Positive polytopes and the full Fock–Goncharov conjecture

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Abstract

We prove the origin-interior form of the polytope criterion of Gross, Hacking, Keel and Kontsevich: over a field of characteristic zero, full Fock–Goncharov for a principal coefficient cluster variety is equivalent to the existence of a bounded full-dimensional positive polytope containing the origin in its interior. Replacing the interior condition by membership of the origin gives a false criterion.

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The initial research manuscript was developed using ChatGPT-6-Pro during a research session lasting approximately 270 minutes. A further ChatGPT-6-Pro session of approximately 14 minutes examined the role and necessity of the origin-interior condition. Codex reviewed the arguments, checked the literature, and revised the exposition. A preceding version of the manuscript passed an AI review using Rethlas-OSV [9], which incorporates the OSVerify framework of Nie, Ye, Liu, Wang, Ermon and Zou [10].

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1 Introduction

Fomin and Zelevinsky introduced cluster algebras through seeds and their mutations [1]. The Laurent phenomenon places all cluster variables in the Laurent polynomial ring of any chosen seed. The intersection of these Laurent rings is the upper cluster algebra. Constructing a basis compatible with mutation requires more than expressing individual cluster variables: its labels and multiplication must also be independent of the initial seed.

Fock and Goncharov placed this question in the geometry of cluster ensembles [2, 6]. Their two spaces, of types $\mathcal{A}$ and $\mathcal{X}$, are obtained by gluing algebraic tori through subtraction-free birational transformations. Duality relates functions on one space to integral tropical points of a dual space. In higher Teichmüller theory, laminations and positive coordinates provide models for such a relation. The duality conjectures in [2, Section 4] and [6, Section 4] propose canonical functions, positive multiplication, and compatible tropical pairings. The latter paper also constructs canonical pairings in finite type. These formulations distinguish the existence of functions from the additional positivity and compatibility properties expected of them.

Gross, Hacking and Keel related cluster varieties to blowups of toric varieties [7]. Their examples show that global regular functions can be too scarce to accommodate all tropical labels. For the rank-two skew form with three copies of each of the directions $(1,0)$, $(0,1)$ and

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$(-1,-1)$, the regular functions on the $\mathcal{X}$-variety come from its base torus. This example will provide the failure of the positive-polytope criterion when the origin is allowed on the boundary. The original duality question consequently separates into constructing formal functions and determining which of them are regular and span the upper algebra.

Gross, Hacking, Keel and Kontsevich constructed theta functions by counting broken lines in a consistent scattering diagram [8]. Their wall positivity gives nonnegative integral multiplication coefficients. The middle algebra is spanned by the theta functions with finite Laurent expansions; their results identify it with the upper algebra under geometric conditions on the cluster complex. In particular, a cluster complex not contained in a half-space yields the full basis theorem for principal coefficients. The enough-global-monomials condition connects this geometry with bounded tropical polytopes. Their compactification construction associates a graded theta algebra to a positive polytope and produces toric degenerations.

Conjecture 0.10 of [8] asks whether the full Fock–Goncharov conjecture for the principal coefficient variety $\mathcal{A}_{\mathrm{prin}}$ is equivalent to the existence of a bounded full-dimensional polytope convex in their sense. It appears as Conjecture 0.15 in the earlier 2014 author manuscript. The preceding construction uses sets $\{f \geq -1\}$ for continuous homogeneous piecewise linear functions f. Each such set contains the origin in its interior. The criterion below explicitly includes this property.

Cheung, Magee and Nájera Chávez proved that positivity of a closed subset is equivalent to broken-line convexity [4]. This identifies the multiplication condition needed for compactification with a condition on paths in the tropical space. Bossinger, Cheung, Magee and Nájera Chávez subsequently constructed positive Newton–Okounkov bodies from boundary divisors on partial minimal models, including applications to flag varieties [3]. These results relate tropical convexity to graded rings of sections. More recently, Cheung, Magee, Mandel and Muller proved valuative independence and finite-value theta reciprocity and developed the deformation of minimizing broken lines into taut ones [5]. We use the same local deformation principle and give the finite-witness argument for the cylindrical formal series arising here.

The proof of the polytope criterion has two algebraic requirements. Every tropical label must define a finite Laurent theta function, and these functions must span the upper algebra. The first follows from a bound on the complete broken-line trajectory, obtained from the gauge of an absorbing positive polytope. For the second, a compactification and its Poisson structure reduce spanning to the exclusion of missing prime divisors. Their valuations produce a tangent scattering diagram; a minimum-weight argument rules out the resulting vanishing of a theta series.

Work over a field $\Bbbk$ of characteristic zero. We use complete principal coefficients: one invertible coefficient variable is added for each seed index, including original frozen indices. Fix a seed and identify the integral tropical points of the dual of $\mathcal{A}_{\mathrm{prin}}$ with a lattice L. All frozen variables, including principal coefficient variables, are invertible. The upper algebra $\mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ is the intersection of the Laurent coordinate rings of the seed tori inside their common function field. For $m \in \mathsf{L}$, let ϑ_m denote the formal theta function. Write $\Theta \subseteq \mathsf{L}$ for the labels whose theta functions are Laurent polynomials in every seed. Their span is $\mathrm{mid}(\mathcal{A}_{\mathrm{prin}})$. When all labels occur and multiplication has finite support, $\mathrm{can}(\mathcal{A}_{\mathrm{prin}})$ denotes the algebra with vector space basis $(\vartheta_m)_{m\in\mathsf{L}}$. In the principal coefficient setting, *full Fock–Goncharov* means that the theta map is injective and

$$\Theta = \mathsf{L}, \qquad \mathrm{can}(\mathcal{A}_{\mathrm{prin}}) = \mathrm{mid}(\mathcal{A}_{\mathrm{prin}}) = \mathrm{up}(\mathcal{A}_{\mathrm{prin}}).$$

The formal multiplication constants are denoted by $\alpha(p,q,r)$; they are nonnegative integers. A convex polytope $\Xi \subset \mathsf{L}_{\mathbb{R}} := \mathsf{L} \otimes_{\mathbb{Z}} \mathbb{R}$ is *positive* if

$$p \in a\Xi \cap \mathsf{L}, \quad q \in b\Xi \cap \mathsf{L}, \quad \alpha(p,q,r) \neq 0 \quad \Longrightarrow \quad r \in (a+b)\Xi \tag{1}$$

for all $a, b \in \mathbb{N} = \{0, 1, \ldots\}$, with $0\Xi = \{0\}$. Cheung, Magee and Nájera Chávez identify this condition with broken-line convexity in the principal coefficient setting [4, Theorem 6.1].

Theorem 1.1. *Let $\mathcal{A}_{\mathrm{prin}}$ be a cluster variety with complete principal coefficients over a field of characteristic zero, for skew-symmetrizable seed data. The following conditions are equivalent:*

1. *The full Fock–Goncharov conjecture holds for $\mathcal{A}_{\mathrm{prin}}$.*
2. *There is a bounded full-dimensional positive polytope $\Xi \subset \mathsf{L}_{\mathbb{R}}$ with $0 \in \operatorname{int} \Xi$.*

When these conditions hold, Ξ may be chosen rational.

The proof is completed in Section 9.

Remark 1.2. The short statement of [8, Conjecture 0.10] does not explicitly impose $0 \in \operatorname{int} \Xi$. Its preceding sets $\{f \geq -1\}$, for continuous homogeneous f, have this property. The dilation argument in [8, Proposition 8.21], numbered Proposition 8.17 in the published version, also uses absorption of every tropical label. Theorem 1.1 gives the criterion with this condition explicit. With general positive polytopes, the condition cannot be weakened to $0 \in \Xi$, as the following proposition shows.

Proposition 1.3. *There is an eighteen-dimensional principal coefficient cluster variety for which full Fock–Goncharov fails, although its tropical dual contains a bounded full-dimensional rational positive polytope containing the origin.*

Proof. For any seed with n mutable indices and no original frozen indices, in the initial coordinates set

$$P_0 = [0, 1]^n \times [-1, 1]^n \subset \mathbb{R}^n \oplus \mathbb{R}^n.$$

Every dilate lies in the closed positive cluster chamber. Theta functions with labels in that chamber are cluster monomials, so their products have the single output $p+q$ [8, Theorem 7.5]. Convexity gives $(p + q)/(a + b) \in P_0$ for $a, b > 0$; zero degrees follow from $\vartheta_0 = 1$. Thus P_0 is positive and $0 \in \partial P_0$.

Take $N = \mathbb{Z}^9$ with basis $e_1, \ldots, e_9$. Set $w_1 = w_2 = w_3 = (1, 0)$, $w_4 = w_5 = w_6 = (0, 1)$ and $w_7 = w_8 = w_9 = (-1, -1)$, and define $\{e_i, e_j\} = \det(w_i, w_j)$. All indices are mutable and all multipliers are one. This is the seed of [7, Example 6.3 and Section 7]. Let $M = \operatorname{Hom}(N, \mathbb{Z})$, and let $p^* : N \to M$ send n to the functional $\{n, \cdot\}$ of this seed. Put $K = \ker p^*$. The map $e_i \mapsto w_i$ is surjective onto $\mathbb{Z}^2$. Nondegeneracy of the determinant form gives

$$K = \left\{ \sum_i a_i e_i : \sum_i a_i w_i = 0 \right\}, \qquad \operatorname{rank} K = 7.$$

For this seed, [7, Section 7] identifies the ring of regular functions on $\mathcal{X}$ with the group algebra of K, that is, the ring of finite sums of characters z^k, $k \in K$. The natural pullback identifies

$$\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \Bbbk[K].$$

We may take $\Bbbk = \mathbb{C}$. The morphism $\widetilde{p} : \mathcal{A}_{\mathrm{prin}} \to \mathcal{X}$ is a torsor under the torus $T_N = \operatorname{Spec} \Bbbk[M]$ [8, Appendix B]. On the initial chart it pulls z^n back to $z^{(p^*n, n)}$. The weight of $\vartheta_{(m,n)}$ under this action is $m - p^*n$ [8, Proposition 7.7]. If $n \notin K$ and $(p^*n, n) \in \Theta$, this theta function would be a regular invariant and hence a finite linear combination of $z^{(0,k)} = \vartheta_{(0,k)}$, $k \in K$. This contradicts linear independence of distinct principal theta functions. Hence $\Theta \neq \mathsf{L}$. The above P_0 with $n = 9$ gives the asserted polytope. □

The condition also cannot be supplied by translating an arbitrary positive polytope. In rank one, $xx' = 1 + y$, and

$$\vartheta_{(m,n)} = \begin{cases} x^m y^n & m \geq 0, \\ (x')^{-m} y^n & m \leq 0. \end{cases}$$

The positive rectangle $[0,1] \times [-1,1]$ translates to $Q = [-1/2, 1/2] \times [-1,1]$. But $(1,2), (-1,2) \in 2Q$ and their theta product is $y^4 + y^5$, whose output $(0,5)$ is outside $4Q$.

2 Principal coefficients and broken lines

For a lattice L, write $L^* = \mathrm{Hom}(L, \mathbb{Z})$ and $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$; the subscript $\mathbb{R}$ similarly denotes extension to $\mathbb{R}$. Its group algebra $\Bbbk[L]$ has basis z^m, $m \in L$, and product $z^m z^{m'} = z^{m+m'}$. Its spectrum is the torus with character lattice L. Pairings between a lattice and its dual are evaluation pairings. Vectors are columns in the bases specified below; a transpose converts a column into a row.

Let N be a finite-rank lattice with basis $(e_i)_{i \in I}$, let $J \subset I$ be the mutable indices, and let $\{\ ,\ \}$ be the rational skew form of the seed. Choose positive integers d_i and let $N^\circ \subset N$ have basis $d_i e_i$. The seed integrality conditions require $d_j\{e_i, e_j\} \in \mathbb{Z}$ and $d_i\{e_i, e_j\} \in \mathbb{Z}$ for $i \in J$, $j \in I$. Mutation takes place only at indices in J. Complete principal coefficients give $2|I|$ invertible variables on each seed chart; the convention adding coefficients only at mutable indices is treated in Section 9. Put $M^\circ = \mathrm{Hom}(N^\circ, \mathbb{Z})$, with dual basis $f_i = d_i^{-1} e_i^*$, and set $\mathsf{L} = M^\circ \oplus N$. Subscript $\bullet J$ below means restriction to columns indexed by J; I in a matrix expression denotes the identity matrix. Write

$$\Delta = \mathrm{diag}(d_i), \quad \Omega_{ij} = \{e_i, e_j\}, \quad B = -\Delta\Omega, \qquad P = \begin{pmatrix} B_{\bullet J} \\ I_{\bullet J} \end{pmatrix}, \quad R = \begin{pmatrix} \Delta^{-1}_{\bullet J} \\ 0 \end{pmatrix}.$$

Here $B_{ji} = d_j\{e_i, e_j\}$, so the columns of B indexed by J are integral. The lower block $I_{\bullet J}$ embeds $\mathbb{Z}^J$ into the second summand of L. Projection onto these mutable second coordinates is a left inverse of P. In particular its image is a direct summand. The map $P : \mathbb{Z}^J \to \mathsf{L}$ is saturated and injective; $R : \mathbb{Q}^J \to \mathsf{L}^*_{\mathbb{Q}}$ gives rational wall normals. The matrix identity

$$R^{\mathsf{T}} P = -\Omega_{JJ} \tag{2}$$

is skew-symmetric. For $n \in \mathbb{N}^J$, its *charge degree* is $|n|_1 = \sum_{i \in J} n_i$. Let $\ell : \mathsf{L}_{\mathbb{R}} \to \mathbb{R}$ be the sum of the mutable coordinates in the second summand. Then $\ell(Pn) = |n|_1$. Write e_i also for the ith standard basis vector of $\mathbb{Z}^J$ when it is an input to P or R. The initial positive chamber is

$$\{q \in \mathsf{L}_{\mathbb{R}} : \langle Re_i, q \rangle > 0 \text{ for every } i \in J\}.$$

The subscript $+$ on $\vartheta_{m,+}$ denotes its expansion at a general point of this chamber. Introduce independent variables t_i, $i \in J$, and write $t^n = \prod_{i \in J} t_i^{n_i}$.

A character with exponent $u \in \mathsf{L}$ is written z^u. Use the standard consistent cluster scattering diagram $\mathfrak{D}$ in $\mathsf{L}_{\mathbb{R}}$, with positive binomial wall functions, as constructed in [8, Theorems 1.12–1.13 and Appendix C]. A wall of charge direction $n \in \mathbb{N}^J \setminus \{0\}$ has a closed rational polyhedral support in $(Rn)^\perp$ and function $(1 + z^{Pn})^c$, $c \in \mathbb{Z}_{>0}$. Normals are normalized to primitive integral covectors when acting on characters; positive integer multiples are absorbed into c. For $K \geq 0$, charge truncation means reduction modulo the monomials t^n with $|n|_1 > K$; only walls acting nontrivially in this quotient are retained. It has finitely many walls. At a transverse crossing with primitive normal ν, the wall automorphism sends z^u to $z^u f^{\nu(u)}$, with the sign of ν chosen according to the crossing direction. A path-ordered

product composes these maps in the order in which the path crosses the walls. Consistency means that this product depends only on the endpoints of a path avoiding the joints, at every finite charge order. The only incoming walls are the initial ones: a wall is incoming when its support contains its positive correction direction.

A broken line of input m and endpoint q is a continuous piecewise affine map $\gamma : (-\infty, 0] \to \mathsf{L}_{\mathbb{R}}$, with $\gamma(0) = q$. On a segment with attached character exponent e, its velocity is $-e$; the first exponent is m. At a wall crossing, the new attached monomial is a term in the old one times the wall function to the positive integer power given by the absolute normal pairing. Thus a nontrivial bend changes e by Pn with $0 \neq n \in \mathbb{N}^J$. The path avoids wall boundaries and intersections with independent normals, collectively called joints. The function $\vartheta_{m,q}$ is the sum of its final attached monomials, with their positive coefficients.

For formal computations replace each correction z^{Pn} by $z^{Pn}t^n$ and complete in total t-degree. For a fixed input, the z-exponent determines the charge, so this bookkeeping preserves coefficients and consistency. For a fixed total charge n, there are at most $|n|_1$ nontrivial bends. The applicable walls, choices of corrections, and ordered records are finite at that charge. A record and an endpoint determine all crossing times by the reverse reconstruction below. Hence every fixed charge has finitely many contributions. All formal operations use the inverse limit of these finite charge quotients. Theta transportation is the path-ordered wall product [8, Theorem 3.5].

We use the near-output multiplication rule [8, Definition–Lemma 6.2 and Proposition 6.4]. For fixed p, q, r, $\alpha(p, q, r)$ is the finite sum of coefficients of pairs of broken lines whose final exponents add to r, with endpoint sufficiently close to r. In particular,

$$\alpha(p, q, r) \neq 0 \implies r = p + q + Pn \ (n \in \mathbb{N}^J), \qquad \alpha(p, q, p + q) = 1. \tag{3}$$

Coefficient extraction here means taking the coefficient of the character z^r in a fixed finite charge truncation. The same extraction is linear and applies to brackets as well as products.

We will choose endpoints outside a countable union of proper affine subspaces. Indeed, a finite rational bend record reconstructs each segment as $Cq - te$, with C invertible and rational. Reversing a bend δ of normal ν replaces C by $(I - \delta\nu/\nu(e))C$, an invertible shear since $\nu(\delta) = 0$. Meeting a joint contained in a linear space U of codimension at least two forces $q \in C^{-1}(U + \mathbb{R}e)$. A segment lying in a wall hyperplane H forces $q \in C^{-1}H$. These are proper subspaces. There are countably many records. This construction also allows countably many prescribed translates of the endpoint and finitely many diagrams. Baire's theorem supplies such endpoints in every nonempty open set.

3 A positive polytope bounds theta functions

Suppose Ξ satisfies condition (2) of Theorem 1.1. Its Minkowski gauge is

$$g(x) = \inf\{t \geq 0 : x \in t\Xi\}.$$

It is finite, continuous, positively homogeneous, and the maximum of finitely many linear functions. Boundedness and the interior condition give $g(x) \geq c\|x\|$ for some $c > 0$ and a fixed norm. Absorption of all lattice points is equivalent to the interior condition: otherwise separation gives a nonzero linear function nonnegative on Ξ, and lattice points on the opposite side belong to no nonnegative dilate.

Lemma 3.1. *If $\alpha(p, q, r) > 0$, then $g(r) \leq g(p) + g(q)$. On every broken line, $g \circ \gamma$ is convex in its affine time parameter.*

Proof. The scaling property [8, Lemma 8.10] gives $\alpha(kp, kq, kr) > 0$ for every positive integer k. Apply (1) in degrees $\lceil kg(p) \rceil$ and $\lceil kg(q) \rceil$, divide by k, and take the limit.

To obtain the scaling property, multiply the decorations and the geometry of a contributing pair by k. Each old selected term gives a nonzero term with k times its exponent in the corresponding kth power. Choosing the original endpoint sufficiently close to r makes the new endpoint sufficiently close to kr. Positive coefficients preserve the contribution.

The balanced-pair construction of [4, Proposition 5.4] applies this inequality to a rational point of a rational broken-line segment, with the ratio of the two degrees equal to its affine time ratio. It gives the Jensen inequality on those segments. To extend it to real endpoints, fix the finite bend record, including all chosen factors and monomials. Reverse reconstruction expresses the crossing times and segment positions as linear functions of the endpoint divided by the fixed nonzero normal pairings. In the charge truncation of this record, support membership, transversality and time order are finite open conditions. At a nearby rational endpoint, keep this fixed record and interpret it as a nongeneric limit segment in the sense of [4, Definition 3.3]. The strict support and time conditions persist. It is the pointwise limit of real general-endpoint paths with the same decorations and coefficients, obtained by avoiding the countable exceptional sets above. Rational endpoints of this kind are dense in the finite open set of admissible records. For three times $s < t < u$ with rational ratios, the CMN construction and the preceding structure-constant inequality give

$$g(\gamma(t)) \le \frac{u-t}{u-s} g(\gamma(s)) + \frac{t-s}{u-s} g(\gamma(u)).$$

The integer degrees in the balanced pair can be chosen in the ratio $(u-t) : (t-s)$ after clearing denominators. Approximation of the three times and of the endpoint extends this inequality to real times. For each fixed rational balanced pair, take its endpoint sufficiently close to its fixed output to apply the near-output rule. Continuity of g passes Jensen's inequality to the given real path. Zero input gives a constant line. A nonzero input cannot first acquire zero exponent at a genuine bend, since the corresponding normal pairing would vanish. Hence possible passages through the origin can be omitted and recovered by continuity. □

Proposition 3.2. *Every label in* L *belongs to* Θ.

Proof. Assume $J \neq \varnothing$ and fix $m \neq 0$ and a general positive-chamber endpoint q. Write the bends of a broken line as Pa_j at times $-u_j < 0$, with $a_j \in \mathbb{N}^J \setminus \{0\}$. Choose $\epsilon > 0$ so that

$$\langle Re_i, q + um \rangle \ge \tfrac{1}{2} \langle Re_i, q \rangle > 0 \quad (0 \le u \le \epsilon,\ i \in J).$$

For $-T < t < 0$, convexity gives

$$g(\gamma(t)) \le \frac{-t}{T} g(\gamma(-T)) + \left(1 + \frac{t}{T}\right) g(q).$$

Homogeneity and $\gamma(-T)/T \to m$ imply $g(\gamma(-T))/T \to g(m)$. Letting $T \to \infty$ yields

$$g(\gamma(t)) \le g(q) - t g(m) \qquad (t < 0). \tag{4}$$

Thus positions at any fixed negative time are uniformly bounded over all these paths.

The function $F = \ell \circ \gamma$ is concave, since its derivative drops by $|a_j|_1$ at a bend. Consequently

$$F'_+(-\epsilon) \ge \frac{F(-\epsilon/2) - F(-\epsilon)}{\epsilon/2}.$$

The right side is uniformly bounded below by (4), whereas the initial derivative is $-\ell(m)$. Hence $\sum_{u_j \ge \epsilon} |a_j|_1 \le K$ for an integer K independent of the path.

The times of these early bends are also uniformly bounded. Their exponents belong to the finite set $\{m + Pn : |n|_1 \le K\}$. Starting from $X = \gamma(-\epsilon)$ and reversing a segment of exponent e to a wall of normal Ra, the elapsed time is

$$\tau = -\frac{\langle Ra, X\rangle}{\langle Ra, e\rangle}, \qquad X' = X + \tau e.$$

At a genuine bend the denominator is nonzero. Only finitely many numerator covectors, exponents and nonzero denominators occur. There are at most K early bends, so iteration bounds all their u_j by a constant U_K.

Integration of the velocity gives the identity

$$\gamma(-u_j) = q + u_j m + P\sum_k \min(u_j, u_k)a_k.$$

Pair with Ra_j and sum over j. The left sides vanish because the bend lies on its wall. The double sum is

$$\sum_{j,k} \min(u_j, u_k)a_j^{\mathsf{T}} R^{\mathsf{T}} P a_k.$$

Its diagonal terms vanish. The terms indexed by (j, k) and (k, j) cancel by (2), since their scalar time factors agree. Therefore

$$\sum_j \langle Ra_j, q + u_j m\rangle = 0. \tag{5}$$

For $u_j < \epsilon$ each summand is at least $\frac{1}{2}\min_i\langle Re_i, q\rangle|a_j|_1$. The absolute sum of the early terms is bounded using K, U_K. Equation (5) bounds the remaining total charge. Explicitly, divide a bound for the absolute sum of the early terms by $\frac{1}{2}\min_i\langle Re_i, q\rangle$ to bound the late charge. Adding the early bound K gives a uniform bound for $|n|_1$ in every final exponent $m + Pn$. At each such charge the finite-record argument of Section 2 applies. There are thus finitely many final exponents and finitely many contributions to each, so $\vartheta_{m,q}$ is Laurent finite. This property propagates to all seed chambers by [8, Proposition 7.1]. Input zero gives 1; if J is empty every theta is a character. □

4 Finite theta functions and rational positive polytopes

Define the rational alternating isomorphism $\Lambda : \mathsf{L}_{\mathbb{Q}} \to \mathsf{L}^*_{\mathbb{Q}}$ by

$$\Lambda = \begin{pmatrix} 0 & \Delta^{-1} \\ -\Delta^{-1} & \Delta^{-1}B \end{pmatrix}, \qquad \Lambda^{-1} = \begin{pmatrix} B\Delta & -\Delta \\ \Delta & 0 \end{pmatrix}. \tag{6}$$

It satisfies $\Lambda P = R$ and $P^{\mathsf{T}}\Lambda = -R^{\mathsf{T}}$. The matrices are inverse because their block products have diagonal blocks I and off-diagonal blocks zero; the lower-right block of Λ is $-\Omega$, which is skew-symmetric. Thus Λ is alternating. For a nonzero finite Laurent polynomial $A = \sum_a c_a z^a$, its tropicalization at $w \in \mathsf{L}^*_{\mathbb{R}}$ is $A^{\mathrm{trop}}(w) = \min_{c_a \neq 0}\langle a, w\rangle$.

Lemma 4.1. *If $\vartheta_{m,+}$ is finite in the initial positive chamber, then*

$$h(p) = \vartheta^{\mathrm{trop}}_{m,+}(\Lambda p)$$

is concave along every broken line, with its affine time parameter.

Proof. Retain auxiliary charge variables and write the finite lift as

$$A = \sum_{n \in E} c_n z^{m+Pn} t^n, \qquad E \subset \mathbb{N}^J \text{ finite}, \quad c_n > 0.$$

Set $K_0 = \max_{n \in E} |n|_1$. For a series, its weight at w is the infimum of $\langle a, w \rangle$ over its nonzero monomials $z^a t^n$. If every correction of a wall has nonnegative w-weight, its automorphism and inverse do not decrease this infimum. Applying both directions gives equality. When correction weights are strictly positive, each input monomial attaining the minimum retains its coefficient: no higher-weight term can contribute to it. These statements hold for path products degree by degree.

For general p, join a general positive-chamber point to p by a straight segment avoiding joints. Every crossed wall of charge v has $\langle Rv, p \rangle < 0$, and hence

$$\langle Pv, \Lambda p \rangle = -\langle Rv, p \rangle > 0.$$

Transportation therefore gives weight $h(p)$ for $\vartheta_{m,p}$, and a minimum attained by a monomial of A of charge at most K_0. For any direction approaching a nongeneric point, a fixed monomial of the finite set E is minimal sufficiently near that point. Degreewise coefficient stabilization and continuity of h preserve the equality in the one-sided limit.

On a straight segment, h is a minimum of finitely many affine functions and is concave. Consider a bend at q with incoming and outgoing exponents a_-, a_+. All walls at q have parallel normals. Since R is injective, their charges are parallel as well. Write their primitive charge as v and their primitive integral normal as $\nu = cRv$, $c > 0$. Normalize other normals by raising their functions to positive integer powers. The combined positive wall function is a series f in $Z = z^{Pv} t^v$. Put $A_0 = \langle a_-, Rv \rangle \neq 0$ and $\sigma = \operatorname{sgn} A_0$. The bend has the form

$$a_+ = a_- + ck\sigma A_0 Pv, \qquad k \geq 0.$$

Choose an actual degree $J_0 \geq k$ in f, together with a finite factor witness for it. This is possible because k is an average of degrees selected from powers of f.

Choose a charge cutoff $K_1 \geq K_0 + J_0 |v|_1 \max_{n \in E} |\nu(m + Pn)|$ containing the witness. In a sufficiently small ball about q, this finite diagram has only the common wall hyperplane. The theta truncations on its two open half-balls are fixed positive finite polynomials A_-, A_+, related by

$$A_+ \equiv E_{\sigma\nu, f}(A_-) \pmod{\text{charge} > K_1}. \tag{7}$$

Here $E_{\sigma\nu,f}(z^a) = z^a f^{\sigma\nu(a)}$. For either half-ball, the monomials common to A and its theta truncation realize h at every general point, by the preceding transportation argument. Every other monomial in that truncation has weight at least h. Both statements are comparisons of finitely many continuous linear functions, so extend to the closed half-ball.

Set $p_- = q + \epsilon a_-$ and $p_+ = q - \epsilon a_+$ for small $\epsilon > 0$. Choose a monomial $z^b t^{n_b}$ common to A and A_- realizing $h(p_-)$ for all sufficiently small ϵ. Then

$$\langle b, \Lambda q \rangle = h(q), \quad |n_b|_1 \leq K_0, \quad h(p_-) = h(q) + \epsilon \langle b, \Lambda a_- \rangle.$$

Write $d = \nu(b) = c\langle b, Rv \rangle$, and let $A_\pm^{(d)}$ be the sum of terms of $A_\pm$ whose exponents a satisfy $\nu(a) = d$. Wall transformations preserve this pairing, so the corresponding block of (7) is multiplication by $f^{\sigma d}$. Its positive-power direction has no cancellation.

We construct an actual exponent b_+ of A_+ with

$$b_+ - b \in \mathbb{R} Pv, \qquad \langle b_+, \Lambda a_+ \rangle \geq \langle b, \Lambda a_- \rangle. \tag{8}$$

If $A_0 \langle b, Rv \rangle > 0$, then $\sigma d = |d| > 0$. For $\sigma = 1$, choose the constant term of $f^{|d|}$, giving $b_+ = b$. For $\sigma = -1$, choose degree $|d| J_0$, giving $b_+ = b + |d| J_0 Pv$. Both charges are at most K_1.

If $A_0\langle b, Rv\rangle < 0$, use the reverse positive block $A_-^{(d)} = A_+^{(d)} f^{|d|}$. A nonzero source of b has $b_+ = b - rPv$ and charge $n_b - rv \in \mathbb{N}^J$, for some integer $r \geq 0$. If $\sigma = -1$, then $\langle Pv, \Lambda p_-\rangle > 0$; its constant-term contribution to A_- would have smaller weight than b unless $r = 0$. If $\sigma = 1$, the same pairing is negative. The degree $|d|J_0$ contribution is within the cutoff, and minimality of b forces $r \geq |d|J_0$.

In all four cases write $b_+ = b + ck'\sigma\langle b, Rv\rangle Pv$. The choices give $k' = 0$ or J_0 in the first pair of cases, and $k' = r/|d|$ in the second pair. The identities $\langle Pv, \Lambda a_+\rangle = -A_0$ and $\langle b, \Lambda Pv\rangle = \langle b, Rv\rangle$ follow from $\Lambda P = R$ and $\langle Rv, Pv\rangle = 0$. They give

$$\langle b_+, \Lambda a_+\rangle - \langle b, \Lambda a_-\rangle = c(k - k')\sigma A_0\langle b, Rv\rangle \geq 0.$$

If $d = 0$, the block is unchanged and $b_+ = b$ gives equality. This proves (8). Since $\langle Rv, q\rangle = 0$, also $\langle b_+, \Lambda q\rangle = h(q)$. The finite half-ball inequality gives

$$h(p_+) \leq h(q) - \epsilon\langle b_+, \Lambda a_+\rangle \leq h(q) - \epsilon\langle b, \Lambda a_-\rangle.$$

Thus the derivative in actual time cannot increase at the bend. This proves concavity on the whole line. Continuity transfers it to fixed-record nongeneric limits. □

Proposition 4.2. *If every theta function is Laurent finite, there is a bounded rational positive polytope with the origin in its interior.*

Proof. Choose an integral basis $a_1, \ldots, a_d$ of L and put

$$f(p) = \min_{j,\,\sigma=\pm 1} \vartheta^{\mathrm{trop}}_{\sigma a_j,+}(\Lambda p).$$

By Lemma 4.1, this is concave on broken lines. The straight-line terms imply

$$f(p) \leq -\max_j |\langle a_j, \Lambda p\rangle|.$$

Since Λ is invertible, $\{f \geq -1\}$ is bounded. It is an intersection of finitely many rational half-spaces, all strict at the origin. Concavity along broken lines implies the decreasing inequality for structure constants and hence positivity by [8, Lemmas 8.4 and 8.7]. Explicitly, apply the final-slope inequality to the two lines in a near-output pair, using a linear piece active at the output; adding gives $f(r) \geq f(p) + f(q)$. □

This proves the implication (1)⇒(2) in Theorem 1.1. For the converse, Proposition 3.2 followed by Proposition 4.2 allows us to replace the original polytope by a rational one. Henceforth Ξ is this rational polytope and $g = -f$ its gauge. Lemma 3.1 and coercivity continue to hold.

5 The affine theta algebra and its symplectic form

We first work over an algebraically closed field $\Bbbk$ of characteristic zero. Positivity and boundedness give finite multiplication on all theta labels. Write $C = \mathrm{can}(\mathcal{A}_{\mathrm{prin}})$ and form the graded algebra

$$S_\Xi = \bigoplus_{d\geq 0} \bigoplus_{m\in d\Xi\cap\mathsf{L}} \Bbbk\vartheta_m T^d,$$

where T is an independent degree-one variable and every $\vartheta_m T^d$ has degree d.

For a normal variety Y, K_Y denotes a canonical Weil divisor. The notation $K_Y + D \sim 0$ means that this divisor is linearly equivalent to zero. Equivalently, its associated rank-one reflexive sheaf is trivial. We use the openness theorem [8, Lemma 8.42]: in a proper flat

family over a regular characteristic-zero base, with flat boundary, the simultaneous conditions of normality and Cohen–Macaulayness, reduced divisorial boundary, log canonicity, trivial $\omega(D)$, and $H^1(\mathcal{O}) = 0$ persist near a fiber satisfying them. Here log canonicity has its usual discrepancy meaning: on a resolution every discrepancy is at least -1. The toric central fiber below satisfies all five conditions.

Lemma 5.1. *The algebra C is a finitely generated normal domain. There is a normal projective Cohen–Macaulay variety Y with reduced boundary divisor D such that*

$$Y \setminus D = \operatorname{Spec} C, \qquad (Y, D) \text{ is log canonical}, \qquad K_Y + D \sim 0.$$

Proof. The lattice semigroup of pairs (m, d) with $m \in d\Xi$ is finitely generated by Gordan's lemma. For each Hilbert basis element take the corresponding $\vartheta_m T^d$. Products have leading term prescribed by addition, with coefficient one; all other labels differ by $Pn \neq 0$ and have strictly larger ℓ-value. Each degree slice is finite. Decompose $(m, d) = \sum_j (m_j, d_j)$ into Hilbert basis elements. The product of their theta monomials equals $\vartheta_m T^d$ plus a finite linear combination of $\vartheta_{m+Pn} T^d$ with $n \neq 0$. Each extra label belongs to $d\Xi$ by positivity and satisfies $\ell(m + Pn) > \ell(m)$. Starting from the largest ℓ-value in the finite set $d\Xi \cap \mathsf{L}$, subtract these already constructed terms. This produces $\vartheta_m T^d$ in the generated algebra and terminates for every label in that slice. Thus S_Ξ is finitely generated. Evaluation at $T = 1$ maps S_Ξ onto C, since absorption places each label in some degree. Its kernel is $(T - 1)$: for a finite expression in the kernel, raise each of its homogeneous terms to a common degree by multiplication by powers of T. The resulting expression is zero by theta independence, and the change of each term is divisible by $T - 1$. Thus $C = S_\Xi/(T - 1)$. Theta independence and Laurent finiteness embed these algebras in the initial Laurent ring and its polynomial extension.

For each output $r = p + q + Pn$, introduce independent mutable parameters t_i and replace the multiplication coefficient by $\alpha(p, q, r)t^n$. In a triple product, both bracketings leading from inputs p, q, s to output r have total charge N determined by $PN = r - p - q - s$. Their parameter monomials are both t^N. Associativity of the original structure constants therefore gives associativity of the new product coefficient by coefficient. Denote the resulting graded algebra by $\mathscr{S}$. The same triangular induction proves finite generation over $\Bbbk[t_i : i \in J]$, while every homogeneous component has the indicated finite free theta basis.

The span $\mathscr{I}$ of $\vartheta_m T^d$ with $g(m) < d$ is an ideal, since $g(r) \leq g(p) + g(q)$. Its quotient is degreewise free as well. Therefore

$$(\mathscr{Y}, \mathscr{B}) = (\operatorname{Proj} \mathscr{S}, \operatorname{Proj}(\mathscr{S}/\mathscr{I})) \longrightarrow \mathbb{A}^J$$

is a proper flat family with flat boundary. For a homogeneous element u, the localization $\mathscr{S}[u^{-1}]$ is a filtered direct limit of the flat base module $\mathscr{S}$, with transition maps multiplication by u. It is flat. Its degree-zero part is a direct summand and hence flat, giving the claim on each chart $D_+(u)$. The same argument applies to $\mathscr{S}/\mathscr{I}$. At $t = 0$ the product is addition of lattice labels; the saturated cone semigroup gives a normal projective toric variety. A lattice point in the cone is interior exactly when it lies on no facet. Consequently the ideal spanned by interior monomials is the intersection of the monomial prime ideals corresponding to the facets. It is radical, and its Proj is the reduced union of the torus-invariant prime divisors. The toric variety is projective because Ξ is bounded; its cone semigroup is saturated because $k(m, d)$ lying in the cone for an integer $k > 0$ forces (m, d) to lie there. Normal toric varieties are Cohen–Macaulay, and the toric boundary cancels the canonical divisor. On a toric log resolution every exceptional toric divisor has discrepancy -1, so the pair is log canonical. Projective toric varieties have $H^1(\mathcal{O}) = 0$.

These are the five central-fiber hypotheses of [8, Lemma 8.42], and the base is regular. Hence all five properties hold on an open neighborhood of zero. The torus weights

$$\operatorname{wt}(\vartheta_m) = m, \quad \operatorname{wt}(t_i) = -Pe_i, \quad \operatorname{wt}(T) = 0$$

give a contracting one-parameter action on the base using the cocharacter $-\ell$. Explicitly, $-\ell$ assigns weight 1 to every t_i, so the orbit of a base point a extends to the map $\mathbb{A}^1 \to \mathbb{A}^J$, $\lambda \mapsto \lambda a$, and sends 0 to the toric fiber. The inverse image of the good open set contains 0 and therefore a nonzero λ. The torus action identifies the fibers at a and λa. Hence the properties hold at $t = (1, \dots, 1)$.

Let $Y = \operatorname{Proj} S_\Xi$ and let D be this fiber of the boundary. The interior ideal has radical equal to the radical of (T). Indeed, for $g(m) < d$, sufficiently large powers of $\vartheta_m T^d$ have all outputs in $(kd-1)\Xi$ and are divisible by T. Conversely $T = \vartheta_0 T$ is in the interior ideal. Thus $Y \setminus D = D_+(T) = \operatorname{Spec} C$. Its normality follows from that of Y. □

Lemma 5.2. *Define the bracket of characters by*

$$\{z^u, z^v\} = -u^\mathsf{T} \Lambda v \, z^{u+v}.$$

It restricts to C. The inverse of its bivector is a regular symplectic form on the smooth locus of $\operatorname{Spec} C$.

Proof. For a wall of charge n, write its primitive normal as $\nu = cRn$, $c > 0$. Then

$$\{z^{Pn}, z^u\} = \langle Rn, u \rangle z^{Pn+u}.$$

The wall map is the exponential of a Hamiltonian derivation: for a binomial wall $(1 + z^{Pn})^a$, take the Hamiltonian

$$ca \sum_{j \geq 1} \frac{(-1)^{j+1}}{j^2} z^{jPn}.$$

Its bracket with z^u is $a\nu(u) z^u \log(1 + z^{Pn})$. Since this derivation fixes z^{Pn}, exponentiation gives $z^u (1 + z^{Pn})^{a\nu(u)}$. Reversing the crossing negates the Hamiltonian. Thus every wall map preserves the bracket. The auxiliary charge variables make these operations continuous: Euler derivatives preserve each charge and a Hamiltonian term raises it by a positive amount. Let $\beta(p, q, r)$ denote the formal theta coefficient of $\{\vartheta_p, \vartheta_q\}$. The positive-charge triangular change from theta functions to characters and its degreewise inverse show that its possible outputs lie in $p + q + P\mathbb{N}^J$. For a fixed output $r = p + q + Pn_0$, only charges at most n_0 componentwise contribute. Choose a finite truncation containing all those charges. At an endpoint Q sufficiently close to r, each relevant normal ν with $\nu(r) \neq 0$ has $\nu(Q)\nu(r) > 0$. A backward final ray $Q + \mathbb{R}_{\geq 0} r$ meets no such hyperplane; it is parallel to the others. Hence a line with final exponent r has no last nontrivial bend in this truncation. Its input must also be r. Consequently the coefficient of ϑ_r equals the coefficient of z^r for any formal theta expansion under consideration, and linear coefficient extraction gives

$$\alpha(p, q, r) = \sum_{e_\gamma + e_\eta = r} c_\gamma c_\eta,$$
$$\beta(p, q, r) = \sum_{e_\gamma + e_\eta = r} c_\gamma c_\eta (-e_\gamma^\mathsf{T} \Lambda e_\eta).$$

Here e_γ, e_η are the final exponents and $c_\gamma, c_\eta > 0$ their coefficients; both sums use the same finite set of pairs. Thus $\alpha(p, q, r) = 0$ implies $\beta(p, q, r) = 0$. In particular the bracket has finite support and preserves C. The formal identities for Jacobi and Leibniz hold degreewise and descend through the injective Laurent realization.

Declare T central. The support inclusion gives $\{(S_\Xi)_d, (S_\Xi)_e\} \subset (S_\Xi)_{d+e}$ and preserves the interior ideal. The bracket extends by the quotient rule on homogeneous charts to Y, with D a Poisson divisor. At the generic point of each boundary component, the terms of the bivector containing a normal direction are divisible by its local equation. Consequently

its top wedge defines a section of the rank-one reflexive sheaf $\mathcal{O}_Y(-K_Y - D)$. By Lemma 5.1 this sheaf is trivial, and its global sections are constants. On the initial torus the top wedge is nonzero because Λ is invertible. It is therefore nowhere zero on the smooth part of $Y \setminus D$. The inverse form is regular there and is closed, as it is the log-symplectic form on a dense torus. Its volume is a nonzero constant multiple of the initial logarithmic volume form. □

6 Divisors omitted by the initial tori

Let $x_i = z^{(f_i,0)}$ and $y_i = z^{(0,e_i)}$ be the initial characters. Initial and adjacent cluster variables belong to C. Laurent finiteness gives

$$C[(\prod_{i\in J} x_i)^{-1}] = \Bbbk[x_i^{\pm 1}, y_i^{\pm 1} : i \in I].$$

For $i \in J$, let x_i' be the adjacent variable obtained by mutation at i. Its exchange relation is $x_i x_i' = F_i$, where

$$F_i = y_i \prod_{j:B_{ji}>0} x_j^{B_{ji}} + \prod_{j:B_{ji}<0} x_j^{-B_{ji}}.$$

Both products range over $j \in I$; an empty product is 1. Invert the other mutable variables. The intersection of the initial and ith adjacent Laurent rings is the rank-one algebra

$$R_i = \Bbbk[\text{other initial characters}^{\pm 1}, x_i, x_i'](x_i x_i' - F_i).$$

To see the equality, the coefficient of each x_i^{-j} must be divisible by F_i^j, which replaces that term by a multiple of $(x_i')^j$. This intersection contains the corresponding localization of C and is generated by elements of it. Principal coefficients make F_i linear in the independent y_i, with unit leading coefficient. Hence R_i is smooth and $x_i = x_i' = 0$ has codimension two. A divisor missed by the initial and all adjacent tori must therefore give positive order to at least two mutable x_i.

Lemma 6.1. *Suppose such a prime divisor exists in* $\operatorname{Spec} C$*, with normalized valuation* v*. For a nonzero rational function,* v *is its integer order of vanishing at the generic point of that divisor;* $v(0) = +\infty$*. Its character valuation is the homomorphism* $a : \mathsf{L} \to \mathbb{Z}$ *defined by* $a(m) = v(z^m)$*. Write* $a = ka_0 \in \mathsf{L}^*$*, where* a_0 *is primitive and* $k \geq 1$*. There are a primitive charge* $n_0 \in \mathbb{N}^J$ *supported on at least two indices, a character* $H = z^b$ *with* $b = Pn_0$*, and* $c \in \Bbbk^*$*, for which the following holds. Choose a character* $Z = z^u$ *with* $\langle a_0, u\rangle = 1$*, and complete* H *to coordinates* $H, S_1, \ldots, S_{d-2}$ *on the torus with character lattice* $\ker a_0$*, where* $d = \operatorname{rank} \mathsf{L}$*. Then*

$$v\left(\sum_r Z^r f_r(H,S)\right) = \min_{f_r \neq 0}\{kr + \operatorname{ord}_{H=c} f_r\} \tag{9}$$

for finite Laurent polynomials. Here $\operatorname{ord}_{H=c}$ *is the multiplicity of the factor* $H - c$ *over* $\Bbbk(S_1, \ldots, S_{d-2})$*.*

Proof. All coefficient and original frozen characters are units, so a is nonnegative and supported in mutable x-coordinates. At the generic point, normality gives a discrete valuation ring. It lies in the smooth locus in characteristic zero. Let τ be a uniformizer. Writing each character as $z^m = \tau^{ka_0(m)} u_m$ with a unit u_m gives

$$d\log z^m = ka_0(m)\frac{d\tau}{\tau} + d\log u_m.$$

The logarithmic expression for the symplectic form is obtained by inverting the character bracket matrix $-\Lambda$. For a valuation-zero function, an overline denotes its image in the residue

field of the valuation ring. The coefficient of $d\tau/\tau$ must vanish after taking residues, since the form is regular at this divisor. Contracting its logarithmic expression with a_0 gives the character direction $\mathbb{Q}\Lambda^{-1}a_0$. Here $(a_0)_x$ is the column of values of a_0 on the characters x_i, and it is zero at frozen indices. Formula (6) identifies the ray with the image under P of the mutable entries of $\Delta(a_0)_x$. Choose primitive n_0 on this ray and set $b = Pn_0$. Saturation of P makes b primitive; skewness gives $\langle a_0, b\rangle = 0$. The residue identity is $d\log \overline{H} = 0$. In characteristic zero over an algebraically closed field, it gives $\overline{H} = c \in \Bbbk^*$.

Primitivity of a_0 splits $\mathsf{L} = \mathbb{Z}u \oplus \ker a_0$. Every character in $\ker a_0$ has valuation zero, whereas $v(Z) = k > 0$. Thus the coordinate ring of the smooth model $X_0 = \mathbb{A}^1_Z \times \operatorname{Spec} \Bbbk[\ker a_0]$ embeds in the valuation ring. The inverse image of its maximal ideal defines the center, which lies in $(Z, H - c)$. Suppose its center has codimension c_0. At its generic point choose regular parameters $Z, t_2, \dots, t_{c_0}$ and lifts of a separating transcendence basis for the center. The torus volume is a valuation-zero unit times $dZ/Z \wedge dt_2 \wedge \cdots \wedge dt_{c_0}$ wedged with the remaining basis differentials. Indeed, if $t_i = \tau^{v(t_i)}u_i$, then

$$dt_i = \tau^{v(t_i)}\left(v(t_i)u_i\frac{d\tau}{\tau} + du_i\right).$$

In a nonzero wedge at most one factor contributes $d\tau/\tau$, since its wedge with itself is zero. The dZ/Z factor has no positive prefactor, while the other center parameters contribute $\tau^{v(t_i)}$. The remaining differentials are regular. This proves the following order bound for the initial logarithmic volume form Ω:

$$0 = \operatorname{ord}_v \Omega \geq \sum_{i=2}^{c_0} v(t_i) - 1 \geq c_0 - 2.$$

Here the left equality is Lemma 5.2. Thus $c_0 = 2$, the center is $(Z, H - c)$, and $F := H - c$ satisfies $v(F) = 1$.

Put $V = Z/F^k$. The exact identity

$$\frac{dZ}{Z} \wedge \frac{dH}{H} = -\frac{dF}{c + F} \wedge \frac{dV}{V}$$

shows that the order-zero coefficient of the volume contains $dF \wedge d\log \overline{V} \wedge d\log \overline{S}_1 \wedge \cdots \wedge d\log \overline{S}_{d-2}$. It is nonzero, so $\overline{V}$ is transcendental over $\Bbbk(\overline{S})$. If M is the minimum in (9), the residue after division by F^M is

$$\sum_{kr + \operatorname{ord}_{H=c} f_r = M} (F^{-\operatorname{ord}_{H=c} f_r} f_r)\Big|_{H=c} \overline{V}^{\,r}.$$

Distinct r give distinct powers of a transcendental element, so this sum is nonzero. This proves the formula. For a quotient of two Laurent polynomials of the same valuation, the displayed leading expressions identify its residue with an element of $\Bbbk(\overline{S}, \overline{V})$. Every valuation-zero rational function is a quotient of Laurent polynomials of the same valuation. Conversely the units S_i and V realize these generators in the residue field. Thus the residue field is exactly $\Bbbk(\overline{S}, \overline{V})$. □

7 Transport to the tangent diagram

Assume the divisor in Lemma 6.1 exists and retain its notation. The construction of n_0 in Lemma 6.1 gives $\Lambda b = \rho^{-1}a_0$ for a positive rational number ρ. Since Λ is alternating and $\Lambda P = R$,

$$\langle a_0, Pn\rangle = \rho(Pn)^{\mathsf{T}}\Lambda b = -\rho b^{\mathsf{T}}\Lambda P n.$$

Thus

$$\langle a_0, Pn\rangle = -\rho\langle Rn, b\rangle \qquad (n \in \mathbb{Q}^J). \tag{10}$$

Let $\mathfrak{D}_b$ be the tangent diagram: retain walls whose closed support W contains b and replace W by $W + \mathbb{R}b$. Its normals annihilate b. For a retained wall, write its supporting inequalities as $\eta_j(x) \geq 0$ within its hyperplane. Those satisfying $\eta_j(b) = 0$ define $W + \mathbb{R}b$; those with $\eta_j(b) > 0$ hold on a fixed compact set after translation by a sufficiently large multiple of b. The finitely many walls not containing b miss that translated set. Thus at every finite charge order a compact path in this diagram translates sufficiently far in the positive b direction to a path in $\mathfrak{D}$ with the same wall product. Applying this to a closed path proves that its product is the identity. Thus $\mathfrak{D}_b$ is consistent and positive. Its charge filtration is the original pointed filtration given by $\mathbb{N}^J$.

Choose a positive-chamber point q_0 simultaneously general for $\mathfrak{D}_b$, for $\mathfrak{D}$ at all endpoints $q_0 + lb$ with integers $l \geq 0$, and for the transport line $q_0 + \mathbb{R}b$. The countable exceptional-set construction in Section 2 applies, adding the proper subspaces $U + \mathbb{R}b$ for joints and hyperplanes parallel to b. Choose $m_* \in \mathsf{L}$ with $\langle a_0, m_*\rangle < 0$, and set

$$m_l = m_* + lb, \qquad q_l = q_0 + lb.$$

Since g is a maximum of finitely many linear functions, one linear piece h is active at every sufficiently large q_l, with $h(b) = g(b) > 0$. There is also a constant c_* such that $g(m_l) = lg(b) + c_*$ for all sufficiently large l.

Lemma 7.1. *For every broken line of input m_l ending at q_l, its total correction Pn satisfies*

$$h(Pn) \leq c_* - h(m_*). \tag{11}$$

There is a nonempty relatively open set of weights in $b^\perp \subset \mathsf{L}^*_{\mathbb{R}}$ for which these supports, after shifting by $-lb$, have uniformly finite sublevel sets.

Proof. Let $e = m_l + Pn$ be the final exponent. The initial asymptotic derivative of $g \circ \gamma$ is $-g(m_l)$. Its left derivative at zero is $-\max_{h' \text{ active at } q_l} h'(e)$. Convexity implies $h(e) \leq g(m_l)$ and gives (11).

Put $w = \ell - \ell(b)h/h(b)$. Then $w(b) = 0$, $\ell(b) > 0$, and

$$w(Pn) \geq |n|_1 - \frac{\ell(b)}{h(b)}(c_* - h(m_*)).$$

A sufficiently small perturbation of w in $b^\perp$ retains a positive linear coefficient in front of $|n|_1$, because $\|Pn\| \leq \|P\||n|_1$. Each bounded sublevel therefore permits only finitely many charges, with a bound independent of l. In particular, a fixed exponent class modulo $\mathbb{Z}b$ has only a fixed finite set of possible full exponents after the shift. □

Lemma 7.2. *For fixed $l > 0$, the wall product along $q_0 + tb$, $0 \leq t \leq l$, and its inverse converge in the completion of the valuation v and preserve v. Their actions on finite Laurent inputs agree with chargewise theta transport.*

Proof. If the path crosses a wall of charge n, then $\langle Rn, q_0\rangle + t\langle Rn, b\rangle = 0$ for some $0 < t \leq l$. Set $\epsilon_0 = \min_{i\in J}\langle Re_i, q_0\rangle > 0$. Then $\langle Rn, q_0\rangle \geq \epsilon_0|n|_1$. Formula (10) gives the positive integer

$$s(n) := \langle a_0, Pn\rangle \geq \frac{\rho\epsilon_0}{l}|n|_1 > 0. \tag{12}$$

Retain $F = H - c$ and $V = Z/F^k$ from Lemma 6.1. The rational function field is $\Bbbk(S, V)(F)$, because $H = c + F$ and $Z = VF^k$. Lemma 6.1 says that $V, S_1, \ldots, S_{d-2}$ have algebraically

independent residues. Consequently every nonzero element of $\Bbbk(S,V)$ has valuation zero, and F has valuation one. Completion is therefore the Laurent series field $\Bbbk(S,V)((F))$. Let $A=\Bbbk[H^{\pm1},S_1^{\pm1},\dots,S_{d-2}^{\pm1}]$. There is an injective map

$$A((Z)) \longrightarrow \Bbbk(S,V)((F)), \qquad \sum_{r\geq\mu} Z^r f_r(H,S) \longmapsto \sum_{r\geq\mu} V^r F^{kr} f_r(c+F,S). \tag{13}$$

Here $A((Z))$ consists of Laurent series bounded below in Z with coefficients in A. Negative powers of H expand as $(c+F)^{-a}=c^{-a}(1+F/c)^{-a}$ for $a>0$, using the formal binomial series; this is defined because $c\neq 0$. Thus the map is defined since each $f_r(c+F,S)$ has nonnegative F-order, and any fixed F-degree receives only finitely many terms. Its valuation is $\min_r(kr+\operatorname{ord}_{H=c} f_r)$: the lowest coefficient is a nonzero Laurent polynomial in the transcendental V. This proves injectivity as well.

In $A((Z))$, the notation $O(Z^j)$ means an element of $Z^jA[[Z]]$. A crossed wall changes Z and each S_i by a unit factor $1+O(Z^{s(n)})$, and sends F to $F+O(Z^{s(n)})$. If $ks(n)>1$, its associated graded action for v is the identity. If $ks(n)=1$, then $k=s(n)=1$ and that action is the invertible shear

$$Z\mapsto Z, \qquad F\mapsto F+\alpha(S)Z,$$

where $\alpha(S)$ is the coefficient obtained by evaluating the linear-in-Z correction to F at $H=c$. On the residue coordinate it sends V to $V/(1+\alpha(S)V)$, whose denominator is nonzero. Thus the wall and its inverse preserve v. Their difference from the identity increases relative valuation by at least $ks(n)-1$. By (12) this tends to infinity with charge. Fix a desired precision F^N. By (12), only bounded-charge walls can alter a rational input modulo that precision relative to its valuation. The remaining maps act identically on the required associated graded pieces. Interspersing the finite lower-charge maps preserves this estimate because those maps preserve v; equivalently one conjugates the higher-order maps by a finite valuation-preserving product. Thus the truncated path products form a Cauchy system. The reverse path gives a Cauchy system of inverses, and their composites are the identity at every finite precision. The limits are inverse valuation-preserving automorphisms.

For a finite Laurent input of minimum Z-degree μ, each output term is an input term plus a sum PN of crossed charges. Formula (12) gives $\langle a_0,PN\rangle \geq (\rho\epsilon_0/l)|N|_1$. Outputs of Z-degree at most R therefore use only bounded total charge. Their complete coefficients stabilize in a finite truncation. Consequently the chargewise output belongs to $A((Z))$, with finite Laurent coefficients and a finite negative-Z part. The same finite truncations define the valuation product; the map (13) sends a high-Z tail to a high-F tail. Their limits agree. □

The initial $\vartheta_{m_l,+}$ is an element of C, so has $v\geq 0$. Lemma 7.2 implies this for its expansion at q_l. In the common series ring its finite negative-Z part must be regular: its nonnegative-Z tail already has nonnegative valuation. Formula (9), or its series version in (13), therefore makes the coefficient of each Z^r, $r<0$, divisible by $(H-c)^{-kr}$. Write that finite coefficient as $f_r(H,S)=\sum_\beta c_{r,\beta}(H)S^\beta$. For $N\geq 1$,

$$A/((H-c)^N) = \big(\Bbbk[H^{\pm1}]/((H-c)^N)\big)[S_1^{\pm1},\dots,S_{d-2}^{\pm1}].$$

The S^β are a free basis over the displayed coefficient ring. Taking $N=-kr$ proves that every $c_{r,\beta}(H)$ is divisible by $(H-c)^{-kr}$.

Lemma 7.3. *For each fixed charge bound K, all sufficiently large integers l give a coefficient-preserving bijection between broken lines of input m_l and endpoint q_l in $\mathfrak{D}$, and broken lines of input m_* and endpoint q_0 in $\mathfrak{D}_b$, with total charge at most K. It is given by*

$$\beta(t)=\gamma_l(t)-l(1-t)b.$$

Proof. Writing the bends as Pa_j at $-u_j$ gives, for all $u \geq 0$,

$$\beta(-u) = q_0 + um_* + P\sum_j \min(u, u_j)a_j, \qquad \frac{\|\beta(-u)\|}{1+u} \leq \|q_0\| + \|m_*\| + \|P\|K.$$

Only finitely many walls affect this cutoff. For a closed wall cone not containing b, its distance from b is positive; the displayed bound shows that $l(1+u)b+\beta(-u)$ misses it for every $u \geq 0$ when l is large. For a wall containing b, write its face inequalities in its supporting hyperplane as $\eta_i(x) \geq 0$. The cylinder $W + \mathbb{R}b$ is defined by those with $\eta_i(b) = 0$. The inequalities with $\eta_i(b) > 0$ hold strictly after adding $l(1+u)b$, uniformly in u, for large l. Thus support membership is exactly the cylindrical support condition.

All retained normals annihilate b, so crossing powers and coefficients agree. Removing or adding the drift gives inverse finite bend records. The simultaneous generality of q_0 ensures that each reconstructed record avoids joints in the complete diagram, including higher-order walls, where one takes constant terms. This proves the full coefficient assertion. □

Every correction of $\mathfrak{D}_b$ has zero a_0-pairing by (10); hence all terms of its theta of input m_* have Z-degree $\langle a_0, m_*\rangle < 0$. Fix a class in $\mathsf{L}/\mathbb{Z}b$ and one of the weights from Lemma 7.1. Every shifted exponent in this class has the same weight. The coercive lower bound on that weight therefore supplies an integer K bounding all charges occurring in this class, uniformly in l. The same bound holds for the star series: each of its individual finite-charge contributions occurs in a sufficiently large l by Lemma 7.3, and hence satisfies the bound. Thus the entire class, on both sides, is decided by a single finite truncation. Lemma 7.3 then identifies the entire finite H-Laurent coefficient of $H^{-l}\vartheta_{m_l,q_l}$ with that of $\vartheta^{\mathfrak{D}_b}_{m_*,q_0}$ for sufficiently large l. Multiplication by H^{-l} preserves the $(H-c)$ divisibility because $c \neq 0$. We obtain

$$\vartheta^{\mathfrak{D}_b}_{m_*,q_0}\Big|_{H=c} = 0 \tag{14}$$

coefficientwise in projected characters. The weight bound also passes to this star theta, so its support has finite sublevel sets for a relatively open nonempty set of weights in $b^\perp$.

Finally, $\mathfrak{D}_b$ has no correction parallel to b. Such a correction would have charge parallel to n_0, by injectivity of P. Its original support contains b, so that original wall is incoming. But n_0 has at least two mutable indices in its support, whereas every incoming direction is initial. The positive representative with this property is supplied by the algorithm in [8, Appendix C]. Proposition 8.1 below contradicts (14).

8 Nonvanishing of proper theta series

For a support in L, a weight has *finite sublevel sets* if, for every real R, only finitely many full exponents have weight at most R. We use the word proper for this property. We retain the original charge filtration even when discussing exponents modulo a primitive direction.

Proposition 8.1. *Let a consistent positive rational conical scattering diagram be finite at each charge order, over a field $\Bbbk$ of characteristic zero, with injective charge map P and wall functions $(1+z^{Pn})^d$, $d \in \mathbb{Z}_{>0}$. Suppose its supports are cylindrical along a primitive $b \in \mathsf{L}$, its normals annihilate b, and no wall correction is parallel to b. Fix a simultaneously general endpoint as in Section 2. If a theta function has proper support for every weight in a nonempty relatively open subset of $b^\perp$, then its specialization at $z^b = c$ is nonzero for every $c \in \Bbbk^*$.*

Proof. Write $H = z^b$. Properness implies finite fibers modulo $\mathbb{Z}b$, since every such fiber has constant weight. Thus specialization is defined coefficientwise in projected characters.

First, positive formal multiplication holds for this diagram. Adjoin t_i, $i \in J$, and give a correction Pn the monomial t^n. At each endpoint,

$$\vartheta_{u,q} = z^u + \sum_{n \neq 0} c_{u,n}(q) z^{u+Pn} t^n, \qquad c_{u,n}(q) \in \mathbb{Z}_{\geq 0}.$$

Triangular elimination in total t-degree gives, for any fixed inputs $u_1, \ldots, u_k$,

$$\prod_{j=1}^{k} \vartheta_{u_j} = \sum_{n \in \mathbb{N}^J} A_n t^n \vartheta_{u_1 + \cdots + u_k + Pn}, \qquad A_0 = 1.$$

At a fixed charge there are finitely many decompositions, so elimination and the displayed sum are defined. Consistent theta transport shows that A_n is independent of the general endpoint. For any fixed product and output label a, choose the endpoint close enough to a that $\nu(q)\nu(a) > 0$ for every normal in the required finite truncation with $\nu(a) \neq 0$. A final backward ray $q + \mathbb{R}_{\geq 0} a$ cannot cross one of these hyperplanes. When $\nu(a) = 0$, the general ray is parallel and disjoint. There can be no final nontrivial bend for output exponent a, so coefficient extraction gives the theta coefficient as the corresponding positive character coefficient. In particular, for every positive integer K, $\vartheta_u^K - \vartheta_{Ku}$ is a theta-positive formal sum, with nonnegative auxiliary degrees.

Fix a weight w for which the support is proper. If ϑ_u has minimum w-weight μ, its Kth power has all weights at least $K\mu$. Positivity implies the same lower bound for ϑ_{Ku}. At a crossing with incoming exponent e, orient the primitive normal ν so that $\kappa = \nu(e) > 0$. Index the binomial factors at this crossing by a, writing them as $(1 + z^{v_a})^{d_a}$ with $d_a \in \mathbb{Z}_{>0}$. A selected correction has a finite witness

$$\delta = \sum_a \lambda_a v_a, \qquad 0 \leq \lambda_a \leq \kappa d_a, \tag{15}$$

where v_a are their correction directions and the λ_a are integers with finite support. A rational broken line allows rational exponents and rational λ_a with the same inequalities. Clearing all denominators by K and replacing $\gamma(t)$ by $\gamma(Kt)$ gives an ordinary line of input Ku and final exponent multiplied by K. Therefore every rational line of input u has weight at least μ. This uses only finitely many factors even at a crossing with infinitely many coplanar walls.

Consider a line with finitely many recorded events at $t_1 < \cdots < t_s < 0$, allowing a constant-term crossing among the events. Let $e_0 = u, e_1, \ldots, e_s$ be the exponents, $\delta_i = e_i - e_{i-1}$ the corrections, and ν_i the crossing-oriented normals. Define

$$\kappa_i = \nu_i(e_{i-1}) = \nu_i(e_i) > 0, \qquad a_i = \delta_i/\kappa_i, \qquad S_i = I + a_i \nu_i.$$

The product $a_i \nu_i$ denotes the rank-one map $x \mapsto a_i \nu_i(x)$. Since $\nu_i(\delta_i) = 0$, its square is zero. Thus $S_i^{-1} = I - a_i \nu_i$, and $e_i = S_i e_{i-1}$. These shears record the change of exponent and the reverse reconstruction of the path. For a final weight w, the weight transported back to event i is

$$w_i = w \circ S_s \cdots S_{i+1}.$$

Suppose an admissible rational correction δ_i' with finite witness satisfies $w_i(\delta_i') < w_i(\delta_i)$. Replace just a_i by

$$a_i(\varepsilon) = (1 - \varepsilon) a_i + \varepsilon \delta_i'/\kappa_i$$

for sufficiently small positive rational ε, and leave the other normalized a_j fixed. Define exponents forward from the same u. At a later event j, replace each witness coefficient by $\lambda_{j,a}(\varepsilon) = \kappa_j(\varepsilon)\lambda_{j,a}/\kappa_j$. The bounds $0 \leq \lambda_{j,a}/\kappa_j \leq d_a$ are unchanged. Continuity and the finitely many inequalities $\kappa_j(0) > 0$ ensure that all new pairings remain positive. Hence (15) continues to hold at every later event.

The endpoint remains exactly q under the following reverse reconstruction:

$$C_j(\varepsilon) = S_{j+1}(\varepsilon)^{-1} \cdots S_s(\varepsilon)^{-1}, \quad \gamma_j^{\varepsilon}(t) = C_j(\varepsilon)q - te_j(\varepsilon), \quad t_i(\varepsilon) = \frac{\nu_i(C_i(\varepsilon)q)}{\nu_i(e_i(\varepsilon))}.$$

Take $t_0(\varepsilon) = -\infty$, $t_{s+1}(\varepsilon) = 0$, and use segment j on $[t_j(\varepsilon), t_{j+1}(\varepsilon)]$ for $0 \le j \le s$, with the first interval open at $-\infty$. Empty products of shears are the identity. At each crossing,

$$\begin{aligned} &C_{i-1}(\varepsilon)q - t_i(\varepsilon)e_{i-1}(\varepsilon) \\ &\quad = C_i(\varepsilon)q - a_i(\varepsilon)\nu_i(C_i(\varepsilon)q) - t_i(\varepsilon)e_{i-1}(\varepsilon) \\ &\quad = C_i(\varepsilon)q - t_i(\varepsilon)\big(\kappa_i(\varepsilon)a_i(\varepsilon) + e_{i-1}(\varepsilon)\big) \\ &\quad = C_i(\varepsilon)q - t_i(\varepsilon)e_i(\varepsilon). \end{aligned}$$

Thus the adjacent segments coincide, the last segment ends at q, and the first retains input u. The denominators are nonzero, so strict time order persists. Take the finite union of the witnesses of all original corrections and of δ_i'. At the original crossing each used wall is met in its relative interior, since the original line avoids joints. Small perturbation preserves all these finitely many support conditions. The fixed endpoint's simultaneous generality makes the new rational record avoid every joint of the full diagram; unrecorded crossings take constant terms. Hence the perturbed path is legal. Its final exponent changes by

$$e_s(\varepsilon) - e_s(0) = \varepsilon S_s \cdots S_{i+1}(\delta_i' - \delta_i),$$

which strictly lowers its weight. It follows that any ordinary line realizing μ minimizes the transported weight at *every* crossing, including constant-term crossings. This is the tautness property used below.

Choose w in the given proper open region, avoiding the following countable family of hyperplanes in $b^{\perp}$. For every finite rational record, every suffix shear product T, and every wall direction v_a, exclude $w(Tv_a) = 0$. Each shear fixes b, so T is invertible, fixes b, and cannot send $v_a \notin \mathbb{R}b$ into $\mathbb{R}b$. These are proper hyperplanes. Also exclude $w(a) = 0$ for all $a \in \mathsf{L} \setminus \mathbb{Z}b$. Primitivity of b makes these proper as well. Then distinct projected lattice exponents have different weights, and all transported weights are nonzero on all wall directions.

Properness ensures that a minimum is attained: take any occurring exponent and minimize over the finite nonempty sublevel set below its weight. Positivity realizes a minimizing exponent by an ordinary broken line. At any crossing of such a line, tautness forces the unique correction

$$\delta_{\min} = \kappa \sum_{a : w_i(v_a) < 0} d_a v_a.$$

Indeed, any positive-weight factor with nonzero choice can be removed to lower the weight, and any negative-weight factor not at its maximum can be increased. Each change uses only one factor. Since the minimizing correction has a finite witness, there can be only finitely many negative-weight factors. The formula therefore defines a finite and unique complete correction.

Take two minimizing lines. Their final exponents differ by jb, $j \in \mathbb{Z}$, because their projected weights are equal. Reverse them from their common endpoint. Before a nontrivial bend, their positions differ by $-tjb$, so their projected paths coincide. Use the finite union of the nontrivial bend times of these two existing lines. Inductively, after all later corrections have been shown equal, the positions at the next such time differ by $-tjb$ and their exponents by jb. Cylindrical supports give the same wall factors; normals annihilating b give the same crossing powers. Their suffix shears, and hence transported weights, are equal. The unique minimizing correction above is therefore the same for both lines, including when one had chosen the constant term. Reversing it preserves the exponent difference jb and equality of suffix

shears. At the initial segments both inputs are u, so $j = 0$. If a final exponent is parallel to b, all normal pairings vanish, and the line must have been straight; this case gives the same conclusion.

Thus the lowest projected coefficient has exactly one complete exponent. It is aH^r with a a positive integer. Its value ac^r is nonzero in characteristic zero. Higher projected terms cannot cancel it, proving the proposition. □

9 Proof of the polytope criterion

Proof of Theorem 1.1. Full Fock–Goncharov implies Laurent finiteness of all theta functions. Proposition 4.2 then constructs a rational positive polytope with the required interior property.

Conversely, a polytope as in condition (2) gives Laurent finiteness by Proposition 3.2. Replace it by the rational polytope of Proposition 4.2. Lemmas 5.1 and 5.2 give a finitely generated normal theta algebra C and its regular symplectic form. If a prime divisor were omitted by the initial and adjacent seed tori, Lemma 6.1 would classify its valuation. Section 7 would then give a proper theta series in the cylindrical diagram $\mathfrak{D}_b$, with no correction parallel to b, whose specialization at $H = c$ vanishes. Proposition 8.1 contradicts this. Thus these seed tori cover every codimension-one point of $\operatorname{Spec} C$.

Every upper-cluster function is regular on these tori and therefore belongs to every height-one local ring of C. A normal Noetherian domain is the intersection of these local rings inside its fraction field, so $\operatorname{up}(\mathcal{A}_{\mathrm{prin}}) \subseteq C$. The reverse inclusion follows from universal Laurent finiteness. Principal theta independence gives the required injectivity, and all labels occur, proving full Fock–Goncharov over an algebraically closed field.

For a general field $\Bbbk$ of characteristic zero, extend scalars to an algebraic closure. Any given $f \in \operatorname{up}_{\Bbbk}$ remains regular on every seed torus and hence has a finite expression $f = \sum_{i=1}^{s} c_i \vartheta_{m_i}$ there. Comparison of initial Laurent coefficients gives a linear system over $\Bbbk$. Independence of the theta functions gives an invertible $s \times s$ coefficient minor, forcing all $c_i \in \Bbbk$. This descends the spanning assertion element by element.

If there are no mutable indices, all functions are characters and an ordinary box proves the statement. When principal coefficients are attached only to mutable indices, the extra coefficient variables in the complete convention are independent Laurent units. All seed tori and theta functions split by that torus factor. Finite Laurent expansion in these extra variables identifies the upper algebras coefficientwise. Positive polytopes transfer by product with a box and by the zero-coordinate section in the reverse direction. Thus the two principal coefficient conventions give the same criterion. □

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