X = \lambda$, the same conclusion follows from a realized $\ell < k$ with $c_\lambda(\ell) < c_\lambda(k)$. However, checking only part of a class-record list may not prove that a class record is a delay record, because a class not yet computed could still provide such a counterexample. To prove delay-record status, one must either rule out all such classes or check the values at every smaller starting integer directly.

Charge Records and G-Plateau Exits. Section 5 showed that the height of an anchored integer is determined by the projected slice value $G(b, K_r)$ together with the residual correction $\delta_G(r)$. We now use the charge-record hierarchy to constrain the first offset for which the coordinate

$$K_r = K^* - \lambda(r)$$

leaves the G-plateau containing K^*. This departure concerns the projected slice value alone and need not produce an actual height change, since a change in G may be cancelled by the residual correction.

Definition 6.8. Fix an anchor $A = 2^\alpha 3^\beta b$, with $\gcd(b, 6) = 1$, and put $K_* = \alpha + 2\beta$. Assume that $G(b, K_*)$ is defined, and let $I = \{K_-, K_- + 1, \ldots, K_+\}$ be the G-plateau containing K_*. Thus $G(b, K)$ is defined and constant for every $K \in I$, and I is maximal among intervals of consecutive integers with these properties.

For each positive terminating offset r, put $K_r = K_* - \lambda(r)$. When the following set is nonempty, its least member,

$$\min\{r \geq 1 : r \in \mathcal{T},\ G(b, K_r) \text{ is defined},\ K_r \notin I\},$$

is called the *first G-plateau exit*.

Since $\lambda(1) = 0$, we have $K_1 = K_* \in I$. Thus the first G-plateau exit is the first terminating offset, among those for which the projected slice is defined, whose coordinate K_r lies outside the plateau containing the initial coordinate K_1.

Lemma 6.9 (Plateau–Charge Interval Correspondence)**.** *Let $K_r = K_* - \lambda(r)$, and let $I = [K_-, K_+] \cap \mathbb{Z}$ contain K_*. Then $K_r \in I$ if and only if*

$$K_* - K_+ \leq \lambda(r) \leq K_* - K_-.$$

Furthermore, the corresponding charge interval contains 0*.*

Proof. The equivalence follows by substituting $K_r = K_* - \lambda(r)$ into $K_- \leq K_r \leq K_+$. Since $K_* \in I$, the resulting charge interval contains zero. □

Corollary 6.10 (First G-Plateau Exit)**.** *Suppose the first G-plateau exit r exists, all positive integers through r terminate, and $G(b, K_m)$ is defined for every $1 \leq m < r$. Then r is either a power of two or a height-delay record, and therefore either a power of two or a height-class record.*

Proof. Define $L = K_* - K_+$, and $U = K_* - K_-$. Since $K_* \in I$, Lemma 6.9 gives $L \leq 0 \leq U$. For every $1 \leq m < r$, the hypotheses ensure that m terminates and that $G(b, K_m)$ is defined. If $K_m \notin I$, then m would satisfy all the conditions in Definition 6.8, contrary to the minimality of r. Therefore $K_m \in I$ for every

$1 \le m < r$. Lemma 6.9 then gives $L \le \lambda(m) \le U$ for every $1 \le m < r$. On the other hand, $K_r \notin I$, so $\lambda(r) \notin [L, U]$. In particular, $\lambda(r) \neq 0$.

If $\lambda(r) > 0$, then necessarily $\lambda(r) > U$. Thus every smaller positive integer has charge strictly less than $\lambda(r)$. In particular, no smaller integer has the same charge, so $r = c_\lambda\big(\lambda(r)\big) = 2^{\lambda(r)}$ by (112).

If $\lambda(r) < 0$, then necessarily $\lambda(r) < L$. Therefore every $m < r$ satisfies $\lambda(m) \ge L > \lambda(r)$. Thus r is a λ-delay record. By Theorem 6.5, it is a height-delay record.

This proves that r is either a power of two or a height-delay record. The final assertion follows from (118), since every height-delay record is a height-class record. □

Corollary 6.10 constrains the first offset whose coordinate K_r leaves the G-plateau I; it does not identify the first actual height change in the anchored sequence. Since K_r need not move through consecutive values of K, it may leave I and land at another point having the same G-value. Even when the projected value does change, that change may be cancelled by the residual correction. More precisely, whenever the relevant residual terms are defined, (103) gives

$$H(A + r) - H(A + r - 1) = \Delta G_r + \Delta\delta_{G,r}.$$

Consequently, the change from $A + r - 1$ to $A + r$ is an actual height change exactly when

$$\Delta G_r + \Delta\delta_{G,r} \neq 0.$$

Thus the record hierarchy restricts where the first G-plateau exit can occur, while deciding whether an actual streak break occurs still requires the residual correction.

6.2 Records Above a Common Tail

The preceding record theory is stated for complete first-arrival paths to 1. Its main arguments, however, use only finite first-arrival paths with a common endpoint and do not depend on the continuation of the orbit beyond that endpoint. This leads to a local version of the record problem: among the starting integers whose trajectories reach a fixed value z, we compare the operation counts accumulated before their first arrival at z.

This extension shows that the record hierarchies arise from the structure of paths that coalesce at a common endpoint, rather than from any special property of the endpoint 1. It also allows the record families above z to be studied without assuming that z terminates. When z does terminate, the relative counts above z differ from the corresponding global counts only by the fixed counts along the path from z to 1. We distinguish these relative counts and record families from their global counterparts by including the parameter z.

Definition 6.11. Call a positive integer z an *admissible tail* if either $z = 1$ or $z > 2$ is nonperiodic under the shortcut map T. Fix an admissible tail z, and let

$$\mathcal{A}_z := \{n \ge 1 : T^j(n) = z \text{ for some } j \ge 0\}.$$

For $n \in \mathcal{A}_z$, let $\mathcal{F}(n, z)$ be the shortcut path that stops at the first arrival at z, and write

$$E_z(n) := E_{\mathcal{F}(n,z)}, \quad O_z(n) := O_{\mathcal{F}(n,z)}, \quad H_z(n) := H_{\mathcal{F}(n,z)}, \quad \lambda_z(n) := \lambda_{\mathcal{F}(n,z)}.$$

Also put

$$\rho_z(n) := \rho_{\mathcal{F}(n,z)} = \frac{2^{E_z(n)} z}{3^{O_z(n)} n}.$$

For $X \in \{H, E, O, \lambda\}$, define the *extended tail-relative class-record function* by

$$c_X(k; z) := \min\{n \in \mathcal{A}_z : X_z(n) = k\},$$

with the convention that the minimum of an empty set is $+\infty$. When this value is finite, it is called the *X-class record above z at relative value k*. Define

$$\mathrm{CR}_X(z) := \{c_X(k; z) : k \text{ is realized in } \mathcal{A}_z\}.$$

For the nonpositive relative charge classes, write

$$\mathrm{CR}_\lambda^{\le 0}(z) := \{c_\lambda(q; z) : q \le 0 \text{ is realized in } \mathcal{A}_z\}.$$

Since $z \in \mathcal{A}_z$, the set $\mathcal{A}_z$ is nonempty. For every $X \in \{H, E, O, \lambda\}$, begin the corresponding ordinal delay list with

$$d_X(0; z) := \min \mathcal{A}_z.$$

For $X \in \{H, E, O\}$, define subsequent terms recursively by

$$d_X(r+1; z) := \min\left\{n \in \mathcal{A}_z : n > d_X(r; z),\ X_z(n) > X_z\big(d_X(r; z)\big)\right\}.$$

For the relative charge, define

$$d_\lambda(r+1; z) := \min\left\{n \in \mathcal{A}_z : n > d_\lambda(r; z),\ \lambda_z(n) < \lambda_z\big(d_\lambda(r; z)\big)\right\}.$$

A next term is defined only when the corresponding set is nonempty. Finally, write

$$\mathrm{DR}_X(z) := \{d_X(r; z) : d_X(r; z) \text{ is defined}\}$$

for the set of tail-relative X-delay records.

We call the elements of $\mathcal{A}_z$ the *sources above z*. For an admissible tail z, every first-arrival path $\mathcal{F}(n, z)$ is nonrepeating and has no input state equal to 1. Indeed, a repeated input would trap the path in a cycle before its first arrival at z. If $z > 2$, an input equal to 1 would instead trap the path in the cycle $1 \leftrightarrow 2$; when $z = 1$, the state 1 is the endpoint and is not an input. Therefore the finite-path four-thirds bound applies to every path considered here. Periodic tails other than the distinguished endpoint $z = 1$ require a separate normalization and are not included below.

Lemma 6.12 (Same-Tail Ratio and Relative Count Structure)**.** *For $x, y \in \mathcal{A}_z$,*

$$\frac{x}{y} = 2^{E_z(x) - E_z(y)} 3^{O_z(y) - O_z(x)} \frac{\rho_z(y)}{\rho_z(x)}. \tag{120}$$

For every realized relative odd count o, define its first division count by

$$e_o(z) := \min\{E_z(n) : n \in \mathcal{A}_z,\ O_z(n) = o\}.$$

The relative odd count 0 is realized, with $e_0(z) = 0$. If o is realized, then every integer j with $0 \le j \le o$ is realized. Whenever $o + 1$ is realized, $e_o(z) < e_{o+1}(z)$. Moreover, for every realized o and every integer $e \ge e_o(z)$, there exists an $n \in \mathcal{A}_z$ such that

$$E_z(n) = e, \qquad O_z(n) = o.$$

Proof. The relative residue identity gives

$$\rho_z(n) = \frac{2^{E_z(n)} z}{3^{O_z(n)} n},$$

and hence

$$n = \frac{2^{E_z(n)} z}{3^{O_z(n)} \rho_z(n)}.$$

Dividing the formulas for x and y cancels the common endpoint z and gives

$$\frac{x}{y} = 2^{E_z(x)-E_z(y)} 3^{O_z(y)-O_z(x)} \frac{\rho_z(y)}{\rho_z(x)}.$$

This proves (120).

Since $z \in \mathcal{A}_z$ and the first-arrival path $\mathcal{F}(z, z)$ has length zero, $E_z(z) = O_z(z) = 0$. Thus the relative odd count 0 is realized and $e_0(z) = 0$.

Now suppose that $n \in \mathcal{A}_z$ has $O_z(n) = o \geq 1$. Let $a \geq 0$ be the number of initial even shortcut inputs before the first odd input u in $\mathcal{F}(n, z)$, and put $v = T(u) = (3u + 1)/2$. After deleting the a initial division steps and the first odd shortcut step, the remaining portion of the path begins at v and ends at z. Since $\mathcal{F}(n, z)$ stops at its first arrival at z, this remaining portion is precisely the first-arrival path $\mathcal{F}(v, z)$. In particular, $v \in \mathcal{A}_z$. Its counts satisfy $O_z(v) = O_z(n) - 1 = o - 1$ and $E_z(v) = E_z(n) - (a + 1) < E_z(n)$. Repeating this deletion proves that every nonnegative odd count below o is realized.

To prove that the first division counts strictly increase, suppose that the relative odd count $o + 1$ is realized. Then the set

$$\{E_z(n) : n \in \mathcal{A}_z,\ O_z(n) = o + 1\}$$

is a nonempty set of nonnegative integers. It therefore has a least member, which by definition is $e_{o+1}(z)$. Choose $n \in \mathcal{A}_z$ attaining this least value, so that $O_z(n) = o + 1$ and $E_z(n) = e_{o+1}(z)$. Apply the operation described in the preceding paragraph: delete the initial even shortcut steps and then the first odd shortcut step from $\mathcal{F}(n, z)$. The remaining first-arrival path begins at some $v \in \mathcal{A}_z$ and satisfies $O_z(v) = o$ and $E_z(v) < E_z(n) = e_{o+1}(z)$. Since $v \in \mathcal{A}_z$ has relative odd count o, its division count is one of the values over which $e_o(z)$ is defined. Hence $e_o(z) \leq E_z(v) < e_{o+1}(z)$, and therefore $e_o(z) < e_{o+1}(z)$.

Finally, let o be realized and choose $n \in \mathcal{A}_z$ with $O_z(n) = o$, and $E_z(n) = e_o(z)$. For $t \geq 0$, the first t shortcut steps from $2^t n$ are division steps through $2^t n,\ 2^{t-1} n,\ \ldots,\ 2n,\ n$. We claim that none of the states preceding n is equal to z. If $z > 2$ and $z = 2^s n$ for some $s \geq 1$, then the trajectory of z would reach n after s division steps. Since $n \in \mathcal{A}_z$, the trajectory of n returns to z, making z periodic. This contradicts the admissibility of the tail. If $z = 1$, the equality $1 = 2^s n$ is impossible for positive integers n and $s \geq 1$.

Thus the first-arrival path from $2^t n$ to z consists of these t initial division steps followed by $\mathcal{F}(n, z)$. Consequently, $E_z(2^t n) = e_o(z) + t$ and $O_z(2^t n) = o$. Taking $t = e - e_o(z)$ realizes every division count $e \geq e_o(z)$ at relative odd count o. □

In analogy with the operation-count cells for complete first-arrival paths to 1 in Definition 4.7, for nonnegative integers e and o define the tail-relative operation-count cell above z by

$$C_{e,o}(z) := \{n \in \mathcal{A}_z : E_z(n) = e,\ O_z(n) = o\}.$$

Thus $C_{e,o}(z)$ consists of the sources whose first-arrival paths to z have division count e and odd count o. We call this cell *occupied* when $C_{e,o}(z) \neq \{\}$.

The global record arguments use four orderings among sources with a fixed value of H, E, O, or λ. The fixed-O and fixed-λ orders appear in Lemma 6.2 and Theorem 4.8, while the fixed-E and fixed-H orders are established in the proof of Theorem 6.3. Because paths above z have a common endpoint and satisfy the same four-thirds residue bound, the same orderings hold for the tail-relative operation-count cells.

Lemma 6.13 (Same-Tail Operation-Count Orders)**.** *For $x, y \in \mathcal{A}_z$, the following strict orders hold:*

- *if* $H_z(x) = H_z(y)$ *and* $O_z(y) > O_z(x)$*, then* $y < x$*;*
- *if* $E_z(x) = E_z(y)$ *and* $O_z(y) > O_z(x)$*, then* $y < x$*;*
- *if* $O_z(x) = O_z(y)$ *and* $E_z(y) < E_z(x)$*, then* $y < x$*;*
- *if* $\lambda_z(x) = \lambda_z(y)$ *and* $O_z(y) < O_z(x)$*, then* $y < x$*.*

Proof. The four comparisons are the tail-relative versions of the global operation-count orders used earlier. For every $n \in \mathcal{A}_z$, the first-arrival path $\mathcal{F}(n, z)$ satisfies the finite-path four-thirds bound. Hence, for any $x, y \in \mathcal{A}_z$,

$$\frac{3}{4} < \frac{\rho_z(x)}{\rho_z(y)} < \frac{4}{3}. \tag{121}$$

Put

$$R = \frac{\rho_z(x)}{\rho_z(y)}.$$

In each of the four cases, let $d \geq 1$ denote the stated positive difference between the relevant operation counts. Equation (120) then gives

$$\frac{y}{x} = \begin{cases} 6^{-d}R, & H_z(x) = H_z(y),\ d = O_z(y) - O_z(x), \\ 3^{-d}R, & E_z(x) = E_z(y),\ d = O_z(y) - O_z(x), \\ 2^{-d}R, & O_z(x) = O_z(y),\ d = E_z(x) - E_z(y), \\ \left(\frac{3}{4}\right)^d R, & \lambda_z(x) = \lambda_z(y),\ d = O_z(x) - O_z(y). \end{cases}$$

Using $R < 4/3$, the first three expressions satisfy

$$6^{-d}R < \frac{2}{9}, \qquad 3^{-d}R < \frac{4}{9}, \qquad 2^{-d}R < \frac{2}{3}.$$

For the fixed-charge case,

$$\left(\frac{3}{4}\right)^d R < \left(\frac{3}{4}\right)^d \frac{4}{3} \leq 1.$$

The final inequality is still strict when $d = 1$ because $R < 4/3$. Thus $y/x < 1$ in every case, and therefore $y < x$. □

Theorem 6.14 (Same-Tail Record Hierarchy)**.** *For every admissible tail z,*

$$\mathrm{CR}_\lambda^{\leq 0}(z) \subseteq \mathrm{CR}_O(z) \subseteq \mathrm{CR}_E(z) \subseteq \mathrm{CR}_H(z), \tag{122}$$

and

$$\mathrm{DR}_\lambda(z) \subseteq \mathrm{DR}_O(z) \subseteq \mathrm{DR}_E(z) = \mathrm{DR}_H(z). \tag{123}$$

The nonnegative relative charge-class records are

$$c_\lambda(q; z) = 2^q z, \qquad q \geq 0. \tag{124}$$

The intersections of the two relative hierarchies satisfy

$$\mathrm{DR}_O(z) = \mathrm{CR}_O(z) \cap \mathrm{DR}_H(z), \tag{125}$$

and

$$\begin{aligned} \mathrm{DR}_\lambda(z) &= \mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{DR}_O(z) \\ &= \mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{DR}_E(z) \\ &= \mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{DR}_H(z). \end{aligned} \tag{126}$$

Proof. For every realized relative odd count o, put

$$b_o(z) := c_O(o; z), \qquad m_o(z) := e_o(z) + o, \qquad q_o(z) := e_o(z) - 2o.$$

At a fixed relative odd count, Lemma 6.13 shows that a smaller division count gives a smaller source. Since $e_o(z)$ is the first division count occurring at odd count o, the least source $b_o(z)$ must occur at that division count. Therefore $E_z\big(b_o(z)\big) = e_o(z)$, and hence $H_z\big(b_o(z)\big) = m_o(z)$ and $\lambda_z\big(b_o(z)\big) = q_o(z)$.

By Lemma 6.12, the first division counts strictly increase. Since they are integers, $e_{o+1}(z) - e_o(z) \ge 1$ whenever $o + 1$ is realized. It follows that

$$m_{o+1}(z) - m_o(z) = e_{o+1}(z) - e_o(z) + 1 \ge 2,$$

so the first relative heights also strictly increase. Similarly,

$$q_{o+1}(z) - q_o(z) = e_{o+1}(z) - e_o(z) - 2 \ge -1.$$

Therefore $q_{o+1}(z) \ge q_o(z) - 1$. Since the relative charges are integers, whenever the sequence decreases, it decreases by exactly one, i.e. $q_{o+1}(z) < q_o(z)$ implies that $q_{o+1}(z) = q_o(z) - 1$. In particular, the sequence cannot skip a relative charge value when it decreases.

We now establish the relative O-, E-, and height-class relations. Since $E_z\big(b_o(z)\big) = e_o(z)$ and $O_z\big(b_o(z)\big) = o$, the source $b_o(z)$ belongs to $C_{e_o(z),o}(z)$, so this cell is occupied.

Now let $j > o$ be a realized relative odd count. By Lemma 6.12, $e_j(z) > e_o(z)$. Because $e_j(z)$ is the least relative division count attained by any source with relative odd count j, no source in $\mathcal{A}_z$ can have both $E_z(n) = e_o(z)$ and $O_z(n) = j$. Equivalently, $C_{e_o(z),j}(z) = \{\}$. Thus, among the sources with relative division count $e_o(z)$, the greatest relative odd count is o. By the fixed-E_z order in Lemma 6.13, the least source with relative division count $e_o(z)$ therefore belongs to $C_{e_o(z),o}(z)$. Since $b_o(z) = c_O(o; z)$ is the least source in $\mathcal{A}_z$ with relative odd count o, it is also the least member of this cell. Consequently, $c_O(o; z) = c_E(e_o(z); z)$.

The same argument applies at height $m_o(z)$. If $j > o$ is realized, then $m_j(z) > m_o(z)$, so no greater odd count is occupied at height $m_o(z)$. The fixed-H_z order therefore gives $c_O(o; z) = c_H(m_o(z); z)$. Hence $c_O(o; z) = c_E(e_o(z); z) = c_H(m_o(z); z)$, which proves $\mathrm{CR}_O(z) \subseteq \mathrm{CR}_E(z)$.

To obtain the full inclusion $\mathrm{CR}_E(z) \subseteq \mathrm{CR}_H(z)$, fix a relative division count e and let $o = \max\{j : j \text{ is realized and } e_j(z) \le e\}$. This maximum exists because $e_0(z) = 0$ and the strictly increasing nonnegative integers $e_j(z)$ satisfy $e_j(z) \ge j$. By Lemma 6.12, the cell $C_{e,o}(z)$ is occupied. No greater realized odd count is occupied at division count e, by the maximality of o. The fixed-E_z order therefore gives $c_E(e; z) = \min C_{e,o}(z)$. Put $h = e + o$. The same relative cell can be written as $C_{e,o}(z) = C_{h-o,o}(z)$. If $j > o$ is realized, then $e_j(z) > e$, and therefore $m_j(z) = e_j(z) + j > e + o = h$. Thus no greater odd count is occupied at height h. The fixed-H_z order gives

$$c_H(h; z) = \min C_{h-o,o}(z) = \min C_{e,o}(z) = c_E(e; z).$$

Therefore $\mathrm{CR}_E(z) \subseteq \mathrm{CR}_H(z)$.

We next identify the relative charge classes. Since $e_0(z) = 0$, we have $q_0(z) = 0$. Let $q < 0$ be a realized relative charge. We first show that the set

$$\{j \ge 0 : q_j(z) \le q\}$$

is nonempty. Because q is realized, choose $n \in \mathcal{A}_z$ with $\lambda_z(n) = q$ and put $j = O_z(n)$. Since $\lambda_z(n) = E_z(n) - 2O_z(n)$, we have $E_z(n) = q + 2j$. By definition, $e_j(z)$ is the least relative division count attained by a source having relative odd count j. Since n is one such source, $e_j(z) \le E_z(n) = q + 2j$. Therefore $q_j(z) = e_j(z) - 2j \le q$, which proves that the displayed set is nonempty.

Let o be the least realized odd count satisfying $q_o(z) \le q$. A relative charge-q cell at odd count j has division count $q + 2j$. By Lemma 6.12, it is occupied exactly when $q + 2j \ge e_j(z)$, or equivalently when $q \ge q_j(z)$. Thus o is the least odd count at which a relative charge-q cell is occupied.

Because $q_0(z) = 0 > q$, we have $o \ge 1$. The minimality of o gives $q_{o-1}(z) > q$. Since these quantities are integers, $q_{o-1}(z) \ge q + 1$. The bound on the downward changes gives $q_o(z) \ge q_{o-1}(z) - 1 \ge q$. Together with $q_o(z) \le q$, this proves $q_o(z) = q$. The first occupied relative charge-q cell is therefore $C_{q+2o,o}(z) = C_{e_o(z),o}(z)$. By the fixed-$\lambda_z$ order, the least source of charge q occurs in this first occupied odd-count cell. Its least member is $b_o(z)$, so $c_\lambda(q; z) = c_O(o; z)$.

For $q = 0$, equation (124), proved below, gives $c_\lambda(0; z) = z$. The zero-length path from z to itself also shows that $c_O(0; z) = z$. Thus every nonpositive relative charge-class record is an O-class record, proving $\mathrm{CR}_\lambda^{\le 0}(z) \subseteq \mathrm{CR}_O(z)$. Together with the preceding class inclusions, this proves (122).

We now prove (124). Let $q \ge 0$. By Lemma 6.12, prepending q division steps to the zero-length path at z gives a first-arrival path from $2^q z$ to z. Its relative counts are $E_z(2^q z) = q$ and $O_z(2^q z) = 0$, so $\lambda_z(2^q z) = q$.

Conversely, if $\lambda_z(n) = q$, then Corollary 4.6 gives $q \le \lfloor \log_2(n/z) \rfloor$. Hence $n \ge 2^q z$. Therefore $c_\lambda(q; z) = 2^q z$, as asserted.

We next prove the delay-record hierarchy. Let $n \in \mathrm{DR}_\lambda(z)$, and let $m < n$ belong to $\mathcal{A}_z$. Since n is a lower relative-charge record, $\lambda_z(m) > \lambda_z(n)$. Suppose that $O_z(m) \ge O_z(n)$. Writing the same-tail ratio in terms of the charge gives

$$\frac{m}{n} = 2^{\lambda_z(m) - \lambda_z(n)} \left(\frac{4}{3}\right)^{O_z(m) - O_z(n)} \frac{\rho_z(n)}{\rho_z(m)}.$$

The first factor is at least 2, the second is at least 1, and $\rho_z(n)/\rho_z(m) > 3/4$. Therefore

$$\frac{m}{n} > 2 \cdot 1 \cdot \frac{3}{4} = \frac{3}{2} > 1,$$

contradicting $m < n$. Hence $O_z(m) < O_z(n)$ for every smaller $m \in \mathcal{A}_z$, and $\mathrm{DR}_\lambda(z) \subseteq \mathrm{DR}_O(z)$.

Now let $n \in \mathrm{DR}_O(z)$. If some $m < n$ in $\mathcal{A}_z$ satisfied $E_z(m) \ge E_z(n)$, then the O_z-delay property would give $O_z(m) \le O_z(n) - 1$. Equation (120) would then imply

$$\begin{aligned} \frac{n}{m} &= 2^{E_z(n) - E_z(m)}\, 3^{O_z(m) - O_z(n)}\, \frac{\rho_z(m)}{\rho_z(n)} \\ &< 1 \cdot \frac{1}{3} \cdot \frac{4}{3} = \frac{4}{9} < 1, \end{aligned}$$

again contradicting $m < n$. Thus $E_z(m) < E_z(n)$ for every smaller $m \in \mathcal{A}_z$, and $\mathrm{DR}_O(z) \subseteq \mathrm{DR}_E(z)$.

It remains to compare the relative E- and height-delay properties. Suppose first that $n \in \mathrm{DR}_H(z)$. If a smaller $m \in \mathcal{A}_z$ satisfied $E_z(m) \ge E_z(n)$, then $H_z(n) > H_z(m)$ would force $O_z(n) \ge O_z(m) + 1$. Consequently,

$$\frac{n}{m} = 2^{E_z(n) - E_z(m)}\, 3^{O_z(m) - O_z(n)}\, \frac{\rho_z(m)}{\rho_z(n)} < 1 \cdot \frac{1}{3} \cdot \frac{4}{3} = \frac{4}{9} < 1,$$

a contradiction. Hence every relative height-delay record is an E-delay record.

Conversely, suppose that $n \in \mathrm{DR}_E(z)$ but that some $m < n$ in $\mathcal{A}_z$ satisfies $H_z(m) \ge H_z(n)$. Put $a = E_z(n) - E_z(m) \ge 1$. The height inequality gives $O_z(m) - O_z(n) \ge a$. By Lemma 6.12, prepending a

division steps to the path from m to z produces a source $p = 2^a m \in \mathcal{A}_z$ with $E_z(p) = E_z(m) + a = E_z(n)$. Moreover, equation (120) gives

$$\frac{p}{n} = \frac{2^a m}{n} = 3^{O_z(n)-O_z(m)} \frac{\rho_z(n)}{\rho_z(m)}.$$

Since $O_z(n) - O_z(m) \le -a$,

$$\frac{p}{n} < \frac{1}{3^a} \cdot \frac{4}{3} \le \frac{4}{9} < 1.$$

Thus $p < n$ while $E_z(p) = E_z(n)$, contradicting the strict E_z-delay property of n. Therefore $\mathrm{DR}_E(z) = \mathrm{DR}_H(z)$. This completes the proof of (123).

We finish with the intersection identities. Every relative O-delay record is an O-class record and, by (123), a relative height-delay record. Hence

$$\mathrm{DR}_O(z) \subseteq \mathrm{CR}_O(z) \cap \mathrm{DR}_H(z).$$

Conversely, suppose $n = c_O(o; z) \in \mathrm{DR}_H(z)$. A smaller source cannot have relative odd count o, by the definition of $c_O(o; z)$. If some $m < n$ in $\mathcal{A}_z$ had $O_z(m) = j > o$, then $c_O(j; z) \le m < n$. But $H_z\big(c_O(j; z)\big) = m_j(z) > m_o(z) = H_z(n)$, contradicting the relative height-delay property of n. Therefore $n \in \mathrm{DR}_O(z)$, proving (125).

Let $a = \min \mathcal{A}_z$. Since $z \in \mathcal{A}_z$, we have $a \le z$. If $a = z$, then $\lambda_z(a) = 0$. If $a < z$, Corollary 4.6 gives

$$\lambda_z(a) \le \lfloor \log_2(a/z) \rfloor < 0.$$

Thus the initial relative delay-record source has nonpositive charge. Every later relative λ-delay record has still lower charge, and every relative delay record is the class record at the value it attains. It follows that

$$\mathrm{DR}_\lambda(z) \subseteq \mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{DR}_O(z).$$

Conversely, suppose $n = c_\lambda(q; z) \in \mathrm{DR}_O(z)$ with $q \le 0$. If $q < 0$, write $n = c_O(o; z)$ and $q_o(z) = q$, as established above. The index o is the least odd count for which $q_o(z) \le q$. If $m < n$ belongs to $\mathcal{A}_z$, then the O_z-delay property gives $j = O_z(m) < o$. Hence $q_j(z) > q$. Since $e_j(z)$ is the first division count at odd count j,

$$\lambda_z(m) = E_z(m) - 2j \ge e_j(z) - 2j = q_j(z) > q.$$

Thus $n \in \mathrm{DR}_\lambda(z)$.

If $q = 0$, then $n = c_\lambda(0; z) = z$ by (124). Since $O_z(z) = 0$, the assumption $z \in \mathrm{DR}_O(z)$ implies that no smaller source belongs to $\mathcal{A}_z$; otherwise its nonnegative odd count could not be strictly less than 0. Hence $z = \min \mathcal{A}_z$, so z is also the initial relative λ-delay record. Therefore $\mathrm{DR}_\lambda(z) = \mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{DR}_O(z)$.

Finally, by (122), $\mathrm{CR}_\lambda^{\le 0}(z) \subseteq \mathrm{CR}_O(z)$. Using (125),

$$\begin{aligned} \mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{DR}_H(z) &= \mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{CR}_O(z) \cap \mathrm{DR}_H(z) \\ &= \mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{DR}_O(z) \\ &= \mathrm{DR}_\lambda(z). \end{aligned}$$

Since $\mathrm{DR}_E(z) = \mathrm{DR}_H(z)$, we also have $\mathrm{CR}_\lambda^{\le 0}(z) \cap \mathrm{DR}_E(z) = \mathrm{DR}_\lambda(z)$. This proves (126). □

Unlike the global record chains (111) and (114), the same-tail chains (122) and (123) are stated with non-strict inclusions. The strictness of the global chains was established using specific starting integers belonging to one global record set but not another. After the comparison is restricted to $\mathcal{A}_z$, those integers

are available only if their trajectories reach the chosen tail z. The global examples therefore do not establish that the same-tail inclusions are proper for every admissible tail.

Although strictness need not carry over to an arbitrary tail, a global record that does reach z retains the corresponding record property when the comparison is restricted to sources in $\mathcal{A}_z$. The next corollary makes this relation precise.

Corollary 6.15 (Global Records Restrict to a Common Tail)**.** *Let z be an admissible tail that terminates, let $X \in \{H, E, O, \lambda\}$, and suppose that $n \in \mathcal{A}_z$. If $n = c_X(k)$ is a global X-class record, then*

$$n = c_X\big(k - X(z); z\big).$$

If $n \in \mathrm{DR}_X$ is a global X-delay record, then $n \in \mathrm{DR}_X(z)$.

Proof. Since z terminates, every source in $\mathcal{A}_z$ also terminates. For any $m \in \mathcal{A}_z$, appending the fixed first-arrival path from z to 1 to the path $\mathcal{F}(m, z)$ gives the complete first-arrival path from m to 1. Consequently,

$$E(m) = E_z(m) + E(z), \quad O(m) = O_z(m) + O(z),$$
$$H(m) = H_z(m) + H(z), \quad \lambda(m) = \lambda_z(m) + \lambda(z).$$

Thus, for every $X \in \{H, E, O, \lambda\}$,

$$X_z(m) = X(m) - X(z). \tag{127}$$

Suppose first that $n = c_X(k)$ is a global X-class record. Equation (127) gives $X_z(n) = k - X(z)$. If some $m < n$ in $\mathcal{A}_z$ had the same relative value, then $X_z(m) = k - X(z)$. Adding $X(z)$ to both sides would give $X(m) = k = X(n)$, contradicting the fact that $n = c_X(k)$ is the least terminating integer with X-value k. Therefore n is the least source in $\mathcal{A}_z$ with relative value $k - X(z)$, and hence $n = c_X\big(k - X(z); z\big)$.

Now suppose that $n \in \mathrm{DR}_X$ is a global delay record. First let $X \in \{H, E, O\}$. If some $m < n$ in $\mathcal{A}_z$ satisfied $X_z(m) \geq X_z(n)$, then (127) would give $X(m) \geq X(n)$. This contradicts the defining inequality $X(m) < X(n)$ for every smaller terminating integer m. Thus $X_z(m) < X_z(n)$ for every smaller $m \in \mathcal{A}_z$, so $n \in \mathrm{DR}_X(z)$.

For $X = \lambda$, suppose that some $m < n$ in $\mathcal{A}_z$ satisfied $\lambda_z(m) \leq \lambda_z(n)$. Adding the fixed value $\lambda(z)$ would give $\lambda(m) \leq \lambda(n)$. This contradicts the defining inequality $\lambda(m) > \lambda(n)$ for every smaller terminating integer m. Therefore $\lambda_z(m) > \lambda_z(n)$ for every smaller $m \in \mathcal{A}_z$, and hence $n \in \mathrm{DR}_\lambda(z)$. □

The corollary concerns the starting integers that retain their record properties, not their ordinal positions in the delay lists. Restricting the global delay list to $\mathcal{A}_z$ can remove earlier global records that do not reach z. It can also produce tail-relative delay records that are not global delay records. Either change can alter the ordinal position of a retained record.

The two implications in Corollary 6.15 also cannot generally be reversed. The tail-relative record definitions compare only sources belonging to $\mathcal{A}_z$. A smaller positive integer that does not reach z is excluded from the tail-relative comparison, even though it may prevent a source from having the corresponding global record property.

For example, take $z = 13$. A first-arrival path to 13 has relative height zero only when it has length zero. Therefore $c_H(0; 13) = 13$. On the other hand, $12 \notin \mathcal{A}_{13}$, but $12 < 13$ and $H(12) = H(13) = 9$. Hence $c_H(9) \leq 12 < 13$, so 13 is not the global height-class record at height 9.

Similarly, $7 \to 11 \to 17 \to 26 \to 13$, and no positive integer smaller than 7 reaches 13. Hence $\min \mathcal{A}_{13} = 7$, so 7 is the initial relative λ-delay record above 13. On the other hand, $\lambda(1) = 0 < 1 = \lambda(7)$, so 7 is not a global λ-delay record. Thus neither implication in Corollary 6.15 has a converse in general.

Global and Same-Tail Record Diagrams. The record sets for complete terminating trajectories satisfy

$$\begin{array}{ccccccc} \mathrm{DR}_\lambda & \subsetneq & \mathrm{DR}_O & \subsetneq & \mathrm{DR}_E & = & \mathrm{DR}_H \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{CR}_\lambda^{\le 0} & \subsetneq & \mathrm{CR}_O & \subsetneq & \mathrm{CR}_E & \subsetneq & \mathrm{CR}_H. \end{array} \tag{128}$$

The horizontal relations are Theorems 6.3 and 6.5. The vertical inclusions express the fact that every delay-record starting integer is the class record at the value it attains. More precisely, for every $X \in \{H, E, O, \lambda\}$, $d_X(r) = c_X\big(X(d_X(r))\big)$ whenever $d_X(r)$ is defined.

Each entry in the diagram denotes a set of starting integers. In particular, a vertical arrow does not assert that $d_X(r) = c_X(r)$: the index r in $d_X(r)$ is an ordinal position in a delay list, whereas the argument of c_X is an attained value of X. The exact intersections between the two rows are given in Corollary 6.6.

For first-arrival paths to an admissible common tail z, Theorem 6.14 gives the corresponding diagram

$$\begin{array}{ccccccc} \mathrm{DR}_\lambda(z) & \subseteq & \mathrm{DR}_O(z) & \subseteq & \mathrm{DR}_E(z) & = & \mathrm{DR}_H(z) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{CR}_\lambda^{\le 0}(z) & \subseteq & \mathrm{CR}_O(z) & \subseteq & \mathrm{CR}_E(z) & \subseteq & \mathrm{CR}_H(z). \end{array} \tag{129}$$

Here the vertical arrows similarly mean that $d_X(r; z) = c_X\big(X_z(d_X(r; z)); z\big)$ whenever $d_X(r; z)$ is defined. The exact intersections of these two rows are given by (125) and (126).

When $z = 1$, one has $\mathcal{A}_1 = \mathcal{T}$, and the tail-relative counts and record sets reduce to their global counterparts. Thus the sets in (129) become those in (128). The explicit integers used in Theorems 6.3 and 6.5 then sharpen the corresponding global inclusions from non-strict to strict.

Examples and Strict Containments. The first terms of the three global delay-record sets are

$$\begin{aligned} \mathrm{DR}_H = \mathrm{DR}_E &= \{1, 2, 3, 6, 7, 9, 18, 25, 27, 54, 73, 97, 129, 171, 231, 313, 327, 649, 703, 871, \ldots\}, \\ \mathrm{DR}_O &= \{1, 3, 7, 9, 25, 27, 73, 97, 129, 171, 231, 313, 327, 703, 871, 1161, 2463, \ldots\}, \\ \mathrm{DR}_\lambda &= \{1, 27, 327, 703, 871, 2463, 2919, 3711, 6171, 13255, \ldots\}. \end{aligned}$$

Reading each list in increasing order, the displayed terms show that the λ-delay list is a subsequence of the O-delay list, which is itself a subsequence of the common E-delay and height-delay list, as asserted by the upper row of (128). For comparison, the initial values of the E- and O-class-record functions, listed in increasing order of their class parameters, are

$$\begin{aligned} (c_E(e))_{e\ge 0} &= (1, 2, 4, 8, 5, 3, 6, 12, 24, 17, 11, 7, 14, 9, 18, 36, 25, 49, 33, \ldots), \\ (c_O(o))_{o\ge 0} &= (1, 5, 3, 17, 11, 7, 9, 25, 33, 43, 57, 39, \ldots). \end{aligned}$$

Equation (109), together with the strict increase of the indices e_o, shows that the O-class sequence occurs as a subsequence of the E-class sequence in the same order. Lemma 6.2 also shows that every O-class record is odd. However, not every odd-valued term of the E-class sequence is an O-class record. For example,

$$c_E(17) = 49, \qquad O(49) = 7, \qquad c_O(7) = 25.$$

Thus one cannot recover the O-class sequence merely by retaining all odd-valued terms of the E-class sequence.

The same delay-record containment pattern is visible above the admissible tail $z = 13$. Here $\min \mathcal{A}_{13} = 7$, and

$$\begin{aligned} \mathrm{DR}_H(13) = \mathrm{DR}_E(13) &= \{7, 9, 18, 25, 33, 39, 78, 105, 123, 169, 219, 295, 379, 447, 703, \ldots\}, \\ \mathrm{DR}_O(13) &= \{7, 9, 25, 33, 39, 105, 123, 169, 219, 295, 379, 447, 703, \ldots\}, \\ \mathrm{DR}_\lambda(13) &= \{7, 447, 703, 2463, 2919, \ldots\}. \end{aligned}$$

Table 11 gives small examples showing that every containment in the global diagram is strict. It also shows that all the corresponding containments are strict for the particular tail $z = 13$. In each entry, the displayed integer belongs to the set on the right of the indicated containment but not to the set on the left.

Table 11: Small examples showing that the horizontal and vertical containments in the two record diagrams are strict. The rightmost column concerns the particular admissible tail $z = 13$; the same-tail containments need not be strict for every admissible tail.

Containment	Global Example	Example Above $z = 13$
$\mathrm{DR}_\lambda \subsetneq \mathrm{DR}_O$	3	9
$\mathrm{DR}_O \subsetneq \mathrm{DR}_E = \mathrm{DR}_H$	2	18
$\mathrm{CR}_\lambda^{\leq 0} \subsetneq \mathrm{CR}_O$	5	9
$\mathrm{CR}_O \subsetneq \mathrm{CR}_E$	2	14
$\mathrm{CR}_E \subsetneq \mathrm{CR}_H$	10	22
$\mathrm{DR}_\lambda \subsetneq \mathrm{CR}_\lambda^{\leq 0}$	31	11
$\mathrm{DR}_O \subsetneq \mathrm{CR}_O$	5	11
$\mathrm{DR}_E \subsetneq \mathrm{CR}_E$	4	11
$\mathrm{DR}_H \subsetneq \mathrm{CR}_H$	4	11

For instance, $31 = c_\lambda(-11)$, but $27 < 31$ and $\lambda(27) = -12$, so $31 \notin \mathrm{DR}_\lambda$. Above the tail 13, the source 11 is a class record for each of its relative values, but the smaller source 9 has greater relative height, division count, and odd count, while the still smaller source 7 has lower relative charge. Hence 11 lies in all four lower-row sets in (129) and in none of the corresponding upper-row sets.

7 Conclusion and Structural Perspective

The central result of this paper is the pair of uniform finite-path bounds

$$\mathcal{R}_\mathcal{F} < 1, \qquad \rho_\mathcal{F} < \frac{4}{3},$$

valid for every finite nonrepeating shortcut path $\mathcal{F}$ whose input states are greater than 1. The proof combines canonical-endpoint counting, verified finite height-delay records and residue bounds, and the finite gap-count estimate proved in Appendix 1. Passing to successively longer initial portions of a forward orbit gives

$$\rho^\circ(n) < \frac{4}{3}, \qquad \mathcal{R}^\circ(n) < 1$$

for every positive starting integer n, when each odd value greater than 1 is counted once. The bounds $\rho^\circ(n) < 4/3$ and $\mathcal{R}^\circ(n) < 1$ apply to terminating and nonterminating orbits alike, including cycles. If n

terminates, its complete first-arrival trajectory has no repeated input states, so the once-counted quantities agree with the complete first-arrival quantities. Consequently,

$$\rho(n) < \frac{4}{3}, \qquad \mathcal{R}(n) < 1.$$

Uniform finiteness proves Roosendaal's Weak Residue Conjecture [14], while the four-thirds bound supplies the stronger inequality $\rho(n) < 2$ used by Luo [10] and proves Ruggiero's proposed height ceiling formula [16], with a nearest-integer refinement. For finite nonrepeating paths that allow 1 as an input, the weaker bound $\rho_{\mathcal{F}} < 16/9$ proves the positive $q = 3$ cases of Luo's Conjectures 5.6(a) and 5.8(a) [10].

For a finite nonrepeating path $\mathcal{F} = (a, \ldots, z)$ whose input states are greater than 1, the residue identity gives

$$3^{O_{\mathcal{F}}}\left(\frac{a}{z}\right) \le 2^{E_{\mathcal{F}}} < \frac{4}{3}\,3^{O_{\mathcal{F}}}\left(\frac{a}{z}\right),$$

or equivalently,

$$O_{\mathcal{F}}\log_2 3 + \log_2(a/z) \le E_{\mathcal{F}} < O_{\mathcal{F}}\log_2 3 + \log_2(a/z) + \log_2(4/3).$$

The above interval for $E_{\mathcal{F}}$ has length $\log_2(4/3) < 1/2$. Thus the residue correction is too small to conceal an integer operation count: for fixed endpoints, either $E_{\mathcal{F}}$ or $O_{\mathcal{F}}$ determines the other exactly, without any assumption about the orbit beyond z. In particular, for a specified terminating integer n, any one of $H(n)$, $E(n)$, and $O(n)$ determines the other two.

The same control compares paths with different starting values. The ratio $W_z(a, b)$ restricts the even–odd step discrepancy of two paths that first arrive at the same endpoint. For adjacent terminating integers it gives a nearest-integer relation between ΔE and ΔO, apart from the stated small exceptions. If two finite nonrepeating paths with input states greater than 1 start at adjacent integers, have a common endpoint, and have equal height, then both operation counts agree. Consequently, an equal-height streak is a consecutive block in a single operation-count class. More generally, if $a < b$, $H(a) = H(b)$, and $d = O(a) - O(b)$, then

$$\frac{3}{4}\,6^d < \frac{b}{a} < \frac{4}{3}\,6^d,$$

so the difference in odd counts controls the possible separation of equal-height starting values.

The charge $\lambda = E - 2O$ places the streak and record phenomena in a common framework. Globally, the delay-record and class-record sets satisfy

$$\mathrm{DR}_\lambda \subsetneq \mathrm{DR}_O \subsetneq \mathrm{DR}_E = \mathrm{DR}_H, \qquad \mathrm{CR}_\lambda^{\le 0} \subsetneq \mathrm{CR}_O \subsetneq \mathrm{CR}_E \subsetneq \mathrm{CR}_H.$$

Thus the height-delay records are exactly the division-count delay records, the λ-delay records form a subsequence of the odd-count delay records, and every realized negative charge-class record is also an odd-count, division-count, and height-class record. The corresponding non-strict hierarchies and intersection identities remain valid for first-arrival paths to any admissible common tail. This shows that the global record relations are one instance of a more general same-tail structure.

Finally, the height and charge slice functions G and Q translate these arithmetic restrictions into information about equal-height streaks. Their projected plateaus coincide apart from the stated exception, and, under the stated hypotheses, the first exit from a projected G-plateau is either a power of two or a λ-delay record. The projected exit identifies where a break may occur, while the residual correction determines whether the actual height changes.

Taken together, these results show that the bound $\rho < 4/3$ is not an isolated numerical estimate. It makes the residue correction small enough to force exact operation-count recovery, synchronization of equal-height paths, rigid record hierarchies, and strong restrictions on the formation and breaking of height streaks.

8 Future Work

Considerable computational resources have been invested in computing long streaks of integers with the same Collatz height. Since the publication of [3], further computations using the same verification algorithm have produced longer streaks. In particular, we have verified a streak of length

$$2^{52} - 1 = 4\,503\,599\,627\,370\,495$$

beginning at $2^{3956} + 1$. Many streak verifications are out of reach of current computational resources when computed directly. The work done herein has been instrumental in closing the gap between theory and computation, potentially allowing for a large skip of direct validation of integer heights to determine lengths of streaks.

The unresolved issue from this document is to determine when $\Delta G_r + \Delta\delta_{G,r} \neq 0$ without explicitly computing the relevant quantities for every integer whose trajectory must be checked to verify a streak. Furthermore, with the introduction of the charge λ, current experimental results suggest that breaks in streaks starting at $2^N + 1$ occur at integers n belonging to the set of λ-class records. The theory developed herein is not yet sufficient to prove this statement.

The bound $\rho(n) < 4/3$ for terminating first-arrival trajectories also appears to leave room for improvement. With additional computational data and sharper analytic estimates, it should possible to reduce this bound to $2^{1/3}$ (or lower). The proof found herein was given to ChatGPT and with some extra work, it was shown that $\rho(n) < 1.3$, but the argument was overly complicated and not worth recreating. We have performed extensive numerical research into locating starting integers n for which $\rho(n)$ is reasonably close to $\rho(993)$ outside the well known values given by [14]. These endeavors have led us to believe that $\rho(993)$ is maximal, in agreement with Roosendaal. This should be the next conjecture to be proven, as opposed to decreasing the upper bound on ρ through gradual, discrete, small steps.

Appendix 1. The Four-Thirds Bound

Appendix 1A. The regions and the finite path

We use ρ° and $\mathcal{R}^\circ$ from Definition 2.18. A cycle is counted once, and termination is not assumed. Put

$$M = 2^{32}, \qquad D = 28\,019\,077\,177\,231\,758\,495.$$

Let $\mathcal{F} = (v_0, \ldots, v_N)$ be a finite shortcut path, where $v_{i+1} = T(v_i)$ and the inputs $v_0, \ldots, v_{N-1} > D$ are distinct. Its endpoint v_N need not exceed D. As in Definition 2.5, write

$$\mathcal{R}_{\mathcal{F}} = \sum_{\substack{0 \le i < N \\ v_i \text{ odd}}} \frac{1}{v_i}.$$

We use n for the original starting integer and $x = v_i$ for an input on $\mathcal{F}$. As usual, $j(x) = \lfloor \log_2 x \rfloor$ and $s(x) = s_{j(x)}(x)$ counts the odd inputs before step $j(x)$.

Overview of the Proof. The proof to establish $\rho^\circ(n) < \frac{4}{3}$ for all positive n will be handled on a case-by-case basis. We first use the two established records to account for the trajectory after entry into a checked range.

1. $n < M$. We know that $\rho(n) \le \rho(993)$ by the established record (35). This also bounds the remaining residue after a later entry below M.
2. After the first entry into $[1, D]$, let y be that entry value and z the first odd iterate below M. If the trajectory starts in $[1, D]$, take $y = n$. We know that $H(y) \le 2456$ by the established record (34). Every odd input from y up to, but not including, z contributes at most $1/M$, and there are at most 2456 such inputs. This remains true if the trajectory rises above D again.

Let $\mathcal{R}_{\text{high}}(n)$ be the sum of odd reciprocals before first entry into $[1, D]$. If there is no entry, use the whole once-counted orbit. To bound this sum, we will prove a bound for every finite path $\mathcal{F}$ described above.

Before we enter the full proof, let us give an overview of what we will do. First, we separately bound the odd inputs too close to the end of $\mathcal{F}$ for their canonical blocks to be included. There can also be at most one odd input divisible by 3, so its contribution is easily bounded. These estimates, together with the terms left at the end of the block calculation, are called the *boundary terms*; see (135) and (149). The remaining inputs are divided according to the regions in Figure 1.

3. $x > D$ and $j(x) \le 1000$. This is the most difficult region. We call the bound for the remaining positive block values in this region the *block estimate*. It is proved in Appendix 1C, using the computation in Appendix 1D.
4. $j(x) > 1000$. The *row and column estimates* in Appendix 1B use geometric series, very similar to (28) and (29) in the proof of uniform boundedness. We break this region into three subregions.
   (a) $1 \le s(x) \le 600$. The number of possible odd inputs in each row is small enough, and j large enough, that a geometric series bound suffices for our purpose.
   (b) $600 < s(x) \le 3j(x)/5$. Similarly, the number of odd inputs in each row is small enough relative to the decrease of their reciprocal weights that a geometric series bound suffices.
   (c) $3j(x)/5 < s(x) \le j(x)$. The number of odd inputs in a row can cause an issue here. However, there are few enough in each column j that a different geometric series bound suffices.

For the block estimate, partition $\mathcal{F}$ into consecutive *blocks*, each consisting of one or more successive shortcut steps. For example, when $N > 8$, one possible partition has endpoints v_0, v_1, v_4, v_8, v_N. A block

beginning at x and containing r steps is assigned the balance $\mathcal{B}(x, r)$. This balance is defined so that the correction terms at a shared endpoint cancel when consecutive blocks are added. Consequently, the sum of the odd reciprocals along the path can be expressed as a sum of block balances together with the two exterior endpoint terms, as shown in (148).

The telescoping identity allows blocks with negative balance to be omitted when forming an upper bound. Lemma 1D.1 gives a test, depending only on a cut gap bracket, that can certify that an initial portion of the inspected path has negative balance. When the test certifies such an initial portion, the first one so certified is chosen as the next block, and the construction resumes at its endpoint. When the test certifies no inspected initial portion as negative, the next block consists of one shortcut step; its balance is at most $15/(4x)$. Repeating this rule from each new endpoint partitions the remainder of the path into consecutive blocks.

The finite computation used to bound the surviving one-step blocks inspects

$$m(x) := \lfloor \log_2(x/12) \rfloor$$

shortcut steps from a candidate start x. This choice places x in the shell

$$12 \cdot 2^{m(x)} < x < 24 \cdot 2^{m(x)}$$

and therefore permits the representation

$$x = b2^{m(x)} + u, \qquad 12 \le b \le 23, \qquad 0 < u < 2^{m(x)}.$$

It also keeps the inspection length close to the canonical length: $j(x) - m(x) \in \{3, 4\}$. Thus the inspected segment remains inside the retained portion of the path, while the twelve possible leading coefficients b make it possible to bound uniformly the starting integers associated with each gap bracket.

For the region $x > D$ and $j(x) \le 1000$, these relations give $61 \le m(x) \le 997$. The computation extends the count through $m = 1000$ only to simplify the finite sum; the three additional values add nonnegative terms to the upper bound and do not reclassify any input in the regions indexed by j in Figure 1. For every retained candidate start, the m-step endpoint remains above D, while the finite transport estimate gives $D < T^m(x) < 24 \cdot 3^{s_m(x)}$. Since $24 \cdot 3^{37} < D$, this forces $s_m(x) \ge 38$. Accordingly, for each $61 \le m \le 1000$, the computation examines every cut gap bracket whose gaps sum to m and whose length is at least 38. The count C_m retains exactly those brackets for which the gap test certifies no negative initial block. It does not assert that the retained balances are positive.

Finally, the sum of the remaining positive one-step block values on one trajectory is at most the sum over all candidate starts in these shells; see (151). For each counted bracket, the possible starting reciprocals sum to less than $1/(2 \cdot 2^m)$. Fortunately, the bound $15/(4x)$ and the count of the remaining brackets suffice for our purpose. The reader is encouraged to keep in mind that the block calculation uses one trajectory, while the later count allows all possible candidates. This gross overcounting still gives the desired bound.

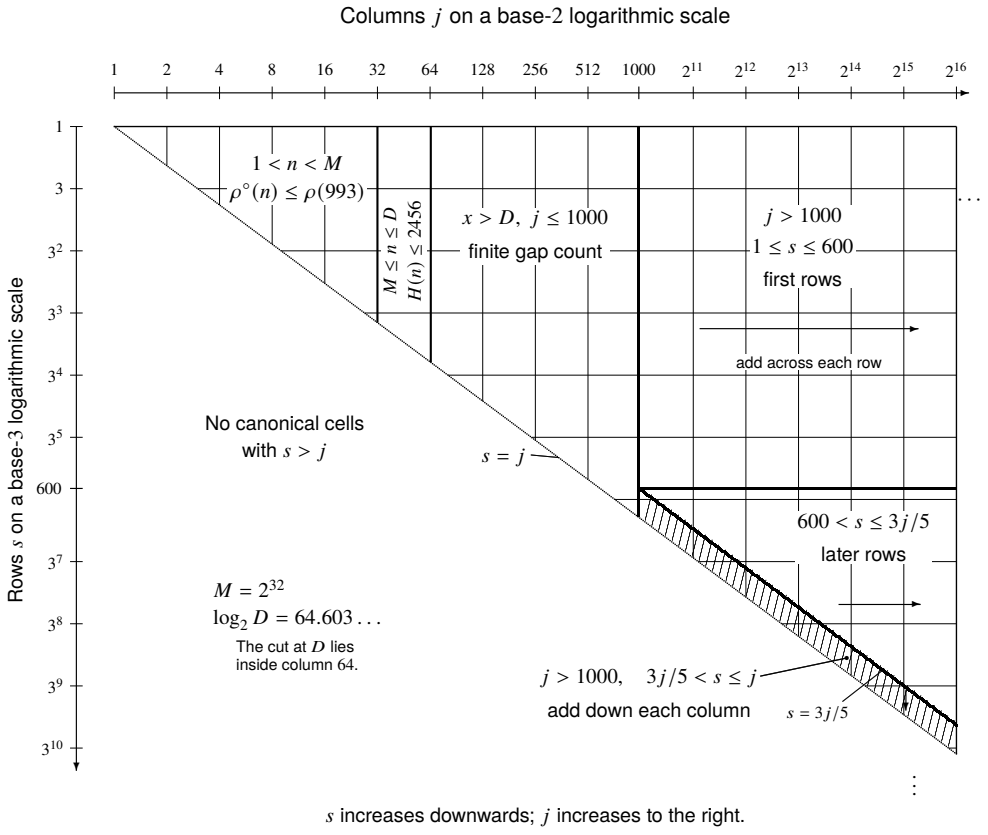


Figure 1. The rows are indexed by s and the columns by j, as in the row and column proof of uniform boundedness. Their spacing is logarithmic. The line $s = 3j/5$ passes through $(j, s) = (1000, 600)$. The first two strips locate a starting or entry value. They do not require its later iterates to stay in that strip. The other four regions describe the estimates for inputs above D after the boundary terms are set aside.

Theorem 1A.1 (Finite Paths Above D)**.** *For every path $\mathcal{F}$ described above,*

$$\mathcal{R}_{\mathcal{F}} < \underbrace{\frac{45}{256}}_{\text{block estimate}} + \underbrace{10^{-6}}_{\text{row and column estimates}} + \underbrace{10^{-15}}_{\text{boundary terms}} < \frac{9}{50}.$$

Reduction to the Stated Bound. The finite records (35) and (34) give

$$\rho(n) \le \rho(993) = \frac{2^{61}}{3^{32} \cdot 993} \qquad (1 \le n < M), \tag{130}$$

$$H(y) \le 2456 \qquad (1 \le y \le D). \tag{131}$$

The height record includes termination. The function H counts ordinary Collatz operations [14, 11]. The portion from y to z in the overview has reciprocal sum at most $2456/M$. Since $\log(1 + a) \le a$ for $a \ge 0$, this gives

$$\rho^\circ(n) \le \rho(993) \exp\left(\frac{2456}{3M}\right) \exp\left(\frac{\mathcal{R}_{\text{high}}(n)}{3}\right). \tag{132}$$

When there is no entry into $[1, D]$, only the last factor is needed, so this inequality still holds. It is enough to show

$$\mathcal{R}_{\text{high}}(n) < \frac{9}{50}, \qquad \rho(993)\exp\left(\frac{9/50 + 2456/M}{3}\right) < \rho(993)e^{61/1000} < \frac{4}{3}. \tag{133}$$

The last numerical comparison is checked in (165).

Where the Finite Path Ends. For $N = 0$ the reciprocal sum is zero, and for $N = 1$ it is less than $1/D$. Assume $N \geq 2$. The endpoint v_N may lie below D, even though every input state of $\mathcal{F}$ exceeds D. Put $Q = N - 1$ and temporarily separate the final step

$$v_Q \longrightarrow v_N.$$

Its odd reciprocal contribution is at most $1/D$. We then partition the shortened path $(v_0, \ldots, v_Q)$. Its input states $v_0, \ldots, v_{Q-1}$ and its endpoint v_Q all exceed D.

Starting from $x = v_i$, the shortened path contains $Q - i$ further steps before reaching v_Q. The condition

$$j(x) < Q - i \qquad \Longleftrightarrow \qquad i + j(x) < Q$$

ensures that the canonical endpoint $T^{j(x)}(x) = v_{i+j(x)}$ is an input state of the shortened path, rather than its endpoint or a state lying beyond it. Define

$$\mathcal{I} = \left\{v_i \mid 0 \leq i < Q,\ v_i \text{ odd},\ 3 \nmid v_i,\ i + j(v_i) < Q\right\}. \tag{134}$$

We call these the *retained inputs*. If $x = v_i \in \mathcal{I}$, then, for every $0 \leq t \leq j(x)$, $i + t \leq i + j(x) < Q$. Consequently,

$$T^t(x) = v_{i+t} > D \qquad (0 \leq t \leq j(x)).$$

Every state used in this conclusion already occurs among the input states of the specified finite path; no assertion is made about the orbit after v_Q. An input satisfying $i + j(x) \geq Q$ is instead bounded separately, regardless of the value of its canonical endpoint. Only the canonical endpoint associated with the starting input x is subject to this retention condition; no corresponding condition is imposed on the canonical endpoints associated with intermediate odd inputs.

Example 1A.2. Take $v_0 = 2^{126} + 19$ and $v_i = T^i(v_0)$ through $N = 131$, so $Q = 130$. Direct iteration gives distinct values above D throughout. Three odd inputs not divisible by 3 illustrate the condition.

Input	Position i	$j(v_i)$	Steps left $Q - i$	Treatment
v_0	0	126	130	Retained
v_5	5	125	125	Bound separately
v_7	7	125	123	Bound separately

For v_0, one has

$$0 + j(v_0) = 126 < 130 = Q,$$

so its canonical endpoint v_{126} is still an input state of the shortened path, and v_0 is retained. For v_5,

$$5 + j(v_5) = 5 + 125 = 130 = Q,$$

so its canonical endpoint is v_Q, the endpoint rather than an input state of the shortened path. It is therefore bounded separately. For v_7,

$$7 + j(v_7) = 7 + 125 = 132 > Q,$$

and its canonical endpoint lies beyond the shortened path and it too is bounded separately. Thus the bound $j(v_i) \leq 1000$ determines which size region contains an input, but the additional inequality $i + j(v_i) < Q$ determines whether that input is retained.

Bound the Inputs Set Aside. For a fixed shell index j, an input satisfying $i + j \geq Q$ lies among the last j input positions before v_Q. There are at most j such positions, and each odd input in shell j has reciprocal below 2^{-j}. There is also at most one odd input divisible by 3, whose reciprocal is below $1/D$. Indeed, an odd step removes that divisibility and halvings do not restore it. Since $x > D$ implies $j(x) \geq 64$,

$$\sum_{\substack{0 \leq i < Q \\ v_i \text{ odd},\ v_i \notin \mathcal{I}}} \frac{1}{v_i} \leq \frac{1}{D} + \sum_{j=64}^{\infty} \frac{j}{2^j} = \frac{1}{D} + \frac{65}{2^{63}} =: b_\partial. \tag{135}$$

The sum follows from identity $\sum_{j=a}^{\infty} j2^{-j} = (a+1)2^{1-a}$. The bound is independent of N and of the values on the path. The computer calculation below needs neither i nor Q. The contribution of these terms to the block bound is included once in B_∂, defined in (149).

Appendix 1B. The three regions with $j > 1000$

For the retained inputs with $j > 1000$, let $N_{j,s}$ count those with $s = s_{j(x)}(x)$, as in Lemma 2.14. Their canonical endpoints are among the distinct inputs of $\mathcal{F}$, so that lemma gives

$$N_{j,s} \leq \binom{j-1}{s-1}, \qquad \sum_j N_{j,s} \leq 2 \cdot 3^{s-1}. \tag{136}$$

Set $N_{j,s} = 0$ outside this range. Each reciprocal in column j is below 2^{-j}. The division used in the earlier row and column proof therefore gives

$$\begin{aligned} \sum_{j=1001}^{\infty} \sum_{s=1}^{j} N_{j,s} 2^{-j} = {} & \underbrace{\sum_{s=1}^{600} \sum_{j=1001}^{\infty} N_{j,s} 2^{-j}}_{\text{first rows}} + \underbrace{\sum_{s=601}^{\infty} \sum_{j=\lceil 5s/3 \rceil}^{\infty} N_{j,s} 2^{-j}}_{\text{later rows}} \\ & + \underbrace{\sum_{j=1001}^{\infty} \sum_{s=\lfloor 3j/5 \rfloor + 1}^{j} N_{j,s} 2^{-j}}_{\text{columns}}. \end{aligned} \tag{137}$$

The Two Row Sums. For $s \leq 600$, give every entry the larger weight 2^{-1001}. For $s > 600$ in the second region, give every entry in row s the larger weight $2^{-5s/3}$. The row bound in (136) gives

$$\underbrace{\sum_{s=1}^{600} \frac{2 \cdot 3^{s-1}}{2^{1001}}}_{\text{first rows}} + \underbrace{\frac{2}{3} \sum_{s=601}^{\infty} r_1^s}_{\text{later rows}} = \frac{3^{600} - 1}{2^{1001}} + \frac{2}{3} \frac{r_1^{601}}{1 - r_1}, \qquad r_1 = \frac{3}{2^{5/3}} < 1. \tag{138}$$

For example, row 600 contributes at most $2 \cdot 3^{599}/2^{1001}$. The later-row sum is the same as (28), now starting at $s = 601$. The condition $j > 1000$ follows there from $s > 600$ and $j \geq 5s/3$.

The Column Sum. For $s > 3j/5$, use (29) without changing its estimate in each column:

$$2^{-j} \sum_{s > 3j/5} N_{j,s} \leq \frac{3}{5} r_2^j, \qquad r_2 = \frac{5}{4} \left(\frac{2}{3} \right)^{3/5} < 1.$$

Compared with (29), only the lower limit of the column summation changes. Summing over $j \geq 1001$ gives

$$\frac{3}{5}\frac{r_2^{1001}}{1-r_2}.$$

Since $r_1 < 19/20$ and $r_2 < 981/1000$, we obtain

$$\begin{aligned}\frac{15}{4}\sum_{\substack{x\in\mathcal{I}\\ j(x)>1000}}\frac{1}{x} &\leq \frac{15}{4}\left(\frac{3^{600}-1}{2^{1001}}+\frac{2}{3}\frac{(19/20)^{601}}{1-19/20}+\frac{3}{5}\frac{(981/1000)^{1001}}{1-981/1000}\right)\\ &< 10^{-6}.\end{aligned} \tag{139}$$

The right-hand side of the first inequality evaluates to $5.42134375718\ldots\times 10^{-7}$, which proves the final numerical bound. The factor $15/4$ converts the reciprocal estimate into a block-balance estimate: by (142), each corresponding one-step block satisfies

$$\mathcal{B}(x,1) \leq \frac{15}{4x}.$$

Appendix 1C. Blocks and the Sum Over Possible Starts

For the retained inputs with $j(x) \leq 1000$, we first use the blocks chosen on $\mathcal{F}$. We then enlarge the remaining positive sum to include all possible starting integers in the same shells.

The Balance of a Block. A block $x, T(x), \ldots, T^r(x)$ has $r \geq 1$ shortcut steps. Its first r values are its inputs. Its endpoint $T^r(x)$ is not an input of that block. Define

$$\mathcal{B}(x,r) = \sum_{\substack{0\leq t<r\\ T^t(x)\text{ odd}}}\frac{1}{T^t(x)}+\frac{8}{x}-\frac{8}{T^r(x)}. \tag{140}$$

If $r, s \geq 1$, then splitting the odd reciprocal sum at step r gives

$$\mathcal{B}(x,r+s) = \mathcal{B}(x,r)+\mathcal{B}(T^r(x),s). \tag{141}$$

Indeed, the term $-8/T^r(x)$ from the first block cancels the term $+8/T^r(x)$ from the second. Repeated use of (141) is the cancellation used in (148). In particular, a block with nonpositive balance may be removed when an upper bound is taken. For even x, $\mathcal{B}(x,1) = -8/x < 0$. For odd $x \geq 21$,

$$0 < \mathcal{B}(x,1) = \frac{9}{x}-\frac{16}{3x+1} = \frac{11x+9}{x(3x+1)} \leq \frac{15}{4x}. \tag{142}$$

Thus the bound $15/(4x)$ applies to the entire balance of the one-step block beginning at x, including its endpoint correction terms, and not merely to the reciprocal term $1/x$.

The Inspected Gap Bracket. For odd $x \geq 25$, put

$$m(x) = \lfloor \log_2(x/12) \rfloor, \qquad 12\cdot 2^{m(x)} < x < 24\cdot 2^{m(x)}. \tag{143}$$

Since $j(x) = \lfloor \log_2 x \rfloor$, the displayed inequalities give

$$m(x)+\log_2 12 < \log_2 x < m(x)+\log_2 24.$$

As $3 < \log_2 12 < 4$ and $4 < \log_2 24 < 5$, it follows that

$$j(x) - m(x) \in \{3, 4\}. \tag{144}$$

Write the gap data of the first $m = m(x)$ shortcut steps starting at x as

$$\mathbf{g}_x = (g_1, \ldots, g_s), \qquad \ell_0 = 0, \qquad \ell_i = g_1 + \cdots + g_i, \qquad \ell_s = m.$$

This cut gap bracket has length $s = s_m(x)$. The first $s - 1$ entries are complete gaps, and the last may be only part of a gap. Each ℓ_i is a number of shortcut steps. A complete gap is one accelerated step. For $x \in \mathcal{I}$, all these steps lie inside the shortened path because $m(x) < j(x)$.

The finite count below uses a zero-or-one function χ that depends only on the gap list $\mathbf{g}_x$. Its formal definition is given in (162), and Lemma 1D.1 proves the implication on which it is based. When all inspected inputs exceed D and $m \le 1000$, its values have the following interpretation:

$$\begin{aligned} \chi(\mathbf{g}_x) = 0 \quad &\Longrightarrow \quad \mathcal{B}(x, \ell_k) < 0 \text{ for some } 1 \le k \le s, \\ \chi(\mathbf{g}_x) = 1 \quad &\Longleftrightarrow \quad \text{the test certifies no such initial block as negative.} \end{aligned}$$

Thus χ acts as the indicator that a cut gap bracket survives the negative-block test: brackets with value 0 are discarded, while those with value 1 are retained for the subsequent upper count. A value of 1 is inconclusive about the actual block balances; it does not assert that any of them is positive. The construction of χ and the proof that $\chi(\mathbf{g}_x) = 0$ certifies a negative initial block are given in Appendix 1D.

Choose the Blocks on the Path. Begin with $u_0 = v_0$. Suppose that the current block start u_i satisfies

$$u_i \in \mathcal{I}, \qquad j(u_i) \le 1000, \qquad \chi(\mathbf{g}_{u_i}) = 0.$$

In this case, let k_i be the least index for which the gap test certifies $\mathcal{B}(u_i, \ell_{k_i}) < 0$. Define the length of the next block and its endpoint recursively by

$$r_i = \begin{cases} \ell_{k_i}, & u_i \in \mathcal{I},\ j(u_i) \le 1000,\ \chi(\mathbf{g}_{u_i}) = 0, \\ 1, & \text{otherwise}, \end{cases}$$

$$u_{i+1} = T^{r_i}(u_i).$$

Here ℓ_{k_i} is a cumulative gap coordinate in the inspected cut gap bracket beginning at u_i.

Apply this rule successively for as long as the current endpoint is different from v_Q. Every selected length r_i is positive, and every selected block lies within the shortened path. The sequence of block endpoints therefore advances along the path and reaches $u_q = v_Q$ after finitely many blocks. These consecutive blocks partition the input states $v_0, \ldots, v_{Q-1}$, so every such input belongs to exactly one block.

When the gap test certifies a negative initial block, the entire block from u_i through its endpoint u_{i+1} is selected, and all odd reciprocals from its input states are included in its balance. The next block begins at u_{i+1}, not at an interior state of the preceding block. In every other case, the one-step block from u_i to $T(u_i)$ accounts for the input u_i, and the construction resumes at $T(u_i)$.

The Two Sets of Starts. For $61 \le m \le 1000$, put

$$\mathcal{A}_m = \{u_h \mid 0 \le h < q,\ u_h \in \mathcal{I},\ j(u_h) \le 1000,\ m(u_h) = m,\ \chi(\mathbf{g}_{u_h}) = 1\}, \tag{145}$$

$$\mathcal{S}_m = \{x \in \mathbb{N} \mid x \text{ odd},\ 12 \cdot 2^m < x < 24 \cdot 2^m,\ 3 \nmid x,\ s_m(x) \ge 38,\ \chi(\mathbf{g}_x) = 1\}. \tag{146}$$

The set $\mathcal{A}_m$ consists of the block starts actually chosen on the fixed path $\mathcal{F}$ for which the gap test finds no negative initial block. The set $\mathcal{S}_m$ is not restricted to the path. It contains all positive integers in the same shell that satisfy the displayed parity, divisibility, odd-input, and gap-test conditions. Thus $\mathcal{S}_m$ is an enlarged set of possible starts.

We verify that every member of $\mathcal{A}_m$ belongs to $\mathcal{S}_m$. Fix $x \in \mathcal{A}_m$. Then $x = u_h$ for some $0 \le h < q$. Since $x \in \mathcal{I}$, Definition (134) gives

$$x \text{ odd}, \qquad 3 \nmid x,$$

and, because x is retained, $T^t(x) > D$ for $0 \le t \le j(x)$. The condition $m(x) = m$ gives

$$12 \cdot 2^m < x < 24 \cdot 2^m.$$

Moreover, (144) gives $m < j(x)$, so the retained-path condition implies $T^m(x) > D$. Finally, $\chi(\mathbf{g}_x) = 1$ follows directly from the definition of $\mathcal{A}_m$.

It remains only to prove that $s_m(x) \ge 38$. Write

$$x = b2^m + u, \qquad 12 \le b \le 23, \qquad 0 < u < 2^m.$$

The same parity induction as in Theorem 2.7 applies with leading coefficient b. Put $u_t = T^t(u)$ and let $s_t(u)$ count its first t odd inputs. For $0 \le t \le m$ it gives

$$T^t(x) = b2^{m-t}3^{s_t(u)} + u_t, \qquad 0 < u_t < 2^{m-t}3^{s_t(u)}.$$

Indeed, for $t < m$ the first term is even, so the two current states have the same parity. The upper bound follows by the induction in Theorem 2.10, starting from $u < 2^m$. In its odd case, use $u_t \le 2^{m-t}3^{s_t(u)} - 1$ before applying $(3u_t + 1)/2$. Thus $s_m(u) = s_m(x)$, and at $t = m$ we obtain

$$T^m(x) = b3^{s_m(x)} + T^m(u), \qquad 0 < T^m(u) < 3^{s_m(x)}.$$

Consequently,

$$D < T^m(x) < 24 \cdot 3^{s_m(x)}.$$

If $s_m(x) \le 37$, then

$$T^m(x) < 24 \cdot 3^{s_m(x)} \le 24 \cdot 3^{37} < D,$$

contrary to $T^m(x) > D$. Hence $s_m(x) \ge 38$.

We have now verified every condition in the definition of $\mathcal{S}_m$. Therefore

$$\mathcal{A}_m \subseteq \mathcal{S}_m. \tag{147}$$

The lower bound $s_m(x) \ge 38$ uses the fact that the retained m-step endpoint satisfies $T^m(x) > D$; it does not follow from $x > D$ alone.

For completeness, $x > D > 24 \cdot 2^{60}$ and $x < 24 \cdot 2^{m(x)}$ imply $m(x) \ge 61$. Also, $j(x) \le 1000$ and $j(x) - m(x) \ge 3$ imply $m(x) \le 997$. Thus $\mathcal{A}_m$ is empty for $998 \le m \le 1000$; retaining those three values in the later finite sum only enlarges the bound.

From All Reciprocals to the Candidate Sum. For the consecutive blocks just chosen,

$$\begin{aligned}
\sum_{\substack{0\le k<Q\\ v_k \text{ odd}}} \frac{1}{v_k} &= \sum_{i=0}^{q-1} \sum_{\substack{0\le t<r_i\\ T^t(u_i) \text{ odd}}} \frac{1}{T^t(u_i)} \\
&= \sum_{i=0}^{q-1} \left(\mathcal{B}(u_i, r_i) - \frac{8}{u_i} + \frac{8}{u_{i+1}} \right) \\
&= \sum_{i=0}^{q-1} \mathcal{B}(u_i, r_i) - \frac{8}{u_0} + \frac{8}{u_q} \qquad (148) \\
&< \frac{8}{D} + \sum_{i=0}^{q-1} \mathcal{B}(u_i, r_i) \\
&\le \frac{8}{D} + \sum_{\substack{0\le i<q\\ \mathcal{B}(u_i,r_i)>0}} \mathcal{B}(u_i, r_i).
\end{aligned}$$

The first equality holds because the chosen blocks are consecutive and partition the input states $v_0, \dots, v_{Q-1}$; hence every odd reciprocal in the shortened path occurs exactly once in the block sums. In the second equality, the definition of $\mathcal{B}(u_i, r_i)$ introduces the endpoint terms

$$-\frac{8}{u_i} + \frac{8}{u_{i+1}}.$$

Because u_{i+1} is both the endpoint of block i and the starting point of block $i+1$, the term $8/u_{i+1}$ from one block cancels the term $-8/u_{i+1}$ from the next. Only the exterior terms $-8/u_0$ and $8/u_q$ remain. Omitting the negative term $-8/u_0$ increases the expression, while $u_q = v_Q > D$ gives $8/u_q < 8/D$. Finally, discarding every nonpositive block balance can only increase the resulting upper bound.

By construction, every block containing more than one step is selected only after the gap test certifies that its balance is negative. A one-step block beginning at an even input also has negative balance, since $\mathcal{B}(x, 1) = -8/x$ in that case. Consequently, every positive balance remaining in (148) comes from a one-step block beginning at an odd input.

These odd one-step starts fall into three groups: the chosen starts in the sets $\mathcal{A}_m$, the retained inputs with $j(x) > 1000$, and the odd inputs excluded from $\mathcal{I}$. The latter two groups are bounded using

$$\mathcal{B}(x, 1) \le \frac{15}{4x}$$

from (142). The final step $v_Q \to v_N$, which was separated before the block partition was formed, contributes at most $1/D$. Combining the excluded-input contribution, the final endpoint correction, and this last step, define

$$B_\partial = \underbrace{\frac{15}{4} b_\partial}_{\text{odd inputs set aside}} + \underbrace{\frac{8}{D}}_{\text{final block endpoint}} + \underbrace{\frac{1}{D}}_{\text{separated final step}} < 10^{-15}. \qquad (149)$$

Then (135), (139), and the inclusion $\mathcal{A}_m \subseteq \mathcal{S}_m$ give

$$\begin{aligned}\mathcal{R}_{\mathcal{F}} &\le \sum_{m=61}^{1000} \sum_{x\in\mathcal{A}_m} \mathcal{B}(x,1) + \frac{15}{4} \sum_{\substack{x\in\mathcal{I}\\ j(x)>1000}} \frac{1}{x} \\ &\quad + \frac{15}{4} \sum_{\substack{0\le i<Q\\ v_i \text{ odd},\ v_i\notin\mathcal{I}}} \frac{1}{v_i} + \frac{9}{D} \qquad (150)\\ &\le B_\partial + 10^{-6} + \sum_{m=61}^{1000} \sum_{x\in\mathcal{A}_m} \mathcal{B}(x,1) \qquad \text{chosen starts}\\ &\le B_\partial + 10^{-6} + \sum_{m=61}^{1000} \sum_{x\in\mathcal{S}_m} \mathcal{B}(x,1) \qquad \text{all candidates}\\ &\le B_\partial + 10^{-6} + \frac{15}{4} \sum_{m=61}^{1000} \sum_{x\in\mathcal{S}_m} \frac{1}{x}. \qquad (151)\end{aligned}$$

The passage from $\mathcal{A}_m$ to $\mathcal{S}_m$ is justified by $\mathcal{A}_m \subseteq \mathcal{S}_m$ and the positivity of $\mathcal{B}(x,1)$ for every $x \in \mathcal{S}_m$. The additional candidates need not be chosen block starts on the original path: some may occur there only as interior inputs of certified negative blocks, while others may not occur on the path at all. Including their positive one-step balances is therefore a deliberate overcount. No additional endpoint or excluded-input corrections are introduced in this enlargement, so the boundary term B_∂ appears only once.

Count All Possible Gap Brackets. For each $61 \le m \le 1000$, put

$$C_m = \sum_{s=38}^{m} \sum_{\substack{g_1+\cdots+g_s=m\\ g_i\ge 1}} \chi(g_1,\dots,g_s). \qquad (152)$$

This considers every cut gap bracket with gap sum m and length at least 38. It contributes 0 if the test proves an initial block negative, and 1 otherwise. No particular trajectory is used in this count. Every $x \in \mathcal{S}_m$ has a cut gap bracket

$$\mathbf{g}_x = (g_1,\dots,g_s), \qquad s = s_m(x),$$

with

$$38 \le s \le m, \qquad g_i \ge 1, \qquad g_1 + \cdots + g_s = m.$$

Hence $\mathbf{g}_x$ occurs among the brackets in (152). Since $\chi(\mathbf{g}_x) = 1$ by the definition of $\mathcal{S}_m$, this bracket contributes 1 to C_m. One bracket fixes at most one odd remainder u modulo 2^m. Indeed, the gap-bracket identity

$$2^m T^m(x) = 3^s x + \sum_{i=0}^{s-1} 3^{s-1-i} 2^{\ell_i}$$

has the same values of s and $\ell_0,\dots,\ell_{s-1}$ for every start with that bracket. Reducing it modulo 2^m gives

$$3^s x + \sum_{i=0}^{s-1} 3^{s-1-i} 2^{\ell_i} \equiv 0 \pmod{2^m}.$$

Since 3^s is invertible modulo 2^m, this congruence fixes one residue class $x \equiv u \pmod{2^m}$. Thus two starts with the same bracket differ by a multiple of 2^m. Choose the unique representative u of this residue class

with $0 \le u < 2^m$. Since the possible starting integers are odd and $m \ge 61$, the remainder u is odd and hence satisfies $0 < u < 2^m$. Moreover, the definition of $m(x)$ gives

$$12 \cdot 2^m < x < 24 \cdot 2^m.$$

Writing $x = b2^m + u$ and using $0 < u < 2^m$ therefore forces $12 \le b \le 23$. Consequently, the only members of the fixed residue class that can lie in shell m are

$$x = b2^m + u, \qquad b \in \{12, 13, \dots, 23\}.$$

These are possible starts only; some may fail the other conditions defining $\mathcal{S}_m$. As b runs through these twelve consecutive values, exactly four of the corresponding starts are divisible by 3. For each remaining candidate, $1/x < 2^{-m}/b$. Keeping the first two numbers in each consecutive triple of b-values gives the largest possible reciprocal sum, so

$$\begin{aligned} \sum_{\substack{\text{possible starts for one bracket} \\ 3 \nmid x}} \frac{1}{x} &\le 2^{-m}\left(\frac{1}{12} + \frac{1}{13} + \frac{1}{15} + \frac{1}{16} + \frac{1}{18} + \frac{1}{19} + \frac{1}{21} + \frac{1}{22}\right) \\ &= 2^{-m}\frac{6719267}{13693680} < \frac{1}{2 \cdot 2^m}. \end{aligned} \tag{153}$$

Adding over the counted brackets yields

$$\begin{aligned} \sum_{x \in \mathcal{S}_m} \frac{1}{x} &\le \frac{C_m}{2 \cdot 2^m}, \\ \mathcal{R}_{\mathcal{F}} &\le B_{\partial} + 10^{-6} + \frac{15}{8} \sum_{m=61}^{1000} \frac{C_m}{2^m}. \end{aligned} \tag{154}$$

Here the factor $15/4$ comes from the one-step balance bound $\mathcal{B}(x, 1) \le 15/(4x)$, while the factor $1/2$ comes from the reciprocal estimate

$$\sum_{x \in \mathcal{S}_m} \frac{1}{x} \le \frac{C_m}{2 \cdot 2^m}.$$

Their product is $15/8$.

Example 1C.1. Let $m = 61$ and consider the cut gap bracket

$$\mathbf{g} = (\underbrace{1, \dots, 1}_{61 \text{ entries}}).$$

The associated congruence modulo 2^{61} fixes the odd remainder $u = 2^{61} - 1$. Thus the possible starts in the shell have the form

$$x = b2^{61} + u = b2^{61} + (2^{61} - 1) = (b + 1)2^{61} - 1, \qquad 12 \le b \le 23.$$

Writing $c = b + 1$, these twelve possible starts are therefore $x = c2^{61} - 1$ for $13 \le c \le 24$. Since $2^{61} \equiv 2 \pmod 3$, one has

$$3 \mid x \iff 2c - 1 \equiv 0 \pmod 3 \iff c \equiv 2 \pmod 3.$$

The four excluded values are consequently $c \in \{14, 17, 20, 23\}$, and the eight candidates not divisible by 3 are

$$x = c2^{61} - 1, \qquad c \in \{13, 15, 16, 18, 19, 21, 22, 24\}.$$

Every one of the first 61 shortcut inputs is odd, and direct induction gives

$$T^t(x) = c2^{61-t}3^t - 1 \qquad (0 \le t \le 61).$$

The quantity $c2^{61-t}3^t = c2^{61}(3/2)^t$ increases with t, and $13 \cdot 2^{61} - 1 > D$. Hence all states in the inspected segment exceed D. By (153), the reciprocals of the eight possible starts have total less than

$$\frac{1}{2 \cdot 2^{61}} = 2^{-62}.$$

Appendix 1D verifies that

$$\chi(\underbrace{1, \ldots, 1}_{61 \text{ entries}}) = 1,$$

so this bracket survives the gap test and contributes one to C_{61}.

Appendix 1D. The Gap Test and its Finite Count

We prove the negative-block test used in Appendix 1C and the finite bound

$$\sum_{m=61}^{1000} \frac{C_m}{2^m} < \frac{3}{32}. \tag{155}$$

The same cut gap bracket may be realized by several different starting integers. As shown in Appendix 1C, a bracket whose gaps sum to m determines the starting integer only modulo 2^m. Within the shell $12 \cdot 2^m < x < 24 \cdot 2^m$, the corresponding starts can have the form

$$x = b2^m + u, \qquad b = 12, 13, \ldots, 23,$$

for the single remainder u fixed by the bracket. Since C_m counts brackets rather than individual starts, a bracket may be discarded only when the same gap data certify a negative initial block for every admissible start realizing that bracket. The test is therefore constructed entirely from the gaps and gives a sufficient condition for negativity that is independent of the particular starting integer.

The Comparison Suggested by the Gaps. Write $x_t = T^t(x)$ and let $s_t = s_t(x)$ count the odd inputs before step t. Define

$$\alpha_t = \frac{2^t}{3^{s_t}}, \qquad \beta_t = \sum_{\substack{0 \le i < t \\ x_i \text{ odd}}} \alpha_i, \qquad V_t = \frac{8 + \beta_t}{\alpha_t}. \tag{156}$$

The affine iterate formula has the form

$$x_t = \frac{3^{s_t}x + J_t}{2^t} = \frac{x}{\alpha_t} + \frac{J_t}{2^t},$$

where $J_t \ge 0$ collects the additive contributions from the odd steps. Therefore

$$x_t \ge \frac{x}{\alpha_t}, \qquad \text{and hence} \qquad \frac{x}{x_t} \le \alpha_t.$$

Summing this inequality over the odd input positions before step r gives

$$x \sum_{\substack{0 \le i < r \\ x_i \text{ odd}}} \frac{1}{x_i} = \sum_{\substack{0 \le i < r \\ x_i \text{ odd}}} \frac{x}{x_i} \le \sum_{\substack{0 \le i < r \\ x_i \text{ odd}}} \alpha_i = \beta_r. \tag{157}$$

Now multiply the block balance by its starting value:

$$x\mathcal{B}(x,r) = x \sum_{\substack{0\le i<r\\ x_i \text{ odd}}} \frac{1}{x_i} + 8 - \frac{8x}{x_r}.$$

The reciprocal-sum term is at most β_r, so the first two terms on the right are at most $8+\beta_r$. If the additive contributions were absent, then $x_r = x/\alpha_r$ and hence $x/x_r = \alpha_r$. In that idealized case,

$$x\mathcal{B}(x,r) \le 8 + \beta_r - 8\alpha_r = \alpha_r\left(\frac{8+\beta_r}{\alpha_r} - 8\right) = \alpha_r(V_r - 8).$$

Since $\alpha_r > 0$, the inequality $V_r < 8$ would then imply $\mathcal{B}(x,r) < 0$.

For the actual shortcut map, the positive additive contributions can make x_r larger than x/α_r. Consequently, x/x_r can be slightly smaller than α_r, making the negative endpoint term $-8x/x_r$ slightly less negative than the idealized value $-8\alpha_r$. The next lemma bounds this loss when all inspected inputs exceed D and $r \le 1000$. It shows that the slightly stronger condition

$$V_r \le \frac{127}{16} < 8$$

still guarantees $\mathcal{B}(x,r) < 0$.

Lemma 1D.1. *Suppose $x_0 = x$ is odd, $x_i > D$ for $0 \le i < r$, and $1 \le r \le 1000$. Then*

$$V_r \le \frac{127}{16} \quad\Longrightarrow\quad \mathcal{B}(x,r) < 0.$$

Proof. For the odd input positions before step r, put

$$I_r := \{i : 0 \le i < r,\ x_i \text{ odd}\}, \qquad a_i := \frac{1}{3x_i} \text{ for } i \in I_r,$$

and write

$$A_r := \sum_{i\in I_r} a_i.$$

There are r input positions $x_0, \dots, x_{r-1}$, so $|I_r| \le r$. Moreover, the hypotheses give $x_i > D$ for every $i \in I_r$ and $r \le 1000$. Therefore

$$A_r < \frac{r}{3D} \le \frac{1000}{3D} < 1.$$

For nonnegative a_i, every product of k distinct terms occurring in the expansion of $\prod_{i\in I_r}(1+a_i)$ is among the nonnegative terms in A_r^k. Hence

$$\prod_{i\in I_r}(1+a_i) \le 1 + A_r + A_r^2 + \cdots = \frac{1}{1-A_r}.$$

Taking reciprocals gives

$$\prod_{i\in I_r}(1+a_i)^{-1} \ge 1 - A_r.$$

Applying this to the finite product identity (18) yields

$$\begin{aligned}\frac{x}{x_r} &= \alpha_r \prod_{\substack{0\le i<r\\ x_i \text{ odd}}} \left(1+\frac{1}{3x_i}\right)^{-1}\\ &\ge \alpha_r \left(1 - \sum_{\substack{0\le i<r\\ x_i \text{ odd}}} \frac{1}{3x_i}\right)\\ &\ge \alpha_r \left(1-\frac{1000}{3D}\right). \end{aligned} \tag{158}$$

Multiplying the definition of $\mathcal{B}(x,r)$ by x and applying (157) gives

$$\begin{aligned} x\mathcal{B}(x,r) &= x \sum_{\substack{0\le i<r\\ x_i \text{ odd}}} \frac{1}{x_i} + 8 - \frac{8x}{x_r}\\ &\le \beta_r + 8 - \frac{8x}{x_r}. \end{aligned}$$

Using the lower bound (158) for x/x_r, we therefore obtain

$$\begin{aligned} x\mathcal{B}(x,r) &\le \beta_r + 8 - 8\alpha_r\left(1-\frac{1000}{3D}\right)\\ &= \beta_r + 8 - 8\alpha_r + \frac{8000\alpha_r}{3D}. \end{aligned} \tag{159}$$

Since $V_r = (8+\beta_r)/\alpha_r$, we have $8+\beta_r = \alpha_r V_r$. Hence

$$\begin{aligned} x\mathcal{B}(x,r) &\le \alpha_r V_r - 8\alpha_r + \frac{8000\alpha_r}{3D}\\ &= \alpha_r\left(V_r - 8 + \frac{8000}{3D}\right). \end{aligned} \tag{160}$$

Now

$$V_r \le \frac{127}{16} = 8 - \frac{1}{16}$$

and

$$\frac{8000}{3D} < \frac{1}{16}.$$

Therefore

$$V_r - 8 + \frac{8000}{3D} \le -\frac{1}{16} + \frac{8000}{3D} < 0.$$

Since $\alpha_r > 0$, it follows that $x\mathcal{B}(x,r) < 0$. Finally, $x > 0$, so $\mathcal{B}(x,r) < 0$. □

To convert Lemma 1D.1 into a test depending only on the gap bracket, it remains to express the evolution of V in terms of the successive gap entries. At an even input, $s_{t+1} = s_t$, so (α,β) becomes $(2\alpha,\beta)$ and V is divided by 2. At an odd input, (α,β) becomes $(2\alpha/3, \beta+\alpha)$, and hence

$$V \longmapsto \frac{8+\beta+\alpha}{2\alpha/3} = \frac{3}{2}(V+1).$$

The following $g_i - 1$ even steps divide this value by 2^{g_i-1}. Consequently, for one gap part,

$$V_0 = 8, \qquad V_{\ell_i} = \frac{3(V_{\ell_{i-1}} + 1)}{2^{g_i}} = \frac{3^i}{2^{\ell_i}} \left(8 + \sum_{k=0}^{i-1} \frac{2^{\ell_k}}{3^k} \right). \tag{161}$$

The +1 in the update records the contribution of the new odd input to β. The first shortcut step in a gap is the odd step, after which each of the remaining $g_i - 1$ even steps divides V by 2. Thus V decreases after the first step and throughout the remainder of that gap. Its value at the gap endpoint is therefore the smallest value attained after that odd step. It is consequently enough to test the values V_{ℓ_i} at the ends of the successive gaps, including the endpoint of the final partial gap.

The recurrence (161) depends only on the gap entries $g_1, \ldots, g_s$, not on the particular starting integer realizing the bracket. Moreover, for each i, the value V_{ℓ_i} corresponds to the initial block consisting of the first ℓ_i shortcut steps. Under the hypotheses of Lemma 1D.1,

$$V_{\ell_i} \le \frac{127}{16} \quad \Longrightarrow \quad \mathcal{B}(x, \ell_i) < 0.$$

We therefore define the zero-or-one survival indicator used in (152) by

$$\chi(g_1, \ldots, g_s) = \begin{cases} 0, & V_{\ell_i} \le 127/16 \text{ for some } 1 \le i \le s, \\ 1, & V_{\ell_i} > 127/16 \text{ for every } 1 \le i \le s. \end{cases} \tag{162}$$

Thus $\chi = 0$ means that the test certifies at least one negative initial block and the bracket can be discarded. The value $\chi = 1$ means only that the bracket survives this one-sided test and must be retained in the upper count. It does not imply that every initial block, or even the full inspected block, has positive balance.

Example 1D.2. For $v_0 = 2^{126} + 19$, the first two gaps are $(1, 3)$, so $\ell_1 = 1$, and $\ell_2 = 4$. All values through $v_4 = T^4(v_0)$ exceed D. With $x = v_0$, the quantities used by the test are

t	$v_t = T^t(v_0)$	s_t	α_t	β_t	V_t
0	$2^{126} + 19$	0	1	0	8
1	$3 \cdot 2^{125} + 29$	1	2/3	1	27/2
4	$9 \cdot 2^{122} + 11$	2	16/9	5/3	87/16

At the end of the first gap,

$$V_{\ell_1} = V_1 = \frac{3(8 + 1)}{2} = \frac{27}{2} > \frac{127}{16},$$

so the test does not certify the one-step initial block as negative. At the end of the second gap,

$$V_{\ell_2} = V_4 = \frac{3(27/2 + 1)}{2^3} = \frac{87}{16} < \frac{127}{16}.$$

Since the hypotheses of Lemma 1D.1 hold with $r = 4$, the four-step initial block satisfies $\mathcal{B}(v_0, 4) < 0$. Hence $\chi(1, 3) = 0$. By the block-addition identity (141),

$$\mathcal{B}(v_0, 4) = \mathcal{B}(v_0, 1) + \mathcal{B}(v_1, 3) < 0.$$

The odd inputs in these four steps are v_0 and v_1, so the block balance includes both reciprocal terms $1/v_0$ and $1/v_1$. Under the block-selection rule, these four steps form the chosen negative block, and the next block begins at its endpoint v_4.

For comparison, suppose that every gap entry equals 1. Then $\ell_i = i$, and the recurrence (161) gives

$$V_i = 11\left(\frac{3}{2}\right)^i - 3 > 8 \qquad (i \geq 1).$$

Thus none of its initial gap endpoints satisfies the rejection threshold. In particular, the bracket in Example 1C.1 has $\chi = 1$ and contributes one to C_{61}.

An Upper-Count Algorithm and Computation. For each fixed m, equation (152) defines C_m by applying the exact gap test to every composition $(g_1, \ldots, g_s)$ of m with $s \geq 38$. The number of such compositions is large, so the following finite-state calculation replaces the exact values of V by rounded upper approximations. Because the approximation is always at least the exact value, the resulting count is an upper bound for C_m.

At the beginning of every gap bracket, $\alpha_0 = 1$ and $\beta_0 = 0$, and hence, by (156),

$$V_0 = \frac{8 + \beta_0}{\alpha_0} = 8.$$

Accordingly, initialize the rounded value at $\widehat{V} = 8$ and store all subsequent approximations in the finite grid

$$\left\{ \frac{h}{64} 2^e \;\middle|\; h, e \text{ integers}, 64 \leq h \leq 127,\ 3 \leq e \leq 23 \right\}. \tag{163}$$

For each gap entry g_i, first apply the odd-step update

$$\widehat{V} \longmapsto \frac{3}{2}(\widehat{V} + 1)$$

and round upward to the least grid value not smaller than the result. Then perform the remaining $g_i - 1$ halvings belonging to that gap. Stop and discard the bracket if one of these halvings produces a value below 8. If an update exceeds the largest grid value, retain the bracket and all of its extensions without performing further tests. As in the definition of C_m, only tuples with total gap sum m and length $s \geq 38$ are counted.

The inequality $\widehat{V} \geq V$ holds initially. The map $z \mapsto 3(z + 1)/2$ and every halving are increasing, and upward rounding can only enlarge a value. Induction over the rounded updates therefore gives $\widehat{V} \geq V$ until the rounded test stops. Halving a listed value at least 16 gives another listed value. If the listed value before a halving is below 16, then the value after that halving is at most $127/16$. Hence a rounded stop implies

$$V \leq \widehat{V} \leq \frac{127}{16},$$

so Lemma 1D.1 proves that the corresponding initial block is negative. If an update exceeds the grid, keeping the bracket and all its extensions can only enlarge the count. Thus the rounded calculation may retain a bracket that the exact test would discard, but it cannot discard a bracket retained by the exact test. Let $\widehat{C}_m$ count the brackets kept by this rounded calculation. Then $C_m \leq \widehat{C}_m$, and exact evaluation gives

$$\sum_{m=61}^{1000} \frac{C_m}{2^m} \leq \sum_{m=61}^{1000} \frac{\widehat{C}_m}{2^m} = 0.093657291203212905564205748 07\ldots < \frac{3}{32}. \tag{164}$$

No floating-point calculation is used to establish this result. The following *Mathematica* procedure was written to calculate the value exactly, with the result then being compared directly to the value of $3/32$. On a laptop with an Intel i7-8650U CPU @ 1.90GHz (2.11 GHz) and 16.0 GB of RAM, the calculation was completed in 100.9 seconds. The code below can be copied directly into *Mathematica* and executed to obtain the result given in (164).

Listing 1. Exact computation of the rounded upper count.

```
(*Compute the EXACT finite sum, and also save each cHat[[m]]*)
{finiteSum, cHat} =  Module[{grid, n, up, even, odd, counts, nextCounts, values},
   (*The grid*)
   grid = Flatten[Table[(h/64) 2^e, {e, 3, 23}, {h, 64, 127}]];
   (*The final state represents overflow above the grid.*)
   n = Length[grid] + 1;
   (*Index of the smallest grid value greater than or equal to y.*)
   up[y_] := 1 + LengthWhile[grid, # < y &];
   (*At an even input, Vhat is divided by 2. State 0 -> bracket is discarded.*)
   even = Append[If[#/2 < 8, 0, up[#/2]] & /@ grid, n];
   (*At an odd input,Vhat maps to 3(Vhat+1)/2.*)
   odd = Append[up[3 (# + 1)/2] & /@ grid, n];
   (*counts[[i,k]] is the number of surviving parity sequences in
      grid state i having k odd inputs,with k capped at 38.*)
   counts = ConstantArray[0, {n, 38}];
   (*The first input is necessarily odd.*)
   counts[[odd[[1]], 1]] = 1;
   values = ConstantArray[0, 1000];
   (*The outer Do loop computes cHat[[m]].*)
   Do[
   values[[m]] = Total[counts[[All, 38]]];
    If[m < 1000, nextCounts = ConstantArray[0, {n, 38}];
     Do[
      If[counts[[i, k]] != 0,
       (*Append an even input,unless it causes rejection.*)
       If[even[[i]] > 0, nextCounts[[even[[i]], k]] += counts[[i, k]]];
       (*Append an odd input.*)
       nextCounts[[odd[[i]], Min[38, k + 1]]] += counts[[i, k]];]
      , {i, 1, n}, {k, 1, 38}
      ];
     counts = nextCounts;],
    {m, 1, 1000}
    ];
   {Sum[values[[m]]/2^m, {m, 61, 1000}], values}
   ];

{N[finiteSum, 30], N[3/32, 10], finiteSum < 3/32}
```

The first values produced by the exact computation and contributing to the weighted sum in (164) are displayed in Table 12. The final output of the *Mathematica* procedure above is:

$$\{0.093657291203212905564205748 0750,\ 0.09375000000,\ \text{True}\}.$$

Table 12. First values of the rounded upper counts $\widehat{C}_m$ entering the weighted sum in (164). For each $61 \le m \le 100$, the quantity $\widehat{C}_m$ counts the gap brackets of total gap sum m, with at least 38 odd inputs, retained by the rounded V-test.

m	$\widehat{C}_m$	m	$\widehat{C}_m$
61	9 397 196 782 201 115	81	3 913 719 046 646 186 634 224
62	17 959 590 990 646 997	82	7 535 655 076 590 518 029 606
63	34 603 440 856 372 965	83	14 307 076 677 132 451 249 490
64	65 492 187 912 674 215	84	27 459 162 788 847 816 757 610
65	124 832 693 239 164 952	85	52 607 116 079 740 183 292 433
66	240 052 124 752 695 401	86	100 092 674 906 715 605 116 914
67	453 717 979 268 762 914	87	192 837 749 144 875 131 652 522
68	870 438 337 611 712 335	88	367 437 587 953 981 791 637 855
69	1 662 206 576 757 259 607	89	702 142 052 206 591 947 211 363
70	3 153 316 232 697 859 645	90	1 352 883 343 658 323 177 074 712
71	6 073 556 823 793 357 126	91	2 572 368 406 472 719 060 210 671
72	11 527 720 553 237 276 487	92	4 938 700 131 576 309 099 805 805
73	21 995 465 220 108 889 569	93	9 473 308 530 014 630 614 048 926
74	42 321 877 234 409 942 033	94	18 042 701 346 798 866 717 558 334
75	80 197 816 821 931 045 480	95	34 772 303 510 967 255 500 920 111
76	153 880 688 059 923 084 019	96	66 346 182 048 106 133 623 724 041
77	294 373 986 820 921 607 127	97	126 859 527 685 600 994 778 622 589
78	559 379 069 423 456 050 748	98	244 594 523 736 184 835 175 574 836
79	1 077 453 098 544 887 483 343	99	465 620 970 914 416 112 234 460 218
80	2 049 497 689 622 316 548 691	100	894 278 252 354 101 043 785 255 555

Conclusion. Equations (154) and (164) give

$$\mathcal{R}_{\mathcal{F}} < B_{\partial} + 10^{-6} + \frac{15}{8}\frac{3}{32} < 10^{-15} + 10^{-6} + \frac{45}{256} < \frac{9}{50}.$$

Together with the cases $N = 0, 1$, this proves Theorem 1A.1.

We next apply the finite-path estimate to an arbitrary forward orbit. If the orbit enters $[1, D]$, apply Theorem 1A.1 to the path ending at its first such entry. If instead a state repeats before the orbit enters $[1, D]$, apply the theorem to the path ending at the first repetition; its input states are distinct, although its endpoint repeats an earlier state. In either case, the theorem gives

$$\mathcal{R}_{\mathrm{high}}(n) < \frac{9}{50}.$$

If neither event occurs, every finite initial path has distinct input states greater than D, so Theorem 1A.1 applies to each one. The corresponding reciprocal sums increase to the once-counted sum over the forward orbit and satisfy

$$\mathcal{R}_{\mathrm{high}}(n) \le \frac{45}{256} + 10^{-6} + 10^{-15} < \frac{9}{50}.$$

Thus the high-orbit estimate holds for every positive starting integer, without any termination assumption.

It remains to convert this reciprocal estimate into the desired residue bound. Since

$$\frac{2456}{M} < 10^{-6}, \qquad \frac{9/50 + 10^{-6}}{3} < \frac{61}{1000},$$

equation (132) gives

$$\rho^{\circ}(n) < \rho(993)e^{61/1000}.$$

Put $a = 61/1000$. For $k \geq 0$, the inequality $(3+k)! \geq 6 \cdot 4^k$ bounds the tail of the exponential series by

$$\sum_{k=0}^{\infty} \frac{a^{3+k}}{(3+k)!} \leq \frac{a^3}{6} \sum_{k=0}^{\infty} \left(\frac{a}{4}\right)^k = \frac{a^3}{6(1-a/4)}.$$

Consequently,

$$\rho(993)e^{a} \leq \rho(993)\left(1 + a + \frac{a^2}{2} + \frac{a^3}{6(1-a/4)}\right) < 1.331964 < \frac{4}{3}. \tag{165}$$

Therefore

$$\rho^{\circ}(n) < \frac{4}{3}$$

for every positive integer n.

Appendix 2. Guide to the Principal Results

This appendix collects the principal identities, bounds, recovery formulas, ordering results, and record relations established in the paper. It is intended as a guide to the results rather than as an independent development of the theory. The hypotheses stated in the referenced results remain in force; in particular, some statements concern arbitrary finite shortcut paths, some require nonrepeating paths whose input states exceed 1, and others concern complete first-arrival paths of terminating integers.

1. Gap Brackets, Affine Formulas, and Residues.

(a) A cut of a finite shortcut path is encoded by its gap bracket. The entries record the division steps represented before the cut, and their sum is the number of shortcut steps in the cut.

(b) If a cut has k odd inputs and endpoint z, its gap or coordinate data give the affine identity

$$2^{\ell_k} z = 3^k n + J,$$

where the intercept J is determined explicitly by the coordinates. This is the Affine Gap Formula, Theorem 1.15.

(c) For every finite shortcut path $\mathcal{F} = (a, \dots, z)$,

$$\rho_{\mathcal{F}} = \frac{2^{E_{\mathcal{F}} z}}{3^{O_{\mathcal{F}} a}} = \prod_{\substack{x \text{ an odd input} \\ \text{of } \mathcal{F}}} \left(1 + \frac{1}{3x}\right).$$

Thus the residue measures the multiplicative correction between the endpoints and the even and odd operation counts; see Proposition 2.6.

(d) For an odd integer $x = 2^j + \bar{x}$, the first j shortcut steps of x and $\bar{x}$ have the same parity pattern, and

$$T^t(x) = 2^{j-t} 3^{s_t} + T^t(\bar{x}) \qquad (0 \le t \le j).$$

In particular, the canonical endpoint has the form

$$T^j(x) = 3^s + q, \qquad 0 < q < 3^s.$$

These are Theorem 2.7, Theorem 2.10, and Corollary 2.11.

(e) If $N_{j,s}$ counts canonical blocks in dyadic shell j having s odd inputs, then

$$N_{j,s} \le \binom{j-1}{s-1}, \qquad \sum_j N_{j,s} \le 2 \cdot 3^{s-1}.$$

The first estimate counts cut signatures within a shell, while the second counts possible canonical endpoints; see Lemma 2.14.

2. Uniform Bounds for Reciprocal Sums and Residues.

(a) Reciprocal sums, residue products, and comparison products are uniformly bounded on finite nonrepeating shortcut paths whose input states exceed 1. The same quantities are uniformly bounded when each distinct odd value in an arbitrary forward orbit is counted once; see Theorem 2.16 and Corollary 2.19.

(b) For every positive integer n, whether or not its orbit terminates or eventually repeats, $\rho^\circ(n) < 4/3$. This is the Once-Counted Four-Thirds Bound, Corollary 2.21.

(c) If $\mathcal{F}$ is a finite nonrepeating shortcut path whose input states are greater than 1, then $\rho_{\mathcal{F}} < 4/3$. No condition is imposed on its endpoint or on the orbit after that endpoint; see Theorem 2.22.

(d) If the input state 1 is allowed, the example $\mathcal{F} = (1, 2)$ shows that equality $\rho_{\mathcal{F}} = 4/3$ can occur. For finite paths with distinct positive input states, the weaker uniform bound $\rho_{\mathcal{F}} < 16/9 < 2$ still holds.

(e) Every terminating positive integer satisfies $\rho(n) < 4/3$. This is Theorem 2.23.

(f) For every finite nonrepeating shortcut path whose input states exceed 1, $\mathcal{R}_{\mathcal{F}} < 1$. For every positive integer n, $\mathcal{R}^\circ(n) < 1$, and for terminating n this specializes to $\mathcal{R}(n) < 1$; see Corollary 2.24.

(g) The three scalar quantities satisfy

$$1 + \frac{\mathcal{R}_{\mathcal{F}}}{3} \le \rho_{\mathcal{F}}, \qquad 3 \log \rho_{\mathcal{F}} \le \mathcal{R}_{\mathcal{F}} \le \log \Pi_{\mathcal{F}} \le 2\mathcal{R}_{\mathcal{F}}.$$

The same inequalities hold for the once-counted quantities and, upon termination, for the complete first-arrival quantities; see Theorem 3.19.

(h) If the forward orbit of n does not terminate, then the stronger bounds

$$\mathcal{R}^\circ(n) \le \frac{9}{50} < \log 2, \qquad \rho^\circ(n) \le e^{3/50} < 2^{1/3}, \qquad \Pi^\circ(n) \le e^{9/25} < 2$$

hold; see Corollary 3.20.

3. Common Endpoints and Recovery of Operation Counts.

(a) For finite paths from a and b to a common endpoint z, the count differences satisfy

$$\frac{2^{E_{\mathcal{F}_a} - E_{\mathcal{F}_b}}}{3^{O_{\mathcal{F}_a} - O_{\mathcal{F}_b}}} = \frac{a}{b} \frac{\rho_{\mathcal{F}_a}}{\rho_{\mathcal{F}_b}}.$$

This is the Common-Endpoint Residue Identity, Theorem 3.2.

(b) Under the nonrepetition and input hypotheses of Theorem 3.3,

$$\left| E_{\mathcal{F}_a} - E_{\mathcal{F}_b} - (O_{\mathcal{F}_a} - O_{\mathcal{F}_b}) \log_2 3 - \log_2(a/b) \right| < \log_2(4/3) < \frac{1}{2}.$$

Consequently, the difference in the even counts is recovered by the nearest-integer formula in Corollary 3.4.

(c) For a single finite path $\mathcal{F} = (a, \ldots, z)$, knowledge of one operation count and the endpoints determines the remaining counts exactly. In particular,

$$H_{\mathcal{F}} = \operatorname{nint}\left(\log_2(6^{O_{\mathcal{F}}} a/z) \right),$$

with the corresponding ceiling and floor formulas given in Theorem 3.7.

(d) If two finite paths to a common endpoint have equal height, then their separation is controlled by the difference $d = O_{\mathcal{F}_a} - O_{\mathcal{F}_b}$:

$$\frac{3}{4} 6^d < \frac{b}{a} < \frac{4}{3} 6^d.$$

In particular, the ratio b/a cannot lie in $[4/3, 9/2]$; see Theorem 3.5.

(e) Adjacent finite paths to a common endpoint having equal height have equal even and odd operation counts; see Corollary 3.6. Consequently, every consecutive equal-height block of terminating integers lies entirely inside one operation-count cell; see Corollary 3.12.

(f) For adjacent terminating integers, if $\Delta E = E(n) - E(n+1)$, and $\Delta O = O(n) - O(n+1)$, then

$$\Delta E = \operatorname{nint}\big((\log_2 3)\Delta O\big).$$

If $\Delta O \neq 0$, the two differences have the same sign and

$$\frac{3}{2} \leq \frac{\Delta E}{\Delta O} \leq 2.$$

This is Theorem 3.14.

(g) The relative residue ratio $W(n)$ satisfies

$$\frac{3}{4} < W(n) < \frac{4}{3}$$

for every terminating adjacent pair outside the explicitly listed small exceptions; see Theorem 3.15 and the discussion following it.

(h) Among terminating integers, the E-delay and height-delay properties are equivalent:

$$\mathrm{DR}_E = \mathrm{DR}_H.$$

This is Theorem 3.18 in its terminating-set form.

4. Charge, Finite Transport, and the Slice Functions.

(a) Height and charge determine the operation counts:

$$E_{\mathcal{F}} = \frac{2H_{\mathcal{F}} + \lambda_{\mathcal{F}}}{3}, \qquad O_{\mathcal{F}} = \frac{H_{\mathcal{F}} - \lambda_{\mathcal{F}}}{3}.$$

(b) Under the finite-path four-thirds bound, the starting value, endpoint, and charge determine the odd count by the exact ceiling formula in Theorem 4.2. For a terminating integer, n and $\lambda(n)$ determine all of $O(n)$, $E(n)$, and $H(n)$; see Corollary 4.10.

(c) For an anchor $A = 2^{\alpha}3^{\beta}b$ with $\gcd(b, 6) = 1$, and offset r, the first α shortcut steps satisfy

$$T^t(A + r) = 2^{\alpha - t}3^{\beta + o_t(r)}b + T^t(r) \qquad (0 \leq t \leq \alpha).$$

This exact finite transport requires no termination assumption; see Theorem 4.12.

(d) The transported boundary data give exact finite-path and terminating formulas for the height and charge of $A + r$; see Theorems 4.13 and 4.14.

(e) The two-variable slices G and Q determine one another exactly on their terminating domain; see Proposition 5.4.

(f) The exact height-break equation is $H(A+r) - H(A+r-1) = \Delta G_r + \Delta\delta_{G,r}$. Thus a projected change in G produces an actual height change exactly when it is not cancelled by the residual correction.

5. Operation-Count Cells and Their Strict Orderings.

The four-thirds residue bound orders not only the least members of operation-count classes, but entire occupied cells. The following properties hold globally for complete first-arrival paths and, with the indicated relative notation, above every admissible common tail.

(a) For every terminating integer $n \in C_{e,o}$,

$$n = \frac{2^e}{3^o \rho(n)},$$

and hence

$$\frac{3}{4}\frac{2^e}{3^o} < n \le \frac{2^e}{3^o}.$$

Thus every occupied cell lies in an interval whose multiplicative width is less than $4/3$.

(b) More generally, if $x \in C_{e_x,o_x}$ and $y \in C_{e_y,o_y}$, then

$$\frac{x}{y} = 2^{e_x - e_y} 3^{o_y - o_x} \frac{\rho(y)}{\rho(x)}, \tag{166}$$

where

$$\frac{3}{4} < \frac{\rho(y)}{\rho(x)} < \frac{4}{3}.$$

The powers of 2, 3, 6, and $4/3$ arising in the following specializations dominate the possible residue ratio.

(c) *Fixed odd count.* If $O(x) = O(y)$ and $E(y) < E(x)$, then $y < x$. Equivalently, at a fixed odd count, increasing the division count moves the entire occupied cell upward. Thus, if $e' < e$, every member of an occupied cell $C_{e',o}$ is smaller than every member of an occupied cell $C_{e,o}$. This ordering is established in Lemma 6.2.

(d) *Fixed division count.* If $E(x) = E(y)$ and $O(y) > O(x)$, then $y < x$. Equivalently, at a fixed division count, increasing the odd count moves the entire occupied cell downward. This is one of the cell orders established in the proof of Theorem 6.3.

(e) *Fixed height.* If $H(x) = H(y)$ and $O(y) > O(x)$, then $y < x$. Thus, among the occupied cells at a fixed height, the cell having the greatest odd count contains the smallest starting integers. This is the second cell order established in the proof of Theorem 6.3.

(f) *Fixed charge.* If $\lambda(x) = \lambda(y)$ and $O(y) > O(x)$, then $y > x$. Thus the fixed-charge order runs in the opposite direction from the fixed-height and fixed-division-count orders: increasing the odd count increases the starting integer. This is Theorem 4.8.

(g) The four global orders may be summarized as follows:

quantity held fixed	count increased	effect on the starting integer
O	E	increases
E	O	decreases
H	O	decreases
λ	O	increases

(h) The fixed-charge ordering also holds for arbitrary nonrepeating finite shortcut paths to a common endpoint. If $\lambda_{\mathcal{F}_a} = \lambda_{\mathcal{F}_b}$ and $d = O_{\mathcal{F}_a} - O_{\mathcal{F}_b}$, then $E_{\mathcal{F}_a} - E_{\mathcal{F}_b} = 2d$, and $H_{\mathcal{F}_a} - H_{\mathcal{F}_b} = 3d$, and

$$\left(\frac{4}{3}\right)^{d-1} < \frac{a}{b} < \left(\frac{4}{3}\right)^{d+1}.$$

In particular, at a fixed finite-path charge, the order of the starting integers agrees with the order of their odd counts, even counts, and heights; see Corollary 4.3.

(i) The fixed-height comparison for common-endpoint paths has the stronger separation property

$$\frac{3}{4}\,6^d < \frac{b}{a} < \frac{4}{3}\,6^d, \qquad d = O_{\mathcal{F}_a} - O_{\mathcal{F}_b} \geq 0,$$

whenever $a < b$ and the two finite-path heights are equal. Hence an equal-height pair satisfies either

$$1 < \frac{b}{a} < \frac{4}{3} \quad \text{or} \quad \frac{b}{a} > \frac{9}{2}.$$

See Theorem 3.5.

(j) As an adjacent specialization, equal finite-path height at a common endpoint forces equality of both operation counts:

$$H_{\mathcal{F}_a} = H_{\mathcal{F}_{a+1}} \quad \Longrightarrow \quad E_{\mathcal{F}_a} = E_{\mathcal{F}_{a+1}}, \qquad O_{\mathcal{F}_a} = O_{\mathcal{F}_{a+1}}.$$

Thus an adjacent equal-height pair cannot lie in two different operation-count cells; see Corollary 3.6.

(k) For every nonnegative odd count o, let

$$b_o = c_O(o), \qquad e_o = E(b_o), \qquad m_o = H(b_o).$$

Then e_o is the least division count occurring at odd count o, and $m_o = e_o + o$ is the least height occurring there. Moreover,

$$C_{e,o} \neq \{\,\} \quad \Longleftrightarrow \quad e \geq e_o.$$

Indeed, every $e \geq e_o$ is realized by $2^{e-e_o} b_o \in C_{e,o}$. Consequently, each fixed-O column of occupied cells begins at e_o and continues through every larger division count; see Lemma 6.2.

(l) The first division counts and first heights satisfy $e_{o+1} > e_o$ and $2 \leq m_{o+1} - m_o \leq 5$. Therefore greater odd counts cannot occur at the first division count or first height belonging to a smaller odd count. Together with the fixed-E and fixed-H orders, this gives $c_O(o) = c_E(e_o) = c_H(m_o)$.

(m) Now fix an admissible common tail z. For $x, y \in \mathcal{A}_z$, the relative count comparison is

$$\frac{x}{y} = 2^{E_z(x)-E_z(y)} 3^{O_z(y)-O_z(x)} \frac{\rho_z(y)}{\rho_z(x)},$$

with

$$\frac{3}{4} < \frac{\rho_z(y)}{\rho_z(x)} < \frac{4}{3}.$$

Equivalently, for $n \in C_{e,o}(z)$,

$$\frac{3}{4}\, z\frac{2^e}{3^o} < n \leq z\frac{2^e}{3^o}.$$

See Lemma 6.12 and equation (121).

(n) The four strict same-tail orders are:

(i) If $H_z(x) = H_z(y)$ and $O_z(y) > O_z(x)$, then $y < x$.
(ii) If $E_z(x) = E_z(y)$ and $O_z(y) > O_z(x)$, then $y < x$.
(iii) If $O_z(x) = O_z(y)$ and $E_z(y) < E_z(x)$, then $y < x$.
(iv) If $\lambda_z(x) = \lambda_z(y)$ and $O_z(y) < O_z(x)$, then $y < x$.

These are precisely the four assertions of Lemma 6.13. In the final assertion, equivalently, at fixed relative charge, increasing the relative odd count increases the source.

(o) The same-tail orderings compare entire occupied relative cells, not merely their least members. In particular:

(i) at fixed O_z, increasing E_z moves the entire cell upward;
(ii) at fixed E_z, increasing O_z moves the entire cell downward;
(iii) at fixed H_z, increasing O_z moves the entire cell downward;
(iv) at fixed λ_z, increasing O_z moves the entire cell upward.

(p) If a relative odd count o occurs above z, then every smaller nonnegative relative odd count also occurs. Let $e_o(z)$ be the least relative division count occurring at odd count o. Whenever $o+1$ is also realized, $e_{o+1}(z) > e_o(z)$. Moreover,

$$C_{e,o}(z) \neq \varnothing \quad \Longleftrightarrow \quad e \geq e_o(z).$$

Thus every realized relative odd-count column begins at a unique first division count and contains every cell above it; see Lemma 6.12.

(q) The same-tail support and ordering properties reproduce the global class and delay arguments above every admissible tail. They are the mechanism behind

$$\mathrm{CR}_\lambda^{\leq 0}(z) \subseteq \mathrm{CR}_O(z) \subseteq \mathrm{CR}_E(z) \subseteq \mathrm{CR}_H(z)$$

and

$$\mathrm{DR}_\lambda(z) \subseteq \mathrm{DR}_O(z) \subseteq \mathrm{DR}_E(z) = \mathrm{DR}_H(z);$$

see Theorem 6.14.

6. Streaks, Plateaus, and Intercepts.

(a) Adjacent height equality always implies adjacent charge equality. Conversely, adjacent charge equality implies adjacent height equality except at the three pairs

$$(2,3), \qquad (4,5), \qquad (5,6).$$

This is Theorem 5.1.

(b) Apart from the single slice exception $(b,K) = (1,1)$, the G- and Q-plateaus coincide. If

$$g = G(b,K+1) - G(b,K), \qquad d = Q(b,K+1) - Q(b,K)$$

and $d \neq 0$, then g and d have opposite signs and $5|d| \leq |g| \leq 8|d|$. Both constants are sharp; see Theorem 5.2.

(c) For fixed E and O, an admissible intercept J determines its coordinate bracket uniquely. The coordinates are recovered successively by taking 2-adic valuations of the decoder remainders; see Theorem 5.5.

(d) A sequence of admissible intercepts

$$J_0,\ J_0 - 3^o,\ \ldots,\ J_0 - (L-1)3^o$$

that passes the decoder and first-arrival checks corresponds to L consecutive integers in the same operation-count cell $C_{e,o}$. They therefore have common values of E, O, H, and λ; see Theorem 5.7.

(e) A class-record offset is only a candidate for a streak break. The actual break is governed by the full equation $\Delta G_r + \Delta\delta_{G,r} \neq 0$, so the residual correction may cancel a nonzero projected change.

7. Class Records, Delay Records, and Their Hierarchies.

(a) Every nonnegative odd count occurs. If

$$b_o = c_O(o), \qquad e_o = E(b_o), \qquad m_o = H(b_o), \qquad q_o = \lambda(b_o),$$

then b_o is odd, e_o and m_o are the least division count and height attained at odd count o, and

$$2 \le m_{o+1} - m_o \le 5, \qquad e_{o+1} > e_o, \qquad q_{o+1} \ge q_o - 1.$$

See Lemma 6.2.

(b) The class-record sets form the strict hierarchy

$$\mathrm{CR}_\lambda^{\le 0} \subsetneq \mathrm{CR}_O \subsetneq \mathrm{CR}_E \subsetneq \mathrm{CR}_H.$$

More precisely, $c_O(o) = c_E(e_o) = c_H(m_o)$. For $q \ge 0$, $c_\lambda(q) = 2^q$, while every realized negative charge class appears at its first downward crossing in the O-class list. These statements comprise Theorem 6.3.

(c) The delay-record sets form the strict hierarchy

$$\mathrm{DR}_\lambda \subsetneq \mathrm{DR}_O \subsetneq \mathrm{DR}_E = \mathrm{DR}_H.$$

See Theorem 6.5.

(d) The intersections between the class and delay rows are exact:

$$\mathrm{DR}_O = \mathrm{CR}_O \cap \mathrm{DR}_H,$$

and

$$\mathrm{DR}_\lambda = \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_O = \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_E = \mathrm{CR}_\lambda^{\le 0} \cap \mathrm{DR}_H.$$

See Corollary 6.6.

(e) The height-class list, together with the E- and O-counts of its entries, determines all four class-record sets. The delay records are then identified by the suffix-minimum conditions in Corollary 6.7.

(f) Every delay-record term is the class record at the value that it attains:

$$d_X(r) = c_X\big(X(d_X(r))\big).$$

Moreover, $d_E(r) = d_H(r)$ for every defined ordinal index r.

(g) Under the hypotheses of Corollary 6.10, the first exit from the G-plateau containing the initial projected coordinate occurs at either a power of two or a height-delay record, and hence at either a power of two or a height-class record.

8. Records Above a Common Tail.

(a) The relative residue identity, support structure, and four strict operation-count orders are collected in Item **5.** above. These properties supply the structural input for the same-tail record hierarchy.

(b) The same-tail class-record hierarchy is

$$\mathrm{CR}_\lambda^{\le 0}(z) \subseteq \mathrm{CR}_O(z) \subseteq \mathrm{CR}_E(z) \subseteq \mathrm{CR}_H(z),$$

and the same-tail delay-record hierarchy is

$$\mathrm{DR}_\lambda(z) \subseteq \mathrm{DR}_O(z) \subseteq \mathrm{DR}_E(z) = \mathrm{DR}_H(z).$$

The corresponding intersection identities also hold; see Theorem 6.14.

(c) For every $q \geq 0$, the nonnegative relative charge classes are explicit: $c_{\mathcal{A}}(q; z) = 2^q z$.

(d) If an admissible tail z terminates, then every global class or delay record that reaches z retains the corresponding tail-relative record property. The converse need not hold because the relative problem compares only sources that reach z; see Corollary 6.15.

(e) When $z = 1$, the same-tail constructions reduce to the global record families, and the non-strict same-tail containments sharpen to the strict global hierarchies above.

The universal terminating bounds in Conjecture 3.22 are not included among the established properties above. Proposition 3.24 proves that the conjectural residue bound $\rho(n) \leq 2^{1/3}$ is equivalent, for each terminating integer, to the corresponding one-third height formula. The assertion that these equivalent statements hold for every terminating positive integer remains conjectural.

Author Contributions and Acknowledgements

Behzad Nikzad developed the gap-bracket formalism, established its structural properties, and investigated its connections with Collatz trajectories. Discussions between Behzad Nikzad and Fatemeh Bagherzadeh led to the study of equal-height streaks. Karl Frinkle contributed computational methods for streak calculations and helped direct the project toward class and delay records. Nikzad and Frinkle subsequently worked together to connect the three principal areas of the paper: bounds on reciprocal sums and residues and their consequences for even and odd operation counts; operation-count cells and the class- and delay-record hierarchies; and equal-height streaks. Although each of these areas is of independent interest, the authors concluded that their connections are best presented within a single cohesive framework. All three authors contributed to the development of the theory and its consequences, the verification of the results, and the organization and presentation of the paper.

The authors would like to thank Chris Moretti for his detailed comments on nearly finished drafts of this document. Having an external party read this material carefully and then make well thought out suggestions on ways to make the material more clear was invaluable. Numerical investigations into some of the properties mentioned herein were performed through supercomputing resources at Southeastern Oklahoma State University and Oral Roberts University. Special thanks to administration at Southeastern Oklahoma State University for continued support. Likewise, Stephen Wheat at Oral Roberts University has been a constant source of compute capabilities in this endeavor, and his sharing of resources has been invalvuable.

Declaration of Generative AI and AI-Assisted Technologies in the Manuscript Preparation Process

During the rather lengthy development and preparation of this work, the authors used OpenAI's ChatGPT to assist in exploring candidate arguments, generating code and examples for testing observations, as well as assisting in organization and maintaining consistency and notation across the many areas of study found herein. ChatGPT was also used to resolve occasional LaTeX formatting issues. The authors independently checked the validity of all conversations, arguments, computations, and code used to help develop the material presented in this paper, and take full responsibility for its content.

References

[1] M. Elia and A. Tucker, "Consecutive Integers and the Collatz Conjecture," *INTEGERS* 15 (2015), Article A54; arXiv:1511.09141.

[2] C. J. Everett, "Iteration of the number-theoretic function $f(2n) = n$, $f(2n + 1) = 3n + 2$," *Advances in Mathematics* 25 (1977), 42–45.

[3] K. Frinkle, S. Rowland, J. Pack, M. Gordon, J. Eskue, S. Egbe, and M. Morris, "Computing streaks of consecutive numbers with the same Collatz height," *The Journal of Supercomputing* 81 (2025), Article 1338, doi:10.1007/s11227-025-07820-8.

[4] G.-G. Gao, "On consecutive numbers of the same height in the Collatz problem," *Discrete Mathematics* 112 (1993), 261–267, doi:10.1016/0012-365X(93)90240-T.

[5] L. E. Garner, "On heights in the Collatz $3n + 1$ problem," *Discrete Mathematics* 55 (1985), 57–64, doi:10.1016/S0012-365X(85)80020-0.

[6] M. D. LaDue, "Clusters of Integers with Equal Total Stopping Times in the $3x + 1$ Problem," *The Fibonacci Quarterly* 56 (2018), 156–162; arXiv:1709.02979.

[7] J. C. Lagarias, "The $3x + 1$ problem and its generalizations," *The American Mathematical Monthly* 92 (1985), 3–23.

[8] J. C. Lagarias, *The* $3x + 1$ *Problem: An Annotated Bibliography (1963–1999)*, arXiv:math/0309224, January 2011 version; see entry 68.

[9] J. C. Lagarias, "The $3x + 1$ Problem: An Overview," arXiv:2111.02635 (2021).

[10] Y. Luo, "Iteration Steps of $3x + 1$ Problem," arXiv:2506.23070v3 (16 July 2026).

[11] OEIS Foundation Inc. (2026). *The On-Line Encyclopedia of Integer Sequences*. Sequence A127789. https://oeis.org/A127789.

[12] OEIS Foundation Inc. (2026). *The On-Line Encyclopedia of Integer Sequences*. Sequence A277109. https://oeis.org/A277109.

[13] OEIS Foundation Inc. (2026). *The On-Line Encyclopedia of Integer Sequences*. Sequence A346775. https://oeis.org/A346775.

[14] E. Roosendaal, "On the $3x+1$ Problem," web resource, https://www.ericr.nl/wondrous/, accessed 24 August 2026.

[15] O. Rozier and C. Terracol, "Paradoxical behavior in Collatz sequences," *Discrete Mathematics* 349 (2026), 115167, doi:10.1016/j.disc.2026.115167; arXiv:2502.00948v5.

[16] R. Ruggiero, "The relationship between stopping time and number of odd terms in Collatz sequences," arXiv:1911.01229 (2019).

[17] G. Scollo, "Looking for Class Records in the $3x + 1$ Problem by means of the COMETA Grid Infrastructure," in R. Barbera (ed.), *Proceedings of GRID Open Days at the University of Palermo*, Consorzio COMETA (2008), 255–263. https://www.dmi.unict.it/~scollo/seminars/gridpa2007/CR3x%2B1paper.pdf.

[18] G. Scollo, “Results and programming techniques from the CR3x+1 project,” in R. Barbera, M. Iacono Manno, and M. Fargetta (eds.), *Proceedings of the Final Workshop of Grid Projects Funded by PON Ricerca 2000–2006 Avviso 1575*, Consorzio COMETA (2010), 102–110. https://www.dmi.unict.it/~scollo/seminars/gridct2009/CR3x%2B1CTpreprint.pdf.

[19] T. Tao, “Almost all orbits of the Collatz map attain almost bounded values,” *Forum of Mathematics, Pi* 10 (2022), e12.

[20] R. Terras, “A stopping time problem on the positive integers,” *Acta Arithmetica* 30 (1976), 241–252.