The ten minimum universal sets of MacMahon cubes

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Abstract

A collection of six-coloured MacMahon cubes is universal if eight of its cubes can be selected to build a $2 \times 2 \times 2$ model of any target variety, with matching exterior colours and no internal matching requirement. We prove that exactly ten labelled collections attain the known minimum size of twelve, even when repeated varieties are allowed, resolving the conjectures of Johnson and Roldán. Usable-cube inequalities force a symmetric multiplicity matrix with row and column sums two; corner coverage then forces two disjoint triangles in Conway's tableau. The written proof is elementary and requires no exhaustive search. The classification and its discrete geometric interpretation are formalised in Lean 4. Exhaustive computation provides a separate check and additional target-count distributions.

AI transparency statement. The classification proof, computational investigation, Lean formalisation and manuscript were produced by AI systems under the author's direction. The systems and their contributions are described at the end of the paper.

1 Introduction

MacMahon's cube puzzles use the thirty varieties obtained by colouring the six faces with six distinct colours, identifying proper rotations but not reflections [1, pp. 42–46]. We study the exterior-matching puzzle known as *Eight Blocks to Madness*: select eight cubes and assemble a $2 \times 2 \times 2$ model of a target variety. Only the exterior colours must match; touching interior faces are unrestricted. A collection is *universal* if it can build every target. Collections may contain repeated varieties; a *set* contains distinct varieties. Face-colour labels remain fixed throughout.

Haraguchi proved that twelve is the minimum size of a universal collection, allowing repeats, and exhibited one attaining it [3, Lemma 2 and Theorem 4]. Johnson and Roldán found nine further sets, bringing the known total to ten [2, Theorem 3]. Their Conjecture 4 asserts that these are all the minimum sets; Conjecture 8 describes them as two triangles in Conway's tableau. This tableau labels varieties by ordered pairs ij of distinct labels in $L = \{A, B, C, D, E, F\}$, as specified in Section 2.

Theorem 1 (Minimum universal collections)**.** *The minimum size of a universal collection is twelve. Exactly ten labelled collections attain that size, even when repeated varieties are permitted. Each unordered partition $L = P \sqcup Q$ into triples gives one such collection: one copy of each variety ij with $i \neq j$ and both labels in the same part. These are all the minimum collections; in particular, none has repeats.*

Our contribution is an elementary proof of this classification, its extension to repeated varieties, and a Lean formalisation of the classification theorems. The written proof first uses usable-cube inequalities to force symmetric multiplicities and row sums two. The resulting

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support graphs are either two triangles or bipartite; corner coverage excludes the latter. An explicit construction proves sufficiency. The formal proof uses a different route for these last two steps, described in Section 6. We also answer the computational questions in [2, Section 6]: for sets of nine, ten and eleven distinct varieties the maximum target counts are respectively 9, 20 and 22, and thirteen is the smallest set size that permits exactly 29 targets.

2 Conventions and tableau facts

Each cube has six different face colours, labelled $1, \ldots, 6$. There are $6!/24 = 30$ rotational varieties. The ten minimum sets are counted separately even though they form one colour-permutation orbit.

Our notation ij corresponds to the conventional upper/lower-case cube name, such as Ab [4, Section 3]. The tableau labels in L are distinct from the face-colour labels $1, \ldots, 6$. At each vertex, the three incident face colours define an *oriented corner type*: an ordered triple modulo cyclic permutation, read clockwise when viewed from outside the cube. Reversing the order gives a different type. Each cube has eight corners, and there are $2\binom{6}{3} = 40$ types in total.

Table 1 lists face colours in the order $+x, -x, +y, -y, +z, -z$. The frame $(+x, +y, +z)$ is left-handed, so the following rule gives clockwise corner triples. Choose one face from each opposite pair and list their colours in axis order x, y, z; reverse the order when an odd number of chosen faces have negative signs. Reduce each triple modulo cyclic permutation. This recovers the eight corner types of every entry directly from the table.

	a	b	c	d	e	f
A	–	152463	162354	142635	132546	123456
B	152436	–	123564	162543	142365	132654
C	162345	123546	–	132456	152634	142563
D	142653	162534	132465	–	123645	152346
E	132564	142356	152643	123654	–	162435
F	123465	132645	142536	152364	162453	–

Table 1: Face-colour representatives of Conway's tableau. Each six-digit entry lists the colours on $+x, -x, +y, -y, +z, -z$ in a left-handed frame. The row and column give the cube name, for example $Ab = 152463$. Their corner types agree with [2, Table 3].

Lemma 2 (Corner–directed-triangles correspondence)**.** *The forty oriented corner types are in bijection with partitions of L into two triples, each equipped with a directed 3-cycle. The six cube varieties containing a given corner are exactly the six arcs of its two cycles. Equivalently, cube ij contains a corner precisely when the corner's two cycles contain the arc $i \to j$.*

Proof. Table 2 records the correspondence explicitly. For a triple written XYZ in increasing label order, write $+$ for the cycle $X \to Y \to Z \to X$ and $-$ for its reverse. The first and second signs refer to the left and right triples respectively.

To verify Table 2, compute the corners of the representatives in Table 1, list the cube names containing each corner, and read each name as an arc. For example, the $(-x, +y, +z)$ corner of Ab has colours $(5, 2, 6)$ in axis order. Reversing gives $(5, 6, 2)$, or 256 modulo cyclic permutation. The full list of cubes with corner 256 is

$$Ab,\ Bc,\ Ca,\ De,\ Ef,\ Fd.$$

These give $A \to B \to C \to A$ and $D \to E \to F \to D$, so 256 occupies the $(+, +)$ column of the $ABC \mid DEF$ row. For every entry the same check gives precisely the six arcs specified by its

row and column. The forty entries are distinct corner types, and the ten rows and four columns exhaust the partitions and direction choices. This verifies the bijection and the incidence rule. In particular, an arc $i \to j$ belongs to eight cycle pairs: four choices of the third label in its cycle and two directions for the complementary cycle, giving the eight corners of ij. □

Partition	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
$ABC \mid DEF$	256	143	134	265
$ABD \mid CEF$	465	132	123	456
$ABE \mid CDF$	345	126	162	354
$ABF \mid CDE$	235	164	146	253
$ACD \mid BEF$	364	125	152	346
$ACE \mid BDF$	356	142	124	365
$ACF \mid BDE$	153	246	264	135
$ADE \mid BCF$	165	243	234	156
$ADF \mid BCE$	136	254	245	163
$AEF \mid BCD$	263	145	154	236

Table 2: Corner types by partition and cycle directions. On each displayed triple XYZ, $+$ means $X \to Y \to Z \to X$ and $-$ means its reverse; the two signs specify the left and right triples. Corner numbers use the fixed face-colour labels of [2], modulo cyclic permutation.

The standard corner-sharing rule [5, Lemma 2.7], in tableau form [3, Propositions 1–2 and Properties 1–2], follows from Lemma 2: two distinct varieties ij and uv share no corner exactly when $i = u$, $j = v$, or $(u, v) = (j, i)$. In every other case they share two corners. Indeed, two distinct arcs in the same directed 3-cycle must form a directed path; this fixes that cycle, leaving two orientations of the other one. For two disjoint arcs, the remaining two labels can be assigned to their respective triples in two ways, each determining both cycle directions. Common tails, common heads, and reversed arcs cannot coexist in directed 3-cycles. The target itself shares all eight corners.

Call a cube *usable* for a target when they share an oriented corner. Each cube is usable for $30 - 9 = 21$ targets, including itself. A target can be built exactly when its eight corners can be assigned distinct physical cubes containing the respective corner types: an oriented-corner match permits a proper rotation matching all three exposed faces. Internal faces impose no further conditions.

3 Necessity

Let a_{ij} be the number of copies of variety ij in a collection of size K, with $a_{ii} = 0$. Write $r_i = \sum_j a_{ij}$ and $c_j = \sum_i a_{ij}$.

Lemma 3 (Row bound)**.** *If every target has at least eight usable physical cubes in the collection, then*

$$r_i + c_j - 2a_{ij} + a_{ji} \le K - 8 \qquad (i \ne j), \qquad 3r_i \le 4K - 40, \quad 3c_i \le 4K - 40.$$

Proof. For target ij, the unusable cubes are those in its row apart from ij, those in its column apart from ij, and the copies of ji. These classes are disjoint and total $r_i + c_j - 2a_{ij} + a_{ji}$. At least eight cubes must remain usable, proving the first inequality. Summing over the five $j \ne i$ gives

$$5r_i + (K - c_i) - 2r_i + c_i = 3r_i + K \le 5(K - 8).$$

The column bound follows by the analogous sum. □

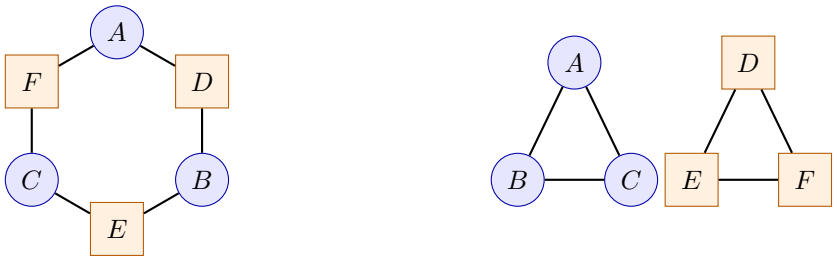
Lemma 4 (Necessary support)**.** *A universal collection of size twelve consists of one copy of every directed edge within each of two disjoint triples of tableau labels.*

Proof. Universality supplies eight usable cubes for each target, so Lemma 3 applies and gives $r_i \leq \lfloor 8/3 \rfloor = 2$. As the six row sums total twelve, all equal two. The same holds for the columns. If $a_{ij} = 0$, the first inequality now gives $4 + a_{ji} \leq 4$, so $a_{ji} = 0$. Thus support is symmetric.

Consider the undirected support graph on the six tableau labels L: an edge $\{i, j\}$ is present when $a_{ij} > 0$. Every vertex has degree one or two. If i has only one neighbour j, its row sum gives $a_{ij} = 2$. By support symmetry, column i has no other nonzero entry, so $a_{ji} = 2$. These entries exhaust both row sums: i and j form an isolated pair, with two copies in each direction. We call this a *double-copy pair*. All other vertices have degree two, forming simple cycles with one copy in each direction on every edge.

On six vertices the only possibilities are two triangles, a 6-cycle, a 4-cycle and a double-copy pair, or three double-copy pairs. All except two triangles are bipartite with parts of size three. Choose these parts as the two triples defining a corner type. Every directed edge of that corner lies within a part, whereas every selected variety joins the parts. Thus no cube in the collection has that corner type. A target having that corner cannot be built, contradicting universality. Only two triangles remain, and their arc multiplicities are all one. □

For distinct varieties, the degree argument leaves only a 6-cycle or two triangles; Figure 1 illustrates why the 6-cycle fails.



C_6

All arcs cross $ABC \mid DEF$.
Four corner types are missing.

$C_3 \sqcup C_3$

Arcs stay within each triple.
Every target can be built.

Figure 1: The two possible degree-two supports for twelve distinct varieties. Each undirected edge represents both cube varieties ij and ji, hence six edges represent twelve cubes. Circles mark $\{A, B, C\}$ and squares mark $\{D, E, F\}$; these are tableau labels, not face colours. The left support misses all four corners for this partition (Table 2); the right support is universal. Repeated varieties allow the additional support types treated in the proof.

4 Sufficiency and classification

Lemma 5 (Two-triangle construction)**.** *For any partition $L = P \sqcup Q$ into triples, the twelve directed edges within P and Q form a universal set.*

Proof. Fix target ij and interchange P, Q if necessary so that $i \in P$. By Lemma 2, its corners are indexed by a third label $k \notin \{i, j\}$ and either direction on the complementary triple; the cycle containing ij is $i \to j \to k \to i$.

If $j \in P$, write $P = \{i, j, p\}$. For each $k \in Q$, the complementary cycle contains one arc between the two labels of $Q \setminus \{k\}$. Assign that cube to the corner. As k and the cycle direction vary, these are the six distinct directed edges within Q. The remaining two corners have $k = p$;

fill them with jp and pi, both of which belong to both corner types. This gives eight distinct cubes.

If $j \in Q$, write $P \setminus \{i\} = \{u, v\}$ and $Q \setminus \{j\} = \{x, y\}$. For the four corners with $k \in \{u, v\}$, the selected varieties containing each such corner are ki and one of xy, yx. Their incidence is described by a $K_{2,2}$ with vertex classes $\{ui, vi\}$ and $\{xy, yx\}$. Here the four corners are the edges, and the four candidate cube varieties are the vertices: each corner joins its two candidates. Orient the resulting 4-cycle cyclically and assign each edge to its head. Every vertex then receives exactly one edge, so each corner receives an incident candidate and all four assigned cubes are distinct. For the other four corners, the analogous four varieties are jx, jy, uv, vu. The two groups of four varieties are disjoint, completing the construction. □

Proof of Theorem 1. Each physical cube is usable for 21 targets, so universality requires $21K \geq 30 \cdot 8 = 240$, hence $K \geq 12$. This lower bound is due to Haraguchi [3, Lemma 2]. The preceding lemmas give necessity and sufficiency at twelve. There are $\frac{1}{2}\binom{6}{3} = 10$ unordered partitions into triples. □

Corollary 6 (Twelve-cube criterion)**.** *A collection of twelve physical cubes, with repeated varieties allowed, is universal if and only if both of the following hold:*

1. *Every target has at least eight usable cubes in the collection, counting physical copies.*
2. *Every oriented corner type occurs on at least one cube in the collection.*

Proof. Universality supplies eight usable cubes for each target. Every oriented corner type belongs to some target, so it must occur in the collection.

Conversely, condition (1) gives the inequalities of Lemma 3. At size twelve these force every row and column sum to equal two and the support to be symmetric, exactly as in the proof of Lemma 4. The same degree argument leaves the four support types listed there. Each bipartite type omits all four corners associated with its balanced bipartition, contradicting condition (2). Only the two-triangle type remains, with one copy of each arc, and Lemma 5 proves it universal. □

The six-cycle collection in Figure 1 shows why condition (2) is needed. Its row and column sums are two and $a_{ij} = a_{ji}$, so target ij has $12 - (r_i + c_j - 2a_{ij} + a_{ji}) = 8 + a_{ij} \geq 8$ usable cubes. Thus condition (1) holds, but the four corners associated with its balanced bipartition are absent.

5 Computational checks and additional results

An exhaustive enumeration of all $\binom{30}{12} = 86,493,225$ subsets returns exactly the ten sets above. The implementation also reproduces the published distribution in [2, Table 2] for all $\binom{30}{8} = 5,852,925$ eight-cube sets. Two algorithms implement Haraguchi's criteria: bipartite matching [3, Proposition 3] and the tree-component test [3, Proposition 4]. The two feasibility tests share the subset enumerator and tableau. The latter test uses a compatibility multigraph on all eight target corners, including isolated vertices. Each usable non-target physical cube is an edge joining its two compatible corners; parallel edges are allowed. Here cubes are edges and corners are vertices, whereas the necessity proof uses a support graph on six tableau labels. A tree component, including an isolated vertex, leaves one corner unfilled by its edges. A component with a cycle can fill all its corners; two parallel edges already form a cycle. Since a copy of the target can fill any corner, the target is buildable exactly when the number of tree components is at most the number of target copies. The subset enumeration precomputes feasibility for all 2^{20} subsets of the twenty usable non-target varieties for each target, and uses no symmetry reduction.

Separate geometric checks generate the 24 proper rotations, 30 varieties and 40 corner types from face-colour permutations. They verify the tableau, the directed-triangle correspondence and physical placements for all 300 minimum-set–target pairs. A separate enumeration of the 70 simple 2-regular graphs gives 60 six-cycles, each building 18 targets, and the ten universal triangle-pair sets.

The single colour-permutation orbit observed in [2, Section 6] also follows from the classification. Colour permutations induce the full symmetric group S_6 on the six tableau rows [4, Lemma 3.1 and Corollary 3.3]. Since colour permutations commute with reflection, which sends ij to ji, rows and columns are permuted identically. Thus the action on entries is $ij \mapsto \pi(i)\pi(j)$, and S_6 acts transitively on unordered partitions into two triples. The stabiliser of a partition has order $2(3!)^2 = 72$, since it may permute each triple and interchange the two triples.

For sets of distinct varieties the exhaustive target-count ranges are:

set size	8	9	10	11	12	13
minimum targets	0	0	1	2	8	10
maximum targets	5	9	20	22	30	30

The lower bound of one target for ten distinct varieties was proved in [4, Lemma 4.1]; the enumeration confirms that it is sharp. No set of size at most twelve builds exactly 29 targets. At size thirteen, 827,250 sets build exactly 29 targets and 53,100 are universal. These enumeration results answer the questions in [2, Section 6] about the maxima for sizes nine, ten and eleven and the possibility of building exactly 29 targets. The earlier distributions for sizes nine through twelve used random samples of 20,000 sets each; our ranges are exhaustive. These target-count bounds rely on the complete enumeration. Full histograms, example sets, source code and checkers are in the accompanying repository.[1]

6 Formal verification and reproducibility

The accompanying repository includes a formalisation using Lean 4.35.0 [6] and mathlib v4.35.0 [7]. The main results are `MacMahon.minimum_size`, `MacMahon.physical_exactly_ten`, and `MacMahon.physical_iff_two_triangles`.

In Lean, an inventory records a natural-number multiplicity for each tableau entry. The stored face-colour representatives are those in Table 1, with the same face order and handedness. A construction assigns eight distinct physical copies to the eight positions, with proper rotations matching the exposed faces. Lean proves that the rotation table consists exactly of determinant-positive signed-axis permutations, that the thirty representatives and their rotations give all 720 face-colour assignments without duplication, and that corner matching is equivalent to exposed-face matching.

The formalisation proves the classification theorems by a different route from the written support-graph and construction arguments. It shares the usable-cube inequalities and symmetry argument, allowing arbitrary multiplicities. For necessity, a label permutation puts the first row into one of two forms: a doubled edge or two distinct neighbours. Arithmetic proofs exclude the doubled edge and force the neighbours to close a triangle; the degree equations then determine the complementary triangle. For sufficiency, Lean checks explicit witnesses for all 300 minimum-set–target pairs instead of formalising the general construction of Lemma 5. These finite checks use kernel reduction. The proof library has no unfinished proofs or added axioms: audits permit only propositional extensionality, classical choice and quotient axioms. The generated tables and construction witnesses are certified by kernel-checked Lean proofs; the data generator itself is not trusted. The target-count enumerations are outside the formalisation, and Corollary 6 has a written proof but no separate Lean statement.

[1] https://github.com/jasondavies/macmahon-min-universal-sets

The development is packaged for the comparator of the Palomar registry. Ten independent statements in `Challenge.lean`, which imports only mathlib, are separated from completed proofs in `Solution.lean`. The comparator checks statement equality, permitted axioms and proof acceptance by Lean's kernel and the independent kernels NanoDa and con-ron. Reproduction commands are in the accompanying repository; local verification does not constitute registry acceptance.

Declaration of AI-generated research and review

GPT-6-Astra (OpenAI), used through Codex, developed the classification proof and computational investigation, wrote the Lean formalisation, and wrote and revised the manuscript. Claude Opus 5.5 (Anthropic), used through Claude Code, contributed to paper review.

The formal claims are checked by Lean and the additional kernels described in Section 6; the target-count computations have separate reproducible checks. These automated checks are not independent human peer review. The literature search was limited to the cited sources and related material located during development; publication priority has not been established by a comprehensive review.

The human author initiated and directed the research and accepts responsibility for the content of this paper and the accuracy of this disclosure.

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