Modular Dvoretzky, Type-Cotype, Khinchin-Kahane and Grothendieck Inequality Problems

K. MAHESH KRISHNA
School of Mathematics and Natural Sciences
Chanakya University Global Campus
NH-648, Haraluru Village
Devanahalli Taluk, Bengaluru North District
Karnataka State, 562 110, India
Email: kmaheshak@gmail.com
Date: November 2, 2026

Abstract: We ask for noncommutative versions of following four: (1) Dvoretzky Theorem, (2) Type-Cotype Definition, (3) Khinchin-Kahane Inequalities, (4) Grothendieck Inequality.
Keywords: Dvoretzky Theorem, Type-Cotype, Khinchin-Kahane Inequality, Grothendieck Inequality, C*-algebra, Hilbert C*-module.
Mathematics Subject Classification (2020): 47B10, 47L20, 46L08, 46L05, 42C15.

hexagon:2611.00024v1

1. Modular Dvoretzky Problem

Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional Banach spaces such that $\dim(\mathcal{X}) = \dim(\mathcal{Y})$. Remember that the **Banach-Mazur distance** between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X},\mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \to \mathcal{Y} \text{ is invertible linear operator}\}.$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the standard Euclidean Hilbert space.

Theorem 1.1. *[1, 18]* ***(John Theorem)*** *If $\mathcal{X}$ is any n-dimensional real Banach space, then*

$$d_{BM}(\mathcal{X}, (\mathbb{R}^n, \langle\cdot,\cdot\rangle)) \leq \sqrt{n}.$$

Theorem 1.2. *[1,6]* ***(Dvoretzky Theorem)*** *There is a universal constant $C > 0$ satisfying the following property: If $\mathcal{X}$ is any n-dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C\frac{\varepsilon^2}{|\log \varepsilon|}\log n,$$

there exists a k-dimensional Banach subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle\cdot,\cdot\rangle)) < 1 + \varepsilon.$$

We refer [7–10, 13, 14, 26–32, 35, 36] for more information on Dvoretzky theorem. Let $\mathcal{A}$ be a unital C*-algebra with invariant basis number property (see [12] for a study on such C*-algebras) and $\mathcal{E}$, $\mathcal{F}$ be finite rank Banach modules over $\mathcal{A}$ such that $\text{rank}(\mathcal{E}) = \text{rank}(\mathcal{F})$. **Modular Banach-Mazur distance** between $\mathcal{E}$ and $\mathcal{F}$ is defined as

$$d_{MBM}(\mathcal{E},\mathcal{F}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{E} \to \mathcal{F} \text{ is invertible module homomorphism}\}.$$

Given a unital C*-algebra $\mathcal{A}$ and $n \in \mathbb{N}$, by $\mathcal{A}^n$ we mean the standard (right) module over $\mathcal{A}$. We equip $\mathcal{A}^n$ with the C*-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}$ defined by

$$\langle (a_j)_{j=1}^n, (b_j)_{j=1}^n \rangle := \sum_{j=1}^n a_j^* b_j, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

Hence the norm on $\mathcal{A}^n$ is given by

$$\|(a_j)_{j=1}^n\| := \left\| \sum_{j=1}^n a_j^* a_j \right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n \in \mathcal{A}^n.$$

Then it is well-known that $\mathcal{A}^n$ is a Hilbert C*-module. We denote this Hilbert C*-module by $(\mathcal{A}^n, \langle \cdot, \cdot \rangle)$.

Problem 1.3. ***(Modular John Problem)*** *Let $\mathcal{A}$ be the set of all unital C*-algebras with invariant basis number property. What is the best function $\Phi : \mathcal{A} \times \mathbb{N} \to (0, \infty)$ satisfying the following property: If $\mathcal{E}$ is any n-rank Banach module over a unital C*-algebra $\mathcal{A}$ with IBN property, then*

$$d_{MBM}(\mathcal{E}, (\mathcal{A}^n, \langle \cdot, \cdot \rangle)) \leq \Phi(\mathcal{A}, n).$$

Problem 1.4. ***(Modular Dvoretzky Problem)*** *Let $\mathcal{A}$ be the set of all unital C*-algebras with invariant basis number property. What is the best function $\Psi : \mathcal{A} \times \mathbb{R} \times \mathbb{N} \to (0, \infty)$ satisfying the following property: If $\mathcal{E}$ is any n-rank Banach module over a unital C*-algebra $\mathcal{A}$ with IBN property and $0 < \varepsilon$, then for every natural number*

$$k \leq \Psi(\mathcal{A}, \varepsilon, n),$$

there exists a k-rank Banach submodule $\mathcal{F}$ of $\mathcal{E}$ such that

$$d_{MBM}(\mathcal{F}, (\mathcal{A}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

2. Modular Type-Cotype and Khinchin-Kahane Problems

Let $\mathcal{H}$ be a Hilbert space, $n \in \mathbb{N}$. Recall that for any n points $h_1, \ldots, h_n \in \mathcal{H}$, we have

$$\frac{1}{2^n} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^n \varepsilon_j h_j \right\|^2 = \sum_{j=1}^n \|h_j\|^2. \tag{1}$$

It is Equality (1) which motivated the definition of type and cotype for Banach spaces.

Definition 2.1. *[1] Let $1 \leq p \leq 2$. A Banach space $\mathcal{X}$ is said to be of* ***(Rademacher) type*** *p if there exists $T_p(\mathcal{X}) > 0$ such that*

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \right)^{\frac{1}{p}} \leq T_p(\mathcal{X}) \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}}, \quad \forall x_1, \ldots, x_n \in \mathcal{X}, \ \forall n \in \mathbb{N}.$$

Definition 2.2. *[1] Let $2 \leq q < \infty$. A Banach space $\mathcal{X}$ is said to be of* ***(Rademacher) cotype*** *q if there exists $C_q(\mathcal{X}) > 0$ such that*

$$\left(\sum_{j=1}^n \|x_j\|^q \right)^{\frac{1}{q}} \leq C_q(\mathcal{X}) \left(\frac{1}{2^n} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^q \right)^{\frac{1}{q}}, \quad \forall x_1, \ldots, x_n \in \mathcal{X}, \ \forall n \in \mathbb{N}.$$

We refer [1, 5, 19, 20, 22–24, 37] for more information on type-cotype for Banach spaces. Let $\mathcal{E}$ be a (right) Hilbert C*-module over a unital C*-algebra $\mathcal{A}$, $n \in \mathbb{N}$. We see that for any n points $x_1, \ldots, x_n \in \mathcal{E}$, we

have

$$\frac{1}{2^n}\sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{-1,1\}}\left\langle\sum_{j=1}^{n}\varepsilon_j x_j,\sum_{k=1}^{n}\varepsilon_k x_k\right\rangle=\sum_{j=1}^{n}\langle x_j,x_j\rangle. \tag{2}$$

Problem 2.3. ***(Modular Type-Cotype Problems)*** *Whether there is a way to define type (we call modular-type) and cotype (we call modular-cotype) for Banach modules over C*-algebras which reduces to Equality (2) for Hilbert C*-modules?*

Problem 2.4. *Whether there is a notion of type and cotype for Banach modules over C*-algebras such that Kwapien theorems hold?, In other words, whether following statements hold?*

(i) *A Banach module $\mathcal{M}$ over a unital C*-algebra $\mathcal{A}$ has modular-type 2 and modular-cotype 2 if and only if $\mathcal{M}$ is isomorphic to a Hilbert C*-module over $\mathcal{A}$.*

(ii) *If $\mathcal{M}$ and $\mathcal{N}$ are Banach modules over a unital C*-algebra $\mathcal{A}$ of modular-type 2 and modular-cotype 2, respectively, then a bounded module morphism $T:\mathcal{M}\to\mathcal{N}$ factors through a Hilbert C*-module over $\mathcal{A}$.*

Problem 2.5. ***(Modular Khinchin-Kahane Inequalities Problems)*** *Whether there is a Khinchin-Kahane inequalities for Banach modules over C*-algebras which reduce to Equality (2) for Hilbert C*-modules?*

3. Modular Grothendieck Inequality Problem

In 1968, Lindenstrauss and Pelczynski derived the following form of Grothendieck inequality [25].

Theorem 3.1. *[1, 3, 4, 11, 15, 25, 34]* ***(Grothendieck Inequality)*** *There is a universal constant K_G satisfying the following: For any Hilbert space $\mathcal{H}$ and any $m,n\in\mathbb{N}$, if a scalar matrix $[a_{j,k}]_{1\leq j\leq m,1\leq k\leq n}$ satisfies*

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}s_jt_k\right|\leq\left(\max_{1\leq j\leq m}|s_j|\right)\left(\max_{1\leq k\leq n}|t_k|\right),\quad\forall s_j,t_k\in\mathbb{K},$$

then

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\langle u_j,v_k\rangle\right|\leq K_G\left(\max_{1\leq j\leq m}\|u_j\|\right)\left(\max_{1\leq k\leq n}\|v_k\|\right),\quad\forall u_j,v_k\in\mathcal{H}.$$

We refer [1–4, 11, 15–17, 21, 33, 34] for more information on Grothendieck inequality.

Problem 3.2. ***(Modular Grothendieck Inequality Problem - 1)*** *Let $\mathcal{A}$ be a unital C*-algebra. Whether there is a universal real constant $R_{\mathcal{A}}>0$ (which can depend upon $\mathcal{A}$) satisfying the following: Let $\mathcal{E}$ be a Hilbert C*-module over $\mathcal{A}$. If $[a_{j,k}]_{1\leq j\leq m,1\leq k\leq n}\in\mathbb{M}_{m\times n}(\mathcal{A})$ satisfies*

$$\left(\sum_{j=1}^{m}\sum_{k=1}^{n}s_j^*a_{j,k}t_k\right)^*\left(\sum_{p=1}^{m}\sum_{q=1}^{n}s_p^*a_{p,q}t_q\right)\leq 1,\quad\forall s_j,t_k\in\mathcal{A},s_j^*s_j=1,\forall 1\leq j\leq m,t_k^*t_k=1,\forall 1\leq k\leq n,$$

then

$$\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\langle u_j,a_{j,k}v_k\rangle\right)^*\left(\sum_{p=1}^{m}\sum_{q=1}^{n}\langle u_p,a_{p,q}v_q\rangle\right)\leq R_{\mathcal{A}},\quad\forall u_j,v_k\in\mathcal{E},\langle u_j,u_j\rangle=1,\forall 1\leq j\leq m,\langle v_k,v_k\rangle=1,\forall 1\leq k\leq n.$$

Problem 3.3. ***(Modular Grothendieck Inequality Problem - 2)*** *Let $\mathcal{A}$ be a unital C*-algebra. Whether there is a universal real constant $R_{\mathcal{A}} > 0$ (which can depend upon $\mathcal{A}$) satisfying the following: Let $\mathcal{E}$ be a Hilbert C*-module over $\mathcal{A}$. If $[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n} \in \mathbb{M}_{m\times n}(\mathcal{A})$ satisfies*

$$\left\| \sum_{j=1}^{m}\sum_{k=1}^{n} s_j^* a_{j,k} t_k \right\| \leq \left(\max_{1\leq j\leq m} \|s_j\| \right) \left(\max_{1\leq k\leq n} \|t_k\| \right), \quad \forall s_j, t_k \in \mathcal{A},$$

then

$$\left\| \sum_{j=1}^{m}\sum_{k=1}^{n} \langle u_j, a_{j,k} v_k \rangle \right\| \leq R_{\mathcal{A}} \left(\max_{1\leq j\leq m} \|u_j\| \right) \left(\max_{1\leq k\leq n} \|v_k\| \right), \quad \forall u_j, v_k \in \mathcal{E}.$$

References

[1] Fernando Albiac and Nigel J. Kalton. *Topics in Banach space theory*, volume 233 of *Graduate Texts in Mathematics*. Springer, 2016.

[2] Ron Blei. The Grothendieck inequality revisited. *Mem. Amer. Math. Soc.*, 232(1093):vi+90, 2014.

[3] Ron C. Blei. An elementary proof of the Grothendieck inequality. *Proc. Amer. Math. Soc.*, 100(1):58–60, 1987.

[4] Joe Diestel, Jan H. Fourie, and Johan Swart. *The metric theory of tensor products: Grothendieck's resume revisited*. American Mathematical Society, Providence, RI, 2008.

[5] Joe Diestel, Hans Jarchow, and Andrew Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[6] Aryeh Dvoretzky. Some results on convex bodies and Banach spaces. In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, pages 123–160. Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961.

[7] T. Figiel. Some remarks on Dvoretzky's theorem on almost spherical sections of convex bodies. *Colloq. Math.*, 24:241–252, 1971/72.

[8] T. Figiel. A short proof of Dvoretzky's theorem. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. XXIII*, pages 6 pp. (erratum, p. 3). 1975.

[9] T. Figiel. A short proof of Dvoretzky's theorem on almost spherical sections of convex bodies. *Compositio Math.*, 33(3):297–301, 1976.

[10] T. Figiel, J. Lindenstrauss, and V. D. Milman. The dimension of almost spherical sections of convex bodies. *Acta Math.*, 139(1-2):53–94, 1977.

[11] Shmuel Friedland, Lek-Heng Lim, and Jinjie Zhang. An elementary and unified proof of Grothendieck's inequality. *Enseign. Math.*, 64(3-4):327–351, 2018.

[12] Philip M. Gipson. Invariant basis number for C^*-algebras. *Illinois J. Math.*, 59(1):85–98, 2015.

[13] Y. Gordon, O. Guedon, and M. Meyer. An isomorphic Dvoretzky's theorem for convex bodies. *Studia Math.*, 127(2):191–200, 1998.

[14] Yehoram Gordon. Gaussian processes and almost spherical sections of convex bodies. *Ann. Probab.*, 16(1):180–188, 1988.

[15] Alexandre Grothendieck. Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.*, 16, 1955.

[16] Uffe Haagerup. A new upper bound for the complex Grothendieck constant. *Israel J. Math.*, 60(2):199–224, 1987.

[17] G. J. O. Jameson. The interpolation proof of Grothendieck's inequality. *Proc. Edinburgh Math. Soc. (2)*, 28(2):217–223, 1985.

[18] Fritz John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, pages 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.

[19] W. B. Johnson and J. Lindenstrauss, editors. *Handbook of the geometry of Banach spaces. Vol. I*. North-Holland Publishing Co., Amsterdam, 2001.

[20] W. B. Johnson and J. Lindenstrauss, editors. *Handbook of the geometry of Banach spaces. Vol. 2*. North-Holland, Amsterdam, 2003.

[21] Sten Kaijser. A simple-minded proof of the Pisier-Grothendieck inequality. In *Banach spaces, harmonic analysis, and probability theory (Storrs, Conn., 1980/1981)*, volume 995 of *Lecture Notes in Math.*, pages 33–44. Springer, Berlin, 1983.

[22] Rafal Latala and Krzysztof Oleszkiewicz. On the best constant in the Khinchin-Kahane inequality. *Studia Math.*, 109(1):101–104, 1994.

[23] Daniel Li and Herve Queffelec. *Introduction to Banach spaces: analysis and probability. Vol. 1*, volume 166 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2018.

[24] Daniel Li and Herve Queffelec. *Introduction to Banach spaces: analysis and probability. Vol. 2*, volume 167 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2018.

[25] Joram Lindenstrauss and Aleksander Pełczyński. Absolutely summing operators in L_p-spaces and their applications. *Stud. Math.*, 29:275–326, 1968.

[26] J. Matousek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[27] Ivan Matsak and Anatolij Plichko. Dvoretzky's theorem by Gaussian method. In *Functional analysis and its applications*, volume 197 of *North-Holland Math. Stud.*, pages 171–184. Elsevier Sci. B. V., Amsterdam, 2004.

[28] V. Milman. Dvoretzky's theorem—thirty years later. *Geom. Funct. Anal.*, 2(4):455–479, 1992.

[29] V. D. Milman. A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies. *Funkcional. Anal. i Priložen.*, 5(4):28–37, 1971.

[30] Vitali D. Milman and Gideon Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[31] Gilles Pisier. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 167–241. Springer, Berlin, 1986.

[32] Gilles Pisier. *The volume of convex bodies and Banach space geometry*, volume 94 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[33] Gilles Pisier. Grothendieck's theorem, past and present. *Bull. Amer. Math. Soc. (N.S.)*, 49(2):237–323, 2012.

[34] Ronald E. Rietz. A proof of the Grothendieck inequality. *Israel J. Math.*, 19:271–276, 1974.

[35] Gideon Schechtman. A remark concerning the dependence on ϵ in Dvoretzky's theorem. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 274–277. Springer, Berlin, 1989.

[36] Andrzej Szankowski. On Dvoretzky's theorem on almost spherical sections of convex bodies. *Israel J. Math.*, 17:325–338, 1974.

[37] S. J. Szarek. On the best constants in the Khinchin inequality. *Studia Math.*, 58(2):197–208, 1976.