On Problem 5 of Vitorino and Vukusic: $\sup_{k\geq 0} B(2^k\alpha) \;\geq\; 12$

David Niedbala Giraudin
ORCID 0009-0009-1526-1178
Technical assistance: Claude (Anthropic)

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Abstract

For an irrational α let $B(\alpha) = \limsup_n a_n(\alpha)$ be the upper limit of its partial quotients. Vitorino and Vukusic proved $\sup_{k\geq 0} B(2^k\alpha) \geq 5$ for every irrational α and asked, as their Problem 5, for an improvement. We prove $\sup_{k\geq 0} B(2^k\alpha) \geq 12$.

The proof is a finite computation organised around an invariant $N(C)$: the least number of doubling levels at which the recurrent kernel of a synchronous product of Raney transducers becomes a finite union of pure cycles. We show that $N(C)$ is intrinsic, that $N(C) < \infty$ for every C would imply the 2-adic Littlewood conjecture, and that $N(C) = \infty$ is equivalent to a rigidity failure; we compute $N(C) = 1, 1, 3, 5, 5, 8, 15, 21, 29, 30, 31$ for $C = 1, \ldots, 11$. In the language of Badziahin's set **LMad** this settles a level-bounded form of his Problem B for $C \leq 11$. The cycles which survive are ideal classes of discriminant $D \equiv 1 \pmod 8$ whose whole $\mathfrak{p}_2$-orbit has partial quotients at most C; we list them, and check the lists by an independent arithmetic route, in both directions for $D \leq 10^5$.

The result concerns the upper limit B only. On the side of the supremum M it gives $\sup_k M(2^k\alpha) \geq 12$, which is weaker than what is known: Vitorino and Vukusic prove 15, and Badziahin's bound $1/25$ yields 24.

Two self-contained Python files, without external dependency, reproduce the table and the lists of discriminants; the table up to $C = 8$ takes about fourteen seconds.

Status labels

PROVED — a complete mathematical proof is given or cited.
VERIFIED — established by a computation which the ancillary files reproduce.
OPEN — neither.

1 Results

Write $\alpha = [a_0; a_1, a_2, \ldots]$, $B(\alpha) = \limsup_n a_n$ and $M(\alpha) = \sup_{n\geq 1} a_n$. Let

$$\mathcal{X}_C^{(n)} \;=\; \Big\{ w \in \{1, \ldots, C\}^{\mathbb{Z}} \;:\; w \text{ is the two-sided partial-quotient word of a configuration with } B(2^k\alpha) \leq C \text{ for } 0 \leq k \leq n \Big\},$$

a subshift (§3 makes this precise). Define

$$N(C) \;=\; \min\{\, n \;:\; \mathcal{X}_C^{(n)} \text{ is a finite union of periodic orbits} \,\} \;\in\; \mathbb{N} \cup \{\infty\}.$$

Theorem 1 (PROVED, conditional on the VERIFIED computation of §6)**.** *For every irrational real number α,*

$$\sup_{k\geq 0} B(2^k\alpha) \;\geq\; 12.$$

Proposition 2 (PROVED)**.** *If $N(C) < \infty$, then $\sup_{k\geq 0} B(2^k\alpha) \geq C+1$ for every irrational α. Hence, if $N(C) < \infty$ for every C, then $\sup_{k\geq 0} B(2^k\alpha) = \infty$ for every irrational α, and since $B \leq M$ the 2-adic Littlewood conjecture follows.*

Lemma 3 (PROVED)**.** *Let G be a finite directed graph, labelled and right-resolving, in which every vertex has in-degree and out-degree at least one. Then all out-degrees equal 1 if and only if the set of bi-infinite label sequences of paths is a finite union of periodic orbits. Consequently $N(C)$ depends only on the subshifts $\mathcal{X}_C^{(n)}$ and not on the presentation used to compute them.*

Lemma 4 (PROVED)**.** *$N(C) = \infty$ if and only if there exist two distinct bi-infinite words admissible at* every *level which agree on their entire left-infinite past. Equivalently, $N(C) < \infty$ is a rigidity statement: in the admissible set, the past determines the future.*

The values obtained (VERIFIED, §6) are

C	1	2	3	4	5	6	7	8	9	10	11
$N(C)$	1	1	3	5	5	8	15	21	29	30	31

and the finiteness of $N(C)$ for $C \geq 12$ is **OPEN**.

2 Setting

The conjecture of de Mathan and Teulié [6] asks whether $\inf_{q\geq 1} q\cdot\|q\alpha\|\cdot|q|_p = 0$ for every real α and every prime p. For $p=2$ it is equivalent to $\sup_{k\geq 0} M(2^k\alpha) = \infty$ for every irrational α (see [9], §2, and Lemma 14 below). Vitorino and Vukusic [9] prove $\sup_k M(2^k\alpha) \geq 15$ (their Theorem 2), isolate the stronger statement obtained by replacing M with the upper limit B, prove $\sup_{k\geq 0} B(2^k\alpha) \geq 5$ (their Theorem 6), and ask for an improvement of this bound (their Problem 5). Their Theorem 4 shows why odd values are hard to pass: for every odd $m \geq 3$ and every K there is an α with $B(2^k\alpha) = m$ for $0 \leq k \leq K$.

On the side of M, Badziahin [3] proved $\liminf_{q\to\infty} q\,|q|_2\,\|qx\| < 1/25$ for every real x; by Lemma 14 this gives $\sup_k M(2^k x) \geq 24$. Nothing here improves on that side; Corollary 15 recovers only $\sup_k M(2^k x) \geq 12$, by an independent route. For recent progress on related inhomogeneous questions see [1].

Throughout, $\mathrm{L} = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\mathrm{R} = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, so that $\mathrm{L}^{a_0}\mathrm{R}^{a_1}\mathrm{L}^{a_2}\cdots$ represents $[a_0; a_1, a_2, \dots]$; the partial quotients are the run lengths of the L/R word, and the condition "all partial quotients $\leq C$" reads "no run longer than C".

3 The transducer model

For $e \in \mathbb{Z}$ let $\varphi_e(x) = 2^e x$, with matrix $D_e = \mathrm{diag}(2^e, 1)$ (projectively). Reading the input word U (a product of L's and R's), the transducer of φ_e in the sense of Raney [8] is in the state M_e determined by

$$D_e\,U \;=\; W_e\,M_e, \tag{1}$$

where W_e is the word emitted so far and M_e is *reduced*: neither $\mathrm{L}^{-1}M_e$ nor $\mathrm{R}^{-1}M_e$ is non-negative. The set of reduced matrices reachable from D_e is finite; its cardinality for $e = 1, \dots, 8$ is 4, 12, 35, 99, 269, 707, 1809, 4523, and the same for $e = -1, \dots, -8$.

The *level-n automaton* reads one partial quotient $a \in \{1, \dots, C\}$ at a time (equivalently one run X^a, with X alternating), drives all the transducers $\varphi_0, \varphi_1, \dots, \varphi_n$ synchronously from the

same input, and carries, for each of them, the letter and the length of the output run currently in progress; a transition is forbidden as soon as one such run exceeds C. Its state is therefore

$$(\text{parity};\ (M_e,\ \mathrm{pl}_e,\ \mathrm{pr}_e)_{0\le e\le n}), \qquad \mathrm{pr}_e \le C.$$

Since B is an upper limit, only finitely many partial quotients of each $2^k\alpha$ may exceed C; the prefix of the word is therefore *free*, and the admissible vertex set is the whole of the joint closure, with the run counters reset at the point of entry. Let K_n be the kernel obtained from the level-n graph by bidirectional pruning (repeatedly deleting vertices of in-degree or out-degree zero). The bi-infinite paths of K_n are exactly $\mathcal{X}_C^{(n)}$.

Remark 5. Every relaxation used below *enlarges* the vertex set. Hence every computed kernel contains the true one, and "the computed kernel is a finite union of pure cycles" implies the same for the true kernel. All conclusions are therefore sound.

4 Two structural lemmas

Proof of Lemma 3. Suppose some vertex s has two outgoing edges; by right-resolvingness their labels $a \neq b$ differ. By pruning there is a left-infinite path into s and, from each successor, a right-infinite path; this produces two bi-infinite words with the same past and different futures. Two distinct periodic points cannot do this: if x has period p, y has period q and they share a left-infinite tail, that tail has period $\gcd(p, q) = d$, whence both x and y equal the d-periodic extension of it. Conversely, if all out-degrees are 1, a bi-infinite path must return to some vertex when followed backwards; from that vertex the forward path is unique, so the path is that cycle. The last assertion follows because the criterion has been expressed purely in terms of the set of bi-infinite words. □

Proof of Lemma 4. Let $Y_n \subset \mathcal{X}_C^{(n)} \times \mathcal{X}_C^{(n)}$ be the set of pairs sharing their past and differing at position 1. By Lemma 3, $Y_n \neq \emptyset$ if and only if K_n is not a union of pure cycles. The Y_n are closed, decreasing and live in a compact space; hence $\bigcap_n Y_n \neq \emptyset$ as soon as every $Y_n \neq \emptyset$, and $\bigcap_n Y_n$ is precisely the set of such pairs inside $\mathcal{X}_C = \bigcap_n \mathcal{X}_C^{(n)}$. □

Proof of Proposition 2. Assume $\sup_{k\ge 0} B(2^k\alpha) \le C$ and put $n = N(C)$. After a finite prefix, all output runs of the transducers $\varphi_0, \ldots, \varphi_n$ have length at most C, so the trajectory of α is a right-infinite path in the level-n graph; the vertices it visits infinitely often span a subgraph with all degrees ≥ 1, hence lie in K_n. Since K_n consists of pure cycles, the partial-quotient word of α is eventually periodic and α is a quadratic irrational. Corollary 6 below then gives $B(2^k\alpha) \to \infty$, a contradiction. (Alternatively: quadratic irrationals satisfy the 2-adic Littlewood conjecture, [9], §2, after [6].) □

Corollary 6 (PROVED; immediate from [5], Theorem 3.3)**.** *If the word $a_{m+1}a_{m+2}\cdots$ is recurrent, then, with q_m the denominator of $[0; a_1, \ldots, a_m]$,*

$$B(n\alpha) \ \ge\ \frac{n}{8\,q_m^2} - 2 \qquad (n \ge 1).$$

In particular a recurrent continued fraction forces $\sup_k B(2^k\alpha) = \infty$.

Proof. Theorem 3.3 of [5] gives $c(n\alpha) \le 8q_m^2/n$ for the Lagrange constant $c(\beta) = \liminf_q q\|q\beta\|$. By inequality (3) of [9], $c(\beta) \ge 1/(B(\beta) + 2)$. Combining gives the stated bound. □

Remark 7. No priority is claimed for Corollary 6. The authors of [9] observe (§2, inequality (3) and the discussion of their Problem 2, taken from [5]) that $\lim_n c(n\alpha) = 0$ would imply their conjecture on B; Theorem 3.3 of [5] supplies that limit for recurrent continued fractions. The corollary is the concatenation of the two.

5 The algorithm

The joint closure of the transducers $\varphi_1, \ldots, \varphi_n$ (all joint states reachable from $(D_1, \ldots, D_n)$ by reading single letters) grows by a factor of about 2.5 per level; measured sizes, which play no role in the proof, are 4, 12, 36, 106, 306, 858, 2346, 6266, 16386, 42018, 105922 and 263026 states for $n = 1, \ldots, 12$. Three observations avoid it.

(i) Free prefix. Because B is an upper limit, the head of the word is unconstrained. The admissible vertex set is therefore the entire joint closure, and joint reachability from a distinguished initial state is no longer required.

(ii) Kernel recursion (PROVED). Let π be the projection forgetting the last transducer. Then $\pi(K_n) \subseteq K_{n-1}$, and

$$K_n \;=\; \mathrm{kernel}\Big(\text{ graph induced on } \pi^{-1}(K_{n-1}) \Big).$$

Indeed the kernel of any subgraph containing K_n is contained in $\mathrm{kernel}(G_n) = K_n$, while K_n itself has all degrees ≥ 1 inside that subgraph. Only $|K_{n-1}| \times (\text{fibre}) \times 2(C+1)$ candidates are ever formed.

(iii) Algebraic generation of the fibre (PROVED). For $e < e'$, applying (1) to $D_{e'} = D_{e'-e} D_e$ and reducing gives

$$\boxed{M_{e'} \;=\; \mathrm{red}(N \cdot M_e)}, \qquad N \text{ a reduced state of the transducer of } 2^{\,e'-e},$$

because the factorisation of a non-negative integer matrix of non-zero determinant as a word in L, R times a reduced matrix is unique. Taking $e' = e + 1$ (four choices of N) and filtering with $e' = e + 2$ (twelve choices) generates a superset of the fibre directly. No closure of any transducer beyond those of 2 and 4 is enumerated; the states are the matrices themselves, reduced by integer division on whole runs rather than by unit subtractions.

With exponents $0, 1, \ldots, n$ — which is literally the range in $\sup_{k \geq 0}$ — the ancillary file `VERIF_pLC_B12.py` (pure Python, one process) runs the chains for $C \leq 8$ in about 14 s, $C \leq 9$ in about 1.5 min, $C \leq 10$ in about 6 min and $C \leq 11$ in about 33 min, with a peak of 31 404 568 candidates and less than 4 GB of memory.

6 Results

C	$N(C)$	window	$\lvert K_{N(C)} \rvert$	discriminants	peak candidates
1	1	2	0	—	24
2	1	2	6	2	40
3	3	4	12	1	294
4	5	6	12	1	972
5	5	6	48	3	2 574
6	8	9	132	4	6 786
7	15	16	824	14	39 660
8	21	22	2 184	26	238 884
9	29	30	7 524	53	1 120 430
10	30	31	13 816	80	4 895 268
11	31	32	24 408	120	31 404 568

Here "window" is the number of exponents $0, \dots, N(C)$, $|K_{N(C)}|$ the number of states of the final kernel, and "discriminants" the number of distinct discriminants of its cycles. The discriminants of the cycles at $C = 2$ are 8 and 12 (the classes of $\sqrt{2}$ and $\sqrt{3}$); at $C = 3, 4$ the only one is 17; at $C = 5$ they are $17, 33, 41$; at $C = 6$ the list adds 1241; at $C = 7$ it is $17, 33, 41, 57, 65, 73, 561, 1241, 1649, 2993, 7913$ together with the multiples $4^m \cdot D$ produced by the one-sided indexing ($260 = 4 \cdot 65$, $1040 = 16 \cdot 65$, $11972 = 4 \cdot 2993$). The lists are nested: there are 11, 19, 40, 59 and 92 odd discriminants at $C = 7, 8, 9, 10, 11$ respectively, the largest being $1\,625\,460\,497$ (present from $C = 9$ on). **All odd discriminants found are** $\equiv 1 \pmod 8$, a constraint that no step of the computation enforces; and every even discriminant in the lists for $C \geq 3$ is 4^m times an odd one of the same list.

Theorem 1 follows from $N(11) = 31 < \infty$ and Proposition 2.

7 The arithmetic shape of the obstruction

(PROVED.) Let w be the period of a cycle of K_n and let P be the corresponding word matrix. Reading w returns every transducer to its state, so by (1) the word V_e emitted at level e over one period satisfies $V_e\, M_e = M_e\, P$, i.e. $V_e = M_e P M_e^{-1}$. Hence $\operatorname{tr} V_e = \operatorname{tr} P$: *every level of a cycle carries one and the same discriminant* D. Preserving the discriminant under multiplication by 2 forces 2 to split in the order $\mathcal{O}_D$ and $\mathfrak{p}_2$ to be invertible, that is $D \equiv 1 \pmod 8$ for odd D.

Conversely, given a class of discriminant $D \equiv 1 \pmod 8$, halving while choosing at each step the child of the same discriminant produces, for every K, a β with $D(2^k\beta) = D$ for $0 \leq k \leq K$, the class of $2^k\beta$ being $[\mathfrak{p}_2]^k$ times that of β. Therefore:

Proposition 8. *The obstruction to a B-bound at level C consists exactly of the pairs (discriminant* $D \equiv 1 \pmod 8$*, ideal class) whose* entire $\mathfrak{p}_2$-orbit in $\mathrm{Pic}(\mathcal{O}_D)$ *has maximal partial quotient* $\leq C$*. Problem 3 of [9] — characterising the classes which contain an element with* $\alpha \sim \alpha/2 \sim (\alpha+1)/2$ *— is the sub-case* $\mathrm{ord}([\mathfrak{p}_2]) = 1$.

Remark 9 (attribution)**.** Neither the framework nor the structural statement is new, and neither is claimed here. Badziahin [4] introduced the set **LMad** of pairs $(w, \mathbf{x}_p)$ of a limit word of a putative counterexample and a p-adic coordinate; under the standard dictionary this is the object called $\mathcal{X}_C$ here, the p-adic coordinate playing the role of the tower of levels. His Theorem 1 states that for a periodic w of minimal period l one has $(w, \mathbf{x}_p) \in$ **LMad** *if and only if* $\mathbf{x}_p$ is an eigenvector of $A_{w_l}^T$; over $\mathbb{Q}_2$ such an eigenvector exists exactly when the discriminant is a square in $\mathbb{Q}_2$, that is $D \equiv 1 \pmod 8$ for odd D. That is the structural content of Proposition 8.

The mechanism behind it is not new either. Remark 4.6 and Definition 4.7 of [2] relate the splitting of p to the existence of a degenerate generalized branch, and Theorem 4.8(3) of [2] shows that along a degenerate generalized branch there are admissible radii along which the orbit lengths stay bounded. Section 8 of [9] reports a communication of Aka along these lines: for every K there are quadratic irrationals β such that $\beta, 2\beta, \dots, 2^K\beta$ all have the same periodic part. What is added here is the identification of the wider family as the obstruction to *their Problem 5*, together with the explicit lists and witnesses below.

Remark 10. For a *fixed* quadratic α, Theorem 1.2 of [2] shows that the period of $2^n\alpha$ has length $c\,2^n + O(2^{15n/16})$. This is why the purely periodic representative of a cycle is never a witness: one must move to another representative at each rung, which is exactly the descending chain. Corollary 6 gives the same conclusion quantitatively, $B(2^k\alpha) \geq 2^k/8 - 2$ for a purely periodic α.

The family is strictly larger than the one described by Problem 3. At $C = 8$, of the nineteen odd discriminants of Section 6, **six have** $\mathrm{ord}([\mathfrak{p}_2]) > 1$ (VERIFIED, order measured as the

period of the class sequence along the chain):

$$65 \text{ (order 2)}, \quad 697 \text{ (3)}, \quad 2993 \text{ (3)}, \quad 4097 \text{ (5)}, \quad 4257 \text{ (3)}, \quad 9425 \text{ (2)},$$

so a full solution of Problem 3 would account for only thirteen of them. The class, not merely the discriminant, matters: the principal class of $D = 561$ has maximal partial quotient 23, while the other class of that discriminant has maximal partial quotient 7, and it is the latter that appears at $C = 7$.

Independent arithmetic verification (VERIFIED). `CHAINES_pLC_B12.py`, the second ancillary file, never touches the automaton. For a primitive form (a, b, c) of discriminant D with root $\beta = (-b+\sqrt{D})/(2a)$, the numbers $\beta/2$ and $(\beta+1)/2$ are roots of the forms $(4a, 2b, c)$ and $(4a, 2b-4a, a-b+c)$ divided by their contents. Since 2 splits, exactly two of $\beta/2$, $(\beta+1)/2$, 2β have discriminant D; the *descending chain* keeps a child of discriminant D, which is unique after the first step because the parent is then one of the two, so that the chain runs through the $\mathfrak{p}_2$-orbit of the class in one direction. The file computes, for every non-square $D \equiv 1 \pmod 8$ with $D \le 10^5$ and every reduced primitive form of discriminant D, the maximum of the periodic partial quotients along sixty halvings, and the least such maximum $m(D)$. The resulting lists $\{D \le 10^5 : \ m(D) \le C\}$ coincide, as sets, with the odd discriminants $\le 10^5$ of the kernel lists, for every $C \le 11$. This is a check in both directions below 10^5. On one side, every discriminant listed by the automaton is carried by an explicit quadratic β with $\sup_{0\le k\le 60} B(2^k\beta) \le C$. On the other, a class whose whole orbit has partial quotients at most C has $m(D) \le C$, so the automaton misses no obstructing class of discriminant at most 10^5. For the nineteen odd discriminants above 10^5 in the lists, the file starts the chain from the class of a cycle word of the kernel (written into the file), and checks that the chain closes and that its maximum is exactly the least C at which the discriminant appears.

Cross-checks against [9]. Their Remark 6 names $[\overline{5; 2, 1, 2}]$, root of $x^2 - 5x - 2$, and $[\overline{5; 1, 2, 2, 1}]$, root of $x^2 - 5x - 4$: discriminants 33 and 41, which together with 17 are exactly the list produced at $C = 5$. Their Lemma 12 concerns the positive root of $x^2 - mx - 2$ for odd $m \ge 3$, of discriminant $m^2 + 8$; the member $m = 7$, $D = 57$, first appears at $C = 7$, $m = 9$, $D = 89$, first appears at $C = 9$, and $m = 11$, $D = 129$, first appears at $C = 11$ — in each case precisely at $C = m$, in agreement with their Theorem 4.

8 Validation

- **Transducers, on the object.** On 60 random words of length 400 over $\{1, \dots, 9\}$ and the eight maps $2^{\pm j}x$, $j \le 4$, the run lengths of the emitted word coincide with the exact continued fraction of $\varphi(x)$, apart from the last four runs, which are affected by the truncation of the input: 480/480.
- **Individual state counts.** $4, 12, 35, 99, 269, 707, 1809, 4523$ for $2^j x$ and the same for $x/2^j$, $j = 1, \dots, 8$; the first six are an auto-test of the ancillary file.
- **Published results of [9], small C.** Lemma 13 ($C = 1$: empty kernel); Theorem 5 ($C = 2$: exactly the classes of $\sqrt{2}$ and $\sqrt{3}$, discriminants 8 and 12); Lemmas 15 and 16 ($C = 3$ and $C = 4$: only the class of $[\overline{3; 1, 1}]$, $D = 17$); Remark 6 ($C = 5$: exactly $D = 17, 33, 41$).
- **Explicit witnesses**, in rational arithmetic with 700 digits of $\sqrt{D}$, of the chains whose existence Lemma 10 of [9] asserts: $\beta = (26513+\sqrt{33})/32768$ gives $\sup_{0\le k\le 12} B(2^k\beta) = 5$, and $\beta = (224485+\sqrt{1241})/557056$ gives $\sup_{0\le k\le 10} B(2^k\beta) = 6$. Their continued-fraction

periods, $[5,2,1,2]$ and $[1,4,3,3,4,1,2,1,2,2,6,1,1,1,1,1,6,2,2,1,2]$, are cycle words of the computed kernels, up to rotation.

- **Arithmetic chains** (Section 7): two-sided agreement for all $D \le 10^5$ and $C \le 11$; closure with the predicted maximum for every larger odd discriminant.
- **Congruence check.** All odd discriminants found are $\equiv 1 \pmod 8$, a constraint that no step of the computation enforces.

What this does not establish. The values $N(C)$ rest on one program, the first ancillary file. The arithmetic check is disjoint from it and two-sided below 10^5, but it only concerns periodic points: the vanishing of the excess, i.e. the absence of non-periodic admissible words at level $N(C)$, is computed by the automaton alone. The ancillary file exists so that an independent check of the table up to $C = 8$ costs a reader fourteen seconds, and of the whole table about half an hour.

9 The finiteness of $N(C)$: what it is, and what is missing

Proposition 11 (PROVED)**.** *$N(C) < \infty$ if and only if $\mathcal{X}_C := \bigcap_n \mathcal{X}_C^{(n)}$ is a finite union of periodic orbits.*

Proof. One direction is trivial. Conversely, suppose $\mathcal{X}_C$ is a finite union of periodic orbits of period at most p, and set $m = 2p+1$. For each m the finite sets $\mathcal{L}_m(\mathcal{X}_C^{(n)})$ decrease with n, hence stabilise, and their stable value is $\mathcal{L}_m(\mathcal{X}_C)$ by a diagonal extraction. Pick n with $\mathcal{L}_m(\mathcal{X}_C^{(n)}) = \mathcal{L}_m(\mathcal{X}_C)$ and let $x \in \mathcal{X}_C^{(n)}$. Every block $x_{i-p}\cdots x_{i+p}$ occurs in some orbit O_i of $\mathcal{X}_C$. Consecutive blocks overlap in $2p$ letters; two periodic points of period $\le p$ agreeing on $2p$ letters agree everywhere, by Fine and Wilf. Hence O_i is independent of i and x lies in it. So $\mathcal{X}_C^{(n)} = \mathcal{X}_C$, and Lemma 3 gives $N(C) \le n$. □

Nothing is therefore lost or gained at finite level: the question is entirely about the limit set. Under the standard correspondence, a bi-infinite partial-quotient word together with its tower of 2-adic levels is a point of

$$X = G/\Gamma, \qquad G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_2), \qquad \Gamma = \mathrm{SL}_2(\mathbb{Z}[1/2]),$$

the real coordinate carrying the geodesic and the 2-adic coordinate the choice of the chain of index-2 sublattices; and "all partial quotients at all positions and all levels are $\le C$" is "the whole orbit under the diagonal group $\psi(\mathbb{R}\times\mathbb{Z})$ stays in the compact set K_δ" in the notation of [5], §5, for a $\delta = \delta(C) > 0$. In this two-sided picture the upper limit B becomes a supremum; note also that a point of $\mathcal{X}_C$ carries a nontrivial 2-adic coordinate and need not come from a real number. Consequently $N(C) < \infty$ for all C is, a priori, *strictly stronger* than the conjecture on B.

Theorem 12 (PROVED, from [5])**.** *For every C, the language of $\mathcal{X}_C$ grows subexponentially and its topological entropy vanishes. Consequently $h(\mathcal{X}_C^{(n)}) \downarrow 0$.*

Proof. Theorem 4.2 of [5] states, for every $\varepsilon > 0$, that the set of pairs $(u,v) \in [0,1]\times\mathbb{Z}_2$ satisfying its lower bound (4.4) with that ε has box dimension zero; for $\varepsilon = \varepsilon(C)$ the real projection of $\mathcal{X}_C$ is contained in it. Covering it by $e^{o(n)}$ intervals of length $(1+C)^{-2n}$ and comparing with the continued-fraction cylinders of depth n — the argument of [5], proof of Theorem 2.1 — bounds the number of admissible words of length n by $e^{o(n)}$. Finally, for a decreasing sequence of subshifts, $\mathcal{L}_m(\mathcal{X}_C^{(n)}) \downarrow \mathcal{L}_m(\mathcal{X}_C)$ for every m, whence $h(\mathcal{X}_C^{(n)}) \to h(\mathcal{X}_C) = 0$. □

Corollary 13 (PROVED)**.** *If some strongly connected component of K_n is not a simple cycle, let N_n be the number of states of the smallest such component; then $h(\mathcal{X}_C^{(n)}) \geq (\log 2)/N_n$. Hence, if $N(C) = \infty$, the branching components of K_n must grow without bound.*

Proof. A strongly connected component which is not a simple cycle has a vertex with two distinct return loops, of lengths at most N_n; freely concatenating them produces at least $2^{\lfloor L/N_n \rfloor}$ distinct closed walks of length at most L, and by right-resolvingness distinct label sequences, so the entropy is at least $(\log 2)/N_n$. The second assertion follows from Theorem 12. □

Measurements (VERIFIED). Let e_n be the number of edges of K_n minus its number of states (the "excess", printed by the ancillary file). It rises, then falls strictly to zero:

$C = 3 : 18, 4, 0$
$C = 4 : 44, 42, 12, 4, 0$
$C = 5 : 84, 146, 74, 16, 0$
$C = 6 : 132, 276, 288, 232, 164, 68, 8, 0$
$C = 7 : 194, 452, 780, 838, 1052, 986, 752, 608, 360, 216, 114, 48, 24, 8, 0$
$C = 8 : 264, \ldots, 3818, 3704, 3284, 2648, 2002, 1000, 564, 296, 134, 66, 40, 16, 8, 0$
$C = 9 : 348, \ldots, 11258, 10434, 8732, 6418, 4216, 3016, 2066, 1490, 1062, 654, 364, 168, 58, 24, 16, 8, 0$
$C = 10 : 440, \ldots, 40446, 40246, 38436, 33722, 25808, 15738, 8352, 3896, 1960, 894, 378, 134, 64, 36, 16, 4, 0$
$C = 11 : 546, \ldots, 205830, 199514, 176416, 134956, 84334, 38506, 10948, 2622, 454, 92, 44, 14, 6, 0$

Problem B of [4]. The question of this section is not new. In the notation recalled above, "every element of **LMad** has periodic w" is Problem B of [4], reformulated there as Problem B′: *are there infinite closed subsets of* **LMad** *which are minimal invariant?* A positive answer to Problem B implies pLC. Proposition 11 and the two formulations below are the effective, two-sided restatement of that question; the partial results towards it in [4] include his Theorem 2 (non-periodic words of complexity $P(w, n) \leq n + c$ are excluded) and his Theorem 3 (limit words of certain recursive concatenations are excluded). Badziahin also notes that he does not know whether the criteria of his Theorems 5 and 6 rule out every infinite word, which would imply pLC, and that this seems not to be the case. The computation below is therefore complementary to those results, not subsumed by them.

What is added here is effectivity. The invariant $N(C)$ makes the question decidable level by level, and its computation for $C \leq 11$ settles the *level-bounded* analogue of Problem B: *over the alphabet $\{1, \ldots, 11\}$, every bi-infinite word all of whose levels have partial quotients at most 11 is periodic*, with the finite list of periods produced explicitly.

Lemma 14 (PROVED)**.** *For every irrational β,*

$$\inf_{q \geq 1} q\,|q|_2\,\|q\beta\| \;=\; \frac{1}{\sup_{k \geq 0} \Lambda(2^k\beta)}, \qquad \Lambda(\gamma) = \sup_{n \geq 0}\big(a_{n+1} + [0; a_{n+2}, \ldots] + [0; a_n, \ldots, a_1]\big),$$

and $M(\gamma) \leq \Lambda(\gamma) \leq M(\gamma) + 2$.

Proof. Write $q = 2^k m$ with m odd; then $q|q|_2\|q\beta\| = m\|m\,(2^k\beta)\|$. For fixed k, the infimum of $m\|m\gamma\|$ over all $m \geq 1$ is the infimum over the convergent denominators (Legendre's theorem, since this infimum is $< 1/2$), and equals $1/\Lambda(\gamma)$, because $q_n\|q_n\gamma\| = \big(a_{n+1} + [0; a_{n+2}, \ldots] + [0; a_n, \ldots, a_1]\big)^{-1}$. If $m = 2m'$ is even, $m\|m\gamma\| = 2\,m'\|m'(2\gamma)\|$ exceeds a value available at level $k + 1$; so the global infimum is the infimum over k of $1/\Lambda(2^k\beta)$. The bracketing of Λ by M is immediate. □

Corollary 15 (PROVED)**.** *For every irrational x, $\sup_{k \geq 0} M(2^k x) \geq 12$, hence $\inf_{q \geq 1} q\,|q|_2\,\|qx\| \leq \frac{1}{12}$.*

Proof. $M \geq B$ and Theorem 1 give $\sup_k M(2^k x) \geq 12$; then $\sup_k \Lambda(2^k x) \geq 12$, and Lemma 14 concludes. □

This M-side statement is weaker than Theorem 2 of [9] and than what Badziahin's interval-subdivision algorithm [3] yields, but it is obtained by an entirely different route and so serves as a cross-check on the whole construction.

The asymmetry. Proving $N(C) < \infty$ for every C would prove the conjecture on B, hence the 2-adic Littlewood conjecture. Refuting it would prove nothing: since a point of $\mathcal{X}_C$ carries a nontrivial 2-adic coordinate and need not come from a real number, $N(C) = \infty$ is compatible with the conjecture and would only close this route. No finite computation can therefore decide the question, in either direction.

Proposition 16 (conditional; sketch)**.** *Assume (i) Margulis' conjecture that every bounded orbit of a maximal split torus of rank at least two on G/Γ is closed, and (ii) that closed orbits of unbounded volume cannot all remain inside a fixed compact set. Then $N(C) < \infty$ for every C.*

Sketch. By (i), the set of points of X corresponding to $\mathcal{X}_C$ is a union of closed ψ-orbits. If there were infinitely many, their volumes would be unbounded by discreteness, contradicting (ii); so $\mathcal{X}_C$ is a finite union of periodic orbits, and Proposition 11 applies. Statement (ii) is in the direction of the equidistribution results of Einsiedler, Lindenstrauss, Michel and Venkatesh, and is itself open in the generality needed. □

What is missing, exactly. Two equivalent formulations of the single remaining gap.

- **Dynamical.** Show that the set of points of X corresponding to $\mathcal{X}_C$ is a finite union of closed ψ-orbits. Box dimension zero is the state of the art [7, 5]; passing from dimension zero to finiteness is a case of the conjecture that every bounded orbit of a higher-rank diagonal action is closed.
- **Combinatorial.** Since e_n is a non-negative integer, it would suffice to prove that $e_{n+1} < e_n$ whenever $e_n > 0$ and $n \geq n_0(C)$. This holds at every level past the peak for every $C = 3, \ldots, 11$, and is open in general.

Finally, a by-product of the computation. For $C \leq 11$, $N(C) < \infty$ is established, so by Proposition 8 the set of ideal classes whose $\mathfrak{p}_2$-orbit has maximal partial quotient $\leq C$ is *finite* and contained in the explicit list of cycles of $K_{N(C)}$: $3, 4, 11, 19, 40, 59, 92$ odd discriminants for $C = 5, \ldots, 11$. Whether this set is finite for every C is **OPEN**.

Two remarks. First, a naive probabilistic count of the obstructing classes is misleading: classes of discriminant $\leq X$ with partial quotients at most C number about X^{δ_C}, and if $\mathfrak{p}_2$ were principal with "probability" $X^{-1/2}$ one would predict $X^{\delta_C - 1/2} \to \infty$ such classes already at $C = 5$, where in fact there are exactly three ($D = 17, 33, 41$). Second, at $C = 11$ the orders of $[\mathfrak{p}_2]$ along the ninety-two obstructing discriminants are $1, 2, 3, 4, 5, 6$ with multiplicities $56, 16, 11, 6, 2, 1$ (measured with the functions of the second ancillary file, which itself checks the orders only at $C = 8$): the order is always small, but no bound $\mathrm{ord}([\mathfrak{p}_2]) \leq f(C)$ is proved here.

10 What is not claimed

- **OPEN.** The finiteness of $N(C)$ for $C \geq 12$, and in general — the level-free form of Problem B of [4]. Section 9 states the asymmetry which makes the question undecidable by computation, gives two equivalent formulations of the gap, and a conditional proof.

- The branch "the kernel retains entropy in the limit" does not occur: Theorem 12.
- Nothing new is claimed on the M side, where the best bound remains the one derived from [3].
- The cycles listed are the periodic orbits of the kernel; the purely periodic representative of such an orbit is never itself a witness, and Corollary 6 explains why.

Ancillary files. `anc/VERIF_pLC_B12.py` — self-contained, pure Python, no external dependency, no intermediate file, parameters fixed in the header; reproduces the table for $C \leq 8$ in about fourteen seconds, and the whole table for $C \leq 11$ with `MAX_C = 11`; also replays the two explicit witnesses of the Validation section. `anc/CHAINES_pLC_B12.py` — same conventions; the arithmetic verification of Section 7, in about three minutes.

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