Illumination of complex 1-unconditional convex bodies

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Abstract

We study ordinary and fractional illumination of complex 1-unconditional convex bodies. In every complex dimension n we give an explicit ordinary bound asymptotic to $3^{n+1}/\sqrt{\pi n}$, and prove $\mathrm{ill}^*(K) \le b_*^n$ and $\mathrm{ill}(K) = O(n\log(n+1)b_*^n)$, where $b_* = 2.4805534697\ldots$ is defined by an explicit scalar equation. For normalized non-polydiscs in dimensions two, three and four, the sharp ordinary bounds are 6, 14 and 30. More generally, a full coordinate $(n-1)$-polydisc gives the sharp bound $2^{n+1}-2$ in every dimension $n \ge 2$. In dimension two we prove the fractional bound 4 and classify equality by the two maximal supporting coefficients at the coordinate axes. If every extreme point has all coordinates nonzero, we obtain the fractional bound 2^n, with equality exactly for the polydisc, and a quantitative strict bound otherwise. Appendices give the exact sizes of simultaneous illuminating families in dimensions one through five, the exact finite verification used in dimension four, and a numerical comparison of the general estimates through dimension 100.

Draft status. The included proofs have been checked internally. The citations were audited on 1 November 2026; external review and a definitive priority assessment remain outstanding. The finite verification used in dimension four is exact. The numerical comparison is supplementary and is not used in the proofs.

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1 Introduction

A complex 1-unconditional convex body in $\mathbb{C}^n$ is invariant under independent rotations of its coordinates. Its ordinary illumination number $\mathrm{ill}(K)$ is the least number of directions that move every boundary point into the interior. The fractional number $\mathrm{ill}^*(K)$ allows a measure on directions, requiring illuminating mass at least one at each boundary point. Both notions concern the underlying real space of dimension $2n$; their precise definitions are given in Section 2.

Fractional illumination was introduced by Naszódi [4] using finitely supported measures. We use the Borel-measure formulation of [5, Section 1.2]. Weighted covering and its relation to ordinary covering were developed in [1]. In real dimension d, the centrally symmetric fractional bound is 2^d; in $\mathbb{C}^n$ this gives 4^n, rather than the complex benchmark 2^n.

Rotem, Schejter and Slomka proved $\mathrm{ill}(\mathbb{D}^n) = 2^{n+1}-1$ and $\mathrm{ill}^*(\mathbb{D}^n) = 2^n$ [5, Theorems 1.3–1.4]. The ordinary formula was stated earlier by Boltyanski and Martini [2]; the correction of their upper-bound construction is explained in [5, Remark 2.8]. We establish general bounds for the unconditional class and sharp results in low dimensions and under a coordinate-section condition. The fractional results distinguish the full-support extreme-point class from the general two-dimensional class, where non-polydisc equality cases also occur.

Sun and Vritsiou proved related real results: bounds for non-parallelepipeds in $\mathbb{R}^3$ and $\mathbb{R}^4$, the full coordinate-hyperplane case, and the case of extreme points with no zero coordinates [7, Theorems 1, 2 and 5]. Their work on 1-symmetric bodies also develops the use of a large inward coordinate with small remaining coordinates [8]. These are real predecessors; the complex assertions below require independent coordinate rotations and estimates in complex dimension

n, whose real dimension is $2n$. The theorem numbers cited from [7] refer to its 2024 arXiv version.

The coauthored manuscript [6] already gives the bound $2 \cdot 3^n - 2^n - 1$ for complex 1-unconditional bodies (Theorem 1.8). Its Theorems 1.5–1.7 treat the 1-symmetric subclass, including the ordinary bound $2^{n+1} - 2$ for non-polydiscs and the planar fractional bound 4. Those results assume coordinate-permutation invariance. We cite its cyclic projection and coloring lemmas where they enter the arguments here.

Our first general estimate is Theorem 3: with $m = \lceil 2n/3 \rceil$,

$$\operatorname{ill}(K) \le C_n = (2^{m+1} - 1)\binom{n}{m} + \sum_{k=m+1}^{n} \binom{n}{k} - (n - m), \qquad C_n \sim \frac{3^{n+1}}{\sqrt{\pi n}}.$$

Its proof shares directions along trees of coordinate supports, improving the separate-support construction of [6, Theorem 1.8]. A different construction gives Theorem 6:

$$\operatorname{ill}^*(K) \le b_*^n, \qquad \operatorname{ill}(K) \le \left\lceil e b_*^n \left[2n \log\left(1 + 32 b_*^2 (n+1)^2\right) + 1 \right] \right\rceil,$$

where $b_* = c_*^{-1}$ and $c_* \in (0, 1/2)$ is determined by

$$\cos(\pi c_*) \cosh(\pi(1 - c_*)) = 1.$$

The two arguments and their auxiliary estimates are kept together in Section 3.

After positive diagonal normalization, a body contains each unit coordinate disc and is contained in $\mathbb{D}^n$. If it contains a full coordinate $(n-1)$-polydisc and is not $\mathbb{D}^n$, Theorem 10 gives $\operatorname{ill}(K) \le 2^{n+1} - 2$. Its sharpness examples are

$$H_n = \mathbb{D}^{n-2} \times \{(z, w) \in \mathbb{C}^2 : |z| + |w| \le 1\}.$$

The coordinate-section hypothesis is automatic in dimension two. Theorems 14 and 15 remove it in dimensions three and four. Consequently the largest ordinary illumination numbers of normalized non-polydiscs in dimensions two, three and four are 6, 14 and 30; when the polydisc is included, they are 7, 15 and 31.

For fractional illumination, Theorem 21 proves $\operatorname{ill}^*(K) \le 4$ for every normalized unconditional body in $\mathbb{C}^2$, with equality exactly when $K = \mathbb{D}^2$ or $a(K)b(K) \ge 1/2$. The coefficients $a(K)$ and $b(K)$ are the maximal coefficients in the supporting inequalities $|z_1| + a(K)|z_2| \le 1$ and $b(K)|z_1| + |z_2| \le 1$ on K. In particular, the already defined body H_2 has fractional number 4 without being a linear image of the bidisc; see the remark after Theorem 21. This contradicts the equality characterization in [5, Conjecture 1.2], while remaining consistent with its proposed upper bound. In arbitrary dimension, Theorem 22 proves that if every extreme point has all coordinates nonzero, then $\operatorname{ill}^*(K) \le 2^n$, with equality exactly for $\mathbb{D}^n$. It also gives an explicit improvement below 2^n for each normalized non-polydisc in that class.

The main estimates allow the directions to depend on the body. Appendix A considers the distinct quantity U_n, the smallest size of one family illuminating every normalized body, and proves $U_1 = 3$, $U_2 = 7$, $U_3 = 21$, $U_4 = 60$ and $U_5 = 155$. Appendix B supplies the finite verification used in dimension four, and Appendix C compares the general bounds numerically.

2 Notation and preliminary facts

Write $[n] = \{1, \dots, n\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$, and $\mathbb{R}_+ = [0, \infty)$. A convex body is a compact convex set with nonempty interior in the underlying real space. It is complex 1-unconditional if it is invariant under independent rotations of its coordinates. Its Minkowski norm is denoted

by $\|\cdot\|_K$. For $z \in \mathbb{C}^n$, write $|z| = (|z_1|, \ldots, |z_n|)$ and $\operatorname{supp}(z) = \{j \in [n] : z_j \neq 0\}$. We also write

$$M_x = \{j \in [n] : |x_j| = \|x\|_\infty\}, \qquad Z_x = \{j \in [n] : x_j = 0\}.$$

A nonzero vector d illuminates $x \in \partial K$ if $x + td \in \operatorname{int} K$ for some $t > 0$. The least cardinality of a family illuminating every boundary point is $\operatorname{ill}(K)$. A finite nonnegative Borel measure μ on $\mathbb{C}^n \setminus \{0\}$ fractionally illuminates K if

$$\mu\{d : d \text{ illuminates } x\} \geq 1 \qquad (x \in \partial K).$$

The infimum of the total masses of such measures is $\operatorname{ill}^*(K)$.

We call K normalized when $e_j \in \partial K$ for every $j \in [n]$. Positive diagonal rescaling gives this normalization, and illumination numbers are invariant under invertible linear maps. Coordinate contractions preserve membership, as proved below; consequently a normalized body satisfies $K \subseteq \mathbb{D}^n$ and contains each unit coordinate disc.

Lemma 1. *For a complex 1-unconditional convex body K, the following statements hold.*

1. *If $|w_j| \leq |z_j|$ for every j, then $\|w\|_K \leq \|z\|_K$.*
2. *For every real linear subspace $E \subseteq \mathbb{C}^n$,*

$$\operatorname{relint}(K \cap E) = E \cap \operatorname{int} K.$$

3. *Suppose $z \in K$, J is a nonempty subset of $\operatorname{supp}(z)$, and the point q obtained from z by setting the coordinates in J to zero belongs to $\operatorname{int} K$. If $w_j = z_j$ for $j \notin J$ and $|w_j| < |z_j|$ for $j \in J$, then $w \in \operatorname{int} K$.*

Proof. Invariance under all coordinate rotations implies central symmetry. Because K has nonempty interior, its center 0 is interior: if an open Euclidean ball of radius r around a lies in K, the same is true around $-a$, and convexity puts the ball of radius r around 0 in K. Its Minkowski functional is therefore a norm, with $\operatorname{int} K = \{z : \|z\|_K < 1\}$.

Every number $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ is a convex combination of two unit complex numbers. Explicitly, if $\lambda = se^{\mathrm{i}\theta}$, $0 \leq s \leq 1$, then

$$\lambda = \frac{1+s}{2} e^{\mathrm{i}\theta} + \frac{1-s}{2}(-e^{\mathrm{i}\theta}).$$

Thus contracting a single coordinate by λ preserves membership in K, by convexity and rotation invariance. Applying these contractions one coordinate at a time proves coordinatewise monotonicity of membership. Apply the same fact to every dilation of K to obtain the norm inequality in part 1. Coordinates with $z_j = 0$ also have $w_j = 0$, so cause no exception.

The inclusion from right to left in part 2 follows by intersecting an ambient open ball with E. For the reverse inclusion, let $y \in \operatorname{relint}(K \cap E)$. If $y = 0$ there is nothing to prove. Otherwise relative interior gives an $s > 0$ with $(1+s)y \in K \cap E$. Consequently $\|y\|_K \leq (1+s)^{-1} < 1$, as required.

For part 3 choose $\rho \in (0, 1)$ such that $|w_j| \leq \rho |z_j|$ for all $j \in J$; there are finitely many strict ratios, so one common ρ exists. Put $y = \rho z + (1-\rho)q$. Then

$$\|y\|_K \leq \rho \|z\|_K + (1-\rho)\|q\|_K < 1.$$

On J we have $y_j = \rho z_j$, and off J we have $y_j = w_j = z_j$. Part 1 therefore gives $\|w\|_K \leq \|y\|_K < 1$. □

We use the real pairing

$$[u,z]=\operatorname{Re}\sum_{j\in[n]}\overline{u_j}z_j,\qquad h_K(u)=\max_{z\in K}[u,z].$$

A direction d illuminates $x\in\partial K$ if and only if

$$[u,d]<0\quad\text{for every nonzero } u \text{ with } [u,x]=h_K(u). \tag{1}$$

Here is a proof that also covers nonsmooth boundary points. If $x+td\in\operatorname{int}K$ for some $t>0$, then $[u,x+td]<h_K(u)=[u,x]$ for every nonzero supporting normal at x, proving necessity. For sufficiency, suppose no positive step enters $\operatorname{int}K$. For each integer $j\ge 1$, separation at $x+j^{-1}d\notin\operatorname{int}K$ supplies a Euclidean unit vector u^j with $[u^j,x+j^{-1}d]\ge h_K(u^j)$. Since $h_K(u^j)\ge[u^j,x]$, we have $[u^j,d]\ge 0$. Pass to a convergent subsequence of these unit vectors, with limit u. The support function is continuous, so $[u,x]\ge h_K(u)$; the reverse inequality holds because $x\in K$. Thus u is a nonzero supporting normal at x with $[u,d]\ge 0$, contradicting (1).

For a normalized unconditional body, put $S=\operatorname{supp}(x)$ and $a_j=|u_j|$ for a supporting normal u at x. Independent coordinate rotations give $u_j=|u_j|x_j/|x_j|$ whenever $j\in S$: the maximum of $[u,\cdot]$ on the coordinate-rotation orbit of x is $\sum_j|u_j||x_j|$, and x already attains that maximum. Thus, writing $h=h_K(u)>0$,

$$h=\sum_{j\in S}a_j|x_j|,\qquad a_j\le h\quad(j\in[n]),\qquad h\le\sum_{j\in S}a_j. \tag{2}$$

The middle inequality follows from the coordinate discs in K; the last uses $K\subseteq\mathbb{D}^n$.

For completeness we include the polydisc calculation from [5, Theorem 1.3].

Lemma 2. *For every $n\ge 1$, $\operatorname{ill}^*(\mathbb{D}^n)=2^n$.*

Proof. Let $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$ and let σ be product normalized arc-length measure on $\mathbb{T}^n$. At a boundary point x of the polydisc, a direction d illuminates x precisely when

$$\operatorname{Re}(\overline{x_j}d_j)<0\qquad\text{for every } j \text{ with } |x_j|=1.$$

Indeed, expansion of $|x_j+td_j|^2$ proves necessity at each active coordinate and sufficiency for all sufficiently small positive t; coordinates with modulus less than one remain interior for such t. If there are k active coordinates, the displayed conditions have σ-probability $2^{-k}\ge 2^{-n}$. Thus $2^n\sigma$ fractionally illuminates the entire polydisc.

For the reverse bound, regard σ as a measure on the boundary torus. A fixed direction illuminates σ-mass 2^{-n} if all its coordinates are nonzero, and zero otherwise, by the same strict coordinate inequalities. If μ is a fractional illuminating measure, Tonelli's theorem for the open illumination relation gives

$$1\le\int_{\mathbb{T}^n}\mu\{d:d\text{ illuminates }x\}\,d\sigma(x)=\int\sigma\{x:d\text{ illuminates }x\}\,d\mu(d)\le 2^{-n}\mu(\mathbb{C}^n\setminus\{0\}).$$

Taking the infimum over μ proves the claim. □

3 General upper bounds

3.1 Directions indexed by coordinate supports

Theorem 3. *For $n\ge 1$, put $m=\lceil 2n/3\rceil$ and*

$$C_n=(2^{m+1}-1)\binom{n}{m}+\sum_{k=m+1}^{n}\binom{n}{k}-(n-m). \tag{3}$$

Every complex 1*-unconditional convex body* $K \subset \mathbb{C}^n$ *satisfies* $\operatorname{ill}(K) \le C_n$*. Moreover,*

$$C_n \sim \frac{3^{n+1}}{\sqrt{\pi n}}.$$

We use the cyclic construction in the proof of Proposition 2.7 of [5]. For $b \ge 1$, put $q_b = 2^{b+1} - 1$. The vectors

$$\left\{ \left(\exp \frac{2\pi \mathrm{i}\, 2^{j-1} t}{q_b} \right)_{j \in [b]} : t \in \mathbb{Z}/q_b\mathbb{Z} \right\} \tag{4}$$

meet every product of b open semicircles, and hence illuminate $\mathbb{D}^b$. We use this published result as an external input.

The following projection property is [6, Lemma 4.3]; we recall its short proof.

Lemma 4. *Append to* (4) *the column of frequency* 2^b*. Every projection of the resulting* $(b+1)$*-column matrix onto* b *columns meets every product of open semicircles.*

Proof. Since $2^{b+1} \equiv 1 \pmod{q_b}$, multiplication of the row parameter t by 2 cyclically permutes the $b+1$ frequencies. It is a permutation of the rows because q_b is odd. Any omitted column can therefore be moved to the last position, reducing the assertion to (4). □

Lemma 5. *Let a finite rooted tree have nonempty coordinate sets as its nodes. Its root is a set* T *of cardinality* b*. Each node has at most two children, and each child is obtained by deleting one coordinate from its parent. Then* q_b *directions suffice, simultaneously for every normalized complex* 1*-unconditional body, to illuminate all boundary points whose supports occur in the tree.*

Proof. Arrange the vectors of (4), with their additional coordinate from Lemma 4, as rows of a matrix. Label b columns by T, leaving one additional column. At a node with support S, let G be the set of rows satisfying fixed open-semicircle conditions on $b - |S|$ columns outside S. Exactly one column outside S remains unrestricted. At the root, G contains every row, and the additional column is unrestricted.

The active columns S and the constrained columns together comprise exactly b columns. Lemma 4 implies that G, projected onto S, meets every product of open semicircles. Indeed, prescribe the desired semicircles on S and retain all the fixed constraints.

Suppose the children are $S \setminus \{i\}$ and $S \setminus \{j\}$. Split G by imposing, in its unrestricted column, respectively the upper and lower open semicircles. These conditions are disjoint. The previously unrestricted column becomes constrained, and the deleted column i or j becomes the unrestricted column of the corresponding child. Thus the invariant persists. For a single child, use just one of the two semicircles. Rows satisfying neither chosen condition stop at the current node; rows at leaves also stop. Every row consequently visits one nested path

$$T = S_0 \supset S_1 \supset \cdots \supset S_r, \qquad |S_{\ell-1} \setminus S_\ell| = 1.$$

The group of a node is exactly the set of rows visiting it. In particular, rows that stop early remain available at that node and all its ancestors.

For a node S, define on the torus of unit complex coordinates

$$g_S(\zeta) = \max_{p \in G} \min_{j \in S} \bigl[-\operatorname{Re}(\overline{\zeta_j} p_j) \bigr].$$

The covering property makes g_S positive everywhere. It is continuous, being a finite maximum of finite minima of continuous functions, so its minimum on the compact torus is positive. Take

$\beta > 0$ no larger than these minima for all the finitely many nodes. Thus, for every node S and every $(\zeta_j)_{j\in S}$ with $|\zeta_j| = 1$, some row $p \in G$ satisfies

$$\mathrm{Re}(\overline{\zeta_j} p_j) \le -\beta \qquad (j \in S). \tag{5}$$

Choose $0 < \delta < \min\{1, \beta/n\}$.

For a row with path $S_0, \dots, S_r$, give amplitude 1 to the coordinates in S_r. The coordinate deleted at step ℓ receives amplitude $\delta^{r-\ell+1}$, and coordinates outside T receive zero. Multiply these positive amplitudes by the corresponding entries of the row to obtain d. At every visited node S_ℓ,

$$|d_j| \ge \delta^{r-\ell} \quad (j \in S_\ell), \qquad |d_j| \le \delta^{r-\ell+1} \quad (j \notin S_\ell). \tag{6}$$

Now let $x \in \partial K$ have support S_ℓ. Select a visiting row satisfying (5) for $\zeta_j = x_j/|x_j|$, and put $a = \delta^{r-\ell}$. For every supporting normal, (2) and (6) give

$$\begin{aligned}[u, d] &\le -\beta a \sum_{j\in S_\ell} |u_j| + \delta a \sum_{j\notin S_\ell} |u_j| \\ &\le a\big[-\beta + (n - |S_\ell|)\delta\big] \sum_{j\in S_\ell} |u_j| < 0.\end{aligned}$$

The last sum is positive since $h_K(u) > 0$. The criterion (1) completes the proof. □

Proof of Theorem 3. By diagonal rescaling we may assume that K is normalized. Write $a_k = \binom{n}{k}$ and $a_{n+1} = 0$. For $k \in [n-1]$, between the k-sets and the $(k+1)$-sets, the Boolean incidence graph has degrees $n-k$ and $k+1$, respectively. We assign a k-set to a containing parent, allowing each parent at most two children. Write $N(\mathcal{A})$ for the set of neighbors of a vertex family $\mathcal{A}$.

If $a_k \le 2a_{k+1}$, edge counting gives, for every family $\mathcal{A}$ of k-sets,

$$|N(\mathcal{A})| \ge \frac{n-k}{k+1}|\mathcal{A}| \ge \frac{|\mathcal{A}|}{2}.$$

Hall's theorem [3], after replacing each upper vertex by two slots, assigns every k-set to a parent. If $a_k \ge 2a_{k+1}$, reverse edge counting gives, for every family $\mathcal{B}$ of upper vertices,

$$|N(\mathcal{B})| \ge \frac{k+1}{n-k}|\mathcal{B}| \ge 2|\mathcal{B}|.$$

To apply Hall's theorem to the duplicated upper vertices, take any set of slots and let $\mathcal{B}$ be their underlying vertices. There are at most $2|\mathcal{B}|$ slots and at least that many lower neighbors by the displayed estimate. Thus Hall's theorem fills both slots of every upper vertex. The unused k-sets become roots. The top level is handled separately: $[n]$ is a root and has no parent. Consequently, including $k = n$ with $a_{n+1} = 0$, the numbers of roots are

$$r_k = \max\{a_k - 2a_{k+1}, 0\}.$$

The choices between successive pairs of layers are independent: a vertex's parent and its children are governed by different matchings. Together they form a binary forest containing every nonempty subset exactly once.

The positive root counts occur precisely at $k \ge m = \lceil 2n/3 \rceil$. Applying Lemma 5 separately to the trees gives a family of at most

$$\begin{aligned}&\sum_{k=m}^{n} (a_k - 2a_{k+1})(2^{k+1} - 1) \\ &= (2^{m+1} - 1)a_m + \sum_{k=m+1}^{n} a_k.\end{aligned}$$

directions. Here the last equality follows from

$$\sum_{k=m}^{n}(a_k-2a_{k+1})2^{k+1}=2^{m+1}a_m,\qquad \sum_{k=m}^{n}(a_k-2a_{k+1})=a_m-\sum_{k=m+1}^{n}a_k.$$

For every $k\ge m$ the number r_k of roots at level k is positive. Prescribe the nested sets $S_k=[k]$, $m\le k\le n$. The set S_n is already the top root. For $m\le k<n$, the matching between levels k and $k+1$ leaves $r_k>0$ lower vertices unmatched. Apply a coordinate permutation to this matching alone so that one unmatched vertex becomes S_k. Containment and all capacities are preserved. Whether a k-set is a root depends only on this matching; its children are assigned by the separate matching between levels $k-1$ and k. The permutations therefore impose no compatibility condition across level pairs. We can make every S_k a root at once.

Use the upper and lower open semicircles at every split in the tree construction. In each root's matrix, the row indexed by $t=0$ has entry 1 in every column. It lies on the boundary of both semicircles, so it stops at the root and is never needed in a descendant's group. Take one common positive margin β for all groups in the forest, and choose $0<\delta<\min\{1,\beta/n\}$.

Remove the $n-m+1$ rows indexed by $t=0$ associated with the roots $S_m,S_{m+1},\dots,S_n$, and replace them by the single direction

$$d_j=\begin{cases}1, & j\le m,\\ \delta^{j-m}, & j>m.\end{cases}$$

At support S_k, its smallest active amplitude is δ^{k-m}, and every inactive amplitude is at most δ^{k-m+1}. Whenever the deleted row supplies the margin β at that root, the replacement has positive real coordinates and the required amplitude gap. Writing $a=\delta^{k-m}$, the same normal estimate gives

$$[u,d]\le a[-\beta+(n-k)\delta]\sum_{j\in S_k}|u_j|<0$$

for every nonzero supporting normal at any point for which that row was selected. The sum is positive by (2). At all other nodes and for all other selected rows the construction is unchanged. This saves $n-m=\lfloor n/3\rfloor$ directions and proves the asserted upper bound. To obtain its asymptotic form, Stirling's formula with $m=2n/3+O(1)$ gives

$$2^{m+1}\binom{n}{m}\sim\frac{3^{n+1}}{\sqrt{\pi n}}.$$

All remaining terms in C_n have absolute value at most $2^{n+1}+n$ and are of smaller order, proving the asymptotic statement. □

Remark 1. The directions just constructed work for every normalized body. Appendix A determines the minimum size under this stronger requirement in dimensions one through five. It does not give a lower bound for the largest illumination number when the directions may be chosen separately for each body.

3.2 An upper bound with a smaller exponential base

Unconditionality here means invariance under independent rotations of the complex coordinates. Let $K\subset\mathbb{C}^n$ be such a convex body, and put

$$L=K\cap\mathbb{R}^n_+.$$

Then L is compact and convex and is a coordinatewise downset: if $y\in L$ and $0\le x_j\le y_j$ for every $j\in[n]$, then $x\in L$. Its n-dimensional volume is positive because $0\in\operatorname{int}K$, and $z\in K$

if and only if $(|z_1|,\dots,|z_n|)\in L$. In particular, coordinatewise domination of moduli implies domination of the Minkowski norm. No normalization of the coordinate radii is needed.

Let $c_*\in(0,1/2)$ be the unique solution of

$$\cos(\pi c_*)\cosh(\pi(1-c_*))=1. \tag{7}$$

The product on the left is strictly decreasing on $(0,1/2)$, and its endpoint values lie on opposite sides of one. Numerically,

$$c_*=0.4031358373055328\dots, \qquad b_*=c_*^{-1}=2.480553469728144\dots.$$

Theorem 6. *For every complex 1-unconditional convex body $K\subset\mathbb{C}^n$,*

$$\mathrm{ill}^*(K)\le b_*^n, \tag{8}$$

and

$$\mathrm{ill}(K)\le\left\lceil eb_*^n\left[2n\log\bigl(1+32b_*^2(n+1)^2\bigr)+1\right]\right\rceil. \tag{9}$$

In particular,

$$\mathrm{ill}(K)=O\bigl(n\log(n+1)b_*^n\bigr).$$

Replacing b_* by $5/2$ gives the simpler explicit estimate

$$\mathrm{ill}(K)\le\left\lceil e(5/2)^n\left[2n\log\bigl(1+200(n+1)^2\bigr)+1\right]\right\rceil. \tag{10}$$

The minimum of this bound and Theorem 3 is valid in every dimension. Numerical comparisons are given in Appendix C.

3.2.1 A one-coordinate inequality

Define a locally finite measure on $\mathbb{C}$ by

$$d\nu_1(z)=\frac{d\rho\,d\theta}{2\pi}, \qquad z=\rho e^{\mathrm{i}\theta}, \quad \rho\ge0, \quad 0\le\theta<2\pi.$$

Equivalently its density with respect to plane area is $(2\pi|z|)^{-1}$. The density is locally integrable. In particular, $\nu_1(\{|z|\le R\})=R$ for $R\ge0$, and ν_1 has no atom at zero. Write $\nu=\nu_1^{\otimes n}$. Then

$$\nu(K)=\mathrm{vol}_n(L), \qquad \nu(aK)=a^n\nu(K) \quad (a>0). \tag{11}$$

The homogeneity degree is n, although the ambient real dimension is $2n$.

Let

$$H(s)=\nu_1\{z:|z|\le1,\ \Re z\le s\}.$$

For $-1<s<1$, direct integration gives

$$H(s)=\frac12+\frac{\arcsin s+s\,\mathrm{arcosh}(1/|s|)}{\pi}, \tag{12}$$

where the second term is interpreted continuously at $s=0$. Also $H(s)=0$ for $s\le-1$, and $H(s)=1$ for $s\ge1$. For $0<s<1$, conditioning on the radius gives

$$H(s)=s+\int_s^1\left(\frac12+\frac{\arcsin(s/\rho)}{\pi}\right)d\rho.$$

Integration by parts yields

$$\int_s^1\arcsin(s/\rho)\,d\rho=\arcsin s-\frac{\pi s}{2}+s\int_s^1\frac{d\rho}{\sqrt{\rho^2-s^2}},$$

which proves (12) for positive s. Symmetry and continuity give the remaining values. Differentiating the resulting expression gives

$$H'(s)=\frac1\pi\int_s^1\frac{d\rho}{\sqrt{\rho^2-s^2}}=\frac{\mathrm{arcosh}(1/s)}{\pi}.$$

Lemma 7. *For every $0 < c \le c_*$,*

$$H(s) \ge c + (1-c)s \qquad (s \le 1). \tag{13}$$

Proof. On $[0,1]$, the function H is concave, so $H(s) \ge (1+s)/2 \ge c + (1-c)s$. On $[-1,0]$, it is convex, and the minimum of $H(s) - (1-c)s$ occurs at

$$s = -\operatorname{sech}(\pi(1-c)).$$

By (12), that minimum equals

$$\frac{1}{2} - \frac{\arcsin(\operatorname{sech}(\pi(1-c)))}{\pi}.$$

It is at least c precisely when $\cos(\pi c)\cosh(\pi(1-c)) \ge 1$, which holds for $c \le c_*$. For $s \le -1$, the right side of (13) is nonpositive. □

The simple choice $c = 2/5$ is certified without numerical approximations. The needed inequality is $\cosh(3\pi/5) \ge 1 + \sqrt{5}$. Using $\pi > 25/8$, the Taylor lower bound gives

$$\cosh(3\pi/5) > 1 + \frac{(15/8)^2}{2} + \frac{(15/8)^4}{24} = \frac{321729}{98304} > \frac{13}{4} > 1 + \sqrt{5}.$$

3.2.2 A comparison of product measures

We first record the measure-comparison fact used below.

Lemma 8. *Let $c > 0$, and let $m_1, \dots, m_n$ be locally finite Borel measures on $[0,\infty)$ such that $m_j([0,s]) \ge cs$ for every $j \in [n]$ and $s \ge 0$. For every bounded Borel coordinatewise downset $A \subset [0,\infty)^n$,*

$$(m_1 \otimes \cdots \otimes m_n)(A) \ge c^n \operatorname{vol}_n(A).$$

Proof. In one dimension the cumulative inequality implies $\int f\, dm_j \ge c \int f\, ds$ for every nonnegative decreasing function f of bounded support. Indeed, apply the layer-cake formula; each superlevel set of f is an initial interval. The claim for open initial intervals follows by taking limits from below, and an initial interval consisting only of zero causes no problem. Apply this comparison successively to the coordinates of 1_A, using Tonelli's theorem. At each stage the function of every remaining coordinate is decreasing. The argument allows atoms at zero. □

Fix $0 < c \le c_*$ and a point $x \in K$. Independent rotations allow us to assume $x = r = (r_1, \dots, r_n) \in L$. Both K and ν are invariant under these rotations. Define $F_r : \mathbb{C}^n \to [0,\infty)^n$ by

$$(F_r(d))_j = \max\left\{0,\ \frac{|d_j| - (1-c)r_j}{c},\ r_j + \frac{\Re d_j}{c}\right\}. \tag{14}$$

When $r_j = 0$, this is simply $|d_j|/c$. Putting $y = F_r(d)$, we have in every coordinate

$$|d_j| \le (1-c)r_j + cy_j, \qquad \Re d_j \le c(y_j - r_j). \tag{15}$$

For $r_j > 0$, let m_{r_j} be the image measure of ν_1 under the corresponding one-coordinate map. For $s \ge 0$, set

$$R = (1-c)r_j + cs, \qquad B = c(s - r_j).$$

Then

$$\{d_j : (F_r(d))_j \le s\} = \{|d_j| \le R,\ \Re d_j \le B\}.$$

The coordinate map is continuous. This preimage is bounded and has finite ν_1-measure, so m_{r_j} is a locally finite Borel measure. Since $R - B = r_j > 0$, Lemma 7 and the degree-one homogeneity of ν_1 imply

$$m_{r_j}([0,s]) = RH(B/R) \geq cR + (1-c)B = cs. \tag{16}$$

For $r_j = 0$, the pushforward is exactly c times Lebesgue measure on $[0,\infty)$. Because F_r acts separately in each coordinate, the image of $\nu_1^{\otimes n}$ under F_r is the product of these coordinate image measures. This equality holds first on rectangles and then on Borel sets by uniqueness of locally finite product measures. Hence Lemma 8, applied to $(1-\varepsilon)L$, gives

$$\nu\big(F_r^{-1}((1-\varepsilon)L)\big) \geq c^n(1-\varepsilon)^n\nu(K), \qquad 0 < \varepsilon < 1. \tag{17}$$

Moreover, this preimage lies in K: if $y \in (1-\varepsilon)L$, then $|d| \leq (1-c)r + cy$, whose K-norm is at most $1 - c\varepsilon$. The closed scaled set $(1-\varepsilon)L$, including its coordinate faces, is intentional; the measures in (16) may have an atom at zero.

3.2.3 A uniform estimate for interior points

Lemma 9. *Let $0 < c \leq c_*$, and let D have probability distribution $\nu|_K/\nu(K)$. For $0 < \varepsilon < 1$, put*

$$\tau = \frac{c\varepsilon}{4}, \qquad \gamma = \frac{c^2\varepsilon^2}{8}, \qquad p = [c(1-\varepsilon)]^n.$$

For every $x \in K$,

$$\mathbb{P}\{x + \tau D \in (1-\gamma)K\} \geq p. \tag{18}$$

The same distribution, step τ, contraction γ, and lower bound p work for every point x.

Proof. Rotate x to $r \in L$ and consider the preimage in (17). We show that every d in this preimage satisfies the desired finite contraction. For $0 < \tau \leq 1$, the two scalar inequalities in (15) imply

$$|r_j + \tau d_j| \leq (1 - \tau c + \tau^2)r_j + (\tau c + \tau^2)y_j. \tag{19}$$

Set $A = r_j + \tau c(y_j - r_j)$, so $A \geq (1-\tau c)r_j \geq r_j/2$. Expanding the square and using (15) gives

$$\begin{aligned} |r_j + \tau d_j|^2 &\leq A^2 + \tau^2\big[(1-2c)r_j^2 + 2cr_jy_j\big] \\ &\leq A^2 + \tau^2 r_j(r_j + y_j). \end{aligned}$$

On the other hand,

$$[A + \tau^2(r_j + y_j)]^2 - A^2 \geq 2A\tau^2(r_j + y_j) \geq \tau^2 r_j(r_j + y_j).$$

Taking square roots proves (19).

Monotonicity of the unconditional norm and its triangle inequality now yield

$$\begin{aligned} \|x + \tau d\|_K &\leq (1 - \tau c + \tau^2)\,\|r\|_K + (\tau c + \tau^2)\,\|y\|_K \\ &\leq 1 - \tau c\varepsilon + \tau^2(2-\varepsilon). \end{aligned}$$

For $\tau = c\varepsilon/4$, the last expression is

$$1 - \frac{c^2\varepsilon^2(2+\varepsilon)}{16} \leq 1 - \frac{c^2\varepsilon^2}{8} = 1 - \gamma.$$

The mass estimate (17), and containment of its preimage in K, prove (18). Rotation invariance returns the statement to the original x. □

For $x \in \partial K$, the event in (18) consists of illuminating directions. Consequently $\mathbb{P}\{D \text{ illuminates } x\} \geq [c(1-\varepsilon)]^n$ for every $\varepsilon > 0$. The event on the left is independent of ε and is Borel: it is the union, over positive rational q, of the open sets $\{d : x + qd \in \operatorname{int} K\}$. Taking the supremum of the lower bounds as $\varepsilon \downarrow 0$, the measure $c^{-n}\nu|_K/\nu(K)$ fractionally illuminates K. Its mass at zero is zero, so it defines a measure on the allowed nonzero directions. Taking $c = c_*$ proves (8).

3.2.4 A finite set of illuminating directions

The passage from a covering measure to finitely many directions uses the standard sampling and net argument; compare the weighted-covering rounding theorem and its proof in [1, Theorem 1.6 and Section 2.6]. We give the argument with the interior margin from Lemma 9, which determines the constants here. Let $\eta = \gamma/2$. Choose a maximal finite η-separated set $\mathcal{N} \subset K$ in the norm of K. Such a set can be formed by successively adding points at distance at least η; the packing estimate below bounds the number of additions. Maximality makes it an η-net, meaning that every point of K has distance at most η from it. Its cardinality is at most

$$N_0 = \left(1 + \frac{2}{\eta}\right)^{2n} = \left(1 + \frac{4}{\gamma}\right)^{2n}. \tag{20}$$

Indeed, the translates $z + (\eta/2) \operatorname{int} K$, for $z \in \mathcal{N}$, are pairwise disjoint and lie in $(1+\eta/2)K$. Comparison of ordinary Lebesgue volumes, in real dimension $2n$, gives (20).

Choose M independent vectors distributed as D in Lemma 9. For each net point, the probability that all of them fail is at most $(1-p)^M \leq e^{-pM}$. Thus

$$M = \left\lceil \frac{\log N_0 + 1}{p} \right\rceil$$

makes the probability that any net point is missed at most $e^{-1} < 1$. There is therefore a choice $d^1, \ldots, d^M$ such that every $z \in \mathcal{N}$ satisfies $\left\|z + \tau d^j\right\|_K \leq 1 - \gamma$ for some $j \in [M]$.

For arbitrary $x \in K$, take $z \in \mathcal{N}$ with $\|x - z\|_K \leq \eta$, and use the vector serving z. Then

$$\left\|x + \tau d^j\right\|_K \leq \|x - z\|_K + \left\|z + \tau d^j\right\|_K \leq \eta + 1 - \gamma = 1 - \gamma/2 < 1.$$

These vectors illuminate all of ∂K. The zero vector has probability zero under ν, so all sampled vectors are nonzero with probability one. Intersecting this event with the positive-probability covering event gives nonzero illuminating directions. We have proved the more flexible bound

$$\operatorname{ill}(K) \leq \left\lceil \frac{2n \log\bigl(1 + 32/(c^2\varepsilon^2)\bigr) + 1}{[c(1-\varepsilon)]^n} \right\rceil, \qquad 0 < \varepsilon < 1. \tag{21}$$

Take $\varepsilon = 1/(n+1)$ and use $(1-\varepsilon)^n = (1 + 1/n)^{-n} \geq e^{-1}$. This proves (9) with $c = c_*$, and (10) with $c = 2/5$.

Remark 2. Choosing $\varepsilon = 1/(n \log n)$, for $n \geq 2$, in (21) gives the sharper asymptotic prefactor

$$\operatorname{ill}(K) \leq (4 + o(1))\, n \log n\, b_*^n.$$

The finite-dimensional parameter ε can also be optimized directly. Neither refinement changes the exponential base.

The constant c_* is optimal for the affine inequality (13): values $c > 1/2$ fail at $s = 0$, and for $c_* < c \leq 1/2$ the negative minimizer used in its proof violates the inequality. This does not assert optimality of the resulting illumination bound.

4 A full coordinate hyperplane section

We now prove a sharp bound for a class defined by an exact coordinate-section condition. Throughout this section, $n \geq 2$ and $K \subset \mathbb{C}^n$ is a normalized complex 1-unconditional convex body. Write $\mathbb{D} = \{x \in \mathbb{C} : |x| \leq 1\}$, $[n] = \{1, \dots, n\}$, and $\mathbf{1} = (1, \dots, 1)$. By a *full coordinate hyperplane section* we mean that, after permuting coordinates,

$$K \cap H = \mathbb{D}^{n-1} \times \{0\}, \qquad H = \{x \in \mathbb{C}^n : x_n = 0\}.$$

Since normalization implies $K \subseteq \mathbb{D}^n$, this is equivalent to $\mathbb{D}^{n-1} \times \{0\} \subseteq K$.

The real counterpart of the following upper bound is [7, Theorem 2]. Here the section is a complex polydisc, and the sharp comparison value is $2^{n+1} - 2$.

Theorem 10. *Let $n \geq 2$ and let $K \subset \mathbb{C}^n$ be a normalized complex 1-unconditional convex body. If $K \neq \mathbb{D}^n$ and K contains a full coordinate $(n-1)$-polydisc, then*

$$\mathrm{ill}(K) \leq 2^{n+1} - 2. \tag{22}$$

This bound is attained, for every $n \geq 2$, by

$$H_n = \mathbb{D}^{n-2} \times \{(x_{n-1}, x_n) \in \mathbb{C}^2 : |x_{n-1}| + |x_n| \leq 1\}.$$

For $n = 2$ the first factor is omitted.

We use Lemma 1 and the cyclic polydisc construction (4). The remaining estimates are proved below.

4.1 Supporting normals

Lemma 11. *Suppose $\mathbb{D}^{n-1} \times \{0\} \subseteq K$, and let $u \neq 0$ be a supporting normal at $x \in \partial K$ for the real pairing $[u, x] = \mathrm{Re} \sum_j \overline{u_j} x_j$. Set $b_j = |u_j|$ and $r_j = |x_j|$. Then*

$$h_K(u) = \sum_{j \in [n]} b_j r_j, \qquad \sum_{j<n} b_j (1 - r_j) \leq b_n r_n. \tag{23}$$

If $x_j \neq 0$, then $u_j = b_j x_j / |x_j|$.

Proof. The orbit of x under independent coordinate rotations lies in K. The maximum of $[u, \cdot]$ on that orbit is $\sum_j b_j r_j$. Since x itself maximizes $[u, \cdot]$ on all of K, equality holds in each of the scalar inequalities $\mathrm{Re}(\overline{u_j} x_j) \leq b_j r_j$. This proves both the first equality and the alignment when $x_j \neq 0$. No alignment assertion is needed when $x_j = 0$. Maximizing the same functional on the contained polydisc gives $h_K(u) \geq \sum_{j<n} b_j$. Subtracting this from the first equality gives the second inequality in (23). □

4.2 Projections onto proper coordinate subsets

Lemma 12. *Put $M = 2^n - 1$ and*

$$p_j^k = \exp\left(\frac{2\pi \mathrm{i}\, 2^{j-1} k}{M}\right), \qquad 0 \leq k < M, \quad j \in [n]. \tag{24}$$

There exists $c_n \in (0, 1)$, depending only on n, such that for every nonempty proper subset $S \subset [n]$ and every family of unit complex numbers $(\zeta_j)_{j \in S}$ there is an index k satisfying

$$\mathrm{Re}(\overline{\zeta_j} p_j^k) > c_n \qquad (j \in S). \tag{25}$$

Proof. First consider a projection onto $n-1$ coordinates. The n column frequencies in (24) are 2^a, $a = 0, \dots, n-1$, taken modulo M. Since $2^n \equiv 1 \pmod{M}$, multiplying all frequencies by a power of 2 cyclically permutes these n exponents. This multiplication also just permutes the rows, because M is odd. Choose the cyclic shift that sends the omitted exponent to $n-1$. The retained frequencies are then $1, 2, \dots, 2^{n-2}$, up to coordinate order. They are exactly the cyclic construction (4) in dimension $n-1$, whose denominator is $2^n - 1$.

Hence every such projection meets every product of $n-1$ open semicircles. Every smaller nonempty projection has the same property: extend its coordinate set to one of size $n-1$ and prescribe arbitrary open semicircles on the added coordinates. In particular, for any $(\zeta_j)_{j\in S}$ one row has strictly positive real pairing with all the ζ_j on S.

To obtain a margin independent of these unit complex numbers, define on the compact torus $\{(\zeta_j)_{j\in S} : |\zeta_j| = 1\}$ the continuous function

$$f_S(\zeta) = \max_{0\le k<M} \min_{j\in S} \operatorname{Re}(\overline{\zeta_j} p_j^k).$$

It is continuous because both extrema are over finite sets, and the preceding paragraph shows $f_S(\zeta) > 0$ at every point. Its attained minimum α_S is therefore positive. There are only finitely many nonempty proper subsets S. Thus

$$c_n = \frac{1}{2} \min_{\varnothing \ne S \subsetneq [n]} \alpha_S$$

is positive and at most $1/2$. A row attaining the maximum in f_S has every pairing at least $\alpha_S \ge 2c_n > c_n$, proving the claim. □

4.3 The illuminating directions and the boundary cases

Proof of the upper bound in Theorem 10. Permute coordinates so that the full section is $\{x_n = 0\}$, and set

$$\gamma = \|\mathbf{1}\|_K^{-1}, \qquad \delta = \frac{c_n(1-\gamma)}{2}. \tag{26}$$

We have $0 < \gamma < 1$. Indeed, if $\mathbf{1} \in K$, coordinatewise monotonicity would imply $\mathbb{D}^n \subseteq K$; normalization would then imply $K = \mathbb{D}^n$, contrary to the hypothesis. Thus $\|\mathbf{1}\|_K > 1$, and in particular $\delta > 0$.

Every $x \in K$ has at least one coordinate with $|x_j| \le \gamma$. To see this, suppose $r_j = |x_j| > \gamma$ for all j. Choose λ with $\max_j \gamma/r_j < \lambda < 1$. Rotation invariance and Lemma 1 give

$$1 = \|\gamma \mathbf{1}\|_K \le \|\lambda(r_1, \dots, r_n)\|_K = \lambda \|x\|_K \le \lambda < 1,$$

a contradiction.

Using the vectors from Lemma 12, define

$$\begin{aligned} d^{k,0} &= -(p_1^k, \dots, p_{n-1}^k, 0), \\ d^{k,1} &= -(\delta p_1^k, \dots, \delta p_{n-1}^k, p_n^k), \qquad 0 \le k < M. \end{aligned} \tag{27}$$

These are $2M$ nonzero vectors, fixed before choosing the boundary point. We show that their union illuminates every $x \in \partial K$.

Case 1: $x_n = 0$. By Lemma 1, x lies on the relative boundary of $K \cap \{x_n = 0\} = \mathbb{D}^{n-1} \times \{0\}$. Consequently $\|x\|_\infty = 1$, and $M_x = \{j \in [n-1] : |x_j| = 1\}$ is nonempty. Choose k with the margin (25) on M_x, taking $\zeta_j = x_j$. For $j \in M_x$ and $0 < t < 2c_n$,

$$|x_j - t p_j^k|^2 = 1 - 2t \operatorname{Re}(\overline{x_j} p_j^k) + t^2 < 1 - 2tc_n + t^2 < 1.$$

For every remaining $j < n$, we initially have $|x_j| < 1$, so $|x_j - tp_j^k| < 1$ for all sufficiently small positive t, by continuity. This includes coordinates that are zero. There are finitely many coordinates, so one $t > 0$ satisfies all these inequalities. The last coordinate remains zero. Hence $x + td^{k,0}$ is in the relative interior of the full section, and therefore in $\operatorname{int} K$ by Lemma 1.

Case 2: $x_n \neq 0$ and $r_j > \gamma$ for all $j < n$. The preceding observation forces $r_n \leq \gamma < 1$. Normalization gives $\|x_n e_n\|_K = r_n < 1$, so $q = x_n e_n$ is interior. All first $n-1$ coordinates of x are nonzero. Choose k with (25) on $[n-1]$, using $\zeta_j = x_j/r_j$. For each $j < n$ we have

$$|x_j - tp_j^k|^2 < r_j^2 - 2tc_n r_j + t^2 < r_j^2 \quad \text{whenever } 0 < t < 2c_n r_j.$$

Choose t smaller than all these finitely many positive upper bounds. The point $x + td^{k,0}$ strictly contracts every coordinate in $[n-1]$ and leaves the last one unchanged. Apply part 3 of Lemma 1 with $J = [n-1]$ and $q = x_n e_n$ to conclude that $x + td^{k,0} \in \operatorname{int} K$.

Case 3: $x_n \neq 0$ and $r_i \leq \gamma$ for some $i < n$. Let

$$E = Z_x \cup \{i\}, \qquad S = [n] \setminus E.$$

Then $\varnothing \neq E \subseteq [n-1]$, $n \in S$, and every coordinate of x on S is nonzero. In particular S is a nonempty proper subset. Also $r_j \leq \gamma$ for every $j \in E$. Choose a single k with (25) on S, using $\zeta_j = x_j/r_j$. This choice uses x and is independent of its supporting normal.

Now let $u \neq 0$ be any supporting normal at x, and put $b_j = |u_j|$. Since $0 \leq r_j \leq 1$, all terms on the left below are nonnegative. Lemma 11 gives

$$(1-\gamma)\sum_{j\in E} b_j \leq \sum_{j\in E} b_j(1-r_j) \leq \sum_{j<n} b_j(1-r_j) \leq b_n r_n \leq b_n. \tag{28}$$

On S the normal aligns with x and the chosen entry has positive pairing at least c_n. On E no alignment or sign estimate is required: the absolute value of each contribution is at most δb_j, including when $x_j = 0$. It follows that

$$\begin{aligned}
[u, d^{k,1}] &\leq -c_n b_n - c_n\delta \sum_{j\in S\setminus\{n\}} b_j + \delta \sum_{j\in E} b_j \\
&\leq -\left(c_n - \frac{\delta}{1-\gamma}\right) b_n - c_n\delta \sum_{j\in S\setminus\{n\}} b_j \\
&= -\frac{c_n}{2} b_n - c_n\delta \sum_{j\in S\setminus\{n\}} b_j < 0.
\end{aligned}$$

For the final strict inequality, if the last expression were zero, positivity of c_n and δ would force $b_n = 0$ and $b_j = 0$ for every $j \in S \setminus \{n\}$. Equation (28) would then force all b_j on E to vanish as well, contradicting $u \neq 0$. Thus the same $d^{k,1}$ pairs negatively with every nonzero supporting normal at x. The illumination criterion (1) proves that $d^{k,1}$ illuminates x.

These three cases exhaust ∂K. The parameter c_n was chosen solely from n, and δ was then fixed from K; neither depends on the boundary point or its supporting normal. We have constructed at most $2M = 2^{n+1} - 2$ illuminating directions. □

4.4 Sharpness and immediate consequences

Proof of sharpness in Theorem 10. The body H_n in the theorem is normalized, complex 1-unconditional, and contains $\mathbb{D}^{n-1} \times \{0\}$. It is not $\mathbb{D}^n$, since $e_{n-1} + e_n \notin H_n$. The upper bound already proved therefore applies to H_n.

For the lower bound consider its two boundary tori

$$T_- = \{x : |x_j| = 1 \ (j < n),\ x_n = 0\},$$
$$T_+ = \{x : |x_j| = 1 \ (j \le n-2 \text{ or } j = n),\ x_{n-1} = 0\}.$$

Let $\mathcal{V}$ be any finite illuminating family for H_n, and let $\mathcal{V}_-$ and $\mathcal{V}_+$ consist of the directions that illuminate at least one point of T_- and T_+, respectively. If $d \in \mathcal{V}_-$ illuminates $x \in T_-$ at some time $t > 0$, then interior membership gives

$$|x_{n-1} + td_{n-1}| + t|d_n| < 1.$$

The reverse triangle inequality gives $|x_{n-1}+td_{n-1}| \ge 1-t|d_{n-1}|$, so $|d_{n-1}| > |d_n|$. Interchanging the last two coordinates shows that $d \in \mathcal{V}_+$ requires $|d_n| > |d_{n-1}|$. Consequently $\mathcal{V}_- \cap \mathcal{V}_+ = \varnothing$.

Let π_- be projection onto coordinates $1, \dots, n-1$. If d illuminates $x \in T_-$ in H_n, then $\pi_-(x + td) \in \operatorname{int} \mathbb{D}^{n-1}$ for the same t: the first $n-2$ moduli are strictly below one, and the displayed inequality also makes the $(n-1)$st modulus strictly below one. Thus $\pi_-(\mathcal{V}_-)$ illuminates every point of $(\partial\mathbb{D})^{n-1}$.

For completeness, illuminating this torus suffices to illuminate the entire polydisc, in any dimension m. Every point $x \in \mathbb{D}^m$ is a finite convex combination $x = \sum_{j\in[N]} \lambda_j y^j$ of torus points: express each scalar coordinate as a convex combination of two unit complex numbers and multiply the coordinatewise weights. Discard terms of zero weight. If a direction d illuminates one y^j, say $y^j + td \in \operatorname{int} \mathbb{D}^m$, then

$$x + \lambda_j td = \lambda_j(y^j + td) + \sum_{a\ne j} \lambda_a y^a \in \operatorname{int} \mathbb{D}^m.$$

The last inclusion follows coordinatewise from a positive weight on a vector all of whose coordinate moduli are strictly below one. The exact polydisc illumination theorem [5] now implies $|\mathcal{V}_-| \ge 2^n - 1$. The projection omitting coordinate $n-1$ gives $|\mathcal{V}_+| \ge 2^n - 1$ in exactly the same way. Since these families are disjoint,

$$|\mathcal{V}| \ge |\mathcal{V}_-| + |\mathcal{V}_+| \ge 2(2^n - 1) = 2^{n+1} - 2.$$

Together with the upper bound, this proves equality. □

Corollary 13. *Every normalized complex* 1*-unconditional body* $K \subset \mathbb{C}^2$ *with* $K \ne \mathbb{D}^2$ *satisfies* $\operatorname{ill}(K) \le 6$*, and this is sharp. In* $\mathbb{C}^3$ *a full coordinate bidisc gives the sharp bound* 14*; in* $\mathbb{C}^4$ *a full coordinate tridisc gives the sharp bound* 30*, in both cases assuming* K *is not the full polydisc.*

Proof. In dimension two every normalized body contains a unit coordinate disc, so the section hypothesis is automatic. Apply Theorem 10 with $n = 2, 3, 4$. Its examples H_2, H_3, H_4 establish the respective sharpness claims. □

Remark 3. The directions (27) depend on K through γ and δ. The directions are chosen separately for each body. In dimensions three and four the corollary only covers the stated section classes. Closeness to $\mathbb{D}^n$ does not by itself imply the required exact section containment.

5 Three complex dimensions

The low-dimensional real counterpart is [7, Theorem 1]. Within the complex 1-symmetric subclass, the following upper bound also follows from [6, Theorem 1.5]; the assertion here requires no coordinate-permutation invariance.

Theorem 14. *Every normalized complex* 1*-unconditional convex body* $K \subset \mathbb{C}^3$ *with* $K \neq \mathbb{D}^3$ *satisfies* $\mathrm{ill}(K) \leq 14$*. The bound is sharp.*

Theorem 10 already handles bodies containing a full coordinate bidisc. We construct fourteen directions whenever at least one coordinate bidisc is missing. The construction uses two copies of a seven-vector family, each with three directions tending to each coordinate axis. The partition of the seven rows comes from an additional coordinate, as in Lemma 5. It is also the two-dimensional case of the coloring in [6, Proposition 4.4], with the omitted row restored.

5.1 Seven directions in a coordinate plane

Set $\eta = e^{2\pi \mathrm{i}/7}$. By Lemma 4, every two-coordinate projection of

$$\{(\eta^t, \eta^{2t}, \eta^{4t}) : t \in \{0, \dots, 6\}\}$$

meets every product of two open semicircles. Replace the last entry by $h_t = \eta^{3-4t}$. Conjugation and rotation preserve open semicircles, so the same projection property holds for the columns η^t, η^{2t}, h_t. Partition the rows by the sign of the imaginary part of the additional coordinate:

$$U = \{t : \operatorname{Im} h_t > 0\} = \{0, 2, 4\}, \qquad V = \{t : \operatorname{Im} h_t < 0\} = \{1, 3, 5\}, \qquad h_6 = 1.$$

The entries η^t, $t \in U$, meet every open semicircle: prescribe that semicircle in the first column and the upper open semicircle in the additional column. Similarly, the entries η^{2t}, $t \in V$, meet every open semicircle, by prescribing the lower open semicircle in the additional column. All seven rows still satisfy arbitrary open-semicircle conditions on the two visible coordinates simultaneously.

For $\varepsilon > 0$, define

$$p^t(\varepsilon) = -(a_t \eta^t, b_t \eta^{2t}), \qquad (a_t, b_t) = \begin{cases} (1, \varepsilon), & t \in U, \\ (\varepsilon, 1), & t \in V, \\ (1, 1), & t = 6. \end{cases} \tag{29}$$

For any $x_1, x_2 \neq 0$, the two semicircle conditions give a row with

$$\operatorname{Re}(\overline{x_j} p_j^t(\varepsilon)) < 0 \qquad (j \in [2]).$$

Positive amplitudes do not change these signs. Since

$$|x_j + s d_j|^2 = |x_j|^2 + 2s \operatorname{Re}(\overline{x_j} d_j) + s^2 |d_j|^2,$$

both moduli strictly decrease for a sufficiently small positive step s. As ε tends to zero, the U-rows tend to three directions on the first axis whose unit entries meet every open semicircle, and the V-rows do the same on the second axis. Thus the additional coordinate determines both the partition and the two axis limits.

5.2 Proof of the bound

Proof of Theorem 14. If K contains a full coordinate bidisc, apply Theorem 10. Otherwise permute the coordinates so that $e_2 + e_3 \notin K$. In fact, the following argument only requires this one missing bidisc. Embed (29) in the two planes by setting

$$A_\varepsilon = \{(p_1^t(\varepsilon), p_2^t(\varepsilon), 0) : t \in \{0, \dots, 6\}\},$$
$$B_\varepsilon = \{(p_1^t(\varepsilon), 0, p_2^t(\varepsilon)) : t \in \{0, \dots, 6\}\}.$$

We choose one $\varepsilon > 0$ for which these fourteen directions illuminate K.

We first record why any three unit entries meeting every open semicircle on each coordinate axis illuminate a normalized complex 1-unconditional body $L \subset \mathbb{C}^2$ different from $\mathbb{D}^2$. At a boundary point x with both coordinates nonzero, their moduli cannot both equal one, since coordinate contractions and rotations would then give $L = \mathbb{D}^2$. Choose a coordinate to decrease so that the unchanged coordinate has modulus less than one. Setting the chosen coordinate to zero leaves an interior point of L, since every axis point of modulus less than one is interior. One of the three directions on the chosen axis strictly decreases that modulus for a small positive step, and Lemma 1(3) puts the new point in $\operatorname{int} L$. If only one coordinate is nonzero, decrease it in the same way; deleting it leaves the interior point zero. This proves the assertion for all boundary points of L.

Now take $L = K \cap \{x_1 = 0\}$, identified with a body in $\mathbb{C}^2$. It is normalized and is not a bidisc. The V-rows from the two families tend to three pure directions on axis 2 and three on axis 3. By the preceding argument, these limits illuminate ∂L in its plane. Lemma 1(2) identifies relative interior in the section with ambient interior intersected with that plane. Thus they also illuminate $\partial K \cap \{x_1 = 0\}$ in K. The U-rows of either family tend to three pure directions on axis 1 that illuminate its unit circle. The limiting directions therefore illuminate the compact set

$$E = (\partial K \cap \{x_1 = 0\}) \cup \{ze_1 : |z| = 1\}.$$

There exists $\varepsilon_0 > 0$ such that $A_\varepsilon \cup B_\varepsilon$ illuminates all of E whenever $0 < \varepsilon < \varepsilon_0$. Indeed, for each $x \in E$, choose one limiting direction $d(0)$ and a step $s > 0$ with $x + sd(0) \in \operatorname{int} K$. Openness supplies neighborhoods of x and $d(0)$ such that the same step enters the interior for every pair in those neighborhoods. Finitely many of the point neighborhoods cover E. The corresponding finitely many directions $d(\varepsilon)$ converge to their limits, so the minimum of their positive admissible upper bounds on ε gives one ε_0 valid on all of E. Fix $\varepsilon \in (0, \varepsilon_0)$.

Let $x \in \partial K \setminus E$. Then $x_1 \neq 0$ and at least one of x_2, x_3 is nonzero. If $x_3 = 0$, a direction in A_ε strictly decreases the first two moduli, and Lemma 1(3), with the interior point zero, shows that it illuminates x. If $x_2 = 0$, use B_ε instead.

It remains to consider three nonzero coordinates. The moduli $|x_2|$ and $|x_3|$ cannot both equal one: deleting the first coordinate and rotating the others would put $e_2 + e_3$ in K. If $|x_3| < 1$, choose a direction in A_ε decreasing the first two moduli. It leaves the third coordinate unchanged, and $x_3 e_3 \in \operatorname{int} K$, so Lemma 1(3) applies. Otherwise $|x_2| < 1$, and a direction in B_ε works by the same argument with the interior point $x_2 e_2$. These cases exhaust the boundary and prove $\operatorname{ill}(K) \leq 14$.

Finally, the body $\mathbb{D} \times \{(z, w) : |z| + |w| \leq 1\}$ has illumination number 14 by the sharpness part of Theorem 10. Hence the bound is sharp. □

6 Four complex dimensions

As in dimension three, related bounds hold for real unconditional bodies [7, Theorem 1] and for the complex 1-symmetric subclass [6, Theorem 1.5]. We treat all complex 1-unconditional bodies, including those without permutation symmetry.

Theorem 15. *Every normalized complex 1-unconditional convex body $K \subset \mathbb{C}^4$ different from $\mathbb{D}^4$ satisfies*

$$\operatorname{ill}(K) \leq 30.$$

The bound is sharp. Consequently every complex 1-unconditional convex body in $\mathbb{C}^4$ has illumination number at most 31, and equality holds precisely when its positive diagonal normalization is $\mathbb{D}^4$.

The proof separates bodies according to which coordinate bidiscs they contain. A full coordinate bidisc on $\{i, j\}$ is present exactly when $e_i + e_j \in K$. Three constructions below

cover every body with a missing bidisc. The remaining case uses a finite covering assertion, verified by the exact integer program in Appendix B. Every geometric argument applies to all boundary points and all their supporting normals.

6.1 Two disjoint missing bidiscs

Proposition 16. *Let $K \subset \mathbb{C}^4$ be normalized and complex 1-unconditional. If $e_1 + e_2 \notin K$ and $e_3 + e_4 \notin K$, then* $\mathrm{ill}(K) \le 30$.

Proof. Put

$$\gamma = \max_{x \in K} \min\{|x_3|, |x_4|\} < 1.$$

For $j \in \{3, 4\}$, use Lemma 5 with root $\{1, 2, j\}$, children $\{1, j\}$ and $\{2, j\}$, singleton children $\{1\}, \{j\}$ below $\{1, j\}$, and singleton child $\{2\}$ below $\{2, j\}$. Write $D_j(\delta)$ for its fifteen directions, embedded with the remaining coordinate zero. Keep the angular entries and the row partitions fixed and let $\delta \in (0, 1)$ be the amplitude parameter of that construction. There is $c > 0$, independent of δ, such that for every node S and every choice of coordinate arguments on S, a row of its group is inward with angular margin c at every coordinate of S; if its smallest amplitude on S is a, every amplitude outside S but inside its root is at most δa.

In each family the rows belonging to the singleton nodes $\{1\}, \{2\}$ converge, as $\delta \to 0$, to directions on their respective coordinate axes, whose unit entries meet every open semicircle. These limiting directions illuminate

$$E = \partial K \cap \{x_3 = x_4 = 0\}.$$

Indeed, the coordinate section on $\{1, 2\}$ is not a bidisc. At a boundary point with both coordinates nonzero, at least one modulus is less than one. Decrease the other coordinate, using an inward axis direction. Deleting that coordinate leaves an axis point of modulus less than one, which is interior to K; strict contraction toward it therefore enters the interior. If only one coordinate is nonzero, decrease it instead, with the interior point zero.

The set E is compact. At each of its points choose one limiting direction and one positive step entering $\operatorname{int} K$. Openness gives neighborhoods of both the point and that direction for which the same step enters the interior. Take a finite cover of E by these point neighborhoods. Convergence of the finitely many corresponding rows gives $\delta_0 > 0$ such that $D_3(\delta) \cup D_4(\delta)$ illuminates E whenever $0 < \delta < \delta_0$. Fix

$$0 < \delta < \min\left\{\delta_0, 1, \frac{c(1-\gamma)}{4}\right\}.$$

Let $x \in \partial K \setminus E$. Choose $j \in \{3, 4\}$ and the other index k so that $x_j \neq 0$ and $|x_k| \le \gamma$. Such a choice exists: if one of x_3, x_4 vanishes, omit that coordinate; otherwise omit a coordinate with the smaller modulus. Put $R = \{1, 2, j\}$ and $S = \operatorname{supp}(x) \cap R$. Every subset of R containing j is a node of the chosen tree, so the row property applies to S.

For a nonzero supporting normal u at x, put $b_i = |u_i|$, $r_i = |x_i|$ and $h = [u, x]$. Normal alignment and normalization give $h = \sum_i b_i r_i > 0$ and $b_i \le h$. Consequently

$$(1 - r_k)h \le \sum_{i \in S} b_i r_i \le \sum_{i \in S} b_i, \qquad b_\ell \le \frac{1}{1-\gamma} \sum_{i \in S} b_i \quad (\ell \in R \setminus S).$$

The sum over S is positive. Choose a row $d \in D_j(\delta)$ having angular margin c on S. This choice depends only on x, not on u. Its omitted coordinate is zero, and there are at most two inactive coordinates inside R, whence

$$[u, d] \le a\left(-c + \frac{2\delta}{1-\gamma}\right) \sum_{i \in S} b_i < 0.$$

The normal criterion proves that d illuminates x. Along with the compact-set argument for E, this proves the bound. □

6.2 Two present bidiscs and a missing bidisc

Proposition 17. *Let $K \subset \mathbb{C}^4$ be a normalized complex 1-unconditional convex body. Suppose that*

$$e_1 + e_3 \in K, \qquad e_1 + e_4 \in K, \qquad e_3 + e_4 \notin K.$$

Then $\mathrm{ill}(K) \le 30$.

Proof. Put

$$\gamma = \max_{x \in K} \min\{|x_3|, |x_4|\}, \qquad c = 1 - \gamma.$$

The maximum exists by compactness. Normalization gives $\gamma \le 1$. Equality would give a point of K with $|x_3| = |x_4| = 1$; coordinate contractions and rotations would then imply $e_3 + e_4 \in K$. Thus $c > 0$. Every $x \in K$ satisfies

$$|x_3| \le \gamma \quad \text{or} \quad |x_4| \le \gamma.$$

Set $\omega = e^{2\pi \mathrm{i}/15}$ and

$$v_j^t = \omega^{2^{j-1}t} \qquad (0 \le t \le 14,\ j \in [4]).$$

By Lemma 4, any three columns meet every product of three open semicircles. Define the following disjoint sets of rows:

$$G = \{t : \operatorname{Im} v_4^t > 0\}, \qquad H = \{t : \operatorname{Im} v_4^t < 0\},$$
$$H_1 = \{t \in H : \operatorname{Im} v_2^t > 0\}, \qquad H_3 = \{t \in H : \operatorname{Im} v_2^t < 0\}.$$

Each of G, H has seven elements. The only row with zero imaginary part in any column is row zero, so $H = H_1 \sqcup H_3$ and $\{0, \dots, 14\} = \{0\} \sqcup G \sqcup H_1 \sqcup H_3$.

The rows G meet every product of two open semicircles in columns $\{1, 2\}$, and also in columns $\{2, 3\}$. To see this, append to the two prescribed conditions the upper-open-semicircle condition on column 4 and use the three-column projection property. Similarly, H meets every product of two open semicircles in columns $\{1, 3\}$. The entries in column 1 of H_1 meet every open semicircle: prescribe that semicircle in column 1, the upper semicircle in column 2, and the lower semicircle in column 4. Prescribing the desired semicircle in column 3 and the lower semicircles in columns 2 and 4 proves the analogous assertion for column 3 of H_3. The entries in column 2 of G also meet every open semicircle. Finally, all fifteen rows meet every product of three open semicircles in columns $\{1, 2, 3\}$.

These finitely many covering statements have a common positive margin. More precisely, choose $\beta > 0$ such that, for each pair

$$(R, S) \in \big\{(\{0, \dots, 14\}, \{1, 2, 3\}), (G, \{1, 2\}), (G, \{2, 3\}),$$
$$(H, \{1, 3\}), (H_1, \{1\}), (G, \{2\}), (H_3, \{3\})\big\}.$$

and every family $(\zeta_j)_{j \in S}$ of unit complex numbers, some $t \in R$ satisfies

$$\operatorname{Re}(\overline{\zeta_j} v_j^t) \ge \beta \qquad (j \in S).$$

Indeed, for each fixed (R, S) the continuous function $\max_{t \in R} \min_{j \in S} \operatorname{Re}(\overline{\zeta_j} v_j^t)$ is positive on the compact torus. The minimum of its positive minima over the displayed finite list gives such a β.

Fix

$$0 < \delta < \min\{1, \beta c/4\}.$$

For each row define the three amplitudes by

$$(a_1^t, a_2^t, a_3^t) = \begin{cases} (\delta^2, 1, \delta), & t \in G, \\ (1, \delta^2, \delta), & t \in H_1, \\ (\delta, \delta^2, 1), & t \in H_3, \\ (1, 1, 1), & t = 0, \end{cases}$$

and set

$$d^t = -(a_1^t v_1^t, a_2^t v_2^t, a_3^t v_3^t, 0), \qquad A = \{d^t : 0 \le t \le 14\}.$$

We prove that A illuminates every $x \in \partial K$ with $|x_4| \le \gamma$.

Fix such an x, put $r_j = |x_j|$ and $S = \{j \in \{1,2,3\} : r_j > 0\}$, and let $u \neq 0$ be any supporting normal at x. Write $b_j = |u_j|$ and $h = h_K(u) > 0$. The normalized-normal estimates give

$$b_j \le h, \qquad h = \sum_{j \in [4]} b_j r_j \le \sum_{j \in S} b_j + \gamma h, \qquad h \le \frac{1}{c} \sum_{j \in S} b_j.$$

In particular $S \neq \varnothing$ and $\sum_{j \in S} b_j > 0$ for every supporting normal. At every nonzero coordinate $u_j = b_j x_j / r_j$. We select the row below using only x, and show its strict inward inequality for every supporting normal u.

If $S = \{1,2,3\}$, take a row having the common margin on all three coordinates. All three amplitudes are positive, so

$$[u, d^t] \le -\beta \sum_{j \in [3]} a_j^t b_j < 0.$$

If $S = \{2,3\}$, select $t \in G$ with the margin on both active coordinates. If $S = \{1,3\}$, select $t \in H$ with the margin on those coordinates. In either case the sole inactive coordinate among the first three has amplitude δ^2, and both active amplitudes are at least δ. Consequently

$$[u, d^t] \le -\beta\delta \sum_{j \in S} b_j + \delta^2 h \le \left(-\beta\delta + \frac{\delta^2}{c} \right) \sum_{j \in S} b_j < 0.$$

Suppose $S = \{1,2\}$. The two full bidiscs in the hypothesis give $b_1 + b_4 \le h$ and $b_1 + b_3 \le h$. Since $r_3 = 0$,

$$(1 - r_4) b_4 \le b_2 r_2 - b_1 (1 - r_1) \le b_2, \qquad b_4 \le \frac{b_2}{c},$$

and hence

$$b_3 \le b_2 r_2 + b_4 r_4 - b_1 (1 - r_1) \le b_2 + \gamma b_4 \le \frac{b_2}{c}.$$

Choose $t \in G$ with the margin on coordinates 1 and 2. Then

$$[u, d^t] \le -\beta\delta^2 b_1 - \beta b_2 + \delta b_3 \le -\beta\delta^2 b_1 + \left(-\beta + \frac{\delta}{c} \right) b_2 < 0,$$

because $b_1 + b_2 > 0$.

Finally, suppose $S = \{i\}$ is a singleton. Choose a row from H_1, G, or H_3 according as $i = 1$, $i = 2$, or $i = 3$, with the margin on coordinate i. Its amplitude there is one, and its other two amplitudes are at most δ. Thus

$$[u, d^t] \le -\beta b_i + 2\delta h \le \left(-\beta + \frac{2\delta}{c} \right) b_i < 0.$$

All nonempty subsets S have now been covered, with a single row chosen before the supporting normal. The normal criterion proves the asserted illumination by A.

Let P interchange coordinates 3 and 4, and put $B = PA$. The body PK satisfies the same hypotheses and has the same value of γ. Applying the preceding argument to PK and then applying P back shows that B illuminates every boundary point of K with $|x_3| \le \gamma$. Every boundary point satisfies one of the two coordinate inequalities. Therefore $A \cup B$ illuminates K, and $|A \cup B| \le 30$. □

6.3 A coordinate triple containing all its bidiscs

Proposition 18. *Let $K \subset \mathbb{C}^4$ be normalized and complex 1-unconditional. If*

$$e_3 + e_4 \notin K, \qquad e_1 + e_2, e_1 + e_4, e_2 + e_4 \in K,$$

then $\mathrm{ill}(K) \le 30$.

Proof. Set

$$\gamma = \max_{x \in K} \min\{|x_3|, |x_4|\} < 1.$$

The strict inequality follows from compactness, coordinate contractions and $e_3 + e_4 \notin K$. For a nonzero supporting normal u at x, write $r_j = |x_j|$, $b_j = |u_j|$ and $h = [u, x]$. The normal alignment and normalization give

$$h = \sum_{j \in [4]} b_j r_j > 0, \qquad b_j \le h \quad (j \in [4]).$$

Whenever $e_i + e_j \in K$, we also have $b_i + b_j \le h$.

Put $\zeta = e^{2\pi \mathrm{i}/15}$ and $v_j^t = \zeta^{2^{j-1}t}$ for $0 \le t \le 14$ and $j \in [4]$. Every three-coordinate projection of these fifteen rows meets every product of three open semicircles. Define

$$G = \{t : \operatorname{Im} v_4^t > 0\}, \qquad H = \{t : \operatorname{Im} v_4^t < 0\}.$$

Both sets have seven elements and their complement is $\{0\}$. Prescribing the upper semicircle in column 4 shows that the G-rows meet every product of two open semicircles in columns $\{1, 3\}$ and in columns $\{2, 3\}$. Prescribing the lower semicircle gives the same property for the H-rows in columns $\{1, 2\}$. These statements also imply their restrictions to single columns.

For a positive parameter δ, define

$$d^{t,A} = \begin{cases} -(\delta v_1^t, \delta v_2^t, v_3^t, 0), & t \in G, \\ -(v_1^t, v_2^t, \delta v_3^t, 0), & t \in H, \\ -(1, 1, 1, 0), & t = 0. \end{cases}$$

Positive amplitudes preserve the three-coordinate covering property of all fifteen rows.

Use Lemma 5 for the tree with root $\{1, 2, 4\}$, children $\{1, 4\}$ and $\{2, 4\}$, singleton children $\{1\}, \{4\}$ below $\{1, 4\}$, and singleton child $\{2\}$ below $\{2, 4\}$. Denote its fifteen directions by $d^{t,B}$, embedded in coordinates $1, 2, 4$, with coordinate 3 zero. The construction supplies the following property for every nonempty $S \subseteq \{1, 2, 4\}$ containing 4: for any coordinate arguments on S, one row has a positive angular margin on S, and its smallest active amplitude a is at least $1/\delta$ times every inactive amplitude within $\{1, 2, 4\}$. Take a common positive margin $c \in (0, 1)$ for these finitely many covering properties and the G, H properties above, and fix

$$0 < \delta < \frac{c(1 - \gamma)}{4}.$$

Here the rows are chosen from their coordinate arguments, independently of the supporting normal.

First suppose $r_3 \le \gamma$ and $x_4 \neq 0$, and put $S = \operatorname{supp}(x) \cap \{1,2,4\}$. Since $b_3 \le h$,

$$(1-r_3)h \le \sum_{j\in S} b_j r_j \le \sum_{j\in S} b_j.$$

The latter sum is positive. Every inactive normal coefficient is therefore at most $(1-\gamma)^{-1}\sum_{j\in S} b_j$. Choose the row $d^{t,B}$ just described. There are at most two inactive coordinates within $\{1,2,4\}$, so

$$[u, d^{t,B}] \le a\left(-c + \frac{2\delta}{1-\gamma}\right)\sum_{j\in S} b_j < 0.$$

This row illuminates x.

Next suppose $x_3 \neq 0$ and $r_4 \le \gamma$. If $x_1, x_2 \neq 0$, choose a direction $d^{t,A}$ inward in all three of its nonzero coordinates. Their normal coefficients have positive sum, since $b_4 \le h$ and $r_4 < 1$. Hence $[u, d^{t,A}] < 0$.

If $x_1 = 0$ and $x_2 \neq 0$, the full bidiscs on $\{2,4\}$ and $\{1,2\}$ yield

$$(1-r_4)b_4 \le b_3 r_3 - b_2(1-r_2) \le b_3,$$
$$b_1 \le b_3 r_3 + b_4 r_4 - b_2(1-r_2) \le \frac{b_3}{1-r_4}.$$

Choose a G-row inward on $\{2,3\}$. Then

$$[u, d^{t,A}] \le \left(-c + \frac{\delta}{1-\gamma}\right) b_3 - c\delta b_2 < 0.$$

For strictness, if $b_3 = 0$ the displayed bounds imply $b_1 = b_4 = 0$, and consequently $b_2 > 0$. If $x_2 = 0$ and $x_1 \neq 0$, interchange 1 and 2, using the full bidiscs on $\{1,4\}$ and $\{1,2\}$.

If $x_1 = x_2 = 0$, the full bidiscs on $\{1,4\}$ and $\{2,4\}$ give

$$b_1, b_2 \le b_3 r_3 - (1-r_4)b_4 \le b_3.$$

Here $b_3 > 0$, since $r_4 < 1$. A G-row inward on coordinate 3 satisfies $[u, d^{t,A}] \le (-c+2\delta)b_3 < 0$.

Finally suppose $x_3 = x_4 = 0$. Choose an H-row inward on all nonzero coordinates among $1, 2$. A zero coordinate among these two has zero normal coefficient. Indeed, if $x_2 = 0$, then $h = b_1 r_1 \ge b_1 + b_2$ and $r_1 \le 1$ force $b_2 = 0$; the other case is identical. Moreover $b_3 \le h \le b_1 + b_2$, so

$$[u, d^{t,A}] \le (-c+\delta)(b_1 + b_2) < 0.$$

These cases exhaust ∂K. If $x_4 \neq 0$ and $r_3 \le \gamma$, the first case applies. If $x_4 \neq 0$ and $r_3 > \gamma$, the definition of γ gives $r_4 \le \gamma$ and $x_3 \neq 0$, so the second case applies. When $x_4 = 0$, either $x_3 \neq 0$ and the second case applies, or the last case applies. Thus the fifteen vectors $d^{t,A}$ and fifteen vectors $d^{t,B}$ illuminate K. □

6.4 All coordinate bidiscs

We first record a finite construction on four coordinates. Put

$$v_j^k = \exp\left(\frac{2\pi \mathrm{i}\, 2^{j-1} k}{31}\right), \qquad k \in [30], \quad j \in [4], \qquad c = \sin(\pi/124).$$

Define two distinct indices $m(k), q(k) \in [4]$ by the following table.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$m(k)$	1	2	1	4	3	2	3	4	1	4	1	4	3	2	4
$q(k)$	4	4	2	3	4	3	2	2	4	2	3	2	1	4	1

k	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$m(k)$	2	1	2	4	2	3	3	1	3	2	3	4	1	2	1
$q(k)$	1	4	1	3	1	2	4	2	1	3	4	3	3	4	4

Lemma 19. *For every nonempty proper $S \subset [4]$ and every family of unit complex numbers $(a_j)_{j\in S}$, there is a row $k \in [30]$ such that*

$$m(k) \in S, \qquad \mathrm{Re}(\overline{a_j} v_j^k) > c \quad (j \in S). \tag{30}$$

For every family $(a_j)_{j\in[4]}$ of unit complex numbers and every two-element set $L \subset [4]$, there is a row $k \in [30]$ satisfying either

$$\mathrm{Re}(\overline{a_j} v_j^k) > c \quad (j \in [4]), \tag{31}$$

or

$$q(k) \in L, \qquad \mathrm{Re}(\overline{a_j} v_j^k) > c \quad (j \neq q(k)). \tag{32}$$

Proof. For a unit complex number a_j, the condition $\mathrm{Re}(\overline{a_j} z) > c$ defines an open arc on the unit circle of length $61/124$ turns. Since $61/124 > 15/31$, this arc contains fifteen consecutive points of the uniform grid with denominator 31. Indeed, the first grid point strictly after the left endpoint lies at most $1/31$ turns after it, so the fifteenth lies at most $15/31$ turns after it, strictly before the right endpoint. If the first residue is $b_j \in \{0, \dots, 30\}$, every retained row in

$$W(j, b_j) = \{k \in [30] : (2^{j-1}k - b_j) \bmod 31 < 15\}$$

satisfies the required strict inequality in coordinate j.

The integer verifier in Appendix B checks

$$\left(\bigcap_{j\in S} W(j, b_j) \right) \cap \{k : m(k) \in S\} \neq \varnothing \tag{33}$$

for every nonempty proper S and every choice of its block starts. These are exactly $4 \cdot 31 + 6 \cdot 31^2 + 4 \cdot 31^3 = 125054$ checks, and they prove (30).

For all $31^4 = 923521$ choices of four block starts, the verifier checks that either the full intersection is nonempty, or

$$\# \left\{ j \in [4] : \left(\bigcap_{i\neq j} W(i, b_i) \right) \cap \{k : q(k) = j\} \neq \varnothing \right\} \geq 3. \tag{34}$$

The first alternative proves (31). In the second alternative, the three or more eligible indices meet every pair L, which proves (32). There are 9341 empty full intersections; the cardinality in (34) is three in 1320 cases and four in 8021 cases. The displayed verifier exhausts these finite sets, uses integer arithmetic only, and returns the stated counts. The open-arc reduction proves the assertions for all continuous arguments, including endpoint configurations. □

Lemma 20. *Suppose $K \subset \mathbb{C}^4$ is a normalized complex 1-unconditional convex body containing every coordinate bidisc. If $K \neq \mathbb{D}^4$, then $\mathrm{ill}(K) \leq 30$.*

Proof. If K contains a full coordinate tridisc, apply Theorem 10. Otherwise define

$$\gamma = \max_{i\in[4]} \|\mathbf{1} - e_i\|_K^{-1} < 1, \qquad \delta = \frac{c(1-\gamma)}{4}, \qquad \tau = \frac{c\delta(1-\gamma)}{2}. \tag{35}$$

Every coordinate triple of every $x \in K$ contains an index j with $|x_j| \leq \gamma$. To see this, if T is a coordinate triple and $r = \min_{j\in T} |x_j|$, monotonicity gives $r \, \|\mathbf{1}_T\|_K \leq \|x\|_K \leq 1$, hence $r \leq \gamma$. In particular, a point of full support has at least two coordinates of modulus at most γ.

Fix the thirty directions

$$d_j^k = -a_j^k v_j^k, \qquad a_j^k = \begin{cases} 1, & j = m(k), \\ \tau, & j = q(k), \\ \delta, & j \notin \{m(k), q(k)\}. \end{cases} \tag{36}$$

Here $0 < \tau < \delta < 1$. We show that they illuminate K.

Fix $x \in \partial K$, put $r_j = |x_j|$, and let $u \neq 0$ be a supporting normal at x. With $b_j = |u_j|$ and $h = h_K(u)$, normal alignment and containment of the coordinate bidiscs give

$$h = \sum_{j \in [4]} b_j r_j > 0, \qquad h \geq b_i + b_j \quad (i \neq j), \qquad h \geq b_j. \tag{37}$$

At a nonzero coordinate, $u_j = b_j x_j / r_j$. Thus a row satisfying $\mathrm{Re}(\overline{x_j/r_j} v_j^k) > c$ contributes at most $-c a_j^k b_j$ to $[u, d^k]$. Any unchecked coordinate contributes at most $a_j^k b_j$. In each case below the row is chosen from x before the supporting normal is considered.

Suppose first that x has full support. Choose a pair L on which $r_j \leq \gamma$ and apply the second assertion of Lemma 19. A row satisfying (31) has negative pairing with every nonzero supporting normal. Otherwise put $q = q(k) \in L$. From $h \geq b_q$ we obtain

$$(1-\gamma) b_q \leq (1 - r_q) b_q \leq \sum_{j \neq q} b_j r_j \leq \sum_{j \neq q} b_j.$$

All amplitudes outside q are at least δ, so

$$[u, d^k] \leq -c\delta \sum_{j \neq q} b_j + \tau b_q \leq -\frac{c\delta}{2} \sum_{j \neq q} b_j < 0.$$

The final sum cannot vanish: otherwise $u \neq 0$ would imply $h = b_q r_q < b_q$, contradicting (37).

Suppose x has exactly three nonzero coordinates. Let i be the missing coordinate, choose $t \in \mathrm{supp}(x)$ with $r_t \leq \gamma$, and put $R = \mathrm{supp}(x) \setminus \{t\}$. Apply (30) to R. Write $m = m(k) \in R$ and let l be the other index of R. The bidisc inequalities for $\{t, l\}$ and $\{i, l\}$ give

$$b_t (1 - r_t) + b_l (1 - r_l) \leq b_m r_m, \qquad b_i \leq b_m r_m + b_t r_t - b_l (1 - r_l).$$

Consequently

$$b_t \leq \frac{b_m}{1 - \gamma}, \qquad b_i \leq b_m + \frac{\gamma b_m}{1 - \gamma} = \frac{b_m}{1 - \gamma}.$$

The unchecked coordinates i, t have amplitude at most δ. It follows that

$$[u, d^k] \leq -c b_m - c a_l^k b_l + \delta (b_i + b_t) \leq -\frac{c}{2} b_m - c a_l^k b_l < 0.$$

For strictness, $b_m = b_l = 0$ would force $b_t = b_i = 0$ by the preceding estimates. The amplitude a_l^k may equal τ; its positivity suffices.

Suppose x has exactly two nonzero coordinates. Apply (30) to their pair, and denote them by $m = m(k)$ and l. For either missing coordinate i, the bidisc inequality for $\{i, l\}$ implies

$$b_i \leq b_m r_m - b_l (1 - r_l) \leq b_m.$$

Therefore

$$[u, d^k] \leq -(c - 2\delta) b_m - c a_l^k b_l < 0.$$

Indeed, $c - 2\delta \geq c/2 > 0$, and $h > 0$ ensures that at least one of the active coefficients b_m, b_l is positive.

Finally, if x has only one nonzero coordinate j, then $r_j = 1$: the normalized axis section is the unit disc and its relative interior lies in $\operatorname{int} K$. Now $h = b_j$, and the bidisc inequality for $\{i, j\}$ forces $b_i = 0$ for every $i \neq j$. The row supplied by (30) for the singleton $\{j\}$ has negative pairing with every nonzero supporting normal. There is no empty-support boundary point because $0 \in \operatorname{int} K$.

In every case one of the fixed directions has negative pairing with all nonzero supporting normals at x. The strict normal criterion proves the claim. □

6.5 Completion of the proof

Proof of Theorem 15. Suppose first that $K \neq \mathbb{D}^4$. If every coordinate bidisc is present, Lemma 20 applies. Otherwise let

$$\mathcal{E} = \{\{i, j\} \subset [4] : i \neq j,\ e_i + e_j \notin K\}.$$

If two members of $\mathcal{E}$ are disjoint, a coordinate permutation puts us in Proposition 16.

It remains that $\mathcal{E}$ is nonempty and its members pairwise intersect. Either all of them contain a common index, or they are exactly the three edges of a triangle. Indeed, if there are at least two edges, write two of them as $\{a, b\}$ and $\{a, c\}$. An edge avoiding a must be $\{b, c\}$. Once all three edges are present, every further edge would be disjoint from one of them. A family with only one edge has a common index as well.

In the common-index case, permute coordinates so that this index is 3 and one missing edge is $\{3, 4\}$. The bidiscs on $\{1, 2\}$, $\{1, 4\}$ and $\{2, 4\}$ are present, so Proposition 18 applies. In the triangle case, label its vertices $2, 3, 4$. The bidiscs on $\{1, 3\}$ and $\{1, 4\}$ are present and the one on $\{3, 4\}$ is missing; Proposition 17 applies. These cases exhaust all possibilities and prove $\operatorname{ill}(K) \leq 30$.

The normalized non-polydisc

$$H_4 = \mathbb{D}^2 \times \{(z_3, z_4) \in \mathbb{C}^2 : |z_3| + |z_4| \leq 1\}$$

has illumination number 30 by the sharpness argument in Theorem 10. Thus the bound is sharp. The exact polydisc theorem gives $\operatorname{ill}(\mathbb{D}^4) = 31$. Finally, positive diagonal rescaling preserves illumination numbers and normalizes every complex 1-unconditional body, which proves the last assertion. □

7 Fractional illumination in two complex dimensions

We give the fractional bound and its complete equality criterion in complex dimension two. The upper bound for the 1-symmetric subclass appears in [6, Theorem 1.7]; the criterion here allows different supporting coefficients at the two axes. All bodies in the following theorem are normalized.

Theorem 21. *Let $K \subset \mathbb{C}^2$ be a normalized complex 1-unconditional convex body, and define*

$$\begin{aligned} a(K) &= \max\{a \geq 0 : |z_1| + a|z_2| \leq 1 \text{ for every } z \in K\}, \\ b(K) &= \max\{b \geq 0 : b|z_1| + |z_2| \leq 1 \text{ for every } z \in K\}. \end{aligned} \tag{38}$$

Then $\operatorname{ill}^*(K) \leq 4$, and

$$\operatorname{ill}^*(K) = 4 \quad \Longleftrightarrow \quad K = \mathbb{D}^2 \ \text{ or } \ a(K)b(K) \geq \frac{1}{2}. \tag{39}$$

Proof. Write $a = a(K)$ and $b = b(K)$. The feasible sets in (38) are closed, contain zero and are contained in $[0, 1]$, by the coordinate discs in K. Thus the maxima exist. Let σ_j denote uniform probability measure on the circle $\{\xi e_j : |\xi| = 1\}$.

A nonzero supporting normal at e_1 has the form (s, w), where $s > 0$ and $|w| \le as$. Indeed, its support value is s; coordinate rotations then give $s|z_1| + |w||z_2| \le s$ throughout K. Conversely, every vector $(1, w)$ with $|w| \le a$ supports K at e_1. The normal criterion, followed by a coordinate rotation, gives

$$\begin{aligned} d \in \mathcal{I}_K(\xi e_1) \quad &\Longleftrightarrow \quad \mathrm{Re}(\bar{\xi} d_1) + a|d_2| < 0, \\ d \in \mathcal{I}_K(\xi e_2) \quad &\Longleftrightarrow \quad \mathrm{Re}(\bar{\xi} d_2) + b|d_1| < 0, \end{aligned} \tag{40}$$

where $\mathcal{I}_K(x)$ is the set of directions illuminating x.

Suppose first that $K \neq \mathbb{D}^2$. We claim that $\mu_0 = 2\sigma_1 + 2\sigma_2$ fractionally illuminates K. A boundary point with one zero coordinate belongs to one of the two unit axis circles, where the inward half of that same direction circle has μ_0-mass one. Now let $x = (r_1\xi_1, r_2\xi_2) \in \partial K$, with $r_1, r_2 > 0$ and $|\xi_1| = |\xi_2| = 1$. At least one coordinate, say r_j, is strictly less than one; otherwise unconditionality would give $K = \mathbb{D}^2$. Write k for the other index. Every nonzero supporting normal has the form $u_l = c_l\xi_l$, $c_l \ge 0$. If $c_k = 0$, then $c_j > 0$ and $h_K(u) = r_j c_j < c_j$, contradicting the coordinate disc in K. Thus $c_k > 0$ for every such normal. Every direction on the kth coordinate circle with $\mathrm{Re}(\overline{\xi_k} d_k) < 0$ illuminates x. This again gives mass one and proves the claim. For $K = \mathbb{D}^2$, the value four follows from Lemma 2.

Assume $ab \ge 1/2$. Regard $\nu = 2\sigma_1 + 2\sigma_2$ as a measure on ∂K. For a direction with both coordinates nonzero, put $t = |d_2|/|d_1|$. By (40), the proportions of the two axis circles illuminated by d are

$$q_1 = \begin{cases} \pi^{-1} \arccos(at), & at < 1, \\ 0, & at \ge 1, \end{cases} \qquad q_2 = \begin{cases} \pi^{-1} \arccos(b/t), & b/t < 1, \\ 0, & b/t \ge 1. \end{cases}$$

If either proportion is zero, their sum is at most $1/2$. Otherwise set $v = at$ and $w = b/t$. Then $v^2 + w^2 \ge 2vw = 2ab \ge 1$, so $\arccos v + \arccos w \le \pi/2$. Hence $q_1 + q_2 \le 1/2$ in every case. If one coordinate of d is zero, it illuminates half of the corresponding axis circle and none of the other one, since $a, b > 0$. Consequently every direction illuminates ν-mass at most one. For any fractional illuminating measure μ, Tonelli's theorem for the open illumination relation gives

$$4 \le \int_{\partial K} \mu(\mathcal{I}_K(x))\, d\nu(x) = \int \nu\{x : d \in \mathcal{I}_K(x)\}\, d\mu(d) \le \mu(\mathbb{C}^2 \setminus \{0\}). \tag{41}$$

Together with the upper bound this proves equality.

It remains to prove strictness when $K \neq \mathbb{D}^2$ and $ab < 1/2$. Choose $t > 0$ such that

$$\sqrt{2}\, b < t < \frac{1}{\sqrt{2}\, a},$$

where the right endpoint means infinity if $a = 0$. Let σ be the probability measure on $d = (\eta_1, t\eta_2)$, with independent uniform unit complex numbers η_1, η_2. The two axis-circle illumination probabilities are

$$p_1 = \pi^{-1} \arccos(at) > \frac{1}{4}, \qquad p_2 = \pi^{-1} \arccos(b/t) > \frac{1}{4}.$$

We obtain a uniform strict improvement at all remaining boundary points. Fix, in this order,

$$\gamma = \max\{s \ge 0 : (s, s) \in K\} < 1, \qquad m = \min\{1, t\}, \qquad M = \max\{1, t\}, \qquad c = \frac{(1-\gamma)m}{2}.$$

The strict inequality for γ follows from $K \neq \mathbb{D}^2$. At a full-support boundary point $x = (r_1\xi_1, r_2\xi_2)$ choose j with $r_j \le \gamma$, which exists by coordinatewise monotonicity, and let $k \neq j$. Every supporting normal as above satisfies

$$c_j \le h_K(u) = r_j c_j + r_k c_k \le \gamma c_j + c_k, \qquad (1-\gamma) c_j \le c_k.$$

In particular $c_k > 0$ for every nonzero normal. Directions with both aligned real parts negative illuminate x, and have σ-probability $1/4$. A disjoint additional set is

$$0 < \operatorname{Re}(\overline{\xi_j} d_j) < c, \qquad \operatorname{Re}(\overline{\xi_k} d_k) < -m/2.$$

On this set, for every nonzero supporting normal,

$$[u, d] < c c_j - \frac{m}{2} c_k \le 0.$$

The first inequality is strict because $c_k > 0$. Its probability is at least

$$\varepsilon = \frac{1}{3\pi} \arcsin\left(\frac{(1-\gamma)m}{2M}\right) > 0 :$$

the first coordinate condition has probability at least $\pi^{-1} \arcsin(c/M)$, and the second at least $1/3$. Both bounds are independent of x and of the chosen index. Thus every boundary point is illuminated with probability at least

$$p = \min\{p_1, p_2, 1/4 + \varepsilon\} > 1/4.$$

The measure $p^{-1}\sigma$ fractionally illuminates K and has mass $p^{-1} < 4$, completing the proof. □

Remark 4. For $H_2 = \{(z, w) : |z| + |w| \le 1\}$, the supporting coefficients are $a(H_2) = b(H_2) = 1$. Theorem 21 therefore gives $\mathrm{ill}^*(H_2) = 4$. This is a counterexample to the equality clause of [5, Conjecture 1.2]. Indeed,

$$\operatorname{ext} H_2 = (\mathbb{T} \times \{0\}) \cup (\{0\} \times \mathbb{T}), \qquad \operatorname{ext}(\mathbb{D}^2) = \mathbb{T}^2, \qquad \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

A boundary point of H_2 with both coordinates nonzero is the nontrivial convex combination, with weights $|z|$ and $|w|$, of $(z/|z|, 0)$ and $(0, w/|w|)$. Every point on either displayed axis circle is extreme: its unit coordinate forces the same coordinate in both terms of any convex decomposition, and then the other coordinate must vanish. In the bidisc, extremality is equivalent to both coordinates being extreme in the disc, which gives the torus. The former extreme-point set is disconnected and the latter is connected. Since an invertible affine map carries extreme points homeomorphically onto extreme points, these bodies are not affinely equivalent. This conclusion concerns the equality clause only.

8 Extreme points with no zero coordinates

The condition considered here is the complex counterpart of the condition in Sun and Vritsiou's Theorem 5 [7]. Extreme points are understood in the underlying real vector space. Having no zero complex coordinate does not mean having no zero real coordinate in $\mathbb{R}^{2n}$, so that real theorem does not directly imply the fractional bound below.

Theorem 22. *Let $K \subset \mathbb{C}^n$ be a normalized complex 1-unconditional convex body such that*

$$x \in \operatorname{ext} K \quad \Longrightarrow \quad \operatorname{supp}(x) = [n]. \tag{42}$$

Then

$$\mathrm{ill}^*(K) \le 2^n.$$

If $K \neq \mathbb{D}^n$, put

$$\gamma = \max\{s \ge 0 : s(1, \dots, 1) \in K\} = \|(1, \dots, 1)\|_K^{-1}.$$

Then $0 < \gamma < 1$ and

$$\mathrm{ill}^*(K) \le \left(2^{-n} + \frac{\arcsin((1-\gamma)/2)}{\pi\, 3^{n-1}}\right)^{-1} < 2^n. \tag{43}$$

Consequently, within the normalized class defined by (42), equality $\mathrm{ill}^*(K) = 2^n$ holds exactly for $K = \mathbb{D}^n$.

Proof. We first reduce the fractional assertion to extreme points, using the convex-combination argument of [5, Lemma 2.1]. Every boundary point y is a finite convex combination of extreme points. Choose one summand x with positive coefficient λ. If $\lambda < 1$, write $y = \lambda x + (1-\lambda)z$ with $z \in K$. Whenever $x + td \in \operatorname{int} K$, convexity gives

$$y + \lambda td = \lambda(x + td) + (1-\lambda)z \in \operatorname{int} K.$$

The case $\lambda = 1$ is immediate. Thus every direction illuminating x illuminates y. In particular, a family or measure illuminating every extreme point illuminates the whole boundary.

Fix an extreme point x and write $x_j = r_j\xi_j$, where $r_j > 0$ and $\xi_j \in \mathbb{T}$, with $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$. Every nonzero supporting normal at x has the form $u_j = b_j\xi_j$, with $b_j \geq 0$.

Let σ be the product of normalized arc-length measures on $\mathbb{T}^n$. For the same extreme point x, the set

$$A_x = \{d \in \mathbb{T}^n : \operatorname{Re}(\overline{\xi_j} d_j) < 0 \text{ for every } j \in [n]\}$$

has measure 2^{-n}. For $d \in A_x$, every nonzero supporting normal satisfies

$$[u, d] = \sum_{j\in[n]} b_j \operatorname{Re}(\overline{\xi_j} d_j) < 0,$$

because its coefficients are nonnegative and not all zero. Thus every direction in A_x illuminates x. Thus $2^n\sigma$ fractionally illuminates K.

Assume now that $K \neq \mathbb{D}^n$. Since $0 \in \operatorname{int} K$ and $K \subseteq \mathbb{D}^n$, we have $0 < \gamma \leq 1$. If $\gamma = 1$, coordinate rotations and contractions of $(1, \ldots, 1) \in K$ give $\mathbb{D}^n \subseteq K$, a contradiction. For every $x \in K$, coordinatewise monotonicity also gives $\min_j |x_j| \leq \gamma$.

For each extreme point x, choose j_0 with $r_{j_0} \leq \gamma$. This choice depends only on x, not on a supporting normal. Since the unit coordinate discs lie in K, every nonzero supporting normal at x satisfies

$$b_{j_0} \leq \sum_{j\in[n]} r_j b_j, \qquad (1-\gamma)b_{j_0} \leq \sum_{j\neq j_0} b_j =: S.$$

Here $S > 0$: if $S = 0$, then $\gamma < 1$ forces $b_{j_0} = 0$, contrary to $u \neq 0$. Define

$$B_x = \left\{ d \in \mathbb{T}^n : \begin{array}{ll} 0 < \operatorname{Re}(\overline{\xi_{j_0}} d_{j_0}) < (1-\gamma)/2, \\ \operatorname{Re}(\overline{\xi_j} d_j) < -1/2 \quad (j \neq j_0) \end{array} \right\}.$$

Every direction in B_x satisfies, for every nonzero supporting normal at x,

$$[u, d] < \frac{(1-\gamma)b_{j_0} - S}{2} \leq 0.$$

Strictness follows from $S > 0$, including when $b_{j_0} = 0$. Thus B_x also consists of illuminating directions. It is disjoint from A_x, and independence of the coordinates gives

$$\sigma(B_x) = \frac{\arcsin((1-\gamma)/2)}{\pi} \left(\frac{1}{3}\right)^{n-1}.$$

Indeed, $\cos\theta < -1/2$ occupies one third of the circle, and $0 < \cos\theta < a$ occupies the proportion $\arcsin(a)/\pi$. This lower bound for the measure of illuminating directions is uniform over all extreme points and therefore over the entire boundary, by the reduction at the start of the proof. The illumination relation is open in (x, d), so all its direction sections are Borel. Scaling σ by the reciprocal of $\sigma(A_x) + \sigma(B_x)$ proves (43). Finally, $\mathrm{ill}^*(\mathbb{D}^n) = 2^n$ follows from Lemma 2, completing the equality assertion. □

A Simultaneous families in dimensions one through five

Let U_n be the least cardinality of a finite family of directions that illuminates every normalized complex 1-unconditional convex body in $\mathbb{C}^n$. The family must be chosen independently of the body. This differs from bounding $\mathrm{ill}(K)$ separately for each K.

Proposition 23.

$$U_1 = 3, \quad U_2 = 7, \quad U_3 = 21, \quad U_4 = 60, \quad U_5 = 155.$$

Proof. Fix $n \in \{1, 2, 3, 4, 5\}$, put $m = \lceil 2n/3 \rceil$, and write $L_n = (2^{m+1} - 1)\binom{n}{m}$. We show that $L_n \le U_n \le C_n$, where C_n is defined in (3).

The directions in the proof of Theorem 3 depend only on n, the finite matchings and cyclic vectors, and their common positive margin. The normal estimates use only normalization of K. Hence the same family works for every normalized K, proving $U_n \le C_n$.

For every m-set $S \subset [n]$, when $m < n$, consider the normalized body

$$K_S = \{z \in \mathbb{C}^n : \|z_S\|_\infty + \|z_{S^c}\|_\infty \le 1\}.$$

At a point x of its coordinate torus

$$|x_j| = 1 \ (j \in S), \qquad x_j = 0 \ (j \notin S),$$

the directional derivative of its norm is

$$\max_{j \in S} \mathrm{Re}(\overline{x_j} d_j) + \|d_{S^c}\|_\infty.$$

Illumination forces this derivative to be negative: for a convex function, an interior point along the ray forces a negative one-sided derivative at its initial point. Consequently

$$\mathrm{Re}(\overline{x_j} d_j) < -\|d_{S^c}\|_\infty \quad (j \in S), \qquad \min_{j \in S} |d_j| > \max_{j \notin S} |d_j|.$$

Thus the m largest coordinate magnitudes of d determine S uniquely. The groups of directions serving these tori for different m-sets are disjoint. Within the group for S, projection must illuminate the m-polydisc torus, and hence the polydisc, requiring at least $2^{m+1} - 1$ directions by the exact polydisc illumination theorem [5]. Summation gives $U_n \ge L_n$. If $m = n$, the polydisc itself gives this conclusion.

For $n = 1, 2$ we have $m = n$, so $C_n = L_n$. For $3 \le n \le 5$ we have $m = n - 1$ and

$$C_n - L_n = \sum_{k=m+1}^{n} \binom{n}{k} - (n - m) = 1 - 1 = 0.$$

Thus $U_n = L_n = C_n$ in these five dimensions. Substitution gives the stated values. □

B The four-coordinate finite verification

The following Python program verifies the finite assertions (33) and (34). The finite reduction to these assertions is proved in Lemma 19. All sets of rows are represented by integer bit masks; every loop ranges over the entire finite set indicated. The program requires only Python's standard library. Running it with Python 3.12.14 gives `125054 923521 9341 [0, 0, 0, 1320, 8021]`.

```
from itertools import combinations, product

m = [1,2,1,4,3,2,3,4,1,4,1,4,3,2,4,
     2,1,2,4,2,3,3,1,3,2,3,4,1,2,1]
q = [4,4,2,3,4,3,2,2,4,2,3,2,1,4,1,
     1,4,1,3,1,2,4,2,1,3,4,3,3,4,4]
assert len(m) == len(q) == 30
assert all(1 <= a <= 4 and 1 <= b <= 4 and a != b
           for a, b in zip(m, q))
w = [[sum(1 << (k-1) for k in range(1,31)
          if ((1 << j)*k-a) % 31 < 15)
      for a in range(31)] for j in range(4)]
main = [sum(1 << k for k in range(30) if m[k] == j+1)
        for j in range(4)]
tiny = [sum(1 << k for k in range(30) if q[k] == j+1)
        for j in range(4)]

tested = 0
for size in (1,2,3):
    for S in combinations(range(4), size):
        allowed = 0
        for j in S:
            allowed |= main[j]
        for starts in product(range(31), repeat=size):
            candidates = (1 << 30)-1
            for j, a in zip(S, starts):
                candidates &= w[j][a]
            assert candidates & allowed
            tested += 1
assert tested == 125054

holes = 0
histogram = [0]*5
for a,b,c,d in product(*w):
    if a & b & c & d:
        continue
    holes += 1
    candidates = [b&c&d, a&c&d, a&b&d, a&b&c]
    count = sum(bool(candidates[j] & tiny[j])
                for j in range(4))
    assert count >= 3
    histogram[count] += 1
assert holes == 9341
assert histogram == [0,0,0,1320,8021]
```

```
print(tested, 31**4, holes, histogram)
```

C Numerical comparison through dimension 100

The table compares the following estimates. Here $c < c_*$ is the positive rational approximation recorded in the accompanying data file, and $b = 1/c$:

$$A_n = 2 \cdot 3^n - 2^n - 1,$$
$$C_n = (2^{m+1} - 1)\binom{n}{m} + \sum_{k=m+1}^{n} \binom{n}{k} - (n - m), \qquad m = \lceil 2n/3 \rceil,$$
$$E_n = \left\lceil eb^n \left[2n \log\bigl(1 + 32b^2(n+1)^2\bigr) + 1 \right] \right\rceil,$$
$$T_n = \left\lceil \frac{2n \log\bigl(1 + 32/(c^2\varepsilon_n^2)\bigr) + 1}{[c(1-\varepsilon_n)]^n} \right\rceil.$$

Here A_n is the separate-support bound from [6, Theorem 1.8], C_n is the bound in Theorem 3, E_n evaluates the explicit formula with $b = 1/c$, and T_n is (21) with a numerically chosen value of ε_n. The chosen parameters approximately minimize the real-valued expression before taking its ceiling; validity needs only $0 < \varepsilon_n < 1$.

The first dimension in this table with $E_n < C_n$ is $n = 51$; the first with $T_n < C_n$ is $n = 47$. Each remains smaller through $n = 100$, the end of this table. The smaller of C_n and T_n is bold. The last column is T_n/C_n: values below one mean $T_n < C_n$. These compare the displayed estimates, not necessarily the best known bounds in each individual dimension.

Numbers below 10^7 are printed exactly. Larger counts are rounded *upward* to six significant digits for display. This rounding does not decrease the computed counts; ratios are ordinary rounded approximations. Admissibility of the rational c was checked by exact rational lower bounds on $\cos(\pi c)\cosh(\pi(1-c))$. The formulas remain valid for this smaller value of c. Integer ceilings agree when evaluated at 120 and 170 decimal digits. Agreement at two precisions is not interval certification of each integer ceiling. The integer evaluations are numerical and are not used in the proofs. Full computed counts and the chosen ε_n are recorded in the accompanying data file.

n	A_n	C_n	E_n	T_n	T_n/C_n
1	3	**3**	97	56	18.67
2	13	**7**	518	308	44
3	45	**21**	2047	1221	58.14
4	145	**60**	7103	4225	70.42
5	453	**155**	22891	13555	87.45
6	1393	**470**	70353	41470	88.23
7	4245	**1329**	209211	122774	92.38
8	12865	**3563**	607225	354861	99.6
9	38853	**10711**	1729986	1007086	94.02
10	117073	**30653**	4856875	2817201	91.91
11	352245	**84379**	1.34743×10^7	7789603	92.32
12	1058785	**253240**	3.70155×10^7	2.13328×10^7	84.24
13	3180453	**731819**	1.00851×10^8	5.79540×10^7	79.19
14	9549553	**2049513**	2.72846×10^8	1.56369×10^8	76.3
15	2.86651×10^7	**6149077**	7.33709×10^8	4.19428×10^8	68.21
16	8.60280×10^7	$\mathbf{1.78895 \times 10^7}$	1.96264×10^9	1.11929×10^9	62.57
17	2.58150×10^8	$\mathbf{5.06892 \times 10^7}$	5.22569×10^9	2.97352×10^9	58.66
18	7.74579×10^8	$\mathbf{1.52071 \times 10^8}$	1.38569×10^{10}	7.86814×10^9	51.74
19	2.32400×10^9	$\mathbf{4.44521 \times 10^8}$	3.66100×10^{10}	2.07461×10^{10}	46.67
20	6.97253×10^9	$\mathbf{1.27008 \times 10^9}$	9.64068×10^{10}	5.45279×10^{10}	42.93
21	2.09187×10^{10}	$\mathbf{3.81023 \times 10^9}$	2.53123×10^{11}	1.42909×10^{11}	37.51
22	6.27580×10^{10}	$\mathbf{1.11768 \times 10^{10}}$	6.62813×10^{11}	3.73568×10^{11}	33.42

Continued on the next page

n	A_n	C_n	E_n	T_n	T_n/C_n
23	1.88278×10^{11}	$\mathbf{3.21332 \times 10^{10}}$	1.73137×10^{12}	9.74214×10^{11}	30.32
24	5.64843×10^{11}	$\mathbf{9.63995 \times 10^{10}}$	4.51251×10^{12}	2.53514×10^{12}	26.3
25	1.69455×10^{12}	$\mathbf{2.83529 \times 10^{11}}$	1.17370×10^{13}	6.58396×10^{12}	23.22
26	5.08367×10^{12}	$\mathbf{8.19082 \times 10^{11}}$	3.04702×10^{13}	1.70679×10^{13}	20.84
27	1.52511×10^{13}	$\mathbf{2.45725 \times 10^{12}}$	7.89653×10^{13}	4.41713×10^{13}	17.98
28	4.57534×10^{13}	$\mathbf{7.24241 \times 10^{12}}$	2.04313×10^{14}	1.14136×10^{14}	15.76
29	1.37261×10^{14}	$\mathbf{2.10030 \times 10^{13}}$	5.27839×10^{14}	2.94491×10^{14}	14.02
30	4.11782×10^{14}	$\mathbf{6.30090 \times 10^{13}}$	1.36176×10^{15}	7.58813×10^{14}	12.04
31	1.23535×10^{15}	$\mathbf{1.86027 \times 10^{14}}$	3.50857×10^{15}	1.95276×10^{15}	10.5
32	3.70604×10^{15}	$\mathbf{5.41168 \times 10^{14}}$	9.02878×10^{15}	5.01937×10^{15}	9.275
33	1.11182×10^{16}	$\mathbf{1.62351 \times 10^{15}}$	2.32076×10^{16}	1.28875×10^{16}	7.938
34	3.33544×10^{16}	$\mathbf{4.79993 \times 10^{15}}$	5.95884×10^{16}	3.30547×10^{16}	6.886
35	1.00064×10^{17}	$\mathbf{1.39998 \times 10^{16}}$	1.52846×10^{17}	8.46976×10^{16}	6.05
36	3.00190×10^{17}	$\mathbf{4.19994 \times 10^{16}}$	3.91676×10^{17}	2.16825×10^{17}	5.163
37	9.00568×10^{17}	$\mathbf{1.24319 \times 10^{17}}$	1.00279×10^{18}	5.54581×10^{17}	4.461
38	2.70171×10^{18}	$\mathbf{3.63392 \times 10^{17}}$	2.56518×10^{18}	1.41730×10^{18}	3.9
39	8.10511×10^{18}	$\mathbf{1.09018 \times 10^{18}}$	6.55652×10^{18}	3.61925×10^{18}	3.32
40	2.43154×10^{19}	$\mathbf{3.23015 \times 10^{18}}$	1.67454×10^{19}	9.23531×10^{18}	2.859
41	7.29460×10^{19}	$\mathbf{9.45971 \times 10^{18}}$	4.27366×10^{19}	2.35493×10^{19}	2.489
42	2.18838×10^{20}	$\mathbf{2.83792 \times 10^{19}}$	1.08994×10^{20}	6.00083×10^{19}	2.115
43	6.56514×10^{20}	$\mathbf{8.41588 \times 10^{19}}$	2.77789×10^{20}	1.52815×10^{20}	1.816
44	1.96955×10^{21}	$\mathbf{2.46866 \times 10^{20}}$	7.07545×10^{20}	3.88916×10^{20}	1.575
45	5.90863×10^{21}	$\mathbf{7.40598 \times 10^{20}}$	1.80107×10^{21}	9.89217×10^{20}	1.336
46	1.77259×10^{22}	$\mathbf{2.19791 \times 10^{21}}$	4.58201×10^{21}	2.51470×10^{21}	1.144
47	5.31777×10^{22}	6.45634×10^{21}	1.16505×10^{22}	$\mathbf{6.38921 \times 10^{21}}$	0.9896
48	1.59533×10^{23}	1.93691×10^{22}	2.96073×10^{22}	$\mathbf{1.62251 \times 10^{22}}$	0.8377
49	4.78599×10^{23}	5.75201×10^{22}	7.52029×10^{22}	$\mathbf{4.11826 \times 10^{22}}$	0.716
50	1.43580×10^{24}	1.69177×10^{23}	1.90924×10^{23}	$\mathbf{1.04481 \times 10^{23}}$	0.6176
51	4.30739×10^{24}	5.07531×10^{23}	4.84487×10^{23}	$\mathbf{2.64950 \times 10^{23}}$	0.522
52	1.29222×10^{25}	1.50809×10^{24}	1.22889×10^{24}	$\mathbf{6.71587 \times 10^{23}}$	0.4453
53	3.87665×10^{25}	4.44049×10^{24}	3.11567×10^{24}	$\mathbf{1.70161 \times 10^{24}}$	0.3832
54	1.16300×10^{26}	1.33215×10^{25}	7.89606×10^{24}	$\mathbf{4.30966 \times 10^{24}}$	0.3235
55	3.48899×10^{26}	3.96044×10^{25}	2.00031×10^{25}	$\mathbf{1.09109 \times 10^{25}}$	0.2755
56	1.04670×10^{27}	1.16729×10^{26}	5.06543×10^{25}	$\mathbf{2.76130 \times 10^{25}}$	0.2366
57	3.14009×10^{27}	3.50186×10^{26}	1.28226×10^{26}	$\mathbf{6.98571 \times 10^{25}}$	0.1995
58	9.42026×10^{27}	1.04158×10^{27}	3.24472×10^{26}	$\mathbf{1.76668 \times 10^{26}}$	0.1696
59	2.82608×10^{28}	3.07266×10^{27}	8.20785×10^{26}	$\mathbf{4.46643 \times 10^{26}}$	0.1454
60	8.47824×10^{28}	9.21797×10^{27}	2.07557×10^{27}	$\mathbf{1.12882 \times 10^{27}}$	0.1225
61	2.54347×10^{29}	2.74291×10^{28}	5.24691×10^{27}	$\mathbf{2.85199 \times 10^{27}}$	0.104
62	7.63041×10^{29}	8.09811×10^{28}	1.32598×10^{28}	$\mathbf{7.20349 \times 10^{27}}$	0.08895
63	2.28913×10^{30}	2.42944×10^{29}	3.34992×10^{28}	$\mathbf{1.81891 \times 10^{28}}$	0.07487
64	6.86737×10^{30}	7.23180×10^{29}	8.46070×10^{28}	$\mathbf{4.59151 \times 10^{28}}$	0.06349
65	2.06022×10^{31}	2.13667×10^{30}	2.13627×10^{29}	$\mathbf{1.15873 \times 10^{29}}$	0.05423
66	6.18064×10^{31}	6.41001×10^{30}	5.39245×10^{29}	$\mathbf{2.92342 \times 10^{29}}$	0.04561
67	1.85419×10^{32}	1.90876×10^{31}	1.36082×10^{30}	$\mathbf{7.37376 \times 10^{29}}$	0.03863
68	5.56257×10^{32}	5.64328×10^{31}	3.43322×10^{30}	$\mathbf{1.85943 \times 10^{30}}$	0.03295
69	1.66878×10^{33}	1.69299×10^{32}	8.65953×10^{30}	$\mathbf{4.68773 \times 10^{30}}$	0.02769
70	5.00632×10^{33}	5.04293×10^{32}	2.18365×10^{31}	$\mathbf{1.18153 \times 10^{31}}$	0.02343
71	1.50190×10^{34}	1.49187×10^{33}	5.50511×10^{31}	$\mathbf{2.97733 \times 10^{31}}$	0.01996
72	4.50568×10^{34}	4.47560×10^{33}	1.38756×10^{32}	$\mathbf{7.50087 \times 10^{31}}$	0.01676
73	1.35171×10^{35}	1.33355×10^{34}	3.49653×10^{32}	$\mathbf{1.88931 \times 10^{32}}$	0.01417
74	4.05512×10^{35}	3.94730×10^{34}	8.80907×10^{32}	$\mathbf{4.75777 \times 10^{32}}$	0.01205
75	1.21654×10^{36}	1.18419×10^{35}	2.21887×10^{33}	$\mathbf{1.19789 \times 10^{33}}$	0.01012
76	3.64961×10^{36}	3.52935×10^{35}	5.58782×10^{33}	$\mathbf{3.01536 \times 10^{33}}$	0.008544
77	1.09489×10^{37}	1.04523×10^{36}	1.40691×10^{34}	$\mathbf{7.58889 \times 10^{33}}$	0.00726

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n	A_n	C_n	E_n	T_n	T_n/C_n
78	3.28465×10^{37}	3.13569×10^{36}	3.54165×10^{34}	$\mathbf{1.90958 \times 10^{34}}$	0.00609
79	9.85393×10^{37}	9.34791×10^{36}	8.91379×10^{34}	$\mathbf{4.80413 \times 10^{34}}$	0.005139
80	2.95618×10^{38}	2.76975×10^{37}	2.24305×10^{35}	$\mathbf{1.20842 \times 10^{35}}$	0.004363
81	8.86853×10^{38}	8.30925×10^{37}	5.64334×10^{35}	$\mathbf{3.03906 \times 10^{35}}$	0.003657
82	2.66056×10^{39}	2.47767×10^{38}	1.41958×10^{36}	$\mathbf{7.64168 \times 10^{35}}$	0.003084
83	7.98168×10^{39}	7.34452×10^{38}	3.57029×10^{36}	$\mathbf{1.92118 \times 10^{36}}$	0.002616
84	2.39451×10^{40}	2.20336×10^{39}	8.97794×10^{36}	$\mathbf{4.82918 \times 10^{36}}$	0.002192
85	7.18351×10^{40}	6.57141×10^{39}	2.25725×10^{37}	$\mathbf{1.21370 \times 10^{37}}$	0.001847
86	2.15506×10^{41}	1.94877×10^{40}	5.67430×10^{37}	$\mathbf{3.04988 \times 10^{37}}$	0.001565
87	6.46516×10^{41}	5.84629×10^{40}	1.42619×10^{38}	$\mathbf{7.66279 \times 10^{37}}$	0.001311
88	1.93955×10^{42}	1.74398×10^{41}	3.58406×10^{38}	$\mathbf{1.92499 \times 10^{38}}$	0.001104
89	5.81865×10^{42}	5.17380×10^{41}	9.00552×10^{38}	$\mathbf{4.83511 \times 10^{38}}$	0.0009345
90	1.74560×10^{43}	1.55214×10^{42}	2.26246×10^{39}	$\mathbf{1.21430 \times 10^{39}}$	0.0007823
91	5.23678×10^{43}	4.63098×10^{42}	5.68314×10^{39}	$\mathbf{3.04916 \times 10^{39}}$	0.0006584
92	1.57104×10^{44}	1.37436×10^{43}	1.42737×10^{40}	$\mathbf{7.65558 \times 10^{39}}$	0.000557
93	4.71311×10^{44}	4.12306×10^{43}	3.58447×10^{40}	$\mathbf{1.92185 \times 10^{40}}$	0.0004661
94	1.41394×10^{45}	1.23038×10^{44}	9.00026×10^{40}	$\mathbf{4.82396 \times 10^{40}}$	0.0003921
95	4.24180×10^{45}	3.65268×10^{44}	2.25959×10^{41}	$\mathbf{1.21069 \times 10^{41}}$	0.0003315
96	1.27254×10^{46}	1.09581×10^{45}	5.67215×10^{41}	$\mathbf{3.03815 \times 10^{41}}$	0.0002773
97	3.81762×10^{46}	3.27055×10^{45}	1.42368×10^{42}	$\mathbf{7.62311 \times 10^{41}}$	0.0002331
98	1.14529×10^{47}	9.71253×10^{45}	3.57292×10^{42}	$\mathbf{1.91251 \times 10^{42}}$	0.0001969
99	3.43586×10^{47}	2.91376×10^{46}	8.96564×10^{42}	$\mathbf{4.79761 \times 10^{42}}$	0.0001647
100	1.03076×10^{48}	8.69779×10^{46}	2.24952×10^{43}	$\mathbf{1.20337 \times 10^{43}}$	0.0001384

References

[1] S. Artstein-Avidan and B. A. Slomka, *On weighted covering numbers and the Levi-Hadwiger conjecture*, Israel J. Math. **209** (2015), no. 1, 125–155. doi:10.1007/s11856-015-1213-5. For the locators used here, see arXiv:1310.7892v2.

[2] V. Boltyanski and H. Martini, *Illumination of direct vector sums of convex bodies*, Studia Sci. Math. Hungar. **44** (2007), no. 3, 367–376. doi:10.1556/SScMath.44.2007.3.6.

[3] P. Hall, *On representatives of subsets*, J. London Math. Soc. **s1-10** (1935), no. 1, 26–30. doi:10.1112/jlms/s1-10.37.26.

[4] M. Naszódi, *Fractional illumination of convex bodies*, Contrib. Discrete Math. **4** (2009), no. 2, 83–88. doi:10.55016/ojs/cdm.v4i2.62022.

[5] L. Rotem, A. Schejter, and B. A. Slomka, *The complex Illumination problem*, Combinatorica **46**, article 3 (2026). doi:10.1007/s00493-025-00195-7. The theorem, proposition and remark numbers cited here refer to the published article.

[6] L. Rotem, A. Schejter and B. Slomka, *Illumination of Real and Complex 1-Symmetric Convex Bodies*, unpublished manuscript, October 2026, 22 pages. The locators used here refer to the version with Theorems 1.5–1.8, Lemma 4.3 and Proposition 4.4.

[7] W. R. Sun and B.-H. Vritsiou, *Illuminating* 1*-unconditional convex bodies in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$*, and certain cases in higher dimensions*, Canad. J. Math., accepted manuscript posted online 4 August 2025. doi:10.4153/S0008414X25101260. The theorem numbers used here refer to arXiv:2407.11331v1 (2024).

[8] W. R. Sun and B.-H. Vritsiou, *On the illumination of* 1*-symmetric convex bodies*, arXiv:2407.10314v1 (2024).