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PARAMETER MONOTONICITY FOR GENERALIZED k-MARKOV NUMBERS

QIYUE TANG

Abstract. We prove that the order reversal asked about in Question 35 of Banaian and Huang cannot occur for generalized k-Markov numbers, answering their question negatively.

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The initial research draft was generated with ChatGPT-6-Pro from references and earlier research materials supplied by the author. The archived log records a complete proof draft within approximately 40 minutes of the session's start. Codex assisted with mathematical review, exact symbolic checks, literature searches, manuscript organization, and English-language editing and polishing. The manuscript passed review by Rethlas-OSV [12, 13, 14].

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1. Introduction

The Markov equation

$$x^2 + y^2 + z^2 = 3xyz$$

arose in Markoff's work of 1879 and 1880 on Diophantine approximation. Its positive integral solutions are called Markov triples, and their coordinates are Markov numbers. Vieta's formulas replace a coordinate z by $3xy - z$ and organize these triples into a tree. Frobenius's uniqueness conjecture, formulated in 1913, asks whether the largest coordinate determines a triple up to permutation. Aigner's monograph [1] develops this history and its connections with continued fractions and combinatorial counting.

The tree also labels Markov numbers by reduced fractions q/p in $[0, 1]$. Neighboring fractions are combined by the Farey sum $q/p \oplus v/r = (q + v)/(p + r)$. Aigner proposed three ordering conjectures concerning fixed numerator, fixed denominator, and fixed sum of numerator and denominator. In the coordinates (p, q) used here, the corresponding comparisons increase p with q fixed, increase q with p fixed, or replace (p, q) by $(p + i, q - i)$ with $i > 0$, respectively. A label (p, q) is primitive if $\gcd(p, q) = 1$; the comparisons are made between admissible primitive labels. Rabideau and Schiffler [2] used continuants and continued fractions to prove the fixed-numerator conjecture. Lee, Li, Rabideau, and Schiffler [3] proved the remaining orderings and obtained broader comparisons controlled by the slope of the segment joining two labels. In particular, slopes at least $-8/7$ give increase with the first coordinate, while slopes at most $-5/4$ give decrease. Their construction also assigns values to noncoprime labels by means of Markov distance and snake graphs. Independently, Lagisquet, Pelantová, Tavenas, and Vuillon [4] proved the three orderings using lattice paths and products of two fixed matrices; their argument also treats an extension to noncoprime labels.

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The sharp boundary slopes were subsequently determined by two approaches. Gaster [5] used the stable norm of the modular torus and calculated its corner slopes to characterize monotonicity on relatively prime pairs. Huang [6] used Markov distance and a Ptolemy inequality to treat all integer labels. In the classical case, his upper and lower thresholds are approximately -1.1432 and -1.2417. Between these thresholds, a line can contain values that first decrease and then increase. Huang's analysis compares successive ratios along a line and relates their limiting behavior to the boundary sequences near slopes zero and one. This extends the ordering problem from three distinguished directions to arbitrary rational directions.

The parameter generalization comes from the equation

$$x^2 + y^2 + z^2 + k(xy + xz + yz) = 3(k+1)xyz,$$

introduced by Gyoda and Matsushita [11], where k is a nonnegative integer. Its coordinates are called k-Markov numbers. The case $k = 0$ is classical; the case $k = 1$ was studied by Banaian and Sen [7] through generalized cluster algebras, continued fractions, and snake graphs. For arbitrary k, Banaian [8] proved the analogues of Aigner's three orderings. Her proof uses generalized Markov distance, skein relations, and fence posets. These combinatorial models make it possible to compare labels while retaining the dependence on k.

Banaian and Huang [9] extended the labels to

$$\mathcal{R} = \{(p, q) \in \mathbb{Z}^2 : p > q > 0\}$$

by counting order ideals in fence posets, and studied monotonicity along lines for each fixed parameter. They denoted the resulting values by $m^{(k)}_{(p,q)}$ and described an intermediate interval of slopes, the gray zone, between the increasing and decreasing regimes. Their concluding discussion asks how comparisons change when the parameter changes. Example 33 compares $(8, 7)$ and $(13, 1)$: the order changes from smaller to larger as k increases from 2 to 3. Question 35 asks whether the reverse change can occur when the connecting slope lies in the gray zone for both parameters. We prove that it cannot.

For noncoprime labels this requires an additional argument: the values on a multiple of a primitive ray satisfy a second-order recurrence rather than a power law. Our parameter comparison therefore includes a uniform estimate for the discrepancy between the logarithm of a counting polynomial and its homogeneous Farey length.

We use the following explicit upper endpoint of that zone. For real $k \geq 2$, define

$$\beta = 3k^2 + 8k + 6, \qquad \Delta = \beta^2 - 4, \qquad \mu = \frac{\beta + \sqrt{\Delta}}{2}, \tag{1}$$

$$C_\partial = \frac{3(k+1)^2}{3k+2}\left(1 + \sqrt{\frac{\beta-2}{\beta+2}}\right), \qquad U(k) = -\frac{\log C_\partial}{\log(\mu/C_\partial)}. \tag{2}$$

Every square root in these formulas is positive, and log denotes the natural logarithm. The denominator in (2) is positive, as shown in Lemma 4.3. The constant C_∂ equals $3(k+1)$ times the diagonal asymptotic coefficient in [9, Section 5.3]. The expressions agree using $\beta - 2 = (k+2)(3k+2)$ and

$$3k^4 + 17k^3 + 34k^2 + 28k + 8 = (k+1)(k+2)(\beta - 2).$$

Thus (2) is their upper endpoint.

For each label $P = (p, q) \in \mathcal{R}$, write $M_P(k) = M_{p,q}(k) = m^{(k)}_{(p,q)}$ at integer parameters. Section 2 defines its polynomial extension to real parameters. Our main result is the following stronger monotonicity statement.

Theorem 1.1. *Let $k_0 \geq 2$ be real and let $P = (p, q), Q = (p', q') \in \mathcal{R}$. If $p < p'$ and*

$$\frac{q' - q}{p' - p} < U(k_0),$$

then $M_P(k)/M_Q(k)$ is strictly increasing on $[k_0, \infty)$.

The proof appears in Section 5. To state the consequence for the original question, let $L(k)$ denote the lower gray-zone endpoint of [9]. Explicitly, it is

$$L(k) = -\frac{\log \dfrac{2k + 3 + \sqrt{4k^2 + 12k + 5}}{2}}{\log \left(\dfrac{3(k+1)^2}{2(2k+1)} \left(1 + \dfrac{2k+1}{\sqrt{4k^2 + 12k + 5}} \right) \right)}.$$

Corollary 1.2. *There do not exist $P = (p, q), Q = (p', q') \in \mathcal{R}$ and integers $1 < k_1 < k_2$ such that $p < p'$, the slope $(q' - q)/(p' - p)$ belongs to $(L(k_i), U(k_i))$ for $i = 1, 2$, and*

$$M_P(k_1) > M_Q(k_1), \qquad M_P(k_2) < M_Q(k_2).$$

Consequently, Question 35 of Banaian and Huang has a negative answer.

Proof. The initial gray-zone condition gives the hypothesis of Theorem 1.1 with $k_0 = k_1 \geq 2$. The ratio, already greater than one at k_1, is larger at k_2. □

The argument compares a homogeneous length associated with the Farey recurrence and the logarithm of the original counting polynomial. A uniform root disk for the fence polynomials gives logarithmic derivative bounds. These imply that the corner masses of a convex Farey interpolation decrease with k. An exponential estimate on Farey gaps accounts for its entire curvature, and hence makes the difference between two parameter values concave. The diagonal boundary then gives a positive margin, which dominates the error for every radial multiple. The Farey-gap estimate follows the approach of Doan, Li, and Nguyen [10, Section 3]; we establish the needed estimate here for the shifted recurrence.

2. Counting polynomials and a uniform root disk

We recall the fence construction of [9, Definitions 2 and 8]. An order ideal of a finite poset is a subset closed under passing to smaller elements; the empty subset is included. A fence poset has a path as its Hasse diagram. Consider the triangulation of the plane by the lines through integer points of slopes $0, -1, \infty$, with each line divided into unit lattice segments. Orient the segment from $(0, 0)$ to (p, q). Perturb its internal intersections with half-integral points slightly to the left, keeping its endpoints fixed. This is the left-biased path of [9]; at a midpoint a straight primitive segment has the same convention.

For each crossed lattice edge τ, introduce a chain of $k + 1$ vertices $(\tau, 0), \ldots, (\tau, k)$. The chain descends in this order when the crossing is closer to the endpoint on the right of the oriented path, and ascends otherwise. For successive crossings τ, τ', connect (τ, k) to $(\tau', 0)$ by a descending edge if their common lattice endpoint is on the right, and by an ascending edge if it is on the left. The number of order ideals is $M_{p,q}(k)$. In particular, the direction of each chain and each connecting edge is independent of k. This construction applies to nonprimitive labels as well.

For later calculations we also give the recursive description. For primitive boundary labels put

$$M_{1,0} = M_{0,1} = 1, \qquad M_{1,1} = k + 2.$$

For vectors $A = (p, q), B = (r, v)$ write $\det(A, B) = pv - qr$. Starting with the interval between $(1, 0)$ and $(1, 1)$, subdivide a neighboring pair A, B of determinant one by inserting $A + B$. If W is the old opposite label, the Vieta recurrence is

$$M_{A+B} = 3(k + 1)M_A M_B - k(M_A + M_B) - M_W. \tag{3}$$

The initial old value is $M_{0,1} = 1$; in the two new subintervals the old label is the other parent. This Stern–Brocot construction assigns every primitive label in $\mathcal{R}$ once. For primitive $A \in \mathcal{R}$, define auxiliary polynomials F_d by

$$F_0 = 0, \quad F_1 = M_A, \quad F_{d+1} = (3(k+1)M_A - k)F_d - F_{d-1}, \quad M_{dA} = F_d \ (d \geq 1). \tag{4}$$

The auxiliary value F_0 is used only in this recurrence. The correspondence of (3) with the primitive fence counts is the Farey labeling of the k-Markov equation; (4) is [9, Proposition 18]. They therefore give the original values at positive integral k.

Lemma 2.1. *For $P = (p, q) \in \mathcal{R}$, the fence count extends to a polynomial with nonnegative integer coefficients. Its degree and leading coefficient are*

$$\deg M_P = p + q - 1, \qquad \operatorname{lc} M_P = 2^{p-q}3^{q-1}. \tag{5}$$

Here the degree is the largest exponent with nonzero coefficient, and the leading coefficient is the coefficient of that exponent. For $k > 0$,

$$0 \leq \frac{M_P'(k)}{M_P(k)} \leq \frac{p+q-1}{k}. \tag{6}$$

Primes denote derivatives with respect to the real parameter k.

Proof. Record whether the last vertex of a partial fence belongs to the ideal. A chain with n edges has one of the two transfer matrices

$$\mathsf{U}(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{L}(n) = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.$$

Start with the two states $(1, 1)$ for the first vertex and sum the two final states. Each variable chain contributes $\mathsf{U}(k)$ or $\mathsf{L}(k)$ and each connecting edge contributes $\mathsf{U}(1)$ or $\mathsf{L}(1)$. Thus the count is a polynomial with nonnegative integer coefficients. Equality with the recurrences at all positive integers makes their polynomial expressions identical.

Along $q = 1$, the initial values $M_{0,1} = 1$ and $M_{1,1} = k + 2$ give

$$M_{p,1} = (2k+3)M_{p-1,1} - M_{p-2,1} - k \quad (p \geq 2).$$

Induction gives degree p and leading coefficient 2^{p-1}. At every other primitive subdivision both parents have positive second coordinate. In (3), the highest term of $3kM_AM_B$ has strictly larger degree than the subtraction terms. Indeed, writing the parent degrees as $D_A, D_B \geq 1$, the product term has degree $D_A + D_B + 1$, whereas kM_A and kM_B have degrees $D_A + 1$ and $D_B + 1$. The old label is a parent from the preceding subdivision, so its coordinate sum is smaller than that of $A + B$. The leading coefficient is consequently $3 \operatorname{lc}(M_A) \operatorname{lc}(M_B)$. When a parent is $(1, 1)$, its degree and leading coefficient are 1 and 1, which agree with the same formula. Adding parent coordinates now gives (5).

If $P = dA$, (4) gives

$$\deg M_{dA} = d(\deg M_A + 1) - 1, \qquad \operatorname{lc} M_{dA} = 3^{d-1}(\operatorname{lc} M_A)^d,$$

which yields (5). Finally, kM_P'/M_P is a weighted average of the exponents of a polynomial with nonnegative coefficients, proving (6). □

Proposition 2.2. *Every complex root ζ of M_P, for $P \in \mathcal{R}$, satisfies $|\zeta + 2| \leq 4$.*

Proof. Partition the oriented edge word of the fence into maximal runs of equal direction, of lengths $n_1, \ldots, n_r$. For complex arguments $x_1, \ldots, x_j$, define the continuant recursively by

$$K() = 1, \quad K(x_1) = x_1, \quad K(x_1, \ldots, x_j) = x_jK(x_1, \ldots, x_{j-1}) + K(x_1, \ldots, x_{j-2}) \quad (j \geq 2).$$

The ideal count is $K(n_1 + 1, n_2, \ldots, n_{r-1}, n_r + 1)$ for $r \geq 2$, and $n_1 + 2$ for $r = 1$. This follows by multiplying the transfer matrices, or from

$$\mathsf{U}(n) = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{L}(n) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}.$$

The initial and final state sums add one to the first and last digits. For example, the digits for $(2,1)$ are $k+2, 2k+2$, giving $2k^2+6k+5$.

Write each digit as $a_i k + b_i$, where a_i, b_i are nonnegative integers. If $a_i > 0$, its run contains a_i variable chains, at most $a_i - 1$ internal connecting edges, and at most two end contributions. At a fence endpoint the added one replaces the missing outside connecting edge. Consequently

$$0 \le b_i \le a_i + 1 \le 2a_i. \tag{7}$$

If $a_i = 0$, the digit is a positive integer.

The continuant is the determinant of the complex symmetric tridiagonal matrix with diagonal $a_i k + b_i$ and both adjacent off-diagonals equal to i, where $\mathrm{i}^2 = -1$. Eliminate every maximal block of constant digits by a Schur complement. The determinant of such a block is a positive integer continuant. Each endpoint diagonal entry of its inverse is the ratio of a shorter continuant to the whole continuant, hence lies in $(0,1]$; the corner entry has modulus the reciprocal of the whole continuant, hence at most one. More explicitly, for a constant block $c_1, \dots, c_s$, cofactor expansion gives endpoint inverse entries

$$\frac{K(c_2,\dots,c_s)}{K(c_1,\dots,c_s)}, \qquad \frac{K(c_1,\dots,c_{s-1})}{K(c_1,\dots,c_s)},$$

and corner entry $(-\mathrm{i})^{s-1}/K(c_1,\dots,c_s)$. Empty continuants equal one. The recurrence, read from either end, bounds both ratios by one because every $c_i \ge 1$. For conformable matrix blocks P, Q, R, S, with S square and invertible, the Schur identity

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det(S)\det(P - QS^{-1}R)$$

applies since each eliminated block S has positive determinant. Each incident coupling is i, so the diagonal correction is $-\mathrm{i}^2 (S^{-1})_{jj} = (S^{-1})_{jj}$.

Thus elimination adds at most two real corrections in $[0,1]$ to each surviving diagonal, and creates adjacent couplings of modulus at most one. Divide the ith surviving row and column by $\sqrt{a_i}$. Up to a positive constant the resulting determinant is $\det(kI + T)$, where every diagonal of T belongs to $[0,4]$, since $(b_i+2)/a_i \le 4$, and every off-diagonal row sum of absolute values is at most two. For an eigenvalue ξ of T, choose an eigenvector component of maximal absolute value, say the jth. Its row equation implies $|\xi - T_{jj}| \le \sum_{i \ne j} |T_{ji}| \le 2$. Since $0 \le T_{jj} \le 4$, it follows that $|\xi - 2| \le 4$. The roots of the determinant are the negatives of the eigenvalues of T, proving the assertion. Its leading coefficient is nonzero, so this accounts for all roots with multiplicity. □

Corollary 2.3. *For $P = (p,q) \in \mathcal{R}$ and $k \ge 2$,*

$$\frac{M_P'}{M_P} \ge \frac{p+q-1}{k+6}, \qquad \frac{M_P'}{M_P} \ge \frac{2}{k+2}. \tag{8}$$

Proof. For a root ζ with $|\zeta + 2| \le 4$ and $k \ge 2$,

$$\begin{aligned}(k+6)(k-\Re\zeta) - |k-\zeta|^2 &= 4(k+2) + (k-2)\Re(\zeta+2) - |\zeta+2|^2 \\ &\ge (4 + \Re(\zeta+2))(k-2) \ge 0.\end{aligned}$$

Since $M_P(k) > 0$, no denominator vanishes. Hence $\Re(1/(k-\zeta)) \ge 1/(k+6)$. Summing over the roots with their algebraic multiplicities gives the first inequality. If the degree is at least four, it implies the second. Degrees two and three occur only at $(2,1)$ and $(3,1)$, respectively, with

$$M_{2,1} = 2k^2 + 6k + 5, \qquad M_{3,1} = 4k^3 + 18k^2 + 26k + 13.$$

The corresponding values of $(k+2)M' - 2M$ are $2(k+1)$ and $2(k+1)(2k^2+10k+13)$, both positive. □

3. Farey lengths and their curvature

Fix a real $k \geq 2$. For a primitive label A in the closed sector $0 \leq q \leq p$, define the shifted value and its algebraic length by

$$t_A = 3(k+1)M_A - k, \qquad \ell_A = \operatorname{arcosh}(t_A/2) = \log \frac{t_A + \sqrt{t_A^2 - 4}}{2}. \tag{9}$$

We call t_A the trace parameter associated with A. The boundary values are $t_{1,0} = 2k+3$ and $t_{1,1} = \beta$. Put $\kappa = k(k+2)$ and $\nu = k^2(2k+3)$. For two parents with trace parameters a, b, and new and old opposite parameters c, d, (3) becomes

$$c + d = ab - \kappa, \qquad a^2 + b^2 + c^2 - abc + \kappa(a+b+c) + \nu = 0. \tag{10}$$

The second polynomial is invariant under $c \mapsto ab-\kappa-c$ and vanishes at $(2k+3, 2k+3, 2k+3)$. One mutation produces the initial triple $(2k+3, \beta, 2k+3)$, proving (10) throughout the tree.

Every new inward trace is larger than its parents. Indeed, if $a \leq b$ and $d \leq b$, then

$$c - b \geq b(a-2) - \kappa \geq (2k+3)(2k+1) - k(k+2) = 3(k+1)^2 > 0.$$

This preserves the induction hypothesis for both new intervals. In particular c is the larger root of the quadratic in (10), and all traces are at least $2k+3$. Since $\kappa < (2k+3)^2/4$,

$$c \geq \frac{ab - \kappa}{2} \geq \frac{3}{8}ab.$$

Set $\gamma = 3(2k+3)/8 > 1$. The normalized parameters $3t_A/8$ are supermultiplicative under subdivision. Both initial labels have first coordinate one, so induction gives

$$t_{p,q} \geq \frac{8}{3}\gamma^p. \tag{11}$$

On rational slopes define $f_k(q/p) = \ell_{p,q}/p$ for primitive (p, q) in the closed sector. We will extend this function continuously. For any integer label $P = dA \in \mathcal{R}$ with A primitive, put

$$\Theta_k(P) = d\ell_A. \tag{12}$$

Lemma 3.1. *Along a Farey fan centered at a primitive label with trace a, neighboring traces can be indexed by $n \in \mathbb{Z}$ as*

$$z_n = \frac{\kappa}{a-2} + A_+\lambda^n + A_-\lambda^{-n}, \qquad \lambda = \frac{a + \sqrt{a^2-4}}{2}, \tag{13}$$

where $A_+, A_- > 0$ and

$$A_+A_- = \frac{(a+k)^2(a+k^2-2)}{(a-2)(a^2-4)} > 1. \tag{14}$$

Here the fan is the two-sided recurrence $z_{n+1} = az_n - z_{n-1} - \kappa$ obtained by keeping the center fixed.

Proof. Subtracting $\kappa/(a-2)$ makes the recurrence homogeneous, with characteristic roots λ, λ^{-1}. If $z_0 = b$ and $z_1 = c$, solving these initial conditions gives

$$A_+ = \frac{c - \kappa/(a-2) - \lambda^{-1}(b - \kappa/(a-2))}{\lambda - \lambda^{-1}}, \quad A_- = \frac{\lambda(b - \kappa/(a-2)) - c + \kappa/(a-2)}{\lambda - \lambda^{-1}}.$$

Their product has numerator

$$a\left(b - \frac{\kappa}{a-2}\right)\left(c - \frac{\kappa}{a-2}\right) - \left(b - \frac{\kappa}{a-2}\right)^2 - \left(c - \frac{\kappa}{a-2}\right)^2$$

and denominator $a^2 - 4$. Using (10) therefore yields

$$A_+A_- = \frac{a^2 + \kappa a + \nu + \kappa^2/(a-2)}{a^2-4},$$

which factors as (14) and exceeds one. Since

$$(2k+3)(2k+1) - 4\kappa = 3,$$

we have $\kappa/(a-2) < (2k+3)/4$. Thus $A_+ + A_- = z_0 - \kappa/(a-2) > 0$. Their positive product proves that both coefficients are positive. □

Proposition 3.2. *The values of f_k on rational slopes extend to a continuous convex function on $[0,1]$, with finite one-sided endpoint derivatives. At a primitive interior label $A = (p,q)$ the derivative jump equals*

$$f'_{k,+}(q/p) - f'_{k,-}(q/p) = pJ(k, t_A), \qquad J(k,t) = \log \frac{(t+k)^2(t+k^2-2)}{(t-2)(t^2-4)}. \tag{15}$$

The signs $+$ and $-$ denote right and left derivatives with respect to the slope variable.

Proof. By Lemma 3.1,

$$\frac{z_n}{2} = \frac{\kappa}{2(a-2)} + \sqrt{A_+ A_-}\cosh\left(n\log\lambda + \tfrac{1}{2}\log(A_+/A_-)\right).$$

Since $\kappa > 0$ and $A_+A_- > 1$, monotonicity of arcosh gives

$$\operatorname{arcosh}(z_n/2) > \left|n\log\lambda + \tfrac{1}{2}\log(A_+/A_-)\right|.$$

Take three consecutive primitive vectors V, U, W in a Farey sequence of order N, namely the ordered reduced fractions in $[0,1]$ with denominator at most N. They satisfy $V + W = mU$ for an integer $m \geq 1$. In the fan centered at U, the outer vectors have indices $0, -m$, up to reversing the orientation. Indeed, if V_0, W_0 are the Stern–Brocot parents of U, the neighbors have forms $V_0 + nU, W_0 + jU$, with $m = n + j + 1$. The triangle inequality therefore gives

$$\ell_V + \ell_W > m\ell_U. \tag{16}$$

Write $V = (p_V, q_V)$, $U = (p_U, q_U)$, and $W = (p_W, q_W)$ in increasing slope order. The determinant-one identities give successive slope distances $1/(p_V p_U)$ and $1/(p_U p_W)$. Hence the two interpolation slopes are $p_V\ell_U - p_U\ell_V$ and $p_U\ell_W - p_W\ell_U$. Their difference is $p_U(\ell_V + \ell_W - m\ell_U) > 0$, since $p_V + p_W = mp_U$. Hence every finite interpolation is convex. Subdivision decreases it pointwise, and it is bounded below by zero and above by the endpoint chord. Its limit is finite and convex in $(0,1)$, hence continuous there, and retains every rational value once that value has entered the sequence.

For an interior $A = (p,q)$ choose an integer $B = (r,v)$ with $\det(A,B) = 1$, so the right-hand fan has labels $B + nA$ for large n. The other side has labels $nA - B$. Here negative fan indices represent the same unoriented labels after reversing the sign of the vector. As $n \to \infty$, formula (13) gives

$$\ell_{B+nA} = n\ell_A + \log A_+ + o(1), \qquad \ell_{nA-B} = n\ell_A + \log A_- + o(1).$$

The slope differences from q/p are exactly $1/[p(np+r)]$ and $-1/[p(np-r)]$. Dividing the differences of the normalized lengths by these quantities, the right secant is $p\ell_{B+nA} - (np+r)\ell_A$ and the left secant is $(np-r)\ell_A - p\ell_{nA-B}$. Thus they tend to

$$f'_{k,+}(q/p) = p\log A_+ - r\ell_A, \qquad f'_{k,-}(q/p) = -p\log A_- - r\ell_A. \tag{17}$$

For a convex function these limits are its one-sided derivatives. Their difference and (14) give (15).

At slope zero, use the fan $A = (1,0)$, $B = (1,1)$. The values at $1/(n+1)$ tend to $\ell_{1,0}$, and the endpoint secants tend to the finite number $\log A_+ - \ell_{1,0}$. At slope one, the fan centered at $(1,1)$ with neighboring labels $(j+1, j)$ gives the same conclusions. Convexity then supplies endpoint continuity and finite one-sided derivatives. □

For a continuous convex function, its distributional second derivative is a nonnegative measure: its mass on (u,v) is $f'_-(v) - f'_+(u)$. The finite endpoint derivatives show that this measure has finite total mass on $(0,1)$. Write δ_x for the unit point mass at x.

Proposition 3.3. *The curvature measure of f_k is purely atomic:*

$$f_k'' = \sum_{\substack{p>q>0 \\ \gcd(p,q)=1}} pJ(k, t_{p,q})\, \delta_{q/p}. \tag{18}$$

The series is an equality of finite measures on $(0,1)$.

Proof. Let a, b be the trace parameters at neighboring Farey labels and c, d the new and old opposite traces. Write $\ell_a = \operatorname{arcosh}(a/2)$ and $\ell_b = \operatorname{arcosh}(b/2)$, and put

$$\widehat{c} = 2\cosh(\ell_a + \ell_b) = \frac{ab + \sqrt{(a^2-4)(b^2-4)}}{2},$$

Substitution into the quadratic (10) gives

$$(\widehat{c} - c)(\widehat{c} - d) = 4 + \kappa(a + b + \widehat{c}) + \nu. \tag{19}$$

The number $\widehat{c}$ lies to the right of the quadratic's vertex, and the right side is positive, so $\widehat{c} > c$. Also

$$\widehat{c} - d \geq \tfrac{1}{2}\sqrt{(a^2-4)(b^2-4)}.$$

Since $a, b \geq 2k+3$ and $\widehat{c} \leq ab$, (19) gives a finite positive bound C_k, depending only on the fixed parameter k, such that $0 < \widehat{c} - c \leq C_k$. For example, $\sqrt{(a^2-4)(b^2-4)} \geq (45/49)ab$ because $a, b \geq 7$. One can take

$$C_k = \frac{98}{45}\left(\frac{4+\nu}{49} + \frac{9}{7}\kappa\right)$$

at this step, by dividing the right side of (19) by ab. Bounds denoted by C_k below may be enlarged.

Write $\lambda_a = (a + \sqrt{a^2-4})/2$ and similarly λ_b. Let $A_{a,b}$ be the growing coefficient of the fan centered at a with initial traces $z_0 = b, z_1 = c$. Solving these two equations gives

$$A_{a,b} = \lambda_b - \frac{\widehat{c} - c + \dfrac{\kappa}{a-2}(1 - \lambda_a^{-1})}{\sqrt{a^2-4}}. \tag{20}$$

It follows that $0 \leq \lambda_b - A_{a,b} \leq C_k/a$. To bound the logarithm uniformly, use $c \geq 3ab/8$, $\lambda_a^{-1} \leq 1/(a-1)$, and $\kappa/(a-2) < (2k+3)/4$ in the expression for the coefficient. The positive term is bounded using $\sqrt{a^2-4} \leq a$ and $\lambda_b \leq b$; the negative terms use $\sqrt{a^2-4} \geq a-1$ and $\lambda_b \geq b-1$. Thus

$$\frac{A_{a,b}}{\lambda_b} \geq \frac{3}{8} - \frac{2k+3}{(2k+2)^3} - \frac{2k+3}{4(2k+2)^2} \geq \frac{3}{8} - \frac{7}{216} - \frac{7}{144} = \frac{127}{432} > \frac{1}{4}.$$

For $0 < x \leq 1$ one has $-\log x \leq (1-x)/x$, by integrating $1/u$ from x to 1. Apply this with $x = A_{a,b}/\lambda_b > 1/4$ and use $\lambda_b \geq b - 1 \geq 6b/7$. Consequently

$$0 \leq \ell_b - \log A_{a,b} \leq \frac{C_k}{ab}. \tag{21}$$

If the neighboring vectors are (p, q) and (r, v) in increasing slope order, the right derivative at q/p is $p \log A_{a,b} - r\ell_a$. The left derivative at v/r is $p\ell_b - r \log A_{b,a}$. Their difference is the curvature mass in the open gap, namely

$$p(\ell_b - \log A_{a,b}) + r(\ell_a - \log A_{b,a}) \leq C_k \frac{p+r}{ab}.$$

For neighbors in the Farey sequence of order N one has $p, r \leq N$ and $p + r > N$. There are at most $(N+1)^2$ gaps. By (11), the sum of their masses is at most $C_k N^3 \gamma^{-N}$, which tends to zero. The gaps exhaust the complement of the finitely many rational slopes of denominator at most N. Hence no residual measure remains after the point masses (15) are removed. □

Proposition 3.4. *If $k_2 > k_1 \geq 2$, then $f_{k_2} - f_{k_1}$ is concave on $[0,1]$.*

Proof. For $t > 2$ and $k \geq 2$, partial differentiation of (15) gives

$$J_t = -\frac{2(k+2)}{(t+k)(t-2)} - \frac{k^2-4}{(t+k^2-2)(t+2)} < 0.$$

For an interior primitive label, Corollary 2.3 implies

$$t'_A = 3M_A + 3(k+1)M'_A - 1 \geq \frac{(3k+4)t_A + 2k^2 + k - 2}{(k+1)(k+2)}.$$

The other partial derivative is $J_k = 2/(t+k) + 2k/(t+k^2-2)$. Substitute the lower bound into $J_k + J_t t'_A$, using $J_t < 0$, and bring these three rational terms to a common denominator. The result is

$$\frac{d}{dk} J(k, t_A(k)) \leq -\frac{(k^2-2)t_A^2 + (6k^3 - 3k^2 + 20)t_A + 4k^3 + 26k^2 - 32}{(k+1)(t_A-2)(t_A+2)(t_A+k^2-2)} < 0. \tag{22}$$

All coefficients in the numerator, viewed as a polynomial in t_A, are positive for $k \geq 2$. Proposition 3.3 now shows that the distributional second derivative of $f_{k_2} - f_{k_1}$ is the difference of two finite atomic measures with strictly smaller mass at every atom. It is nonpositive, which proves concavity. □

4. The diagonal boundary and the slope margin

The scalar functions in (1)–(2) determine both the diagonal boundary derivatives and the margin needed to control finite labels. In this section set

$$h = \frac{3k+4}{\sqrt{\Delta}}, \qquad r = \frac{6k}{(k+1)(3k+2)(\beta+2)}. \tag{23}$$

Lemma 4.1. *The diagonal endpoint values are*

$$f_k(1) = \log \mu, \qquad f'_{k,-}(1) = \log \mu - \log C_\partial. \tag{24}$$

Moreover,

$$(\log C_\partial)' = h + r, \qquad (\log(\mu/C_\partial))' = h - r. \tag{25}$$

Proof. The initial values $M_{1,0} = 1$ and $M_{2,1} = 2k^2 + 6k + 5$, together with (3), give

$$M_{j+1,j} = \beta M_{j,j-1} - M_{j-1,j-2} - k(k+2) \quad (j \geq 2).$$

The constant particular solution is $k/(3k+2)$ because $\beta - 2 = (k+2)(3k+2)$. The homogeneous roots are μ, μ^{-1}. Solving the two initial equations, using $\mu + \mu^{-1} = \beta$, gives coefficients $C_\partial/[3(k+1)]$ and $C_\partial/[3(k+1)\mu]$. Thus

$$M_{j+1,j} = \frac{k}{3k+2} + \frac{C_\partial}{3(k+1)}(\mu^j + \mu^{-j-1}). \tag{26}$$

The trace therefore has leading term $C_\partial \mu^j$, and

$$\ell_{j+1,j} = j \log \mu + \log C_\partial + o(1) \qquad (j \to \infty).$$

Divide by $j+1$ and compute the endpoint secant slope to obtain (24).

Directly from (1), $(\log \mu)' = 2h$ and

$$\log \frac{C_\partial}{\sqrt{\mu}} = \log \frac{6(k+1)^2}{(3k+2)\sqrt{\beta+2}}, \qquad \left(\log \frac{C_\partial}{\sqrt{\mu}}\right)' = r. \tag{27}$$

For the second identity, logarithmic differentiation of its middle expression gives

$$\frac{2}{k+1} - \frac{3}{3k+2} - \frac{3k+4}{\beta+2} = \frac{6k}{(k+1)(3k+2)(\beta+2)} = r.$$

Together with $(\log \mu)' = 2h$, this proves (25). □

Proposition 4.2. *Let $P=(p,q), Q=(p+b,q-a)\in\mathcal{R}$, where $a,b>0$. For $k_2>k_1\geq 2$,*

$$[\Theta_{k_2}(P)-\Theta_{k_1}(P)]-[\Theta_{k_2}(Q)-\Theta_{k_1}(Q)]$$
$$\geq \int_{k_1}^{k_2}\{a(h-r)-b(h+r)\}\,dk. \tag{28}$$

Proof. Put $g=f_{k_2}-f_{k_1}$. It is concave by Proposition 3.4. By Lemma 4.1,

$$g'_-(1)=\int_{k_1}^{k_2}(h-r)\,dk, \qquad g(1)-g'_-(1)=\int_{k_1}^{k_2}(h+r)\,dk.$$

For almost every $u\in(0,1)$, concavity and the endpoint supporting line give

$$g'(u)\geq g'_-(1), \qquad g(u)-ug'(u)\leq g(1)-g'_-(1).$$

The homogeneous function $(x,y)\mapsto xg(y/x)$, defined for $x>y>0$, is locally Lipschitz. Integrate its first partial derivative along the horizontal segment from (p,q) to $(p+b,q)$, and its second partial derivative along the vertical segment from $(p+b,q)$ to $(p+b,q-a)$. Both segments remain in this sector. The substitutions $u=q/x$ and $u=y/(p+b)$ preserve null sets on these compact segments, so the almost-everywhere inequalities can be integrated there. At integer labels $xg(y/x)=\Theta_{k_2}(x,y)-\Theta_{k_1}(x,y)$. The horizontal change is at most $b(g(1)-g'_-(1))$ and the vertical change at most $-ag'_-(1)$, proving (28). □

Lemma 4.3. *For $k\geq 2$, $U(k)<-1$ and $U'(k)>0$. If positive integers a,b satisfy $-a/b<U(k_0)$ for some $k_0\geq 2$, then for every $k\geq k_0$,*

$$a(h-r)-b(h+r)>\frac{h}{10}. \tag{29}$$

Proof. Define the dimensionless functions

$$\omega=\frac{2\log(C_\partial/\sqrt{\mu})}{\log\mu}, \qquad z=\frac{r}{h}.$$

By (27),

$$\omega'=\frac{2h}{\log\mu}(z-\omega), \qquad U=-\frac{1+\omega}{1-\omega}. \tag{30}$$

We prove $0<\omega<1$ and $z/\omega<9/10$. First,

$$\Delta-(3k+4)^2(k+1)^2=16+40k+27k^2+6k^3>0,$$

so $h<1/(k+1)$. Also $\log\mu$ increases and $\log\mu(2)=\log(17+12\sqrt{2})>3$. Thus $2h/\log\mu<1/(k+1)$. The expression

$$z=\frac{6k\sqrt{k+2}}{(k+1)\sqrt{3k+2}(3k+4)\sqrt{\beta+2}}$$

has logarithmic derivative

$$\frac{z'}{z}=\frac{1}{k}+\frac{1}{2(k+2)}-\frac{1}{k+1}-\frac{3}{2(3k+2)}-\frac{3}{3k+4}-\frac{3k+4}{\beta+2}.$$

Putting these terms over a common denominator gives

$$\left(-\frac{z'}{z}-\frac{1}{k+1}\right)k(k+2)(3k+2)(3k+4)(\beta+2)$$
$$=3232+7552(k-2)+6056(k-2)^2$$
$$+2244(k-2)^3+396(k-2)^4+27(k-2)^5>0.$$

Hence $-z'/z>2h/\log\mu$. The numerator in the definition of ω is positive, since its value at 2 is $2\log(9/8)>0$ and its derivative is $2r>0$. Differentiating the ratio gives

$$\left(\frac{\omega}{z}\right)'=\frac{2h}{\log\mu}+\frac{\omega}{z}\left(-\frac{z'}{z}-\frac{2h}{\log\mu}\right)>0.$$

At $k=2$, the bounds $\log(9/8) > 1/9$, $\log\mu(2) < 4$, and $z(2) = 1/(15\sqrt{2}) < 1/20$ imply $\omega(2)/z(2) > 10/9$. Therefore $z/\omega < 9/10$ throughout. Equation (30) now implies $\omega' < 0$, and $0 < \omega(2) < 1$ gives $0 < \omega < 1$. This proves the asserted properties of U and the positivity of $\log(\mu/C_\partial) = (1-\omega)\log\mu/2$.

The slope assumption implies $a > b$ and

$$\frac{a}{b} > -U(k_0) \geq -U(k) = \frac{1+\omega}{1-\omega}, \qquad a+b < \frac{a-b}{\omega}.$$

Since $a-b \geq 1$ is an integer,

$$a(h-r) - b(h+r) = (a-b)h - (a+b)r > (a-b)h\left(1 - \frac{z}{\omega}\right) > \frac{h}{10}. \qquad \square$$

5. Radial estimates and parameter monotonicity

Fix a primitive label $A \in \mathcal{R}$ and let $D = \deg M_A$. The characteristic roots of (4) are e^{ℓ_A} and $e^{-\ell_A}$, since $t_A = 2\cosh\ell_A$. The initial values $F_0 = 0$ and $F_1 = M_A$ therefore give

$$M_{dA} = M_A \frac{\sinh(d\ell_A)}{\sinh\ell_A} \qquad (d \geq 1). \tag{31}$$

Using $2\sinh\ell_A = \sqrt{t_A^2 - 4}$ and $3(k+1)M_A = t_A + k$, we obtain

$$M_{dA} = \frac{t_A + k}{3(k+1)\sqrt{t_A^2-4}} e^{d\ell_A}(1 - e^{-2d\ell_A}).$$

Taking logarithms gives

$$\log M_{dA} = d\ell_A - \log(3(k+1)) + \eta + \log(1 - e^{-2d\ell_A}), \qquad \eta = \log\frac{t_A+k}{\sqrt{t_A^2-4}}. \tag{32}$$

Define its derivative after removing the common parameter factor by

$$E_{d,A} = (\log M_{dA})' - d\ell_A' + \frac{1}{k+1} = \eta' + \frac{2d\ell_A'}{e^{2d\ell_A} - 1}. \tag{33}$$

Proposition 5.1. *For every primitive $A \in \mathcal{R}$, every integer $d \geq 1$, and every real $k \geq 2$,*

$$-h/10 < E_{d,A} < 0. \tag{34}$$

Proof. By Corollary 2.3,

$$t_A' - t_A/k = 3(k+1)M_A' - 3M_A/k \geq \frac{3M_A(2k^2+k-2)}{k(k+2)} > 0.$$

Thus $\ell_A' > 0$. The function $d/(e^{2d\ell_A} - 1)$ is strictly decreasing for real $d > 0$, as follows by differentiating and using $e^x(1-x) < 1$ for $x > 0$. Consequently $\eta' < E_{d,A} \leq E_{1,A}$. Direct simplification of (33) at $d = 1$ gives

$$E_{1,A} = \frac{\sqrt{t_A^2-4} - t_A'(t_A + k - \sqrt{t_A^2-4})}{(t_A+k)\sqrt{t_A^2-4}} < 0,$$

since $t_A' > t_A/k$ and $t_A + k - \sqrt{t_A^2-4} > k$. Hence $\eta' < 0$. Another form of its derivative is

$$\eta' = \frac{t_A - (t_A k + 4)\{M_A'/M_A + 1/(k+1)\}}{t_A^2 - 4}.$$

Using (6), we obtain

$$|\eta'| < \left(\frac{D}{k} + \frac{1}{k+1}\right)\frac{t_A k + 4}{t_A^2 - 4}. \tag{35}$$

If $D \geq 3$, Lemma 2.1 gives $M_A \geq 2k^D$. Thus

$$3(k+1)M_A \geq 144, \quad \frac{k}{3(k+1)M_A} \leq \frac{1}{72}, \quad t_A \geq 142, \quad kt_A \geq 284.$$

If $D = 2$, the primitive label is $(2,1)$, and its explicit polynomial gives $3(k+1)M_A \geq 225$ and $M_A \geq 2(k+1)^2$, so the same four bounds hold. Therefore

$$\begin{aligned} \frac{t_A k + 4}{t_A^2 - 4} &= \frac{k}{t_A} \frac{1 + 4/(t_A k)}{1 - 4/t_A^2} \\ &\leq \left(\frac{72}{71}\right)^2 \frac{5041}{5040} \frac{k}{3(k+1)M_A} = \frac{36}{35} \frac{k}{3(k+1)M_A}. \end{aligned} \tag{36}$$

For $D \geq 3$, (35)–(36) yield

$$|\eta'| < \frac{6}{35(k+1)} \frac{D+1}{k^D} \leq \frac{24}{35(k+1)k^3} < \frac{h}{10}.$$

Here $(D+1)/k^D \leq 4/k^3$ follows by comparing successive integers $D \geq 3$. The last strict inequality follows from $\sqrt{\Delta} < \beta$ and

$$\begin{aligned} 7(3k+4)(k+1)k^3 - 48\beta = 48 &+ 2624(k-2) + 2880(k-2)^2 \\ &+ 1260(k-2)^3 + 259(k-2)^4 + 21(k-2)^5 > 0. \end{aligned}$$

For $D = 2$, the stronger bound on M_A instead gives

$$|\eta'| < \frac{18}{35(k+1)^3} < \frac{h}{10},$$

where the last inequality follows in the same way from

$$7(3k+4)(k+1)^3 - 36\beta = 666 + 1737(k-2) + 1089(k-2)^2 + 259(k-2)^3 + 21(k-2)^4 > 0.$$

Together with $\eta' < E_{d,A} < 0$, these estimates prove (34). □

We now combine the length comparison with the radial estimate.

Proof of Theorem 1.1. Write $b = p' - p > 0$ and $a = q - q'$. The hypothesis and Lemma 4.3 imply that $a > b > 0$ are integers. For $k_2 > k_1 \geq k_0$, Proposition 4.2 and Lemma 4.3 give

$$\begin{aligned} &[\Theta_{k_2}(P) - \Theta_{k_1}(P)] - [\Theta_{k_2}(Q) - \Theta_{k_1}(Q)] \\ &\qquad > \int_{k_1}^{k_2} \frac{h}{10}\, dk. \end{aligned} \tag{37}$$

For an arbitrary label $X = dA$ with A primitive, (33) gives

$$\frac{d}{dk}(\log M_X - \Theta_k(X)) = -\frac{1}{k+1} + E_{d,A}.$$

Write $P = d_P A_P$ and $Q = d_Q A_Q$ for the primitive decompositions. Apply this identity to P and Q and subtract. The common term cancels, and Proposition 5.1 gives $E_{d_P,A_P} - E_{d_Q,A_Q} > -h/10$. Integrating and adding (37) yields

$$\log \frac{M_P(k_2)}{M_Q(k_2)} - \log \frac{M_P(k_1)}{M_Q(k_1)} > 0.$$ □

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Zhili College, Tsinghua University, Beijing 100084, China
Email address: tangqy24@mails.tsinghua.edu.cn