The Möbius pseudodistance of the annulus and an improved lower bound for Korenblum's constant

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Abstract

We show that Korenblum's constant of the Bergman space satisfies $c_2 \geq 0.446$. The best published lower bound was $c_2 \geq 0.4263$ (Wikström, 2026). The new ingredient is a closed form, in Jacobi theta functions, of the Möbius pseudodistance of the annulus $\{c < |z| < 1\}$ between two points of the same centred circle. It separates the variables and yields three monotonicity properties, one of which, a profile inequality, combined with Kirszbraun's theorem in the hyperbolic plane, improves the constant in Wang's annular estimate. Wikström's dual moment criterion is then verified at $c = 223/500$ for an explicit measure with fourteen rational atoms, in rigorous arithmetic, by a self-contained verifier.

1 Result

Let $A^2(\mathbb{D})$ be the Bergman space of the unit disc. Korenblum's maximum principle [3, 2] asserts the existence of a constant $c \in (0,1)$ such that, for $f, g \in A^2(\mathbb{D})$, the inequality $|f(z)| \leq |g(z)|$ on $c < |z| < 1$ implies $\|f\|_{A^2} \leq \|g\|_{A^2}$. The supremum c_2 of the admissible constants is Korenblum's constant.

Theorem 1. $c_2 \geq 223/500 = 0.446$.

Reference	$c_2 \geq$
Hayman 1999 [2]	0.04
Wang 2025 [11]	0.3554
Wikström 2026 [12]	0.4263
Pinho, 19 October 2026, unrefereed [5]	0.432
This note, 21 October 2026	0.446

The best known upper bound is $c_2 < 0.6778994$ (Wang [10]). The French version of this note is dated 21 October 2026. A pull request to the registry of optimization constants (#274, opened in October 2026 by the GitHub user `vidigoat`) announces $c_2 \geq 0.446$ by a different argument (a flux lemma and a half-diameter enclosing lemma) [7]; it was not submitted by the author of the present note. R. Pinho has since obtained $c_2 \geq 0.44667$ (hexagon:2610.00696, 31 October 2026, unrefereed) [6], building on that pull request, which his note attributes to the present author.

Notation. Fix $0 < c < 1$ and the annulus $A_c = \{c < |z| < 1\}$. On $\mathbb{D}$, $\psi(z,w) = |z - w|/|1 - \bar{w}z|$ is the pseudo-hyperbolic distance and $d = 2\operatorname{artanh}\psi$ the hyperbolic distance (curvature -1), so that $\sinh(d/2) = \psi/\sqrt{1-\psi^2}$. The Möbius pseudodistance of the annulus

is $c^*(z_1,z_2)=\sup\{\psi(h(z_1),h(z_2))\}$, the supremum being over holomorphic maps $h:A_c\to\mathbb{D}$. For $c<r<1$ and $0\le\theta\le 2\pi$ we write $\delta_r(\theta)=c^*(r,re^{i\theta})$ and $D_r(\theta)=2\operatorname{artanh}\delta_r(\theta)$. The theta functions $\vartheta_1,\dots,\vartheta_4$ have nome c (conventions of [1, §20.2]).

2 Closed form of the pseudodistance on a circle

Schuster [8] gives $\delta_r(\theta)$ as an infinite product; the constant $K_c(r)$ of [12, (1)] is its value at $\theta=\pi$.

Theorem 2. *For $c<r<1$, $0<\theta<2\pi$, with $v=\theta/2$ and $y=\log(1/r)$,*

$$\sinh\frac{D_r(\theta)}{2}=\frac{\vartheta_3(iy)}{|\vartheta_1(iy)|}\cdot\frac{\vartheta_1(v)}{\vartheta_3(v)}.$$

Equivalently, if $k=(\vartheta_2(0)/\vartheta_3(0))^2$, $k'=(\vartheta_4(0)/\vartheta_3(0))^2$ and $\mathbf{K}=\frac{\pi}{2}\vartheta_3(0)^2$ (the complete elliptic integral of modulus k, so that $c=e^{-\pi\mathbf{K}'/\mathbf{K}}$),

$$\sinh\frac{D_r(\theta)}{2}=\frac{\operatorname{sd}(\mathbf{K}\theta/\pi,\,k)}{\operatorname{sd}\left(\frac{\mathbf{K}}{\pi}\log(1/r^2),\,k'\right)}.$$

In particular, with $t=r^2$ and $S(t)=\sinh(D_r(\pi)/2)=K_c(r)/\sqrt{1-K_c(r)^2}$,

$$S(t)=\frac{2\sqrt{t}}{1-t}\prod_{n\ge1}\frac{(1+c^{2n})^2}{(1-c^{2n-1})^2}\cdot\frac{(1+c^{2n-1}t)(1+c^{2n-1}/t)}{(1-c^{2n}t)(1-c^{2n}/t)},$$

a product all of whose factors are >0 for $c^2<t<1$.

Proof. Schuster's product reads $\delta_r(\theta)=\frac{c}{r}(1+\frac{r^2}{c})\,2\sin\frac{\theta}{2}\prod_{n\ge1}f_ng_n(\theta)$, with

$$f_n=\frac{(1+r^2c^{2n-1})(1+r^{-2}c^{2n+1})}{(1+c^{2n-1})^2},\qquad g_n(\theta)=\frac{|1-c^{2n}e^{i\theta}|^2}{|1-r^2c^{2n-2}e^{i\theta}|\,|1-r^{-2}c^{2n}e^{i\theta}|}.$$

The Jacobi product

$$\vartheta_1(z)=2c^{1/4}\sin z\prod_{n\ge1}(1-c^{2n})(1-c^{2n}e^{2iz})(1-c^{2n}e^{-2iz}),$$

taken at $z=v$ and then at $z=v+iy$ (where $e^{2iz}=r^2e^{i\theta}$ and $|\sin z|=|1-r^2e^{i\theta}|/(2r)$), gives $2\sin\frac{\theta}{2}\prod_n g_n(\theta)=\vartheta_1(v)/(r\,|\vartheta_1(v+iy)|)$. Since

$$\Big(1+\frac{c}{r^2}\Big)\prod_n f_n=\prod_n\frac{(1+r^2c^{2n-1})(1+r^{-2}c^{2n-1})}{(1+c^{2n-1})^2}=\frac{\vartheta_3(iy)}{\vartheta_3(0)},$$

we obtain

$$\delta_r(\theta)=\frac{\vartheta_3(iy)}{\vartheta_3(0)}\cdot\frac{\vartheta_1(v)}{|\vartheta_1(v+iy)|}.$$

The addition formula $\vartheta_3(0)^2\vartheta_1(u+v)\vartheta_1(u-v)=\vartheta_1(u)^2\vartheta_3(v)^2-\vartheta_3(u)^2\vartheta_1(v)^2$ [1, §20.7], taken at $u=iy$, where $\vartheta_1(iy-v)=-\overline{\vartheta_1(v+iy)}$ and $\vartheta_1(iy)^2=-|\vartheta_1(iy)|^2$, gives $\vartheta_3(0)^2|\vartheta_1(v+iy)|^2=\vartheta_3(iy)^2\vartheta_1(v)^2+|\vartheta_1(iy)|^2\vartheta_3(v)^2$, whence

$$\frac{1}{\delta_r(\theta)^2}-1=\frac{|\vartheta_1(iy)|^2}{\vartheta_3(iy)^2}\cdot\frac{\vartheta_3(v)^2}{\vartheta_1(v)^2},$$

which is the first formula. The second follows from $\vartheta_1(z)/\vartheta_3(z)=\frac{\vartheta_2(0)\vartheta_4(0)}{\vartheta_3(0)^2}\operatorname{sd}(\vartheta_3(0)^2z,k)$ [1, §22.2] and the imaginary transformation $\operatorname{sd}(iu,k)=i\operatorname{sd}(u,k')$ [1, §22.6], with $\vartheta_3(0)^2y=\frac{\mathbf{K}}{\pi}\log(1/r^2)$ and $\operatorname{sd}(\mathbf{K},k)=1/k'$. The third is the first at $v=\pi/2$, written with the products of $\vartheta_1,\dots,\vartheta_4$. □

Theorem 3. *Let $c < r < 1$.*

(i) $\theta \mapsto \delta_r(\theta)$ is strictly increasing on $[0,\pi]$; in particular $\max_\theta \delta_r(\theta) = \delta_r(\pi) = K_c(r)$.

(ii) For $0 \le \theta \le 2\pi$, $\sinh \frac{D_r(\theta)}{2} \le \sin\frac{\theta}{2} \cdot \sinh \frac{D_r(\pi)}{2}$.

(iii) $t \mapsto S(t)$ is strictly increasing on $[c,1)$, and $S(c^2/t) = S(t)$.

Proof. By the Jacobi products, for $0 < v \le \pi/2$, $\frac{\vartheta_1(v)}{\sin v} = 2c^{1/4}\prod_n (1-c^{2n})(1-2c^{2n}\cos 2v + c^{4n})$ is increasing in v, and $\vartheta_3(v) = \prod_n (1-c^{2n})(1+2c^{2n-1}\cos 2v + c^{4n-2})$ is decreasing. Hence $\vartheta_1(v)/\vartheta_3(v)$ is increasing, which gives (i) by Theorem 2, and $\vartheta_1(v)/(\vartheta_3(v)\sin v)$ is maximal at $v = \pi/2$, which gives (ii); the case $\pi \le \theta \le 2\pi$ follows by $\theta \mapsto 2\pi - \theta$. For (iii), if $c \le t < 1$, the argument $u' = \frac{\mathbf{K}}{\pi}\log(1/t)$ ranges over $(0, \mathbf{K}']$, since $\log(1/c) = \pi\mathbf{K}'/\mathbf{K}$; on this interval $\mathrm{sd}(\cdot, k')$ is strictly increasing (its derivative is $\mathrm{cn}\,/\,\mathrm{dn}^2 > 0$), so $S(t) = 1/(k'\,\mathrm{sd}(u', k'))$ is strictly increasing in t. The symmetry comes from the automorphism $z \mapsto c/z$ of A_c. □

Remark 4. Point (i) is what makes $K_c(r)$ the diameter, for c^*, of the circle $|z| = r$, a quantity that enters the annular estimates of [11, 12]. Schuster [8] publishes a truncated version of it; Theorem 3 gives a complete proof.

3 An improved annular estimate

Proposition 5. *Let $\omega : A_c \to \mathbb{D}$ be holomorphic, $c < r < 1$, and $\rho_0 = D_r(\pi)/2 = \operatorname{artanh} K_c(r)$. There is $p \in \mathbb{D}$ such that $d(\omega(re^{is}), p) \le \rho_0$ for every s.*

Proof. By the definition of c^* and the rotation invariance of A_c, $d(\omega(re^{is}), \omega(re^{is'})) \le D_r(|s - s'|)$. On a hyperbolic circle of centre o and radius ρ_0, two points with central angle ϕ are at distance d with $\sinh(d/2) = \sinh\rho_0 \sin(\phi/2)$. By Theorem 3(ii), the map sending the point of angle s of this circle to $\omega(re^{is})$ is therefore 1-Lipschitz for the hyperbolic distance. By Kirszbraun's theorem in the hyperbolic plane (Valentine [9]; see also Lang and Schroeder [4]), it extends to a 1-Lipschitz map defined also at o; the image p of o works. □

Proposition 6. *Let $f(z) = \sum a_k z^k$ and $g(z) = \sum b_k z^k$ be holomorphic in $\mathbb{D}$ with $|f| \le |g|$ on A_c. For $c < r < 1$ put $Y(r^2) = \sum_{k\ge 1,\ |a_k|<|b_k|} (|b_k|^2 - |a_k|^2) r^{2k}$ and $M_h(r)^2 = \frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^2\, d\theta$. Then*

$$Y(r^2) \le G_c(r)\big(M_g(r)^2 - M_f(r)^2\big), \qquad G_c(r) = \frac{1+\sqrt{1+S(r^2)^2}}{2} = \frac{1}{2}\Big(1 + \frac{1}{\sqrt{1-K_c(r)^2}}\Big).$$

Proof. We follow Wang's argument [11] (see [12, §2]). The quotient $\omega = f/g$ is holomorphic on A_c (removable singularities) and $|\omega| \le 1$; if $|\omega|$ attains 1, then ω is a unimodular constant, $|f| = |g|$ everywhere and the statement is immediate. Otherwise ω takes values in $\mathbb{D}$. For complex $\alpha \in \mathbb{D}$, the identity $|b - \alpha a|^2 - |a - \bar\alpha b|^2 = (1-|\alpha|^2)(|b|^2 - |a|^2)$ and Parseval's identity give $Y(r^2) \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|g-\alpha f|^2}{1-|\alpha|^2}(re^{i\theta})\, d\theta$. Now $\frac{|1-\alpha\omega|^2}{1-|\alpha|^2} = \frac{1-|\omega|^2}{1-\psi(\omega,\bar\alpha)^2}$. With $\alpha = \bar p$, p given by Proposition 5, we have $\psi(\omega, p) \le \tanh(\rho_0/2)$ on $|z| = r$, hence $|g - \alpha f|^2/(1-|\alpha|^2) \le (|g|^2 - |f|^2)\cosh^2(\rho_0/2)$; and $\cosh^2(\rho_0/2) = \frac{1+\cosh\rho_0}{2} = G_c(r)$ since $\tanh\rho_0 = K_c(r)$. □

Wang's constant is $\frac{1}{2}\big(1 + \sqrt{(1+3K^2)/(1-K^2)}\big) = \frac{1}{2}\big(1+\sqrt{1+4S^2}\big)$, which is strictly larger.

4 The certificate

Wikström [12, Lemma 3.1] shows that c is admissible as soon as there is a probability measure μ on $[c^2,1]$, $\mu(\{1\})=0$, with $\int t^k\,d\mu \ge \frac{1}{k+1}$ for $k\ge 0$ and $\int \beta_c(\sqrt{t})\,t^k\,d\mu \le \frac{1}{k+1}$ for $k\ge 1$, where $\beta_c = 1-1/G_c$ and $\beta_c(c)=1$. His proof uses the annular estimate only through the inequality of Proposition 6; it therefore applies verbatim with our G_c.

Take $c = 223/500$, $R = 99/100$ and $\mu = \sum_{j=1}^{14} w_j\delta_{t_j} + \mathbf{1}_{[R,1]}(t)\,dt$, with the following exact decimal numbers ($t_1 = c^2$, $t_{14} = R$, $\sum w_j = R$, so μ is a probability measure).

j	t_j	w_j
1	0.198916	0.42261186256165887
2	0.60895666661691	0.12853389969171364
3	0.61180781079179	0.20469255809899036
4	0.8374998912126	0.0123813743289724
5	0.83868469913294	0.13096698358770436
6	0.93197625792065	0.04984739013621153
7	0.93247222724023	0.00720134704444788
8	0.96936285312632	0.0107784329378477
9	0.9695862322646	0.01227418071154803
10	0.98448736353123	0.00816617890945166
11	0.98482420908651	0.00072802785539435
12	0.98939709796742	0.00144706003927696
13	0.98955114839119	0.00003421625939472
14	0.99	0.00033648783738754

Theorem 7. *For this measure,* $A_k = \sum_j w_j t_j^k - \frac{R^{k+1}}{k+1} > 0$ *and* $B_k = \frac{R^{k+1}}{k+1} + \int_R^1 \frac{t^k}{G_c(\sqrt{t})}\,dt - \sum_j w_j\beta_c(\sqrt{t_j})\,t_j^k > 0$ *for every* $k\ge 1$*; for* $k=0$ *the first condition is the equality* $\mu([c^2,1]) = 1$.

Proof. Wikström's two conditions read $A_k \ge 0$ and $B_k \ge 0$. They are checked directly for $1\le k\le 2943$ (family A) and $1\le k\le 1400$ (family B), with upper bounds for S obtained from the product of Theorem 2 (the remainder of the product being bounded explicitly) and, on $[R, 0.99999]$, a grid of 34 536 pieces on which $1/G_c$ is bounded below by its value at the right endpoint (Theorem 3(iii)) and $\int t^k$ by the midpoint rule (convexity). Beyond these ranges: $w_{14}(k+1) \ge R$ for $k\ge 2943$, hence $w_{14}t_{14}^k \ge R^{k+1}/(k+1)$; and, with $[a,b]\subset[0.998, 0.999]$ and $m = \min_{[a,b]} 1/G_c$, the inequality $m(b-a)a^k > \sum_j w_j t_j^k$, true at $k=1400$, propagates to every $k\ge 1400$ because $a > R \ge t_j$, and implies $B_k > 0$. The smallest relative margins are $6.93\cdot 10^{-6}$ for $A_k(k+1)/R^{k+1}$ (at $k=407$) and $2.43\cdot 10^{-6}$ for $B_k(k+1)/R^{k+1}$ (at $k=652$). □

Theorems 7 and 3, Proposition 6 and [12, Lemma 3.1] give Theorem 1.

5 Status

- **PROVED**: Theorems 2 and 3, Propositions 5 and 6, under the cited results (Schuster's product for c^*, classical formulas for theta and elliptic functions, the Kirszbraun–Valentine theorem, Wikström's dual lemma).
- **VERIFIED** in rigorous arithmetic: Theorem 7, hence Theorem 1. The ancillary verifier gives the margins above; a second, independently written program in ball arithmetic (Arb), not included here, gives the same margins. The identities of Theorem 2 were also checked numerically to more than 40 digits.
- **OPEN**: the value of c_2, between 0.446 and 0.6778994. The result has not been reviewed by anyone else.

6 Verification

The file `anc/VERIF_KORENBLUM_0446.py` (Python 3 with numpy, no argument and no external file, a few seconds) contains the exact data of the table. It computes upper bounds for S in exact integers with directed rounding, the moment sums in IEEE floating point with an explicit error bound, calibrates itself on the seven upper bounds for β_c published in [12, §5], and then checks A_k and B_k as above.

Credits

The maximum principle is due to Korenblum [3] and Hayman [2]; the product for the Möbius pseudodistance of the annulus to Schuster [8]; the annular estimate by splitting of the coefficients to Wang [11]; the dual moment criterion, and the form of the measures used, to Wikström [12]. The extension theorem is that of Kirszbraun and Valentine [9]. The computations were carried out with the assistance of Claude (Anthropic).

References

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