Projected stated SL_n-skein algebras and the Huang–Wang conjectures

Qiyue Tang

Abstract

We prove Conjectures 1.5 and 1.6 of Min Huang and Zhihao Wang: the projected stated SL_n-skein algebra equals its quantum cluster and upper cluster algebras, and the specified exchangeable variables generate it over the frozen Laurent coefficient algebra. The results hold for triangulable unpunctured marked surfaces with at least two boundary marked points per connected component, over every coefficient domain and parameter in the conjectures.

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The main results of this paper were generated in a ChatGPT-6-Pro conversation lasting approximately 262 minutes of elapsed time. The author supplied the research materials. Codex subsequently reviewed the arguments, checked references, and revised the exposition. The manuscript was also reviewed using Rethlas–OSV [OSV], which implements the OSVerify framework of Nie, Ye, Liu, Wang, Ermon, and Zou [NLYWEZ].

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1 Introduction

Cluster coordinates and skein theory give two descriptions of functions on moduli spaces of surface local systems. Cluster coordinates describe these functions in overlapping algebraic tori, whereas skein theory represents them by diagrams and local relations. For a quantum cluster algebra, the Laurent phenomenon gives an inclusion in the intersection of all quantum seed tori, its upper cluster algebra. Identifying this intersection with a diagram algebra requires both a construction of cluster variables by diagrams and a converse statement expressing every universally Laurent element by diagrams. This paper establishes that identification for the projected stated SL_n-skein algebras in the conjectures of Huang and Wang.

Fomin and Zelevinsky introduced cluster algebras to study canonical bases and total positivity [FZ]. Berenstein, Fomin, and Zelevinsky introduced upper cluster algebras and proved equality with cluster algebras under acyclicity hypotheses [BFZ]. Berenstein and Zelevinsky subsequently developed quantum seeds, compatible skew forms, quantum mutation, and the quantum Laurent phenomenon [BZ]. These constructions distinguish the algebra generated by cluster variables from the intersection of Laurent coordinate algebras; equality is a further assertion about the seed.

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Fock and Goncharov constructed coordinate systems on moduli spaces of decorated and framed local systems, extending the coordinates of Teichmüller theory to higher rank [FG]. For SL_n, an ideal triangulation is refined into an n-triangulation. Its small vertices index coordinates, and its directed small edges give the exchange matrix. A flip of an ideal edge is represented by a sequence of mutations rather than a single mutation when $n > 2$. For SL_2, Fomin, Shapiro, and Thurston described the cluster combinatorics of marked surfaces in terms of arcs and triangulations [FST]. These geometric coordinate systems supply the seeds considered here.

Bonahon and Wong constructed the SL_2 quantum trace, relating the skein quantization of the character variety to quantum Teichmüller coordinates [BW]. Muller introduced a skein algebra whose diagrams may end at marked boundary points and proved that, after inverting boundary elements, it lies between the quantum cluster algebra and the quantum upper cluster algebra. For a triangulable surface with at least two marked points in each connected component, he proved equality of all three algebras [Muller]. This result is the SL_2 precedent for the equality studied below.

Stated skein algebras place endpoints on open boundary intervals and label them by states. The states make cutting along ideal arcs compatible with finite sums over the newly created endpoints. Lê established triangular decomposition for the stated SL_2 theory [Le]; Higgins developed the corresponding stated SL_3 theory, including injective splitting maps and a triangular decomposition [Higgins]. Lê and Sikora defined stated SL_n-skein modules and algebras for arbitrary n, together with the boundary relations, quantum-group coactions, and decomposition along saturated arc systems [LS]. Their ordinary and transmuted quantum matrix factors will provide the finite filtration used in our proof.

Lê and Yu constructed higher-rank quantum traces in both $\mathcal{X}$- and $\mathcal{A}$-coordinates [LY]. Their full quantum trace is injective on the unpunctured surfaces considered here. They also defined the reduced skein algebra by killing bad corner arcs and constructed its quantum trace, whose image is the projected algebra. Initial $\mathcal{A}$-coordinates are represented by explicit weighted arcs and tripods. The distinction between the full algebra and this projected image is part of the proof: the full algebra supplies matrix-factor coordinates, while the projected image is the object in the Huang–Wang conjectures.

Several results address the cluster side of this comparison. Ishibashi and Yuasa constructed quantum cluster structures related to $\mathfrak{sl}_3$ webs and proved the inclusion of the boundary-localized skein algebra in the corresponding quantum cluster algebra [IY]. In the classical setting, Ishibashi, Oya, and Shen proved $\mathcal{A} = \mathcal{U}$ for the relevant moduli spaces of G-local systems on unpunctured surfaces with at least two marked points, and identified the upper algebra with regular functions on the generic decorated moduli space [IOS]. Their use of pinnings and Wilson lines is also suited to the classical full-skein comparison. Goncharov and Shen's work on local-system coordinates and Donaldson–Thomas transformations supplies the coordinate conventions and the mutation sequence used below [GS19, GS16].

Huang and Wang studied the projected stated algebra inside the $\mathcal{A}$-quantum torus and constructed cluster expressions for elementary stated arcs [HW]. Their Theorems 1.2 and 1.3 establish its inclusions in the quantum cluster and upper cluster algebras, respectively, under the stated surface hypotheses. In Conjecture 1.5 they asked whether the projected algebra, the quantum cluster algebra, and the quantum upper cluster algebra coincide. Conjecture 1.6 specifies generators of the upper algebra over the Laurent algebra of frozen variables: the exchangeable variables appearing in their elementary-arc mutation formulas. Both conjectures concern triangulable surfaces without interior punctures and with at least two boundary marked points in each connected component. Their coefficient ring is an arbitrary commutative domain with the indicated invertible quantum parameter.

Cao, Huang, and Wang subsequently proved the projected and full, boundary-localized equalities for polygons, and constructed rotation-invariant bases in that setting. They also established the full-skein equality for $n = 2$ on surfaces with the corresponding marked-point hypothesis [CHW]. These results settle substantial special cases and give the full-seed matrices used in our comparison. Kim and Wang proved naturality of the higher-rank quantum traces under changes of triangulation [KW].

Their network transformations, together with the complete flip word of [SS, FG], allow local formulas to be compared across adapted triangulations.

We prove both Huang–Wang conjectures in their original range. The marked-point requirement is imposed on each connected component of the surface; an individual boundary circle may have a single marked point. All frozen variables in the projected cluster algebra are inverted.

Theorem 1.1. *Let $n \geq 2$ and let Σ be a triangulable marked bordered surface without interior punctures, with at least two boundary marked points in each connected component. Let R be a commutative domain, choose $v \in R^{\times}$, and set $\omega = v^2$. Then*

$$\widetilde{\mathscr{S}}_{\omega}(\Sigma) = \mathcal{A}_{\omega}(\Sigma) = \mathcal{U}_{\omega}(\Sigma).$$

Over the Laurent algebra of frozen variables, this algebra is generated by the exchangeable variables specified in Huang–Wang Conjecture 1.6.

Here $\widetilde{\mathscr{S}}_{\omega}(\Sigma)$ is the image of the reduced quantum trace, and $\mathcal{A}_{\omega}(\Sigma)$ and $\mathcal{U}_{\omega}(\Sigma)$ are the cluster and upper cluster algebras of its projected seed. Section 2 defines these objects and fixes the normalizations. Section 8 identifies the generating family with the one in the original conjecture, including the extremal arc entry recovered by a reverse-arc cofactor identity.

Proof strategy. The first part of the proof expresses elementary arc coefficients by mutation words on triangles and quadrilaterals. Cutting and gluing then give the inclusion of the projected skein algebra in the cluster algebra. The local calculations retain all endpoint cases and all occurrences of a boundary mark. They also produce the frozen units needed for the passage to the original generating family.

For the reverse inclusion, we use the full stated skein algebra. Attaching a triangle to each boundary interval embeds its initial torus into a larger projected torus. In the classical fiber, a collar correspondence with decorated local systems identifies the frozen localization of the full algebra with the full upper algebra. Primality of the individual corner minors removes the frozen poles. A fixed saturated arc system then gives finite free filtered pieces. We prove that the initial mutable webs retain their degrees at the classical specialization, using reduced arc and tripod representatives and a nonvanishing calculation on products of determinant cones. This strictness makes each bounded-degree upper lattice finite, so Nakayama's lemma lifts the classical equality to the generic quantum parameter.

Finally, injectivity of the full trace in prime fibers gives integral descent. Positive quantum theta functions [DM], the surface Donaldson–Thomas transformation [GS16], and the pointed-basis theorem of [Qin, OQY] give a basis of the projected upper algebra over arbitrary coefficient rings. Specializing finite skein preimages of this basis proves the asserted equality for every coefficient domain and parameter in the conjectures. Sections 3–8 carry out these steps; Appendix A records the complete local exchange rows.

2 Skein algebras and quantum seeds

2.1 Surface and seed conventions

A marked bordered surface is obtained from a compact oriented surface by deleting a finite set of boundary points meeting every boundary circle. A boundary interval is a connected component of the remaining boundary. Ideal arcs join deleted points; boundary arcs have endpoints on boundary intervals. An ideal triangulation cuts the surface into triangles with their vertices deleted. We exclude the monogon and bigon, as in [HW]. The ends of a based ideal arc, and the occurrences of a marked point in different triangle corners, remain distinct incidences.

For a triangle whose vertices are clockwise ordered, its small vertices have coordinates

$$(a,b,c)\in\mathbb{Z}^3_{\geq 0},\qquad a+b+c=n,\qquad (a,b,c)\notin\{(n,0,0),(0,n,0),(0,0,n)\}.$$

Draw arrows with increments $(-1,1,0),(0,-1,1),(1,0,-1)$ whenever both ends are small vertices. An arrow on the triangle boundary has weight $1/2$; all other arrows have weight one. Glue the small vertices on identified ideal edges and add signed arrow weights. Let V_λ be the resulting vertex set. Its mutable subset consists of vertices off the physical boundary, and the remaining vertices are frozen. Write

$$Q_\lambda(u,w)=\mathrm{wt}(u\longrightarrow w)-\mathrm{wt}(w\longrightarrow u).$$

The mutable rows and columns are integral. Define $H_\lambda(u,w)=Q_\lambda(u,w)$ when u,w are not on the same boundary interval. On a common boundary interval set $H_\lambda(u,u)=1$, set $H_\lambda(u,w)=-1$ if the boundary arrow goes from w to u, and set it to zero otherwise. With inverses taken over $\mathbb{Q}$, put

$$K_\lambda=nH_\lambda^{-1},\qquad P_\lambda=2K_\lambda Q_\lambda K_\lambda^T.$$

The matrix K_λ is integral, P_λ is integral and skew-symmetric, and $P_\lambda Q_\lambda(-,k)=-2n^2e_k^T$ for mutable k [LY, Lemma 11.9 and Theorem 11.7]. Here e_k is the row basis vector at k, $Q(-,k)$ is a column, and T denotes transpose. These formulas specify the original projected seed, including its frozen block.

For a finite seed index set V_t and its skew form P_t, the based quantum torus is the free R-module with basis $\{M_t(a):a\in\mathbb{Z}^{V_t}\}$ and multiplication

$$M_t(a)M_t(b)=v^{aP_tb^T}M_t(a+b).$$

We denote this torus by $T_{t,R}$. Its coordinate $x_j=M_t(e_j)$ is invertible. A fixed order on V_t gives

$$M_t(a)=v^{-\sum_{i<j}a_ia_jP_t(i,j)}\prod_j x_j^{a_j}.$$

Square brackets around a product of coordinates of one seed mean this Weyl normalization. More generally, if $XY=v^{2d}YX$, then $[XY]=v^{-d}XY$. For a row $q_k=Q_t(k,*)$, write $q_k=q_k^+-q_k^-$ with disjoint nonnegative parts. Mutation at k replaces x_k by

$$x_k'=M_t(-e_k+q_k^+)+M_t(-e_k+q_k^-)$$

and fixes the other coordinates. Matrix mutation changes the sign of row and column k and, for $i,j\neq k$, replaces $Q_t(i,j)$ by

$$Q_t(i,j)+[Q_t(i,k)]_+[Q_t(k,j)]_+-[-Q_t(i,k)]_+[-Q_t(k,j)]_+,\qquad [r]_+=\max(r,0).$$

Let C be the identity matrix with row k replaced by $-e_k+q_k^+$. The mutated skew form is CP_tC^T. The notation $t\sim\lambda$ means that t is reached from the initial seed by a finite sequence of mutable mutations. All tori are regarded inside their common division ring over $\mathrm{Frac}(R)$. The cluster algebra is generated by all mutable cluster variables and the frozen coordinates and their inverses. The upper algebra is the intersection of all these seed tori over R.

The based X-torus has monomials $M_{\mathcal{X}}(b)$, $b\in\mathbb{Z}^{V_\lambda}$, with multiplication $M_{\mathcal{X}}(b)M_{\mathcal{X}}(c)=v^{2bQ_\lambda c^T}M_{\mathcal{X}}(b+c)$. Its balanced lattice consists of vectors whose restriction to each triangle belongs to the subgroup generated by the three barycentric-coordinate functions and $n\mathbb{Z}^{V_\tau}$. The A-torus monomials may also be denoted $M_{\mathcal{A}}(a)$. The lattice isomorphism in (3.1) identifies the A-torus with the balanced X-subtorus.

For integers a,b, $[a,b]$ denotes the increasing list of integers from a to b, empty if $a>b$. If I,J are increasing lists of the same size, $X_{I,J}$ is the submatrix with those rows and columns; its classical

minor is $\det X_{I,J}$. The empty minor is one. For a quantum matrix with parameter q (in our skein convention, $q = q_{\mathrm{LY}} = v^{-2n^2}$), the row-ordered quantum minor is

$$\det_q X_{I,J} = \sum_{\sigma \in \mathfrak{S}_d} (-q)^{\ell(\sigma)} X_{i_1, j_{\sigma(1)}} \cdots X_{i_d, j_{\sigma(d)}}, \qquad d = |I| = |J|,$$

where $\ell(\sigma)$ counts inversions. The group $\mathfrak{S}_d$ consists of the permutations of $\{1, \dots, d\}$. The notation $M_J^I(c)$ for a stated arc denotes its quantum minor with states I at the row side and J at the column side. We set $\bar{i} = n + 1 - i$. The indicator $\mathbf{1}_{\mathcal{C}}$ is one when condition $\mathcal{C}$ holds and zero otherwise. The Kronecker symbol δ_{ab} is one for $a = b$ and zero otherwise. Throughout, $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

2.2 Three skein algebras

Fix an integer $n \geq 2$ and a surface Σ as above. Choose an ideal triangulation λ. A stated n-web is a framed oriented graph in $\Sigma \times (-1, 1)$ with univalent boundary vertices and n-valent sources or sinks, cyclically ordered at each interior vertex. Endpoints on each boundary wall have distinct heights and carry states in $\{1, \dots, n\}$. Closed oriented components are allowed. The full stated skein algebra is the free R-module on their isotopy classes, modulo the crossing, framing, determinant, evaluation, coevaluation, and boundary exchange relations of [LS, Section 4]; multiplication stacks the first web above the second. We use precisely the normalization of [LY, Section 4.2, (64)–(71)], with the parameter conversion in (2.4).

A corner arc cuts off one deleted boundary point. Choose an orientation-preserving planar picture of its corner sector with the marked point at the top and the surface below it. The positional indices i, j are the states at the left and right sides, respectively, for both orientations of the arc. The forward arc points left to right and the reverse arc right to left; thus positional indices need not be start–end indices. This is the convention of [LY, Section 7.1, Figure 7]. A bad corner arc has $i < j$. Let I_{bad} be the two-sided ideal generated by these arcs at all corners. Killing them makes both positional corner matrices lower triangular. The reduced quantum trace $\mathrm{tr}^{\mathcal{A}}_{\lambda,R}$ is the R-algebra map of [LY, Theorem 13.1], computed by cutting a web into triangles, summing over matched states on the cuts, and applying the triangle trace and the balanced $\mathcal{A}/\mathcal{X}$ coordinate map. Its target is the based torus $T_{\lambda,R}$ defined above. We write

$$B_R(\Sigma) = \mathscr{S}_R(\Sigma), \qquad \mathscr{S}_R^{\mathrm{red}}(\Sigma) = \mathscr{S}_R(\Sigma)/I_{\mathrm{bad}}, \qquad S_R(\Sigma) = \mathrm{im}\big(\mathrm{tr}^{\mathcal{A}}_{\lambda,R} : \mathscr{S}_R^{\mathrm{red}}(\Sigma) \to T_{\lambda,R}\big). \tag{2.1}$$

Here B_R is the **full stated skein algebra**, whereas S_R is the **projected reduced trace image**, denoted $\widetilde{\mathscr{S}_\omega}$ by HW.

Denote the quantum cluster algebra and upper cluster algebra of the projected seed by

$$\mathcal{A}_R(\Sigma), \qquad \mathcal{U}_R(\Sigma) = \bigcap_{t \sim \lambda} T_{t,R}. \tag{2.2}$$

All frozen variables are inverted. Thus $\mathbb{B}_{\mathrm{fr},R} = R\langle x_f^{\pm 1} : f \text{ frozen}\rangle$ is their common coefficient algebra.

The *full seed* of [LY, CHW] is constructed by attaching a triangle to each boundary interval. In an attached triangle, label the original interface e_1, the exterior side omitted from the full seed e_2, and the other exterior side e_3. Delete the small vertices on e_2; the remaining small vertices form the full index set. Write Mut for its mutable subset. Its frozen set H consists of the vertices on the sides e_3. The coordinate shear (4.1) specifies its torus inside the enlarged projected torus, and the mutable exchange columns are the columns transported by that shear. For this seed and all its mutations, let V_t denote the full index set and M_t its based monomials. Define

$$T^{H+}_{t,R} = \bigoplus_{a\in\mathbb{Z}^{V_t},\ a_h\geq 0\ (h\in H)} R\,M_t(a), \qquad U^{+}_{\mathrm{full},R} = \bigcap_t T^{H+}_{t,R}, \qquad U^{\mathrm{fr}}_{\mathrm{full},R} = \bigcap_t T_{t,R}. \tag{2.3}$$

The full frozen set H differs from the frozen set of the projected seed. Some elements of H vanish in the reduced quotient, so localization at H cannot precede this quotient.

2.3 Integral parameters

For universal formulas, v is an indeterminate; specialization sends it to the chosen unit in the coefficient domain. Set

$$A = \mathbb{Z}[v^{\pm 1}], \qquad v = \omega^{1/2} = \widehat{q}_{\mathrm{LY}}^{-1}, \qquad q_{\mathrm{LY}} = v^{-2n^2}. \tag{2.4}$$

The convention $q = \widehat{q}^{\,2n^2}$ is [LY, Section 2.2]. We use HW's matrix conventions. For the barred reduced matrices of LY, the dictionary is

$$Q_{\mathrm{HW}} = -\bar{Q}_{\mathrm{LY}}/2, \qquad P_{\mathrm{HW}} = -\bar{P}_{\mathrm{LY}}, \qquad K_{\mathrm{HW}} = \bar{K}_{\mathrm{LY}}, \qquad H_{\mathrm{HW}} = \bar{H}_{\mathrm{LY}}. \tag{2.5}$$

In particular, H receives no further sign or scaling. Away from a common boundary edge, LY has $H = -\bar{Q}_{\mathrm{LY}}/2$; on that edge the diagonal entries are one, with -1 on an arrow $u \to v$. HW reverses the arrows and places -1 on $v \to u$, giving the same H. Consequently $HK = nI$, and the mutable rows satisfy $H = Q_{\mathrm{HW}}$. The based A-multiplication $\widehat{q}_{\mathrm{LY}}^{\bar{P}_{\mathrm{LY}}}$ becomes $v^{P_{\mathrm{HW}}}$, and the based X-multiplication is $v^{2Q_{\mathrm{HW}}} = \omega^{Q_{\mathrm{HW}}}$. These conventions follow [LY, Section 11.7, Lemma 11.9] and [HW, Remarks 2.4 and 2.7]. HW PDF equations (51)–(53) are HTML equations (57)–(59); we use the PDF numbering.

In BZ conventions we take $q_{\mathrm{BZ}}^{1/2} = v$ and $\Lambda = P$, with compatibility diagonal $2n^2 I$. When a source uses the opposite exchange matrix, we specify the corresponding skew form and parameter simultaneously.

The filtered lifting argument uses only

$$R_0 = \mathbb{C}[v^{\pm 1}]_{(v-1)}, \qquad p = v - 1, \qquad K_0 = \mathbb{C}(v). \tag{2.6}$$

The denominators of normalized exterior projectors are units in this discrete valuation ring. Integral descent and upper-algebra base change will give the result over arbitrary coefficient domains, including positive characteristic and roots of unity.

2.4 Conventions for maps and specializations

The bar map on the universal based torus sends v to v^{-1}, fixes $M_t(a)$, and reverses products. The corresponding skein reflection switches crossings and reverses boundary heights [LY, Proposition 4.11]. A based map sends each based monomial to a based monomial with coefficient one. For a lattice matrix E acting on row exponents, its action is $M(a) \mapsto M(aE)$. A pullback of a function by a map f is composition with f; pulling an exponent back to a triangle means evaluation at its image small vertex.

When the coefficient ring is clear, subscripts on algebras and traces are omitted. A subscript 0 denotes the classical fiber $v = 1$ over $\mathbb{C}$; a subscript K_0 denotes extension to $\mathbb{C}(v)$. The notation Frac means the ordinary fraction field in the commutative setting and the Ore division ring in the noncommutative setting. For a Laurent expansion with finitely many terms, its support is the set of exponents with nonzero coefficients; coefficient extraction means taking the coefficient of the specified based monomial. These conventions also apply to the auxiliary Laurent variables in Section 4.3.

3 Local formulas and projected skein elements

3.1 Cutting and mutation

Proposition 3.1. *The canonical cutting map is an integral based embedding of the initial A-tori. Along every mutation word avoiding the cut directions it carries each cluster variable to the corresponding cut-surface variable times a based monomial in the cut frozen variables. The A-coordinate systems are natural under complete flips.*

Proof. The A/X correspondence and the cutting map. By [LY, Theorem 11.7, Lemma 11.9(c), Proposition 11.10], the integral matrices K_λ, H_λ satisfy

$$H_\lambda K_\lambda = K_\lambda H_\lambda = nI, \qquad \psi_\lambda(M_{\mathcal{A}}(a)) = M_{\mathcal{X}}(aK_\lambda). \tag{3.1}$$

The balanced X-lattice consists precisely of exponents for which bH_λ/n is integral. The inverse is $M_{\mathcal{X}}(b) \mapsto M_{\mathcal{A}}(bH_\lambda/n)$. The balanced-lattice condition makes the displayed exponent integral over every coefficient ring.

Cut Σ along a collection of ideal edges. Let D duplicate the X-exponent at each cut edge and preserve every other exponent. Balancedness is defined triangle by triangle, so duplication preserves it. The X-cutting map is the monomial map $S_{\mathcal{X}}(M_{\mathcal{X}}(b)) = M_{\mathcal{X},\mathrm{cut}}(bD)$. The skein state sum of [LY, Theorem 12.1] is compatible with X-cutting. Hence

$$\Phi = \psi_{\mathrm{cut}}^{-1} S_{\mathcal{X}} \psi_\Sigma, \qquad E = K_\Sigma D H_{\mathrm{cut}}/n \tag{3.2}$$

is an integral, based, injective map of A-tori, compatible with projected skein splitting in the initial tori.

Uncut columns. For an uncut small vertex w,

$$DH_{\mathrm{cut}}(*, w) = H_\Sigma(*, w), \qquad E(*, w) = e_w^T. \tag{3.3}$$

Indeed, by the definition of H, mutable columns are sums of directed triangle-adjacency contributions. Original boundary columns have additional diagonal and adjacent terms only on their own boundary interval. The new boundary corrections lie in the two copies of cut-edge columns and therefore leave column w unchanged. Equation (3.1) gives the second identity. This calculation retains corner occurrences separately and thus permits based loops and repeated occurrences of a marked point.

For an uncut mutable vertex i, the whole row satisfies $H(i, *) = Q(i, *)$, whence

$$Q_\Sigma(i, *)E = Q_{\mathrm{cut}}(i, *). \tag{3.4}$$

The exchange submatrices on uncut vertices also coincide. These identities concern uncut columns; cut columns can have additional contributions when an edge is a based loop.

Common frozen factors along mutation words. Freeze the cut directions and mutate only in uncut directions. We use row vectors. Let q_i, r_i be the current source and target exchange rows, and let E be the based-map matrix. Inductively, the uncut columns are identity columns, the uncut exchange submatrices agree, and $q_i E = r_i$.

Mutate at k. Write $q_k = q_k^+ - q_k^-$ and $r_k = r_k^+ - r_k^-$, and set

$$C = q_k^- E - r_k^- = q_k^+ E - r_k^+.$$

The vector C is supported on the target cut-edge frozens. The images of both source exchange exponents differ from their target counterparts by

$$d = e_k - E_k + C, \tag{3.5}$$

which has the same support restriction. Compatibility gives $(r_k^+ - r_k^-)P_{\text{target}}d^T = 0$, so the two exchange monomials have identical Weyl pairings with $M(d)$. Therefore

$$\Phi(x'_k) = [x'_{k,\text{target}}M(d)]. \tag{3.6}$$

Only row k of the map matrix changes:

$$E'_k = 2e_k - E_k + C. \tag{3.7}$$

Its uncut columns remain identity columns. For $i \neq k$ set $h = q_i(k) = r_i(k)$. Including the sign change of the k entry, row mutation reads

$$q'_i = q_i + [h]_+q_k^+ - [-h]_+q_k^- - 2he_k.$$

Since $q'_i(k) = -h$, substitution gives

$$\begin{aligned} q'_iE' &= r_i + [h]_+(r_k^+ + C) - [-h]_+(r_k^- + C) - 2hE_k - h(2e_k - 2E_k + C) \\ &= r_i + [h]_+r_k^+ - [-h]_+r_k^- - 2he_k = r'_i. \end{aligned} \tag{3.8}$$

The identity for $i = k$ follows by sign reversal. Thus the induction continues through every finite word, including repeated pivots. The cutting map sends each source cluster variable to the corresponding target variable times a cut-frozen based monomial. Intermediate seeds are handled by the transported matrices; they need not come from triangulations with geometric H, K.

Triangulation naturality. The A-seeds of all adapted triangulations define the same cluster algebra. We obtain this from the complete flip argument of [KW, v2], the balanced tori of [LY], and the following finite A/X mutation identities.

In a mutable direction put $X = A_k$ and $Y = A^{Q(k,*)}$, with parameter $q = v^{2n^2}$. Compatibility gives $YX = q^{-2}XY$. For every rational function f defined at Y, the commutation relation gives $f(Y)X = Xf(q^{-2}Y)$ and $f(Y)X^{-1} = X^{-1}f(q^2Y)$. Put $T = X(1+q^{-1}Y)^{-1}$; then $T^{-1} = X^{-1}(1+qY)$. Moving the next X, respectively X^{-1}, to the left adds the factor indexed by $r = s$ in the following formulas. Starting with the empty product at $s = 0$, induction gives

$$T^s = X^s\prod_{r=0}^{s-1}(1 + q^{-(2r+1)}Y)^{-1}, \qquad T^{-s} = X^{-s}\prod_{r=0}^{s-1}(1 + q^{2r+1}Y), \quad s \geq 0. \tag{3.9}$$

The left mutation matrix C has the identity columns away from k, with $C_{kk} = -1$ and $C_{ik} = [Q(i,k)]_+$ for $i \neq k$. Together with $C^2 = I, H' = CHC^T$, this proves commutativity of the monomial and nonmonomial A/X squares over the universal domain. For negative indices we use the finite products in (3.9).

For a directed path p in the triangle network, ℓ_p is the indicator vector of the small vertices in its left region. Let p_i be the diagonal paths and put $\kappa = \sum_i \ell_{p_i}$. Let Θ be the X-coordinate change for the square move; primes refer to the network after that move. We use the path-sum formula of [KW, Lemma 4.5] with its printed normalization sign corrected to $M_{\mathcal{X}}(-\kappa)$, consistently with [LY, Theorem 10.5]. The exponent there is $n\ell_p - \kappa$; the determinant-one diagonal also forces this sign. The unnormalized square-move identity KW4.7 is unchanged. Inverting $\Theta(M_{\mathcal{X}}(\kappa')) = M_{\mathcal{X}}(\kappa)$ from KW4.8 gives the required negative-exponent identity. Its Weyl pairing changes sign throughout. The remaining factor is a unit v^a; reflection sends it to v^{-a} while fixing the two based expressions. Thus $a = 0$ over the universal ring.

We apply [SS, Proposition 4.2] along the standard complete flip word. Layer s has $s(n-s)$ mutually nonadjacent four-valent alternating square centers. Equivalently, for $a+b+c+d=n-2$ the old and new poles are $(a,b,c,d)+e_2+e_4$ and $(a,b,c,d)+e_1+e_3$; the equatorial vertices are obtained by adding $e_1+e_2, e_2+e_3, e_3+e_4, e_4+e_1$. The rectangle mutation lemma of [FG, Section 10.3] gives the four-valent condition and the next-layer update. X-cutting reduces a general surface to the ordinary quadrilateral. This yields naturality at the universal and generic parameters; subsequent specializations concern finite Laurent identities.

Boundary identifications preserve this complete flip. Write the octahedral coordinates as $x=(a,b,c,d)\in\mathbb{N}^4$. An equatorial vertex on physical side 12 requires $c=d=0$, whereas one on opposite side 34 requires $a=b=0$. Both occur only when $x=0, n=2$, and then their exchange contributions have the same sign. The other opposite pair is identical. Gluing adjacent sides oppositely within the two triangles would close the link of their common corner and create an interior puncture. Repetition of a compact marked point does not itself identify these sides. Consequently cutting to the quadrilateral never combines opposite exchange contributions into a cancellation.

The terminal H-matrix is also determined. Put $S=H+H^T$, so $H=Q+S/2$ and S is supported on the physical boundary frozen block. For a mutable congruence matrix $C=I+ue_k^T$, one has $e_k^TS=Se_k=0$ and hence $CSC^T=S$. Thus H is transported with Q. The terminal geometric Q and unchanged boundary block S determine the terminal geometric H.

□

3.2 Elementary arcs

Proposition 3.2. *For every allowed coefficient domain, $S_R(\Sigma)\subseteq\mathcal{A}_R(\Sigma)$.*

Proof. We first calculate the mutation words on ordinary polygons. Section 3.2 then glues their neutral-support expressions to a general surface.

Quantum lifting for coefficient-one arcs. Suppose a mutation word visits each pivot at most once. At each visit the denominator is the unchanged initial variable, while all other variables enter the exchange monomials with nonnegative exponents. Recursive expansion uses addition, multiplication, and inverses of initial monomials. Every Laurent coefficient therefore lies in $\mathbb{N}[v^{\pm1}]$, and the initial frozen exponents are nonnegative before normalization.

By LY10.5, either orientation of a simple triangle corner arc is a finite sum of based X-monomials with coefficient one. Distinct paths have distinct left-region vectors: closing them by the same boundary path expresses their difference as the directed boundary of bounded faces, each containing an undeleted small vertex. Equal indicators force the face-chain difference, and hence the path difference, to vanish. The based A/X isomorphism preserves distinct exponents and coefficient one.

Suppose an expression obtained from such a word, normalized by an incident frozen monomial, equals the arc at $v=1$. Positivity excludes additional support. Each supported coefficient has value one at specialization and is therefore v^a. The expression is a based monomial in the terminal seed, hence bar-invariant, giving $a=0$. This proves the quantum identity for these coefficient-one arc expressions.

The two triangle orientations. In the first triangle write

$$A_{pq}=g_{n-p,q,p-q},\quad 0\le q\le p\le n,$$

Here $g_{a,b,c}$ is the reflection-normalized seed web of [LY, Section 13.2], whose trace is the coordinate at (a,b,c). For the oppositely oriented triangle use its own coordinates F_{qp}, indexed by $(q,n-p,p-q)$, for $0\le q\le p\le n$. Corner labels equal one. On the common side put $y_s=A_{ss}=F_{ss}$, with $y_0=y_n=1$. Let $C=(C_{ij})$ and $\bar{C}=(\bar{C}_{ij})$ be the two positional corner matrices: C is the forward corner arc at

the first triangle corner and $\bar{C}$ the opposite-oriented corner at the second, with the fixed-side state convention of Section 2.2. Their entries above the diagonal vanish in the reduced algebra. Classical formulas use their $v = 1$ specializations; quantum formulas use the corresponding projected stated entries.

LY(179) and complementary minors give the classical coordinates

$$A_{pq} = A_{p0}A_{qq} \det C_{[p-q+1,p],[1,q]}, \qquad A_{p0} = (C_{11} \cdots C_{pp})^{-1}, \tag{3.10}$$

$$F_{qp} = F_{pp} \det \bar{C}_{[n-p+q+1,n],[q+1,p]}. \tag{3.11}$$

To check (3.10) at $v = 1$, use the reversal matrix $E_{ab} = (-1)^{n-a}\delta_{n+1-a,b}$. Let R be the reverse arc at the same corner as C, with the fixed left/right indices of Section 2.2. Its start–end matrix is $R^T = EC^{-1}E^{-1}$. The reverse-arc minor in [LY, (179)], with $(i, j, k) = (n - p, q, p - q)$, has row set $[n - p + 1, n - q]$ and column set $[n - p + q + 1, n]$ in R^T. Reversing these indices gives

$$\det(R^T)_{[n-p+1,n-q],[n-p+q+1,n]} = (-1)^{(p-q)q} \det(C^{-1})_{[q+1,p],[1,p-q]}.$$

Since C is lower triangular, the leading $p \times p$ block of C^{-1} is the inverse of that block of C. Jacobi complementation therefore gives

$$\det(C^{-1})_{[q+1,p],[1,p-q]} = (-1)^{(p-q)(p+1)} \frac{\det C_{[p-q+1,p],[1,q]}}{\det C_{[1,p],[1,p]}}.$$

The combined sign is $(-1)^{p(p+1)-q(q+1)} = 1$, and the denominator is A_{p0}^{-1}. Multiplication by the other factor A_{qq} in [LY, (179)] proves (3.10). Empty minors give the same calculation at $q = 0$ and $q = p$. The positional matrices $C, \bar{C}$ retain the conventions above; equation (3.11) is the transpose of the corresponding LY minor.

Fix i and mutate in the order $p = i - 1, i - 2, \ldots, 2$, with $q = 1, \ldots, p - 1$ in each row. Appendix A.1 lists the complete exchange rows. Their classical recurrence is

$$A_{pq}Z_{pq} = A_{p-1,q}Z_{p+1,q} + A_{p,q+1}Z_{p,q-1}, \tag{3.12}$$

The boundary values are $Z_{i,q} = A_{i-1,0}A_{i,q+1}$ and $Z_{p,0} = A_{i1}A_{p-1,0}$. The solution is

$$Z_{pq} = A_{p-1,0}A_{i0}A_{q+1,q+1} \det C_{[p-q,p-1]\cup\{i\},[1,q+1]}. \tag{3.13}$$

The boundary values follow from (3.10). To check the recurrence, use increasing row lists and write $\Delta_{I,J} = \det C_{I,J}$ within this calculation. For $S = [p - q + 1, p - 1]$, $a = p - q$, $b = p$, and $c = i$, the three-term identity is

$$\begin{aligned}&\Delta_{S\cup\{b\},[1,q]}\Delta_{S\cup\{a,c\},[1,q+1]} \\ &\quad = \Delta_{S\cup\{a\},[1,q]}\Delta_{S\cup\{b,c\},[1,q+1]} + \Delta_{S\cup\{c\},[1,q]}\Delta_{S\cup\{a,b\},[1,q+1]}.\end{aligned}$$

It follows by expanding the last column, or by the Plücker relation for these nested column sets. Substituting (3.10) and (3.13) into (3.12), all three terms have the common factor

$$A_{p0}A_{p-1,0}A_{i0}A_{qq}A_{q+1,q+1}.$$

Canceling it leaves exactly the displayed identity. This verifies every interior recurrence, including $q = 1$, when S is empty. Lower triangularity of the target $p = k, q = k - 1$ gives $Z_{k,k-1} = A_{i0}A_{kk}C_{ik}$. Section 3.2 lifts the identity to the quantum setting.

For $i \geq 2$ define $U_1 = A_{i1}, U_k = Z_{k,k-1}$ $(1 < k < i), U_i = A_{i-1,0}$. Then

$$C_{ik} = \begin{cases} [U_k A_{i0}^{-1} A_{kk}^{-1}], & k < i, \\ [U_i A_{i0}^{-1}], & k = i. \end{cases} \tag{3.14}$$

For $i = 1$ set $C_{11} = [A_{10}^{-1}]$ separately, so that $U_1 = U_i$ has a single definition.

For the reverse orientation fix $j < n$. Mutate at $F_{k,k+t}$ in the order $k = j-1, \dots, 1$, with $t = n-j, \dots, 1$ in each row. Denote the changed values by $W_{k,t}$. Appendix A.2 supplies the complete rows. The recurrence and its solution are

$$F_{k,k+t} W_{k,t} = W_{k,t+1} F_{k,k+t-1} + W_{k+1,t} F_{k-1,k+t-1}, \tag{3.15}$$

$$\begin{aligned} W_{k,n-j+1} &= F_{jn} F_{k-1,k+n-j}, \qquad W_{j,t} = F_{j-1,n} F_{j,j+t-1}, \\ W_{k,t} &= F_{k+t-1,k+t-1} F_{jn} \det \bar{C}_{\{j\} \cup [n-t+2,n],[k,k+t-1]}. \end{aligned} \tag{3.16}$$

For the reverse recurrence take the ordered row list $R = \{j\} \cup [n-t+1, n]$, the column list $J = [k, k+t]$, and let $a = j$, $b = n-t+1$, $c = k$, $d = k+t$. Here $a < b$ since $t \le n-j$. With $\Delta_{I,L} = \det \bar{C}_{I,L}$ in this calculation, Desnanot–Jacobi gives

$$\begin{aligned} &\Delta_{R \setminus a, J \setminus c} \Delta_{R \setminus b, J \setminus d} \\ &\quad = \Delta_{R,J} \Delta_{R \setminus \{a,b\}, J \setminus \{c,d\}} + \Delta_{R \setminus a, J \setminus d} \Delta_{R \setminus b, J \setminus c}. \end{aligned}$$

After (3.11) and (3.16) are substituted, the common scalar factor in (3.15) is $F_{k+t,k+t} F_{k+t-1,k+t-1} F_{jn}$, and the remaining minors are precisely the four complementary minors in this identity. At $t = 1$ the middle minor is empty and equals one. At $k = j$ or $t = n-j+1$, the prescribed boundary values are obtained by triangularity and (3.11). At $t = 1$ it gives $W_{k,1} = F_{kk} F_{jn} \bar{C}_{jk}$. Set $V_k = W_{k,1}$ $(k < j)$ and $V_j = F_{j-1,n}$. In the boundary case $j = n$, define

$$V_k = F_{k-1,k}, \qquad 1 \le k \le n. \tag{3.17}$$

Both cases are expressed by

$$\bar{C}_{jk} = \begin{cases} [V_k F_{kk}^{-1} F_{jn}^{-1}], & k < j, \\ [V_j F_{jn}^{-1}], & k = j. \end{cases} \tag{3.18}$$

These finite identities invert only the incident sides. The initial frozen exponents on the unused third side are nonnegative, and bad triangle states vanish.

The quadrilateral seam. Let D_{ij} be the stated coefficient of the oriented arc crossing the diagonal of the quadrilateral, with state i on the first endpoint side and state j on the second. The side convention is chosen so that cutting gives $\sum_k C_{ik} \otimes \bar{C}_{jk}$. Fix $1 \le i, j \le n$ and let $m = \min(i, j) \ge 2$. First perform in the first triangle

$$p = i-1, \dots, m, q = 1, \dots, m-1; \qquad p = m-1, \dots, 2, q = 1, \dots, p-1,$$

then the reverse rectangle word, and finally mutate $y_1, \dots, y_{m-1}$ in order. The two triangle words have disjoint pivot sets with no arrows between them, so they commute. Truncation preserves the relevant targets. For a deleted pivot with $q \ge m \ge 2$, the only smaller-q mutable neighbor in the complete rows of Appendix A.1 is the already visited predecessor $Z_{p,q-1}$. The additional top-row neighbor $A_{i-1,0}$ is frozen and remains in the exchange monomial. No deleted pivot is adjacent to a later retained pivot or to y_s with $s < m$. Mutations at distinct vertices with zero exchange entry commute, including their coefficient rows. Starting with the last deleted pivot, commute its mutation

past all later retained mutations; repeating this operation moves every deleted mutation to the end. Thus the retained variables and their exchange rows are unchanged.

After the two words the complete seam exchange **column** is the following; under the conventions of Appendix A.4 it is the negative of the seam row:

$$Q(-, y_s) = U_s + V_s - U_{s+1} - V_{s+1} - \mathbf{1}_{s>1} y_{s-1} + (1 - \mathbf{1}_{i=s+1} - \mathbf{1}_{j=s+1}) y_{s+1}. \tag{3.19}$$

The letters here denote coordinate incidence vectors: for example, U_s in this column means the unit vector at the vertex carrying the current coordinate U_s. A coefficient of two records two arrows. corner coordinates with index n equal one and contribute zero to a row. Appendix A.3 derives this column from the complete triangle rows. Each seam mutation removes the preceding U, V pair from the next exchange row. Thus the final changed value X is

$$X = \left[\frac{U_1 V_1}{y_1}\right] + \sum_{k=2}^{m-1} \left[\frac{U_k V_k}{y_{k-1} y_k}\right] + \left[\frac{y_m^{\mathbf{1}_{i=j}} U_m V_m}{y_{m-1}}\right]. \tag{3.20}$$

Each quantum coefficient equals one. The preceding changed seam coordinate occurs linearly at each step, so each summand is produced once with a parameter power. The V_k are distinct coordinates of the pre-seam seed, and the other factors contain none of these markers. Thus the summands have distinct exponents, and bar invariance fixes each parameter power to one. In particular, the last term for $i = j < n$ retains y_i; for $i = j = n$ this factor equals one.

Cutting and normalization in a quadrilateral. The four physical corners of an ordinary quadrilateral are distinct. Appendix A.4 verifies $HE_0 = DH_{\mathrm{cut}}$ directly and proves that based cutting satisfies

$$S(A_{pq}) = [A^U_{pq} \otimes F^V_{pp}], \qquad S(F_{qp}) = [A^U_{qq} \otimes F^V_{qp}]. \tag{3.21}$$

These formulas concern the ordinary polygon. The triangle exchange rows are homogeneous. Starting from the initial degrees $\deg A_{pq} = e_p$ and $\deg F_{qp} = e_q$, induction gives

$$\begin{aligned} &S(U_k) = [U_k \otimes F_{k-1,k-1} F_{ii}], \quad k < i, \\ &S(U_i) = [U_i \otimes F_{i-1,i-1}], \\ &S(V_k) = [A_{k-1,k-1} A_{jj} \otimes V_k], \quad k < j, \\ &S(V_j) = [A_{j-1,j-1} \otimes V_j]. \end{aligned} \tag{3.22}$$

The two original endpoint-side factors satisfy $S([A_{i0} F_{jn}]) = [A_{i0} A_{jj} \otimes F_{ii} F_{jn}]$. Substitution in (3.20) identifies each normalized summand with $C_{ik} \otimes \bar{C}_{jk}$. At the double terminal, y_m supplies the diagonal factors missing on both sides. Hence

$$S([X A_{i0}^{-1} F_{jn}^{-1}]) = \sum_{k=1}^{\min(i,j)} C_{ik} \otimes \bar{C}_{jk} = S(D_{ij}).$$

The right-hand side is the skein state sum. Injectivity of initial cutting from Section 3.1 gives

$$D_{ij} = [X A_{i0}^{-1} F_{jn}^{-1}], \qquad m \geq 2. \tag{3.23}$$

The boundary cases $m = 1$ are

$$D_{i1} = [A_{i1} A_{i0}^{-1} F_{1n}^{-1}], \qquad D_{1j} = [V_1 A_{10}^{-1} F_{jn}^{-1}], \quad j > 1. \tag{3.24}$$

In particular, $D_{1n} = [F_{01} A_{10}^{-1}]$, replacing the printed F_{12}/A_{10}.

Gluing on neutral support. An essential arc has endpoints on two distinct open boundary intervals. A narrow neighborhood of the arc and its endpoint sides is an ordinary triangle for a corner arc and a

quadrilateral strip otherwise. Its interior is embedded. The unused lateral sides can be two copies of one global ideal edge, and marked corners can repeat. Retain a lateral side that is a physical boundary side; if it degenerates to a corner, use the triangle case. The remaining sides and strip diagonal extend to a triangulation. Cutting the lateral sides gives an ordinary patch P and a complementary surface R. Small vertices and pivots off the cut sides remain distinct.

Let J send the A-exponents of P to global labels, adding exponents when labels coincide. For a local exponent a put $r = aK_P$. Neutrality means that r vanishes at every small vertex on every cut side. Extend such an exponent by zero to the global vector $r_g = rJ$. For rows off the cut sides,

$$H_\Sigma(u, *) = H_P(u, *)J.$$

Indeed, a nonseam u has a unique preimage and all its incident face germs lie in the patch. Since Q is a sum of triangle quivers, $e_u Q_P J = e_{p(u)} Q_\Sigma$ remains valid when adjacent columns merge. The correction $H - Q$ concerns entries on the same boundary edge. Nonseam boundary vertices remain on unchanged physical boundary edges, so this correction is preserved. The possibly nonlocal cut rows have zero coefficients in r. Therefore

$$aJ = \frac{1}{n} r_g H_\Sigma, \qquad aJE = \frac{1}{n} r_g D H_{\mathrm{cut}} = a \oplus 0. \tag{3.25}$$

This identity holds on the neutral sublattice and proves that J is injective there.

The local arc has no endpoints on a cut side. Boundary X-homogeneity [LY, Proposition 10.4, Corollary 12.2] makes every X-exponent vanish on the cut sides. Thus each term of its finite A-expansion $h_P = \sum_a c_a M_P(a)$ satisfies (3.25), and $\sum_a c_a M_\Sigma(aJ)$ maps under Φ to $h_P \otimes 1$. The global arc has the same image by the state sum.

On the other hand, Section 3.1 applied to the global mutation word shows that the normalized cluster expression c satisfies

$$\Phi(c) = [h_P M_{\mathrm{cut}}(d)] \tag{3.26}$$

where d is an integral cut-frozen vector. We show $d = 0$.

At $v = 1$, label amalgamation J commutes with the finite initial Laurent expressions of these words. The complete row tables control possible cancellation after identifying the lateral sides: a forward pivot meets only the first side, and a reverse pivot only the second. Both sides occur at a seam pivot only at the first pivot when $i = j = n$, with equal signs. Subsequent rows contain only the preceding changed coordinate and the next U, V pair. Mutating inside the patch creates no outside neighbor. The classical global expression is consequently the label substitution of the local finite Laurent polynomial.

By (3.25), the two sides of (3.26) at $v = 1$ are $h_{P,1} \otimes 1$ and $h_{P,1} x^d$. The polynomial $h_{P,1}$ is nonzero with finite support. A nonempty finite lattice set cannot be invariant under a nonzero translation: an integral linear functional nonzero on d would have different maxima before and after translation. Therefore $d = 0$. Basedness removes the remaining quantum scalar, giving the identity over the formal parameter and, by torus injectivity, the global arc formula.

The final denominators involve only actual endpoint boundary variables. Each numerator is a global cluster variable, or a frozen variable already in the coefficient algebra.

From arcs to $S_A \subseteq \mathcal{A}_A$. The full stated skein algebra is generated by the finitely many state-labeled arcs of a saturated system [LY, Theorem 6.2(3)]. If a generator has both endpoints on an open interval c, choose another interval c' in the same component. After cutting the arc, the component containing c' has a copy of the arc. Choose an embedded connector from this copy to c'. A narrow finger isotopy along the connector, followed by boundary cutting, expresses the generator as a finite sum of products of essential arcs from c to c'. The boundary relations handle both orientations and height orders, with

parameter-unit coefficients. This is the elementary argument of HW7.18 and uses two total marks per component.

The coefficient is $\mathbb{c}_{\bar{i}}^{-1}$ in HW PDF(23), or HTML(29); each $\mathbb{c}_i$ in HW(1) is a signed parameter monomial. For the opposite height order, LS v3(57) has coefficient $\mathbb{c}_i$. LY6.2(3) uses the saturated decomposition of LS v3 Theorem 8.8 and Corollary 8.15, including transmuted factors for endpoints on one interval. The optional invertibility of $[n-2]!$ in LS Proposition 7.5 is used only to remove a defining relation that is retained here. The resulting finite diagram identities specialize through the skein presentation to every allowed coefficient ring.

By Section 3.2 every such essential arc state belongs to $\mathcal{A}_A$ in an adapted triangulation. Section 3.1 identifies these embedded cluster algebras, and generation passes to the projected quotient. Hence

$$S_A(\Sigma) \subseteq \mathcal{A}_A(\Sigma) \subseteq \mathcal{U}_A(\Sigma). \tag{3.27}$$

The second inclusion is the quantum Laurent phenomenon [BZ]. The local identities are finite Laurent identities over the universal ring, with denominators given by allowed boundary monomials. The same inclusions therefore hold after every allowed specialization.

□

3.3 Boundary units

Proposition 3.3. *Every projected frozen coordinate has its inverse in $S_R(\Sigma)$.*

Proof. When the incident boundary intervals are distinct, [LY, Lemma 7.6] applies. If a compact boundary circle has one mark, its corner endpoints lie on the same open interval; we treat this case directly.

Let $F = \mathcal{O}_q(SL_n)$, and let $\check{F}$ be the punctured-monogon, or transmutation, algebra of a same-interval corner. Let $\kappa : F \to \check{F}$ be the linear isomorphism of [LY, Section 5.2]. Let $I^-, \check{I}^-$ be generated by the forward bad entries u_{ij} $(i < j)$. By [LY, Proposition 5.5],

$$\kappa(I^-) = \check{I}^-. \tag{3.28}$$

Take $x = u_{ii}$ and $y = \overleftarrow{u}_{\bar{i}\bar{i}}$. In the ordinary reduced bigon, [LY(138)] gives $xy = yx = 1$. Formula [LY(108)] compares $\kappa(xy)$ with a unit multiple of $\kappa(x)\kappa(y)$. Each strict correction raises the right state of x and thus contains a bad entry $\check{u}_{i,j'}$ $(j' > i)$. In the reverse product yx, a strict correction lowers the left state of the second factor x, producing $\check{u}_{k',i}$ $(k' < i)$. Modulo the bad ideal,

$$1 = \lambda \check{x}\check{y}, \qquad 1 = \mu \check{y}\check{x}, \qquad \lambda, \mu \in A^\times. \tag{3.29}$$

A left and a right inverse in a unital ring coincide. Hence every diagonal entry is a unit. This uses the linear formula for κ and remains valid without a domain assumption on the reduced algebra.

For an ordered subset K, the ordinary diagonal quantum minor equals $\prod_{a\in K} u_{aa}$ modulo I^-, because every nonidentity permutation term has a bad entry with row less than column. Apply (3.28) and [LY(110)] to this diagrammatic minor. Every strict word correction is again bad, giving

$$\overline{\kappa}(\Delta_{K,K}) = \lambda_K \prod_{a\in K} \check{u}_{aa}, \qquad \lambda_K \in A^\times. \tag{3.30}$$

Thus these diagrammatic principal minors are units over the original coefficient ring.

The projected boundary seed web is such a minor. The $k = 0$ source diagram of [LY, Lemma 10.3] identifies the side vertex $i + j = n$ with $\Delta_{[n-j+1,n],[n-j+1,n]}(C(v_2))$. Left-turn elongation [LY, Section 13.2] follows the finite link interval of a boundary mark. Combining the successive triangle sectors is a ribbon isotopy in the corner collar, preserving parallel strands and ports. Even if the

two link endpoints lie on the same open boundary interval, they remain distinct physical ports. The resulting diagram is the minor of κ.

The skein map from the local monogon collar sends its bad ideal into the global bad ideal. These local units therefore give global projected units, and

$$\mathbb{B}_{\mathrm{fr},A} \subseteq S_A, \tag{3.31}$$

All these identities specialize to the allowed coefficient rings.

□

4 The full algebra and its classical fiber

4.1 The full seed and mixed upper algebra

Proposition 4.1. *The full trace identifies $B_{R_0}(\Sigma)$ with a subalgebra of U^+_{full,R_0}.*

Proof. Coordinates of the attached triangles. Attach a triangle to every boundary interval of Σ and denote the enlarged surface by Σ^*. In each triangle, let e_1 be the original interface, e_2 the exterior side retained by the collar embedding, and e_3 the other exterior side. The full frozen vertices H lie on e_3; the omitted vertices W lie on e_2.

Formula [LY(240)] and [CHW, Lemmas 4.3–4.4] give a based monomial embedding L from the full initial torus to the projected torus of Σ^*. Write $\bar{V}_\lambda$ for the original projected vertex set and $V'_\lambda = V_{\Sigma^*} \setminus W$ for the full vertex set. On each attached triangle define $p(i,j,k) = (0, n-k, k) \in e_2$. For source coordinates a_v and target coordinates A_v,

$$a_v \longmapsto \begin{cases} A_v, & v \in \bar{V}_\lambda, \\ [A_v A_{p(v)}^{-1}], & v \in V'_\lambda \setminus \bar{V}_\lambda, \end{cases} \qquad y_w = A_w \quad (w \in W). \tag{4.1}$$

The coordinate at W remains A_w, replacing the expression $A_w/A_{p(w)} = 1$ in CHW(78).

The full and enlarged projected seeds have the same mutable set. Adjoin W to the full seed with zero mutable exchange rows. Their frozen–frozen skew pairings can be nonzero, so y_w need not be central. The exponent matrix $\widehat{C}$ has rows $e_v, e_v - e_{p(v)}, e_w$ and is unimodular. The added exchange entries are $\widehat{Q}(w,k) = 0$ for $w \in W$ and mutable k. The full skew form transported from the target is $\widehat{P} = \widehat{C} P_{\Sigma^*} \widehat{C}^T$, including its frozen block. The integral matrix identity CHW4.1 gives its restriction to the full sublattice and the mutable-column correspondence. We verify the latter directly below.

Give the attached triangle clockwise vertices $(n,0,0), (0,n,0), (0,0,n)$ and sides $e_1 = \{k = 0\}, e_2 = \{i = 0\}, e_3 = \{j = 0\}$. Use $Q(z,y) = \mathrm{wt}(z \to y) - \mathrm{wt}(y \to z)$ with arrow increments $(-1,1,0), (0,-1,1), (1,0,-1)$. Define the complete row

$$R_r = \{(i, n-r-i, r) : 0 \le i \le n-r\}, \qquad 1 \le r < n.$$

whose first point $w_r = (0, n-r, r)$ is omitted. A mutable vertex $y = (a,b,c)$ satisfies $a, b \ge 1, c \ge 0, a+b+c = n$. The neighbors that can contribute to $\sum_{z \in R_r} Q(z,y)$ are:

Row layer	Positive neighbor	Negative neighbor
$r = c > 0$	$(a+1, b-1, c)$	$(a-1, b+1, c)$
$r = c+1$	$(a-1, b, c+1)$	$(a, b-1, c+1)$
$r = c-1 \ge 1$	$(a, b+1, c-1)$	$(a+1, b, c-1)$

Every pair lies in the allowed coordinate region and avoids the deleted ideal corners. All relevant small edges have weight one: even if a neighbor lies on e_2 or e_3, the other endpoint y does not, so the small edge is interior. Each pair contributes $+1, -1$. For $c = 0$ only the pair $r = 1$ occurs; for $c = 1$ the entire pair $r = 0$ is absent. At the top layer $r = n - 1$, a possible pair still corresponds to $c = n - 2, a = b = 1$. Thus all boundary cases have row sum zero.

Every row R_r satisfies $r > 0$ and lies off the gluing side e_1, so outside mutable vertices have no adjacency with this row. A mutable y on e_1 also receives the indicated contributions only from the attached triangle. Repeated compact marks and singleton circles do not identify distinct physical small vertices of e_2, e_3. Hence for every global mutable y,

$$\sum_{z \in R_r} Q_{\Sigma^*}(z, y) = 0.$$

Here $c = 0$ denotes the interface containing the mutable neighbor. The omitted-row layers are $1 \leq r < n$; there is no omitted row R_0.

Let C be the shear matrix obtained from $\widehat{C}$ by removing the added W rows. The row-sum identity gives

$$C^T Q_{\mathrm{full}}(:, \mathrm{Mut}) = Q_{\Sigma^*}(:, \mathrm{Mut}).$$

Restoring the e_w rows makes $\widehat{C}$ block unitriangular.

The exchange and common-factor induction of Section 3.1 shows that, at every pair of corresponding seeds, L is the sublattice embedding of the full torus into the target. Adjoining $y_w^{\pm 1}$ gives the entire target torus. The exponents and compatibility identities are fixed over the universal ring before specialization.

The finite trace square for the collar embedding followed by reduction sends B_{R_0} into $S_{R_0}(\Sigma^*)$, which lies in the target upper algebra by (3.27). This supplies the map and coordinate identities needed below.

Descent from the enlarged torus. We use the following lattice fact. If $M \hookrightarrow N$ induces an inclusion of compatible based quantum tori $T_M \subset T_N$, then in their ambient division ring,

$$T_N \cap \mathrm{Frac}(T_M) = T_M. \tag{4.2}$$

Grade T_N by N/M. For z on the left, take a nonzero right denominator $d \in T_M$ such that $zd \in T_M$. In every nonzero quotient degree c, one has $z_c d = 0$. The torus is a domain, so $z_c = 0$. The remaining exponents lie in M.

Apply (4.2) in each corresponding seed to obtain

$$B_{R_0} \subseteq U^{\mathrm{fr}}_{\mathrm{full}, R_0}. \tag{4.3}$$

Nonnegative full frozen exponents. Represent a full skein element by a finite web. In a thin collar, its portions in the attached triangles can be chosen as simple corner tails from e_1 to e_2. All crossings and vertices stay on the original side, and there are no endpoints on e_3. Apply the finite state sum along every interface.

By Section 3.2, each nonzero triangle tail is a cluster coordinate obtained without repeated pivots, divided by an incident e_1, e_2 monomial. Its unused-side e_3 exponents are nonnegative. Products and finite sums preserve this property; the original-side factors have no e_3 exponent. Section 3.1 preserves every e_3 column of the cutting matrix. Injectivity of the lattice map prevents distinct original monomials from canceling after cutting. Nonnegative cut e_3 support therefore implies nonnegative support before cutting.

Formula (4.1) changes only the e_2 columns. The full $h \in H$ exponent is exactly the target e_3 exponent. Hence

$$B_{R_0} \subseteq T^{H+}_{\mathrm{full},0,R_0}. \tag{4.4}$$

Mutation invariance of the minimum frozen exponent. In a fully Laurent torus, fix a frozen variable f. For a nonzero Laurent polynomial, let w_f be the minimum exponent of f among its nonzero terms, and set $w_f(0) = +\infty$. The coefficient torus in the remaining variables is a domain, so leading products do not cancel and the valuation extends to the division ring.

Mutate at $k \neq f$. The two f exponents in the exchange relation are $[q_{fk}]_+, [-q_{fk}]_+$, at least one of which is zero. If $q_{fk} \neq 0$, the lowest part contains one monomial; otherwise it contains two distinct monomials. Compatibility makes the exchange column nonzero, so the residue P is nonzero. In the residue division ring,

$$\overline{x'_k} = x_k^{-1} P,$$

where P belongs to the coefficient division ring with f, x_k removed. Distinct integral powers $(x_k^{-1}P)^a$ have distinct x_k-degrees. For a fixed power a, move all powers of x_k to the left. The remaining coefficient contains a common nonzero product of conjugates of P, which is invertible in the coefficient division ring. Canceling this factor reduces any linear relation to a relation among distinct monomials in the unchanged coordinates, whose coefficients are zero. Thus the lowest terms of the new Laurent monomials are independent, and their minimum f exponent computes the same valuation.

In each seed, mixed membership is consequently equivalent to $w_h \geq 0$ for all frozen variables. Together with (4.3)–(4.4), this gives

$$B_{R_0} \subseteq U^{+}_{\mathrm{full},R_0}. \tag{4.5}$$

The same argument applies over K_0 and to the classical fiber at $v = 1$.

□

4.2 Filtration and frozen divisors

Choose a saturated oriented boundary arc system $\mathcal{E} = \{a_1, \ldots, a_r\}$: its arcs are pairwise disjoint, have distinct physical endpoints on the boundary intervals, and cut the surface into ideal monogons, each containing one original marked point [LS, Theorem 8.8]. Fix the order and orientation of these arcs throughout the degree argument. Give each of the n^2 state assignments on a_e degree one. For $D \geq 0$, let $F_D B_{R_0}$ be the span of products of at most D such coefficients; put $F_D = 0$ for $D < 0$. Thus F_0 contains the scalar multiples of the empty diagram. The associated graded algebra is $\bigoplus_{D \geq 0} F_D / F_{D-1}$.

Proposition 4.2. *The word filtration associated with a fixed saturated arc system has finite free layers and a domain as its associated graded algebra. In the classical fiber, each full frozen variable generates a prime ideal, and its divisibility order equals its minimum Laurent exponent.*

Proof. The saturated word filtration. By [LY, Proposition 3.3, Lemma 3.5], the ordinary matrix-factor basis is indexed by

$$\Gamma = \mathrm{Mat}_n(\mathbb{N}) / \mathbb{Z} I_n$$

Here I_n is the identity matrix, and two nonnegative matrices represent the same class when their difference is an integral multiple of I_n. Choose the unique minimal nonnegative representative $\widehat{m}$, characterized by $\min_i \widehat{m}_{ii} = 0$, and put $\ell(m) = \sum_{ij} \widehat{m}_{ij}$. The transmuted factor has the same basis

and word-length control by [LY, Theorem 5.2]. For $m \in \Gamma$, write $b_e(m)$ for this basis element in the factor associated with a_e. Theorem 6.2 gives the global multiplication-module isomorphism for a fixed ordering of these factors.

Let G_e be the set of fundamental stated coefficients of a_e. Factors can be reordered without changing length: the proof of Theorem 6.2 uses Lemma 4.8 to give $\operatorname{span}_{R_0}(G_e)\operatorname{span}_{R_0}(G_f) = \operatorname{span}_{R_0}(G_f)\operatorname{span}_{R_0}(G_e)$. Each step reorders boundary heights and states while preserving the number of occurrences of every arc. Hence

$$F_D B_{R_0} = \operatorname{span}_{R_0}\{b_1(m_1)\cdots b_r(m_r) : \sum_e \ell(m_e) \le D\}. \tag{4.6}$$

This is finite free, and the successive quotient has the basis elements of length exactly D. Reduction therefore commutes with the filtration and its associated graded. The length ℓ is not additive on Γ: the determinant-one relation can lower word length.

At $v = 1$, cross-factor braiding and transmuted multiplication become ordinary multiplication, so

$$B_0 \simeq \mathcal{O}(SL_n)^{\otimes r}, \qquad \operatorname{gr}_F B_0 \simeq \bigotimes_{e=1}^{r} \mathbb{C}[M_n]/(\det X_e). \tag{4.7}$$

For one factor the highest homogeneous part of $\det - 1$ is det, and the highest part of every multiple is divisible by det. This proves the graded identification. The determinant cones are geometrically integral, as is their product.

Each quantum graded layer is finite free and hence p-adically separated. If two nonzero homogeneous factors had zero product, divide each by its largest common power of p. Their nonzero classical residues would have nonzero product, a contradiction. Thus

$$\operatorname{gr}_F B_{R_0}, \quad \operatorname{gr}_F B_{K_0} \quad \text{are domains.} \tag{4.8}$$

It follows that $\delta_K(b) = \min\{D : b \in F_D B_{K_0}\}$ is additive on nonzero products. The classical degree $\delta_0(b) = \min\{D : b \in F_D B_0\}$ has the same property. Set both degrees of zero to $-\infty$. The filtration also gives $\delta(a+b) \le \max\{\delta(a), \delta(b)\}$, for either degree.

To extend degree to fractions, let P be the quantum polynomial algebra generated by all initial variables and T its Laurent torus. We have $P \subseteq B \subseteq T$. The left and right Ore conditions for P follow by a dimension count. Over the coefficient field, use the auxiliary total degree of nonnegative monomials. Let V_0 be the initial full index set. Ordered monomials form a basis, so

$$\dim P_{\le N} = \binom{N + |V_0|}{|V_0|}, \qquad \frac{\dim P_{\le N+d}}{\dim P_{\le N}} \longrightarrow 1 \quad (N \longrightarrow \infty)$$

for fixed $d \ge 0$. For two nonzero elements a, b of degree at most d, choose N such that $2\dim P_{\le N} > \dim P_{\le N+d}$. The map $(r, s) \mapsto ra - sb$ has nonzero kernel. Since the algebra is a domain, both multipliers are nonzero, giving a common left multiple; the right-handed argument is identical. Over R_0, clear the finitely many central coefficient denominators. For nonzero $a, b \in B$ choose a positive seed monomial d such that $da, db \in P$. In P choose nonzero r, s with $rda = sdb$. Then $(rd)a = (sd)b$ is a common left multiple in B. Right multiples are obtained in the same manner. Hence B is left and right Ore. Since B contains the initial torus generators and embeds in that torus, their division rings coincide. Consequently

$$\delta_K(ab^{-1}) = \delta_K(a) - \delta_K(b) \tag{4.9}$$

is well defined on the common division ring. Indeed, if $ab^{-1} = cd^{-1}$, choose nonzero $r, s \in B$ with $br = ds$, using the right Ore condition. Rewriting both fractions with this denominator gives $ar = cs$.

Additivity on products yields

$$\delta_K(a)+\delta_K(r)=\delta_K(c)+\delta_K(s),\qquad \delta_K(b)+\delta_K(r)=\delta_K(d)+\delta_K(s).$$

Subtracting proves that the two degree differences agree. To check additivity on fractions, choose $ct=bu\neq 0$; then $b^{-1}c=ut^{-1}$ and

$$(ab^{-1})(cd^{-1})=au(dt)^{-1},\qquad \delta_K(u)-\delta_K(t)=\delta_K(c)-\delta_K(b).$$

This gives the required sum of degrees. A common right denominator similarly reduces subadditivity for a sum of fractions to subadditivity in B. Below, specialization is applied to finite fraction identities whose denominators have nonzero residue.

Geometric integrality of the extreme-minor quotient. We prove the required primality in the classical fiber. Over any field k, put

$$d=\det X_{[1,i],[n-i+1,n]},\qquad 1\le i<n. \tag{4.10}$$

The ring $k[M_n]/(d)$ is geometrically integral. Indeed, d is the determinant of an independent generic $i\times i$ matrix, with the other entries free. For $i=1$ the quotient is a polynomial ring. For $i>1$, matrices of rank at most $i-1$ are parametrized by products of an $i\times(i-1)$ matrix and a $(i-1)\times i$ matrix, so their geometric zero locus is irreducible. At rank $i-1$ a cofactor is nonzero, making the determinant hypersurface reduced on this dense open set. Thus over an algebraic closure the determinant has one irreducible factor of multiplicity one.

The determinant of X is nonzero in this quotient, since the identity matrix satisfies $d=0$ and $\det X=1$. Hence the open subscheme $Y=\operatorname{Spec}((k[M_n]/(d))[(\det X)^{-1}])$ is geometrically integral. For a unit t, let $D_{i+1}(t)$ be the identity matrix with diagonal entry $i+1$ replaced by t. The following scheme maps are inverse on every test algebra:

$$\begin{aligned} Y &\longrightarrow Z\times\mathbb{G}_m, & X &\longmapsto (D_{i+1}((\det X)^{-1})X,\det X),\\ Z\times\mathbb{G}_m &\longrightarrow Y, & (Z_0,t) &\longmapsto D_{i+1}(t)Z_0, \end{aligned}\qquad Z=\{X\in SL_n : d(X)=0\}. \tag{4.11}$$

Scaling this row preserves d and multiplies the determinant by the scaling factor. Thus the maps are inverse and $k[Y]\simeq k[Z][t^{\pm1}]$. The left-hand side remains a domain after every field extension, and therefore so does $k[Z]$:

$$k[SL_n]/(d)\text{ is geometrically integral.} \tag{4.12}$$

Primality of each full frozen variable. Each full frozen variable f_h is, up to a nonzero scalar, the extreme minor of a corner at an original marked point. Distinct endpoint intervals give an ordinary matrix factor. If the endpoints share an interval, the factor is transmuted, but at $v=1$ its multiplication is that of the ordinary SL_n coordinate algebra, and the diagram $\kappa(d)$ represents the same classical minor.

For this individual corner choose a saturated system containing it. Cutting off the corner disk containing its original mark leaves another mark of the component, so the system can be completed [LS, Theorem 8.8]. The construction also applies to a singleton compact boundary circle, whose endpoints are distinct physical ports.

In the classical decomposition for this auxiliary system, the chosen corner is an independent factor. Section 4.2 therefore gives

$$B_0/(f_h)\simeq(\mathbb{C}[SL_n]/(d))\otimes_{\mathbb{C}}\mathbb{C}[SL_n]^{\otimes(r-1)}, \tag{4.13}$$

which is a domain by geometric integrality. Thus each (f_h) is a prime ideal of B_0. The auxiliary system proves an intrinsic ideal property and does not change the filtration fixed in Section 4.2.

Prime order and the minimum Laurent exponent. Combine the nonnegative-support inclusion of Section 4.1 with the primality from Section 4.2. In the initial classical torus,

$$P_0 = \mathbb{C}[x_v : v \in V] \;\subseteq\; B_0 \;\subseteq\; T_0^{H+}, \qquad B_0/(f) \text{ is a domain}, \quad f = x_h. \tag{4.14}$$

For every nonzero $b \in B_0$ and $r \geq 0$,

$$b \in f^r B_0 \quad \Longleftrightarrow \quad w_f(b) \geq r. \tag{4.15}$$

The forward implication follows from $B_0 \subseteq T_0^{H+}$. Conversely, take a positive initial monomial m not involving f and clear the other negative Laurent exponents of b. This gives $mb = f^r P$, where P is a finite sum of positive monomials, hence $P \in P_0 \subseteq B_0$. Every other seed variable x_j lies outside fB_0: otherwise $x_j/f \in B_0$ would contradict the nonnegative f exponents imposed by (4.14). Primality yields $m \notin (f)$, so $mb \in (f)$ implies $b = fb_1$. Remove one factor f and repeat to obtain (4.15). Equivalently, after inverting the other initial variables, both ends of (4.14) become $\mathbb{C}[f, x_j^{\pm 1} : j \neq h]$, and localization of the prime quotient stays injective. This identifies the geometric order with the torus order.

Writing $v_f(b) = \max\{r \geq 0 : b \in f^r B_0\}$, we obtain $v_f(b) = w_f(b)$ and $v_{f_i}(f_j) = \delta_{ij}$. For comparison, in [Francone, Example 4.1.1], the ring $\mathbb{C}[y, z, X, (y+z)/X]$ has prime X, whereas $y+z$ has geometric X-order one and minimum original Laurent X-exponent zero. It fails (4.14), since $(y+z)/X$ has a negative X exponent.

□

4.3 Classical coordinates and the collar correspondence

Proposition 4.3. *In the classical fiber,* $B_0 = U^+_{\mathrm{full},0}$.

Proof. Work over $\mathbb{C}$, with commutative $\mathbb{C}$-algebras as test algebras.

Fully pinned twisted local systems. Write B^+ and B^- for the upper and lower triangular subgroups of SL_n, and U^+ for the upper unitriangular subgroup. A decoration is a coset in SL_n/U^+; its underlying flag is its image in SL_n/B^+. A positive pinning is a pair consisting of a decoration and a transverse opposite flag, in the orbit of (U^+, B^-). Its stabilizer is $U^+ \cap B^-$. A local system assigns fibers and parallel-transport isomorphisms to paths, with composition under concatenation; a twisted local system is such a system on the nonzero tangent bundle with the fiber-loop holonomy specified below.

The fixed saturated arcs give a free fundamental groupoid graph with boundary intervals as objects. The tangent bundle of an oriented surface with nonempty boundary is trivializable. Require the tangent-fiber loop Ω to have holonomy sI, where $s = (-1)^{n-1}$. After choosing lifts of the graph edges, a fully pinned twisted representation is specified by r independent SL_n matrices. We write P_e for the resulting frame at boundary interval e, and T_γ for parallel transport along a path γ. A positive pinning has stabilizer group scheme $U^+ \cap B^- = 1$, trivial also over nonreduced test algebras. It therefore supplies a full frame and removes gauge freedom at each boundary fiber.

The ordinary map in [Wang, Section 3.5] from the surface $SO(2)$-frame groupoid to the thickened $SO(3)$-frame groupoid has fiber map $\mathbb{Z} \to \mathbb{Z}/2$ and kernel generated by Ω^2. The relevant equivalence is between twisted SL_n-representation functors: the prescribed $\Omega \mapsto sI$ satisfies $s^2 = 1$, so every such representation kills the kernel and factors uniquely through the quotient. The free-graph description gives this equivalence directly.

Section 4.2 gives $B_0 = \mathcal{O}(SL_n)^{\otimes r}$. The classical stated evaluation map is [Wang, Theorem 3.15]. Put

$$J_{ab} = \delta_{\bar{a},b}, \qquad E_{ab} = (-1)^{n-a}\delta_{\bar{a},b}. \tag{4.16}$$

so that $\det E = 1, E^2 = sI, E^{-1} = sE$. Let ρ_W be the parallel-transport matrix of an oriented saturated arc in Wang's framed lift. Both state indices are reflected in Wang's formula on PDF page 12. Recording the start index as the row and the end index as the column, the independent arc matrix is

$$M = J\rho_W^T EJ, \qquad \rho_W = (JMJE^{-1})^T. \tag{4.17}$$

This is a coordinate automorphism of each SL_n matrix factor. Thus evaluation is an isomorphism of schemes, including their functors on nonreduced test algebras.

We fix its relation to the IOS lift. IOS Definition 3.2 starts at the negative object E^-, uses the left-normal lift, and ends with a clockwise hook from E^+ through π to E^-; denote it by β_-. Our lift starts at the positive object E^+, uses the right normal, and ends with a clockwise hook from E^- through π to E^+; denote it by $\beta = \beta_+$. The fiber half-turn $R_\pi(x, v) = (x, -v)$ gives precisely $R_\pi(\beta_-) = \beta_+$.

Let T be the positive vertical direction in the thickening and J_Σ positive rotation through $\pi/2$ in the surface. The two frame maps are

$$j_+(v) = (T, v, J_\Sigma v), \qquad j_-(v) = (T, -v, -J_\Sigma v) = j_+(R_\pi v). \tag{4.18}$$

Under j_+ the positive boundary tangent t_e is Wang's frame $(T, t_e, J_\Sigma t_e)$. Wang's good-position hooks add a positive $\pi/2$ at each end. Relative to our right-normal lift with clockwise terminal half-turn, the total angle increases by 2π. Hence $\rho_W = s\rho_\beta$. Passing to the dual local system $\sigma(g) = Jg^{-T}J$ in (4.17) gives

$$ME^{-1} = J\rho_\beta^T J = \sigma(\rho_\beta)^{-1}. \tag{4.19}$$

This fixes the transpose, dual, inverse, and central sign s in the coordinate comparison.

A lift half-turn must also be distinguished from a change of canonical pinning. For a boundary interval denoted by E, let E^- and E^+ be its negative and positive tangent objects. In its negative pinning frame write the adjacent decorations as $\mathsf{A}_- = U^+$ and $\mathsf{A}_+ = E^{-1}h_E U^+$, where h_E is the unique diagonal determinant-one matrix giving the opposite decoration in this frame. In this expression E^{-1} is the inverse of the reversal matrix in (4.16). If $r_E : E^- \to E^+$ is the outward half-turn, the positive pinning exchanges the decorated and underlying endpoints. Thus [IOS, Section 3.3, Example 3.20] gives the gauge

$$P_{E^+} = T_{r_E}(P_{E^-})E^{-1}h_E. \tag{4.20}$$

The transformation R_π concerns lifts; comparison of the canonical pinnings also uses $E^{-1}h_E$. We use positive pinnings throughout and the specified triangle branch cut to compute decorations, thereby retaining this gauge in the comparison with the negative IOS Wilson convention.

At an original mark let e_-, e_+ denote the intervals of its two physical boundary ports, and let γ_p be the peripheral path transporting the latter port to the common fiber of the former. In the dual twisted local system, (4.19) gives the corner formula

$$C_p = P_{e_-}^{-1}\, T_{\gamma_p}(P_{e_+})\, E. \tag{4.21}$$

After transporting both frames to the same fiber this can be abbreviated as $C_p = P_-^{-1}P_+E$. Even when $e_- = e_+$, the transport γ_p retains the peripheral holonomy of the singleton circle. The physical ports remain distinct.

Raw volumes and normalized triangle coordinates. In a reduced triangle the bad relations make the corner matrices C_v lower triangular, and determinant one makes their diagonal entries units. Adjacent frames $P' = PC_vE^{-1}$ satisfy $P'B^+ = PB^-$, the matching condition for the underlying flags of adjacent decorations. Conversely, generic decorations determine unique positive pinnings [IOS, Remark 3.10], giving the same decorated-representation functor.

Take clockwise vertices v_1, v_2, v_3. Use the positive E^+ frame on e_3 as reference, and place the IOS triangle branch cut at v_3. Set $C_1 = C_{v_1}$ and $C_2 = C_{v_2}$. Clockwise boundary transport pulls the edge frames to one fiber as

$$P_3 = I, \qquad P_1 = C_1E^{-1}, \qquad P_2 = C_1E^{-1}C_2E^{-1}. \tag{4.22}$$

The IOS counterclockwise order is (v_3, v_2, v_1). Crossing the fixed cut changes the transport of each v_1, v_2 flag by one full fiber loop relative to clockwise pullback. The flags are therefore (U^+, sP_2U^+, sP_1U^+). Let $D = \operatorname{diag}(C_1)$ and $H = J\operatorname{diag}(C_2)J$ be the diagonal matrices of the corner weights. Lower triangularity of C_1, C_2 gives $sP_1U^+ = C_1EU^+ = DEU^+$ and $sP_2U^+ = C_1(E^{-1}C_2E)U^+ = C_1HU^+$. The transported flags are consequently

$$(\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3) = (U^+, C_1HU^+, DEU^+). \tag{4.23}$$

For a matrix X, write $X_{[1,d]}$ for its first d columns; vertical bars concatenate these column blocks. For $i, j, k \geq 0$ with $i + j + k = n$, (3.10) and exterior expansion give

$$g_{ijk} = \det\left[I_{[1,k]} \mid (C_1H)_{[1,j]} \mid (DE)_{[1,i]}\right]. \tag{4.24}$$

Here $\Lambda^i E(e_1 \wedge \cdots \wedge e_i) = e_{n-i+1} \wedge \cdots \wedge e_n$: the anti-diagonal coefficients and the reversal permutation contribute two factors $(-1)^{\binom{i}{2}}$, which cancel. Thus (4.24) has no additional sign.

For $0 \leq a \leq n$, let ω_a be the weight $\operatorname{diag}(t_1, \ldots, t_n) \mapsto t_1 \cdots t_a$; the empty product and the determinant are one, so $\omega_0 = \omega_n = 0$ in additive weight notation. The admissible fundamental triple invariant of [GS19(350)] satisfies

$$\Delta_{\omega_a,\omega_b,\omega_c} = (-1)^{bc}\,\delta_{abc}, \qquad a + b + c = n, \tag{4.25}$$

where δ is the raw block volume in the same order. On (A, U^+, EU^+), the chamber minor has row set $[b + 1, a + b]$ and Chevalley shuffle coefficient one. Moving b standard columns in the raw volume contributes $(-1)^{ab}$, while the GS normalization contributes $(-1)^{(n-1)b}$. Their ratio is $(-1)^{bc}$.

We use the cyclic argument order explicitly specified in [IOS, Remark 4.14]:

$$x_{ijk} := \Delta_{\omega_i,\omega_k,\omega_j}(\mathsf{A}_3, \mathsf{A}_1, \mathsf{A}_2).$$

Moving the last i block to the front gives

$$x_{ijk} = (-1)^{ij+ik+jk} g_{ijk}. \tag{4.26}$$

[GS19, Definitions 8.5–8.6, Theorem 8.7, Example 9.21] allow this distinguished vertex and standard reduced word in each face, followed by amalgamation. These are standard triangle seeds. The edge sign is $(-1)^{i(n-i)}$, invariant under exchanging the endpoints, so adjacent faces rescale their common coordinate consistently.

A global Cartan action realizes this symmetric sign. Choose $z^n = (-1)^{\binom{n}{2}}$ and put

$$h = \operatorname{diag}(z, -z, z, -z, \ldots) \in SL_n, \qquad \chi_a(h) = z^a(-1)^{\binom{a}{2}}.$$

The determinant of h is $z^n(-1)^{\binom{n}{2}} = 1$. Since $i+j+k=n$, the identity $\binom{n}{2} = \binom{i}{2} + \binom{j}{2} + \binom{k}{2} + ij + ik + jk$ gives

$$\chi_i(h)\chi_j(h)\chi_k(h) = (-1)^{ij+ik+jk}. \tag{4.27}$$

Apply the same h at every physical marked point. Weights add at repeated marks, preserving (4.27). By the proof of [IOS, Proposition 6.2], each mutable column has boundary weight zero; equivalently, its exchange monomials have equal Cartan weight. This property persists under mutation. The global action therefore rescales coordinates by scalars in every seed and preserves every Laurent ring. The normalized IOS and raw LY upper algebras are the same subalgebra of the common function field.

On a general surface, the elongated Y turns left along each mark's finite link interval. Corner triangularity multiplies an extremal exterior vector by its diagonal weight. These are exactly the factors of the unique surviving state in LY13.3. Thus the global raw seed is still the transported-flag volume (4.24), and the single global h gives the required normalization.

The inverse collar construction. The original surface here carries a full pinned system: its C_p are arbitrary SL_n matrices. Lower triangularity in Section 4.3 was a reduced-triangle condition. We first restrict the full system to its F-open locus, then compare it with generic decorations on the enlarged surface Σ^*. This localization is taken in the full algebra, before reduction can kill the top-right minors.

Let N be the number of original boundary intervals, and obtain Σ^* by attaching a triangle to each. For an old clockwise interval $r \to p$, label the attached triangle clockwise by (p, r, q_e). The retained side e_2 is $r \to q_e$ and the bridge e_3 is $q_e \to p$. The positive pinning on the retained side supplies a decorated flag at the old r and an opposite underlying flag at the new q_e.

Let H_{SL_n} be the diagonal torus. Here $\chi_k(\mathrm{diag}(t_1,\dots,t_n)) = t_1\cdots t_k$, with $\chi_0 = \chi_n = 1$, is the fundamental character of the diagonal torus. A pinning frame P_e gives the canonical decoration of its opposite flag

$$P_e E^{-1} U^+.$$

Every decoration of this underlying flag is uniquely $P_e E^{-1} U^+ \cdot h_e$ with $h_e \in H_{SL_n}$. The pinning trivializes the H-torsor $G/U^+ \to G/B^+$ naturally over every commutative test algebra.

Put

$$d_{p,i} = \det C_p[1,\dots,i;\ n-i+1,\dots,n], \quad F = \{d_{p,i} : p, 1 \le i < n\}. \tag{4.28}$$

By (4.21), in the common frame of the preceding pinning, the underlying flags at the bridge endpoints are $(B^-, C_p B^-)$. They are transverse exactly when all $d_{p,i}$ are units: the matrix of the relevant quotient projection is the top-right block in (4.28), and a square matrix over a test ring is invertible precisely when its determinant is a unit.

Starting with an F-open full pinned system and parameters h_e, extend the local system along the twisted tangent-groupoid equivalence induced by the specified collar embedding, using the lifts above. Assign the stated decorations to the old and new marks. The retained pairs are generic by the pinning, and the bridges by the F-open condition. Conversely, a generically decorated Σ^* system determines a canonical positive pinning on each retained e_2. Restrict it to the original surface and read off the unique h_e relative to the canonical opposite decoration. These constructions are inverse on objects and morphisms over all commutative $\mathbb{C}$-algebras. Their matrix words, inverses, and inverses of unit minors are regular, and the pinnings remove gauge automorphisms. Hence

$$\mathcal{A}^{\times}_{SL_n,\Sigma^*} \simeq \mathrm{Spec}\, B_0[F^{-1}] \times H^N_{SL_n}. \tag{4.29}$$

The two transported ports of a singleton circle and the T_{γ_p} holonomy in (4.21) are retained. Even if $p = r$ is one physical old mark, the calculation uses different lifts related by peripheral transport. The forward construction assigns one decoration to that mark; its transported column blocks need not coincide. The inverse reads the retained pinning and the decoration torsor. These constructions are inverse also for singleton boundary circles, proving (4.29) in that case.

We identify the initial coordinates of this functor with (4.1). In the attached triangle of clockwise order (p, r, q_e), retain the same IOS branch cut $v_3 = q_e$ as above and use the retained positive frame $P_e = I$ as common reference. The retained edge gives $\mathsf{A}_r = U^+$ and $\mathsf{A}_{q_e} = E^{-1}h_eU^+$. At the old p flag, changing from clockwise positive-pinning transport to the counterclockwise IOS path adds one fiber loop. Since $C_p = P_e^{-1}P_+E$, we obtain

$$(\mathsf{A}_p, \mathsf{A}_r, \mathsf{A}_{q_e}) = (C_pEU^+,\ U^+,\ E^{-1}h_eU^+), \qquad s(P_e^{-1}P_+)U^+ = C_pEU^+. \tag{4.30}$$

For $i + j + k = n$ the raw volume is therefore

$$\begin{aligned} g_{ijk} &= \det\left[(E^{-1}h_e)_{[1,k]} \mid I_{[1,j]} \mid (C_pE)_{[1,i]}\right] \\ &= \det(C_p)_{[j+1,j+i],[n-i+1,n]}\, \chi_k(h_e). \end{aligned} \tag{4.31}$$

The second equality has no additional s^i or scalar factor. Indeed, $Ee_a = (-1)^{a-1}e_{n+1-a}$ and $E^{-1} = sE$. Reordering the descending columns within the two exterior blocks gives canceling signs. Moving the last k rows of the first k block to the end leaves the total sign $s^k(-1)^{k(n-k)} = (-1)^{k(2n-k-1)} = 1$. This proves the determinant identity (4.31) for every rank.

For $i = 0$ the omitted retained-edge vertex $w_k = (0, n-k, k)$ has coordinate

$$y_{w_k} = g_{0,n-k,k} = \chi_k(h_e). \tag{4.32}$$

Dividing (4.31) by this $y_{p(v)}$ gives the LY/CHW consecutive minor $\det(C_p)_{[j+1,j+i],[n-i+1,n]}$, the full coordinate in (4.1). In particular, $j = 0$ gives the full frozen variable $d_{p,i}$. In [CHW, v1, Section 2.6], M_J^I uses rows I and columns J, agreeing with this order. Projected indices on the original surface retain their coordinates under restriction. The extra $s^k = (-1)^{k(n-k)}$ in the normalized IOS retained-edge invariant has already been accounted for by the global Cartan rescaling (4.26).

The $\chi_1, \dots, \chi_{n-1}$ form an integral basis of $X^*(H_{SL_n})$, since $t_1 = \chi_1, t_a = \chi_a/\chi_{a-1}, t_n = \chi_{n-1}^{-1}$. Thus (4.29) identifies the coordinate rings as

$$\mathcal{O}(\mathcal{A}^{\times}_{SL_n,\Sigma^*}) = B_0[F^{-1}][y_w^{\pm 1} : w \in W]. \tag{4.33}$$

Classical upper equality and removal of frozen poles. On the independent saturated matrix generators, the map in (4.33) is (4.17)–(4.19), hence is the classical skein collar map. Its initial-coordinate formula is (4.1). The augmented $\widehat{C}$ is unimodular, and the mutable-column identity gives the same y monomial for both exchange terms. Induction along an arbitrary mutation word yields

$$T_{\Sigma^*,t,0} = T_{\mathrm{full},t,0}[y_w^{\pm 1} : w \in W]. \tag{4.34}$$

All algebras here are commutative. An element of the initial enlarged torus has a unique finite expansion $\sum_{a\in\mathbb{Z}^W} u_a y^a$, with coefficients in the initial full torus. The y_w remain unchanged under mutation and are algebraically independent over the full fraction field. By (4.34), this element belongs to the enlarged torus at t exactly when every u_a belongs to $T_{\mathrm{full},t,0}$. Taking this condition for every t gives

$$\mathcal{U}_0(\Sigma^*) = U^{\mathrm{fr}}_{\mathrm{full},0}[y^{\pm 1}]. \tag{4.35}$$

By [IOS, Theorem 4.3] and the seed comparison in Section 4.3, the left side is the ring in (4.33). Comparing y^0 coefficients gives

$$B_0[F^{-1}] = U^{\mathrm{fr}}_{\mathrm{full},0}. \tag{4.36}$$

For $u \in U^+_{\mathrm{full},0}$, equation (4.36) gives a presentation $u = (f^r)^{-1}a$ with $a \in B_0$. Since all $w_{f_i}(u) \geq 0$, (4.15) makes a divisible by each corresponding $f_i^{r_i}$. Removing these factors successively does not change the orders at the other frozen variables, because $w_{f_i}(f_j) = \delta_{ij}$. Thus $u \in B_0$. The reverse inclusion is the classical case of (4.5), so

$$B_0 = U^+_{\mathrm{full},0}. \tag{4.37}$$

□

5 Degrees of the initial full seed webs

Proposition 5.1. *For every initial full mutable web g, its generic and classical degrees in the fixed saturated word filtration satisfy $\delta_K(g) = \delta_0(g_0)$.*

Proof. Both degrees use the saturated oriented arc system fixed in Section 4.2.

Embedded arcs and tripods. [LY, Section 11.3] constructs embedded elongated weighted Y-graphs on surfaces without interior punctures, and [LY, Section 13.2] gives the initial seed webs. Boundary small vertices give colored arcs; interior small vertices give colored tripods. Full variables from attached triangles are the corner consecutive minors $M_{[\bar{i},n]}^{[j+1,j+i]}(c)$ already defined on the original surface. All these webs are filtered by the fixed arcs on Σ. Every nonzero color lies in $1, \ldots, n-1$. By [LY, Lemma 13.3], the initial trace is a nonzero monomial, so each g_0 is nonzero.

The physical ports are distinct even when they correspond to the same compact marked point or open boundary interval. An embedded arc or tripod has a disk as regular neighborhood. The following argument concerns these webs.

To handle repeated compact corners, blow up each boundary mark p to its finite ordered link interval. Write the local half-disk as $0 < r < \varepsilon, 0 \leq \theta \leq \pi$, assigning distinct sectors to the triangle-corner occurrences. Choose the angular coordinate so that left-turn continuation approaches the terminal boundary side $\theta = 0$. For arms entering in the order $\theta_1 < \cdots < \theta_s$ choose $0 < r_s < \cdots < r_1 < \varepsilon$. Arm h travels radially along θ_h to r_h and then turns toward the terminal side at that radius. For $a < b$, the circular segment of arm a does not reach angle θ_b, while the circular segment of arm b has radius $r_b < r_a$ and lies inside the radial segment of arm a. Thus radial and circular segments do not cross; distinct circular segments occupy distinct radii. Repeated sectors entering the original triangle remain in the corner collar, away from the source and the original stems.

These nested tracks preserve the source cyclic order and produce distinct ordered terminal ports. A singleton boundary circle can identify both compact edge endpoints with p, but the endpoints of its link interval remain separate. Each continuation stops at the first terminal side, introducing no new cycle or twist. Diagrammatic endpoint heights are separate from these planar positions. LY Lemma 4.12 allows two or three blocks on one boundary edge, and Lemma 4.8 handles their height reordering.

Proper walls and a median representative. By [LS, Theorem 8.8], the complementary regions of the fixed saturated system $\mathcal{E}$ are ideal monogons, each a standard disk with one boundary ideal point. In each monogon choose disjoint spokes from that point to its arc-sides. Joining corresponding spokes across the two copies of a_e gives a proper ideal wall c_e crossing a_e once. All wall endpoints are actual marked ideal points.

Cutting along the walls gives disk cells, each containing exactly one original open boundary interval. Indeed, the spokes divide each ideal monogon into sectors containing segments of original boundary intervals. Pairings across arc-sides join their half-sides in the linear order along the interval. Since the boundary component is an open interval, the sector chain cannot close and forms a disk. The dual ribbon graph G has these intervals as vertices and the fixed a_e as edges; hence G is a ribbon spine of the surface. Homotopic walls, or a removable trivial wall, would cut off a cell containing no original boundary interval, contrary to construction. Even if a based wall has the same compact mark at both ends, its lifted endpoints $p, \gamma p$ are distinct.

Choose one metric for all seed webs. If the compact surface has negative Euler characteristic, take a compact hyperbolic core with geodesic boundary. Its universal cover $\Omega \subset \mathbb{H}^2$ is an intersection of boundary half-planes, hence convex. For fixed $\varepsilon > 0$, the outward neighborhood $\Omega_\varepsilon = \{x : d(x, \Omega) \leq \varepsilon\}$ is convex and has outward hypercycles as its actual boundary. Place the marked points on this boundary. Represent each wall and its deck translates by geodesic chords joining their actual marked lifts. Their endpoints do not alternate, so the chords remain disjoint. The core boundary geodesics serve only to define the metric.

For an annulus use the ε-tube of a hyperbolic axis modulo its translation, with the two hypercycles as boundary. For a disk use a strictly convex hyperbolic disk. Again represent walls by chords between actual marked endpoints. To preserve the fixed wall isotopy classes, truncate small collars near the marks, separate all wall incidences into distinct boundary endpoints, and align their orders in the collars. Apply [Epstein, Theorem 3.1] successively to the finite disjoint arc family, obtaining a simultaneous proper-arc isotopy from the walls to the chords, and extend it through the ideal collars. Pull the metric back by this isotopy. The original walls are then these chords, with the original a_e and cell data unchanged. This metric is fixed before choosing a web.

Let Y be an embedded tripod. Its three leaves are distinct physical points, even if they lie on one interval or share a compactification mark. Taper its disk neighborhood at the leaves to obtain three proper side arcs. In a lift, replace these sides by geodesic chords joining the same lifted leaves. Chords in a convex region cross exactly when their boundary endpoints alternate. Since the original sides and their deck translates are disjoint, the straightened sides remain disjoint from all translates. A small generic adjustment within the boundary intervals, preserving their order, makes the leaves noncollinear and gives a finite-area convex triangular disk D.

The boundary of D projects injectively: a nontrivial translate cannot share a lifted vertex without identifying distinct physical leaves or fixing a lifted actual point. The whole disk D also projects injectively. If the interiors of D and a nontrivial translate γD met, their disjoint boundaries would force one to contain the other. Their equal finite hyperbolic areas exclude strict containment. Equality would make a nontrivial deck transformation preserve a compact disk and have a fixed point, contradicting the free deck action. This also gives the present polygonal instance of [Epstein, Lemma 1.6].

The lifted wall cells are intersections of Ω_ε with chosen wall half-planes, hence convex; their intersections with D are connected. Only finitely many walls meet D, since D is compact and avoids marked endpoints. Each wall meets D in at most one chord. The chords divide D into finitely many cells whose dual is a finite tree, embedded without folding in the global dual tree. Place the source in the median of the three leaf cells and join it to the leaves by disjoint thin arms along tree geodesics, preserving cyclic order. Each wall word is reduced, and the nonempty arms have distinct first halfedges. The covering is injective on vertex stars, so their projected first halfedges are also distinct. Later portions of the projected arms may revisit edges, labels, and vertices.

This median tripod is ribbon-isotopic to Y. The original and new leaf-1-to-leaf-2 paths are proper simple arcs with the same endpoints and relative homotopy class. By [Epstein, Theorem 3.1] they are ambient-isotopic. Slide the trivalent vertex along the fixed path to the chosen source, then cut the surface along this path. Cyclic order determines the same side and cut component for the third arms. The fundamental group of that component injects into that of the surface by van Kampen for

a proper-arc cut. The third arms are therefore relatively homotopic in the cut component. Apply the same theorem there, fixing the first two arms. Gluing gives an isotopy preserving the three leaves, boundary order, and ribbon framing. The chord and proper-arc argument likewise gives a reduced representative of a colored arc.

The cost in the next subsection is computed on the median representative. The isotopy identifies this representative with the original skein element.

The determinant-cone lower bound. Write the fixed oriented-edge variables of Section 4.2 as independent X_e, retaining the orientation of each a_e : start $\to$ end. The dictionary in Section 4.3 gives

$$X_e = M_e E^{-1} = \sigma(\rho_e)^{-1}, \qquad \sigma(g) = Jg^{-T}J.$$

Here M_e is the stated-coefficient matrix of a_e, and ρ_e is its positive-lift holonomy as in (4.19). The map ρ_e uses column-vector holonomy, and $\sigma(\rho_e)$ transports dual columns from start to end. Its inverse X_e therefore transports dual row covectors forward by right multiplication. This agrees with the row-start, column-end stated-matrix convention. If a is traversed before b, column holonomy satisfies $\rho_{ab} = \rho_b\rho_a$, so

$$X_{ab} = \sigma(\rho_b\rho_a)^{-1} = X_a X_b.$$

The fixed endpoint basis change E^{-1} identifies state contraction with ordinary row-matrix multiplication. Changing the framed lift contributes only a fixed central scalar $s = (-1)^{n-1}$ of degree zero. The transformation $M_e \mapsto X_e$ is invertible and linear in the degree-one entries, preserving the fixed word filtration. Thus forward passages use $\Lambda^i X_e$ and reverse passages use $\Lambda^i X_e^{-1}$. The degrees i and $n-i$ of complementary minors have the same directional convention as the quantum count.

A forward traversal of color i uses $F_i(X) = \Lambda^i X$ of degree i, where $\Lambda^i X$ is the induced map on the i-th exterior power and its matrix entries in the increasing-subset basis are the size-i minors of X. A reverse traversal uses the polynomial cofactor operator

$$H_i(X) = \det(X)\Lambda^i(X^{-1}), \tag{5.1}$$

For increasing size-i subsets $I, J \subseteq \{1, \dots, n\}$, Jacobi's complementary-minor formula reads

$$(H_i(X))_{I,J} = (-1)^{\sum_{a\in I} a + \sum_{b\in J} b} \det X_{J^c, I^c},$$

where complements are increasing lists in $\{1, \dots, n\}$. Thus each entry is a polynomial of degree $n-i$. The formula defines H_i also on singular matrices and agrees with $\Lambda^i(X^{-1})$ when $\det X = 1$. Set

$$C(g) = \sum_{\text{forward traversals}} i + \sum_{\text{reverse traversals}} (n-i). \tag{5.2}$$

Tensor contraction along the reduced arms gives a homogeneous polynomial representative P_g of g_0 of degree $C(g)$. To prove that the filtration degree is $C(g)$, we show that P_g does not vanish identically on the product of determinant cones.

Let η be an indeterminate. For each edge choose independent matrix parameters $L_e, R_e \in SL_n$ and put

$$X_e = L_e \operatorname{diag}(\eta^1, \dots, \eta^{n-1}, 0) R_e. \tag{5.3}$$

For an increasing subset $I = \{i_1 < \cdots < i_d\}$, $|I\rangle = e_{i_1} \wedge \cdots \wedge e_{i_d}$ and $\langle I|$ is its dual coefficient functional. Among the nonzero size-i diagonal minors, the unique least exponent is $1 + \cdots + i = i(i+1)/2$, attained at $[1, i]$. The complementary-minor formula gives the least exponent $(n-i)(n-i+1)/2$

for H_i. Thus the lowest terms are the rank-one operators below; the omitted terms have strictly larger η-degree:

$$F_i(X_e) = \eta^{i(i+1)/2}(\Lambda^i L_e|[1,i]\rangle)(\langle[1,i]|\Lambda^i R_e) + \cdots, \tag{5.4}$$

$$H_i(X_e) = \eta^{(n-i)(n-i+1)/2}(\Lambda^i R_e^{-1}|[n-i+1,n]\rangle)(\langle[n-i+1,n]|\Lambda^i L_e^{-1}) + \cdots. \tag{5.5}$$

The only structurally forced zero between consecutive lowest factors is an immediate reversal on the same edge: its pairing is $\langle[1,i]|[n-i+1,n]\rangle = 0$, since $0 < i < n$. Reduced arms exclude it. Different edges have independent flags. Consecutive forward traversals of one loop give a minor of $R_e L_e$; consecutive reverse traversals give a minor of the corresponding inverse-order product. Both are nonzero polynomials.

The three first halfedges at a median are distinct, so the central volume uses independent edge-end flags. The two ends of a single loop arise from R_e and L_e^{-1} independently. Endpoint pairings with fixed decomposable states can also be nonzero. A zero-length arm retains its fixed covector. If the resulting fixed wedge vanished, the whole classical web would vanish, contrary to its nonzero trace in LY13.3. Otherwise the remaining generic flags complete it to a nonzero volume.

The parameter scheme of all L_e, R_e is integral. The product of the finitely many nonzero transition, endpoint, and central-pairing polynomials is nonzero, even when some flags are reused. Thus the lowest η coefficient of P_g is nonzero, and P_g does not vanish on the determinant cones. By (4.7),

$$\delta_0(g_0) = C(g). \tag{5.6}$$

The quantum upper bound. Work over $R_0 = \mathbb{C}[v^{\pm 1}]_{(v-1)}$ using finite skein identities. Each fixed wall vertex cell meets a genuine open boundary interval. A finite tangle in that cell with l strip ports can be placed in a boundary cube adjacent to that interval. The D-relation [LS, Section 7.2, equation (82)] gives a finite state sum with the same l external ports, assigning states to them and multiplying by counit or RT-matrix coefficients. It also permits $k = 0$ original endpoints on the marking. Here the counit evaluates a quantum coordinate function at the identity matrix, and the Reshetikhin–Turaev operator of the finite tangle is the linear map between the tensor products of its input and output state spaces. Proposition 6.6 of [LS] identifies its matrix coefficients with these counit values.

For the exterior projectors put $q = q_{\mathrm{LY}} = v^{-2n^2}$. If β_s is the positive fundamental braid crossing, use

$$T_s = q^{1/n}\beta_s = v^{-2n}\beta_s.$$

The defining relations [LY(64), LS(43)] give $T_s - T_s^{-1} = (q - q^{-1})1$, equivalently $(T_s - q)(T_s + q^{-1}) = 0$. The same relation follows directly from [LS, (9)]: the equal-state block is q, and for each $a < b$ the block on $(e_a \otimes e_b, e_b \otimes e_a)$ is

$$\begin{pmatrix} q - q^{-1} & 1 \\ 1 & 0 \end{pmatrix},$$

It satisfies the quadratic relation, with $(1, -q)$ of eigenvalue $-q^{-1}$. These blocks cover arbitrary n.

Let T_σ be the product along any reduced expression of $\sigma \in \mathfrak{S}_i$; the braid relations make it independent of that expression. For this Hecke convention,

$$A_i = \sum_{\sigma \in \mathfrak{S}_i} (-q)^{-\ell(\sigma)} T_\sigma, \qquad d_i = \sum_{\sigma \in \mathfrak{S}_i} q^{-2\ell(\sigma)}, \qquad P_i = d_i^{-1} A_i. \tag{5.7}$$

Fix an adjacent transposition s. Partition the permutations into pairs $\{\sigma, s\sigma\}$ with $\ell(s\sigma) = \ell(\sigma) + 1$. Their contribution to A_i is $(-q)^{-\ell(\sigma)}(T_\sigma - q^{-1}T_{s\sigma})$. Since $T_s T_\sigma = T_{s\sigma}$ and $T_s T_{s\sigma} = T_\sigma + (q - q^{-1})T_{s\sigma}$,

$$T_s(T_\sigma - q^{-1}T_{s\sigma}) = -q^{-1}T_\sigma + q^{-2}T_{s\sigma} = -q^{-1}(T_\sigma - q^{-1}T_{s\sigma}).$$

Summing proves $T_s A_i = -q^{-1} A_i$, hence $A_i^2 = \sum_\sigma (-q)^{-\ell(\sigma)} (-q^{-1})^{\ell(\sigma)} A_i = d_i A_i$, proving the idempotent normalization for every i. Since $d_i(1) = i! \neq 0$, one has $d_i \in R_0^\times$ and $P_i^2 = P_i$. These projectors are finite linear combinations of diagrams over the DVR.

Perform the following operations in order.

1. For a tripod, use the D-relation in the boundary-adjacent source cell to compare its external matrix coefficients and insert P_i in each color block. The volume tensor is alternating on every block, so the local skein element is unchanged. The Hecke identity $P_i^2 = P_i$ distributes copies of the projector along each embedded ribbon arm, placing an exterior block at every strip traversal. A source cell with no original endpoints is allowed by $k = 0$. For a colored arc, the initial minor is already an exterior block, from which the same distribution starts.
2. Resolve every vertex cell by the D-relation. Removing its actual boundary interval leaves an ordered interval. Connect the external ports to the true boundary by noncrossing fans in reverse order, following the fixed halfedge order of a_e. The relation assigns states to the original ports and adds no strip strands. Both ends of every strip now carry fans reaching actual boundary intervals.
3. Replace a reverse color block using $i \mapsto n - i$ only after these resolutions. The replacement region consists of the strip and both endpoint fans, forming a proper tubular bigon along the fixed a_e with its marked sides on genuine boundary intervals. Different traversals use parallel copies. If both sides lie on one open interval, the physical ports are still distinct; use the linear diagram transport of the ordinary bigon.

This last operation can be realized on the fixed generating arc. For a wall c_e, compress the finitely many web crossings in order toward its unique intersection with a_e, preserving order on each side and fixing the intersection. Extend the compression to a wall-collar isotopy preserving the transverse a_e and the other generators. Color blocks remain consecutive and disjoint. In each wall cell, the fixed half-arcs of $\mathcal{E}$ already form noncrossing fans from wall incidences to the physical boundary interval. Choose the D-resolution fans as parallel copies of these half-arcs. Their reverse order is the actual boundary order.

For each passage, its first fan leg, strip, and second fan leg now form an embedded ribbon rectangle along the same a_e. Repeated passages use distinct parallel subbands in that tubular neighborhood. If a_e is a loop, its two halfedge incidences are distinct sides in the cut cell and occupy distinct slots on the true interval, still forming a proper bigon. Changes in endpoint number occur within these consecutive slots, away from other bands. The operation therefore uses the tubular bigon of the fixed a_e.

In an ordinary bigon, the matrix coefficients of $W_i = \operatorname{im} P_i$ are quantum minors of size i. A forward block is therefore a finite expression with exactly i fundamental strands. Exterior multiplication gives the perfect comodule pairing

$$W_i \otimes W_{n-i} \longrightarrow W_n = R_0. \tag{5.8}$$

In the subset basis, only complementary subsets pair nontrivially, with value $\pm q^a$, so the pairing matrix is invertible over R_0. The identity $\det_q = 1$ makes the top exterior coaction trivial, giving $W_i^* \simeq W_{n-i}$. Thus reverse-block coefficients are finite R_0-linear combinations of quantum minors of degree $(n - i)$. The change in endpoint number takes place in the proper marked bigon constructed above.

Transport these ordinary-bigon identities linearly back to the tubular neighborhoods. The result uses parallel copies of the fixed generating arcs. The endpoint fans already follow the halfedge order on the original intervals, so these are the generators a_e of Section 4.2. If their heights differ from the chosen product order, [LY, Lemma 4.8] changes states and boundary heights without changing

the number of copies of an arc. Crossings, source coefficients, projectors, and state permutations contribute finite coefficients in R_0.

A forward traversal consequently uses at most i copies of fixed fundamental arcs, and a reverse traversal at most $n-i$ copies. Hence

$$g \in F_{C(g)} B_{R_0}. \tag{5.9}$$

The free successive layers imply that reduction cannot increase degree. Equations (5.6)–(5.9) give $C(g) = \delta_0(g_0) \le \delta_K(g) \le C(g)$, proving the proposition.

□

6 Quantum lifting and integral descent

6.1 Finite filtered lifting

Proposition 6.1. *One has* $B_{R_0} = U^+_{\mathrm{full},R_0}$ *and* $B_{K_0} = U^+_{\mathrm{full},K_0}$.

Proof. The common parameter valuation. For $0 \neq c \in \mathbb{C}(v)$, define $\mathrm{ord}_p(c) = r$ by writing $c = p^r g(v)/h(v)$, with $g, h \in \mathbb{C}[v]$ and $g(1)h(1) \neq 0$. On the initial torus T_{K_0} define

$$\nu_p\Big(\sum_a c_a M(a)\Big) = \min_{c_a \neq 0} \mathrm{ord}_p(c_a). \tag{6.1}$$

Set $\nu_p(0) = +\infty$. The residue torus is a classical Laurent domain, so products of lowest coefficients remain nonzero. Common Ore denominators extend ν_p to the division ring. To identify its residue field, write a finite numerator and denominator in the form $p^r a_0, p^s b_0$, with a_0, b_0 in the initial R_0-torus and at least one unit coefficient in each. Their quotient has valuation $r-s$. When this is zero, its residue is the classical rational function $\bar{a}_0/\bar{b}_0$ with nonzero denominator. Every classical rational function lifts in this way. Hence the residue field is the specified initial classical rational function field.

At $v = 1$, quantum mutation reduces to classical birational mutation. The residue coordinates of each new seed form an algebraically independent frame of the same function field. Their Laurent monomials are independent, so the minimum coefficient valuation in the new torus is again ν_p. Induction proves this for every finite mutation word.

Consequently $U^+_{R_0}$ is p-saturated in the initial $T^{H+}_{R_0}$: if $pu \in U^+_{R_0}$ and $u \in T^{H+}_{R_0}$, the minimum u coefficient valuation is nonnegative in every seed, and scalar division preserves frozen support. Thus $u \in U^+_{R_0}$. The reduction of u belongs to every classical mixed torus. By (4.37),

$$\overline{U^+_{R_0}} \subseteq U^+_0 = B_0. \tag{6.2}$$

Injectivity of LY13.1 at $v = 1$ identifies B_0 with the actual residue image of the full trace.

Finite rank of bounded-degree upper lattices. For an integer D define

$$E_D = \{0\} \cup \{u \in U^+_{R_0} : \delta_K(u) \le D\}. \tag{6.3}$$

Subadditivity of δ_K makes E_D an R_0-module. Scalars p have degree zero, so the preceding saturation makes E_D a p-saturated submodule of the initial free torus module. Therefore

$$E_D/pE_D \hookrightarrow T^{H+}_{R_0}/pT^{H+}_{R_0}. \tag{6.4}$$

Take $u \in E_D$ with $\bar{u} \neq 0$. A positive initial mutable monomial b clears the negative initial Laurent exponents of u. Frozen exponents are already nonnegative, so the product $a = bu$ is a positive initial polynomial and belongs to B_{R_0}. Section 5 and the graded-domain property give

$$\delta_0(\bar{b}) = \delta_K(b).$$

We have $\bar{b} \neq 0$ and, in the residue torus, $\bar{a} = \bar{b}\bar{u} \neq 0$. By (6.2), $\bar{u} \in B_0$, whence

$$\begin{aligned} \delta_0(\bar{u}) &= \delta_0(\bar{a}) - \delta_0(\bar{b}) \\ &\leq \delta_K(a) - \delta_K(b) = \delta_K(u) \leq D. \end{aligned} \tag{6.5}$$

To recover an integral filtered bound from the generic one, use the fixed basis slices in (4.6): $F_d B_{R_0} = B_{R_0} \cap F_d B_{K_0}$ inside B_{K_0}. Thus $\delta_K(a) = d$ implies $a \in F_d B_{R_0}$ and then $\delta_0(\bar{a}) \leq d$. Applied to a, this gives the numerator inequality. For the denominator b we use the strictness proved in Section 5. Hence the image in (6.4) lies in the finite-dimensional space $F_D B_0$.

Enumerate the initial mixed-torus monomials as $e_1, e_2, \ldots$, and let $L_m = \bigoplus_{j=1}^{m} R_0 e_j$, with $L_0 = 0$. The coordinate projection $\mathrm{pr}_m : L_m \to R_0$ gives an exact sequence

$$0 \longrightarrow E_D \cap L_{m-1} \longrightarrow E_D \cap L_m \xrightarrow{\mathrm{pr}_m} \mathrm{pr}_m(E_D \cap L_m) \longrightarrow 0.$$

The image is an ideal of the DVR R_0. If it is zero, the first inclusion is an equality. Otherwise choose an element whose m-th coordinate generates that ideal. Its span is free of rank one and splits the sequence: subtracting its appropriate multiple from any element leaves zero m-th coordinate. Induction therefore gives nested free bases of $E_D \cap L_m$. Every torus element has finite support, so their union is a basis of E_D.

Reduction of this direct-sum basis modulo p is a basis of E_D/pE_D. By (6.4)–(6.5), that vector space embeds in $F_D B_0$. It follows that the basis has at most $\dim_{\mathbb{C}} F_D B_0$ elements. Thus E_D is finite free.

Nakayama and generic equality. By (4.5), $F_D B_{R_0} \subseteq E_D$. Its reduction image is $F_D B_0$, and (6.5) places the residue image of E_D in that same space. The images therefore coincide. For $u \in E_D$, choose $b \in F_D B_{R_0}$ with the same residue. Then $u - b = pw$ in the initial torus. Since $u - b \in E_D$ and E_D is p-saturated there, $w \in E_D$. Hence

$$E_D = F_D B_{R_0} + pE_D. \tag{6.6}$$

The quotient $E_D/F_D B_{R_0}$ is finitely generated and equals its multiple by p. Choose generators $z_1, \ldots, z_l$ and express $z_i = p\sum_j a_{ij} z_j$, with $a_{ij} \in R_0$. Multiplying by the adjugate of the matrix $(\delta_{ij} - pa_{ij})$ shows that its determinant annihilates every z_i. That determinant is congruent to one modulo p, hence is a unit of R_0. Thus the quotient is zero. Thus for all D,

$$E_D = F_D B_{R_0}, \qquad B_{R_0} = U^{+}_{\mathrm{full},R_0}. \tag{6.7}$$

For $D < 0$ set $F_D = 0$; the same finite-module argument gives $E_D = 0$. Every upper element has finite fraction degree, so taking the union exhausts the upper algebra.

For $u \in U^{+}_{\mathrm{full},K_0}$, the common valuation ν_p supplies one $m \geq 0$ such that $p^m u$ has Laurent coefficients in R_0 in every seed, with unchanged frozen support. Hence $p^m u \in U^{+}_{R_0}$, and (6.7) gives

$$B_{K_0} = U^{+}_{\mathrm{full},K_0}. \tag{6.8}$$

For the fully frozen-inverted version, the equality $B_{K_0} = U^{+}$ first supplies the needed Ore property. Each frozen h conjugates monomials in every seed torus by units; conjugation and its inverse preserve mixed support, giving $hB_{K_0} = B_{K_0}h$. The frozen elements pairwise commute up to units and generate a regular Ore set. The common valuations w_h of Section 4.1 then give one finite monomial f^N making all frozen valuations of any $u \in U^{\mathrm{fr}}$ nonnegative. Therefore

$$U^+[H^{-1}] = U^{\mathrm{fr}}, \qquad B_{K_0}[H^{-1}] = U^{\mathrm{fr}}_{\mathrm{full},K_0}. \tag{6.9}$$

□

6.2 Integral descent and the projected algebra

Proposition 6.2. *Over $A = \mathbb{Z}[v^{\pm 1}]$, the projected skein algebra equals the upper cluster algebra.*

Proof. Descent from $\mathbb{C}(v)$ to $\mathbb{Q}(v)$. Put $K = \mathbb{Q}(v)$ and $L = \mathbb{C}(v)$. The skein presentation and finite trace state sum commute with coefficient extension. Full-trace injectivity gives

$$B_L = B_K \otimes_K L \subseteq T_K \otimes_K L = T_L.$$

For $u \in U^+_{\mathrm{full},K}$, each finite seed Laurent expression is also defined over L. By (6.8) the element belongs to B_L. Apply faithful flatness of field extension to the vector-space exact sequence $0 \to B_K \to T_K \to T_K/B_K \to 0$. The quotient class of u vanishes after extension and therefore already vanishes. Hence

$$U^+_{\mathrm{full},K} \subseteq B_K. \tag{6.10}$$

Full-trace saturation at scalar primes. Continue with $A = \mathbb{Z}[v^{\pm 1}]$. For an irreducible $\pi \in A$, the ring $A/(\pi)$ is an allowed coefficient domain and the image of v is invertible. The full trace is injective over this domain by [LY, Theorem 13.1(a)].

The skein presentation is the free module on finite diagrams modulo Laurent-coefficient relations, so it commutes with coefficient reduction. The trace uses finite state sums, and the balanced A/X inverse is an integral monomial-lattice map. The coefficient square commutes. If $b \in B_A$ has trace in πT_A, its class in $B_A/\pi B_A$ has zero trace over $A/(\pi)$. Injectivity in that fiber gives $b \in \pi B_A$. The reverse inclusion follows from linearity, proving

$$B_A \cap \pi T_A = \pi B_A. \tag{6.11}$$

For $u \in U^+_{\mathrm{full},A}$, equation (6.10) provides nonzero $d \in A$ such that $du \in B_A$. Factor d in the UFD A. If $\pi \mid d$, then $du \in B_A \cap \pi T_A$. By (6.11), write $du = \pi b_1$ and cancel π in the free A-module T_A, obtaining $(d/\pi)u = b_1 \in B_A$. Repeating finitely many times yields

$$U^+_{\mathrm{full},A} \subseteq B_A. \tag{6.12}$$

The finite-neighbor criterion. For a compatible quantum seed over A, [BZ, Theorem 5.1, (5.2)–(5.3)] gives

$$U^{\mathrm{fr}} = T_0 \cap \bigcap_{k \in \mathrm{Mut}} T_{\mu_k 0}. \tag{6.13}$$

Compatibility gives full column rank of the extended exchange matrix, the condition in the BZ coprimality argument. The $P, 2n^2$ normalization of Section 2.3 makes the theorem applicable.

By the common frozen valuations of Section 4.1, mixed membership is equivalent to (6.13) together with nonnegative initial H exponents. Thus

$$U^+ = T_0^{H+} \cap \bigcap_{k \in \mathrm{Mut}} T_{\mu_k 0}^{H+}. \tag{6.14}$$

The valuation comparison is established over the fraction field. Requiring coefficients in A leaves support unchanged, so (6.14) also holds integrally. General specializations will be treated separately in Section 7.

Clearing denominators before taking the quotient. Partition the full initial indices as

$$I \sqcup F \sqcup E_{\mathrm{mut}} \sqcup H. \tag{6.15}$$

Here I is the projected mutable set. The original boundary small vertices F are frozen in the projected seed and mutable in the full seed. The other indices are the attached mutable vertices E_{mut} and full frozens H.

Formula [LY(208)] and the CHW initial construction give two facts. The full P restricts on $I \cup F$ to the projected commutation form. For each $i \in I$, its full exchange column vanishes on $E_{\mathrm{mut}} \cup H$ and restricts on $I \cup F$ to the projected column: an original mutable vertex and an attached vertex lie in no common triangle. Arrows between original boundary vertices can cancel after triangle attachment, but they are frozen–frozen entries of the projected seed.

Take $u \in \mathcal{U}_A(\Sigma)$ and use the restricted P to view it in the full initial torus with variables only in $I \cup F$. Choose a positive ordered F-monomial m so that

$$b = mu \tag{6.16}$$

has nonnegative initial F exponents. We check the finitely many tori in (6.14):

- In the initial mixed torus, the H exponents are zero.
- At a neighbor in $i \in I$, the exchange relations coincide, u lies in the projected neighbor torus, and the factors of m are unchanged. Hence b lies in the full mixed neighbor.
- At a neighbor in $e \in E_{\mathrm{mut}}$, every variable of u, m is unchanged.
- At a neighbor in $f \in F$, the expression b has only nonnegative powers of x_f. The reverse exchange relation writes each x_f as $x_f'^{-1}$ times the sum of two exchange monomials. Their H exponents are nonnegative, so the finite expansion lies in this mixed neighbor torus.

Equations (6.14) and (6.12) give $b \in B_A$. Expression (6.16) may still contain mutable inverses. Choose a positive I-monomial c such that

$$cb = P(x_v : v \in I \cup F) \tag{6.17}$$

is a finite linear combination of positive monomials. The three elements c, b, P belong to B_A, so (6.17) is an identity in that algebra.

Now apply the algebra map $\pi : B_A \to S_A \subseteq T_{\mathrm{proj},A}$ obtained from the full-to-reduced quotient followed by the projected trace. The initial $I \cup F$ webs map to the corresponding projected variables [LY, Section 13.2, Lemma 13.3]. Hence

$$\pi(c)\pi(b) = P_{\mathrm{proj}} = \pi(c) m_{\mathrm{proj}} u.$$

In the projected torus, $\pi(c)$ is a nonzero invertible monomial, so left cancellation gives $\pi(b) = m_{\mathrm{proj}} u$. By Section 3.3, m_{proj} is a unit in S_A itself, also for singleton boundary circles. Therefore

$$\mathcal{U}_A(\Sigma) \subseteq S_A(\Sigma). \tag{6.18}$$

Together with (3.27), this gives

$$S_A = \mathcal{A}_A = \mathcal{U}_A. \tag{6.19}$$

The two denominator clearings keep the argument inside the algebra on which the quotient is defined: full frozen variables H are not inverted, and π is applied before passing to fractions.

□

7 Coefficient specialization

Proposition 7.1. *For every coefficient domain R and unit v_R, one has $S_R(\Sigma) = \mathcal{A}_R(\Sigma) = \mathcal{U}_R(\Sigma)$.*

Proof. To pass from (6.19) to nonflat coefficient changes, let $M = \mathbb{Z}^{V_\lambda}$ be the A-exponent lattice, let I be its mutable index set, so that the mutable lattice is $\mathbb{Z}^I$. We use a basis theorem for the upper algebra.

Conventions and reddening. For the actual projected seed P, Q, take

$$B_{\mathrm{DM}} = -Q, \qquad \Lambda_{\mathrm{DM}} = -P/(2n^2), \qquad t = v^{-2n^2}. \tag{7.1}$$

Then $\mathbb{Z}[t^{\pm 1/(2n^2)}] = A$ and $t^{\Lambda_{\mathrm{DM}}(a,b)} = v^{P(a,b)}$. In column conventions the DM matrix $B_1(e_i)$ is $Q(-,i)$, and compatibility $PQ(-,i) = -2n^2 e_i^T$ gives its normalized relation. Differences of theta-support exponents belong to the integral monoid $+Q\mathbb{N}^I$, matching OQY dominance. Each scattering bend adds an actual integral degree in $B_1(\mathbb{Z}^I)$, so all support differences lie in this integral monoid. DM's permitted half-integral frozen–frozen entries are compatible with these parameters.

More explicitly, DM mutation changes the basis to $e'_k = -e_k$ and $e'_r = e_r + [-Q(r,k)]_+ e_k$ for $r \neq k$, and changes the lattice vector to $T_k(m) = m + [m_k]_+ Q(-,k)$. Pairing with the new basis gives the additional term $Q(r,k)[m_k]_+ + [-Q(r,k)]_+ m_k$ in coordinate r. Simplifying separately for $m_k \geq 0$ and $m_k \leq 0$ gives the tropical coordinates

$$m'_k = -m_k, \qquad m'_r = m_r + [Q(r,k)]_+ [m_k]_+ - [-Q(r,k)]_+ [-m_k]_+, \tag{7.2}$$

These are the coordinates of [OQY, Section 3.3]. Thus both the torus product and compatible pointedness use the same conventions.

[GS16, Theorems 1.3 and 1.12] give a cluster DT transformation, and hence reddening, for admissible surface principal quivers. Here g is the genus and μ the number of punctures and boundary special points; in our unpunctured setting, μ is the number of boundary marked points. Admissibility requires $g + \mu \geq 3$, or the exceptional annulus with one mark on each boundary circle. If $g \geq 1$, then $\mu \geq 2$ gives the inequality. If $g = 0$ and $\mu = 2$, every boundary circle being marked forces the surface to be a bigon or this annulus; triangulability excludes the bigon. The case $g = 0, \mu \geq 3$ satisfies the inequality.

By GS16(260), deleting boundary frozen vertices from the A-quiver gives the X-quiver, hence the required principal mutable quiver. For the c/g-vector comparison, use principal coefficients, obtained by adjoining an identity frozen block to the principal exchange matrix. Let C_i be the row c-vector in the GS convention; initially $C_i = e_i$. Sign coherence makes its nonzero entries have one sign, denoted $\mathrm{sgn}(C_i)$. Let C be the transpose of the matrix of these rows, and let G collect the principal g-vectors as columns. GS uses $\varepsilon = -Q_{\mathrm{prin}} = Q_{\mathrm{prin}}^T$, with row c-vector recurrence at $i \neq k$ given by

$$C'_i = C_i + [\mathrm{sgn}(C_k)\varepsilon_{ik}]_+ C_k, \qquad C'_k = -C_k. \tag{7.3}$$

After transposing the row matrix, $\varepsilon_{ik} = Q_{ki}$ identifies this with the Fomin–Zelevinsky/Qin column c-vector recurrence for exchange columns Q. The terminal negative permutation in GS is therefore the negative permutation in these conventions. Skew-symmetric tropical duality $G^T C = I$ [Qin, Theorem 2.2.6] makes the terminal principal g-vectors a negative basis permutation. Extended labels add only integral frozen coordinates, and the principal g-vectors are independent of the added frozen rows. GS

Theorem 3.10 supplies the corresponding result for the opposite quiver. A seed is injective-reachable when a finite mutation sequence gives mutable variables whose principal g-vectors, the mutable parts of their pointed labels in the initial seed, are the negatives of the standard basis up to permutation. The actual full-rank seed therefore has this property. For the $n = 2$ triangle with no mutable directions, the scattering diagram is empty and the conclusion is immediate.

Finiteness of all theta functions. A Laurent series is m-pointed if its coefficient at m is one and its support is contained in $m + Q\mathbb{N}^I$, with $Q\mathbb{N}^I = \{\sum_{i\in I} c_i Q(-,i) : c_i \geq 0\}$, viewed as exponent vectors. Compatible pointedness means that the label is transported by (7.2) in every seed under consideration. For this lattice, [DM, Theorems 1.1–1.2, Corollaries 4.13 and A.4] construct formal ϑ_m with leading coefficient one and support in $m + Q\mathbb{N}^I$. Laurent coefficients and structure constants are positive Laurent polynomials. In $\vartheta_p\vartheta_q$, the coefficient of the sum-label term ϑ_{p+q} is a monomial unit. Localized cluster monomials are theta functions; a theta function Laurent-finite in one seed belongs to the upper algebra and is compatibly pointed after mutation.

Let $\Theta \subseteq M$ be the Laurent-finite labels. If $p, q \in \Theta$, their product is a finite Laurent polynomial. In its formal theta expansion, $t^{\Lambda_{\mathrm{DM}}(p,q)}\vartheta_{p+q}$ cannot be canceled by other positive summands. Hence the support of ϑ_{p+q} lies in the finite support of the product, and $p + q \in \Theta$. This argument is over universal positive coefficients.

The initial variables give $e_i \in \Theta$, and frozen Laurent monomials give both signs of every frozen basis vector. Injective reachability supplies cluster variables with extended g-vectors $-e_{\sigma(i)} + c_i$, where c_i is supported on frozen indices. Adding the finite frozen label $-c_i$ gives each mutable $-e_i \in \Theta$. Closure under addition of these signed basis vectors yields

$$\Theta = M. \tag{7.4}$$

Every formal theta function is therefore a finite element of $\mathcal{U}_A$.

The pointed basis theorem over arbitrary rings. We use Section 3 of [OQY, 2512.05228v2, 2026-01-29]. Full rank in its Section 3.3 is an integral-lattice condition, satisfied here by compatibility. Theorem 3.17 states that, for an injective-reachable full-rank seed, an upper-algebra family indexed by all lattice labels and compatibly pointed along a reaching sequence is a basis over the coefficient ring.

The restriction involving the possible zero divisor 2 in Assumption 3.1 concerns isolated extended mutable columns. Our extended matrix has full column rank over the integers, hence no zero mutable column. Lemma 3.15 applies: the exchange products are monic regular polynomials. Through [Qin, v4, Proposition 4.2.1, Theorem 4.3.1; Lemmas 3.1.2 and 3.4.12], Theorem 3.17 obtains finite spanning from monic leading terms, a truncated nilpotent-ideal Nakayama argument, and finite intervals between dominance bounds. The nilpotence of that ideal makes the argument valid over the coefficient ring in Theorem 3.17.

By (7.4), our theta family satisfies this theorem. For a commutative unital ring R with a chosen $v_R \in R^\times$, OQY Lemma 3.20 makes finite mutation expressions compatible with specialization and preserves leading coefficient one and pointedness. Applying Theorem 3.17 again, or Corollary 3.21 directly, gives

$$\mathcal{U}_A \otimes_A R \;\simeq\; \mathcal{U}_R, \qquad \{\vartheta_{m,R} : m \in M\} \text{ is an } R\text{-basis.} \tag{7.5}$$

All frozen variables are inverted here. We apply (7.5) to the coefficient domains allowed in the conjectures.

Specializing finite skein preimages. By (6.19), every universal ϑ_m has a finite stated-diagram preimage with coefficients in A. Specializing this expression and its finite trace state sum gives $\vartheta_{m,R}$ in S_R. These images span $\mathcal{U}_R$ by (7.5), so

$$\mathcal{U}_R \subseteq S_R.$$

The finite local formulas of Section 3.2 and the quantum Laurent identities specialize as well, giving $S_R \subseteq \mathcal{A}_R \subseteq \mathcal{U}_R$. Hence

$$S_R(\Sigma) = \mathcal{A}_R(\Sigma) = \mathcal{U}_R(\Sigma) \tag{7.6}$$

for every allowed coefficient domain and parameter.

For disconnected surfaces, apply the argument directly to the block seed of a finite disjoint union. Take the union of the saturated systems, the sum of word lengths, and the product of determinant cones. The full classical pinned and collar functors are componentwise products; the y support is still finite and unique. Over the characteristic-zero field, the classical tensor identity can also be obtained by intersecting coefficientwise in the finite Laurent support of the other component, and then interchanging the components. Integral saturation, finite-neighbor transfer, concatenated reddening, and the all-label theta argument apply to the entire block seed.

□

8 The Huang–Wang generating family

Proposition 8.1. *The algebra generated over the frozen Laurent coefficient algebra by the variables in Huang–Wang Conjecture 1.6 equals $S_R(\Sigma)$.*

Proof. The coefficient algebra in the conjecture is $\mathbb{B}_{\mathrm{fr},R}$. A frozen diagonal numerator is therefore already a coefficient. The correction in (3.24) at $(i,j) = (1,n)$ replaces the printed numerator F_{12} by F_{01}. We must consequently compare with the originally specified generating family.

Ordered quantum cofactors of the reverse arc. In the ordinary full bigon, index u_{ij} by fixed physical sides, and index the reverse arrow $\vec{u}$ by those same sides. Let S be the Hopf-algebra antipode of the quantum coordinate algebra of the bigon, characterized by inversion of its fundamental matrix. Formulas [LY(40), (83)] give

$$S(u_{ij}) = (-q)^{i-j} \det_q(u^{ji}) = (-q)^{i-j} \vec{u}_{\bar{j},\bar{i}}, \tag{8.1}$$

where $q = q_{\mathrm{LY}} = v^{-2n^2}$ and u^{ji} deletes row j and column i. Cancel the common unit and apply the full-skein orientation-reversal algebra automorphism [LY, Proposition 4.4] to obtain

$$u_{\bar{j},\bar{i}} = \det_q(\vec{u}^{ji}). \tag{8.2}$$

For an oriented essential arc γ, its tubular bigon has physical sides on distinct open boundary intervals. The strict embedding of [LY, Section 4.12] induces an algebra homomorphism. Apply it to (8.2) in the full surface skein algebra and then project.

Let U_{ij} be the entries of γ indexed by its own start and end states, and let W_{ab} be the analogous entries of the oppositely oriented arc. The fixed-side and intrinsic indices satisfy $W_{ab} = \vec{U}_{ba}$. With $i = 1, j = n$ in (8.2), the row-ordered determinant becomes the following column-ordered polynomial in W:

$$U_{1n} = \sum_{\sigma \in \mathfrak{S}_{n-1}} (-q)^{\ell(\sigma)} W_{\sigma(1)+1,1}\, W_{\sigma(2)+1,2} \cdots W_{\sigma(n-1)+1,n-1}. \tag{8.3}$$

Each first index on the right is at least two and each second index is at most $n-1$. Thus all entries avoid the bad endpoint $(1,n)$. For $n = 2$ the expression is $U_{12} = W_{21}$.

The original generating family and S_R. Let G_R be the algebra over $\mathbb{B}_{\mathrm{fr},R}$ generated by the exchangeable variables appearing in the numerator words as printed in HW PDF equations (9), (11), and (12), corresponding to HTML equations (15), (17), and (18). Take their union over the oriented essential arcs and adapted triangulations allowed in the original definition.

The corrections of Section 3.2 retain the original mutation words for all $m \geq 2$ and change only their expansions and terminal factors. The other valid arc formulas retain their numerators. Except for D_{1n}, every essential arc entry therefore belongs to G_R through these numerators and actual frozen inverses; a frozen numerator is already a coefficient. For D_{1n} use (8.3). Each reverse-oriented entry on its right has an unchanged valid formula and belongs to G_R. Essential-arc generation from Section 3.2 gives $S_R \subseteq G_R$.

Conversely, multiply the correct arc formula of each unchanged numerator by its actual boundary factors to place that numerator in S_R. The extra printed term F_{12} is an initial variable of an adapted triangulation, represented by the web in [LY, Lemma 13.3] and [HW, Theorem 2.11(a)]. If frozen, it is already in the coefficient algebra. Section 3.3 supplies all frozen units. Hence

$$G_R = S_R. \tag{8.4}$$

Together with (7.6), this proves Conjecture 1.6 for its original generating family.

□

Proof of Theorem 1.1. Proposition 7.1 gives the equality of the projected skein, cluster, and upper cluster algebras over every allowed coefficient domain. Proposition 8.1 identifies their generators over the frozen Laurent algebra with the family in the original Conjecture 1.6. The surface and parameter assumptions in both propositions are those of the theorem. □

A Local mutation rows and quadrilateral cutting

We give the complete exchange-row calculations used in Section 3.2, for arbitrary integral indices in the stated ranges.

A.1 Forward complete rows

Lemma A.1. *The forward mutation word of Section 3.2 has exactly the four exchange-neighbor lists in the table below. Deleting pivots with second index at least m leaves the rows and variables used in the truncated word unchanged.*

Proof. Fix i. Visit $p = i-1, \dots, 2$ in order, and $q = 1, \dots, p-1$ within each row. A coordinate Z_{pq} has already been mutated. The table gives the neighbors in the two exchange monomials; reversing every arrow exchanges the two sides without changing the relation.

Pivot condition	First-side neighbors	Second-side neighbors
$p = i-1,\ q = 1$	$A_{i-1,0}, A_{i-2,1}, A_{i,2}$	$A_{i-2,0}, A_{i,1}, A_{i-1,2}$
$p = i-1,\ q > 1$	$A_{i-1,0}, A_{i-2,q}, A_{i,q+1}$	$Z_{p,q-1}, A_{p,q+1}$
$p < i-1,\ q = 1$	$Z_{p+1,1}, A_{p-1,1}$	$A_{p-1,0}, A_{i,1}, A_{p,2}$
$p < i-1,\ q > 1$	$Z_{p+1,q}, A_{p-1,q}$	$Z_{p,q-1}, A_{p,q+1}$

Corner labels equal one in products and zero as incidence vectors. When boundary labels coincide, their row contributions are added or canceled.

Describe mutation at k by incoming and outgoing incidence vectors I_k, O_k. Let S_k be diagonal, with entry -1 at position k and one elsewhere. Matrix mutation is

$$Q_{\text{new}} = S_k(Q_{\text{old}} + I_k \wedge O_k)S_k, \qquad (I \wedge O)_{uv} = I_u O_v - O_u I_v, \tag{A.1}$$

with conventions $I_u = [Q(u,k)]_+$ and $O_u = [Q(k,u)]_+$. The opposite convention exchanges the two sides of the table.

If previous pivots have each been visited once, let $S^{(r)}$ take value -1 on the visited indices. Expanding (A.1) gives

$$S^{(r)}Q^{(r)}S^{(r)} = Q^{(0)} + \sum_{\ell<r}(S^{(\ell)}I_\ell) \wedge (S^{(\ell)}O_\ell). \tag{A.2}$$

In a previously verified complete row, every interior neighbor visited later is an axial grid neighbor; exceptional terms lie on unvisited boundary indices. Hence the row at (p,q) receives earlier contributions only from the axial pivots $(p+1,q)$ and $(p,q-1)$. Substituting these known rows in (A.2), in the four cases determined by the top row and first column, gives the table, with all other entries zero. Bottom, top, and diagonal coincidences cancel in the same integral formula. This induction over the chronological word proves completeness of the neighbor lists.

Substituting the top boundary values

$$Z_{i,q} = A_{i-1,0}A_{i,q+1}, \qquad Z_{p,0} = A_{i1}A_{p-1,0}$$

in (3.12) recovers the extra top-row and first-column factors in the table.

Under truncation at $q \geq m \geq 2$, a deleted pivot is outside the first column. In the top-row case, its neighbors are $A_{i-1,0}, A_{i-2,q}, A_{i,q+1}, Z_{p,q-1}, A_{p,q+1}$; the smaller-$q$ neighbor $A_{i-1,0}$ is frozen. Below the top row, its neighbors are $Z_{p+1,q}, A_{p-1,q}, Z_{p,q-1}, A_{p,q+1}$. Thus the only smaller-$q$ mutable neighbor is the already visited $Z_{p,q-1}$. Every other unvisited pivot neighbor has second index at least m, and no neighbor is y_s with $s < m$. Consequently the last deleted mutation commutes with every later retained mutation. Moving it to the end leaves the exchange rows at those retained pivots unchanged. Induction on the number of deleted mutations gives the truncated word used in Section 3.2, with all frozen terms retained.

□

A.2 Reverse complete rows

Lemma A.2. *The reverse mutation word of Section 3.2 has the four exchange-neighbor lists below and yields recurrence (3.15).*

Proof. Fix $j < n$. Visit $k = j-1, \dots, 1$ in decreasing order, and within each row visit $t = n-j, \dots, 1$. Before mutation at $F_{k,k+t}$, the visited positions in the next row and preceding column carry $W_{k+1,t}$ and $W_{k,t+1}$, respectively. The table uses these original coordinates.

Condition	First-side neighbors	Second-side neighbors
$k = j-1,\ t = n-j$	$F_{j,n-1}, F_{j-1,n}, F_{j-2,n-2}$	$F_{j,n}, F_{j-1,n-2}, F_{j-2,n-1}$
$k = j-1,\ t < n-j$	$F_{j,j+t-1}, F_{j-1,n}, F_{j-2,j+t-2}$	$W_{j-1,t+1}, F_{j-1,j+t-2}$
$k < j-1,\ t = n-j$	$W_{k+1,n-j}, F_{k-1,k+n-j-1}$	$F_{j,n}, F_{k,k+n-j-1}, F_{k-1,k+n-j}$
$k < j-1,\ t < n-j$	$W_{k+1,t}, F_{k-1,k+t-1}$	$W_{k,t+1}, F_{k,k+t-1}$

In (A.2), the earlier axial pivots contributing to this row are $(k+1,t)$, if $k < j-1$, and $(k,t+1)$, if $t < n-j$. Their four presence/absence cases give the four rows above; all other boundary terms are unmutated coordinates. With

$$W_{j,t} = F_{j-1,n}F_{j,j+t-1}, \qquad W_{k,n-j+1} = F_{j,n}F_{k-1,k+n-j},$$

the first-side product in every row is $W_{k+1,t}F_{k-1,k+t-1}$, and the second-side product is $W_{k,t+1}F_{k,k+t-1}$. This gives (3.15), including both boundary rows. If $j = 1$, the mutation word is empty. If $n - j = 1$, only the rows with $t = n - j$ occur. Corner coordinates are one, and coincident incidence labels are combined as in Appendix A.1.

□

A.3 Seam columns and endpoint coincidences

Lemma A.3. *After the two triangle sweeps, the seam column is* (A.4)*. Successive seam mutations give the expansion* (3.20)*, including the single and double terminal cases.*

Proof. Record exchange adjacency by the actual matrix column $Q(-, y_s)$, the negative of row $Q(y_s, -)$. Passing between them exchanges the two summands of mutation. Substituting the coordinates of $a = n - s, b = s$ in (A.9) gives the initial seam column

$$Q(-, y_s) = +A_{s+1,s} - A_{s,s-1} + F_{s-1,s} - F_{s,s+1}. \tag{A.3}$$

The four terms of (A.9) are the row terms $A_{s,s-1}, -A_{s+1,s}, F_{s,s+1}, -F_{s-1,s}$, fixing the row/column sign for every integer n, s in range.

We compute the rows chronologically through the two triangle sweeps. In the forward table, the only pivots adjacent to y_s are $A_{s+1,s}$ when $s + 1 < i$, followed by $A_{s,s-1}$ when $s > 1$. At the first, $h = Q(y_s, k) = -1$, so row mutation gives $\mathrm{row}(y_s)' = \mathrm{row}(y_s) - \mathrm{row}(k)^- + 2e_k$. For $s > 1$ the subtracted negative side is $Z_{s+1,s-1} + y_{s+1}$; for $s = 1$ it is $A_{10} + U_1 + y_2$, including the top-row boundary case.

At the second pivot, $h = +1$ and the row update is $\mathrm{row}(y_s)' = \mathrm{row}(y_s) + \mathrm{row}(k)^+ - 2e_k$. If $s < i - 1$, its positive side is $Z_{s+1,s-1} + y_{s-1}$, canceling the previous Z. At the terminal $s = i - 1$ it is $U_i + y_{s-1} + A_{i,s}$, whose last term cancels the original neighbor. For $s = 1, i = 2$ there is no forward pivot, and the initial row already has the same form. Thus the U-sector row contribution is always

$$-U_s + U_{s+1} + \mathbf{1}_{s>1}y_{s-1} - (1 - \mathbf{1}_{i=s+1})y_{s+1}.$$

For $j < n$ the only effective reverse-rectangle seam pivot is $F_{s,s+1}$, namely $k = s, t = 1$, with $h = +1$. Its positive side is $V_{s+1} + F_{s-1,s}$ if $s + 1 < j$ and $V_j + y_j + F_{s-1,s}$ if $s = j - 1$. The term $F_{s-1,s}$ cancels the initial term, so its seam incidence is zero if it is visited later. The complete table in Appendix A.2 gives the same expressions at $n - j = 1$. If $j = n$ there is no reverse rectangle, $V_s = F_{s-1,s}$, and the initial V-sector already has the required form; y_n is a zero incidence vector. Hence the V-sector row contribution is

$$-V_s + V_{s+1} + \mathbf{1}_{j=s+1}y_{s+1}.$$

Add these row contributions and negate to obtain the complete exchange column

$$Q(-, y_s) = U_s + V_s - U_{s+1} - V_{s+1} - \mathbf{1}_{s>1}y_{s-1} + (1 - \mathbf{1}_{i=s+1} - \mathbf{1}_{j=s+1})y_{s+1}. \tag{A.4}$$

Each terminal condition $i = s + 1$ or $j = s + 1$ removes one copy of y_{s+1}, and a double terminal reverses its sign. We now use the rows to compute the seam mutations.

Set $\alpha_s = 1 - \mathbf{1}_{i=s+1} - \mathbf{1}_{j=s+1}$. For $s < m - 1$ one has $\alpha_s = 1$. At the last index $s = m - 1$ the value is zero when $i \neq j$ and -1 when $i = j$. The first row is $-U_1 - V_1 + U_2 + V_2 - \alpha_1 y_2$, giving the nonterminal exchange $y_1' = [U_2V_2y_1^{-1}] + [U_1V_1y_2y_1^{-1}]$. Each preceding seam mutation cancels the old U_s, V_s pair from the next row, leaving

$$Q_{\mathrm{current}}(y_s, -) = U_{s+1} + V_{s+1} - e_{y'_{s-1}} - \alpha_s y_{s+1}.$$

Thus the nonterminal recurrence is $y'_s = [U_{s+1}V_{s+1}y_s^{-1}] + [y'_{s-1}y_{s+1}y_s^{-1}]$. At the last mutation, read both terms from the terminal row. For $m \geq 3$ this gives

$$y'_{m-1} = [U_m V_m y_m^{\mathbf{1}_{i=j}} y_{m-1}^{-1}] + [y'_{m-2}y_{m-1}^{-1}].$$

The second term no longer contains y_m. If $m = 2$, replace that second term by $[U_1V_1y_1^{-1}]$, reading the first term by the same terminal rule. Expansion yields (3.20). Every bracket contains coordinates of one current seed and uses its Weyl normalization; in particular, products of y_s, y'_s from different seeds are not assigned a common-seed normalization.

For a uniform termwise formula, let $1 \leq k \leq m$. Term k of the final expansion is

$$\left[U_k V_k\, y_{k-1}^{-1} y_k^{\mathbf{1}_{k=i}+\mathbf{1}_{k=j}-1}\right], \qquad y_0 = 1. \tag{A.5}$$

A nonterminal term has two denominator factors, a single terminal one, and a double terminal acquires y_m. The previous changed coordinate enters linearly at every step, and the V_k markers separate the terms, so the bar argument of Section 3.2 fixes every coefficient to one. The row/column convention preserves the two exchange-neighbor sets and their equal-sign lateral-side identification used in neutral-support gluing.

□

A.4 Quadrilateral cutting

Lemma A.4. *The exponent map defined in* (A.6) *is the canonical quadrilateral cutting map* (3.21).

Proof. Glue triangles U, V along the side with third barycentric coordinate zero, identifying seam $U(a, b, 0)$ with $V(b, a, 0)$. Corner row vectors are zero. For a source vertex represented in U or V define

$$\begin{aligned} e_{(a,b,c)}E_0 &= e_U(a,b,c) + e_V(n-a,a,0), \quad \text{representative in } U, \\ e_{(a,b,c)}E_0 &= e_V(a,b,c) + e_U(n-a,a,0), \quad \text{representative in } V. \end{aligned} \tag{A.6}$$

The two definitions agree on the seam as the sum of its two copies. We verify $HE_0 = DH_{\mathrm{cut}}$. Since $H^{-1} = K/n$ by (3.1), this makes E_0 the canonical exponent matrix.

In [LY, Appendix A(260)–(261)], an interior H-row has value $+1$ at

$$(a+1,b,c-1), (a-1,b+1,c), (a,b-1,c+1)$$

and value -1 at

$$(a-1,b,c+1), (a+1,b-1,c), (a,b+1,c-1).$$

The boundary row $a = 0$ is

$$e_{(0,b,c)} + e_{(1,b,c-1)} - e_{(0,b+1,c-1)} - e_{(1,b-1,c)}. \tag{A.7}$$

Cyclic rotation gives the cut-triangle H-row on the seam side $c = 0$ as

$$H_{(a,b,0)} = e_{(a,b,0)} + e_{(a,b-1,1)} - e_{(a+1,b-1,0)} - e_{(a-1,b,1)}. \tag{A.8}$$

For a nonseam source row, its original triangle retains the same H-row. The other component is the signed sum of $e(n-a, a, 0)$ determined by the first coordinates of its neighbors. The two interior contributions of each $a-1, a, a+1$ cancel. On physical side $a = 0$ the corner term is zero and the two first-coordinate-one terms cancel. On the other physical side, cyclic rotation of (A.7) gives two canceling pairs with equal first coordinates. Thus the row is DH_{cut}.

On seam $U(a,b,0) = V(b,a,0)$, one has $a,b>0, a+b=n$. The glued half-arrows cancel and the source H-row has the four terms

$$+e_U(a,b-1,1) - e_U(a-1,b,1) + e_V(b,a-1,1) - e_V(b-1,a,1). \tag{A.9}$$

By (A.6), its U-component is

$$e_U(a,b-1,1) - e_U(a-1,b,1) + e_U(a,b,0) - e_U(a+1,b-1,0),$$

exactly (A.8); its V-component similarly gives the other cut H-row. Therefore every row satisfies $HE_0 = DH_{\mathrm{cut}}$, including the corner-zero cases $a=1, b=1, n=2$.

With $A_{pq} = g_U(n-p,q,p-q)$ and $F_{qp} = g_V(q,n-p,p-q)$ from Section 3.2, formula (A.6) becomes (3.21). The canonical map is based, so its Weyl coefficient is one. This proves (3.21). □

A.5 Endpoint and topology cases

Case	Formula or argument
Forward $i=1$	Use $C_{11} = A_{10}^{-1}$ separately.
Forward $k=i$	The diagonal numerator $A_{i-1,0}$ may be frozen.
Reverse $j=n$	Use $V_k = F_{k-1,k}$.
Quadrilateral $m=1$	Equation (3.24), including $D_{1n} = F_{01}/A_{10}$.
$i=j<n$	The final seam term contains y_i.
$i=j=n$	The same factor is $y_n = 1$.
Identified lateral sides	Each triangle sweep sees one side; only the first seam at $i=j=n$ sees both, with equal signs.
Repeated marks or based-loop cuts	Retain physical ports and small-vertex occurrences; use the uncut H-rows and neutral lattice.
Singleton boundary circles	Use the projected units of Section 3.3 and the individual classical primes of Section 4.2.
The original Conjecture 1.6 family	The reverse minor (8.3) recovers the corrected entry.

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Zhili College, Tsinghua University, Beijing 100084, China
tangqy24@mails.tsinghua.edu.cn