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Improved and sharp bounds for illumination

Working research draft

1 November 2026

Abstract

We prove improved ordinary illumination bounds and exact ordinary and fractional values for several classes of convex bodies. Every full-dimensional complex zonotope in $\mathbb{C}^2$ has illumination number five, except complex affine images of the bidisc, whose number is seven. For complex 1-symmetric non-polydiscs in $\mathbb{C}^3$, an explicit construction gives twelve illuminating directions. For real 1-symmetric non-cubes in $\mathbb{R}^n$, connected domination in the cube graph gives the upper bound $2^{n-1}+\gamma_c(Q_{n-1})+n-1 = 2^{n-1}(1+(1+o(1))/n)$; an explicit family has illumination number 2^{n-1}. We also give a product boundary measure proving fractional lower bounds 2^n for bodies between specified inner and outer bodies, and attain these bounds along Minkowski interpolations with a positive inner summand. Finally, the least size U_n of a single family illuminating every normalized complex 1-unconditional body satisfies $U_n \sim 3^{n+1}/\sqrt{\pi n}$.

Draft status. The proofs included here have been checked internally; external review and literature priority remain outstanding. The finite covering assertions have exact, reproducible certificates. Bounds for an individual body and for a single family serving all normalized bodies are kept separate. The citation audit identifies established inputs and earlier special cases; it does not certify novelty of the remaining results.

1 Introduction

Illumination asks for directions that move every boundary point of a convex body into its interior. The ordinary illumination number $\mathrm{ill}(K)$ counts directions. Its fractional counterpart $\mathrm{ill}^*(K)$ minimizes the mass of a measure giving at least unit mass to the directions illuminating each boundary point. We study exact values, improvements under coordinate symmetry, and the additional cost of choosing directions independently of the body.

Naszódi introduced fractional illumination with measures of finite support [8, Definition 1]. We use the Borel-measure formulation of Rotem, Schejter, and Slomka [1, Section 1.2]. Their polydisc values $\mathrm{ill}(\mathbb{D}^n) = 2^{n+1}-1$ and $\mathrm{ill}^*(\mathbb{D}^n) = 2^n$ are established inputs [1, Theorems 1.3–1.4]. The ordinary formula was asserted earlier by Boltyanski and Martini [5]; we use the corrected construction in [1, Proposition 2.7 and Remark 2.8].

Theorem 2 determines the ordinary number for every full-dimensional complex zonotope in $\mathbb{C}^2$:

$$\mathrm{ill}(Z) = 7 \quad \text{for a complex affine bidisc,} \qquad \mathrm{ill}(Z) = 5 \quad \text{otherwise.}$$

The proof reduces to three generating discs. Five explicit directions illuminate that model; the only nonsmooth boundary points lie on three circles, on which illumination is verified by a finite table of open intervals. In particular, six never occurs in this class. The previously published [1, Theorem 1.8] gives at most six directions in this dimension; the five-direction construction below sharpens that bound.

Theorem 3 gives twelve directions for every complex 1-symmetric non-polydisc in $\mathbb{C}^3$. The construction uses twelve rows of thirteenth roots of unity, with one distinguished coordinate in

each row. A finite covering check gives a common strictly inward margin, and a supporting-normal argument determines the coordinate magnitudes. The maximum is six in dimension two and lies in $[9, 12]$ in dimension three (Corollary 4); sharpness of twelve is not established. The earlier bound fourteen and the geometric reduction are in the coauthored manuscript [2, Theorem 1.5 and Proposition 2.8].

For real 1-symmetric bodies, write M_n for the supremum of $\mathrm{ill}(K)$ over bodies in $\mathbb{R}^n$ that are not affine cubes. Tikhomirov proved $M_n < 2^n$ in sufficiently large dimensions [10, Theorem 1]. Sun and Vritsiou obtained $M_n \le 2^n - 2$ for every $n \ge 3$ [9, Theorem 23]. The coauthored manuscript [2, Theorem 1.4] gives $M_n \le 3 \cdot 2^{n-2} + 1$ for $n \ge 4$. Proposition 5 proves

$$M_n \le 2^{n-1} + \gamma_c(Q_{n-1}) + n - 1,$$

where γ_c is the connected domination number of the cube graph. A connected dominating set in Q_{n-1} gives a vertex cover in Q_n with an adjacent successor assignment, and this produces illuminating directions. The asymptotic estimate of Griggs [3] yields $M_n \le 2^{n-1}(1 + (1 + o(1))/n)$. Proposition 8 gives the explicit non-cubes

$$P_n = \left\{ x \in [-1,1]^n : \sum_{j=1}^{n} |x_j| \le n - 1 \right\}, \qquad \mathrm{ill}(P_n) = 2^{n-1}.$$

The fractional statements concern convex bodies between $H = \mathbb{D}^p \times B^r$ and $E = \mathbb{D}^p \times \prod_{j=1}^{r} L(a_j, b_j)$, where $p + 2r = n$, B is the complex two-dimensional ℓ_1 ball, and the sets $L(a_j, b_j)$ are defined in (23). If $a_j b_j \ge 1/2$, then every $H \subseteq C \subseteq E$ satisfies $\mathrm{ill}^*(C) \ge 2^n$, without a symmetry assumption on C. Proposition 9 attains this lower bound for $(1-t)H + tC$ whenever $0 \le t < 1$. The upper bound for an arbitrary endpoint C is not asserted. The inner bodies are the Hanner equality cases in the companion paper [11]; the planar envelopes also occur in the planar classification in [4, Section 7]. Remark 1 explains why these examples contradict the equality clause, but not the upper bound, of [1, Conjecture 1.2].

Appendix A treats the distinct quantity U_n: the least number of directions in one family that illuminates every normalized complex 1-unconditional body. A lower bound obtained from different bodies matches the support construction asymptotically. This proves $U_n \sim 3^{n+1}/\sqrt{\pi n}$, without giving the same lower bound for any one body's illumination number. The common families in [9, Theorem D] and [2, Theorem 1.6] serve the real and complex 1-symmetric classes, respectively; they do not give this result for all normalized complex 1-unconditional bodies.

2 Notation and preliminary facts

Write $[n] = \{1, \dots, n\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$, and $\mathbb{R}_+ = [0, \infty)$. A convex body is a compact convex set with nonempty interior in the underlying real space. It is complex 1-unconditional if it is invariant under independent rotations of its coordinates. For such a body, its Minkowski norm is denoted by $\|\cdot\|_K$. For $z \in \mathbb{C}^n$, write $|z| = (|z_1|, \dots, |z_n|)$ and $\mathrm{supp}(z) = \{j \in [n] : z_j \ne 0\}$. We also write

$$M_x = \{j \in [n] : |x_j| = \|x\|_\infty\}, \qquad Z_x = \{j \in [n] : x_j = 0\}.$$

A nonzero vector d illuminates $x \in \partial K$ if $x + td \in \mathrm{int}\, K$ for some $t > 0$. The least cardinality of a family illuminating every boundary point is $\mathrm{ill}(K)$. For real bodies the same definition uses $\mathbb{R}^n$. A finite nonnegative Borel measure μ on $\mathbb{C}^n \setminus \{0\}$ fractionally illuminates K if

$$\mu\{d : d \text{ illuminates } x\} \ge 1 \qquad (x \in \partial K).$$

The infimum of the total masses of such measures is $\mathrm{ill}^*(K)$.

We call K normalized when $e_j \in \partial K$ for every $j \in [n]$. Positive diagonal rescaling gives this normalization, and illumination numbers are invariant under translations and invertible real linear maps. Coordinate contractions preserve membership, as proved below; consequently a normalized body satisfies $K \subseteq \mathbb{D}^n$ and contains each unit coordinate disc.

Lemma 1. *For a complex 1-unconditional convex body K, the following statements hold.*

1. *If $|w_j| \le |z_j|$ for every j, then $\|w\|_K \le \|z\|_K$.*
2. *For every real linear subspace $E \subseteq \mathbb{C}^n$,*

$$\operatorname{relint}(K \cap E) = E \cap \operatorname{int} K.$$

Proof. Invariance under all coordinate rotations implies central symmetry. Because K has nonempty interior, its center 0 is interior: if an open Euclidean ball of radius r around a lies in K, the same is true around $-a$, and convexity puts the ball of radius r around 0 in K. Its Minkowski functional is therefore a norm, with $\operatorname{int} K = \{z : \|z\|_K < 1\}$.

Every number $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$ is a convex combination of two unit complex numbers. Explicitly, if $\lambda = se^{\mathrm{i}\theta}$, $0 \le s \le 1$, then

$$\lambda = \frac{1+s}{2} e^{\mathrm{i}\theta} + \frac{1-s}{2}(-e^{\mathrm{i}\theta}).$$

Thus contracting a single coordinate by λ preserves membership in K, by convexity and rotation invariance. Applying these contractions one coordinate at a time proves coordinatewise monotonicity of membership. Apply the same fact to every dilation of K to obtain the norm inequality in part 1. Coordinates with $z_j = 0$ also have $w_j = 0$, so cause no exception.

The inclusion from right to left in part 2 follows by intersecting an ambient open ball with E. For the reverse inclusion, let $y \in \operatorname{relint}(K \cap E)$. If $y = 0$ there is nothing to prove. Otherwise relative interior gives an $s > 0$ with $(1+s)y \in K \cap E$. Consequently $\|y\|_K \le (1+s)^{-1} < 1$, as required.

□

We use the real pairing

$$[u, z] = \operatorname{Re} \sum_{j \in [n]} \overline{u_j} z_j, \qquad h_K(u) = \max_{z \in K} [u, z].$$

A direction d illuminates $x \in \partial K$ if and only if

$$[u, d] < 0 \quad \text{for every nonzero } u \text{ with } [u, x] = h_K(u). \tag{1}$$

Here is a proof that also covers nonsmooth boundary points. If $x + td \in \operatorname{int} K$ for some $t > 0$, then $[u, x + td] < h_K(u) = [u, x]$ for every nonzero supporting normal at x, proving necessity. For sufficiency, suppose no positive step enters $\operatorname{int} K$. For each integer $j \ge 1$, separation at $x + j^{-1}d \notin \operatorname{int} K$ supplies a Euclidean unit vector u^j with $[u^j, x + j^{-1}d] \ge h_K(u^j)$. Since $h_K(u^j) \ge [u^j, x]$, we have $[u^j, d] \ge 0$. Pass to a convergent subsequence of these unit vectors, with limit u. The support function is continuous, so $[u, x] \ge h_K(u)$; the reverse inequality holds because $x \in K$. Thus u is a nonzero supporting normal at x with $[u, d] \ge 0$, contradicting (1).

For a normalized unconditional body, put $S = \operatorname{supp}(x)$ and $a_j = |u_j|$ for a supporting normal u at x. Independent coordinate rotations give $u_j = |u_j| x_j / |x_j|$ whenever $j \in S$: the maximum of $[u, \cdot]$ on the coordinate-rotation orbit of x is $\sum_j |u_j||x_j|$, and x already attains that maximum. Thus, writing $h = h_K(u) > 0$,

$$h = \sum_{j \in S} a_j |x_j|, \qquad a_j \le h \quad (j \in [n]), \qquad h \le \sum_{j \in S} a_j. \tag{2}$$

The middle inequality follows from the coordinate discs in K; the last uses $K \subseteq \mathbb{D}^n$.

The illumination relation is open in the boundary point and the direction: for fixed $t > 0$, the condition $x + td \in \operatorname{int} K$ is open, and illumination is its union over $t > 0$. Consequently all integrals of this relation below are measurable, and Tonelli's theorem applies. The normal criterion has the same proof over $\mathbb{R}^n$, using the usual real inner product.

3 Complex zonotopes in two dimensions

The result in this section concerns complex zonotopes and does not assume invariance under independent coordinate rotations. Up to translation, a complex zonotope in $\mathbb{C}^2$ has the form

$$Z = \sum_{j \in [N]} \mathbb{D} v^j, \qquad v^j \in \mathbb{C}^2.$$

It is full-dimensional precisely when its generators span $\mathbb{C}^2$ over $\mathbb{C}$.

Theorem 2. *Let $Z \subset \mathbb{C}^2$ be a full-dimensional complex zonotope. Then*

$$\operatorname{ill}(Z) = \begin{cases} 7, & \text{if } Z \text{ is a translate of a complex-linear image of } \mathbb{D}^2, \\ 5, & \text{otherwise.} \end{cases}$$

In particular, no full-dimensional complex zonotope in $\mathbb{C}^2$ has illumination number 6.

Proof. Translation does not affect illumination, so assume that Z is centered at the origin. The bidisc case follows from the exact polydisc theorem [1, Theorem 1.4] and linear invariance. We prove the other case by constructing five directions for

$$K_0 = \mathbb{D}e_1 + \mathbb{D}e_2 + \mathbb{D}(e_1 + e_2). \tag{3}$$

The reduction to this model is the two-dimensional instance of [1, Lemma 3.4], using the numbering of the published version. We include the argument to specify how the directions are transferred.

First, every set illuminating a full-dimensional convex body K also illuminates $K + L$ for any nonempty compact convex set L. Indeed, write $x = y + z \in \partial(K + L)$ with $y \in K$ and $z \in L$. Then $y \in \partial K$, since $\operatorname{int} K + L \subseteq \operatorname{int}(K + L)$. If $y + td \in \operatorname{int} K$, then $x + td = (y + td) + z \in \operatorname{int}(K + L)$.

Next, positive rescaling of individual generators preserves the same illuminating sets. For $r_j > 0$ and $r = \min_{j \in [N]} r_j$, write

$$\sum_{j \in [N]} r_j \mathbb{D} v^j = r \sum_{j \in [N]} \mathbb{D} v^j + \sum_{j \in [N]} (r_j - r) \mathbb{D} v^j.$$

The preceding observation and dilation invariance prove one implication; applying it to the rescaled generators with factors r_j^{-1} proves the reverse implication. Multiplication of a generator by a unit complex number leaves its disc unchanged, so the assertion also holds for nonzero complex rescalings. These are Fact 3.2 and Lemma 3.3 of the published version of [1].

Discard zero generators and merge generators on the same complex line. If exactly two lines remain, full dimensionality makes Z a complex-linear image of $\mathbb{D}^2$. Otherwise choose generators v^1, v^2, v^3 on three distinct complex lines and write

$$v^3 = \alpha v^1 + \beta v^2, \qquad \alpha\beta \neq 0.$$

The complex-linear map T defined by $Te_1 = \alpha v^1$ and $Te_2 = \beta v^2$ is invertible, and

$$TK_0 = \mathbb{D}(\alpha v^1) + \mathbb{D}(\beta v^2) + \mathbb{D} v^3.$$

Thus, if a set V illuminates K_0, the set TV illuminates TK_0, then $\mathbb{D}v^1+\mathbb{D}v^2+\mathbb{D}v^3$ by generator rescaling, and finally Z by Minkowski addition.

For $x \in \partial K_0$, define its normal cone by

$$N_{K_0}(x) = \{u \in \mathbb{C}^2 : [u, y - x] \le 0 \text{ for every } y \in K_0\}.$$

We call x smooth if this cone is a single ray. The support function is

$$h_{K_0}(u) = |u_1| + |u_2| + |u_1 + u_2|. \tag{4}$$

Indeed, writing $x = (z_1 + z_3, z_2 + z_3)$ with $z_j \in \mathbb{D}$, the three independent maxima in $[u, x]$ are the three terms on the right-hand side.

Fix such a representation of a boundary point x. Each term is at most its corresponding support value, so equality in (4) holds exactly when equality holds in all three terms. Consequently $u \in N_{K_0}(x)$ if and only if

$$u_1 = t_1 z_1, \qquad u_2 = t_2 z_2, \qquad u_1 + u_2 = t_3 z_3, \qquad t_j \ge 0, \tag{5}$$

where $t_j = 0$ is required if $|z_j| < 1$.

If $|z_1| < 1$, then $u_1 = 0$ and u_2 lies on the fixed ray $\mathbb{R}_+ z_2$; the case $|z_2| < 1$ is analogous. If $|z_3| < 1$, then $u_2 = -u_1$ and u_1 lies on $\mathbb{R}_+ z_1$. In each case the normal cone is contained in one ray. Since a boundary point has a nonzero normal, it is smooth.

Suppose now that $|z_j| = 1$ for every $j \in [3]$. The parameters in (5) satisfy

$$t_1 z_1 + t_2 z_2 - t_3 z_3 = 0.$$

If the three numbers z_j do not lie on one real line through the origin, the real-linear map on the left has rank two. Its kernel is one-dimensional, and its intersection with $\mathbb{R}^3_+$ is at most a ray. This again makes x smooth.

In the remaining case, write $z_3 = q$, $z_1 = s_1 q$, and $z_2 = s_2 q$, where $|q| = 1$ and $s_1, s_2 \in \{-1, 1\}$. The choice $s_1 = s_2 = -1$ gives $-t_1 - t_2 = t_3$, so all three parameters vanish; this is impossible at a boundary point. The other choices give

$$A(q) = q(2, 2), \qquad B(q) = q(2, 0), \qquad C(q) = q(0, 2).$$

Writing $\operatorname{pos}\{u, v\} = \{au + bv : a, b \ge 0\}$, equation (5) yields

$$\begin{aligned} N_{K_0}(A(q)) &= q \operatorname{pos}\{e_1, e_2\}, \\ N_{K_0}(B(q)) &= q \operatorname{pos}\{e_1, e_1 - e_2\}, \\ N_{K_0}(C(q)) &= q \operatorname{pos}\{e_2, e_2 - e_1\}. \end{aligned} \tag{6}$$

Each cone has dimension two. These three circles are therefore exactly the nonsmooth boundary points of K_0.

Choose the five directions

$$\begin{aligned} &d^1 = (1 - \mathrm{i}, -\mathrm{i}), \quad d^2 = (1, 1 + \mathrm{i}), \qquad d^3 = (-1 - \mathrm{i}, -1), \\ &d^4 = (\mathrm{i}, -1 + \mathrm{i}), \quad d^5 = (-1 + \mathrm{i}, 1 - \mathrm{i}). \end{aligned} \tag{7}$$

They satisfy $\sum_{j\in[5]} d^j = 0$, and the determinant of their first four real coordinate rows is

$$\det \begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} = 5.$$

For every nonzero $u \in \mathbb{C}^2$, some $j \in [5]$ therefore satisfies $[u, d^j] < 0$. Otherwise all five inner products would be nonnegative with sum zero, and hence all would vanish, contradicting real spanning. By (1), these five directions illuminate every smooth boundary point.

For $d = (a, b)$, equations (1) and (6) give the following necessary and sufficient conditions:

$$\begin{array}{c|ll} & \text{conditions for illumination by } d & \\ A(q) & \operatorname{Re}(\overline{q}a) < 0 & \operatorname{Re}(\overline{q}b) < 0 \\ B(q) & \operatorname{Re}(\overline{q}a) < 0 & \operatorname{Re}(\overline{q}(a-b)) < 0 \\ C(q) & \operatorname{Re}(\overline{q}b) < 0 & \operatorname{Re}(\overline{q}(b-a)) < 0 \end{array} \tag{8}$$

Put $q = \exp(\mathrm{i}\pi s/4)$, where $s \in \mathbb{R}/8\mathbb{Z}$. For each direction in (7), the sets of parameters satisfying (8) are the following open intervals, interpreted modulo 8:

	A	B	C
d^1	$(1,4)$	$(2,5)$	$(0,2)$
d^2	$(3,6)$	$(2,4)$	$(4,7)$
d^3	$(-1,2)$	$(0,3)$	$(6,8)$
d^4	$(5,8)$	$(4,6)$	$(6,9)$
d^5	$\varnothing$	$(5,9)$	$(1,5)$

(9)

Every entry follows from

$$\operatorname{Re}(e^{-\mathrm{i}t}z) < 0 \quad \Longleftrightarrow \quad t \in (\arg z + \pi/2, \arg z + 3\pi/2) \pmod{2\pi} \qquad (z \neq 0).$$

For example, the two inequalities for $A(q)$ and d^1 give $s \in (1,5)$ and $s \in (0,4)$, whose intersection is $(1,4)$. For the fifth direction these two half-circles are opposite, so their intersection is empty.

In column A, the intervals in rows $3, 1, 2, 4$ have union $(-1, 8)$ on the real line. In column B, rows $3, 1, 4, 5$ have union $(0, 9)$, and in column C, rows $1, 5, 2, 4$ have union $(0, 9)$. Each union has length $9 > 8$ and therefore maps onto $\mathbb{R}/8\mathbb{Z}$. Successive intervals overlap strictly, so their endpoints cause no exception. Hence the five directions illuminate every point of all three circles. Together with the smooth case, this proves $\operatorname{ill}(K_0) \leq 5$. The reduction above gives $\operatorname{ill}(Z) \leq 5$.

Finally, any illuminating set of a full-dimensional convex body in $\mathbb{R}^4$ has at least five members. For every nonzero u, choose a point maximizing $[u, x]$; a direction illuminating that point must have negative inner product with u. By separation, the origin must therefore lie in the full-dimensional interior of the convex hull of the directions. The convex hull of at most four points has affine dimension at most three, which is impossible. Thus $\operatorname{ill}(Z) \geq 5$, completing the proof. □

The value 5 in Theorem 2 concerns complex zonotopes. It does not replace the sharp bound 6 for general complex 1-unconditional bodies in $\mathbb{C}^2$ from the proof of Corollary 4.

4 1-symmetric bodies in dimension three

A complex 1-unconditional body is 1-*symmetric* if it is also invariant under every coordinate permutation. The bound $\operatorname{ill}(K) \leq 14$ for non-polydiscs in this class follows from [2, Theorem 1.5]. The following construction improves it. It uses the reduction underlying [2, Proposition 2.8]; we include a complete geometric proof using supporting normals.

Theorem 3. *Every complex* 1*-symmetric convex body* $K \subset \mathbb{C}^3$ *which is not a complex affine image of* $\mathbb{D}^3$ *satisfies*

$$\operatorname{ill}(K) \leq 12.$$

Proof. Put $\zeta = \exp(2\pi\mathrm{i}/13)$ and define

$$v^r = (\zeta^r, \zeta^{3r}, \zeta^{4r}) \quad (r \in [12]), \qquad (f(1), \dots, f(12)) = (2,1,1,3,1,2,2,1,3,1,3,2). \tag{10}$$

We first verify that, with $\beta = \sin(\pi/26) > 0$, for every nonempty proper $S \subset [3]$ and every $(z_j)_{j\in S}$ with $|z_j| = 1$, there is $r \in [12]$ such that

$$f(r) \in S, \qquad \mathrm{Re}(\overline{z_j} v_j^r) \le -\beta \quad (j \in S). \tag{11}$$

All residues in the following verification are taken modulo 13. Set $(a_1, a_2, a_3) = (1, 3, 4)$ and $B_s = \{s, s+1, \dots, s+5\}$. A nonempty residue set meets every B_s if and only if its largest cyclic gap is at most six: a gap of length g leaves exactly $g-1$ consecutive missing residues. For singleton supports, the sets $\{a_j r : f(r) = j\}$ and their largest gaps are

j	$\{a_j r : f(r) = j\}$	largest gap
1	$\{2,3,5,8,10\}$	5
2	$\{3,5,8,10\}$	6
3	$\{3,5,10\}$	6

For $i < j$, define

$$R_{ij}(s) = \{a_j r : a_i r \in B_s,\ f(r) \in \{i,j\},\ r \in [12]\}.$$

Their largest cyclic gaps, calculated directly from (10), are

$(i,j)\backslash s$	0	1	2	3	4	5	6	7	8	9	10	11	12
$(1,2)$	6	6	6	4	4	4	6	6	6	6	6	6	6
$(1,3)$	4	4	4	5	6	5	4	4	4	5	6	6	5
$(2,3)$	6	6	6	6	6	4	6	6	6	6	6	6	6

Every entry is at most six. Thus for any prescribed blocks B_{s_j}, $j \in S$, a row satisfies $f(r) \in S$ and $a_j r \in B_{s_j}$ for every $j \in S$. This finite assertion comprises $3 \cdot 13 + 3 \cdot 13^2 = 546$ cases; the accompanying verifier checks all of them with integer arithmetic.

For each unit z, the closed arc $\{w : |w| = 1,\ \mathrm{Re}(\overline{z}w) \le -\beta\}$ has angular length $12\pi/13$, exactly six spacings between successive thirteenth roots of unity. It therefore contains six consecutive roots. Applying the finite assertion to one such block in each coordinate proves (11), including all endpoint cases.

Scale K so that $e_j \in \partial K$ for $j \in [3]$. This uses one common factor, since K is 1-symmetric. Then $K \subset \mathbb{D}^3$ and there is a unique $\rho > 0$ with $\rho(1,1,1) \in \partial K$. Coordinate contractions give $\rho\mathbb{D}^3 \subset K$, so $0 < \rho < 1$: equality would give $K = \mathbb{D}^3$. Fix

$$\Delta = \frac{\beta}{6}, \qquad \delta = \min\left\{\frac{\Delta}{2}, \frac{\beta\Delta(1-\rho)}{2\rho}\right\}. \tag{12}$$

For every $r \in [12]$ define a direction by

$$d_j^r = \begin{cases} v_j^r, & j = f(r), \\ \delta v_j^r, & j = 3 \ne f(r), \\ \Delta v_j^r, & j \ne f(r),\ j \ne 3. \end{cases} \tag{13}$$

These twelve nonzero directions are fixed before choosing a boundary point.

Let $x \in \partial K$. For any nonzero supporting normal u at x, write $b_j = |u_j|$ and $t_j = |x_j|$. Maximizing independently over coordinate rotations shows that

$$u_j = b_j \frac{x_j}{|x_j|} \quad (x_j \ne 0), \qquad h_K(u) = \mathrm{Re}\sum_{j=1}^{3} \overline{u_j} x_j = \sum_{j=1}^{3} b_j t_j > 0.$$

There is no restriction on the argument of u_j when $x_j = 0$. Swapping any two coordinate moduli of x and choosing their arguments to maximize the same functional gives

$$(b_i - b_j)(t_i - t_j) \geq 0. \tag{14}$$

Indeed, the swapped point belongs to K, and subtracting its functional value from $h_K(u)$ gives the displayed inequality.

Suppose first that $M_x \neq [3]$. Apply (11) with $S = M_x$ and $z_j = x_j/|x_j|$, and put $k = f(r) \in M_x$ for the chosen row. Write $\alpha_j = |d_j^r|$. Then $\alpha_k = 1$ and $0 < \alpha_j \leq \Delta$ for $j \neq k$. For $j \notin M_x$, (14) gives $b_j \leq b_k$, including when $x_j = 0$. Consequently

$$\begin{aligned} \operatorname{Re} \sum_{j=1}^{3} \overline{u_j} d_j^r &\leq -\beta b_k - \beta \sum_{j \in M_x \setminus \{k\}} \alpha_j b_j + \Delta \sum_{j \notin M_x} b_j \\ &\leq -(\beta - 3\Delta) b_k - \beta \sum_{j \in M_x \setminus \{k\}} \alpha_j b_j < 0. \end{aligned}$$

For the strict inequality, $\beta - 3\Delta = \beta/2 > 0$. If $b_k = 0$, all coefficients outside M_x vanish, and some coefficient on $M_x \setminus \{k\}$ is positive because $u \neq 0$. The row was chosen using x alone, so this holds for every supporting normal at x.

If $M_x = [3]$, all coordinate moduli equal ρ. Apply (11) with $S = \{1, 2\}$ and $z_j = x_j/|x_j|$ for $j \in \{1, 2\}$. Then $k = f(r) \in \{1, 2\}$, $\alpha_1, \alpha_2 \geq \Delta$, and $\alpha_3 = \delta$. The third unit coordinate disc is contained in K, hence

$$b_3 \leq h_K(u) = \rho(b_1 + b_2 + b_3), \qquad b_3 \leq \frac{\rho}{1 - \rho}(b_1 + b_2).$$

In particular $b_1 + b_2 > 0$. By (12),

$$\operatorname{Re} \sum_{j=1}^{3} \overline{u_j} d_j^r \leq -\beta\Delta(b_1 + b_2) + \delta b_3 \leq -\frac{\beta\Delta}{2}(b_1 + b_2) < 0.$$

Again the chosen row works for every supporting normal. The normal criterion (1) now shows that the directions (13) illuminate K. □

Corollary 4. *The largest illumination number among complex* 1*-symmetric bodies that are not complex affine polydiscs is six in* $\mathbb{C}^2$*, and lies between nine and twelve in* $\mathbb{C}^3$*.*

Proof. First let $K \subset \mathbb{C}^2$ be any normalized complex 1-unconditional body with $K \neq \mathbb{D}^2$. Three equally spaced unit directions on each coordinate axis illuminate K. Indeed, if x has only one nonzero coordinate, then it belongs to the unit circle on that axis. For every nonzero supporting normal the coefficient on that axis is positive, since its support value is positive. Thus a strictly inward direction on that axis illuminates x. If $x = (r_1\xi_1, r_2\xi_2)$ has $r_1, r_2 > 0$, at least one modulus, say r_j, is less than one: otherwise coordinate contractions imply $K = \mathbb{D}^2$. Let $k \neq j$. Write each supporting normal as $u_l = c_l\xi_l$, $c_l \geq 0$. If $c_k = 0$, then $c_j > 0$ and $h_K(u) = r_j c_j < c_j$, contradicting the unit coordinate disc in K. Hence $c_k > 0$ for every nonzero normal. Any direction on axis k with $\operatorname{Re}(\overline{\xi_k} d_k) < 0$ illuminates x. Three equally spaced points meet every open semicircle, proving the upper bound six.

For the lower examples, put $B_1^n = \{z \in \mathbb{C}^n : \sum_j |z_j| \leq 1\}$. At ξe_j, $|\xi| = 1$, illumination by d is equivalent to

$$\operatorname{Re}(\bar{\xi} d_j) + \sum_{k \neq j} |d_k| < 0.$$

Necessity follows by differentiating the norm along the ray; sufficiency follows from the strict negative derivative. This requires $|d_j| > \sum_{k \neq j} |d_k|$, so a direction serves at most one axis

circle. Its illuminated subset of that circle is contained in an open semicircle. At least three directions are needed for each circle, giving $\operatorname{ill}(B_1^n) \geq 3n$. Conversely, take three equally spaced unit directions on each axis. At any boundary point select a nonzero coordinate and a strictly inward direction on that axis; the sum of moduli then decreases for a sufficiently small positive step. Hence $\operatorname{ill}(B_1^n) = 3n$. These bodies are 1-symmetric. For $n = 2, 3$, their values $3n$ differ from the polydisc value $2^{n+1} - 1$ [1], so they are not complex affine polydiscs. The planar bound and Theorem 3 finish the proof. □

The interval $[9, 12]$ does not establish that twelve is sharp.

5 Real 1-symmetric bodies

A real 1-symmetric convex body is invariant under all coordinate sign changes and permutations. Let

$$M_n = \sup\{\operatorname{ill}(K) : K \subset \mathbb{R}^n \text{ is 1-symmetric and is not an affine cube}\}.$$

Let Q_d be the graph on $\{0,1\}^d$ whose edges join vertices differing in exactly one coordinate. A set dominates a graph if every vertex outside it has a neighbor in it. Write $\gamma_c(Q_d)$ for the minimum size of a dominating set whose induced graph is connected.

Proposition 5. *For every $n \geq 3$,*

$$M_n \leq 2^{n-1} + \gamma_c(Q_{n-1}) + n - 1. \tag{15}$$

Consequently,

$$M_n \leq 2^{n-1}\left(1 + \frac{1 + o(1)}{n}\right) \qquad (n \longrightarrow \infty). \tag{16}$$

The direction construction builds on Sun–Vritsiou's large-coordinate and small-coordinate argument [9, Lemma 12]. Lemma 6 is the real reduction already stated in [2, Proposition 2.9]; we give a supporting-normal proof. The subsequent cube assignment supplies the connected-domination bound. For normalized bodies satisfying $\frac{1}{2}(1, \dots, 1) \notin \operatorname{int} K$, Tikhomirov's construction already has $2^{n-1} + O(2^n/n)$ directions in large dimensions [10, Section 4, Proposition 10]. That subclass estimate does not cover every body defining M_n.

We first give the geometric and graph arguments needed to construct the directions.

Lemma 6. *Let $n \geq 2$ and let $\mathcal{B} \subset \{-1,1\}^n \times [n]$ be finite. Suppose that for every nonempty proper $S \subset [n]$ and every $a \in \{-1,1\}^S$ there is $(b,k) \in \mathcal{B}$ with*

$$k \in S, \qquad b_j = -a_j \quad (j \in S). \tag{17}$$

If $K \subset \mathbb{R}^n$ is 1-symmetric and is not an affine cube, then $\operatorname{ill}(K) \leq |\mathcal{B}|$.

Proof. Scale K by one common factor so that $e_j \in \partial K$ for every $j \in [n]$. Coordinate contractions follow from convexity and sign invariance. Thus $K \subset [-1,1]^n$, and there is a unique $\rho > 0$ with $\rho(1, \dots, 1) \in \partial K$. Moreover, $\rho[-1,1]^n \subset K$, whence $0 < \rho < 1$; equality would give $K = [-1,1]^n$. Set

$$\Delta = \frac{1}{2n}, \qquad \delta = \min\left\{\frac{\Delta}{2}, \frac{\Delta(1-\rho)}{2\rho}\right\} > 0. \tag{18}$$

For every $(b,k) \in \mathcal{B}$ define $d^{b,k} \in \mathbb{R}^n$ by

$$d_j^{b,k} = \begin{cases} b_j, & j = k, \\ \delta b_j, & j = n \neq k, \\ \Delta b_j, & j \neq k,\ j \neq n. \end{cases} \tag{19}$$

These directions are fixed before choosing a boundary point.

Fix $x \in \partial K$. For a nonzero supporting normal u at x, put $c_j = |u_j|$ and $t_j = |x_j|$. Maximizing over the coordinate sign changes gives

$$u_j = c_j \operatorname{sign}(x_j) \quad (x_j \neq 0), \qquad h_K(u) = \sum_{j=1}^{n} c_j t_j > 0.$$

At a zero coordinate of x, the sign of u_j is unrestricted. Exchanging the moduli of coordinates i, j of x and then choosing signs that maximize the same functional gives

$$(c_i - c_j)(t_i - t_j) \geq 0. \tag{20}$$

Indeed, the exchanged point belongs to K, and subtracting its functional value from $h_K(u)$ gives this inequality.

Suppose $M_x \neq [n]$. Choose (b, k) using (17) with $S = M_x$ and $a_j = \operatorname{sign}(x_j)$. Write $\alpha_j = |d_j^{b,k}|$. Then $\alpha_k = 1$, $0 < \alpha_j \leq \Delta$ for $j \neq k$, and (20) gives $c_j \leq c_k$ for $j \notin M_x$, including the zero coordinates of x. Consequently

$$\begin{aligned} \langle u, d^{b,k} \rangle &\leq -c_k - \sum_{j \in M_x \setminus \{k\}} \alpha_j c_j + \Delta \sum_{j \notin M_x} c_j \\ &\leq -(1 - n\Delta) c_k - \sum_{j \in M_x \setminus \{k\}} \alpha_j c_j < 0. \end{aligned}$$

For strictness, $1 - n\Delta = 1/2$. If $c_k = 0$, every coefficient outside M_x vanishes, and some coefficient on $M_x \setminus \{k\}$ is positive because $u \neq 0$.

If $M_x = [n]$, then every $t_j = \rho$. Choose (b, k) using (17) with $S = [n-1]$ and $a_j = \operatorname{sign}(x_j)$. Thus $k \neq n$, $\alpha_j \geq \Delta$ for $j < n$, and $\alpha_n = \delta$. Since $[-e_n, e_n] \subset K$,

$$c_n \leq h_K(u) = \rho \sum_{j=1}^{n} c_j, \qquad c_n \leq \frac{\rho}{1 - \rho} \sum_{j=1}^{n-1} c_j.$$

The sum on the right is positive. Hence

$$\langle u, d^{b,k} \rangle \leq -\Delta \sum_{j=1}^{n-1} c_j + \delta c_n \leq -\frac{\Delta}{2} \sum_{j=1}^{n-1} c_j < 0.$$

In both cases the pair (b, k) depends on x alone, and the inequality holds for every nonzero supporting normal at x. The supporting-normal criterion for illumination therefore shows that the directions (19) illuminate K. □

Write $v \oplus e_j$ for the vertex obtained by changing coordinate j of v. A vertex cover meets every edge; equivalently, its complement is independent.

Lemma 7. *Let $A \subset \{0,1\}^n$ be a vertex cover, and let $T : A \to A$ satisfy*

$$T(v) = v \oplus e_{k(v)} \qquad (v \in A)$$

for some $k(v) \in [n]$. Suppose every directed cycle of T uses every coordinate direction. Then the family

$$\mathcal{B}(A, T) = \big\{ \big((2v_j - 1)_{j \in [n]}, k(v) \big) : v \in A \big\}$$

satisfies (17) and has cardinality $|A|$.

Proof. Fix S and a as in (17). The vertices satisfying $2v_j - 1 = -a_j$ for every $j \in S$ form a proper face F of positive dimension. It contains an edge, so $A \cap F \neq \varnothing$. If $k(v) \notin S$ for every $v \in A \cap F$, then $T(A \cap F) \subset A \cap F$. Iterating T from any vertex of this finite nonempty set would give a directed cycle contained in F. Such a cycle cannot use any coordinate in S, a contradiction. Thus some $v \in A \cap F$ has $k(v) \in S$, which proves the assertion. □

Proof of Proposition 5. For a binary vector y, let $p(y)$ be the sum of its coordinates modulo two, and let $O_n = \{v \in \{0,1\}^n : p(v) = 1\}$. Choose a minimum connected dominating set D in Q_{n-1}. Translate it by a binary vector so that $0 \in D$, and put

$$E_D = \{(y, p(y)) : y \in D\} \subset Q_n.$$

Every odd vertex is $(z, 1-p(z))$. If $z \in D$, its last-coordinate neighbor belongs to E_D. Otherwise choose a neighbor $y \in D$ of z. Then $p(y) = 1 - p(z)$, so $(y, p(y))$ is adjacent to the given odd vertex. Adjacent elements of D lift to even vertices at distance two, joined by a path with an odd intermediate vertex. Since D is connected and every odd vertex has a neighbor in E_D, the induced graph on $A_0 = O_n \cup E_D$ is connected.

In the ordinary coordinate order, define

$$v^i = \sum_{h=1}^{i} e_h \quad (0 \le i \le n), \qquad v^{n+i} = \sum_{h=i+1}^{n} e_h \quad (1 \le i \le n-1). \tag{21}$$

The vertices $v^0, \dots, v^{2n-1}$ are distinct. In this order, closed by $v^{2n} = v^0$, they form a cycle with successive coordinate directions $1, \dots, n, 1, \dots, n$. Let V be its vertex set and put $A = A_0 \cup V$. The cycle has n even vertices and contains $0 \in E_D$. Every odd cycle vertex already belongs to O_n. Thus

$$|A| \le 2^{n-1} + |D| + n - 1.$$

The complement of A consists of even vertices and is independent, so A is a vertex cover. Each added even vertex has an odd neighbor in A_0, so $Q_n[A]$ is connected.

For $u \in A$, let $h(u)$ be its graph distance to V in $Q_n[A]$. Define

$$\begin{aligned} T(v^i) &= v^{i+1} && (i \bmod 2n), \\ T(u) &= \min\{w \in A : w \sim u,\ h(w) = h(u) - 1\} && (u \in A \setminus V), \end{aligned} \tag{22}$$

where the minimum uses binary order. Connectedness makes each choice possible. Distances decrease until the specified cycle is reached, so it is the only directed cycle of T; it uses every coordinate. Lemmas 7 and 6 therefore give $\mathrm{ill}(K) \le |A|$ for every body in the definition of M_n. This proves (15).

Griggs proves

$$\gamma_c(Q_d) = (1 + o(1))\frac{2^d}{d} \qquad (d \longrightarrow \infty)$$

in [3, Theorems 2 and 4]. Substitute $d = n - 1$ and use $n - 1 = o(2^{n-1}/n)$ to obtain (16). □

For $n = 3$, the body P_n below is the regular cuboctahedron: its vertices are the signed permutations of $(1, 1, 0)$. Kiss and de Wet give a four-light-source construction for this body [7, Theorem 3.4, pp. 66–67]. Together with the general lower bound four in $\mathbb{R}^3$, this already yields $\mathrm{ill}(P_3) = 4$. Their illumination *parameter* is a different quantity; here we use the number of sources in their construction. The following proof treats the whole family directly.

Proposition 8. *For $n \ge 3$, the real 1-symmetric body*

$$P_n = \left\{ x \in [-1,1]^n : \sum_{j=1}^{n} |x_j| \le n - 1 \right\}$$

is not an affine cube and satisfies $\mathrm{ill}(P_n) = 2^{n-1}$.

Proof. The vertices of P_n are exactly the signed permutations of $(1,\dots,1,0)$. To see this, a point with $\sum_j |x_j| < n-1$ has a coordinate of modulus less than one that can be perturbed in both directions while preserving both constraints. If equality holds and two coordinate moduli lie strictly between zero and one, transfer a small amount from one modulus to the other, retaining the signs; again the point is not extreme. There cannot be exactly one nonintegral modulus, since the sum of the moduli is the integer $n-1$. The remaining possibility is precisely $n-1$ coordinates of modulus one and one zero. Conversely, in a convex decomposition of such a point, the cube constraints fix all its unit coordinates in every term, and the sum constraint then forces the remaining coordinate to be zero. Thus each displayed point is a vertex.

Fix a direction d and the index j of the zero coordinate of one of these vertices. Illumination requires d_i to have sign opposite to the vertex coordinate for every $i \neq j$, by the active cube constraints. These requirements determine at most one vertex with zero coordinate j; if a required d_i is zero there is no such vertex. Consequently d illuminates at most n vertices. There are $n2^{n-1}$ vertices, so $\mathrm{ill}(P_n) \geq 2^{n-1}$.

For the reverse inequality take

$$D = \left\{ d \in \{-1,1\}^n : \prod_{j=1}^{n} d_j = 1 \right\}.$$

At a vertex x with $x_j = 0$, prescribe $d_i = -x_i$ for $i \neq j$ and choose the unique remaining sign making $d \in D$. For $0 < t < 1$,

$$\|x + td\|_\infty < 1, \qquad \sum_{i=1}^{n} |x_i + td_i| = n - 1 - (n-2)t < n - 1.$$

Thus d illuminates x, and the 2^{n-1} members of D illuminate P_n. The body P_n is 1-symmetric directly from its definition. Its $n2^{n-1}$ vertices differ in number from the 2^n vertices of an affine cube for $n \geq 3$. To justify the passage from vertices to the boundary, express any boundary point as a convex combination of vertices and move one vertex of positive weight into the interior along an illuminating direction. The resulting convex combination is interior and is a positive step from the original point. □

These examples give the lower bound $2^{n-1} \leq M_n$. Together with (16), they show that the upper bound has the correct leading term. They do not determine the smaller-order difference $M_n - 2^{n-1}$.

6 Fractional lower bounds and equality families

The polydisc case of the proposition below is the published fractional formula [1, Theorem 1.3]. Its extension to the specified inner bodies agrees with the Hanner formulas in [11]. We retain a direct proof to establish the lower bound for every intermediate convex body, including bodies without coordinate symmetry.

On $\mathbb{C}^2$ put

$$B = \{z : |z_1| + |z_2| \leq 1\}, \qquad L(a,b) = \{z : |z_1| + a|z_2| \leq 1,\ b|z_1| + |z_2| \leq 1\}, \tag{23}$$

where $0 < a, b \leq 1$. Then $B \subseteq L(a,b)$, and both bodies meet each coordinate axis in its unit disc.

Proposition 9. *Fix $p, r \in \mathbb{Z}_{\geq 0}$ with $p + 2r = n \geq 1$, and choose $a_j, b_j \in (0,1]$ with $a_j b_j \geq 1/2$. On disjoint coordinate blocks put*

$$H = \mathbb{D}^p \times B^r, \qquad E = \mathbb{D}^p \times \prod_{j \in [r]} L(a_j, b_j). \tag{24}$$

If a convex body C satisfies $H \subseteq C \subseteq E$, then

$$\mathrm{ill}^*(C) \geq 2^n.$$

Moreover, for every $0 \leq t < 1$,

$$\mathrm{ill}^*((1-t)H + tC) = 2^n. \tag{25}$$

If C is complex 1-unconditional, all these bodies are complex 1-unconditional. Invertible complex linear images preserve the conclusions.

Proof. We construct the boundary measure giving the lower bound. On a disc take twice uniform probability measure on its unit circle. Any fixed nonzero direction illuminates exactly an open semicircle, so the illuminated mass is one; a zero direction illuminates no point.

On $L(a,b)$ let σ_1, σ_2 be uniform probability measures on the two unit axis circles, and put $\nu_{a,b} = 2\sigma_1 + 2\sigma_2$. At ξe_1, the first defining inequality in (23) is active. Hence illumination by d requires

$$\mathrm{Re}(\overline{\xi} d_1) + a|d_2| < 0.$$

Indeed, if a positive step enters the interior, convexity makes the one-sided derivative of the active convex function strictly negative. At ξe_2, illumination similarly requires $\mathrm{Re}(\overline{\xi} d_2) + b|d_1| < 0$. For $d_1 d_2 \neq 0$ put $s = |d_2|/|d_1|$. The proportions illuminated on the two circles are therefore at most

$$q_1 = \begin{cases} \pi^{-1} \arccos(as), & as < 1, \\ 0, & as \geq 1, \end{cases} \qquad q_2 = \begin{cases} \pi^{-1} \arccos(b/s), & b/s < 1, \\ 0, & b/s \geq 1. \end{cases}$$

If one is zero, their sum is at most $1/2$. Otherwise set $v = as$ and $w = b/s$. Since $v^2 + w^2 \geq 2vw = 2ab \geq 1$, we have $w \geq \sqrt{1-v^2}$ and hence $\arccos v + \arccos w \leq \pi/2$. Thus $q_1 + q_2 \leq 1/2$ in all cases. If exactly one coordinate of d is zero, only half of its own axis circle can be illuminated, and none of the other one, since $a, b > 0$. The zero vector illuminates neither circle. Every fixed direction consequently illuminates $\nu_{a,b}$-mass at most one.

Take the product ν of these measures over the p disc factors and the r two-dimensional factors of E. Its total mass is $2^p 4^r = 2^n$, and its support lies in $H \cap \partial E$. At every point in that support all factor points are boundary points. A direction illuminating E there must illuminate each factor; the product measure therefore assigns mass at most one to its illuminated set. The points supporting ν also belong to ∂C. Illumination in C implies illumination in E, because $\operatorname{int} C \subseteq \operatorname{int} E$. For any fractional illuminating measure μ for C, Tonelli's theorem now gives

$$2^n \leq \int \mu\{d : d \text{ illuminates } x\}\, d\nu(x) = \int \nu\{x : d \text{ illuminates } x\}\, d\mu(d) \leq \mu(\mathbb{C}^n \setminus \{0\}).$$

This proves the lower bound. It also applies to $(1-t)H + tC$, which lies between H and E.

For the upper bound we give a fractional illuminating measure of mass 2^n for H. On each disc use twice uniform circle measure on directions. On B use $2\sigma_1 + 2\sigma_2$, now as a direction measure. At a boundary point of B, choose any nonzero coordinate; the inward open semicircle on that axis decreases $|z_1| + |z_2|$ for small positive steps and has measure one. Thus this direction measure illuminates every boundary point of B. Take the product of the factor direction measures. A boundary point of H has illuminating mass at least one: at boundary factors the mass is at least one, and at interior factors every direction works for a sufficiently small step. The finitely many steps can be replaced by their minimum, since a segment from a body point to an interior point is interior except possibly at its initial point. This product measure has mass 2^n.

Finally, a fractional illuminating measure for a full-dimensional convex body A also illuminates $A + D$ for any compact convex set D. Given $x \in \partial(A + D)$, choose a decomposition

$x = y + z$ with $y \in A$ and $z \in D$. Then $y \in \partial A$, since an interior y would make x interior. Whenever $y + sd \in \operatorname{int} A$, one has $x + sd \in \operatorname{int} A + D \subseteq \operatorname{int}(A + D)$. Hence the illuminating mass at x is at least the mass at y. This is also the Minkowski-sum fact in [1, Fact 3.2]. Apply it to $A = (1 - t)H$ and $D = tC$. The restriction $t < 1$ ensures that A is full-dimensional, and dilation preserves the fractional upper bound just proved for H. This gives equality in (25). Coordinate-rotation invariance is preserved by Minkowski addition, and invertible complex linear maps preserve illumination and the corresponding measures. □

Remark 1. Taking $p = 0$, $r = 1$, and $C = H = B$ gives $\mathrm{ill}^*(B) = 4$. This body is not an affine image of $\mathbb{D}^2$. Writing $\operatorname{ext} K$ for the set of extreme points of K, we have

$$\operatorname{ext} B = (\partial\mathbb{D} \times \{0\}) \cup (\{0\} \times \partial\mathbb{D}), \qquad \operatorname{ext}(\mathbb{D}^2) = \partial\mathbb{D} \times \partial\mathbb{D}.$$

For the first identity, a boundary point with both coordinates nonzero is the convex combination of its two normalized axis points, with weights $|z_1|$ and $|z_2|$; interior points are not extreme. Each unit axis point is extreme, since a convex decomposition forces equality in its coordinate-disc constraint, fixing that coordinate, and the defining norm then makes the other coordinate zero. For the second identity, strict convexity of each disc makes a point with both coordinates of modulus one extreme, while any coordinate of modulus less than one admits a nontrivial symmetric perturbation within its disc. The first extreme-point set has two connected components and the second is connected. Invertible affine maps preserve extreme points and connectedness. Thus this example, also discussed in [11, 4], contradicts the equality clause of [1, Conjecture 1.2]. It does not contradict the proposed upper bound 2^n.

For an arbitrary endpoint C between H and E, the lower bound is proved, but the upper bound $\mathrm{ill}^*(C) \le 2^n$ is not established here. The equalities for $t < 1$ alone do not justify passing to $t = 1$.

A One family for all normalized bodies

Let U_n be the least cardinality of a finite family of directions that illuminates every normalized complex 1-unconditional convex body in $\mathbb{C}^n$. The family must be chosen independently of the body. This is a stronger requirement than bounding $\mathrm{ill}(K)$ separately for each K.

Proposition 10. *Put* $m = \lceil 2n/3 \rceil$ *and*

$$L_n = (2^{m+1} - 1)\binom{n}{m}, \qquad B_n = L_n + \sum_{k=m+1}^{n} \binom{n}{k}.$$

Put $C_n = B_n - (n - m)$. *Then*

$$L_n \le U_n \le C_n \le B_n, \tag{26}$$

and

$$U_n \sim \frac{3^{n+1}}{\sqrt{\pi n}}. \tag{27}$$

Proof. The directions in the proof of Lemma 11 depend only on n, the finite matchings and cyclic vectors, and their common positive margin. The normal estimates use only normalization of K. Hence the same family works for every normalized K, proving $U_n \le C_n$.

For every m-set $S \subset [n]$, when $m < n$, consider the normalized body

$$K_S = \{z \in \mathbb{C}^n : \|z_S\|_\infty + \|z_{S^c}\|_\infty \le 1\}.$$

At a point x of its coordinate torus

$$|x_j| = 1 \ (j \in S), \qquad x_j = 0 \ (j \notin S),$$

the directional derivative of its norm is

$$\max_{j\in S} \operatorname{Re}(\overline{x_j} d_j) + \|d_{S^c}\|_\infty.$$

Illumination forces this derivative to be negative: for a convex function, an interior point along the ray forces a negative one-sided derivative at its initial point. Consequently

$$\operatorname{Re}(\overline{x_j} d_j) < -\|d_{S^c}\|_\infty \quad (j \in S), \qquad \min_{j\in S} |d_j| > \max_{j\notin S} |d_j|.$$

Thus the m largest coordinate magnitudes of d determine S uniquely. The groups of directions serving these tori for different m-sets are disjoint. Within the group for S, projection must illuminate the m-polydisc torus, and hence the polydisc, requiring at least $2^{m+1}-1$ directions by the exact polydisc illumination theorem [1]. Summation gives $U_n \geq L_n$. If $m = n$, the polydisc itself gives this conclusion.

Finally, $B_n - L_n \leq 2^n$. Stirling's formula, with $m = 2n/3 + O(1)$, yields

$$2^{m+1}\binom{n}{m} \sim \frac{3}{\sqrt{\pi n}}\, 3^n.$$

The same asymptotic holds for $C_n = B_n - (n-m)$; squeezing between the lower and upper bounds proves the claim.

□

A.1 The common upper-bound construction

We include the coordinate-support argument needed for the upper bound, so that the simultaneous assertion has its full proof here. Its use as an individual-body upper bound is also recorded in the companion manuscript [4].

Lemma 11. *For every $n \geq 1$, a single family of at most*

$$C_n = (2^{m+1}-1)\binom{n}{m} + \sum_{k=m+1}^{n} \binom{n}{k} - (n-m), \qquad m = \lceil 2n/3 \rceil, \tag{28}$$

directions illuminates every normalized complex 1-unconditional body in $\mathbb{C}^n$.

We use the cyclic construction in the proof of Proposition 2.7 of [1]. For $b \geq 1$, put $q_b = 2^{b+1} - 1$. The vectors

$$\left\{ \left(\exp \frac{2\pi \mathrm{i}\, 2^{j-1} t}{q_b} \right)_{j\in[b]} : t \in \mathbb{Z}/q_b\mathbb{Z} \right\} \tag{29}$$

meet every product of b open semicircles, and hence illuminate $\mathbb{D}^b$. We use this published result as an external input.

The following projection property is already [2, Lemma 4.3]; we include its short proof.

Lemma 12. *Append to (29) the column of frequency 2^b. Every projection of the resulting $(b+1)$-column matrix onto b columns meets every product of open semicircles.*

Proof. Since $2^{b+1} \equiv 1 \pmod{q_b}$, multiplication of the row parameter t by 2 cyclically permutes the $b+1$ frequencies. It is a permutation of the rows because q_b is odd. Any omitted column can therefore be moved to the last position, reducing the assertion to (29). □

Lemma 13. *Let a finite rooted tree have nonempty coordinate sets as its nodes. Its root is a set T of cardinality b. Each node has at most two children, and each child is obtained by deleting one coordinate from its parent. Then q_b directions suffice, simultaneously for every normalized complex 1-unconditional body, to illuminate all boundary points whose supports occur in the tree.*

Proof. Arrange the vectors of (29), with their additional coordinate from Lemma 12, as rows of a matrix. Label b columns by T, leaving one additional column. At a node with support S, let G be the set of rows satisfying fixed open-semicircle conditions on $b - |S|$ columns outside S. Exactly one column outside S remains unrestricted. At the root, G contains every row, and the additional column is unrestricted.

The active columns S and the constrained columns together comprise exactly b columns. Lemma 12 implies that G, projected onto S, meets every product of open semicircles. Indeed, prescribe the desired semicircles on S and retain all the fixed constraints.

Suppose the children are $S \setminus \{i\}$ and $S \setminus \{j\}$. Split G by imposing, in its unrestricted column, respectively the upper and lower open semicircles. These conditions are disjoint. The previously unrestricted column becomes constrained, and the deleted column i or j becomes the unrestricted column of the corresponding child. Thus the invariant persists. For a single child, use just one of the two semicircles. Rows satisfying neither chosen condition stop at the current node; rows at leaves also stop. Every row consequently visits one nested path

$$T = S_0 \supset S_1 \supset \cdots \supset S_r, \qquad |S_{\ell-1} \setminus S_\ell| = 1.$$

The group of a node is exactly the set of rows visiting it. In particular, rows that stop early remain available at that node and all its ancestors.

For a node S, define on the torus of unit complex coordinates

$$g_S(\zeta) = \max_{p \in G} \min_{j \in S} \bigl[-\operatorname{Re}(\overline{\zeta_j} p_j) \bigr].$$

The covering property makes g_S positive everywhere. It is continuous, being a finite maximum of finite minima of continuous functions, so its minimum on the compact torus is positive. Take $\beta > 0$ no larger than these minima for all the finitely many nodes. Thus, for every node S and every $(\zeta_j)_{j \in S}$ with $|\zeta_j| = 1$, some row $p \in G$ satisfies

$$\operatorname{Re}(\overline{\zeta_j} p_j) \le -\beta \qquad (j \in S). \tag{30}$$

Choose $0 < \delta < \min\{1, \beta/n\}$.

For a row with path $S_0, \dots, S_r$, give amplitude 1 to the coordinates in S_r. The coordinate deleted at step ℓ receives amplitude $\delta^{r-\ell+1}$, and coordinates outside T receive zero. Multiply these positive amplitudes by the corresponding entries of the row to obtain d. At every visited node S_ℓ,

$$|d_j| \ge \delta^{r-\ell} \quad (j \in S_\ell), \qquad |d_j| \le \delta^{r-\ell+1} \quad (j \notin S_\ell). \tag{31}$$

Now let $x \in \partial K$ have support S_ℓ. Select a visiting row satisfying (30) for $\zeta_j = x_j/|x_j|$, and put $a = \delta^{r-\ell}$. For every supporting normal, (2) and (31) give

$$\begin{aligned} [u, d] &\le -\beta a \sum_{j \in S_\ell} |u_j| + \delta a \sum_{j \notin S_\ell} |u_j| \\ &\le a\bigl[-\beta + (n - |S_\ell|)\delta\bigr] \sum_{j \in S_\ell} |u_j| < 0. \end{aligned}$$

The last sum is positive since $h_K(u) > 0$. The criterion (1) completes the proof. □

Proof of Lemma 11. All choices below are made independently of the normalized body K. Write $a_k = \binom{n}{k}$ and $a_{n+1} = 0$. For $k \in [n-1]$, between the k-sets and the $(k+1)$-sets, the Boolean incidence graph has degrees $n-k$ and $k+1$, respectively. We assign a k-set to a containing parent, allowing each parent at most two children. Write $N(\mathcal{A})$ for the set of neighbors of a vertex family $\mathcal{A}$.

If $a_k \le 2a_{k+1}$, edge counting gives, for every family $\mathcal{A}$ of k-sets,

$$|N(\mathcal{A})| \ge \frac{n-k}{k+1}|\mathcal{A}| \ge \frac{|\mathcal{A}|}{2}.$$

Hall's matching theorem [6], after replacing each upper vertex by two slots, assigns every k-set to a parent. If $a_k \ge 2a_{k+1}$, reverse edge counting gives, for every family $\mathcal{B}$ of upper vertices,

$$|N(\mathcal{B})| \ge \frac{k+1}{n-k}|\mathcal{B}| \ge 2|\mathcal{B}|.$$

To apply Hall's theorem to the duplicated upper vertices, take any set of slots and let $\mathcal{B}$ be their underlying vertices. There are at most $2|\mathcal{B}|$ slots and at least that many lower neighbors by the displayed estimate. Thus Hall's theorem fills both slots of every upper vertex. The unused k-sets become roots. The top level is handled separately: $[n]$ is a root and has no parent. Consequently, including $k = n$ with $a_{n+1} = 0$, the numbers of roots are

$$r_k = \max\{a_k - 2a_{k+1}, 0\}.$$

The choices between successive pairs of layers are independent: a vertex's parent and its children are governed by different matchings. Together they form a binary forest containing every nonempty subset exactly once.

The positive root counts occur precisely at $k \ge m = \lceil 2n/3 \rceil$. Applying Lemma 13 separately to the trees gives a family of at most

$$\begin{aligned}&\sum_{k=m}^{n}(a_k - 2a_{k+1})(2^{k+1}-1)\\ &= (2^{m+1}-1)a_m + \sum_{k=m+1}^{n} a_k.\end{aligned}$$

directions. Here the last equality follows from

$$\sum_{k=m}^{n}(a_k - 2a_{k+1})2^{k+1} = 2^{m+1}a_m, \qquad \sum_{k=m}^{n}(a_k - 2a_{k+1}) = a_m - \sum_{k=m+1}^{n} a_k.$$

For every $k \ge m$ the number r_k of roots at level k is positive. Prescribe the nested sets $S_k = [k]$, $m \le k \le n$. The set S_n is already the top root. For $m \le k < n$, the matching between levels k and $k+1$ leaves $r_k > 0$ lower vertices unmatched. Apply a coordinate permutation to this matching alone so that one unmatched vertex becomes S_k. Containment and all capacities are preserved. Whether a k-set is a root depends only on this matching; its children are assigned by the separate matching between levels $k-1$ and k. The permutations therefore impose no compatibility condition across level pairs. We can make every S_k a root at once.

Use the upper and lower open semicircles at every split in the tree construction. In each root's matrix, the row indexed by $t = 0$ has entry 1 in every column. It lies on the boundary of both semicircles, so it stops at the root and is never needed in a descendant's group. Take one common positive margin β for all groups in the forest, and choose $0 < \delta < \min\{1, \beta/n\}$.

Remove the $n-m+1$ rows indexed by $t=0$ associated with the roots $S_m, S_{m+1}, \dots, S_n$, and replace them by the single direction

$$d_j = \begin{cases} 1, & j \le m, \\ \delta^{j-m}, & j > m. \end{cases}$$

At support S_k, its smallest active amplitude is δ^{k-m}, and every inactive amplitude is at most δ^{k-m+1}. Whenever the deleted row supplies the margin β at that root, the replacement has positive real coordinates and the required amplitude gap. Writing $a = \delta^{k-m}$, the same normal estimate gives

$$[u,d] \le a[-\beta + (n-k)\delta] \sum_{j \in S_k} |u_j| < 0$$

for every nonzero supporting normal at any point for which that row was selected. The sum is positive by (2). At all other nodes and for all other selected rows the construction is unchanged. This saves $n-m = \lfloor n/3 \rfloor$ directions. All matchings, rows, margins, and magnitudes depend only on n. Thus the same family works for every normalized body and has at most C_n directions. □

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