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GERSTEN INJECTIVITY AND MOREL'S PURITY CONJECTURE

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Abstract. Morel conjectured that the $\mathbb{A}^1$-homotopy sheaves of an S^1-spectrum built from suspension spectra of smooth projective schemes are pure; one condition in purity is injectivity of restriction to dense open subschemes. Over a regular base S, we show that this injectivity condition, for the homotopy sheaves of algebraic K-theory with coefficients in an arbitrary spectrum, is equivalent to Gersten injectivity at the henselizations of all local rings of smooth S-schemes; and that these K-theory spectra are pure in Morel's sense. Hence every henselian regular local ring at which Gersten injectivity fails, integrally, rationally, or with finite coefficients, gives a counterexample to Morel's conjecture. The author's finite-coefficient counterexample over $\mathbb{Z}_3[[x,y]]/(3+x^2-y^3)$ therefore disproves the conjecture, already over the regular scheme $\operatorname{Spec}\mathbb{Z}[1/2][x,y]/(3+x^2-y^3)$. Recently announced integral and rational Gersten counterexamples, including unpublished work of the author, would show, if confirmed and if they persist after henselization, that the conjecture fails already for the K-theory spectrum itself.

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1. Introduction

Morel's conjectures over a base. Let S be a separated noetherian scheme of finite Krull dimension, and let Sm_S be the category of smooth separated S-schemes of finite type. In [Mor05], Morel developed the stable $\mathbb{A}^1$-homotopy theory of S^1-spectra on Sm_S. Over a field he proved the stable $\mathbb{A}^1$-connectivity theorem [Mor05, Theorem 6.1.8, p. 45]. Over a general base he formulated two conjectures.

The first, the stable $\mathbb{A}^1$-connectivity conjecture [Mor05, Conjecture 2, p. 1], predicts that $\mathbb{A}^1$-localization preserves connective S^1-spectra when S is regular. Ayoub disproved it over bases of dimension at least two [Ayo06]; see also [Mor05, footnote 1, p. 1]. A shifted form, in which connectivity may drop by $\dim S$, was formulated by Schmidt and Strunk [SS18] and proved in increasing generality by Schmidt–Strunk, Deshmukh–Hogadi–Kulkarni–Yadav [DHKY21] and Druzhinin [Dru22]; an unstable refinement is due to Bouis and Kundu [BK26].

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*See the AI disclosure in Section 9.

The second conjecture concerns purity [Mor05, Conjecture 6.4.12, p. 59]. A Nisnevich sheaf of abelian groups on Sm_S is *pure* if Zariski and Nisnevich cohomology agree on smooth S-schemes and if cohomology does not change, in a range of degrees, when one removes a closed subset of high codimension (Definition 2.4). An S^1-spectrum is *pure* if it lies in the smallest triangulated subcategory of the stable $\mathbb{A}^1$-homotopy category containing the suspension spectra of smooth *projective* S-schemes and closed under arbitrary coproducts and retracts (Definition 2.5). Morel conjectured that the $\mathbb{A}^1$-homotopy sheaves of pure S^1-spectra are pure (Conjecture 2.6).

Morel showed that pure sheaves satisfy the Gersten conjecture at local rings of smooth S-schemes [Mor05, Corollary 6.4.15 and Remark 6.4.16, pp. 61–62]. Using the Grassmannian model of algebraic K-theory [MV99], he observed that his two conjectures would imply Gersten's conjecture for algebraic K-theory of regular local rings [Mor05, pp. 7 and 62], and he expected the Nisnevich sheaves associated to Quillen's K-groups to be pure over a regular base [Mor05, Remark 6.4.13, p. 60].

Results. For a scheme X write $K(X)$ for the nonconnective algebraic K-theory spectrum of perfect complexes [TT90, BGT13]. For a spectrum M put $K(X;M) = K(X) \wedge M$ and $K_n(X;M) = \pi_n K(X;M)$. The cases of interest are $M = \mathbb{S}$ (integral K-theory), $M = \mathbb{S}/N$ (coefficients $\mathbb{Z}/N$), and $M = H\mathbb{Q}$, for which $K_n(X;H\mathbb{Q}) = K_n(X) \otimes \mathbb{Q}$. When S is regular, $X \mapsto K(X;M)$ defines an S^1-spectrum $K_S \wedge M$ over S (Section 4). For $X \in \mathrm{Sm}_S$ and $x \in X$, write $\mathcal{O}^h_{X,x}$ for the henselization of the local ring $\mathcal{O}_{X,x}$.

Theorem A (Theorem 4.10)**.** *Let S be regular. For every spectrum M, the S^1-spectrum $K_S \wedge M$ is pure.*

For the integral spectrum K_S this is the argument indicated by Morel [Mor05, Remark 6.4.13], based on the Grassmannian model of algebraic K-theory [MV99, §4, Theorem 3.13], [ST15]. We give the details. The infinite loop spaces $\Omega^\infty\Sigma^r K$ are reconstructed from finite products of Grassmannians by Čech nerves and geometric realizations; no explicit model of the E_∞-structure is needed. Arbitrary coefficients then follow formally, because the pure spectra are closed under colimits.

The second result identifies one of the conditions in purity with Gersten injectivity. It holds for any presheaf of abelian groups, and is proved by comparing stalks at points of the Nisnevich site (Section 3).

Theorem B (Proposition 3.7 and Theorem 5.2)**.** *Let S be regular, let M be a spectrum, and let $n \in \mathbb{Z}$. The following are equivalent:*

(i) *for every $X \in \mathrm{Sm}_S$ and every dense open $U \subseteq X$, the restriction map $\underline{\pi}_n^{\mathbb{A}^1}(K_S \wedge M)(X) \to \underline{\pi}_n^{\mathbb{A}^1}(K_S \wedge M)(U)$ is injective;*

(ii) *for every $X \in \mathrm{Sm}_S$ and every $x \in X$, the map $K_n(\mathcal{O}^h_{X,x};M) \to K_n(\operatorname{Frac}\mathcal{O}^h_{X,x};M)$ is injective.*

Condition (i) *is the case $d = 1$, $q = 0$ of condition (ii) in Morel's definition of pure sheaves.*

Theorem C (Theorem 5.2)**.** *Let S be regular. If Morel's Conjecture 2.6 holds over S, then for every spectrum M, every $n \in \mathbb{Z}$, every $X \in \mathrm{Sm}_S$ and every $x \in X$, the map*

$$K_n(\mathcal{O}^h_{X,x};M) \longrightarrow K_n(\operatorname{Frac}\mathcal{O}^h_{X,x};M)$$

is injective. In particular, if R is a regular local ring whose henselization fails Gersten injectivity integrally, rationally, or with coefficients $\mathbb{Z}/N$ in some degree, then Conjecture 2.6 fails over $\operatorname{Spec} R$.

The second sentence follows from the first with $X = S = \operatorname{Spec} R$ and x the closed point. If R is henselian, the failure can be made explicit: restriction from S to any dense open on which a Gersten kernel class vanishes is not injective (Corollary 5.3).

Corollary D (Theorem 6.4)**.** *Let V be a complete discrete valuation ring of mixed characteristic $(0,3)$ with uniformizer 3, for instance $V=\mathbb{Z}_3$, and put $A=V[[x,y]]/(3+x^2-y^3)$, $S=\operatorname{Spec}A$, $U=\operatorname{Spec}A[1/y]$. Then $K_S/3$ is pure, and $\underline{\pi}_2^{\mathbb{A}^1}(K_S/3)(S)\to\underline{\pi}_2^{\mathbb{A}^1}(K_S/3)(U)$ is not injective. Hence Conjecture 2.6 fails over the regular local scheme S.*

The arithmetic input is the author's construction of a nonzero class in $K_2(A;\mathbb{Z}/3)$ supported on the divisor $V(y)$ [Fel26a, Theorem 1.1 and Proposition 4.3]. Since this class is defined before completion, the conjecture also fails over the regular affine scheme $\operatorname{Spec}\mathbb{Z}[1/2][x,y]/(3+x^2-y^3)$, of finite type over $\mathbb{Z}$ (Corollary 6.5).

Integral and rational coefficients. Preprints of OpenAI announce counterexamples to the integral Gersten conjecture for two-dimensional ramified regular local rings of mixed characteristic $(0,5)$, in degrees five and three [OAI26a, OAI26b]. A preprint prepared with ChatGPT/Codex and posted by the GitHub user HayamaKazuma announces infinite-order rational Gersten kernel classes, including for complete regular local rings [HK26]. Independently, in work prepared with GPT-6 Astra and not yet published, the author has obtained a rational counterexample of Adams weight three over the three-dimensional regular local ring

$$N_0=\left(\mathbb{Z}_{(3)}[x,u,v]/(uv-x^2-3)\right)_{(x,u,v)},$$

and has shown that it persists after henselization and completion [Fel26c, Fel26d]. The substitution $x\mapsto u$, $y\mapsto -v$, $z\mapsto x$ identifies N_0 with the ring $\left(\mathbb{Z}_{(3)}[x,y,z]/(3+xy+z^2)\right)_{(x,y,z)}$ of [HK26, Theorem 2.1]. None of these texts has been refereed, and the present paper does not use them as proved results; Theorem C shows what they would imply for Morel's conjecture.

The rings A in [OAI26a, OAI26b] are localizations of finitely generated algebras, not henselian. By Theorem B, the codimension-one purity condition for $K_{\operatorname{Spec}A}$ fails if and only if Gersten injectivity fails at the henselization of some local ring of a smooth $\operatorname{Spec}A$-scheme; whether the announced classes survive henselization is not addressed in those texts (Remark 7.4). By contrast, [HK26, Theorem 2.1] and [Fel26c, Fel26d] both assert that the generic kernel of $K_3(-)\otimes\mathbb{Q}$ is nonzero for the henselization and the completion of N_0. If this holds, Conjecture 2.6 fails for the integral spectrum K_S and for $K_S\wedge H\mathbb{Q}$, over $\operatorname{Spec}N_0$, over its henselization and over its completion (Corollary 7.2). This would also contradict Morel's expectation that the sheaves associated to Quillen's K-groups are pure.

The proof and its dependencies.

(i) Theorem A uses the Grassmannian model of K-theory [MV99, ST15], Nisnevich descent and homotopy invariance for regular schemes [TT90, Qui73], and formal properties of colimits. It does not use the stable $\mathbb{A}^1$-connectivity conjecture, which fails in general [Ayo06]: all connectivity statements are applied objectwise, before motivic localization.

(ii) Theorem B uses only the description of points of the Nisnevich site by henselizations, and the continuity of K-theory.

(iii) Corollary D uses, in addition, the nonvanishing theorem of [Fel26a].

Relation to previous work. Over a field, every strictly $\mathbb{A}^1$-invariant sheaf is pure [Mor05, Lemma 6.4.11, p. 59]; the proof uses Gabber's presentation lemma [Gab94] in the form of Colliot-Thélène–Hoobler–Kahn [CTHK97]. Over a general base, Morel pointed out that strictly $\mathbb{A}^1$-invariant sheaves need not be pure, for instance the pushforward $i_*\mathbb{Z}$ along a closed immersion of positive codimension [Mor05, pp. 6–7 and 59], and stressed that the generators of pure spectra must be smooth and *projective*. Deshmukh, Kulkarni and Yadav used Ayoub's construction to produce, over the regular base $\mathbf{P}^3_k$, an $\mathbb{A}^1$-local S^1-spectrum whose Nisnevich Gersten complex is not exact [DKY22, Example 3.8 and Remark 3.9]; that spectrum is supported on a singular surface and is not known to be pure. We are not aware of a previously published counterexample to Conjecture 2.6; see Remark 8.1.

Gersten injectivity with finite coefficients is known for discrete valuation rings [Gil86, GL00], for regular local rings containing a field [Qui73, Pan03], for unramified regular local rings [Ska17, Corollary 5.7], and, for coefficients invertible in the ring, for noetherian henselian regular local rings [Fel26b, Corollary 6.13]. By Theorem B, the last result shows that the injectivity condition holds for $K_S \wedge \mathbb{S}/N$ when N is invertible on S (Remark 5.5).

Organization. Section 2 fixes conventions and recalls Morel's definitions. Section 3 studies stalks of Nisnevich sheaves and proves the characterization behind Theorem B. Section 4 proves Theorem A. Section 5 proves Theorems B and C. Section 6 proves Corollary D, and Section 7 discusses integral and rational coefficients. Section 8 contains complements and questions.

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2. Conventions and Morel's definitions

2.1. **Sites and ∞-categories.** Throughout, S is a separated noetherian scheme of finite Krull dimension, and Sm_S is the category of smooth separated S-schemes of finite type, as in [Mor05, p. 8]. The identity $S \to S$ is an object of Sm_S; no finiteness assumption is made on S over $\operatorname{Spec}\mathbb{Z}$. A family of étale morphisms $\{V_i \to X\}$ in Sm_S is a *Nisnevich covering* if every point $x \in X$ has a preimage in some V_i with the same residue field. For a presheaf of sets or abelian groups F on Sm_S, we write $F^{\#}$ for its associated Nisnevich sheaf and $\theta\colon F \to F^{\#}$ for the canonical map; we use the construction $F^{\#} = F^{++}$ of [Sta26, Section Tag 00W1].

We write $\mathcal{S}$ for the ∞-category of spaces and Sp for the stable ∞-category of spectra [HTT, HA], and $\mathrm{PSh}(\mathrm{Sm}_S, \mathcal{S})$, $\mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$ for the ∞-categories of presheaves of spaces and of spectra on Sm_S. Limits and colimits in these ∞-categories are computed objectwise.

2.2. **The stable $\mathbb{A}^1$-homotopy category of S^1-spectra.**

Definition 2.1. A presheaf $E \in \mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$ is *motivic* if:

(i) (Brown–Gersten property) $E(\varnothing) \simeq 0$, and for every distinguished Nisnevich square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\;j\;} & X \end{array}$$

in Sm_S, that is, a cartesian square with j an open immersion, p étale, and p inducing an isomorphism $p^{-1}(X \setminus U)_{\mathrm{red}} \to (X \setminus U)_{\mathrm{red}}$, the square of spectra obtained by applying E is cartesian [Mor05, Definition 3.1.6, p. 20], [MV99, §3, Definition 1.3];

(ii) ($\mathbb{A}^1$-invariance) for every $X \in \mathrm{Sm}_S$, the projection induces an equivalence $E(X) \to E(X \times \mathbb{A}^1)$.

Let $\mathrm{SH}^{S^1}(S) \subseteq \mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$ be the full subcategory of motivic presheaves. Motivic presheaves of spaces are defined in the same way, with 0 replaced by the point.

Lemma 2.2. *The full subcategory* $\mathrm{SH}^{S^1}(S)$ *of* $\mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$ *is closed under small limits and small colimits. It is a stable accessible localization; we write* L_{mot} *for its left adjoint.*

Proof. Both conditions in Definition 2.1 ask that certain squares of spectra be cartesian and certain maps be equivalences. Limits of cartesian squares are cartesian. In the stable ∞-category Sp a square is cartesian if and only if it is cocartesian [HA, Proposition 1.1.3.4], and colimits of cocartesian squares are cocartesian. Limits and colimits of equivalences are equivalences. Since limits and colimits in $\mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$ are computed objectwise, both conditions are preserved. The subcategory is defined by locality with respect to a set of maps, namely the maps representing the two conditions, so it is an accessible localization [HTT, §5.5.4]. □

The homotopy category of $\mathrm{SH}^{S^1}(S)$ is Morel's stable $\mathbb{A}^1$-homotopy category $\mathrm{SH}^{S^1}_{\mathbb{A}^1}(\mathrm{Sm}_S)$ [Mor05, §4]: by [Mor05, Remark 3.1.8, Lemma 4.1.4 and Remark 4.2.6], for every presheaf of spectra E with the two properties of Definition 2.1 and every $X \in \mathrm{Sm}_S$, the map $E(X) \to L_{\mathbb{A}^1}(a_{\mathrm{Nis}}E)(X)$ to the $\mathbb{A}^1$-localization of the associated sheaf of spectra is a stable weak equivalence. Under this identification, cofiber sequences in $\mathrm{SH}^{S^1}(S)$ are Morel's exact triangles and coproducts are his wedges. For a presheaf of spaces Y we write $\Sigma^\infty_+ Y = L_{\mathrm{mot}}\Sigma^\infty_+ Y \in \mathrm{SH}^{S^1}(S)$, and for $X \in \mathrm{Sm}_S$ we write $\Sigma^\infty_+ X$ for the case of the represented presheaf.

Definition 2.3 ([Mor05, Definition 4.2.2, p. 30])**.** Let $E \in \mathrm{SH}^{S^1}(S)$ and $n \in \mathbb{Z}$. The n-th $\mathbb{A}^1$-homotopy sheaf $\underline{\pi}^{\mathbb{A}^1}_n(E)$ is the Nisnevich sheaf associated to the presheaf of abelian groups $X \mapsto \pi_n E(X)$. For arbitrary $E \in \mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$ put $\underline{\pi}^{\mathbb{A}^1}_n(E) = \underline{\pi}^{\mathbb{A}^1}_n(L_{\mathrm{mot}}E)$.

In particular, if E is motivic, then $\underline{\pi}^{\mathbb{A}^1}_n(E)$ is the sheaf associated to $X \mapsto \pi_n E(X)$; in Morel's model this is [Mor05, Remark 4.2.6, p. 31].

2.3. **Pure sheaves and pure spectra.**

Definition 2.4 ([Mor05, Definition 6.4.9, p. 59])**.** A Nisnevich sheaf of abelian groups $\mathcal{F}$ on Sm_S is *pure* if:

(i) for every $X \in \mathrm{Sm}_S$ and every $q \geq 0$, the map $H^q_{\mathrm{Zar}}(X; \mathcal{F}) \to H^q_{\mathrm{Nis}}(X; \mathcal{F})$ is an isomorphism;

(ii) for every $X \in \mathrm{Sm}_S$, every integer $d \geq 0$ and every open subscheme $U \subseteq X$ whose complement is everywhere of codimension at least d, the map $H^q_{\mathrm{Nis}}(X; \mathcal{F}) \to H^q_{\mathrm{Nis}}(U; \mathcal{F})$ is an isomorphism for $q \leq d-2$ and injective for $q = d-1$.

We only use the case $d = 1$, $q = 0$ of condition (ii). An open subscheme $U \subseteq X$ has complement of codimension at least one everywhere if and only if U contains every generic point of X, that is, if and only if U is dense. So this case reads:

(2.1)
for every $X \in \mathrm{Sm}_S$ and every dense open $U \subseteq X$, the map $\mathcal{F}(X) \to \mathcal{F}(U)$ is injective.

Definition 2.5 ([Mor05, p. 59 and footnote 14])**.** Let $\mathcal{P}(S) \subseteq \mathrm{SH}^{S^1}(S)$ be the smallest full subcategory which contains $\Sigma^\infty_+ X$ for every smooth projective S-scheme X and is closed under equivalences, cofibers, the shifts $\Sigma^{\pm 1}$, retracts and arbitrary coproducts. Its objects are called *pure*.

Morel calls the corresponding subcategory of the homotopy category the thick triangulated subcategory generated by the suspension spectra of smooth projective S-schemes; his footnote 14 [Mor05, p. 59] specifies closure under arbitrary wedges, retracts, cones and suspensions. Since shifts and cofibers in $\mathrm{SH}^{S^1}(S)$ induce those of its homotopy category, Definition 2.5 defines the same class of objects.

Conjecture 2.6 (Morel, [Mor05, Conjecture 6.4.12, p. 59])**.** For every pure S^1-spectrum E over S and every $n \in \mathbb{Z}$, the sheaf $\underline{\pi}^{\mathbb{A}^1}_n(E)$ is pure.

The conjecture is stated under the standing hypotheses above, without a regularity assumption; Morel's discussion of its consequences concerns regular bases, and all our bases are regular.

Lemma 2.7. *The full subcategory $\mathcal{P}(S)$ is closed under all small colimits in $\mathrm{SH}^{S^1}(S)$.*

Proof. In a stable ∞-category the pushout $B \sqcup_A C$ is the cofiber of the map $A \to B \oplus C$ whose components are the first given map and the negative of the second [HA, Proposition 1.1.3.4]; it lies in $\mathcal{P}(S)$ when A, B, C do. Hence $\mathcal{P}(S)$ admits small coproducts and pushouts, computed in $\mathrm{SH}^{S^1}(S)$. By [HTT, Proposition 4.4.2.6] it admits all small colimits, and by [HTT, Proposition 4.4.2.7] the inclusion, which preserves coproducts and pushouts, preserves all small colimits. □

3. Stalks at points of the Nisnevich site

In this section S is in addition normal. Every $X \in \mathrm{Sm}_S$ is then normal [Sta26, Tag 033C], and a connected affine $X \in \mathrm{Sm}_S$ is integral, since a noetherian normal ring is a finite product of normal domains [Sta26, Tag 030C].

3.1. Nisnevich neighbourhoods.

Definition 3.1. Let $X \in \mathrm{Sm}_S$ and $x \in X$. A *Nisnevich neighbourhood* of x is a pair (V, v) where V is a connected affine scheme with an étale morphism $V \to X$, and $v \in V$ is a point mapping to x with $\kappa(v) = \kappa(x)$. A morphism $(V', v') \to (V, v)$ is an X-morphism $V' \to V$ sending v' to v. We write $\mathrm{Nbh}(X, x)$ for this category. For a presheaf F of abelian groups on Sm_S put

$$F_x = \operatorname*{colim}_{(V,v)\in \mathrm{Nbh}(X,x)^{\mathrm{op}}} F(V), \qquad F_x^{\mathrm{gen}} = \operatorname*{colim}_{(V,v)\in \mathrm{Nbh}(X,x)^{\mathrm{op}}} \operatorname*{colim}_{W\subseteq V \text{ dense open}} F(W),$$

and let $F_x \to F_x^{\mathrm{gen}}$ be induced by restriction.

Lemma 3.2. *Let $X \in \mathrm{Sm}_S$ and $x \in X$.*

(i) *The category $\mathrm{Nbh}(X, x)$ is nonempty, has at most one morphism between any two objects, and any two objects receive morphisms from a common object. Hence the colimits in Definition 3.1 are filtered.*

(ii) *Every object (V, v) of $\mathrm{Nbh}(X, x)$ has V integral, and every morphism $(V', v') \to (V, v)$ induces an injection $\mathcal{O}(V) \to \mathcal{O}(V')$.*

(iii) *The ring $\operatorname{colim}_{(V,v)} \mathcal{O}(V)$ is the henselization $\mathcal{O}^h_{X,x}$ of $\mathcal{O}_{X,x}$; it is a domain, and $\operatorname{Frac} \mathcal{O}^h_{X,x} = \operatorname{colim}_{(V,v)} \operatorname{Frac} \mathcal{O}(V)$.*

(iv) *If X is affine and connected, x is a closed point and $\mathcal{O}(X)$ is a henselian local ring, then (X, x) is an initial object of $\mathrm{Nbh}(X, x)$. Hence $F_x = F(X)$ for every presheaf F.*

Proof. (i) Since X is locally noetherian, x has a connected affine open neighbourhood, which is an object. Let $a, b\colon V' \to V$ be two morphisms $(V', v') \to (V, v)$. The étale morphism $V \to X$ is unramified, so its diagonal is an open immersion [Sta26, Tag 02GE]; it is also a closed immersion since V is affine and X is separated. The locus where a and b agree is the preimage of the diagonal under $(a, b)\colon V' \to V \times_X V$, hence open and closed in V'. It contains v': the points of $V \times_X V$ lying over v under both projections correspond to the prime ideals of $\kappa(v) \otimes_{\kappa(x)} \kappa(v) = \kappa(x)$, so there is exactly one such point, the diagonal point $\Delta(v)$. Since $a(v') = b(v') = v$, the point $(a, b)(v')$ is of this kind, so it lies on the diagonal. As V' is connected, $a = b$. Given (V_1, v_1) and (V_2, v_2), the point (v_1, v_2) of $V_1 \times_X V_2$ has residue field $\kappa(x)$; a connected affine open neighbourhood of it in the étale X-scheme $V_1 \times_X V_2$ is a common source.

(ii) The scheme V is connected, affine, and étale, hence smooth, over the normal scheme X; so it is normal [Sta26, Tag 033C] and, being connected, integral. An étale morphism $V' \to V$ is open, so it maps the generic point of V' to that of V. A dominant morphism of integral affine schemes induces an injection on rings.

(iii) The henselization $\mathcal{O}^h_{X,x}$ is the colimit of $\mathcal{O}(U)$ over the étale neighbourhoods (U, u) of (X, x) with $\kappa(u) = \kappa(x)$ [Sta26, Tag 04HY]; compare [Mor05, Example 5.1.3, p. 36]. Every such neighbourhood contains a connected affine open neighbourhood of u, which is an object of $\mathrm{Nbh}(X, x)$, so these objects are cofinal and the colimit may be computed over $\mathrm{Nbh}(X, x)$. By (ii) it is a filtered colimit of domains along injective maps, hence a domain. Every fraction f/g with $g \neq 0$ has numerator and denominator in a common $\mathcal{O}(V)$, which gives the formula for the fraction field.

(iv) Let (V, v) be an object. Choose an affine open $\operatorname{Spec} B \subseteq V$ containing v. The étale ring map $\mathcal{O}(X) \to B$ and the prime of v, which lies over the maximal ideal with the same residue field, admit a section $B \to \mathcal{O}(X)$ because $\mathcal{O}(X)$ is henselian [Sta26, Tag 04GG, item (8)]. This gives a morphism $(X, x) \to (V, v)$, unique by (i). □

Lemma 3.3. *Let $V \in \mathrm{Sm}_S$ be integral with generic point η, and let $g\colon W' \to V$ be an étale morphism with W' integral and a point $w \in W'$ such that $g(w) = \eta$ and $\kappa(w) = \kappa(\eta)$. Then there are dense opens $W_1 \subseteq W'$ and $W_2 \subseteq V$ such that g restricts to an isomorphism $W_1 \to W_2$.*

Proof. Since g is flat with discrete fibers, $\dim \mathcal{O}_{W',w} = \dim \mathcal{O}_{V,\eta} = 0$ [Sta26, Tag 00ON], so w is a generic point of the integral scheme W', namely its unique one. The local rings at the generic points are $\kappa(w)$ and $\kappa(\eta)$, and g induces an isomorphism between them. Thus g is a birational morphism of integral schemes, locally of finite type, with reduced target. The conclusion is [Sta26, Tag 0BAC]. □

Remark 3.4. Lemma 3.3 shows that the dense open subschemes of an integral $V \in \mathrm{Sm}_S$ are cofinal among the Nisnevich neighbourhoods of its generic point η. Thus $\operatorname{colim}_W F(W)$, over dense opens $W \subseteq V$, is the stalk F_η, and F_x^{gen} is the colimit of the stalks of F at the generic points of the neighbourhoods of x.

3.2. Local vanishing and stalks of sheafifications.

Lemma 3.5. *Let F be a presheaf of abelian groups on Sm_S and $Y \in \mathrm{Sm}_S$.*

(i) *For $s \in F(Y)$, one has $\theta(s) = 0$ in $F^{\#}(Y)$ if and only if there is a Nisnevich covering $\{Y_j \to Y\}$ with $s|_{Y_j} = 0$ for all j.*

(ii) *For $\alpha \in F^{\#}(Y)$ there are a Nisnevich covering $\{Y_j \to Y\}$ and $s_j \in F(Y_j)$ with $\alpha|_{Y_j} = \theta(s_j)$.*

Proof. Write $\theta = \theta_2 \circ \theta_1$ with $\theta_1\colon F \to F^+$ and $\theta_2\colon F^+ \to F^{++}$. The presheaf F^+ is separated, so θ_2 is injective [Sta26, Tag 00WB]. Thus $\theta(s) = 0$ if and only if $\theta_1(s) = 0$. By construction $F^+(Y)$ is the filtered colimit of the zeroth Čech groups $H^0(\mathcal{U}, F)$ over coverings $\mathcal{U}$ of Y, and the image of s in $H^0(\mathcal{U}, F)$ is the family of its restrictions; this proves (i). Part (ii) is [Sta26, Tag 00WK]. □

Lemma 3.6. *Let F be a presheaf of abelian groups on Sm_S, let $X \in \mathrm{Sm}_S$ and $x \in X$. The map $F_x \to (F^{\#})_x$ induced by θ is bijective.*

Proof. Injectivity: if $s \in F(V)$ satisfies $\theta(s)|_V = 0$ for some neighbourhood (V, v), then by Lemma 3.5(i) there is a covering $\{V_j \to V\}$ with $s|_{V_j} = 0$; some V_j has a point v_j over v with $\kappa(v_j) = \kappa(x)$, and a connected affine open neighbourhood V'' of v_j in V_j gives a neighbourhood $(V'', v_j) \to (V, v)$ with $s|_{V''} = 0$. Surjectivity: for $\alpha \in F^{\#}(V)$, by Lemma 3.5(ii) there are a covering $\{V_j \to V\}$ and $s_j \in F(V_j)$ with $\alpha|_{V_j} = \theta(s_j)$; choosing j and V'' as before, the image of α in $(F^{\#})_x$ is the image of $s_j|_{V''}$. □

3.3. The characterization.

Proposition 3.7. *Let S be normal and let F be a presheaf of abelian groups on Sm_S. The following are equivalent:*

(i) *for every $X \in \mathrm{Sm}_S$ and every dense open $U \subseteq X$, the map $F^{\#}(X) \to F^{\#}(U)$ is injective;*

(ii) *for every $X \in \mathrm{Sm}_S$ and every $x \in X$, the map $F_x \to F_x^{\mathrm{gen}}$ is injective.*

Proof. (i)⇒(ii). Let $\xi \in F_x$ map to zero in F_x^{gen}. Represent ξ by $s \in F(V)$ for a neighbourhood (V, v). Since the colimit defining F_x^{gen} is filtered (Lemma 3.2(i)), after replacing (V, v) by a neighbourhood mapping to it we may assume that $s|_W = 0$ for a dense open $W \subseteq V$. Then $\theta(s) \in F^{\#}(V)$ restricts to $\theta(s|_W) = 0$ in $F^{\#}(W)$, so $\theta(s) = 0$ by (i). Hence the image of ξ in $(F^{\#})_x$ vanishes, and $\xi = 0$ by Lemma 3.6.

(ii)⇒(i). Let $\alpha \in F^{\#}(X)$ with $\alpha|_U = 0$. Fix $x \in X$. By Lemma 3.5(ii), and choosing a point over x with the same residue field as in the proof of Lemma 3.6, there are a neighbourhood (V, v) of x and $s \in F(V)$ with $\alpha|_V = \theta(s)$. Let η be the generic point of the integral scheme V (Lemma 3.2(ii)). The étale morphism $V \to X$ is open, so it maps η to a generic point

of X, which lies in U; hence $W = V \times_X U$ is a dense open of V. Since $\theta(s|_W) = \alpha|_W = 0$, Lemma 3.5(i) gives a covering $\{W_j \to W\}$ with $s|_{W_j} = 0$. Some W_j contains a point w over η with $\kappa(w) = \kappa(\eta)$; let W' be a connected affine open neighbourhood of w in W_j, which is integral. By Lemma 3.3, there are dense opens $W_1 \subseteq W'$ and $W_2 \subseteq V$ with an isomorphism $W_1 \to W_2$ over V. Pulling back $s|_{W_2}$ along it gives $s|_{W_1} = 0$, so $s|_{W_2} = 0$. Thus the image of s in F_x^{gen} vanishes, and by (ii) its image in F_x vanishes: there is a neighbourhood $(V_x, v_x) \to (V, v)$ with $s|_{V_x} = 0$, hence $\alpha|_{V_x} = 0$.

The family $\{V_x \to X\}_{x \in X}$ is a Nisnevich covering of X, and α restricts to zero on each member. Since $F^{\#}$ is a sheaf, $\alpha = 0$. □

Corollary 3.8. *Let $S = \operatorname{Spec} R$ with R a normal henselian noetherian local ring of finite Krull dimension, and let F be a presheaf of abelian groups on Sm_S. Then $\theta\colon F(S) \to F^{\#}(S)$ is bijective.*

Proof. Let s be the closed point. By Lemma 3.2(iv), $F_s = F(S)$ and $(F^{\#})_s = F^{\#}(S)$. Apply Lemma 3.6. □

Remark 3.9. Corollary 3.8 expresses that $\operatorname{Spec} R$ is a point of the Nisnevich topos when R is henselian; compare [Mor05, Example 5.2.8, p. 41]. Proposition 3.7 concerns only the case $d = 1$, $q = 0$ of condition (ii) in Definition 2.4, and uses henselian local rings. It may be compared with Morel's equivalence between purity and the Gersten conjecture [Mor05, Corollary 6.4.15 and Remark 6.4.16], which uses condition (i) of Definition 2.4, that is [Mor05, Definition 6.4.9(1)], to pass to Zariski local rings.

4. Algebraic K-theory spectra are pure

In this section S is in addition regular.

4.1. The K-theory presheaves.

Lemma 4.1. *Let S be regular and let M be a spectrum. The presheaf $X \mapsto K(X; M)$ on Sm_S is motivic, and $K(X)$ is connective for every $X \in \mathrm{Sm}_S$. The functor $M \mapsto (X \mapsto K(X; M))$ from Sp to $\mathrm{SH}^{S^1}(S)$ preserves small colimits and shifts.*

Proof. Every $X \in \mathrm{Sm}_S$ is smooth over the regular scheme S, hence regular [Sta26, Tag 07NF]; it is also noetherian, separated and of finite Krull dimension. For such X the negative K-groups vanish [TT90, Proposition 6.8], so $K(X)$ is connective. Moreover $K(X)$ agrees with the K-theory $G(X)$ of coherent sheaves [TT90, Theorem 3.21], [Wei13, Chapter V, Theorem 3.4], and $G(X) \to G(X \times \mathbb{A}^1)$ is an equivalence by Quillen's homotopy theorem for flat morphisms whose fibres are affine spaces [Qui73, §7, Proposition 4.1]. This proves $\mathbb{A}^1$-invariance. The Brown–Gersten property is Nisnevich descent for K-theory of perfect complexes [TT90, Theorem 10.8]; also $K(\varnothing) = 0$.

Smashing with M is an exact functor on Sp, so it preserves cartesian squares and equivalences; hence $X \mapsto K(X; M)$ is motivic. Smashing with $K(X)$ preserves colimits and shifts in M. Colimits in $\mathrm{SH}^{S^1}(S)$ are computed objectwise by Lemma 2.2, which proves the last assertion. □

We write $K_S \wedge M \in \mathrm{SH}^{S^1}(S)$ for this presheaf, K_S for $K_S \wedge \mathbb{S}$, and K_S/N for $K_S \wedge \mathbb{S}/N$, the cofiber of multiplication by N on K_S.

4.2. Recollections on spectra and spaces.

For a spectrum E and $r \geq 0$ put $Y_r(E) = \Omega^\infty \Sigma^r E$, pointed by the base point. The equivalence $Y_r(E) \simeq \Omega Y_{r+1}(E)$ is adjoint to a map $\Sigma Y_r(E) \to Y_{r+1}(E)$, which induces maps

$$\Sigma^{-r}\Sigma^\infty Y_r(E) \longrightarrow \Sigma^{-r-1}\Sigma^\infty Y_{r+1}(E) \tag{4.1}$$

compatible with the maps $\varepsilon_r\colon \Sigma^{-r}\Sigma^\infty Y_r(E) \to E$ obtained by desuspending the counits $\Sigma^\infty\Omega^\infty\Sigma^r E \to \Sigma^r E$.

Lemma 4.2. *Let E be a connective spectrum. The maps ε_r induce an equivalence $\operatorname{colim}_{r\geq 0}\Sigma^{-r}\Sigma^\infty Y_r(E)\xrightarrow{\sim} E$, natural in E, where the colimit is taken along (4.1).*

Proof. Fix $r\geq 1$. Since E is connective, $Y_r(E)$ is $(r-1)$-connected. For an $(r-1)$-connected pointed CW complex Y and $k\geq 0$, the suspension $\pi_j(\Sigma^k Y)\to\pi_{j+1}(\Sigma^{k+1}Y)$ is an isomorphism for $j\leq 2(r-1+k)$ by the Freudenthal suspension theorem [Hat02, Corollary 4.24], since $\Sigma^k Y$ is $(r-1+k)$-connected. Passing to the colimit over k, the unit $\eta\colon Y\to\Omega^\infty\Sigma^\infty Y$ is an isomorphism on π_j for $0\leq j\leq 2r-2$.

Take $Y=Y_r(E)$. The triangle identity $\Omega^\infty(\varepsilon)\circ\eta_{\Omega^\infty\Sigma^r E}=\mathrm{id}$ shows that Ω^∞ of the counit $\Sigma^\infty Y_r(E)\to\Sigma^r E$ is an isomorphism on π_j for $0\leq j\leq 2r-2$, hence so is the counit itself. For $j<0$ both sides have vanishing π_j. Thus $\pi_n(\Sigma^{-r}\Sigma^\infty Y_r(E))\to\pi_n E$ is an isomorphism whenever $r\geq n+2$. Homotopy groups of spectra commute with sequential colimits, which proves the claim; naturality follows from naturality of the counits. □

Lemma 4.3. *Let $\mathcal{C}\subseteq\mathrm{Sp}$ be a full subcategory containing the sphere $\mathbb{S}$ and closed under equivalences, small colimits and Σ^{-1}. Then $\mathcal{C}=\mathrm{Sp}$.*

Proof. Every space Y is the colimit of the constant diagram with value the point indexed by Y itself [HTT, §3.3.4]; since Σ^∞_+ preserves colimits, $\Sigma^\infty_+ Y\in\mathcal{C}$. For pointed Y, the reduced suspension spectrum $\Sigma^\infty Y=\operatorname{cofib}(\Sigma^\infty_+ *\to\Sigma^\infty_+ Y)$ lies in $\mathcal{C}$. By Lemma 4.2, every connective spectrum lies in $\mathcal{C}$. An arbitrary spectrum E is the sequential colimit of its connective covers $\tau_{\geq -k}E=\Sigma^{-k}\tau_{\geq 0}(\Sigma^k E)$, $k\geq 0$, as one checks on homotopy groups; hence $E\in\mathcal{C}$. □

Lemma 4.4. *Let $\mathcal{X}$ be an ∞-topos, for instance $\mathrm{PSh}(\mathrm{Sm}_S,\mathcal{S})$, let $Y\in\mathcal{X}$ be pointed, and assume that $*\to Y$ is an effective epimorphism. Let $\check{C}_\bullet$ be its Čech nerve, $\check{C}_p=*\times_Y\cdots\times_Y *$ with $p+1$ factors. Then:*

(i) *$\check{C}_p\simeq(\Omega Y)^p$ for $p\geq 0$, where $\Omega Y=*\times_Y *$;*

(ii) *the map $\operatorname{colim}_{\Delta^{\mathrm{op}}}\check{C}_\bullet\to Y$ is an equivalence;*

(iii) *for every $q\geq 0$, the map $\operatorname{colim}_{[p]\in\Delta^{\mathrm{op}}}\check{C}_p^q\to Y^q$ is an equivalence.*

In $\mathrm{PSh}(\mathrm{Sm}_S,\mathcal{S})$, the map $*\to Y$ is an effective epimorphism if $Y(X)$ is connected for every $X\in\mathrm{Sm}_S$.

Proof. (i) Induction on p, the case $p=0$ being $\check{C}_0=*$. For $p\geq 1$, $\check{C}_p\simeq\check{C}_{p-1}\times_Y *$, and $\check{C}_{p-1}\to Y$ factors through $*\to Y$; pasting cartesian squares gives $\check{C}_p\simeq\check{C}_{p-1}\times(*\times_Y *)$.

(ii) This is the definition of an effective epimorphism [HTT, §6.2.3].

(iii) The diagonal $\Delta^{\mathrm{op}}\to(\Delta^{\mathrm{op}})^q$ is cofinal because Δ^{op} is sifted [HTT, Definition 5.5.8.1 and Lemma 5.5.8.4]. In an ∞-topos, finite products commute with colimits in each variable, since colimits are universal [HTT, Theorem 6.1.0.6]. Hence $Y^q\simeq\operatorname{colim}_{(\Delta^{\mathrm{op}})^q}\check{C}_{p_1}\times\cdots\times\check{C}_{p_q}\simeq\operatorname{colim}_{[p]\in\Delta^{\mathrm{op}}}\check{C}_p^q$.

For the last assertion: limits and colimits in $\mathrm{PSh}(\mathrm{Sm}_S,\mathcal{S})$ are objectwise, so a map is an effective epimorphism if it is so objectwise, and a map of spaces is an effective epimorphism if and only if it is surjective on π_0 [HTT, §7.2.1]. □

4.3. **The Grassmannian model.** For $n\geq 0$, the Grassmannian $\mathrm{Gr}_n(\mathcal{O}_S^{2n})$ of rank-n subbundles of $\mathcal{O}_S^{2n}$ which are locally direct summands is smooth and projective over S. For $n=0$ it is S. For $n\geq 1$, taking the quotient by the subbundle identifies it with the Grassmannian of rank-n locally free quotients of $\mathcal{O}_S^{2n}$, which is represented by the base change to S of a scheme over $\operatorname{Spec}\mathbb{Z}$ [Sta26, Tag 089T and Tag 089U]. Over $\operatorname{Spec}\mathbb{Z}$ this scheme is smooth and proper, and the Plücker map is a closed embedding into a projective space [Nit05, §1, Construction of Grassmannian]. Following [MV99, §4] and [ST15, §1], let $\mathrm{Gr}=\operatorname{colim}_{n\geq 0}\mathrm{Gr}_n(\mathcal{O}_S^{2n})$ be the infinite Grassmannian, the colimit in presheaves of sets on Sm_S along the standard closed immersions. We regard Gr and the constant presheaf $\mathbb{Z}$ as presheaves of spaces.

Theorem 4.5 (Morel–Voevodsky; Schlichting–Tripathi)**.** *Let S be regular. There is a morphism $\mathbb{Z}\times\mathrm{Gr}\to\Omega^\infty K$ in $\mathrm{PSh}(\mathrm{Sm}_S,\mathcal{S})$ whose motivic localization is an equivalence.*

Proof. This is [MV99, §4, Theorem 3.13, p. 140]. Schlichting and Tripathi gave a corrected proof of the step relating $\mathbb{Z} \times BGL$ to K-theory [ST15, introduction, Theorem 8.2 and Remark 8.5]. The presheaf $\Omega^\infty K$ is motivic, since Ω^∞ preserves limits and K is motivic (Lemma 4.1); thus the statement in the unstable $\mathbb{A}^1$-homotopy category of [MV99] is equivalent to the displayed one. In [MV99], $\mathbb{Z}$ denotes the constant Nisnevich sheaf; the constant presheaf maps to it by a Zariski-local equivalence, which motivic localization inverts. □

Lemma 4.6. *Let $f\colon A \to B$ be a morphism in $\mathrm{PSh}(\mathrm{Sm}_S, \mathcal{S})$ whose motivic localization is an equivalence. Then for every $q \geq 0$ the motivic localization of $f^q\colon A^q \to B^q$ is an equivalence.*

Proof. It suffices to treat $f \times \mathrm{id}_C$ for a presheaf C, since f^q is a composite of such maps. Let Z be a motivic presheaf of spaces, and let Z^C be the internal mapping object, so that $\mathrm{Map}(D \times C, Z) \simeq \mathrm{Map}(D, Z^C)$. Writing C as a colimit of representable presheaves h_T gives $Z^C \simeq \lim_T Z^{h_T}$ with $Z^{h_T}(X) = Z(X \times_S T)$. The functor $X \mapsto X \times_S T$ preserves distinguished squares and commutes with $- \times \mathbb{A}^1$, so each Z^{h_T} is motivic, and so is the limit Z^C. Hence $\mathrm{Map}(B \times C, Z) \to \mathrm{Map}(A \times C, Z)$ is identified with $\mathrm{Map}(B, Z^C) \to \mathrm{Map}(A, Z^C)$, an equivalence. □

4.4. **Purity.** The functor $Y \mapsto \Sigma^\infty_+ Y = L_{\mathrm{mot}} \Sigma^\infty_+ Y$ from $\mathrm{PSh}(\mathrm{Sm}_S, \mathcal{S})$ to $\mathrm{SH}^{S^1}(S)$ preserves small colimits, being a composite of left adjoints. It inverts morphisms whose motivic localization is an equivalence, because its right adjoint Ω^∞ takes values in motivic presheaves of spaces. If Y is pointed, its reduced suspension spectrum $\Sigma^\infty Y$ is the cofiber of $\Sigma^\infty_+(*) \to \Sigma^\infty_+ Y$, where $*$ is the terminal presheaf, represented by S.

Definition 4.7. A presheaf $Y \in \mathrm{PSh}(\mathrm{Sm}_S, \mathcal{S})$ is *pure-generated* if $\Sigma^\infty_+(Y^q) \in \mathcal{P}(S)$ for every $q \geq 0$.

For $q = 0$ the condition says that $\Sigma^\infty_+ S$ is pure, which holds because $S \to S$ is smooth and projective.

Lemma 4.8. *Let S be regular. The presheaf $\Omega^\infty K$ is pure-generated.*

Proof. By Theorem 4.5 and Lemma 4.6, $\Sigma^\infty_+((\Omega^\infty K)^q) \simeq \Sigma^\infty_+((\mathbb{Z} \times \mathrm{Gr})^q)$. In presheaves of sets, filtered colimits commute with finite products, so $(\mathbb{Z} \times \mathrm{Gr})^q \simeq \coprod_{\mathbb{Z}^q} \mathrm{colim}_n \mathrm{Gr}_n(\mathcal{O}_S^{2n})^q$, with q-fold fiber products over S. This colimit of presheaves of sets is also the colimit in presheaves of spaces: colimits are objectwise, and a sequential colimit of sets computed in $\mathcal{S}$ is discrete, since filtered colimits of 0-truncated spaces are 0-truncated [Ker, Tag 0631]. Each $\mathrm{Gr}_n(\mathcal{O}_S^{2n})^q$ is smooth and projective over S, as a fiber product of smooth projective S-schemes. So $\Sigma^\infty_+((\mathbb{Z} \times \mathrm{Gr})^q)$ is a coproduct of sequential colimits of pure objects, hence pure by Lemma 2.7. □

Lemma 4.9. *Let Y be a pointed presheaf of spaces with $Y(X)$ connected for every $X \in \mathrm{Sm}_S$. If ΩY is pure-generated, then so is Y.*

Proof. By Lemma 4.4, $Y^q \simeq \mathrm{colim}_{[p] \in \Delta^{\mathrm{op}}} (\Omega Y)^{pq}$ for every $q \geq 0$. Since Σ^∞_+ preserves colimits, $\Sigma^\infty_+(Y^q)$ is a geometric realization of the pure objects $\Sigma^\infty_+((\Omega Y)^{pq})$; apply Lemma 2.7. □

Theorem 4.10. *Let S be a separated regular noetherian scheme of finite Krull dimension. For every spectrum M, the S^1-spectrum $K_S \wedge M$ is pure. In particular K_S, K_S/N for $N \geq 1$, and $K_S \wedge H\mathbb{Q}$ are pure.*

Proof. The case $M = \mathbb{S}$. For $r \geq 0$ let Y_r be the pointed presheaf $X \mapsto \Omega^\infty \Sigma^r K(X)$. Then $Y_0 = \Omega^\infty K$, $\Omega Y_{r+1} \simeq Y_r$, and $Y_{r+1}(X)$ is connected because $\pi_0 Y_{r+1}(X) = K_{-r-1}(X) = 0$ (Lemma 4.1). By Lemma 4.8 and induction using Lemma 4.9, every Y_r is pure-generated; in particular $\Sigma^\infty_+ Y_r$ and its cofiber $\Sigma^\infty Y_r$ are pure. Since each $K(X)$ is connective, Lemma 4.2 applied objectwise gives an equivalence $\mathrm{colim}_r \Sigma^{-r} \Sigma^\infty Y_r \xrightarrow{\sim} K$ in $\mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$, with objectwise suspension spectra. Applying L_{mot}, which preserves colimits and shifts and fixes the motivic presheaf K, gives $K_S \simeq \mathrm{colim}_r \Sigma^{-r} \Sigma^\infty Y_r$ in $\mathrm{SH}^{S^1}(S)$. By Lemma 2.7, K_S is pure.

General M. Let $\mathcal{C} \subseteq \mathrm{Sp}$ be the full subcategory of spectra M with $K_S \wedge M \in \mathcal{P}(S)$. It contains $\mathbb{S}$ by the first case, and is closed under small colimits and Σ^{-1} by Lemmas 4.1 and 2.7. By Lemma 4.3, $\mathcal{C} = \mathrm{Sp}$. □

Remark 4.11. The proof uses the E_∞-structure of K-theory only through the deloopings $\Omega^\infty\Sigma^r K$, which are part of the spectrum; Čech nerves replace bar constructions. The same argument applies to any motivic presheaf of spectra E such that $E(X)$ is connective for every $X \in \mathrm{Sm}_S$ and $\Omega^\infty E$ is motivically equivalent to a filtered colimit of smooth projective S-schemes, compatibly with finite products. Connectivity of the homotopy sheaves of E would not suffice: the Čech nerve argument needs each $\Omega^\infty\Sigma^{r+1}E(X)$ to be connected. Morel suggests that orthogonal Grassmannians play this role for Hermitian K-theory [Mor05, Remark 6.4.13]; the finite stages of the orthogonal Grassmannians of Schlichting and Tripathi are smooth and affine [ST15, Lemma 2.2 and Theorem 1.1], so additional input would be needed.

5. Gersten injectivity and the purity conjecture

Lemma 5.1. *Let S be regular, M a spectrum, $n \in \mathbb{Z}$, and let F be the presheaf $X \mapsto K_n(X; M)$ on Sm_S. For $X \in \mathrm{Sm}_S$ and $x \in X$, the map $F_x \to F_x^{\mathrm{gen}}$ of Definition 3.1 is identified with*

$$K_n(\mathcal{O}^h_{X,x}; M) \longrightarrow K_n(\operatorname{Frac}\mathcal{O}^h_{X,x}; M).$$

Proof. Nonconnective K-theory commutes with filtered colimits of rings [TT90, Proposition 3.20], [BGT13]; so does smashing with M and taking homotopy groups. For each neighbourhood (V, v) the dense opens $D(f)$, $0 \neq f \in \mathcal{O}(V)$, are cofinal among dense opens of V, and $\operatorname{colim}_f \mathcal{O}(V)[1/f] = \operatorname{Frac}\mathcal{O}(V)$. Hence $\operatorname{colim}_W F(W) = K_n(\operatorname{Frac}\mathcal{O}(V); M)$, and Lemma 3.2(iii) gives $F_x = K_n(\mathcal{O}^h_{X,x}; M)$ and $F_x^{\mathrm{gen}} = K_n(\operatorname{Frac}\mathcal{O}^h_{X,x}; M)$, compatibly with restriction. □

Theorem 5.2. *Let S be a separated regular noetherian scheme of finite Krull dimension, M a spectrum and $n \in \mathbb{Z}$. Put $\mathcal{M} = \underline{\pi}_n^{\mathbb{A}^1}(K_S \wedge M)$.*

(i) *The sheaf $\mathcal{M}$ satisfies (2.1) if and only if $K_n(\mathcal{O}^h_{X,x}; M) \to K_n(\operatorname{Frac}\mathcal{O}^h_{X,x}; M)$ is injective for every $X \in \mathrm{Sm}_S$ and $x \in X$.*

(ii) *If Conjecture 2.6 holds over S, then the maps in (i) are injective for every M, n, X and x.*

(iii) *If R is a regular local ring whose henselization R^h satisfies $\ker\big(K_n(R^h; M) \to K_n(\operatorname{Frac} R^h; M)\big) \neq 0$ for some M and n, then Conjecture 2.6 fails over $\operatorname{Spec} R$, for the pure spectrum $K_{\operatorname{Spec} R} \wedge M$.*

Proof. (i) By Lemma 4.1 and Definition 2.3, $\mathcal{M}$ is the sheaf associated to $X \mapsto K_n(X; M)$. A regular scheme is normal, so Proposition 3.7 applies; combine it with Lemma 5.1.

(ii) By Theorem 4.10, $K_S \wedge M$ is pure, so Conjecture 2.6 implies that $\mathcal{M}$ is pure, in particular that it satisfies (2.1). Apply (i).

(iii) Apply (ii) with $S = X = \operatorname{Spec} R$ and x the closed point, for which $\mathcal{O}^h_{X,x} = R^h$. The ring R is regular, noetherian, of finite Krull dimension, and $\operatorname{Spec} R$ is separated. □

Corollary 5.3. *Let R be a regular henselian noetherian local ring, $S = \operatorname{Spec} R$, M a spectrum and $n \in \mathbb{Z}$. Let $a \in K_n(R; M)$ be nonzero, and suppose that a restricts to zero in $K_n(U; M)$ for a dense open $U \subseteq S$. Then $\underline{\pi}_n^{\mathbb{A}^1}(K_S \wedge M)(S) \to \underline{\pi}_n^{\mathbb{A}^1}(K_S \wedge M)(U)$ is not injective; hence the pure sheaf condition (2.1) and Conjecture 2.6 fail over S.*

Proof. Let $\mathcal{M} = \underline{\pi}_n^{\mathbb{A}^1}(K_S \wedge M)$. The square

$$\begin{array}{ccc} K_n(R; M) & \longrightarrow & K_n(U; M) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ \mathcal{M}(S) & \longrightarrow & \mathcal{M}(U) \end{array}$$

commutes, and the left vertical map is bijective by Corollary 3.8. So $\theta(a) \neq 0$ while $\theta(a)|_U = 0$. The spectrum $K_S \wedge M$ is pure by Theorem 4.10. □

Remark 5.4. Under the hypotheses of Corollary 5.3, a class a in the kernel of $K_n(R; M) \to K_n(\operatorname{Frac} R; M)$ vanishes on some principal open $D(f)$, $f \neq 0$, by continuity of K-theory, as in the proof of Lemma 5.1. So the explicit open U always exists.

Remark 5.5 (Invertible coefficients). If N is invertible on S, the rings $\mathcal{O}^h_{X,x}$ are noetherian henselian regular local rings in which N is invertible (henselization preserves regularity [Sta26, Tag 06LN]), and $K_n(\mathcal{O}^h_{X,x}; \mathbb{Z}/N) \to K_n(\operatorname{Frac} \mathcal{O}^h_{X,x}; \mathbb{Z}/N)$ is injective by [Fel26b, Corollary 6.13], using primary decomposition of $\mathbb{Z}/N$. By Theorem 5.2(i), the sheaves $\underline{\pi}_n^{\mathbb{A}^1}(K_S/N)$ then satisfy (2.1). So, with coefficients $\mathbb{Z}/N$, the method produces counterexamples only when N is not invertible on S.

6. The finite-coefficient counterexample

Let V be a complete discrete valuation ring of mixed characteristic $(0, 3)$ in which 3 is a uniformizer, for instance $V = \mathbb{Z}_3$. Put

$$A = V[[x, y]]/(3 + x^2 - y^3), \qquad S = \operatorname{Spec} A, \qquad U = \operatorname{Spec} A[1/y].$$

Fix a primitive cube root of unity ζ in an algebraic closure of $\operatorname{Frac} V$, put $O = V[\zeta]$, and let σ be the nontrivial automorphism of O over V.

Lemma 6.1. (i) *The ring A is a complete, hence henselian, regular local domain of dimension two with maximal ideal (x, y).*

(ii) *The element y is prime, and $x \mapsto \zeta^2 - \zeta = -1 - 2\zeta$ induces an isomorphism of V-algebras $A/yA \simeq O$; the ring O is a discrete valuation ring.*

(iii) *The closed subset $V(y)$ is irreducible of codimension one, and U is dense in S.*

Proof. (i) The ring $V[[x, y]]$ is a complete regular local ring of dimension three with maximal ideal $\mathfrak{n} = (3, x, y)$, and $f = 3 + x^2 - y^3 \equiv 3 \not\equiv 0$ modulo $\mathfrak{n}^2$ since 3 is a uniformizer of V. Hence $A = V[[x, y]]/(f)$ is regular local of dimension two [Sta26, Tag 00NQ], a domain [Sta26, Tag 00NP], with maximal ideal (x, y) generated by a regular system of parameters. It is complete as a quotient of a complete noetherian local ring, and complete local rings are henselian [Sta26, Tag 04GM]; compare [Fel26a, Theorem 1.1].

(ii) This is [Fel26a, Lemma 4.1]: one has $(\zeta^2 - \zeta)^2 = -3$, so $A/yA \simeq V[[x]]/(x^2 + 3) \simeq O$, and $x^2 + 3$ is an Eisenstein polynomial. In particular yA is prime.

(iii) The prime yA is nonzero and minimal over (y), so it has height one by Krull's principal ideal theorem. Thus $V(y)$ is irreducible of codimension one in the integral scheme S, and $U = D(y)$ is nonempty, hence dense. □

Theorem 6.2 ([Fel26a, Theorem 1.1, equations (7)–(8) and Proposition 4.3])**.** *Let $i\colon \operatorname{Spec}(A/yA) \to \operatorname{Spec} A$ be the closed immersion. Choose $\beta_0 \in K_2(O; \mathbb{Z}/3)$ whose image under the coefficient boundary $K_2(O; \mathbb{Z}/3) \to K_1(O)$ is the class of ζ; it exists because $\zeta^3 = 1$. Put $\beta = 2(\beta_0 - \sigma^*\beta_0)$, regarded as a class of $K_2(A/yA; \mathbb{Z}/3)$ via Lemma 6.1(ii). Then $a = i_*\beta \in K_2(A; \mathbb{Z}/3)$ is nonzero.*

In [Fel26a] the class is written $\beta = \frac{1}{2}(\beta_0 - \sigma^*\beta_0)$; since $K_2(O; \mathbb{Z}/3)$ is killed by 3, multiplication by $\frac{1}{2}$ is multiplication by 2. The pushforward i_* is restriction of scalars on G-theory, and $G_2(A; \mathbb{Z}/3) = K_2(A; \mathbb{Z}/3)$ by the resolution theorem [Wei13, Chapter V, Theorem 3.3]. The proof that $a \neq 0$ passes to the normal surface obtained by adjoining ζ, where the supporting divisor splits into two conjugate components and the class becomes the product of a mod-3 class lifting ζ with the class of a line bundle of order three; this product is detected in weight two of the motivic filtration [Fel26a, §§2–4].

Lemma 6.3. *The image of a in $K_2(U; \mathbb{Z}/3) = K_2(A[1/y]; \mathbb{Z}/3)$ is zero.*

Proof. Quillen's localization theorem gives a fiber sequence $G(A/yA) \xrightarrow{i_*} G(A) \to G(A[1/y])$ [Qui73, §7, Proposition 3.2]; see also [TT90, Theorem 7.4]. Smashing with $\mathbb{S}/3$ preserves fiber sequences, so the composite $G_2(A/yA;\mathbb{Z}/3) \to G_2(A;\mathbb{Z}/3) \to G_2(A[1/y];\mathbb{Z}/3)$ is zero. The last two groups are $K_2(A;\mathbb{Z}/3)$ and $K_2(A[1/y];\mathbb{Z}/3)$, compatibly with restriction, by the resolution theorem [Wei13, Chapter V, Theorem 3.3], since A and $A[1/y]$ are regular. This argument is contained in the proof of [Fel26a, Proposition 4.3]. □

Theorem 6.4. *The S^1-spectrum $K_S/3$ is pure, and the restriction map $\underline{\pi}_2^{\mathbb{A}^1}(K_S/3)(S) \to \underline{\pi}_2^{\mathbb{A}^1}(K_S/3)(U)$ is not injective. Hence $\underline{\pi}_2^{\mathbb{A}^1}(K_S/3)$ is not pure, and Conjecture 2.6 fails over the two-dimensional regular local scheme S.*

Proof. Apply Corollary 5.3 with $R = A$, $M = \mathbb{S}/3$, $n = 2$, the dense open U of Lemma 6.1(iii), and the class a of Theorem 6.2, which vanishes on U by Lemma 6.3. □

The class of Theorem 6.2 is already defined before completion, which gives a base of finite type over $\mathbb{Z}$.

Corollary 6.5. *Put $f = 3 + x^2 - y^3$,*

$$B = \mathbb{Z}[1/2][x,y]/(f), \qquad R_0 = B_{(3,x,y)} = \big(\mathbb{Z}_{(3)}[x,y]/(f)\big)_{(x,y)}.$$

The ring B is regular, and Gersten injectivity $K_2(R_0^h;\mathbb{Z}/3) \to K_2(\operatorname{Frac} R_0^h;\mathbb{Z}/3)$ fails for the henselization R_0^h of R_0. Consequently Conjecture 2.6 fails, for the pure spectrum $K/3$, over $\operatorname{Spec} R_0$ *and over the regular affine scheme* $\operatorname{Spec} B$ *of finite type over $\mathbb{Z}$.*

Proof. Regularity. The ring $P = \mathbb{Z}[1/2][x,y]$ is regular [Sta26, Tag 07NF]. Let $\mathfrak{p} \subset P$ be a prime containing f, with maximal ideal $\mathfrak{m}$ in $P_\mathfrak{p}$, and suppose $f \in \mathfrak{m}^2$. The derivations $\partial/\partial x$ and $\partial/\partial y$ of $P_\mathfrak{p}$ map $\mathfrak{m}^2$ into $\mathfrak{m}$, so $2x, 3y^2 \in \mathfrak{p}$, hence $x \in \mathfrak{p}$. If $3 \notin \mathfrak{p}$, then $y \in \mathfrak{p}$ and $3 = f - x^2 + y^3 \in \mathfrak{p}$, a contradiction. So $3 \in \mathfrak{p}$, then $y^3 = 3 + x^2 - f \in \mathfrak{p}$, and $\mathfrak{p} = (3,x,y)$. The regular local ring $P_\mathfrak{p}$ has dimension three and maximal ideal generated by $3, x, y$; hence $3 \notin \mathfrak{m}^2$, and $f \equiv 3$ modulo $\mathfrak{m}^2$, a contradiction. Thus $f \notin \mathfrak{m}^2$ for every such $\mathfrak{p}$, and $B_\mathfrak{p} = P_\mathfrak{p}/(f)$ is regular [Sta26, Tag 00NQ]. Since 2 is a unit in $\mathbb{Z}_{(3)}$ and $3 = y^3 - x^2$ in B, the two descriptions of R_0 agree.

The class. The ring $C_0 = R_0/yR_0 = \mathbb{Z}_{(3)}[x]/(x^2+3)$ is a discrete valuation ring with uniformizer x, since x^2+3 is Eisenstein. The element $\zeta_0 = (-1-x)/2$ satisfies $\zeta_0^2 + \zeta_0 + 1 = (x^2+3)/4 = 0$, and $\sigma_0 \colon x \mapsto -x$ is an automorphism of C_0 with $\sigma_0(\zeta_0) = \zeta_0^2$. As in Theorem 6.2, choose $b_0 \in K_2(C_0;\mathbb{Z}/3)$ with coefficient boundary $[\zeta_0]$, and put $a_0 = i_{0*}(2(b_0 - \sigma_0^* b_0)) \in K_2(R_0;\mathbb{Z}/3)$, where $i_0 \colon \operatorname{Spec} C_0 \to \operatorname{Spec} R_0$ and the pushforward is taken on G-theory, as in Theorem 6.2. As in Lemma 6.3, a_0 vanishes in $K_2(R_0[1/y];\mathbb{Z}/3)$.

Comparison with Theorem 6.4. The completion of R_0 is the ring A of Section 6 for $V = \mathbb{Z}_3$, and $R_0 \to A$ is flat [Sta26, Tag 00MC]. For a finitely generated C_0-module M one has $M \otimes_{R_0} A = M \otimes_{C_0} A/yA$, naturally in M, and these functors are exact; so the image of a_0 in $K_2(A;\mathbb{Z}/3)$ is the pushforward of the image of $2(b_0 - \sigma_0^* b_0)$ in $K_2(A/yA;\mathbb{Z}/3)$. Under $A/yA \simeq O$ of Lemma 6.1(ii), x maps to $\zeta^2 - \zeta$, so $\zeta_0 \mapsto \zeta$ and σ_0 induces σ. By naturality of the coefficient boundary, the image of b_0 in $K_2(O;\mathbb{Z}/3)$ is an admissible choice of β_0 in Theorem 6.2, so the image of a_0 in $K_2(A;\mathbb{Z}/3)$ is nonzero.

Conclusion. The map $R_0 \to A$ factors through R_0^h, because A is identified with the completion of R_0^h [Sta26, Tag 06LJ]. Hence the image of a_0 in $K_2(R_0^h;\mathbb{Z}/3)$ is nonzero. Since $y \neq 0$ in the domain R_0^h, this image vanishes in $K_2(\operatorname{Frac} R_0^h;\mathbb{Z}/3)$. As R_0^h is the henselization of the local ring of $\operatorname{Spec} B$ at $(3,x,y)$, and of $\operatorname{Spec} R_0$ at its closed point, Theorem 5.2(ii), applied with $X = S$, gives the assertion. □

7. Integral and rational coefficients

Remark 7.1 (The integral sheaf in Theorem 6.4). By [Fel26a, Theorem 1.1], $K_1(A)[3] = 0$, so the coefficient sequence identifies $K_2(A;\mathbb{Z}/3)$ with $K_2(A)/3$; the class a lifts to $c \in$

$K_2(A) \setminus 3K_2(A)$ whose image in $K_2(\operatorname{Frac} A)$ is divisible by 3 [Fel26a, Corollary 5.1]. On the other hand $K_2(A) \to K_2(\operatorname{Frac} A)$ is injective by van der Kallen [vdK76]. By Corollary 3.8, $\underline{\pi}_2^{\mathbb{A}^1}(K_S)(S) = K_2(A)$. A section vanishing on U has zero stalk at the generic point of S, and by Remark 3.4 and Lemma 3.6 this stalk is the image of the section in $K_2(\operatorname{Frac} A)$; so the integral sheaf $\underline{\pi}_2^{\mathbb{A}^1}(K_S)$ satisfies (2.1) for $U \subseteq S$. The example therefore does not by itself contradict Morel's expectation that the sheaves associated to Quillen's K-groups are pure [Mor05, Remark 6.4.13].

Integral and rational Gersten counterexamples have recently been announced [OAI26a, OAI26b, HK26], and the author has obtained a rational one in unpublished work [Fel26c, Fel26d]. We have not refereed these texts; the following statements record what Theorem 5.2 would give.

Corollary 7.2. *Put*

$$N_0 = \big(\mathbb{Z}_{(3)}[x,u,v]/(uv - x^2 - 3)\big)_{(x,u,v)}, \qquad \widehat{N}_0 = \mathbb{Z}_3[[x,u,v]]/(uv - x^2 - 3),$$

and let N_0^h be the henselization of N_0. Let R be N_0^h or $\widehat{N}_0$, and assume that there is a nonzero $a \in K_3(R) \otimes \mathbb{Q}$ whose image in $K_3(\operatorname{Frac} R) \otimes \mathbb{Q}$ is zero. Then, for $M = \mathbb{S}$ and for $M = H\mathbb{Q}$, Conjecture 2.6 fails for the pure spectrum $K_S \wedge M$ over $S = \operatorname{Spec} R$. If $R = N_0^h$, it also fails over $S = \operatorname{Spec} N_0$.

Proof. The ring N_0 is a regular local ring of dimension three, essentially of finite type over $\mathbb{Z}$: the ambient ring $\mathbb{Z}_{(3)}[x,u,v]_{(3,x,u,v)}$ is regular local of dimension four, the element $uv - x^2 - 3$ has nonzero image -3 in the cotangent space, and the maximal ideal of the quotient is (x,u,v) since $3 = uv - x^2$ [Sta26, Tag 00NQ]. Its henselization is a noetherian [Sta26, Tag 06LJ] henselian regular local ring [Sta26, Tag 06LN]. Its completion is $\widehat{N}_0$, since completion of finite modules over a noetherian local ring is exact; it is a regular local ring [Sta26, Tag 07NY], henselian because it is complete [Sta26, Tag 04GM].

Integral classes. Choose $b \in K_3(R)$ and an integer $d \geq 1$ with $b \otimes 1 = da$. The image of b in $K_3(\operatorname{Frac} R)$ is torsion, say of order $m \geq 1$. Then $c = mb$ satisfies $c|_{\operatorname{Frac} R} = 0$ and $c \otimes 1 = mda \neq 0$, so c has infinite order.

Conclusion. By Remark 5.4, the classes c and a vanish on a dense open $D(f) \subseteq \operatorname{Spec} R$. Since R is a henselian regular noetherian local ring, Corollary 5.3, applied to c with $M = \mathbb{S}$ and to a with $M = H\mathbb{Q}$ (recall that $K_3(R; H\mathbb{Q}) = K_3(R) \otimes \mathbb{Q}$), gives the assertion over $\operatorname{Spec} R$. If $R = N_0^h$, then Gersten injectivity fails at the henselization of N_0, integrally and rationally, and Theorem 5.2(iii) gives the assertion over $\operatorname{Spec} N_0$. □

Remark 7.3 (Sources for the hypothesis). The hypothesis of Corollary 7.2 is asserted in two independent texts.

(i) By [HK26, Theorem 2.1], transported along the isomorphism $x \mapsto u$, $y \mapsto -v$, $z \mapsto x$, an explicit product of three unit symbols in $K_3(N_0)$ has infinite order in $K_3(\widehat{N}_0)$ and vanishes in $K_3(N_0[1/u])$. Its image in $K_3(N_0^h)$ then has infinite order as well, because $N_0 \to \widehat{N}_0$ factors through N_0^h.

(ii) In [Fel26c], a nonzero class of Adams weight three in the generic kernel of $K_3(N_0) \otimes \mathbb{Q}$ is obtained by pushing a cyclotomic weight-two class forward from the divisor $u = 0$; in [Fel26d] it is shown to remain nonzero in $K_3(N_0^h) \otimes \mathbb{Q}$ and in $K_3(\widehat{N}_0) \otimes \mathbb{Q}$.

Remark 7.4 (Classes over non-henselian local rings). The preprints [OAI26a, OAI26b] announce nonzero kernels of $K_n(A) \to K_n(\operatorname{Frac} A)$, for $n = 5$ and $n = 3$, where A is a two-dimensional ramified regular local ring of mixed characteristic $(0,5)$ obtained by localizing a finitely generated algebra. By Theorem 5.2, such a class c contradicts Conjecture 2.6 for $K_{\operatorname{Spec} A}$ as soon as its image in $K_n(A^h)$ is nonzero. Conversely, if Gersten injectivity holds at the henselizations of all local rings of smooth $\operatorname{Spec} A$-schemes, then the injectivity condition (2.1) holds for $\underline{\pi}_n^{\mathbb{A}^1}(K_{\operatorname{Spec} A})$ by Theorem 5.2(i), even though $c \neq 0$. Thus the relevant question is whether these classes survive henselization; the cited texts do not discuss henselization.

Remark 7.5 (Morel's route to the Gersten conjecture). Morel observed that his two conjectures, applied to K_S, would imply Gersten's conjecture for algebraic K-theory of regular local rings [Mor05, p. 62]. Theorem 5.2 isolates the part of this implication that involves only Conjecture 2.6: its codimension-one injectivity condition for $K_S \wedge M$ is equivalent to Gersten injectivity at henselizations, and does not involve the stable $\mathbb{A}^1$-connectivity conjecture.

8. Complements and questions

Remark 8.1 (Search for prior counterexamples). For this version we searched for counterexamples to Morel's Conjecture 6.4.12 in the work citing [Mor05], in particular on stable $\mathbb{A}^1$-connectivity over a base [Ayo06, SS18, DHKY21, Dru22, BK26], on Gersten complexes over a general base [DKY22], and in the recent Gersten counterexamples [Fel26a, OAI26a, OAI26b, HK26, Fel26c, Fel26d], none of which mentions Morel's purity conjecture. We did not find one. This is evidence, not a proof, of novelty.

Question 8.2. Which modification of Conjecture 2.6 could hold over regular bases? By Theorem 5.2 and Remark 5.5, the codimension-one condition for K-theory spectra holds with coefficients invertible on S. Morel also expected purity for étale cohomology with torsion coefficients prime to the residue characteristics [Mor05, pp. 6 and 60]. One may ask whether Conjecture 2.6 holds for pure spectra whose homotopy sheaves are annihilated by some positive integer invertible on S, such as K_S/N with N invertible on S; the results above establish only the codimension-one injectivity condition for these. Restricting instead to spectra on which all residue characteristics act invertibly would still include $K_S \wedge H\mathbb{Q}$, and would therefore be ruled out by the rational counterexamples of Corollary 7.2 if its hypothesis is confirmed.

Question 8.3. Do the integral classes of [OAI26a, OAI26b] remain nonzero in $K_n(A^h)$?

9. AI disclosure

An earlier version of this manuscript was drafted with OpenAI's GPT-6 Astra under the author's direction, including the deduction from the finite-coefficient counterexample. The present version was revised with Anthropic's Claude (Opus 5.5), which checked Morel's definitions against [Mor05], searched the literature for prior counterexamples, proposed the characterization by henselian stalks and the extension to arbitrary coefficients, added references, and proofread the final text. An adversarial referee report produced with GPT-6 Astra suggested the corrections in Remark 4.11 and Question 8.2 and the finite-type base of Corollary 6.5. All suggestions were checked by the author, who assumes full responsibility for the statements, proofs and references.

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