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DEEP LOCI OF DOUBLE BOTT–SAMELSON VARIETIES

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Abstract. We prove that the deep locus of a complex double Bott–Samelson variety associated with a positive braid is its stabilizer locus. Consequently, the deep locus is empty if and only if any two components of the braid closure are unlinked. This resolves Conjecture 4.45 of Castronovo, Gorsky, Simental, and Speyer.

AI use and scholarly credit

The proof was developed in a ChatGPT-6-Pro research conversation whose recorded continuation lasted approximately 480 minutes, building on earlier project materials. The research included AI dialogue review and exact computations. Codex subsequently audited the arguments and prepared and revised the English manuscript. An earlier version passed an AI review using Rethlas-OSV [12], based on the OSVerify framework of Nie et al. [11].

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1. Introduction

Fomin and Zelevinsky introduced cluster algebras in their study of total positivity and dual canonical bases [5]. A cluster gives a transcendence basis in which the algebra has Laurent expansions, and mutation relates these coordinate systems. Geometrically, each cluster determines an algebraic torus inside the spectrum of the cluster algebra. The resulting torus charts provide coordinates on an open subset; understanding their complement is a separate geometric problem.

The distinction between the union of charts and its affine completion already appears in the local theory of cluster algebras. Muller developed cluster localizations and locally acyclic covers [9]. A locally acyclic cluster algebra admits a finite cover by acyclic cluster localizations. Such covers give finite generation, normality, and equality with the upper cluster algebra. The local pieces can themselves contain points outside every cluster torus, so local acyclicity naturally leads to the question of how much of the affine variety its torus charts see. Muller called their complement the deep locus.

Beyer and Muller studied deep points systematically for type A, rank two, the Markov cluster algebra, and unpunctured surface cluster algebras [2]. Their computations relate the complement of cluster charts to the algebra and combinatorics of particular mutation classes. A different description uses symmetries: an automorphism that rescales every cluster variable acts on each cluster torus by translations. Its effective action on that torus is free, so a point with a nontrivial stabilizer must be deep. This gives an inclusion between two loci defined in quite different ways, one by all cluster charts and the other by an algebraic group action.

Castronovo, Gorsky, Simental, and Speyer proposed equality of these loci for skew-symmetric locally acyclic cluster algebras [4, Conjecture 1.1], following a question of Shende and Speyer for open positroid varieties. They established it for simply laced finite type and for a family containing the maximal open positroid strata in $\mathrm{Gr}(2,n)$ and $\mathrm{Gr}(3,n)$. Their geometric approach uses braid

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varieties and Demazure weaves. They also formulated the positive-braid empty-deep-locus criterion studied here [4, Conjecture 4.45].

The general conjecture and its geometric cases have since developed differently. Neville and Simental proved the equality when a seed has a mutable quiver that is a tree without multiple edges, and constructed acyclic examples with deep points having trivial stabilizer [10]. Thus local acyclicity alone does not imply the equality. Tyler subsequently established the no-mysterious-points statement for locally acyclic surface type and the remaining finite types [15]. These results distinguish the role of a particular geometric realization from properties shared by much larger classes of cluster algebras.

Double Bott–Samelson varieties have a concrete realization by products of elementary braid matrices. Shen and Weng constructed cluster structures on decorated double Bott–Samelson cells and described their reflection and Donaldson–Thomas transformations [14]. Gao, Shen, and Weng related the half-decorated models to augmentations and fillings [6]. Casals, Gorsky, Gorsky, Le, Shen, and Simental constructed cluster structures on general braid varieties using weave calculus, proved local acyclicity and equality with the upper cluster algebra, and obtained cluster structures on open Richardson varieties [3]. This supplies both a geometric model and operations that carry cluster charts between different braid words.

Two further developments clarify the role of these operations. Gorsky, Kim, Scroggin, and Simental constructed splicing maps that identify suitable open subsets with products of braid varieties; in the double Bott–Samelson case, they proved the corresponding compatibility with cluster structures [7]. Quan studied deep loci for the reduced-word Berenstein–Fomin–Zelevinsky atlas on type-A double Bruhat cells [13]. The choice of atlas is part of that problem: the complement of a specified subcollection of charts can be larger than the complement of the full mutation class. Here the deep locus always uses the full cluster atlas.

There is also a mirror-symmetry motivation. In the framework of [4], cluster charts on a Landau–Ginzburg mirror correspond to candidates for Lagrangian tori, and the deep locus records the part of the mirror missed by those charts. An empty deep locus is therefore a geometric coverage condition relevant to generation questions. The result below determines this condition for all positive-braid double Bott–Samelson varieties.

Let β be a positive braid word on n strands. We use the cluster structure on $\mathrm{BS}(\beta)$ of [4], with all frozen variables inverted. Write $T_{\mathbf{s}}$ for its cluster tori, where $\mathbf{s}$ ranges over the entire mutation class, and put

$$\mathcal{D}(\mathrm{BS}(\beta)) = \mathrm{BS}(\beta) \setminus \bigcup_{\mathbf{s}} T_{\mathbf{s}}.$$

The diagonal torus rescales a crossing coordinate by the ratio of the two strand parameters at that crossing. Divide this torus by the subgroup acting trivially on the entire variety, and denote the resulting effective torus by T_β^{eff}. Its nonfree locus is

$$\mathcal{S}(\mathrm{BS}(\beta)) = \{z \in \mathrm{BS}(\beta) : \mathrm{Stab}_{T_\beta^{\mathrm{eff}}}(z) \neq \{1\}\}.$$

Section 2 gives the action and its kernel explicitly.

Theorem 1.1. *For every positive braid word β, one has*

$$\mathcal{D}(\mathrm{BS}(\beta))(\mathbb{C}) = \mathcal{S}(\mathrm{BS}(\beta))(\mathbb{C}).$$

Equivalently, the associated reduced closed loci are equal.

The closure of β joins each bottom strand endpoint to the corresponding top endpoint outside the braid diagram, with the orientation induced by the strands. Its components correspond to the cycles of the strand permutation. For two distinct components, their linking number is half the signed number of their mutual crossings. Since the braid is positive, this number is zero exactly when the diagram has no crossings between those components; in this case the two components are unlinked.

Corollary 1.2. *The deep locus of* $\mathrm{BS}(\beta)$ *is empty if and only if any two distinct components of the braid closure of* β *are unlinked.*

Proof. Combine Theorem 1.1 with the stabilizer criterion in [4, Corollary 4.43]. □

This proves [4, Conjecture 4.45].

The proof starts with a hypothetical deep point at which the effective torus acts freely, chosen with minimal word length and then minimal strand number. Opening any nonzero crossing gives a shorter double Bott–Samelson variety. If this child point were free, a cluster torus through it would lift to one through the original point. Consequently every opening is nonfree. The resulting fixed-point equations control the full ambient cotangent space and force the coordinates, after a torus normalization, to lie in $\{0, 1, -1\}$.

Reduction modulo two then yields a binary point with a condition at every cyclic cut of the word. We prove that this condition survives braid relations and deletion of adjacent equal letters. The word property and the Red-Min theorem reduce the word to a Coxeter word. The last deletion contradicts an explicit entry of its lower triangular factor. The reduction modulo two is applied to integer matrix identities obtained from the minimal complex point.

2. Braid matrices and stabilizers

For $n \geq 1$, a positive braid word is a finite sequence $\beta = i_1 \cdots i_m$ with $1 \leq i_j < n$. Its letters represent the positive generators σ_i; the relations are $ij = ji$ for $|i - j| > 1$ and $i(i+1)i = (i+1)i(i+1)$. Let $s_i = (i, i+1) \in S_n$. Permutations act on labels with products composed from right to left.

For a scalar x, let $B_i(x)$ be the identity matrix except for its block in rows and columns $i, i+1$, which is

$$B_i(x)\big|_{\{i,i+1\}} = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}.$$

Set $S_i = B_i(0)$. For a vector v, $\mathrm{diag}(v)$ is the diagonal matrix with diagonal entries $v_1, \ldots, v_n$. For $z = (z_1, \ldots, z_m)$ put $M_\beta(z) = B_{i_1}(z_1) \cdots B_{i_m}(z_m)$. Write $[r] = \{1, \ldots, r\}$, with $[0] = \varnothing$. For ordered index sets R, C, $M_{R,C}$ denotes the corresponding submatrix, with both sets in increasing order. Define

$$f_r(z) = \det M_\beta(z)_{[r],[r]} \quad (1 \leq r < n), \qquad f_0 = f_n = 1.$$

The signed matrix model for the variety is

$$\mathrm{BS}(\beta) = \{z \in \mathbb{A}^m : \ f_1(z) \cdots f_{n-1}(z) \neq 0\}.$$

It has a unique factorization $M_\beta = LDU$, with L, U unit lower and upper triangular and $D = \mathrm{diag}(d_1, \ldots, d_n)$, where $d_r = f_r / f_{r-1}$. These constructions are defined over $\mathbb{Z}$: each entry is a polynomial divided by a product of the f_r. We use the same formulas over a field k and write $\mathrm{BS}(\beta)(k)$ for their k-valued points. The empty word gives a point.

We write U^+ and U^- for the upper and lower unitriangular subgroups, and B^+ and B^- for the invertible upper and lower triangular subgroups. For a permutation v, $\dot{v}$ denotes the signed permutation representative obtained from a reduced product of the matrices $B_i(0)$. The permutation w_0 reverses all n indices. The Demazure product of a word is obtained by multiplying a simple reflection when it increases Coxeter length and retaining the current permutation otherwise.

For a word γ of length N with Demazure product w_0, the braid variety used below is

$$X(\gamma) = \{z \in \mathbb{C}^N : \dot{w}_0^{-1} M_\gamma(z) \in B^+\}.$$

Here $\mathbb{G}_m$ denotes the multiplicative algebraic group, and E_{rs} will denote the matrix with a single 1 in row r, column s. For $n = 2$, the equation for $X(11)$ is $z_1 z_2 = 1$, and projection to z_2 identifies it with $\mathrm{BS}(1) = \mathbb{G}_m$.

More generally, choose a reduced positive lift Δ of w_0. The map M_Δ parametrizes $U^+\dot{w}_0$. To see this, write $B_i(z) = (I + zE_{i,i+1})B_i(0)$ and move all signed permutation factors to the right. Reducedness makes the successive pairs of strand labels distinct and increasing; for Δ, they exhaust all pairs $a < b$. The remaining product contains one factor $I \pm z_j E_{ab}$ for each such pair. Its (a, b) entry is $\pm z_j$ plus a polynomial in coordinates whose row-column distances are smaller than $b - a$: every other contribution is a product along a chain $a < a_1 < \cdots < b$. Increasing induction on $b - a$ recovers all coordinates polynomially. If $M_\beta = LDU$, the unique buffer coordinates above $z \in \mathrm{BS}(\beta)$ are determined by

$$M_\Delta = \dot{w}_0 L^{-1}, \qquad \dot{w}_0^{-1} M_\Delta M_\beta = DU.$$

Indeed, $\dot{w}_0 L^{-1} \dot{w}_0^{-1} \in U^+$. Conversely, the defining equation for $X(\Delta\beta)$ gives an LDU factorization of M_β by reversing these equalities. This proves the coordinate identification $X(\Delta\beta) \simeq \mathrm{BS}(\beta)$ used below.

A seed consists of algebraically independent coordinates $x_1, \dots, x_N$, a set of mutable indices, and an integral exchange matrix (b_{jk}), with rows indexed by $1, \dots, N$ and columns by the mutable indices, whose mutable square submatrix is skew-symmetric. The remaining indices are frozen. Mutation at a mutable index k replaces x_k by

$$x_k' = x_k^{-1} \left(\prod_{b_{jk}>0} x_j^{b_{jk}} + \prod_{b_{jk}<0} x_j^{-b_{jk}} \right)$$

and changes the matrix by $b'_{j\ell} = -b_{j\ell}$ if $j = k$ or $\ell = k$, and otherwise by

$$b'_{j\ell} = b_{j\ell} + [b_{jk}]_+[b_{k\ell}]_+ - [-b_{jk}]_+[-b_{k\ell}]_+, \qquad [a]_+ = \max(a, 0).$$

All frozen coordinates are inverted in the coordinate ring. The cluster torus of a seed is the open subset on which all its coordinates are nonzero. The atlas consists of these tori for all finite mutation sequences. A diagonal cluster automorphism rescales each cluster variable by a nonzero constant. The cluster quasi-isomorphisms used here send mutable seed coordinates to corresponding mutable coordinates multiplied by frozen monomials, and frozen coordinates to invertible Laurent monomials in frozen coordinates. Their inverses have the same property, so they carry cluster tori to cluster tori.

In the standard seed for $\mathrm{BS}(\beta)$, the coordinate attached to position j is the i_j-th principal minor of the prefix product $B_{i_1}(z_1) \cdots B_{i_j}(z_j)$. The last occurrence of each color is frozen, and its coordinate equals the full-product minor f_{i_j}. Thus a word in which each color occurs once has no mutable coordinates.

We use the following cluster-theoretic inputs. The identification $\mathrm{BS}(\beta) \simeq X(\Delta\beta)$, where Δ is a positive lift of the longest permutation, identifies its cluster structure with the weave structure [4, Lemma 4.34]. Braid moves induce cluster isomorphisms. A trivalent projection at a pair ii, defined where the right coordinate is nonzero, lifts every child cluster torus to a parent cluster torus [4, Theorem 4.15(e), Lemma 4.18, Corollary 4.19]. Dilation preserves every cluster torus, and parabolic decomposition gives a cluster product [4, Proposition 3.4, Corollary 4.26, Lemma 4.41]. The reflection maps on decorated double Bott–Samelson cells are quasi-cluster transformations [14, Proposition 4.11]; their half-decorated versions are described in [6, Appendix B]. We identify the reflection used here explicitly in Subsection 4.1.

Put $p_0 = 1$, $p_j = s_{i_1} \cdots s_{i_j}$, and $w = p_m$. Thus p_j sends a position after the first j crossings to its original strand label. The ordered endpoints of crossing j are

$$a_j = p_{j-1}(i_j), \qquad b_j = p_{j-1}(i_j + 1).$$

Let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{Z}^n$. The full diagonal torus $\widetilde{T} = (\mathbb{G}_m)^n$ acts with weight $e_{a_j} - e_{b_j}$ on z_j. Indeed,

$$\mathrm{diag}(t) B_i(x) = B_i((t_i/t_{i+1})x)\, \mathrm{diag}(s_i(t)).$$

In the expression $w(t)$ we use $(w(t))_r = t_{w(r)}$. For a homogeneous function g, $\mathrm{wt}(g) \in \mathbb{Z}^n$ denotes the exponent vector of its dilation character. Iteration gives

$$M_\beta(tz) = \mathrm{diag}(t) M_\beta(z) \,\mathrm{diag}(w(t))^{-1}, \qquad \mathrm{wt}(f_r) = \lambda_r := \sum_{a=1}^{r} (e_a - e_{w(a)}). \tag{1}$$

The active set of z is $I(z) = \{j : z_j \neq 0\}$. Its support graph has vertices $1, \dots, n$ and one edge $\{a_j, b_j\}$ for each $j \in I(z)$; repeated crossings give repeated edges.

Lemma 2.1. *The stabilizer of z in $\widetilde{T}$ consists of the diagonal elements constant on every connected component of its support graph. Moreover,*

$$\mathrm{Stab}_{\widetilde{T}}(z) \subseteq \mathrm{Fix}(w) := \{t : t_a = t_{w(a)} \text{ for all } a\}.$$

Proof. The first assertion follows directly from the crossing weights. If t fixes z, each nonzero $f_r(z)$ and (1) give $\prod_{a \le r} t_a = \prod_{a \le r} t_{w(a)}$. Dividing successive equations yields the second assertion. These are equations of group schemes over any field. □

Let K_β be the kernel of this action. The graph of all crossing roots has the same components as the graph of the simple generators occurring in β: the subgroup generated by the successive conjugate transpositions equals the subgroup generated by these simple transpositions. Its components are consecutive intervals, called the parabolic blocks of β. The opening isomorphism in Lemma 3.1 also shows by induction that $\mathrm{BS}(\beta)$ is nonempty. As a nonempty principal open in affine space it is dense. Hence K_β consists exactly of the elements constant on each parabolic block.

The effective dilation torus is $\widetilde{T}/K_\beta$, identified with the diagonal cluster-automorphism torus by [4, Lemma 4.41]. We call z relatively free when $\mathrm{Stab}_{\widetilde{T}}(z) = K_\beta$. If every generator occurs, then K_β is the scalar torus and relative freeness means that the support graph is connected. The nonfree locus is a finite union of loci obtained by setting all crossing coordinates across a partition of a parabolic block to zero, and is therefore closed.

Lemma 2.2. *If w is an n-cycle, then $\lambda_1, \dots, \lambda_{n-1}$ are an integral basis of*

$$Q_{n-1} = \{(u_1, \dots, u_n) \in \mathbb{Z}^n : \textstyle\sum u_a = 0\}.$$

Every point of $\mathrm{BS}(\beta)$ has at least $n-1$ active coordinates. If it has exactly $n-1$, its support graph is a tree and the active crossing roots are an integral basis of Q_{n-1}.

Proof. The roots $e_a - e_{w(a)}$ for $a < n$ are the edge roots of a cycle with one edge removed. For any tree on $[n]$, summing its signed edge roots along the unique path from a to n gives $e_a - e_n$. Thus its $n-1$ edge roots generate the rank-$n-1$ lattice Q_{n-1} and form an integral basis. Taking successive partial sums is unimodular, with inverse $e_a - e_{w(a)} = \lambda_a - \lambda_{a-1}$ for $a < n$, where $\lambda_0 = 0$. This proves the assertion about λ_r.

For each nonzero $f_r(z)$, some nonzero monomial of that polynomial uses only active coordinates. Its weight is λ_r. Thus the active roots generate Q_{n-1}. At least $n-1$ are needed. If their graph were disconnected, all these roots would have coordinate sum zero on each component, and could not generate Q_{n-1}. With exactly $n-1$ edges the connected graph is a tree, which proves the remaining assertions. □

In particular, when w is an n-cycle, there is a unique effective dilation sending all $f_r(z)$ to 1. We will use this normalization only to obtain an integer point. For a regular unit g, the notation $d\log g$ means the algebraic differential $g^{-1}dg$.

3. Opening a nonzero crossing

An opening removes one letter while retaining a nonzero parameter. We need both its matrix formula and its compatibility with every child cluster torus.

Lemma 3.1. *Let β' be obtained from β by deleting letter j, and set $x = z_j$. There is an isomorphism*

$$\Omega_j^B : \mathrm{BS}(\beta) \cap \{z_j \neq 0\} \longrightarrow \mathrm{BS}(\beta') \times \mathbb{G}_m.$$

Writing its child coordinate as z^B and the ordered original endpoints as $(a, b) = (a_j, b_j)$, its diagonal factors satisfy

$$d_r^B = d_r x^{-\mathbf{1}_{r=a}+\mathbf{1}_{r=b}}. \tag{2}$$

Here $\mathbf{1}_E$ is 1 when condition E holds and 0 otherwise.

Proof. Factor the active block as

$$B_i(x) = L_i(x)U_i(x), \qquad L_i(x) = \begin{pmatrix} x & 0 \\ 1 & x^{-1} \end{pmatrix}, \quad U_i(x) = \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix},$$

with identity entries off this block. Slide L_i to the left and U_i to the right. For a lower triangular C and upper triangular E, the equations

$$B_h(z)C = C'B_h(v), \qquad v = \frac{C_{hh}z - C_{h+1,h}}{C_{h+1,h+1}}, \tag{3}$$

$$EB_h(z) = B_h(v)E', \qquad v = \frac{E_{hh}z + E_{h,h+1}}{E_{h+1,h+1}} \tag{4}$$

To verify these statements, set $C' = B_h(z)CB_h(v)^{-1}$ and $E' = B_h(v)^{-1}EB_h(z)$. The only entries that can violate triangularity are, respectively, $(h, h+1)$ and $(h+1, h)$. They equal

$$C_{hh}z - C_{h+1,h} - vC_{h+1,h+1}, \qquad -E_{hh}z - E_{h,h+1} + vE_{h+1,h+1}.$$

The displayed choices of v make them zero. The diagonal pairs become $(C_{h+1,h+1}, C_{hh})$ and $(E_{h+1,h+1}, E_{hh})$, respectively. All diagonal entries remain powers of x. Starting with the same initial triangular matrices, the inverse scan recovers

$$z = \frac{C_{h+1,h+1}v + C_{h+1,h}}{C_{hh}}, \qquad z = \frac{E_{h+1,h+1}v - E_{h,h+1}}{E_{hh}}.$$

In the inverse construction the initial triangular factor is prescribed by x. Recover the first original coordinate from the displayed inverse formula, compute the next triangular factor from the matrix identity, and continue. This recursively inverts both scans. All entries and all inverse entries are Laurent polynomials in x and polynomial in the other coordinates, so the maps are regular over $\mathbb{Z}[x, x^{-1}]$.

The final factorization is $M_\beta = L_{\mathrm{out}} M_{\beta'}(z^B) U_{\mathrm{out}}$. The diagonal of L_{out} has entries x, x^{-1} at a, b, and all other entries 1; U_{out} is unit upper triangular. For lower triangular C and upper triangular E, the leading r-minor of CME is the leading r-minor of M times $\prod_{a \le r} C_{aa}E_{aa}$. This follows by restricting the first r rows of C and the first r columns of E in the matrix product. Applying it here proves (2), so the open conditions are equivalent. Starting from the empty word and applying the inverse with $x = 1$ also proves nonemptiness by word length. □

Proposition 3.2. *For every cluster torus $\mathcal{T}$ of $\mathrm{BS}(\beta')$, the inverse image $(\Omega_j^B)^{-1}(\mathcal{T} \times \mathbb{G}_m)$ is a cluster torus of $\mathrm{BS}(\beta)$.*

Proof. First use the unit-left factorization

$$B_i(x) = \ell_i(x)V_i(x), \qquad \ell_i(x) = I + x^{-1}E_{i+1,i}, \quad V_i(x)\big|_{\{i,i+1\}} = \begin{pmatrix} x & -1 \\ 0 & x^{-1} \end{pmatrix},$$

Let z^0 be its output after the corresponding left and right scans. The identities

$$B_i(y)\ell_i(x) = B_i(y - x^{-1}), \qquad B_i(y)B_i(x) = B_i(y - x^{-1})V_i(x)$$

give the trivalent projection of [4, Lemma 4.9].

To realize this projection at the specified crossing, work in $X(\Delta\beta_1 i\beta_2)$. The braid identity $\Delta\beta_1 = \tau(\beta_1)\Delta$, with $\tau(i) = n - i$, and a reduced expression for Δ ending in i produce an adjacent pair ii immediately before the pivot x. These moves lie to the left of the pivot and do not change it. Contract the pair and reverse the braid moves. The buffer Δ ensures that all braid varieties used have Demazure product w_0.

We check its output against the scan. For a reduced word u, the factorization $M_u(v) = N(v)\dot{w}$, with $N(v) \in U^+ \cap \dot{w}U^-\dot{w}^{-1}$, has unique reduced-cell parameters. If

$$C_1 M_u(v_1) = C_2 M_u(v_2), \qquad C_1, C_2 \in B^-,$$

then $C_2^{-1}C_1 \in B^- \cap U^+ = \{1\}$. Thus lower sliding commutes with each reduced braid block. For right sliding, write $M_u(v) = \dot{w}\widetilde{N}(v)$, where $\widetilde{N}(v) \in U^- \cap \dot{w}^{-1}U^+\dot{w}$. An equality $M_u(v_1)E_1 = M_u(v_2)E_2$ gives $E_1E_2^{-1} \in B^+ \cap U^- = \{1\}$. This gives the corresponding uniqueness for right sliding. Every commuting or three-letter braid block is reduced. Moving ℓ_i through the chosen sequence therefore gives the same parameters of β_1 as the direct scan; its remaining unit lower factor is absorbed into the unique Δ parameters since $M_\Delta \ell_{\text{out}} \in U^+\dot{w}_0$. Both routes then slide the same $V_i(x)$ through β_2. Their child output is z^0. The trivalent lifting theorem now applies to every child torus.

It remains to compare the two scans. Let $D_{ab}(x)$ have diagonal entries x, x^{-1} at a, b. From

$$B_h(v)D = s_h(D)B_h((D_{hh}/D_{h+1,h+1})v), \qquad DB_h(v) = B_h((D_{hh}/D_{h+1,h+1})v)s_h(D)$$

move the diagonal part of $V_i(x)$ through the factors already scanned. At a crossing h, its entries exchange and the coordinate is multiplied by their ratio. Thus the diagonal entries follow the two strands from the pivot to their original labels a, b. The accumulated ratios are exactly the dilation weights of the child word. This proves

$$z^0 = D_{ab}(x) \cdot z^B. \tag{5}$$

Every child cluster variable is homogeneous under dilation. Hence this parameter-dependent dilation preserves $\mathcal{T} \times \mathbb{G}_m$, with an invertible integral monomial change of its coordinates. Its inverse images under the two opening maps agree. □

Lemma 3.3. *If z is relatively free and an active crossing joins different cycles of w, then its opening child is relatively free, and $K_{\beta'} = K_\beta$.*

Proof. Write $s_{ab} = (a, b)$ for the transposition of the two original labels, and put $\chi(t) = t_a/t_b$. The unit-left scan is equivariant, while (5) and $D_{ab}(\chi(t))s_{ab}(t) = t$ give

$$\Omega_j^B(tz) = (s_{ab}(t) \cdot z^B, \chi(t)x).$$

An element t fixes z exactly when $\chi(t) = 1$ and $s_{ab}(t)$ fixes z^B. On $\ker\chi$, the two entries t_a, t_b agree, so $s_{ab}(t) = t$. Consequently $\mathrm{Stab}(z) = \mathrm{Stab}(z^B) \cap \ker\chi$. The child permutation is $w' = s_{ab}w$. When a, b belong to different cycles of w, they belong to the same cycle of w'. Lemma 2.1 implies $\mathrm{Stab}(z^B) \subseteq \ker\chi$, so the two stabilizers coincide. The kernel is the torus constant on the connected components of the simple-generator graph. Deleting a letter can only remove an edge of this graph, so $K_\beta \subseteq K_{\beta'}$. Thus

$$K_\beta = \mathrm{Stab}(z^B) \supseteq K_{\beta'} \supseteq K_\beta,$$

as required. □

4. Minimal deep points

Suppose that a relatively free complex deep point exists. Choose one for which (m, n), word length followed by strand number, is lexicographically minimal. If a generator is absent, parabolic decomposition expresses the variety as a cluster product. A product point is deep exactly when some factor is deep, and relative freeness holds factor by factor. A deep factor has smaller word

length unless the other factors are empty strands; in that case it has the same word length and fewer strands. Thus the minimal word has full support and $n \geq 2$.

Its support graph is connected. If w had more than one cycle, an active edge would join two cycles. Lemma 3.3 and Proposition 3.2 would give a shorter relatively free deep point. Hence w is an n-cycle. The same argument shows that every active opening has a child which is not relatively free; we call such an opening bad.

The word has length $m > n-1$. Indeed, a word representing an n-cycle must contain every simple generator, so $m \geq n-1$. Equality means each generator occurs once. In that case f_i is the coordinate at the unique occurrence of i: all other factors act by determinant-one operations within $[i]$ or its complement. The whole variety is then a cluster torus with only frozen variables.

All differentials in this section are taken in the ambient cotangent space of the principal open $\mathrm{BS}(\beta) \subseteq \mathbb{A}^m$. Define the subspaces

$$\mathcal{F}_z = \mathrm{span}\{df_1|_z, \ldots, df_{n-1}|_z\}, \qquad \mathcal{A}_z = \mathrm{span}\{dz_j|_z : j \in I(z)\}.$$

We write $(H)_z$ for the equality $\mathcal{F}_z = \mathcal{A}_z$.

Proposition 4.1. *At the minimal point, for each active crossing j there is a cycle A of the child permutation and a sign $\eta_j \in \{1,-1\}$ such that*

$$z_j = \eta_j \prod_{r\in A} d_r, \qquad dz_j = z_j \sum_{r\in A} d\log d_r. \tag{6}$$

Moreover, $|I(z)| = n-1$, the support graph is a tree, and $(H)_z$ holds.

Proof. Let (a,b) be the crossing endpoints and $x = z_j$. The child permutation $w' = s_{ab}w$ has two cycles A, A^c, with $a \in A$, $b \notin A$. If the deleted generator occurred only once, the child would split into two consecutive parabolic blocks. Since w' has two cycles, each block would have a cyclic permutation and hence every point in each block would be relatively free by Lemma 2.1. This contradicts badness. The child therefore still has full support.

Each cycle of w' lies in a component of the child support graph. Badness says this graph is disconnected, so its components are exactly A, A^c. All crossing coordinates joining the two sets vanish.

Let $S_{\beta'}$ be the product of the matrices S_i in the order of the child word, and let a_k, b_k be its crossing endpoints. Moving these signed permutation matrices to the right gives

$$M_{\beta'}(z') = \left(\prod_k (I + \epsilon_k z'_k E_{a_k b_k})\right) S_{\beta'}, \qquad \epsilon_k \in \{1,-1\}.$$

On the fixed locus just described, each surviving elementary factor acts within one block and has determinant one on it. The final signed permutation preserves both blocks. Hence

$$\det((M_{\beta'})_{A,A}) = \det((S_{\beta'})_{A,A}) =: \eta_j \in \{1,-1\}.$$

Gaussian elimination creates no entries between the blocks, even if A is not consecutive. Uniqueness of the triangular factorization yields

$$\prod_{r\in A} d'_r = \eta_j. \tag{7}$$

By (2), the left side is $x^{-1}\prod_{r\in A} d_r$. This proves the first equation in (6).

To differentiate, let T_A be the subtorus constant on A and on A^c. The child point is fixed by T_A. The function $\prod_{r\in A} d'_r$ is a T_A-invariant regular unit, constant on the entire fixed locus by (7). At a fixed point the tangent space splits into crossing-character spaces. The differential vanishes on the zero-character space because of constancy, and on every nontrivial character space because it is

invariant. Thus its differential vanishes on the full child tangent space. Its pullback is independent of the additional $\mathbb{G}_m$ coordinate, and gives

$$0 = d\left(x^{-1}\prod_{r\in A} d_r\right),$$

which proves the second equation.

The $|I(z)|$ independent active covectors lie in $\mathcal{F}_z$, so $|I(z)| \le n-1$. Lemma 2.2 gives the reverse inequality and the tree assertion. Equality of dimensions gives $(H)_z$. □

Lemma 4.2. *After the unique dilation making every f_r equal to 1, the minimal point is integral with coordinates in $\{0, 1, -1\}$. Its reduction modulo two has $n-1$ active coordinates, tree support, and satisfies (H) in characteristic two.*

Proof. Dilation preserves every cluster torus, and equivariance of the opening preserves badness. Now all $d_r = 1$, so (6) gives $z_j = \eta_j$ at an active crossing. At this integral point, for its associated cycle A_j the differential identity reads

$$dz_j = \sum_{r=1}^{n-1} z_j(\mathbf{1}_{r\in A_j} - \mathbf{1}_{r+1\in A_j})df_r. \tag{8}$$

Let J be the $(n-1)\times m$ integer Jacobian matrix of $(f_1, \dots, f_{n-1})$ at z, and let E_I be the matrix whose rows are the coordinate rows at the active positions. Equation (8) gives an integer matrix C with $CJ = E_I$. After reduction modulo two, E_I still has rank $n-1$. Therefore the row space of J is exactly the active coordinate span. The support graph is unchanged. □

4.1. **Rotation.** If $\beta = \gamma i$ and x is the final coordinate, factor $M_\beta = LDU$ and set $y = L_{i+1,i}$. Define

$$r_i(z_1, \dots, z_{m-1}, x) = (y, z_1, \dots, z_{m-1}) \in \mathrm{BS}(i\gamma). \tag{9}$$

Proposition 4.3. *The map (9) is a regular isomorphism preserving the full cluster atlas in both directions. Its effect on principal minors is*

$$f'_r = f_r \quad (r \ne i), \qquad f'_i = f_{i-1}f_{i+1}/f_i. \tag{10}$$

Its inverse moves the first coordinate to the end and replaces it by $-U_{i,i+1}$ of the current product.

Proof. Write $M = M_\beta(z)$. The new product is $M' = B_i(y)MB_i(x)^{-1}$. Eliminate the first $i-1$ pivots and write the relevant Schur block as $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, where $a = d_i$, $ad - bc = d_id_{i+1}$ and $y = c/a$. Direct multiplication gives

$$\begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} = \begin{pmatrix} (ad-bc)/a & -x(ad-bc)/a \\ -b & a+bx \end{pmatrix}.$$

The new first pivot in this block is d_{i+1}, so $f'_i = f_{i-1}d_{i+1}$, and division of its upper-right entry by that pivot gives $U'_{i,i+1} = -x$. The other principal minors are unchanged because their index sets contain both affected indices or neither. For the inverse, factor the new product and put $x = -U'_{i,i+1}$. Reversing the displayed two-by-two multiplication gives a block with lower multiplier y, recovers M, and removes the initial $B_i(y)$ while appending $B_i(x)$. Thus the two coordinate maps are inverse. Both divide only by the inverted principal minors.

For cluster compatibility, use the half-decorated model of [6, Definition B.1 and Propositions B.6, B.9]. Its standard representatives are the signed matrices above, and its open conditions are precisely the nonvanishing of the principal minors. The identification with the weave structure is recorded after [4, Lemma 4.34]. In the fully decorated simply connected model, the right reflection is quasi-cluster: it fixes the left frozen variables and changes one right frozen variable by the Cartan monomial [14, Proposition 4.11]. Its inverse has the same property. Thus the pinning which

sets the left frozen variables to 1 is compatible with the torus transformations and their inverses; this is the half-decorated specialization of [6, Remark B.11]. The left reflection is cluster-compatible by [6, Proposition B.12].

Right reflection followed by left reflection moves the final letter to the front. The intermediate flags keep the coordinates $z_1, \dots, z_{m-1}$. To restore the pinned initial flag, its representative $B_i(y)LS_i^{-1}$ must lie in U^-. All entries above the diagonal except possibly $(i, i+1)$ are already zero, its diagonal entries are 1, and its $(i, i+1)$ entry is $y - L_{i+1,i}$. This uniquely gives $y = L_{i+1,i}$ and identifies the composition with (9). The simply connected reflection theorem applies over $\mathbb{C}$ to these pinned representatives. Each composition and its inverse preserve the full cluster atlas. □

Starting from the normalized minimal point, repeat this rotation. Every image has the same (m, n), an n-cycle permutation, and is deep. Lemma 2.1 makes it relatively free. It is therefore another global minimal point. Equations (10) preserve $f_r = 1$, so Proposition 4.1 and Lemma 4.2 apply to every image without further normalization. Each image has exactly $n-1$ active coordinates, each equal to 1 or -1.

Only the moving coordinate changes, so $x = 0$ if and only if $y = 0$. All denominators in the normalized rotation are 1. Hence reduction modulo two commutes with every rotation and the moving bit remains unchanged.

Until the proof of Theorem 1.1, matrices and differentials are over $\mathbb{F}_2$, so the active block is $\left(\begin{smallmatrix} x & 1 \\ 1 & 0 \end{smallmatrix}\right)$. A cut is an ordinary simultaneous cyclic shift of the word and its bit string. Define the following conditions on a binary point q:

$$(P) : \text{every cut lies in BS, and at a cut ending in } (i, x),\ L_{i+1,i} = x;$$
$$(Q) : (P) \text{ holds and } \mathcal{F}_q = \mathcal{A}_q \text{ at every cut.}$$

At each cut, the product, its triangular factors, and the two cotangent subspaces are computed for that cyclically shifted word and point. Condition (P) says that every actual rotation agrees with ordinary cyclic shift. We have proved the following implication.

Proposition 4.4. *A relatively free complex deep point would give a binary point satisfying (Q) whose permutation is an n-cycle, whose active set has size $n-1$, and whose word length is greater than $n-1$.*

Proof. Apply lexicographic minimality, Lemma 4.2, and the preceding rotation argument. The integer identity (8) supplies (H) at each cut. □

5. Binary braid words

5.1. **Local restrictions.** We next prove restrictions that use only (H) and $n-1$ active coordinates at the current point. The identities in this section are valid over any field, using the signed matrices from Section 2; the application is over $\mathbb{F}_2$.

Let

$$A_j = L^{-1}B_{i_1}(z_1)\cdots B_{i_j}(z_j), \quad A_0 = L^{-1}, \quad A_m = DU.$$

Differentiating $M_\beta = LDU$ gives

$$L^{-1}(dM_\beta)U^{-1}D^{-1} = L^{-1}dL + (dD)D^{-1} + D(dU)U^{-1}D^{-1}.$$

The first and last terms have zero diagonal. Also $(\partial_x B_i(x))B_i(x)^{-1} = E_{i,i+1}$, so taking the diagonal for the derivative in coordinate z_j gives

$$d_r^{-1}\partial_{z_j}d_r = (A_{j-1}E_{i_j,i_j+1}A_{j-1}^{-1})_{rr}, \qquad 1 \le r \le n. \tag{11}$$

For an active position e, condition (H) supplies a diagonal matrix H_e such that

$$\sum_r (H_e)_{rr}\, d\log d_r = z_e^{-1}dz_e. \tag{12}$$

Indeed, $d\log f_i = \sum_{r\le i} d\log d_r$ and these span $\mathcal{F}_z$. Define $K_j^e = A_j^{-1} H_e A_j$. Pairing (11) with H_e shows that $(K_{j-1}^e)_{i_j+1,i_j}$ vanishes unless $j = e$, in which case it equals z_e^{-1}.

Lemma 5.1. *Before its active event e, the matrix K_j^e is lower triangular; after that event it is upper triangular. Immediately before and after, with $i = i_e$ and $x = z_e$, it has the form*

$$K_{e-1}^e = D_- + x^{-1}E_{i+1,i}, \qquad K_e^e = D_+ - x^{-1}E_{i,i+1},$$

where D_-, D_+ are diagonal.

Proof. For every matrix K,

$$\left(B_i(x)^{-1} K B_i(x)\right)_{i,i+1} = -K_{i+1,i}.$$

If K is lower triangular and this lower neighboring entry is zero, conjugation preserves lower triangularity. Reversing the assertion preserves upper triangularity. Since $K_0^e = LH_eL^{-1}$ is lower and $K_m^e = U^{-1}H_eU$ is upper, the vanishing furnished by (12) proves the two assertions by forward and backward induction.

At the event itself, all lower entries outside the active two-by-two block must vanish: entries outside its rows and columns are unchanged, and the remaining pairs are mixed by invertible two-by-two matrices. The only possible off-diagonal term before the event is therefore $cE_{i+1,i}$, with $c = x^{-1}$. In the active block,

$$\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d + cx & -c \\ x(d-a) + cx^2 & a - cx \end{pmatrix}.$$

Upper triangularity gives $a - d = cx = 1$ and the claimed upper entry. □

Lemma 5.2. *Suppose the permutation is an n-cycle, there are $n-1$ active coordinates, and (H) holds. An adjacent pair ii has only the pattern 00. A three-letter braid block $i(i+1)i$ or $(i+1)i(i+1)$ has at most one active coordinate.*

Proof. For ii with pattern 10, the first event produces a nonzero upper root entry by Lemma 5.1. The following zero crossing takes it to a nonzero lower entry, contrary to upper triangularity. Read backwards for 01. Pattern 11 has two edges with the same endpoints, contrary to the tree conclusion of Lemma 2.2.

It suffices first to consider colors 121 on three neighboring indices. For pattern 101, the first event produces $D + bE_{12}$, with $b \neq 0$. After the middle zero its form is $D' - bE_{13}$. Just before the last event its own impulse matrix has form $D'' + cE_{21}$, with $c \neq 0$. Both matrices at this time are conjugates, by the same matrix, of diagonal matrices. They must commute. Their commutator has a nonzero E_{23} coefficient bc, whereas terms involving a diagonal matrix occur only at E_{13} or E_{21}. This is impossible.

For 110, the first impulse starts after its event as $D + bE_{12}$. Write h_2, h_3 for the second and third diagonal entries. The $(3,2)$ entry after conjugation through $B_2(y)$ is $y(h_3 - h_2)$. Here $y \neq 0$, so upper triangularity forces $h_2 = h_3$. The identity

$$B_2(y)^{-1} E_{12} B_2(y) = yE_{12} - E_{13}$$

therefore gives $D' + byE_{12} - bE_{13}$. Conjugation by $B_1(0)$ sends E_{12} to $-E_{21}$ and E_{13} to $-E_{23}$, so the resulting $(2,1)$ entry is $-by \neq 0$. For 011, the impulse just before the last event is $D + cE_{21}$. Moving backwards through the middle crossing gives

$$B_2(y) E_{21} B_2(y)^{-1} = yE_{21} + E_{31}.$$

Lower triangularity again forces equality of the two diagonal entries involved in the middle crossing. Its off-diagonal part becomes $cyE_{21} + cE_{31}$. Moving backwards through the first zero sends E_{21} to $-E_{12}$, while E_{31} becomes a multiple of E_{32}. The resulting upper entry $-cy$ is nonzero before the impulse event, a contradiction. Pattern 111 has three strand edges forming a triangle and is excluded by the tree condition.

The 212 calculations follow by reversing the three indices and reading the conjugation chain backwards. If J is their anti-diagonal permutation matrix, then $JB_i(x)J = B_{3-i}(x)^{-1}$, and lower and upper triangularity are interchanged. Thus all three double-active patterns are excluded in this case as well. □

5.2. **Preservation at every cut.** We return to $\mathbb{F}_2$. Throughout this section the point has an n-cycle permutation, $n-1$ active coordinates, and satisfies (Q).

Lemma 5.3. *If an actual rotation has old and new moving bit zero, then (H) holds before it if and only if it holds after it.*

Proof. The active coordinate functions are unchanged. Formula (10) gives an invertible integral monomial transformation of the principal minors, with determinant -1 on their exponent lattice. Its logarithmic differential is invertible in characteristic two. Thus the pullbacks of the frozen differential spans agree, as do the active spans. □

Lemma 5.4. *The three-letter braid transformation preserves (Q).*

Proof. The coordinate identity in characteristic two is

$$121(a,b,c) \longleftrightarrow 212(c, ac+b, a). \tag{13}$$

We first check (P), including the internal cuts. Put the block at the end of the word, using the ordinary cyclic shifts supplied by (P). After eliminating earlier indices, write the relevant unit triangular factors as

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ q & r & 1 \end{pmatrix}, \qquad U_0 = \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}.$$

The diagonal factor is the identity over $\mathbb{F}_2$. If a final color 1 rotates with its bit unchanged, then that bit is p, and the new factors are

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ r+qu & q & 1 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 1 & p & w \\ 0 & 1 & v+uw \\ 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

For a stable final color 2, its bit is r, and the new factors are

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ q+pr & 1 & 0 \\ p & w & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & v & u+rv \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Multiplication of the displayed factors gives

$$L_1U_1 = B_1(p)L_0U_0B_1(p)^{-1}, \qquad L_2U_2 = B_2(r)L_0U_0B_2(r)^{-1}.$$

Their triangularity and unit diagonals identify them as the unique factors. To apply this inside an $n \times n$ matrix, partition it as $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A is the invertible principal block preceding the three indices in question. The remaining Schur complement is $D - CA^{-1}B$. Left and right block operations supported on the three indices send this complement to the same left and right transforms of $D - CA^{-1}B$. Entries of its leading three-by-three triangular factors depend only on that three-by-three block. If there are no preceding indices, the complement is the original matrix. Thus the formulas apply at either boundary as well.

Record each state by its lower and upper entries in the order

$$(L_{21}, L_{31}, L_{32};\ U_{12}, U_{13}, U_{23}).$$

For the rotations through 121, the three successive states are

$$(u, r+qu, q;\ p, w, v+uw),$$
$$(r, u, v+uw;\ w, p+qw, q),$$
$$(w, v, u;\ r, q, p).$$

For rotations through 212, they are

$$(q+pr, p, w;\ v, u+rv, r),$$
$$(v, w+pv, p;\ q+pr, r, u),$$
$$(w, v, u;\ r, q, p).$$

Iterating (14)–(15), rotation from right to left through 121 extracts (p, q, r), whereas rotation through 212 extracts $(r, q+pr, p)$. The original stable bits require $(c, b, a) = (p, q, r)$, and the transformed bits require $(a, ac+b, c) = (r, q+pr, p)$, which is exactly (13). After all three rotations, the matrices are conjugated by the same block product, so subsequent cuts outside the block agree. This proves (P) in both directions and at every position.

By Lemma 5.2, the possible block patterns and their images are

$$000 \mapsto 000, \qquad 100 \mapsto 001, \qquad 010 \mapsto 010, \qquad 001 \mapsto 100.$$

At cuts not splitting the block, the principal minors are identical functions under the braid coordinate isomorphism. The active differential span also agrees: an outer active coordinate exchanges outer positions, and at a sole middle active coordinate $a = c = 0$ gives $d(ac+b) = db$. Thus (H) holds at these cuts. Each internal cut can be reached from one boundary of the new block by crossing only zero letters. The already proved (P) and Lemma 5.3 carry (H) to it. Hence (Q) is preserved. □

For commuting letters it is necessary to check a cut separating two active coordinates. We first record an identity on the whole principal open, so that it can be differentiated.

Lemma 5.5. *If $|i-j| > 1$, then wherever the triangular factors exist,*

$$L_{i+1,i}(B_j(y) M B_j(x)^{-1}) = L_{i+1,i}(M).$$

The corresponding assertion holds for $U_{i,i+1}$.

Proof. The lower entry is the minor with row set $[i-1] \cup \{i+1\}$ and column set $[i]$, divided by f_i. Each relevant row or column set contains both indices $j, j+1$ or neither. The determinant-one left and right operations therefore preserve both numerator and denominator. Transposing the argument proves the upper assertion. □

Lemma 5.6. *Commutation of disjoint letters preserves (Q).*

Proof. Put the commuting block at the end and write it as $ij(a, b)$. In the old order rotate j, then i. Lemma 5.5 says that the coordinates extracted in the new order, i then j, are the same functions. Thus (P) is preserved. After both steps the matrix states agree because the two braid matrices commute.

Cuts outside the block preserve (H) by coordinate interchange. If at least one bit is zero, Lemma 5.3 also proves (H) at the new internal cut. Suppose both bits are 1. Denote the new coordinate functions in the old two-step order by y_j, y_i. Pull back (H) from the cut after both old rotations. Formula (10) preserves the frozen differential span, so

$$dy_i, dy_j \in \mathcal{F}_q.$$

In the new order the first coordinate function is the same y_i by Lemma 5.5. The active covectors at its intermediate cut pull back to

$$dy_i, \quad db, \quad dz_s \quad (s \text{ active outside the block}).$$

All belong to $\mathcal{F}_q$. There are $n-1$ of them and they are independent, since the commuting coordinate map followed by actual rotation is an isomorphism of the principal open. They therefore span $\mathcal{F}_q$. Pulling the frozen span back by (10) proves (H) at this cut. □

Lemma 5.7. *Ordinary cyclic shift preserves (Q). Deleting an adjacent pair ii preserves (Q), the permutation, and the number of active coordinates.*

Proof. Cyclic shift reindexes the full collection of cuts in (Q). By Lemma 5.2, an adjacent pair has bits 00. In characteristic two, $B_i(0)^2 = I$. Deleting it therefore preserves the total matrix and permutation. As functions of the remaining coordinates, the child principal minors are the restrictions of the parent principal minors to the subspace on which the deleted coordinates are zero.

Each child cut comes from a parent cut not splitting the pair. The new seam can be represented by either end of the pair, with the same total matrix. The extracted neighboring lower entry and the moving bit are unchanged, so (P) holds. Restriction of (H) deletes no active covector, and gives (H) for the child. No active coordinate is removed. □

After each of these operations, an n-cycle and $n-1$ active coordinates again imply tree support by Lemma 2.2. Thus the local restrictions can be applied at every subsequent stage.

5.3. **Word reduction.** We use the word property in its deletion form: an expression in a Coxeter group can be reduced using braid moves and deletion of adjacent equal simple reflections, and any two reduced expressions are related by braid moves [1, Theorem 3.3.1]. We also use Red-Min: an element of a finite Coxeter group can be conjugated to a minimal-length element of its conjugacy class by successive simple reflections without increasing length [8, Theorem 3.1(1)].

Lemma 5.8. *Every word in adjacent transpositions whose permutation is an n-cycle can be changed into a word of length $n-1$ by braid moves, ordinary cyclic shifts, and deletion of adjacent equal letters, without increasing word length.*

Proof. First reduce the word by the deletion form of the word property. Consider a Red-Min step $w \mapsto s_i w s_i$. Write ℓ for Coxeter length. If w preserves $\{i, i+1\}$, it commutes with s_i and the step can be omitted. Otherwise, swapping positions $i, i+1$ does not change the relative order of the inverse images of $i, i+1$: a change would require both inverse images to be these two positions. Consequently

$$\ell(s_i w s_i) - \ell(w) = \big(\ell(w s_i) - \ell(w)\big) + \big(\ell(s_i w) - \ell(w)\big).$$

Each term on the right is 1 or -1. At least one is -1 in a nonincreasing step.

If s_i is a left descent, a reduced expression iv exists. Braid moves take the current reduced word to iv; a cyclic shift produces vi, an expression of $s_i w s_i$ of unchanged word length. If necessary, reduce it by braid moves and deletions. A right descent is treated by shifting the last letter to the front. Thus every Red-Min step is realized by the permitted operations.

The minimal length in the conjugacy class of an n-cycle is $n-1$. Every such expression must contain every simple generator, since omission of i would preserve $[i]$; the word $12\cdots(n-1)$ attains the bound. Hence the resulting word has length $n-1$, and each simple generator occurs once. □

Theorem 5.9. *Let $q \in \mathrm{BS}(\beta)(\mathbb{F}_2)$ satisfy (Q). If its permutation is an n-cycle and it has $n-1$ active coordinates, then the word length is $n-1$.*

Proof. Apply Lemma 5.8. Lemmas 5.4, 5.6, and 5.7 preserve (Q), the cycle condition, and the active count. Every deletion is a 00 deletion by Lemma 5.2.

If the original word were longer than $n-1$, the path would contain a deletion. Consider its last deletion. The child has length $n-1$, since all subsequent steps preserve length. Rotate the parent so that the deleted pair is final, giving γii with bits 00 on the pair. The word γ represents an n-cycle and has length $n-1$, so each simple generator occurs once. Its $n-1$ bits are all 1.

The total matrices of γii and γ coincide. For the latter, use

$$B_j(1) = \ell_j u_j, \qquad \ell_j = I + E_{j+1,j}, \qquad u_j = I - E_{j,j+1}$$

over $\mathbb{F}_2$. If $i \neq j$, then ℓ_i commutes with u_j. As each color occurs only once, all lower factors can be moved to the left of all upper factors. Their products give the unique unit triangular factorization. Higher products of distinct simple lower entries lie at distance at least two from the diagonal. Thus

$$L_{j+1,j} = 1 \qquad (1 \leq j < n).$$

On the parent, however, (P) at its final letter $(i, 0)$ requires $L_{i+1,i} = 0$ for the same matrix. This contradiction proves the result. □

Proof of Theorem 1.1. A relatively free complex deep point would, by Proposition 4.4, produce a binary point satisfying the assumptions of Theorem 5.9 with length greater than $n - 1$. This is impossible. Thus every complex deep point has a nontrivial effective stabilizer, proving $\mathcal{D} \subseteq \mathcal{S}$. Conversely, the effective dilation torus acts freely on each cluster torus, giving $\mathcal{S} \subseteq \mathcal{D}$.

Parabolic factors and empty strands were included in the lexicographic minimality argument. The empty word, including $n = 1$, is a point and its own zero-dimensional cluster torus. The equality therefore holds for every positive braid word. Equality on complex points gives equality of the reduced closed loci. □

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