Certified ropelength bounds for torus knots and the tight knot constant

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Abstract

The tight knot constant α_0 is the largest constant with $L(K) \geq \alpha_0 \operatorname{Cr}(K)^{3/4}$ for every knot K, where L is the ropelength (unit-radius rope) and Cr the crossing number. We exhibit an explicit $C^{1,1}$ embedding of the torus knot $T(19,24)$, with $\operatorname{Cr} = 432$, of thickness at least 1 and length at most 939.984. Hence $\alpha_0 \leq 9.9200$. The published bound is 10.76. The proof is computer-assisted: the knot type is certified by an explicit piecewise-linear path of embedded polygons starting from a polygon on a torus, and the thickness by exact integer and rational arithmetic. Two separately written verifiers and the certificate are supplied. Bounds for $T(14,17)$ and $T(11,14)$, obtained in the same way, are also reported.

1 Introduction

For a closed $C^{1,1}$ curve γ of thickness $\Delta(\gamma)$ (the infimum of the radii of circles through three distinct points of γ), the ropelength of a knot type K is $L(K) = \inf\{\text{length}(\gamma)/\Delta(\gamma)\}$ over curves γ of type K [4]. Buck and Simon [3] proved $L(K) \geq c \operatorname{Cr}(K)^{3/4}$ for an absolute $c > 0$, and the exponent is sharp [4]. Let α_0 be the best constant (entry 22a of [9]). The smallest published upper bound is $\alpha_0 \leq 10.76$, from a numerically tightened $T(25,26)$ [5]. The value 10.02 for $T(24,19)$ appears in [9], attributed to [6] (where it is credited to an earlier online post of the author) and marked there as not yet verified; no coordinates are available.

Theorem 1. *There is a $C^{1,1}$ curve of knot type $T(19,24)$ with thickness ≥ 1 and length ≤ 939.984. Consequently $L(T(19,24)) \leq 939.984$ and, since $\operatorname{Cr}(T(19,24)) = 432$,*

$$\alpha_0 \;\leq\; \frac{939.984}{432^{3/4}} \;<\; 9.9200.$$

In the same way $L(T(14,17)) \leq 584.481$ and $L(T(11,14)) \leq 416.652$ (their certificates are not part of this deposit, because of its size, and are available from the author).

The crossing number of the torus knot $T(p,q)$ is $\min\{p(q-1), q(p-1)\}$ [8]. Each curve is obtained from an explicit polygon F by rounding its corners; Sections 2–3 prove that it has the required type, thickness and length. All statements below refer to one pair (p,q) at a time.

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2 Knot type

The starting polygon. Let Σ be the polyhedral torus with vertices $V(a,b) = \big((R+rc_b)c_a,\ (R+rc_b)s_a,\ rs_b\big)$, $a,b \in \mathbb{Z}/4$, where $(c_k, s_k) = (\cos\frac{k\pi}{2}, \sin\frac{k\pi}{2})$, $R = 2q+6$, $r = q+2$, where $2 \le p < q$ are coprime. Since $R > r > 0$, the faces $V(a,b)V(a+1,b)V(a+1,b+1)V(a,b+1)$ are planar convex trapezoids, and Σ is an embedded torus bounding an unknotted solid torus. On the chart $(\alpha,\beta) \in (\mathbb{R}/\mathbb{Z})^2$ of Σ (bilinear on each face, mapping the boundary of each grid square homeomorphically onto the boundary of the face) consider the line $t \mapsto (pt, qt)$. It meets the grid corners only at $t \in \frac{1}{4}\mathbb{Z}$, at four distinct corners. Cutting it at the grid lines $\alpha, \beta \in \frac{1}{4}\mathbb{Z}$ gives points on the boundaries of the faces; two consecutive cut points lie on the boundary of a common face, and P_0 is formed by the straight segments joining them, each lying in that face. Distinct passes of the line through a grid square are parallel chords, so their endpoints do not interleave on the boundary of the square, hence not on the boundary of the face; two straight chords of a convex polygon with non-interleaving endpoints are disjoint. So P_0 is a simple closed curve on Σ, isotopic on Σ (face by face: two systems of disjoint arcs in a disk with the same non-interleaving endpoints are isotopic rel boundary) to the image of the line, i.e. a curve of class (p,q), and therefore a (p,q) torus knot [10, §3.C]. Each edge of P_0 is subdivided into 16 equal parts. All vertices are rational; in the unit $1/U$, with $U = 16pq\kappa$ ($\kappa = 16, 32, 64$ for $T(19,24), T(14,17), T(11,14)$), they are integers. The verifiers recompute P_0 exactly and reject any other choice of R, r, the subdivision or (p,q).

Lemma 2 (linear steps)**.** *Let C, C' be polygons with the same number of vertices, $\delta \ge \max_v |C'_v - C_v|$, and for each $s \in [0,1]$ let $C_s = (1-s)C + sC'$. Suppose (i) every two non-adjacent edges of C are at distance $> 2\delta$, and (ii) at each vertex, with adjacent edges e_0, e and turning angle θ, $2\delta < \min(|e_0|, |e|)\cos(\theta/2)$. Then every C_s is an embedded polygon; in particular C and C' are ambient isotopic.*

Proof. Points of an edge move by at most δ, so non-adjacent edges of C_s stay at positive distance. An edge vector moves by at most $2\delta < |e|$, so its direction turns by less than $\arcsin(2\delta/|e|)$; the turning angle between two adjacent edges therefore stays below $\theta + 2\arcsin(\cos(\theta/2)) = \pi$, and adjacent edges meet only at their common vertex. So every C_s is embedded. By compactness there is $\eta > 0$ such that, for all s, non-adjacent edges of C_s are at distance $> \eta$, edges are longer than η and turning angles are below $\pi - \eta$. Cut $[0,1]$ into intervals on which every vertex moves by less than ε, and on each of them move the vertices one at a time. For ε small in terms of η, the two triangles swept by one such move (moving v to v' between its neighbours u, w: the triangles uvv' and $v'vw$) meet the polygon only along the two edges being moved, so each move is an elementary (Δ-) move; hence C and C' are equivalent knots [2, Ch. 1]. □

In verifier (i) both conditions are checked exactly, in integers. Write D for the largest squared displacement of a vertex (so $\delta^2 = D$), $d_U = \lfloor\sqrt{D}\rfloor + 1 > \delta$ and $\ell_i = \lfloor\sqrt{|e_i|^2}\rfloor + 1 > |e_i|$. Condition (i) follows from $\operatorname{dist}(e_i, e_j) \ge |m_i - m_j| - \frac{1}{2}(|e_i| + |e_j|)$ (m the midpoints) when $4|m_i - m_j|^2 > (4d_U + \ell_i + \ell_j)^2$, and otherwise from the exact rational distance between the two segments; pairs whose midpoints are farther apart are excluded by an exact integer grid. For (ii), with $n_0 = |e_0|^2$, $n = |e|^2$, $m = \min(n_0, n)$ and $P = \lfloor\sqrt{n_0 n}\rfloor$, the condition $4\delta^2 < m(1+\cos\theta)/2$ is implied by the integer inequality $8D(P+1) < m(P + e_0 \cdot e)$. Coordinates are asserted to be below 2^{24}.

The certificate lists integer polygons $X_0 = P_0, X_1, \ldots, X_K$ (the states of a tightening process) and the final polygon F. In the certificate for $T(19,24)$ every edge of X_K is then split at its midpoint, which leaves the polygon unchanged and is exact in the unit $1/(2U)$, and the path continues in that unit. Each step $X_k \to X_{k+1}$ (and $X_K \to F$) is subdivided by the integer points $X_k + \lfloor j(X_{k+1} - X_k)/m \rfloor$, and every sub-step satisfies Lemma 2. Hence F is a (p,q) torus knot.

3 Thickness and length

Let F have vertices v_i, edges $e_i = v_{i+1} - v_i$, turning angles $\theta_i < \pi/2$. Put $t_i = \tan(\theta_i/2)$ and suppose $\mathrm{MinRad}(v_i) = \min(|e_{i-1}|, |e_i|)/(2t_i) \geq 1$ for all i. Replace the corner at v_i by the arc of the unit circle tangent to e_{i-1} and e_i; the tangency points lie at distance $t_i \leq \frac{1}{2}\min(|e_{i-1}|, |e_i|)$ from v_i, so the arcs do not overlap. The result is a $C^{1,1}$ curve γ with curvature ≤ 1 and $\mathrm{length}(\gamma) \leq \mathrm{length}(F)$. Every point of the arc at v_i is within $\tau_i = \sec(\theta_i/2) - 1 \leq t_i^2/2$ of the two half-edges at v_i (since $\sec x - 1 = \tan^2 x/(\sec x + 1)$).

Lemma 3. *Let γ be a $C^{1,1}$ curve parametrized by arclength with curvature ≤ 1, and let $s < t$ with $t - s < \pi$. Then $\langle \gamma'(s), \gamma(t) - \gamma(s)\rangle > 0$. In particular a pair of points at which the chord is orthogonal to both tangents is separated by arc length $\geq \pi$ in both directions.*

Proof. The unit tangent moves on the sphere with speed ≤ 1, so the angle between $\gamma'(s)$ and $\gamma'(x)$ is at most $x - s$, and $\langle \gamma'(s), \gamma(t) - \gamma(s)\rangle = \int_s^t \langle \gamma'(s), \gamma'(x)\rangle\, dx \geq \int_0^{t-s} \cos y\, dy = \sin(t-s) > 0$. □

For a $C^{1,1}$ closed curve the thickness equals the minimum of the minimal radius of curvature and half the minimal distance between doubly critical pairs (pairs at which the chord is orthogonal to both tangents) [4, Lemma 1]; see also [7]. By Lemma 3, $\Delta(\gamma) \geq 1$ as soon as every two points of γ at arc distance $\geq \pi$ (in both directions) are at distance ≥ 2.

Proposition 4. *Suppose that every edge is shorter than π, that $\mathrm{MinRad}(v_i) \geq 1$ for all i, and that $\mathrm{dist}(e_i, e_j) \geq 2 + \bar\tau_i + \bar\tau_j$ for every pair of distinct edges containing points at polygonal arc distance $\geq \pi$ in both directions, where $\bar\tau_i = \max(\tau_i, \tau_{i+1})$. Then $\Delta(\gamma) \geq 1$.*

Proof. Let $x, y \in \gamma$ be at arc distance $\geq \pi$ along γ in both directions. If x lies on a straight part of γ, put $x^* = x$. If x lies on the arc at v_i, at angle φ from the nearer tangency point, let x^* be its radial projection from the centre of the arc onto the half-edge on that side; then $|x - x^*| = \sec\varphi - 1 \leq \tau_i$, and the piece of arc of length φ is replaced by a segment of length $\tan\varphi \geq \varphi$. Define y^* in the same way. Then $x^* \in e_i$, $y^* \in e_j$ for edges with $|x - x^*| \leq \bar\tau_i$, $|y - y^*| \leq \bar\tau_j$. Between x^* and y^*, each complete arc of γ is replaced by its two tangent segments, which are longer, and the partial arcs at the ends are replaced by longer paths (through the nearer tangency point an arc of length φ becomes a segment of length $\tan\varphi$; through the other one an arc of length $\theta_i - \varphi$ becomes a path of length $2\tan(\theta_i/2) - \tan\varphi$, and $2\tan(\theta_i/2) - \tan\varphi - (\theta_i - \varphi)$ decreases in $\varphi \in [0, \theta_i/2]$ to $\tan(\theta_i/2) - \theta_i/2 \geq 0$); hence the polygonal arc distance between x^* and y^* is at least π in both directions. As edges are shorter than π, the edges e_i and e_j are distinct, and $|x - y| \geq \mathrm{dist}(e_i, e_j) - \bar\tau_i - \bar\tau_j \geq 2$. In particular γ is injective: points at arc distance $< \pi$ are distinct by Lemma 3, and the others are at distance ≥ 2; so γ is an embedded curve, to which [4, Lemma 1] applies. □

The verifiers check the distance condition on larger sets of pairs: verifier (i) uses the threshold $\pi - \frac{2}{5}$ and edges shorter than $\pi - \frac{2}{5}$, verifier (ii) the threshold $\pi - 2\max_i t_i$.

Isotopy of the smoothing. Since $\Delta(\gamma) \geq 1$, the closed normal disks of radius $\frac{1}{2}$ of γ are pairwise disjoint and fill a solid torus N with core γ. On straight parts $F = \gamma$. Near v_i, the two half-edges lie in the plane of the arc, within $\tau_i < \frac{1}{2}$ (in fact $\tau_i \leq 0.003$ here) of it, and the radial projection from the centre of the arc maps them homeomorphically onto the arc; this projection is the nearest-point projection to γ. Hence $F \subset N$ meets each normal disk in exactly one point, and the homotopy moving each point of F along the segment to its projection stays in its own normal disk, so it is an isotopy from F to γ inside N. So γ is a (p, q) torus knot.

Exact checks. With integer coordinates all hypotheses of Proposition 4 reduce to integer inequalities: t_i is bounded above by $(\lfloor\sqrt{|e_{i-1}\times e_i|^2}\rfloor+1)/(\lfloor\sqrt{|e_{i-1}|^2|e_i|^2}\rfloor+e_{i-1}\cdot e_i)$, edge lengths are bracketed by integer square roots, arc distances by sums of these brackets (with π replaced by $333/106<\pi$), and the squared distance between two segments is computed exactly as a rational number.

4 The certificates and their verification

	$T(19,24)$	$T(14,17)$	$T(11,14)$
vertices of F	5376	1920	1536
path steps / certified sub-steps	461 + 49 / 4086 + 98	365 / 2137	189 / 1759
$\max_i t_i$	0.0663	0.0811	0.0676
length$(F)\le$	939.984	584.481	416.652
Cr	432	221	140
length/Cr$^{3/4}\le$	9.9200	10.1971	10.2372

Verification. The certificate for $T(19,24)$ is the set of files `anc/C22a_T19_24_*.csv`: exact integer coordinates, one file per array, the first line giving its shape; the two sequences of polygons are stored as the first polygon followed by the differences between consecutive polygons, and are split into parts `_p01`, `_p02`, ... (files of at most 4 MiB). (i) `anc/verify_C22a.py`, with `certlib.py` and `p0.py` (Python 3 with numpy): every test is an exact integer or rational test, as described above; floating point only chooses how finely a step is subdivided. (ii) `anc/verify2_C22a.py` (numpy, scipy, python-flint), written separately: a KD-tree search for candidate pairs; for the path, double precision with explicit margins and exact rational distances for close pairs; for the thickness, Arb balls [1] for lengths, angles and arc distances, and exact rational segment distances. The two programs share only the definition of P_0 (`p0.py`); they implement the same reductions, Lemma 2 and Proposition 4, so their agreement guards against programming errors, not against an error in these lemmas. Both run without argument, in a few minutes. Verifier (ii) gives the length bounds 939.9716, 584.4727 and 416.6468. On the certificate for $T(19,24)$, both verifiers reject the modified files obtained by shrinking the final polygon by 0.1%, by moving one vertex of one intermediate state (of the first or of the refined part of the path) onto another strand, by exchanging (p,q), and a file for $T(2,3)$ with other torus parameters.

References

[1] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), 1281–1292.

[2] G. Burde, H. Zieschang, *Knots*, de Gruyter, 1985.

[3] G. Buck, J. Simon, *Thickness and crossing number of knots*, Topology Appl. **91** (1999), 245–257.

[4] J. Cantarella, R. B. Kusner, J. M. Sullivan, *On the minimum ropelength of knots and links*, Invent. Math. **150** (2002), 257–286.

[5] A. R. Klotz, M. Maldonado, *The ropelength of complex knots*, J. Phys. A **54** (2021), 445201.

[6] A. R. Klotz, *Tight bounds for tight links: ropelength of $T(Q, Q)$ torus links*, arXiv:2603.02416v2 (13 March 2026).

[7] R. A. Litherland, J. Simon, O. Durumeric, E. Rawdon, *Thickness of knots*, Topology Appl. **91** (1999), 233–244; doi:10.1016/S0166-8641(97)00210-1.

[8] K. Murasugi, *On the braid index of alternating links*, Trans. Amer. Math. Soc. **326** (1991), 237–260.

[9] D. Davis, P. Ivanisvili, T. Tao and contributors, *Optimization constants in mathematics*, entry 22a, `https://teorth.github.io/optimizationproblems/constants/22a.html` (consulted 1 November 2026).

[10] D. Rolfsen, *Knots and Links*, Publish or Perish, 1976.