MINIMAL RECURRENCES FOR BINOMIAL POWER SUMS

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Abstract. For every positive integer r, we prove that the least order of a polynomial-coefficient recurrence for $F_r(n) = \sum_{k=0}^{n} \binom{n}{k}^r$ is $\lfloor (r+1)/2 \rfloor$. This establishes the unrestricted minimal-order assertion arising from Franel's work of 1894–1895. We also determine the least order $\lfloor r/2 \rfloor$, for $r \geq 2$, of the alternating sums $(-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^r$. The lower bounds follow from real moment representations: every nonzero singularity of the representing measure forces a root of the leading characteristic polynomial of the recurrence. A common reflection argument gives telescoping recurrences attaining both bounds. For the ordinary sums, we combine the deformation of Straub and Zudilin with a shifted prime-power congruence to prove that the minimal shift operator is irreducible over $\mathbb{C}(n)$.

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1. Introduction

For a positive integer r, the sums

$$F_r(n) = \sum_{k=0}^{n} \binom{n}{k}^r, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

form a classical family in the theory of polynomial-coefficient recurrences. The first two cases are $F_1(n) = 2^n$ and $F_2(n) = \binom{2n}{n}$. Franel found second-order recurrences for F_3 and F_4 in 1894 and 1895 and proposed a general form for the recurrences [8, 9]. The order suggested by those examples is

$$m_r = \left\lfloor \frac{r+1}{2} \right\rfloor.$$

The present paper proves that this order is minimal for every r. It also determines the corresponding orders for alternating binomial power sums and proves irreducibility and arithmetic rigidity for the ordinary Franel recurrences.

This long-standing problem is supported by a rich history of ideas and prior work. Perlstadt [13] computed recurrences for $r = 5, 6$, and McIntosh [10] treated $r \leq 10$. Cusick [5] proposed a general construction; Stoll [16] identified a gap in its nonvanishing argument and proved the upper bound m_r. Minimality is a separate question: a recurrence obtained from a given summand need not be the shortest recurrence of the sum. Earlier minimality results for $r = 3, 4, 5, 6$ are attributed to Schmidt and Yuan (1995) by Yuan Jin and Dickinson [21, p. 350]. Yuan, Lu and Schmidt [22] ruled out first-order recurrences for every $r \geq 3$. Straub and Zudilin [17, Theorem 1.1] proved the lower bound m_r for recurrences obtained by creative telescoping from (1.1). They also verified unrestricted minimality computationally for $r \leq 20$ [17, Remark 4.1]. Van Hoeij [19, §5, Remark 6] subsequently obtained the matching telescoping upper bound by polynomial reduction and reflection. For more information, see Tonien's paper [18].

To state the results clearly, we use the following conventions throughout. A recurrence of order M for a sequence u is an identity

$$\sum_{j=0}^{M} p_j(n) u(n+j) = 0 \qquad (n \text{ sufficiently large}), \qquad p_j \in \mathbb{C}[n], \quad p_0 p_M \neq 0. \tag{1.2}$$

Rational-function coefficients give the same notion after clearing denominators. Note that the order is the width of the shifts, not the degree of the coefficient polynomials. Two sequences

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define the same *eventual germ* if they agree for all sufficiently large indices. All solution spaces below are spaces of such germs.

For a nonzero recurrence, put

$$D = \max_j \deg p_j, \qquad \chi(X) = \sum_j [n^D] p_j(n) X^j. \tag{1.3}$$

We call χ its *leading characteristic polynomial*. This definition refers to the highest degree in n; its degree in X can be smaller than the order of the recurrence.

The second family considered here is

$$T_r(n) = (-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^r = \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^r, \qquad r \geq 2. \tag{1.4}$$

Write $a_r = \lfloor r/2 \rfloor$ and define the monic polynomials

$$H_r(X) = \prod_{0 \leq k < r/2} (X - \lambda_{r,k}), \qquad \lambda_{r,k} = (-1)^k \left(2 \cos \frac{\pi k}{r}\right)^r, \tag{1.5}$$

$$\widetilde{H}_r(X) = \prod_{h=0}^{a_r - 1} (X - \nu_{r,h}), \qquad \nu_{r,h} = \left(2 \cos \frac{(2h+1)\pi}{2r}\right)^{2r}. \tag{1.6}$$

The formula for H_r also gives $H_1(X) = X - 2$.

Theorem 1.1. *The least recurrence orders of F_r for $r \geq 1$ and T_r for $r \geq 2$ are respectively m_r and a_r. Both bounds are attained over $\mathbb{Q}(n)$ by rational telescoping of the displayed binomial summands.*

Moreover, the polynomials H_r and $\widetilde{H}_r$ belong to $\mathbb{Z}[X]$. For every polynomial-coefficient recurrence of F_r, one has $H_r \mid \chi$; for every such recurrence of T_r, one has $\widetilde{H}_r \mid \chi$.

In particular, at the minimum order the corresponding leading characteristic polynomial is a nonzero scalar multiple of H_r or $\widetilde{H}_r$. The formula for H_r determines the characteristic polynomial discussed in [19, §7, Question 1]. The divisibility assertion applies to every recurrence, regardless of how it is obtained.

For the alternating sums, McIntosh [11, Theorem 1] gave a construction of recurrences of order at most a_r and established sharpness when r is prime or a power of 2 [11, Theorem 2]. His lower bound uses the algebraic degree of the dominant growth constant. The proof here uses every nonzero critical value of a real torus symbol, including values outside the Galois orbit of the dominant one. This gives the same conclusion for all r.

Let S denote the forward shift, $Su(n) = u(n+1)$, and let

$$\mathbb{C}(n)[S; \sigma], \qquad \sigma(c(n)) = c(n+1), \qquad Sc(n) = c(n+1)S,$$

be the shift Ore ring. Let $L_r \in \mathbb{Q}(n)[S; \sigma]$ be the monic minimal operator of F_r. Here monic means that the coefficient of S^{m_r} is 1.

Theorem 1.2 (Irreducibility)**.** *For every $r \geq 1$, the operator L_r is irreducible in $\mathbb{C}(n)[S; \sigma]$.*

Furthermore, one can show that every nonzero complex solution germ of L_r has the same monic minimal annihilator L_r, and every annihilator of that germ is a left multiple of L_r. For $r \geq 3$, no nonzero solution germ is hypergeometric over $\mathbb{C}(n)$.

We first establish minimality and then use the arithmetic congruence to deduce irreducibility. The coefficient field in Theorem 1.2 is $\mathbb{C}(n)$; the automorphism fixes the complex constants and sends n to $n+1$. Hypergeometric means $U(n+1) = a(n)U(n)$ on a tail for some $a(n) \in \mathbb{C}(n)$.

The arithmetic of the other solutions is also rigid. A rational sequence is *globally bounded* on a tail if there are positive integers B, C such that $BC^n u(n)$ is integral for all sufficiently large n. For a rational function $R(x_1, \ldots, x_d)$ regular at the origin, its diagonal is

$$\operatorname{Diag} R(z) = \sum_{n \geq 0} [x_1^n \cdots x_d^n] R(x_1, \ldots, x_d) z^n.$$

Theorem 1.3 (Rational solutions)**.** *Let V_r be the rational-valued solution space of L_r. The limit*

$$\mathcal{L}(U) = \lim_{n\to\infty} \frac{U(n)}{F_r(n)}$$

exists for every $U \in V_r$ and gives a $\mathbb{Q}$-linear isomorphism

$$V_r \xrightarrow{\mathcal{L}} \mathbb{Q} \oplus \mathbb{Q}\pi^2 \oplus \cdots \oplus \mathbb{Q}\pi^{2m_r-2}. \tag{1.7}$$

Note that within V_r, each of the following conditions describes exactly the line $\mathbb{Q}F_r$: bounded denominators on a tail; denominators involving only a fixed finite set of primes; global boundedness; and a generating series which, modulo an element of $\mathbb{Q}[z]$, is the diagonal of a rational function over $\mathbb{Q}$ regular at the origin. For $r \geq 2$, the integer-valued solution germs form the group $\frac{1}{2}\mathbb{Z}F_r$.

The explicit deformation and the values underlying (1.7) are due to Straub and Zudilin [17, Theorem 1.3]. Together with van Hoeij's order bound, their independence already makes these solutions a basis of the minimal telescoper. The unrestricted minimality proved here identifies that operator with L_r. A congruence at shifted prime powers then separates the basis vectors arithmetically. The denominator growth can be read from the limit itself: if a nonzero solution has $\mathcal{L}(U) = P_U(\pi^2)$, with $P_U \in \mathbb{Q}[X]$ of degree less than m_r, its common denominators through index N have logarithm $2\deg(P_U)N + o(N)$ (Theorem 6.1). This yields the classification in Theorem 1.3. For $r = 3$, it also answers the integer-sequence question of Schulte, VanSchalkwyk and Yang [15, Conjecture 4.5]; a short independent proof is given in Corollary 6.2.

The two mechanisms. The analytic part begins with a general observation. If $u(n) = \int x^n \, \mathrm{d}\mu(x)$ for a compactly supported measure, then a recurrence for u becomes a differential equation for μ. Away from zero, each point where μ fails to have a smooth density must be a zero of χ (Proposition 2.1). Moment recurrences and differential equations have long been studied together; relevant forms appear in Batenkov's work on piecewise D-finite functions [1, Theorems 2.9 and 2.13] and in the work of Bostan, Elvey Price, Guttmann and Maillard on Stieltjes moment sequences [2, Theorem 3]. The distributional formulation used here applies directly to eventual recurrences and retains the exact polynomial χ.

Franel already derived the $r = 3$ recurrence from a double integral [8, p. 46]. Here the measures are pushforwards of Haar measure under explicit real Laurent polynomials. Every critical point lying over a nonzero value is a nondegenerate extremum. At each such value the local density contributions have the same sign and orientation, so they cannot cancel. Counting these values proves the required lower bounds. Essam and Guttmann [7, §III B] developed the lattice-walk interpretation of F_r and related its generating function to Green functions of projected lattices. Our measure is the spectral measure at the origin of a symmetric lattice-walk operator. Its critical-value singularities are the density-of-states phenomenon studied by van Hove [20]. The recurrence consequence uses the interior critical values as well as the spectral edge.

The arithmetic part uses a different feature. After multiplication by a suitable power of p, each deformation coefficient at $p + b$ reduces to a fixed nonzero multiple of $F_r(b)$. An annihilating recurrence for any nonzero linear combination can therefore be transferred to F_r. A finite-dimensional descent over $\mathbb{Q}$ makes this argument work for arbitrary complex coefficients (Proposition 5.4). It proves irreducibility without factoring the recurrence operator.

Organization. Sections 2–4 prove Theorem 1.1. Section 5 develops the congruence and proves Theorem 1.2; Section 6 proves the arithmetic classification. Appendix A determines the exact dependence of a minimal telescoping certificate on its summation variable, resolving that part of the patterns recorded in [17, Remark 4.1].

2. A singular-support bound for moment recurrences

A distribution is smooth on an interval if it is given there by a C^∞ function times Lebesgue measure. Its singular support is the complement of the largest open set on which it is smooth. For a compactly supported distribution μ, its moments $\langle \mu, x^n \rangle$ are well-defined by inserting any smooth cutoff equal to 1 near its support.

Proposition 2.1. *Let μ be a compactly supported distribution on $\mathbb{R}$ and put $u(n) = \langle \mu, x^n \rangle$. Suppose*

$$\sum_{j=0}^{M} p_j(n) u(n+j) = 0 \qquad (n \geq N), \qquad p_j \in \mathbb{C}[n], \quad (p_0, \dots, p_M) \neq 0. \tag{2.1}$$

Here $N \in \mathbb{Z}_{\geq 0}$. With D and χ as in (1.3),

$$\operatorname{sing\,supp} \mu \setminus \{0\} \subseteq \{x \in \mathbb{R} : \chi(x) = 0\}. \tag{2.2}$$

In particular, M is at least the number of distinct nonzero points of $\operatorname{sing\,supp} \mu$.

Proof. Use bilinear distribution pairing. The Euler operator $E = x\,\mathrm{d}/\mathrm{d}x$ has formal transpose $E^* = -x\,\mathrm{d}/\mathrm{d}x - 1$. The compactly supported distribution

$$T = \sum_{j=0}^{M} p_j(E^*)(x^j \mu)$$

satisfies $\langle T, x^n \rangle = 0$ for $n \geq N$, since $Ex^n = nx^n$. Every moment of $x^N T$ therefore vanishes. The Fourier–Laplace transform of a compactly supported distribution is entire, and its derivatives at zero are its moments, up to nonzero constants. Thus the transform of $x^N T$ vanishes identically, and uniqueness of the Fourier transform gives $x^N T = 0$. Consequently $T = 0$ away from 0.

The coefficient of $\mu^{(D)}$ in T is

$$(-1)^D x^D \chi(x).$$

Indeed, the highest derivative in $(E^*)^D(x^j \mu)$ has coefficient $(-1)^D x^{D+j}$. By definition of D, the polynomial χ is nonzero. Near a point $x_0 \neq 0$ where $\chi(x_0) \neq 0$, the equation $T = 0$ is a regular linear ordinary differential equation for μ. Its distributional solutions are smooth: for $D > 0$, form the first-order system for $(\mu, \mu', \dots, \mu^{(D-1)})$ and multiply by the inverse of a smooth fundamental matrix. The resulting distributional vector has derivative zero and is constant. For $D = 0$, multiplication by a nonvanishing function forces μ to vanish locally. This proves (2.2). The last assertion follows from $\deg \chi \leq M$. □

The exclusion of 0 is necessary: a distribution supported at 0 can have all sufficiently high moments equal to zero. Positivity is not required in the proposition. It enters through the following criterion for a critical value to belong to the singular support.

Lemma 2.2. *Let P be a smooth real function on a compact smooth manifold without boundary of dimension $d \geq 1$, equipped with a positive smooth density. Let μ be its pushforward under P. Suppose that the critical points in the fiber $P^{-1}(\lambda)$ form a nonempty finite set, are all nondegenerate, and are all local maxima or all local minima. Then $\lambda \in \operatorname{sing\,supp} \mu$.*

Proof. Choose disjoint neighborhoods of these critical points and smooth nonnegative cutoffs equal to 1 near them. The Morse lemma gives coordinates in which $P = \lambda + \eta |y|^2$, with the same $\eta \in \{1, -1\}$ at every point. Polar integration shows that the local pushforward density, with $t = \eta(x - \lambda)$, has the form

$$\mathbf{1}_{\{t>0\}} t^{d/2-1} h(t), \qquad h \in C^\infty([0, \epsilon)), \quad h(0) > 0. \tag{2.3}$$

The spherical average of the smooth coordinate density is a smooth even function of the radius, hence a smooth function of its square. This proves the asserted smoothness of h. The contribution from the remainder of the manifold is smooth near λ, by integration along regular

level sets. Compactness and nondegeneracy ensure that a sufficiently small neighborhood of λ contains no further critical points outside the chosen neighborhoods.

Adding the local terms preserves the form (2.3) with positive $h(0)$. A one-sided power with exponent $d/2-1$ is not smooth at zero: for an integer exponent a derivative has a jump, and for a noninteger exponent a derivative is unbounded. A smooth remainder cannot remove this defect. Hence μ is not smooth at λ. □

3. The real torus symbols and their critical values

Fix $r \geq 2$ and put $d = r-1$. On the torus $\mathbb{T}^d$, write $z_j = e^{i\theta_j}$, $Z = z_1 \cdots z_d$, and

$$A(z) = \prod_{j=1}^{d}(1+z_j), \qquad P = A + A(z^{-1}) = A(1+Z^{-1}), \qquad Q = \frac{A - A(z^{-1})}{i} = \frac{A(1-Z^{-1})}{i}.$$

Both P and Q are real-valued on the torus. With $s = \theta_1 + \cdots + \theta_d$, they become

$$P(\theta) = 2^r \prod_{j=1}^{d} \cos(\theta_j/2)\cos(s/2), \qquad Q(\theta) = 2^r \prod_{j=1}^{d} \cos(\theta_j/2)\sin(s/2). \tag{3.1}$$

These products are periodic, despite the separate half-angle factors. Let μ_r be the pushforward of normalized Haar measure under P, and let $\widetilde{\mu}_r$ be its pushforward under $R = Q^2$.

Lemma 3.1. *For every $n \geq 0$,*

$$F_r(n) = \operatorname{CT} P^n = \int x^n \, \mathrm{d}\mu_r(x), \qquad T_r(n) = \operatorname{CT} Q^{2n} = \int x^n \, \mathrm{d}\widetilde{\mu}_r(x). \tag{3.2}$$

Proof. Expanding $(1+Z^{-1})^n$ gives

$$\operatorname{CT} P^n = \sum_{k=0}^{n} \binom{n}{k} \operatorname{CT}(A^n Z^{-k}) = \sum_{k=0}^{n} \binom{n}{k}^r.$$

Likewise,

$$\operatorname{CT} Q^{2n} = (-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \operatorname{CT}(A^{2n} Z^{-k}) = T_r(n).$$

Constant terms are torus integrals against normalized Haar measure. □

Lemma 3.2. *The distinct nonzero critical values of P are exactly $\lambda_{r,k}$, $0 \leq k < r/2$. At each such value all critical points are nondegenerate extrema of the same type. The distinct nonzero critical values of $R = Q^2$ are exactly $\nu_{r,h}$, $0 \leq h < a_r$. All critical points at these values are nondegenerate local maxima.*

Proof. At a point where $P \neq 0$, all factors in (3.1) are nonzero, and

$$\partial_{\theta_j} \log|P| = -\frac{1}{2}\tan(\theta_j/2) - \frac{1}{2}\tan(s/2).$$

Thus all θ_j at a nonzero critical point are equal modulo 2π, say to θ, and $r\theta \equiv 0 \pmod{2\pi}$. The possible points are $\theta_j = 2\pi k/r$, $0 \leq k < r$; the point with $\theta_j = \pi$, when present, has value zero and is excluded. The value at every other point is

$$P = (-1)^k \left(2\cos\frac{\pi k}{r}\right)^r.$$

The indices k and $r-k$ give the same value. The representatives $0 \leq k < r/2$ have strictly decreasing absolute values, so they give all the distinct nonzero critical values. At these points,

$$\operatorname{Hess} \log|P| = -\frac{1}{4}\sec^2\left(\frac{\pi k}{r}\right)(I_d + J_d), \tag{3.3}$$

where J_d is the all-ones matrix. This matrix is negative definite. Since $\operatorname{Hess} P = P \operatorname{Hess} \log|P|$ at a critical point, a positive critical value is a nondegenerate maximum and a negative one is a nondegenerate minimum.

For $Q \neq 0$, the critical equations instead read

$$\partial_{\theta_j} \log|Q| = -\frac{1}{2}\tan(\theta_j/2) + \frac{1}{2}\cot(s/2) = 0.$$

They force $\theta_j = (2h+1)\pi/r$, $0 \leq h < r$. Writing $\alpha = (2h+1)\pi/(2r)$, one obtains

$$Q = (-1)^h (2\cos\alpha)^r, \qquad \operatorname{Hess}\log|Q| = -\frac{1}{4}\sec^2\alpha\,(I_d + J_d).$$

When r is odd, the point with $\theta_j = \pi$ has value zero and is excluded. Since $\nabla R = 2Q\nabla Q$, there are no other nonzero critical points of R. At every such point, $\operatorname{Hess} R = 2Q^2 \operatorname{Hess}\log|Q|$ is negative definite. Pairing h with $r-1-h$ leaves precisely the distinct values $\nu_{r,h}$ in (1.6). □

Lemmas 2.2 and 3.2 show that each $\lambda_{r,k}$ belongs to $\operatorname{sing\,supp}\mu_r$ and each $\nu_{r,h}$ belongs to $\operatorname{sing\,supp}\widetilde{\mu}_r$. Proposition 2.1 therefore proves the divisibilities in Theorem 1.1 and the lower bounds m_r and a_r. The case $r = 1$ follows directly from $F_1(n) = 2^n$: its recurrence implies $\sum_j p_j(n)2^j = 0$, hence $\chi(2) = 0$.

Lemma 3.3. *One has $H_r, \widetilde{H}_r \in \mathbb{Z}[X]$.*

Proof. For $z^r = 1$, the values $(1+z)^r$ are precisely the critical values of P, with the possible additional value zero. Inversion pairs the equal values; the value at $z = 1$ is unpaired. Thus the distinct nonzero values form the root set of H_r. They are algebraic integers and this set is stable under the Galois group of the cyclotomic field. The coefficients of H_r are consequently rational algebraic integers.

Similarly, for $z^r = -1$, the values $-(1+z)^{2r}$ are precisely the roots of $\widetilde{H}_r$, with each nonzero value counted twice and the additional value zero when r is odd. The same argument proves the assertion for $\widetilde{H}_r$. □

Some small cases illustrate the two spectra:

r	$H_r(X)$	$\widetilde{H}_r(X)$
2	$X - 4$	$X - 4$
3	$(X-8)(X+1)$	$X - 27$
4	$(X-16)(X+4)$	$X^2 - 136X + 16$
5	$(X-32)(X^2+11X-1)$	$X^2 - 625X + 3125$
6	$(X-64)(X+27)(X-1)$	$(X-64)(X^2-2702X+1)$

The positive roots of $\widetilde{H}_r$ also give a sparsity bound: every recurrence of T_r has at least $a_r + 1$ coefficient polynomials of maximal degree. Indeed, a nonzero real or imaginary part of χ has the a_r distinct positive roots $\nu_{r,h}$, so Descartes' rule of signs requires at least $a_r + 1$ nonzero monomials. In particular, every coefficient of a minimum-order alternating recurrence has maximal degree. For a minimum-order ordinary recurrence, both endpoint coefficients have maximal degree, since H_r is monic and has nonzero constant coefficient.

Remark 3.4. The representation of T_r makes it a Stieltjes moment sequence with support exactly $[0, \nu_{r,0}]$. Indeed, the continuous image of the connected torus under Q^2 is this interval, and the inverse image of every open interval meeting it has positive Haar measure. Consequently all its Hankel minors with strictly increasing row and column indices are positive. To see the strictness, the determinant integration identity expresses such a minor as

$$\frac{1}{s!}\int \det[x_\ell^{a_i}]_{i,\ell=1}^{s} \det[x_\ell^{b_j}]_{j,\ell=1}^{s} \prod_{\ell=1}^{s} \mathrm{d}\widetilde{\mu}_r(x_\ell),$$

where a_i and b_j are strictly increasing nonnegative integers. Both generalized Vandermonde determinants are positive when $0 < x_1 < \cdots < x_s$, and their product is nonnegative everywhere. Full support gives strict positivity of the integral. In particular, the ratios $T_r(n+1)/T_r(n)$ increase strictly to $\nu_{r,0}$. These are standard consequences of the moment representation; the recurrence lower bound requires its additional interior singularities.

4. A common telescoping construction

We give a common proof of the matching upper bounds. The ordinary case follows the reduction and reflection argument of van Hoeij [19, §5, Remark 6]. The signed formulation below also treats the alternating sums and verifies the certificate boundary terms. In particular, a minimum-order recurrence of F_r telescopes the original summand, as required for the deformation theorem used in Section 5. Write $(x)_j = x(x+1)\cdots(x+j-1)$, with $(x)_0 = 1$.

Lemma 4.1. *Let $r \geq 2$ and $\varepsilon \in \{1, -1\}$, and put*

$$q_\varepsilon = \begin{cases} 1, & \varepsilon = 1, \\ 2, & \varepsilon = -1, \end{cases} \qquad m_\varepsilon = \begin{cases} \lfloor (r+1)/2 \rfloor, & \varepsilon = 1, \\ \lfloor r/2 \rfloor, & \varepsilon = -1. \end{cases}$$

There exist $p_0, \ldots, p_{m_\varepsilon} \in \mathbb{Q}[n]$, not all zero, and $C \in \mathbb{Q}(n,k)$ such that, for $f_\varepsilon(n,k) = \varepsilon^k \binom{n}{k}^r$,

$$\sum_{a=0}^{m_\varepsilon} p_a(n) f_\varepsilon(n + q_\varepsilon a, k) = C(n, k+1) f_\varepsilon(n, k+1) - C(n,k) f_\varepsilon(n,k). \tag{4.1}$$

For all sufficiently large nonnegative integers n, the certificate product has well-defined, finitely supported values on $k \in \mathbb{Z}$, with removable singularities interpreted by continuation. Its bilateral forward difference therefore sums to zero.

Proof. Work over $K = \mathbb{Q}(n)$ and set

$$\mathcal{D}_\varepsilon C(k) = \varepsilon \left(\frac{n-k}{k+1} \right)^r C(k+1) - C(k), \qquad Q_j(k) = \frac{(n+1)_j^r}{(n-k+1)_j^r}.$$

Brackets denote classes modulo $\mathcal{D}_\varepsilon K(k)$. We will bound only the span of the selected shift classes $[Q_{q_\varepsilon a}]$.

First remove the poles of Q_j from right to left. If the rightmost remaining pole is $k = n + h$, write its principal part as $A(k)/(n-k+h)^r$, with $\deg_k A < r$. Since k^r is invertible modulo $(n-k+h)^r$ in $K[k]$, choose $H \in K[k]$, $\deg H < r$, with

$$k^r H(k) \equiv A(k) \pmod{(n-k+h)^r}.$$

For $V_h(k) = k^r H(k)/(n-k+h)^r$ one has

$$\mathcal{D}_\varepsilon V_h(k) = \varepsilon \frac{(n-k)^r H(k+1)}{(n-k+h-1)^r} - \frac{k^r H(k)}{(n-k+h)^r}.$$

Adding this expression cancels the pole at $n+h$ and introduces at most a pole at $n+h-1$; for $h = 1$ its first term is polynomial. Thus $P_j = Q_j + \mathcal{D}_\varepsilon U_j$ is polynomial, where U_j is a finite sum of such V_h. For $j = 0$ take $P_0 = 1$ and $U_0 = 0$.

Each V_h is $O(k^{r-1})$ at infinity. If $\varepsilon(-1)^r = 1$, the multiplier in $\mathcal{D}_\varepsilon$ is $1 + O(k^{-1})$, so the leading terms cancel and $\mathcal{D}_\varepsilon V_h = O(k^{r-2})$. Otherwise it is $-1 + O(k^{-1})$, giving $\mathcal{D}_\varepsilon V_h = O(k^{r-1})$. Since Q_j is proper for $j \geq 1$, in both cases

$$\deg P_j < b_\varepsilon, \qquad b_\varepsilon = r - \mathbf{1}_{\{\varepsilon(-1)^r = 1\}}.$$

Let $JC(k) = C(n-k)$. Direct substitution gives

$$J\mathcal{D}_\varepsilon C = \mathcal{D}_\varepsilon C^\star, \qquad C^\star(k) = -\varepsilon \left(\frac{k}{n-k+1} \right)^r C(n-k+1).$$

Hence J induces an involution on the quotient. Moreover,

$$\varepsilon^j J Q_j - Q_j = \mathcal{D}_\varepsilon B_j, \qquad B_j(k) = \sum_{\ell=0}^{j-1} \frac{\varepsilon^\ell (n+1)_j^r}{(k+1)_\ell^r (n-k+1)_{j-\ell}^r}.$$

Indeed, after multiplication by $f_\varepsilon(n,k)$, this is the telescoping sum of the j consecutive shifts in k of $f_\varepsilon(n+j,k)$. For $j = q_\varepsilon a$ one has $\varepsilon^j = 1$. Consequently

$$S_j = \frac{P_j + J P_j}{2}, \qquad Q_j - S_j = -\frac{1}{2} \mathcal{D}_\varepsilon (U_j + U_j^\star + B_j).$$

The polynomial S_j is even in $k-n/2$ and has degree less than b_ε. Such polynomials form a space of dimension $\lceil b_\varepsilon/2\rceil = m_\varepsilon$, with basis

$$1, (k-n/2)^2, \dots, (k-n/2)^{2(m_\varepsilon-1)}.$$

Thus the $m_\varepsilon+1$ representatives $S_0, S_{q_\varepsilon}, \dots, S_{q_\varepsilon m_\varepsilon}$ are linearly dependent over K. The displayed certificate identity, followed by clearing denominators in n, gives (4.1).

For the boundary assertion, specialize $n = N$ to a sufficiently large nonnegative integer, avoiding all parameter poles in the finitely many rational functions used above. Put $\delta = (1-\varepsilon)/2$ and use the entire continuation in k

$$f_\varepsilon(N,k) = e^{i\pi\delta k}\left\{\frac{\Gamma(N+1)}{\Gamma(k+1)\Gamma(N-k+1)}\right\}^r.$$

It has zeros of order r at every integer outside $[0,N]$. Therefore $V_h f_\varepsilon$ has a removable singularity at its only possible pole $k = N+h$, and its integer values have finite support. The reflected certificate is explicitly

$$V_h^\star(k) = -\varepsilon\frac{k^r H(N-k+1)}{(k+h-1)^r}.$$

It is polynomial when $h=1$; otherwise its sole possible pole is the negative integer $1-h$, again canceled by a zero of order r. Finally,

$$B_j(k) f_\varepsilon(N,k) = \sum_{\ell=0}^{j-1} f_\varepsilon(N+j, k+\ell),$$

whose integer values also have finite support. Every certificate in the construction is a finite linear combination of these products. Their bilateral forward differences consequently sum to zero. □

Completion of the proof of Theorem 1.1. For $\varepsilon = 1$, bilateral summation in Lemma 4.1 gives a nonzero recurrence of order at most m_r for F_r. For $\varepsilon = -1$, specialize the parameter to $2n$ and use

$$\sum_k f_{-1}(2n+2j, k) = (-1)^{n+j} T_r(n+j).$$

The resulting recurrence is $\sum_{j=0}^{a_r}(-1)^j p_j(2n) T_r(n+j) = 0$ and has order at most a_r. Nonzero polynomial coefficients remain nonzero under the substitution $n \mapsto 2n$. If either endpoint vanished, translating the index would give a recurrence of smaller width, contrary to the lower bounds proved in Section 3. The divisibility already proved then gives $\chi = cH_r$ or $c\widetilde{H}_r$ at the minimum order. For $r=1$, the identity $F_1(n+1) = 2F_1(n)$ supplies the upper bound. Its normalized certificate is $-k/(n-k+1)$, whose product with $\binom{n}{k}$ is $-\binom{n}{k-1}$ and has finite support on the integer lattice. □

5. The deformation and its arithmetic

5.1. **A deformation theorem.** Define the even rational function

$$\mathcal{A}_r(n,t) = \sum_{k=0}^{n}\binom{n}{k}^r\left\{\prod_{h=1}^{k}\left(1-\frac{t}{h}\right)\prod_{h=1}^{n-k}\left(1+\frac{t}{h}\right)\right\}^{-r} = \sum_{j\ge 0} A_{r,j}(n) t^{2j}. \tag{5.1}$$

Empty products are 1. Evenness follows by replacing k by $n-k$, and every $A_{r,j}(n)$ is rational. In particular, $A_{r,0} = F_r$. We use the following result with its original attribution.

Theorem 5.1 (Straub–Zudilin, [17, Theorems 1.3 and 3.1])**.** *Every recurrence over $\mathbb{Q}(n)$ obtained by rational telescoping from $\binom{n}{k}^r$ is solved, for all sufficiently large n, by $A_{r,j}$ for $2j < r$. For every $j \ge 0$,*

$$\lim_{n\to\infty}\frac{A_{r,j}(n)}{F_r(n)} = \phi_{r,j}\pi^{2j}, \qquad \phi_{r,j} = [t^{2j}]\left(\frac{t}{\sin t}\right)^r \in \mathbb{Q}_{>0}. \tag{5.2}$$

The telescoping construction in Section 4 and minimality show that $A_{r,0},\dots,A_{r,m_r-1}$ solve L_r. The case $r=1$ is immediate from $A_{1,0}(n)=2^n$. Independence follows from these limits and the transcendence of π, as in [17, proof of Theorem 1.1]. We give an arithmetic proof as part of a congruence that will control denominators and annihilators.

5.2. **One congruence at shifted prime powers.** Write v_p for the valuation normalized by $v_p(p)=1$, with $v_p(0)=+\infty$, and $\mathbb{Z}_{(p)}=\{x\in\mathbb{Q}: v_p(x)\ge 0\}$. Congruences of formal power series are coefficientwise. Put $\ell_n=\operatorname{lcm}(1,\dots,n)$ for $n\ge 1$, and $\ell_0=1$.

Lemma 5.2. *For every $r\ge 1$ and $n\ge 0$,*

$$\mathcal{A}_r(n,\ell_n t)\in\mathbb{Z}[[t]]. \tag{5.3}$$

For any prime p and integers $\alpha\ge 1$ and $0\le b<p$, the series $\mathcal{A}_r(p^\alpha+b,p^\alpha t)$ lies in $\mathbb{Z}_{(p)}[[t]]$ and

$$\mathcal{A}_r(p^\alpha+b,p^\alpha t)\equiv F_r(b)\big((1-t)^{-r}+(1+t)^{-r}\big)\pmod p. \tag{5.4}$$

Consequently, for every $j\ge 0$,

$$\ell_n^{2j}A_{r,j}(n)\in\mathbb{Z},\qquad p^{2j\alpha}A_{r,j}(p^\alpha+b)\equiv C_{r,j}F_r(b)\pmod p,\quad C_{r,j}=2\binom{r+2j-1}{2j}. \tag{5.5}$$

For fixed r,j,b, one has $v_p(A_{r,j}(p^\alpha+b))=-2j\alpha$ for all sufficiently large primes p, uniformly in $\alpha\ge 1$.

Proof. Replacing t by $\ell_n t$ in (5.1) makes every factor $(1\pm\ell_n t/h)^{-r}$ an integral formal power series. This proves (5.3).

Put $N=p^\alpha+b<2p^\alpha$. For $1\le h\le N$ one has $v_p(h)\le\alpha$. The only index with valuation α is $h=p^\alpha$. After replacing t by $p^\alpha t$, every factor therefore lies in $\mathbb{Z}_{(p)}[[t]]$, and every factor except the one with $h=p^\alpha$ is congruent to 1 modulo p. The identity

$$(1+x)^N\equiv(1+x^{p^\alpha})(1+x)^b\pmod p$$

shows that the summands with $b<k<p^\alpha$ vanish modulo p. For $0\le k\le b$, the exceptional factor occurs in the second product and gives $(1+t)^{-r}$, while $\binom{N}{k}\equiv\binom{b}{k}\pmod p$. For $k=p^\alpha+q$, $0\le q\le b$, it occurs in the first product and gives $(1-t)^{-r}$, while $\binom{N}{p^\alpha+q}\equiv\binom{b}{q}\pmod p$. Summing proves (5.4). Coefficient extraction gives (5.5). Its right side is a fixed nonzero integer, so it is a unit at all sufficiently large primes. This proves the valuation assertion. □

Proposition 5.3. *The eventual germs $A_{r,0},\dots,A_{r,m_r-1}$ form a basis of the rational solution space of L_r over $\mathbb{Q}$ and of the complex solution space over $\mathbb{C}$. The limit map in (1.7) is a $\mathbb{Q}$-linear isomorphism.*

Proof. In fact, every finite collection $A_{r,0},\dots,A_{r,J}$ is linearly independent over $\mathbb{Q}$ as eventual germs. In a nonzero proposed relation, let J be the highest index with nonzero coefficient. For every sufficiently large prime p, all its nonzero rational coefficients are p-adic units, and Lemma 5.2 with $b=0$, $\alpha=1$ gives distinct valuations $-2j$ for the summands at index p. The term with smallest valuation, at $j=J$, cannot cancel.

Choose N beyond the poles and the zeros of the endpoint coefficients of L_r, and beyond the index from which all $A_{r,j}$ with $j<m_r$ satisfy that recurrence. Its order m_r recurrence determines a solution on this tail from m_r consecutive values. The matrix

$$\big[A_{r,j}(N+i)\big]_{0\le i,j<m_r}$$

is nonsingular over $\mathbb{Q}$: otherwise a nonzero rational combination would have m_r consecutive zero values and would vanish on a tail, contradicting independence. Its inverse has rational entries. Thus the same sequences give both asserted bases.

For $U=\sum_j c_jA_{r,j}$, $c_j\in\mathbb{Q}$, Theorem 5.1 gives $\mathcal{L}(U)=\sum_j c_j\phi_{r,j}\pi^{2j}$. The numbers $\phi_{r,j}$ are nonzero rationals, and the powers $1,\pi^2,\dots,\pi^{2m_r-2}$ are linearly independent over $\mathbb{Q}$ by the transcendence of π. This proves the isomorphism. □

5.3. **Transferring annihilators.** The shifted congruence supplies more than independence. It transfers an annihilator of any nonzero combination of the deformation coefficients to the undeformed sequence. The following proof treats complex constants directly.

Proposition 5.4. *Fix $r \geq 1$ and suppose, on a tail,*

$$U = \sum_{j=0}^{J} c_j A_{r,j}, \qquad c_j \in \mathbb{C}, \quad c_J \neq 0.$$

If $Q = \sum_{\ell=0}^{d} q_\ell(n) S^\ell$, $q_\ell \in \mathbb{C}[n]$, annihilates U eventually, then

$$\sum_{\ell=0}^{d} q_\ell(b) F_r(b+\ell) = 0 \qquad (b \in \mathbb{Z}_{\geq 0}). \tag{5.6}$$

Every annihilator of U in $\mathbb{C}(n)[S;\sigma]$ therefore also annihilates F_r eventually.

Proof. The case $Q = 0$ is immediate. Otherwise choose a $\mathbb{Q}$-basis $\beta_1, \ldots, \beta_t$ of the finite-dimensional $\mathbb{Q}$-vector space spanned by all products $c_j[n^h]q_\ell(n)$. Write

$$c_j q_\ell(n) = \sum_{i=1}^{t} \beta_i q_{i,j,\ell}(n), \qquad q_{i,j,\ell} \in \mathbb{Q}[n].$$

For every sufficiently large integer n, the identity $QU(n) = 0$ becomes

$$\sum_{i=1}^{t} \beta_i \left(\sum_{j=0}^{J} \sum_{\ell=0}^{d} q_{i,j,\ell}(n) A_{r,j}(n+\ell) \right) = 0.$$

Each parenthesized expression is rational. Independence of the β_i over $\mathbb{Q}$ makes each expression zero separately.

Fix $b \geq 0$. Evaluate these rational identities at $n = p + b$, where $p > b + d$ is a sufficiently large prime avoiding all coefficient denominators. Multiply by p^{2J} and reduce modulo p. By (5.5), the terms with $j < J$ vanish, leaving

$$C_{r,J} \sum_{\ell=0}^{d} q_{i,J,\ell}(b) F_r(b+\ell) \equiv 0 \pmod{p}.$$

Here $q_{i,j,\ell}(p+b) \equiv q_{i,j,\ell}(b) \pmod{p}$. The displayed rational number is independent of p and is divisible by infinitely many primes, so it is zero. Multiplying by β_i and summing over i, then canceling $C_{r,J} c_J \neq 0$, gives (5.6). For rational-function coefficients, first clear their polynomial denominators in n, which preserves annihilation on a tail. □

Proof of Theorem 1.2. Suppose that $L_r = BR$ in $\mathbb{C}(n)[S;\sigma]$ with $0 < \operatorname{ord} R < m_r$. The constant coefficient of R is nonzero, because the constant coefficient of L_r is nonzero. Beyond the finitely many poles and endpoint zeros of R, nonzero initial data give a nonzero solution germ U of R: forward and backward uniqueness prevent it from vanishing eventually. Then $L_r U = 0$, so Proposition 5.3 expresses U as a nonzero complex combination of $A_{r,0}, \ldots, A_{r,m_r-1}$. Proposition 5.4 gives $RF_r = 0$, contrary to the minimal order m_r. This proves irreducibility.

For any nonzero solution germ U of L_r and any annihilator Q of U, Proposition 5.4 gives $QF_r = 0$. Right division by the monic operator L_r writes $Q = HL_r + R$ with $\operatorname{ord} R < m_r$; this is the usual division algorithm in the shift Ore ring [12]. Since $RF_r = 0$, minimality forces $R = 0$. Finally, a nonzero hypergeometric germ has an annihilator $S - a(n)$ with $a(n) \in \mathbb{C}(n)$, impossible when $r \geq 3$ and $m_r \geq 2$. □

5.4. **The rate of convergence of solution ratios.** The characteristic roots also give a uniform exponential rate for the limits. This concerns complex solutions as well as rational ones.

Corollary 5.5. *For $r \geq 3$ and every complex solution germ U of L_r, the limit $\mathcal{L}(U) = \lim U(n)/F_r(n)$ exists and*

$$\limsup_{n\to\infty} \left| \frac{U(n)}{F_r(n)} - \mathcal{L}(U) \right|^{1/n} \leq \cos^r\left(\frac{\pi}{r}\right). \tag{5.7}$$

For $r = 1, 2$, every solution germ is a constant multiple of F_r.

Proof. After clearing denominators, divide a minimum-order recurrence by its coefficient of S^{m_r}. Theorem 1.1 shows that the resulting coefficients tend to those of H_r, and the trailing coefficient is nonzero on a sufficiently late tail. The roots $\lambda_{r,k}$ are nonzero with distinct absolute values. Perron's theorem [14], in the form stated in [4, §4], therefore gives a basis of solutions with successive ratios tending to $\lambda_{r,k}$. The elementary bounds

$$\frac{2^{rn}}{(n+1)^{r-1}} \leq F_r(n) \leq 2^{rn}$$

show that $F_r(n)^{1/n} \to 2^r = \lambda_{r,0}$, so F_r has a nonzero component in the dominant basis direction. Subtracting the corresponding multiple of F_r from U leaves only the other directions, whose largest exponential modulus is $(2\cos(\pi/r))^r$. This proves (5.7). □

6. Denominators, diagonals and integer solutions

6.1. **Exact denominator growth.** For $x \in \mathbb{Q}$, let $\mathrm{den}(x)$ be its positive denominator in lowest terms, with $\mathrm{den}(0) = 1$. Fix an initial index $N_0 \geq 1$ from which the displayed basis expansion of the rational germ holds, and put

$$D_U(N) = \mathrm{lcm}\{\mathrm{den}(U(n)) : N_0 \leq n \leq N\}. \tag{6.1}$$

Changing finitely many terms changes $\log D_U(N)$ by at most a constant for all sufficiently large N, and hence does not affect its exponential growth rate. Write $\psi(N) = \log \ell_N = \sum_{p^\alpha \leq N} \log p$.

Theorem 6.1. *Let*

$$U(n) = \sum_{j=0}^{J} c_j A_{r,j}(n), \qquad c_j \in \mathbb{Q}, \quad c_J \neq 0,$$

on the chosen tail. There are a finite set of primes $\mathcal{S}$ and an integer $B \geq 1$ such that

$$v_p(U(p^\alpha)) = -2J\alpha \qquad (p \notin \mathcal{S},\ \alpha \geq 1), \tag{6.2}$$

$$v_p(D_U(N)) = 2J\lfloor \log_p N \rfloor \qquad (p \notin \mathcal{S},\ N \geq N_0), \tag{6.3}$$

and

$$D_U(N) \mid B\ell_N^{2J}, \qquad \log D_U(N) = 2J\psi(N) + O_{U,N_0}(\log N). \tag{6.4}$$

In particular,

$$\lim_{N\to\infty} \frac{\log D_U(N)}{N} = 2J. \tag{6.5}$$

Proof. Choose B with $Bc_j \in \mathbb{Z}$ for every j. By (5.3), $B\ell_n^{2J}U(n)$ is integral, which gives the divisibility in (6.4). Let $\mathcal{S}$ contain all primes at most N_0, the divisors of B and $C_{r,J}$, and the prime divisors of the numerator of c_J in lowest terms. Outside this finite set, c_J is a unit and the other c_j are integral. At $n = p^\alpha$, the term with $j = J$ has valuation exactly $-2J\alpha$ by (5.5); every lower term has strictly larger valuation. This proves (6.2).

For $p \notin \mathcal{S}$, the divisibility bound gives $v_p(D_U(N)) \leq 2J\lfloor \log_p N \rfloor$. If $p \leq N$, choose $\alpha = \lfloor \log_p N \rfloor$. Then $N_0 < p \leq p^\alpha \leq N$, so (6.2) gives the matching lower bound. If $p > N$, both sides are zero. This proves (6.3). Only the fixed exceptional primes can contribute to $\log D_U(N) - 2J \log \ell_N$, and each contributes $O(\log N)$. Finally, the prime number theorem gives $\psi(N) \sim N$. □

For the solution space of L_r, Proposition 5.3 shows that the possible nonzero-solution rates in (6.5) are exactly $0, 2, \dots, 2m_r - 2$. More precisely, the solutions with rate at most $2J$ form

$$\operatorname{span}_{\mathbb{Q}}\{A_{r,0}, \dots, A_{r,J}\}, \qquad 0 \le J < m_r.$$

Equivalently, write the Apéry limit uniquely as $\mathcal{L}(U) = P_U(\pi^2)$, where $P_U \in \mathbb{Q}[X]$ and $\deg P_U < m_r$. For every nonzero $U \in V_r$,

$$\lim_{N\to\infty} \frac{\log D_U(N)}{N} = 2 \deg P_U. \tag{6.6}$$

Indeed, (5.2) gives $P_U(X) = \sum_j c_j \phi_{r,j} X^j$, and each $\phi_{r,j}$ is nonzero. Thus the arithmetic filtration agrees with the polynomial degree of the Apéry limit.

6.2. **The globally bounded direction.**

Proof of the arithmetic assertions in Theorem 1.3. Write a nonzero rational solution in the basis of Proposition 5.3, and let J be its highest nonzero index. If $J = 0$, it is a rational multiple of F_r and has bounded denominators. If $J > 0$, equation (6.2) with $\alpha = 1$ shows that its denominator at index p is divisible by p^{2J} for every sufficiently large prime p. It therefore has neither bounded denominators nor a fixed finite set of denominator primes. It cannot be globally bounded, since for $p \nmid BC$ multiplication by BC^p does not change this negative valuation.

Every diagonal of a rational function over $\mathbb{Q}$ regular at the origin is globally bounded. For completeness, write that function as $P(\boldsymbol{x})/Q(\boldsymbol{x})$ with integral polynomials and $Q(0) = a \neq 0$. The series $H = 1 - Q/a$ has zero constant term, and

$$\frac{P}{Q} = \frac{P}{a} \sum_{q\ge 0} H^q.$$

A monomial of total degree h receives terms only from $q \le h$, so its coefficient has denominator dividing a^{h+1}. A diagonal in d variables is consequently made integral by multiplying its nth coefficient by $a(a^d)^n$; absolute values may be used to make these scaling integers positive. A polynomial change affects only a finite common denominator. Hence a diagonal solution germ belongs to $\mathbb{Q}F_r$. Conversely,

$$\sum_{n\ge 0} F_r(n) z^n = \operatorname{Diag} \frac{1}{(1-x_1)\cdots(1-x_r) - x_1 \cdots x_r}. \tag{6.7}$$

Indeed, geometric expansion gives $\sum_{k\ge 0} (x_1 \cdots x_r)^k \prod_i (1-x_i)^{-k-1}$, whose diagonal coefficient is $\sum_{k=0}^{n} \binom{n}{k}^r$. Rational multiples are again rational diagonals.

It remains to determine the integer lattice for $r \ge 2$. For every $N \ge 1$,

$$\gcd\{F_r(n) : n \ge N\} = 2. \tag{6.8}$$

Pairing the summands with indices k and $n-k$ shows that $F_r(n)$ is even for $n \ge 1$: when $n = 2h$, the unpaired coefficient $\binom{2h}{h} = 2\binom{2h-1}{h-1}$ is also even. For every prime p, all interior coefficients $\binom{p^\alpha}{k}$ are divisible by p. Thus $F_r(p^\alpha) \equiv 2 \pmod p$, excluding all odd primes from the gcd, and $F_r(2^\alpha) \equiv 2 \pmod 4$ because $r \ge 2$. This proves (6.8). An integer-valued solution germ is already a rational multiple cF_r; the gcd says exactly that the denominator of c divides 2. □

Theorem 1.3 has an equivalent interpretation for binomial sums. Bostan, Lairez and Salvy [3, Theorem 3.5] characterize univariate multiple binomial sums over $\mathbb{Q}$ as diagonals of rational power series. Within their class, the solution germs of L_r are therefore exactly $\mathbb{Q}F_r$. The secondary deformation solutions cannot be rewritten as multiple binomial sums in that class. This conclusion concerns rational functions and sums over $\mathbb{Q}$; these hypotheses specify the class of diagonals used here.

6.3. **The classical integer-sequence question.** The preceding theorem classifies eventual integer solutions for $r \geq 2$. In the classical case $r = 3$, a much shorter argument answers the question about entire integer sequences. This is the converse asked in [15, Conjecture 4.5], where the initial values were assumed positive, and in the conclusion of [18]. The Casoratian formula used in the proof is also present in [18, proof of Theorem 6].

Corollary 6.2. *An integer sequence $U(0), U(1), \ldots$ satisfies*

$$(n+1)^2 U(n+1) = (7n^2+7n+2)U(n) + 8n^2 U(n-1), \qquad n \geq 1, \tag{6.9}$$

if and only if $U(n) = cF_3(n)$ for some $c \in \mathbb{Z}$. More generally, a rational solution of (6.9) with bounded denominators on a tail is a rational multiple of F_3 on the entire sequence.

Proof. The Casoratian $W_n = F_3(n)U(n+1) - F_3(n+1)U(n)$ satisfies

$$W_n = -\frac{8n^2}{(n+1)^2} W_{n-1} = \frac{(-8)^n}{(n+1)^2}(U(1) - 2U(0)).$$

If $b = U(1) - 2U(0) \neq 0$, choose arbitrarily large odd primes dividing neither the numerator nor denominator of b nor a common denominator for the tail of U. At $n = p - 1$, the formula gives $v_p(W_n) = -2$, contradicting bounded denominators of W_n. Thus $U(1) = 2U(0)$. The recurrence uniquely propagates these initial values, giving $U = U(0)F_3$. Integrality at $n = 0$ forces $U(0) \in \mathbb{Z}$; the converse follows from the defining binomial sum. □

6.4. **Further questions.** The minimal orders and the leading characteristic polynomials leave several finer questions. Straub and Zudilin [17, Remark 4.1] record experimental formulas for the degree in n of a primitive minimum-order Franel recurrence, and additional factorization patterns in its certificate. Appendix A settles the denominator and degree in the summation variable k. The degree in n and the possible factors depending only on n require further arguments.

The analytic method applies more broadly to constant terms $u(n) = \mathrm{CT}\, P^n$ of real Laurent polynomials on a torus. To prove minimality, it is enough to exhibit sufficiently many distinct nonzero values at which the pushforward measure is not smooth and to construct a matching recurrence. Extrema of a common type give a direct noncancellation criterion. For symbols with saddle points or several types of critical points in the same fiber, the local contributions must be compared more carefully.

Related extensions of the Straub–Zudilin deformation to other binomial sums were studied by Dougherty-Bliss and Zeilberger [6]. The arithmetic method here requires a rational basis whose scaled reductions at $p + b$ all recover the same distinguished sequence at b, with distinct scaling exponents. Establishing an analogous deformation and congruence for the alternating family would give information about its entire solution space. The moment argument alone determines its minimal order and characteristic roots, but does not supply that arithmetic structure.

Appendix A. The summation variable in a minimal certificate

We determine the k-dependent denominator and numerator-degree patterns observed in Remark 4.1 of Straub and Zudilin [17]. It does not address possible denominator factors depending only on n, the predicted degree in n, or the further factorization patterns in that remark.

Every minimum-order recurrence is a scalar multiple over $\mathbb{Q}(n)$ of the telescoper constructed in Section 4: after normalizing the leading coefficients, the difference of two such operators would be an annihilator of smaller order. Hence every recurrence in the following proposition has a rational certificate.

Proposition A.1. *Let $r \geq 1$, $m = \lfloor (r+1)/2 \rfloor$, and let $\sum_{j=0}^{m} p_j(n) S^j$ be a minimal-order Franel recurrence, with $p_j \in \mathbb{Q}[n]$ and $p_0 p_m \neq 0$. Its rational certificate $C \in \mathbb{Q}(n,k)$ is characterized by*

$$\mathcal{D}C = \sum_{j=0}^{m} p_j(n) Q_j(n,k), \qquad \mathcal{D}C = \frac{(n-k)^r}{(k+1)^r} C(n,k+1) - C(n,k), \quad Q_j = \frac{(n+1)_j^r}{(n-k+1)_j^r}.$$

Put $K = \mathbb{Q}(n)$ and $B_m(k) = \prod_{h=1}^{m} (n-k+h)^r$. Then the reduced denominator of C in $K[k]$ is B_m, up to a unit of K. Equivalently,

$$C(n,k) = \frac{A(n,k)}{B_m(k)}, \qquad A \in K[k], \qquad \gcd_{K[k]}(A, B_m) = 1.$$

Moreover,

$$k^r \mid A(n,k), \qquad \deg_k A = rm + \mathbf{1}_{\{r \text{ even}\}}.$$

For a fixed recurrence its rational certificate is unique.

Proof. Write $T = \sum_j p_j Q_j$. Its only possible finite poles in k are $n+1, \ldots, n+m$, each of order at most r. Work over an algebraic closure of K and group the finitely many poles of C into orbits under integer translation. In each occupied orbit choose a pole α for which none of $\alpha - 1, \alpha - 2, \ldots$ is a pole. At $k = \alpha - 1$, the function $C(k+1)$ has a pole and $C(k)$ is regular. The multiplier $(n-k)^r/(k+1)^r$ cannot remove this pole unless $\alpha - 1 = n$; its pole at -1 only increases the order. Otherwise $\alpha - 1$ must be a pole of T. Thus every occupied orbit is $n + \mathbb{Z}$, and its leftmost pole has the form $n + h$ with $h \geq 1$. At the rightmost pole $n + h$, the shifted term is regular, so T has a pole there. Hence $h \leq m$. Descending from this rightmost pole in $\mathcal{D}C = T$ shows that every pole of C has order at most r. Therefore $B_m C \in K[k]$.

There is no pole of C at -1 or 0, and T is regular at -1. The order-r pole of the multiplier there forces $C(k+1)$ to vanish to order at least r. Hence $k^r \mid B_m C$, since $B_m(0) \neq 0$ in K.

It remains to show that no factor of B_m cancels. By Theorem 1.1,

$$\deg p_m = D := \max_j \deg p_j.$$

For $r = 1$, the same equality follows directly from $p_0 + 2p_1 = 0$. For a sufficiently large nonnegative integer N, interpret $f(N,k) = \binom{N}{k}^r$ by gamma-function continuation, as in the telescoping construction. The pole bounds just proved imply that $b(N,k) = C(N,k) f(N,k)$ is finite at every integer k and has finite support there. Summing its forward difference from $k = N + h$ to infinity gives

$$b(N, N+h) = -\sum_{j=h}^{m} p_j(N) \sum_{\ell=0}^{j-h} \binom{N+j}{\ell}^r, \qquad 1 \leq h \leq m. \tag{A.1}$$

The right side is a nonzero polynomial in N: its $j = m$ summand has degree $D + r(m-h)$, with leading coefficient $-[n^D]p_m/((m-h)!)^r$, whereas every $j < m$ summand has smaller degree. Since $f(N,k)$ has a zero of exact order r at $k = N + h$, equation (A.1) is incompatible with a pole of order less than r for C at $k = n + h$. This proves the exact denominator assertion.

Finally, let $C = c(n) k^a (1 + O(k^{-1}))$ at infinity, where $a \in \mathbb{Z}$ and $c \neq 0$. Since $Q_0 = 1$ and $Q_j = O(k^{-rj})$ for $j \geq 1$, one has $T = p_0 + O(k^{-r})$. Direct expansion gives

$$\mathcal{D}C = \begin{cases} -2c(n) k^a (1 + O(k^{-1})), & r \text{ odd}, \\ c(n)(a - r(n+1)) k^{a-1} (1 + O(k^{-1})), & r \text{ even}. \end{cases}$$

The displayed leading coefficients are nonzero in K. It follows that $a = 0$ for odd r and $a = 1$ for even r, proving the numerator degree. The same expansion shows that $\ker \mathcal{D} = 0$ on $K(k)$, which proves uniqueness of the rational certificate. □

Remark A.2. The argument proves the same conclusion for any polynomial-coefficient telescoping relation of width m satisfying $p_0 p_m \neq 0$ and $\deg p_m = \max_j \deg p_j$. Minimality is used only to supply this last degree equality. All denominator statements above are in $K[k]$; elements of $K^\times$ are units and are not controlled by this proposition.

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