Illumination of several families of complex convex bodies

Working research draft

1 November 2026

Abstract

We determine ordinary and fractional illumination numbers for complex Hanner bodies and for bodies obtained as convex hulls of coordinate polydiscs indexed by the maximal independent sets of an arbitrary finite simple graph. The formulas are sums over coordinate supports. For Hanner bodies we classify all equality cases in the ordinary and fractional dimension bounds, including the sharp ordinary bound for non-polydiscs. For graph bodies, fractional equality 2^n occurs exactly when the graph is a disjoint union of isolated vertices and edges; a vertex of degree at least two gives the strict bound $3 \cdot 2^{n-2}$. We also prove the fractional bound 2^n for bodies defined by nondecreasing submodular constraints on coordinate moduli. For finite systems of positive weighted inequalities satisfying an explicit condition on tight sets, we obtain the same fractional bound and an ordinary bound of $(2 + o(1))n \log n\, 2^n$, together with an explicit estimate in every dimension. This includes bounded systems on laminar families with arbitrary positive coefficients. We conclude with a conjectural classification of fractional equality for normalized complex 1-unconditional bodies.

Draft status. The proofs included here have been checked internally; external review and literature priority remain outstanding. The final equality classification is explicitly a conjecture and is not used in any proof. The present literature audit does not certify priority for the results.

hexagon:2611.00007v1

1 Introduction

A direction illuminates a boundary point of a convex body if a positive displacement in that direction enters the interior. The ordinary illumination number $\mathrm{ill}(K)$ is the least number of directions illuminating all boundary points. Fractional illumination replaces a finite family by a measure on directions, requiring mass at least one of illuminating directions at each boundary point; the infimum of its total mass is $\mathrm{ill}^*(K)$. All bodies in this paper are considered in their underlying real vector spaces. Complex 1-unconditionality means invariance under independent coordinate rotations.

Naszódi introduced fractional illumination using measures of finite support [10, Definition 1]. We use the Borel-measure formulation of Rotem, Schejter, and Slomka [12, Section 1.2]. Their complex illumination conjectures [12, Conjectures 1.1–1.2] motivate the bounds considered here. The ordinary and fractional polydisc values are published results [12, Theorems 1.3–1.4]; the formulas below extend these values to specified families. The non-polydisc fractional equality examples also require a change to the equality clause of their fractional conjecture, as explained after Corollary 5.

We first study complex Hanner bodies, constructed from the complex unit disc by Cartesian products and ℓ_1 sums on disjoint coordinate blocks. If $\mathcal{S}(H)$ consists of the distinct supports of their extreme points, put $F_H(t) = \sum_{S \in \mathcal{S}(H)} t^{|S|}$. Theorem 2 proves

$$\mathrm{ill}(H) = 2F_H(2) - F_H(1), \qquad \mathrm{ill}^*(H) = F_H(2).$$

The support polynomial multiplies under products and adds under ℓ_1 sums. Corollary 5 gives the ordinary and fractional dimension bounds and all their equality cases within the recursively

defined class. In particular, the polydisc uniquely attains the ordinary maximum $2^{n+1}-1$, while fractional equality 2^n holds precisely for coordinate permutations of $\mathbb{D}^p \times B^r$, where $p+2r=n$ and $B=\{(z,w):|z|+|w|\le 1\}$. Among non-polydiscs the sharp ordinary bound is $2^{n+1}-2$, attained precisely by $\mathbb{D}^{n-2}\times B$ up to coordinate permutation.

The Hanner formulas are special cases of the graph formula below, as shown in Remark 2.

Next, for a graph G on $[n]$ with maximal independent sets $\mathcal{M}(G)$, define $K_G=\operatorname{conv}\bigcup_{S\in\mathcal{M}(G)}\mathbb{D}^S$. Theorem 6 gives

$$\operatorname{ill}(K_G)=\sum_{S\in\mathcal{M}(G)}(2^{|S|+1}-1),\qquad \operatorname{ill}^*(K_G)=\sum_{S\in\mathcal{M}(G)}2^{|S|}.$$

An edge between two distinct maximal supports prevents a direction from illuminating points on both associated tori. This makes their contributions additive. A weighted count of maximal independent sets then proves that fractional equality 2^n occurs exactly for disjoint unions of isolated vertices and edges. The counting bound $W(G)\le 2^n$ alone also follows from the classical antipodal-set bound of Danzer and Grünbaum [4]; Remark 3 gives this deduction explicitly. It is distinct from the exact illumination identities.

For a nondecreasing submodular function f with $f(\varnothing)=0$ and $f(\{j\})>0$, let

$$K_f=\left\{z\in\mathbb{C}^n:\sum_{j\in A}|z_j|\le f(A)\text{ for every }A\subseteq[n]\right\}.$$

Theorem 8 proves $\operatorname{ill}^*(K_f)\le 2^n$. Intersection of tight sets permits each zero coordinate of a boundary point to be assigned to a nonzero coordinate present in the same tight sets. Random orders and arguments, with suitably decreasing magnitudes, then give a uniform illumination probability.

Proposition 9 uses that assignment condition for positive weighted inequalities. It proves the fractional bound 2^n, an explicit ordinary bound in every dimension, and the asymptotic estimate $\operatorname{ill}(K)\le(2+o(1))n\log n\,2^n$. Laminar families satisfy the condition for arbitrary positive coefficients and right sides. The proof includes the finite approximation and covering argument needed for the ordinary bounds.

Section 7 formulates the selected higher-dimensional equality conjecture using products of discs and two-dimensional constraint bodies. The exact classifications proved for Hanner and graph bodies do not establish that conjecture for arbitrary unconditional bodies.

2 Notation and preliminary facts

Write $[n]=\{1,\dots,n\}$, $\mathbb{D}=\{z\in\mathbb{C}:|z|\le 1\}$, and $\mathbb{R}_+=[0,\infty)$. A convex body is a compact convex set with nonempty interior in the underlying real space. It is complex 1-unconditional if it is invariant under independent rotations of its coordinates. Its Minkowski norm is denoted by $\|\cdot\|_K$. For $z\in\mathbb{C}^n$, write $|z|=(|z_1|,\dots,|z_n|)$ and $\operatorname{supp}(z)=\{j\in[n]:z_j\ne 0\}$. We also write

$$Z_x=\{j\in[n]:x_j=0\}.$$

A nonzero vector d illuminates $x\in\partial K$ if $x+td\in\operatorname{int}K$ for some $t>0$. The least cardinality of a family illuminating every boundary point is $\operatorname{ill}(K)$. A finite nonnegative Borel measure μ on $\mathbb{C}^n\setminus\{0\}$ fractionally illuminates K if

$$\mu\{d:d\text{ illuminates }x\}\ge 1\qquad(x\in\partial K).$$

The infimum of the total masses of such measures is $\operatorname{ill}^*(K)$.

We call K normalized when $e_j\in\partial K$ for every $j\in[n]$. Positive diagonal rescaling gives this normalization, and illumination numbers are invariant under invertible linear maps. Coordinate

contractions preserve membership, as proved below; consequently a normalized body satisfies $K \subseteq \mathbb{D}^n$ and contains each unit coordinate disc.

Lemma 1. *For a complex 1-unconditional convex body K, the following statements hold.*

1. *If $|w_j| \le |z_j|$ for every j, then $\|w\|_K \le \|z\|_K$.*
2. *For every real linear subspace $E \subseteq \mathbb{C}^n$,*

$$\operatorname{relint}(K \cap E) = E \cap \operatorname{int} K.$$

Proof. Invariance under all coordinate rotations implies central symmetry. Because K has nonempty interior, its center 0 is interior: if an open Euclidean ball of radius r around a lies in K, the same is true around $-a$, and convexity puts the ball of radius r around 0 in K. Its Minkowski functional is therefore a norm, with $\operatorname{int} K = \{z : \|z\|_K < 1\}$.

Every number $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$ is a convex combination of two unit complex numbers. Explicitly, if $\lambda = se^{\mathrm{i}\theta}$, $0 \le s \le 1$, then

$$\lambda = \frac{1+s}{2} e^{\mathrm{i}\theta} + \frac{1-s}{2}(-e^{\mathrm{i}\theta}).$$

Thus contracting a single coordinate by λ preserves membership in K, by convexity and rotation invariance. Applying these contractions one coordinate at a time proves coordinatewise monotonicity of membership. Apply the same fact to every dilation of K to obtain the norm inequality in part 1. Coordinates with $z_j = 0$ also have $w_j = 0$, so cause no exception.

The inclusion from right to left in part 2 follows by intersecting an ambient open ball with E. For the reverse inclusion, let $y \in \operatorname{relint}(K \cap E)$. If $y = 0$ there is nothing to prove. Otherwise relative interior gives an $s > 0$ with $(1+s)y \in K \cap E$. Consequently $\|y\|_K \le (1+s)^{-1} < 1$, as required.

□

The illumination relation is open: for each fixed positive t, the condition $x + td \in \operatorname{int} K$ is open in (x, d), and illumination is the union of these conditions over $t > 0$. Its sections are therefore Borel measurable. Invertible linear maps preserve ordinary illumination and push fractional illuminating measures forward without changing their mass.

We use the exact ordinary polydisc value $\operatorname{ill}(\mathbb{D}^k) = 2^{k+1} - 1$ from [12, Theorem 1.4]. Boltyanski and Martini had asserted this formula earlier [2]; [12, Remark 2.8] identifies an error in that construction and supplies a corrected proof. Our ordinary arguments use the latter. The value $\operatorname{ill}^*(\mathbb{D}^k) = 2^k$ is also known [12, Theorem 1.3]; its direct proof is included in the fractional arguments below. All numbered citations to [12] refer to the published version. The value seven for a product of two smooth planar bodies is also recorded in [1, Theorem 2], with attribution to [2]. The direct vector sums in these references are Cartesian products on complementary subspaces; they are different from the ℓ_1 sums used below.

3 Complex Hanner bodies

The construction below replaces the intervals in the real Hanner construction [6] by complex discs. For the real recursive description and its geometry, see also [7, Sections 1 and 3]. These references concern the underlying family. Complete proofs of the complex illumination formulas and equality cases are given below.

A *complex Hanner body* is obtained recursively from the unit disc $\mathbb{D} \subset \mathbb{C}$ by taking Cartesian products and ℓ_1 sums on disjoint coordinate blocks. For bodies A and B with Minkowski norms

$\|\cdot\|_A$ and $\|\cdot\|_B$, these operations are

$$A \times B = \{(x,y) : \max\{\|x\|_A, \|y\|_B\} \le 1\},$$
$$A \oplus_1 B = \{(x,y) : \|x\|_A + \|y\|_B \le 1\}.$$

Every resulting body $H \subset \mathbb{C}^n$ is normalized and complex 1-unconditional. Write $\operatorname{ext} H$ for its extreme points and define

$$\mathcal{S}(H) = \{\operatorname{supp}(x) : x \in \operatorname{ext} H\}, \qquad F_H(t) = \sum_{S \in \mathcal{S}(H)} t^{|S|}.$$

These definitions depend on H, not on a choice of recursive description. For a nonempty $S \subseteq [n]$, put

$$E_S = \{z \in \mathbb{C}^n : z_j = 0 \text{ for } j \notin S\},$$
$$\mathbb{D}^S = \{z \in E_S : |z_j| \le 1 \text{ for } j \in S\},$$

and let

$$T_S = \{z \in E_S : |z_j| = 1 \text{ for } j \in S\}$$

be its coordinate torus.

Theorem 2. *For every complex Hanner body H,*

$$\operatorname{ill}(H) = 2F_H(2) - F_H(1), \qquad \operatorname{ill}^*(H) = F_H(2). \tag{1}$$

The polynomial is computed recursively by

$$F_{\mathbb{D}}(t) = t, \qquad F_{A\times B}(t) = F_A(t)F_B(t), \qquad F_{A\oplus_1 B}(t) = F_A(t) + F_B(t). \tag{2}$$

We first describe the extreme points and explain why directions illuminating different tori must be counted separately.

Lemma 3. *For a complex Hanner body H,*

$$\operatorname{ext} H = \bigsqcup_{S \in \mathcal{S}(H)} T_S, \qquad H = \operatorname{conv}(\operatorname{ext} H).$$

Every point of H is a finite convex combination of these extreme points. Moreover, $H \cap E_S = \mathbb{D}^S$ for every $S \in \mathcal{S}(H)$, and (2) holds.

Proof. The extreme points of a disc are its unit circle, and each point of the disc is a convex combination of two circle points. We prove the remaining assertions inductively, using

$$\operatorname{ext}(A \times B) = \operatorname{ext} A \times \operatorname{ext} B, \tag{3}$$
$$\operatorname{ext}(A \oplus_1 B) = (\operatorname{ext} A \times \{0\}) \cup (\{0\} \times \operatorname{ext} B). \tag{4}$$

For (3), if one coordinate block is not extreme, a nontrivial convex decomposition in that block gives one in the product. Conversely, a convex decomposition of a point whose two blocks are extreme must be constant in each block.

For (4), an extreme point lies on the boundary, where $\|x\|_A + \|y\|_B = 1$. If both blocks are nonzero, the identity

$$(x,y) = \|x\|_A \left(\frac{x}{\|x\|_A}, 0\right) + \|y\|_B \left(0, \frac{y}{\|y\|_B}\right)$$

is a nontrivial convex decomposition. Thus an extreme point has one block zero, and its other block must be extreme in the corresponding factor. Conversely, suppose $x \in \operatorname{ext} A$ and $(x,0)$ is a convex combination, with both weights positive, of $(x^1,y^1),(x^2,y^2) \in A \oplus_1 B$. Since $x^1,x^2 \in A$, extremality gives $x^1 = x^2 = x$. Now $\|x\|_A = 1$, so the sum constraint forces $y^1 = y^2 = 0$. This proves extremality of $(x,0)$; the other branch is identical.

Finite convex representations also persist under both operations. For a product, multiply the coefficients in finite convex representations of its two blocks. For a sum, use

$$A \oplus_1 B = \operatorname{conv}\big((A \times \{0\}) \cup (\{0\} \times B)\big)$$

and then the representations in the factors. The displayed identity follows directly from the norm inequality: for a point with $\alpha = \|x\|_A$, $\beta = \|y\|_B$ and $\alpha+\beta \le 1$, use weights $\alpha, \beta, 1-\alpha-\beta$ on the two normalized blocks and zero, omitting zero weights. The reverse inclusion is convexity. The induction therefore proves the finite convex-hull assertion as well as the extreme-point identities.

At a product, the supports are precisely $S \cup R$ with $S \in \mathcal{S}(A)$ and $R \in \mathcal{S}(B)$; their intersections with the two coordinate blocks recover S and R uniquely. At a sum, the support families are united without repetitions, since their nonempty sets lie in disjoint blocks. The two identities above now give the asserted disjoint union of tori and the polynomial recursion. The supports themselves may intersect; the tori for distinct supports are disjoint.

Finally, $T_S \subset H$ implies $\mathbb{D}^S = \operatorname{conv} T_S \subset H \cap E_S$. The convex-hull identity follows by taking products of finite circle representations in the coordinates of S. The reverse inclusion follows from $H \subseteq \mathbb{D}^n$. □

Lemma 4. *If $S, R \in \mathcal{S}(H)$ are distinct, no direction illuminates both a point of T_S and a point of T_R.*

Proof. We induct on the recursive construction. The assertion is vacuous for a disc. For $H = A \times B$, a direction illuminating (x,y) has projections illuminating x in A and y in B, since interior membership in the product requires interior membership in both factors. Distinct supports differ in at least one factor, where the induction hypothesis rules out a common direction.

For $H = A \oplus_1 B$, a direction $d = (v,w)$ illuminating $(x,0)$, with $\|x\|_A = 1$, satisfies for some $s > 0$

$$1 > \|x + sv\|_A + s\|w\|_B \ge 1 - s\|v\|_A + s\|w\|_B.$$

Thus $\|v\|_A > \|w\|_B$. Illuminating a point $(0,y)$ in the other branch requires $\|w\|_B > \|v\|_A$, which is impossible for the same direction. If both tori lie in the same branch, projection of an illuminating direction illuminates the corresponding points of that factor. The induction hypothesis again applies. □

Proof of Theorem 2. The recursion follows from Lemma 3. We first justify the reduction to extreme points for both illumination numbers. The ordinary reduction is the argument of [12, Lemma 2.1]; we give it explicitly, including the inclusion of direction sets needed for measures. For $x \in \partial H$, let

$$\mathcal{I}_H(x) = \{d \in \mathbb{C}^n \setminus \{0\} : d \text{ illuminates } x\}.$$

By Lemma 3, write $x = \sum_{k\in[M]} \lambda_k y^k$, where $y^k \in \operatorname{ext} H$, $\lambda_k > 0$ and $\sum_k \lambda_k = 1$. Fix one index k before choosing a direction. If $y^k + sd \in \operatorname{int} H$, then

$$x + \lambda_k sd = \lambda_k(y^k + sd) + \sum_{h\neq k} \lambda_h y^h \in \operatorname{int} H.$$

Indeed, its H-norm is at most $\lambda_k\|y^k + sd\|_H + 1 - \lambda_k < 1$. Consequently

$$\mathcal{I}_H(y^k) \subseteq \mathcal{I}_H(x). \tag{5}$$

A set illuminating every extreme point therefore illuminates the boundary. Likewise, a measure assigning mass at least one to $\mathcal{I}_H(y)$ for each extreme point y assigns mass at least one to $\mathcal{I}_H(x)$ for every boundary point x. No choice of the extreme point depending on the direction is needed.

For the ordinary lower bound, take any finite illuminating set D and, for each $S \in \mathcal{S}(H)$, let D_S consist of its directions illuminating at least one point of T_S. These groups are pairwise disjoint by Lemma 4. If d illuminates $x \in T_S$, then $H \subseteq \mathbb{D}^n$ implies $|x_j + sd_j| < 1$ for $j \in S$ at some positive step. Expanding the square gives

$$\operatorname{Re}(\overline{x_j} d_j) < 0 \qquad (j \in S). \tag{6}$$

Thus the projections of D_S illuminate every point of T_S in the polydisc $\mathbb{D}^S$. They also illuminate its entire boundary by the finite convex-combination argument above, applied to that polydisc. Every such projection is nonzero; even if some projections coincide, their number is at most $|D_S|$. The exact polydisc theorem [12, Theorem 1.4] therefore gives

$$|D| \geq \sum_{S \in \mathcal{S}(H)} |D_S| \geq \sum_{S \in \mathcal{S}(H)} (2^{|S|+1} - 1).$$

For the matching upper bound, embed an optimal illuminating set of $\mathbb{D}^S$ in E_S for each $S \in \mathcal{S}(H)$, with zero coordinates outside S. It has $2^{|S|+1} - 1$ directions. Lemma 3 gives $H \cap E_S = \mathbb{D}^S$, and Lemma 1(2) identifies the relative interior of this section with $E_S \cap \operatorname{int} H$. The embedded directions therefore illuminate T_S in H. Their union illuminates all extreme points, and hence all of ∂H by (5). This proves

$$\operatorname{ill}(H) = \sum_{S \in \mathcal{S}(H)} (2^{|S|+1} - 1) = 2F_H(2) - F_H(1).$$

For the fractional lower bound, define the direction sets

$$R_S = \bigcup_{x \in T_S} \mathcal{I}_H(x) \qquad (S \in \mathcal{S}(H)).$$

They are open and pairwise disjoint. Openness follows because, for each fixed positive step, the condition $x + sd \in \operatorname{int} H$ is open jointly in (x, d); taking the union over steps gives an open illumination relation. Its sections $\mathcal{I}_H(x)$ and their unions R_S are open. Disjointness is Lemma 4.

Let σ_S be Haar probability measure on T_S, that is, the product of the uniform probability measures on its coordinate circles. For any fixed direction d, the necessary conditions (6) hold on a set of σ_S-measure $2^{-|S|}$ if all d_j, $j \in S$, are nonzero, and on an empty set otherwise. Each nonzero coordinate imposes one open semicircle, of measure $1/2$. It follows that

$$\sigma_S\{x \in T_S : d \in \mathcal{I}_H(x)\} \leq 2^{-|S|} \mathbf{1}_{R_S}(d).$$

Here $\mathbf{1}_{R_S}$ is the indicator function of R_S. For a fractional illuminating measure μ, the open relation is Borel measurable, so Tonelli's theorem yields

$$\begin{aligned} 1 &\leq \int_{T_S} \mu(\mathcal{I}_H(x))\, d\sigma_S(x) \\ &= \int_{\mathbb{C}^n \setminus \{0\}} \sigma_S\{x \in T_S : d \in \mathcal{I}_H(x)\}\, d\mu(d) \\ &\leq 2^{-|S|} \mu(R_S). \end{aligned}$$

Summing over the disjoint R_S gives $\mu(\mathbb{C}^n \setminus \{0\}) \geq \sum_S 2^{|S|} = F_H(2)$.

For the upper bound, view each σ_S instead as a probability measure on directions in E_S, still with unit moduli on S and zeros outside S, and put

$$\mu = \sum_{S \in \mathcal{S}(H)} 2^{|S|} \sigma_S.$$

All these directions are nonzero. At any $x \in T_S$, independent uniform coordinates satisfy (6) with probability exactly $2^{-|S|}$. For directions supported on S, these inequalities are sufficient: a sufficiently small positive step enters the relative interior of $\mathbb{D}^S$, hence the interior of H. Therefore $2^{|S|}\sigma_S(\mathcal{I}_H(x)) = 1$. The measure μ illuminates every extreme point and, by (5), every boundary point. Its total mass is $F_H(2)$, proving the fractional formula. □

Corollary 5. *Let $H \subset \mathbb{C}^n$ be a complex Hanner body, and put*

$$B = \mathbb{D} \oplus_1 \mathbb{D} = \{(z, w) \in \mathbb{C}^2 : |z| + |w| \le 1\}.$$

Then $\mathrm{ill}^*(H) \le 2^n$*, with equality precisely when, after a coordinate permutation,*

$$H = \mathbb{D}^p \times B^r, \qquad p, r \in \mathbb{Z}_{\ge 0}, \quad p + 2r = n.$$

Factors with exponent zero are omitted. For these bodies,

$$F_H(t) = 2^r t^{p+r}, \qquad \mathrm{ill}(H) = 2^{n+1} - 2^r. \tag{7}$$

Moreover, $\mathrm{ill}(H) \le 2^{n+1} - 1$*, with equality precisely for $H = \mathbb{D}^n$. For $n \ge 2$ and $H \ne \mathbb{D}^n$,*

$$\mathrm{ill}(H) \le 2^{n+1} - 2,$$

with equality precisely for coordinate permutations of $\mathbb{D}^{n-2} \times B$.

Proof. We prove $F_H(2) \le 2^n$ by induction, retaining the equality conditions. It holds for a disc. A product of factors A_1, A_2 of dimensions $a, b \ge 1$ satisfies $F_{A_1 \times A_2}(2) = F_{A_1}(2)F_{A_2}(2) \le 2^{a+b}$, with equality if and only if both factors attain equality. At a sum,

$$F_{A_1 \oplus_1 A_2}(2) = F_{A_1}(2) + F_{A_2}(2) \le 2^a + 2^b \le 2^{a+b}.$$

The last inequality is equivalent to $2^{-a} + 2^{-b} \le 1$. Both terms are at most $1/2$, so equality occurs exactly when $a = b = 1$. In that case the factors are discs and their sum is B. Inductively, equality is possible exactly for products of discs and copies of B, up to coordinate permutation. Conversely, every such product attains equality. The recursion gives $F_B(t) = 2t$ and then (7).

Let $c_H = F_H(1) = |\mathcal{S}(H)|$. Its recursion multiplies at a product and adds at a sum, starting from one for a disc. Thus $c_H = 1$ if and only if $H = \mathbb{D}^n$: a sum has at least two supports, while a product has one only when both factors do. The exact formula therefore gives

$$\mathrm{ill}(H) = 2F_H(2) - c_H \le 2^{n+1} - c_H.$$

This proves the ordinary bound and its polydisc equality case. If $H \ne \mathbb{D}^n$, then $c_H \ge 2$, proving the stricter bound. Equality in that bound implies

$$2\big(2^n - F_H(2)\big) + c_H = 2.$$

The first term is nonnegative and $c_H \ge 2$, so $F_H(2) = 2^n$ and $c_H = 2$. The classification already proved gives $H = \mathbb{D}^p \times B^r$, after a coordinate permutation, with $c_H = 2^r$. Consequently $r = 1$ and $p = n - 2$. Conversely, this body attains the stated value by (7). □

All numerical conclusions are preserved by invertible complex linear maps: directions are mapped bijectively and illuminating measures are pushed forward with unchanged total mass. The coordinate descriptions in Corollary 5 refer to the recursively defined class above; for its linear images they become statements of complex linear equivalence.

Remark 1. The equality clause of [12, Conjecture 1.2] proposes that only linear images of the polydisc have fractional illumination number 2^n. Corollary 5 gives $\mathrm{ill}^*(B) = 4$ in complex dimension two, although B is not a linear image of $\mathbb{D}^2$. Indeed, Lemma 3 gives

$$\operatorname{ext} B = (\partial\mathbb{D} \times \{0\}) \cup (\{0\} \times \partial\mathbb{D}), \qquad \operatorname{ext}(\mathbb{D}^2) = \partial\mathbb{D} \times \partial\mathbb{D}.$$

The first set has two connected components, whereas the second is connected. An invertible real-linear map preserves extreme points and connectedness, so even real-linear equivalence is impossible. Thus the published equality clause fails; its proposed upper bound is unaffected. Section 7 states a replacement equality conjecture for the unconditional class.

4 Bodies defined by graphs

Let G be a finite simple graph on $[n]$, where $n \geq 1$, and let $\mathcal{M}(G)$ be the family of its inclusion-maximal independent vertex sets. Using the coordinate polydiscs and tori defined in Section 3, put

$$K_G = \operatorname{conv} \bigcup_{S \in \mathcal{M}(G)} \mathbb{D}^S, \qquad W(G) = \sum_{S \in \mathcal{M}(G)} 2^{|S|}.$$

Every singleton extends to a maximal independent set, so K_G contains every unit coordinate disc. It is a normalized complex 1-unconditional convex body and is contained in $\mathbb{D}^n$. Moreover,

$$K_G = \operatorname{conv} \bigcup_{S \in \mathcal{M}(G)} T_S, \qquad K_G \cap E_S = \mathbb{D}^S \quad (S \in \mathcal{M}(G)).$$

The first identity follows from $\mathbb{D}^S = \operatorname{conv} T_S$; the second follows from the defining containment and $K_G \subseteq \mathbb{D}^n$. Every point of K_G is a finite convex combination of points from the indicated tori.

The positive real section is the stable-set polytope of G, a classical graph polytope studied, for example, by Chvátal [3]:

$$K_G \cap \mathbb{R}^n_+ = \operatorname{conv}\{\mathbf{1}_I : I \subseteq [n] \text{ is independent in } G\}.$$

Here $\mathbf{1}_I$ has coordinate one on I and zero elsewhere. To check the identity, put $Q_S = \mathbb{D}^S \cap \mathbb{R}^n_+$ and $P = \operatorname{conv} \bigcup_{S \in \mathcal{M}(G)} Q_S$. The vertices of Q_S are the vectors $\mathbf{1}_I$ with $I \subseteq S$, and every independent set extends to a maximal one. Also, P is closed under decreasing coordinates. Indeed, if $q = \sum_k \lambda_k q^k \in P$, where $q^k \in Q_{S_k}$, and $0 \leq u \leq q$, put

$$w^k_j = \begin{cases} (u_j/q_j) q^k_j, & q_j > 0, \\ 0, & q_j = 0. \end{cases}$$

Then $w^k \in Q_{S_k}$ and $u = \sum_k \lambda_k w^k \in P$. For a convex combination $z = \sum_k \lambda_k z^k$ of points in the complex polydiscs, the triangle inequality gives $|z| \leq \sum_k \lambda_k |z^k| \in P$. Hence $K_G \cap \mathbb{R}^n_+ \subseteq P$; the reverse inclusion follows from $Q_S \subseteq K_G$. The full real section $K_G \cap \mathbb{R}^n$ is the unconditional extension of this stable-set polytope, as studied in [8, Sections 3–4]. Graph descriptions of real spaces with unconditional bases were studied by Reisner [11]. In the real Hanner case, supports of vertices are maximal independent sets of the associated graph; see [7, Section 6]. The illumination formulas below are proved directly for every finite simple graph.

Remark 2. The graph class contains the complex Hanner class. To see this, let G_1, G_2 have disjoint nonempty vertex sets. Write $G_1 \sqcup G_2$ for their disjoint union and $G_1 \vee G_2$ for their join, obtained by adding every edge between the two sets. Then

$$K_{G_1 \sqcup G_2} = K_{G_1} \times K_{G_2}, \qquad K_{G_1 \vee G_2} = K_{G_1} \oplus_1 K_{G_2}.$$

Indeed, a maximal independent set in the disjoint union is exactly $S_1 \cup S_2$ with $S_i \in \mathcal{M}(G_i)$: independence and maximality can be tested separately in each component. Its coordinate polydisc is $\mathbb{D}^{S_1} \times \mathbb{D}^{S_2}$. Taking convex hulls gives the product identity, since multiplying the coefficients in two finite convex representations represents each point of the product. In the join, an independent set lies entirely in one vertex set, and it is maximal precisely when it is maximal there. Taking convex hulls therefore gives $\operatorname{conv}((K_{G_1} \times \{0\}) \cup (\{0\} \times K_{G_2})) = K_{G_1} \oplus_1 K_{G_2}$. A single vertex gives $\mathbb{D}$. Recursion now identifies bodies of graphs built from single vertices by these two operations (the cographs) with complex Hanner bodies. The support families agree by Lemma 3, so the graph formula specializes to (1). The corresponding real product and sum identities are recorded in [8, Section 4].

Theorem 6. *For every finite simple graph G on $[n]$,*

$$\operatorname{ill}(K_G) = \sum_{S \in \mathcal{M}(G)} (2^{|S|+1} - 1), \qquad \operatorname{ill}^*(K_G) = \sum_{S \in \mathcal{M}(G)} 2^{|S|}. \tag{8}$$

Proof. First we show that no direction illuminates points from two distinct tori T_S, T_R, with $S, R \in \mathcal{M}(G)$. Distinct maximal independent sets are incomparable, so choose $r \in R \setminus S$. Maximality of S supplies a neighbor $s \in S$ of r. Since R is independent, $s \notin R$. Every independent set contains at most one of s, r. Thus

$$|z_s| + |z_r| \le 1 \qquad (z \in K_G). \tag{9}$$

If d illuminates $x \in T_S$, then for some $t > 0$,

$$|x_s + t d_s| + t|d_r| < 1.$$

The inequality is strict at an interior point of K_G. The reverse triangle inequality and $|x_s| = 1$ give $|d_s| > |d_r|$. Illuminating any point of T_R instead forces $|d_r| > |d_s|$. This proves the claimed separation.

Write $\mathcal{I}_{K_G}(x)$ for the nonzero directions illuminating x, and define the open direction sets

$$R_S = \bigcup_{x \in T_S} \mathcal{I}_{K_G}(x) \qquad (S \in \mathcal{M}(G)).$$

They are pairwise disjoint by the preceding argument. They are open because the illumination relation is open, as in the proof of Theorem 2. If d illuminates $x \in T_S$, the containment $K_G \subseteq \mathbb{D}^n$ implies

$$\operatorname{Re}(\overline{x_j} d_j) < 0 \qquad (j \in S). \tag{10}$$

Indeed, at an interior step $|x_j + t d_j|^2 < 1$, and expanding the square with $|x_j| = 1$ gives this strict inequality.

Let D be a finite illuminating family and put $D_S = D \cap R_S$. Projection of D_S onto E_S illuminates the torus of $\mathbb{D}^S$ by (10), hence its entire boundary by the convex-combination argument in (5). The projected directions are nonzero and their number is at most $|D_S|$. The exact polydisc theorem [12, Theorem 1.4] gives

$$|D| \ge \sum_{S \in \mathcal{M}(G)} |D_S| \ge \sum_{S \in \mathcal{M}(G)} (2^{|S|+1} - 1).$$

For the reverse inequality, embed an optimal illuminating family for each $\mathbb{D}^S$ in E_S, setting coordinates outside S to zero. A direction in this family illuminates the corresponding torus point in K_G, since an interior step in the coordinate polydisc belongs to $\operatorname{int} K_G$ by Lemma 1(2). Their union illuminates all the tori. For any $x \in \partial K_G$, choose a finite representation $x = \sum_k \lambda_k y^k$ with $\lambda_k > 0$, $\sum_k \lambda_k = 1$, and y^k on these tori. Fix one k. If $y^k + td \in \operatorname{int} K_G$, then

$$x + \lambda_k td = \lambda_k(y^k + td) + \sum_{h \neq k} \lambda_h y^h \in \operatorname{int} K_G.$$

Thus the union illuminates the entire boundary, proving the ordinary formula. This same argument transfers fractional illumination from the tori to the boundary, with k fixed before choosing d.

For each S, let σ_S be product uniform probability measure on T_S. A fixed direction satisfies (10) on σ_S-mass $2^{-|S|}$ if all its coordinates in S are nonzero, and on zero mass otherwise. Hence

$$\sigma_S\{x : d \in \mathcal{I}_{K_G}(x)\} \leq 2^{-|S|}\mathbf{1}_{R_S}(d).$$

For every fractional illuminating measure μ, Tonelli's theorem for the open illumination relation yields

$$1 \leq \int_{T_S} \mu(\mathcal{I}_{K_G}(x))\, d\sigma_S(x) \leq 2^{-|S|}\mu(R_S).$$

Summing over the disjoint R_S gives $\mu(\mathbb{C}^n \setminus \{0\}) \geq W(G)$. Conversely, regard σ_S as a measure on directions supported on S. At $x \in T_S$, its directions satisfy (10) with probability $2^{-|S|}$. Such a direction enters the relative interior of $\mathbb{D}^S$ and hence the interior of K_G. Consequently

$$\mu = \sum_{S \in \mathcal{M}(G)} 2^{|S|}\sigma_S$$

fractionally illuminates every torus and then the whole boundary. Its mass is $W(G)$, proving the fractional formula. □

Corollary 7. *For every graph G on $[n]$, $\operatorname{ill}^*(K_G) \leq 2^n$. Equality holds if and only if G is a disjoint union of isolated vertices and edges. If G has a vertex of degree at least two, then*

$$\operatorname{ill}^*(K_G) \leq 3 \cdot 2^{n-2}. \tag{11}$$

Proof. By Theorem 6, it suffices to estimate $W(G)$. For the graph with no vertices, set $\mathcal{M}(G) = \{\varnothing\}$ and $W(G) = 1$. We induct on the number of vertices. For a vertex v, let $N[v]$ be its closed neighborhood, and let $G - A$ denote the induced graph after deleting a vertex set A. The maximal independent sets containing v are exactly

$$\{v\} \cup R, \qquad R \in \mathcal{M}(G - N[v]).$$

To check this, vertices adjacent to v are excluded, and maximality on the remaining vertices is exactly maximality in $G - N[v]$. Every maximal independent set omitting v is still maximal in $G - \{v\}$: a vertex that could be added there could also be added in G. Therefore

$$W(G) \leq W(G - \{v\}) + 2W(G - N[v]).$$

If v has degree $d \geq 1$, induction gives

$$W(G) \leq 2^{n-1} + 2^{n-d} \leq 2^n.$$

For $d \geq 2$ the right side is at most $3 \cdot 2^{n-2}$. If v is isolated, every maximal independent set contains it and $W(G) = 2W(G - \{v\}) \leq 2^n$. This proves the bound for all graphs.

If every degree is at most one, write p for the number of isolated vertices and r for the number of edges, so $p+2r=n$. A maximal independent set contains every isolated vertex and exactly one endpoint of each edge. There are 2^r such sets, each of size $p+r$, and hence

$$W(G) = 2^r 2^{p+r} = 2^n.$$

After arranging the connected components on disjoint coordinate blocks, these graphs also satisfy $K_G = \mathbb{D}^p \times B^r$. Indeed, an isolated vertex contributes a disc, an edge contributes $\operatorname{conv}((\mathbb{D}\times\{0\}) \cup (\{0\}\times\mathbb{D})) = B$, and convex hull commutes with finite Cartesian products by multiplying the coefficients of convex combinations in the factors. If any degree is at least two, (11) is strictly smaller than 2^n. These cases prove the classification. □

Remark 3. The estimate $W(G) \le 2^n$ is also a consequence of the classical bound for antipodal sets [4]. Here is a direct verification of this application. Put

$$V_G = \{v \in \{-1,0,1\}^n : \operatorname{supp}(v) \in \mathcal{M}(G)\}, \qquad P = \operatorname{conv} V_G.$$

Then $|V_G| = W(G)$ and P has nonempty interior: for each j, choose a maximal independent set containing j and average the signs on its other coordinates to obtain $\pm e_j \in P$. For distinct $u, v \in V_G$, we exhibit a linear functional f with $f(u) = 1$, $f(v) = -1$, and $|f| \le 1$ on P. If the supports agree, some coordinate has opposite signs; take $f(x) = u_j x_j$ there. If the supports are $S \ne R$, choose $r \in R \setminus S$ and a neighbor $s \in S \setminus R$ as in the graph-formula proof. Then

$$f(x) = u_s x_s - v_r x_r$$

has the required values. Its bound follows from $|x_s| + |x_r| \le 1$ on P, since every independent set contains at most one endpoint of an edge. Thus V_G is antipodal. For completeness, the sets $(u+P)/2$, $u \in V_G$, lie in P and have pairwise disjoint interiors. Otherwise $u + a = v + b$ for $a, b \in \operatorname{int} P$; the functional just constructed would give $2 = f(b) - f(a) < 2$. Taking volumes yields

$$|V_G|\, 2^{-n} \operatorname{vol}(P) \le \operatorname{vol}(P),$$

and hence $W(G) \le 2^n$. This argument alone supplies neither the complex illumination identities nor the graph-specific strict bound; those are established in the preceding proofs.

5 Bodies defined by submodular constraints

Let $f : 2^{[n]} \to [0,\infty)$ satisfy $f(\varnothing) = 0$, $f(\{j\}) > 0$ for every $j \in [n]$, and

$$f(A) \le f(B) \quad (A \subseteq B), \qquad f(A) + f(B) \ge f(A \cup B) + f(A \cap B) \quad (A, B \subseteq [n]).$$

Thus f is nondecreasing and submodular. Define

$$K_f = \left\{ z \in \mathbb{C}^n : \sum_{j \in A} |z_j| \le f(A) \text{ for every } A \subseteq [n] \right\}. \tag{12}$$

The positive real section is the independence polytope of a polymatroid, in the sense of Edmonds [5]. Thus the defining polyhedral construction is classical; the conclusion below concerns its complex illumination number. The singleton inequalities make K_f bounded, and positivity of f on nonempty sets gives a neighborhood of zero in K_f. In particular, K_f is a complex 1-unconditional convex body. This class includes the bodies associated with loopless matroid independence polytopes and does not require permutation symmetry.

Theorem 8. *For every function f as above,*

$$\mathrm{ill}^*(K_f) \le 2^n. \tag{13}$$

Proof. Fix $x \in \partial K_f$, put $r_j = |x_j|$, and write $S = \mathrm{supp}(x)$ and $Z_x = [n] \setminus S$. The set S is nonempty. Call $A \subseteq [n]$ tight when $r(A) = f(A)$, where $r(A) = \sum_{j\in A} r_j$. Tight sets are closed under intersection and union. We include the elementary submodular argument. For tight A, B,

$$f(A) + f(B) = r(A) + r(B) = r(A \cup B) + r(A \cap B) \le f(A \cup B) + f(A \cap B) \le f(A) + f(B),$$

so equality holds throughout.

For each $j \in Z_x$ which belongs to a nonempty tight set, let T_j be the intersection of all tight sets containing j. It is tight, and

$$r(T_j) = f(T_j) \ge f(\{j\}) > 0.$$

Choose $i(j) \in S \cap T_j$. For any remaining $j \in Z_x$, choose $i(j)$ arbitrarily in S. These choices are made from x before choosing a random direction. They have the property

$$j \in A \cap Z_x, \quad A \text{ tight} \quad \Longrightarrow \quad i(j) \in A \cap S. \tag{14}$$

Let $\pi : [n] \to [n]$ be a uniformly random bijection, interpreted as the ranks of the coordinates. Independently choose uniform angles $\theta_1, \dots, \theta_n \in [0, 2\pi)$, and set

$$d_j = \varepsilon^{\pi(j)-1} e^{\mathrm{i}\theta_j}, \qquad j \in [n], \qquad 0 < \varepsilon < 1. \tag{15}$$

Consider the event

$$\pi(i(j)) < \pi(j) \quad (j \in Z_x).$$

For $i \in S$ let d'_i be the number of $j \in Z_x$ assigned to i. The sets consisting of a parent and its assigned coordinates are disjoint. Their relative orders in a uniform permutation are independent, as is seen by assigning independent uniform real numbers to all coordinates and ranking them. Therefore the displayed event has probability

$$\prod_{i\in S} \frac{1}{d'_i + 1} \ge \prod_{i\in S} 2^{-d'_i} = 2^{-|Z_x|}. \tag{16}$$

Here $d + 1 \le 2^d$ for every nonnegative integer d.

Fix $s \in (0, 1)$ and require in addition

$$\cos(\theta_j - \arg x_j) \le -s \quad (j \in S). \tag{17}$$

Its probability is $(\arccos(s)/\pi)^{|S|}$, independently of the order event. We show that the two events imply illumination whenever $\varepsilon/(1-\varepsilon) < s$.

For a nonempty tight set A, its first coordinate j_0 in the order π belongs to S: otherwise (14) would give an earlier coordinate of A. The right derivative at $t = 0$ of $\sum_{j\in A} |x_j + t d_j|$ equals

$$\sum_{j\in A\cap S} \varepsilon^{\pi(j)-1} \cos(\theta_j - \arg x_j) + \sum_{j\in A\cap Z_x} \varepsilon^{\pi(j)-1}.$$

By (17), it is at most

$$\varepsilon^{\pi(j_0)-1} \left(-s + \frac{\varepsilon}{1-\varepsilon} \right) < 0. \tag{18}$$

The geometric series bounds all subsequent coordinates, irrespective of which belong to A. All tight inequalities therefore become strict for sufficiently small positive t. The remaining

nonempty inequalities are already strict and remain so by continuity. There are finitely many inequalities, so one positive t makes all of them strict. Their strict intersection is open, and hence $x + td \in \operatorname{int} K_f$. This argument includes zero coordinates and nonsmooth boundary points.

The distribution of d in (15) depends on n, ε only. With $\varepsilon = s/(2+s)$, the illumination probability at every boundary point is at least

$$2^{-|Z_x|} \left(\frac{\arccos s}{\pi}\right)^{|S|} \geq 2^{-n} \left(\frac{2 \arccos s}{\pi}\right)^{n}.$$

Scale this probability measure by the reciprocal of the last expression. It fractionally illuminates every K_f, and all its directions are nonzero. Taking the infimum as $s \downarrow 0$ proves (13); no limit measure is asserted to illuminate.

□

6 Positive weighted inequalities

The assignment property (14) also applies to certain weighted systems without submodularity. The coefficients now affect the magnitudes of the chosen directions, while the number of directions in the bounds below depends only on dimension. The finite covering argument is related to the classical passage from fractional to integral covers [9]. We give the sampling and union-bound proof, with all constants, directly.

Proposition 9. *Let $\mathcal{F}$ be a finite family of nonempty subsets of $[n]$, let $c_{j,A} > 0$ for $A \in \mathcal{F}$ and $j \in A$, and let $b_A > 0$. Suppose that*

$$K = \left\{ z \in \mathbb{C}^n : \sum_{j \in A} c_{j,A} |z_j| \leq b_A \quad (A \in \mathcal{F}) \right\}$$

is bounded. At $x \in \partial K$, write $S = \operatorname{supp}(x)$ and call $A \in \mathcal{F}$ tight when its inequality is an equality at x. Assume that for every such x and every $j \notin S$ belonging to a tight set, there exists $i(j) \in S$ such that

$$j \in A, \quad A \text{ tight} \quad \Longrightarrow \quad i(j) \in A. \tag{19}$$

Then $\operatorname{ill}^*(K) \leq 2^n$, and, writing $m_n = \lceil 16\pi n \rceil$,

$$\operatorname{ill}(K) \leq \left\lceil \frac{4}{3}\, 2^n \left[n \log\big((n+1)m_n\big) + 1 \right] \right\rceil. \tag{20}$$

The ordinary estimate can be sharpened asymptotically to

$$\operatorname{ill}(K) \leq (2 + o(1)) n \log n \, 2^n.$$

In particular, these conclusions hold for arbitrary positive coefficients and right sides when $\mathcal{F}$ is laminar: any two members are disjoint or one contains the other.

Proof. The finitely many positive right sides give a neighborhood of zero inside K. Together with boundedness this makes K a complex 1-unconditional convex body. Put

$$c_- = \min_{A \in \mathcal{F},\, j \in A} c_{j,A} > 0, \qquad c_+ = \max_{A \in \mathcal{F},\, j \in A} c_{j,A}.$$

For a fixed $s \in (0,1)$ choose

$$\varepsilon = \frac{s c_-}{2c_+ + s c_-}$$

before choosing a boundary point. Let π be a uniform rank permutation and choose independent uniform angles θ_j. Use $d_j = \varepsilon^{\pi(j)-1} e^{i\theta_j}$, as in (15).

Fix $x \in \partial K$, put $S = \operatorname{supp}(x)$ and $Z_x = [n] \setminus S$, and choose the assignments in (19). Assign each remaining zero coordinate arbitrarily to a member of the nonempty set S. By (16), the event $\pi(i(j)) < \pi(j)$ for every $j \in Z_x$ has probability at least $2^{-|Z_x|}$. On this event the first coordinate j_0 of each tight set belongs to S. Require also $\cos(\theta_j - \arg x_j) \le -s$ for $j \in S$. The right derivative at zero of a tight inequality's left side is

$$\sum_{j \in A \cap S} c_{j,A} \varepsilon^{\pi(j)-1} \cos(\theta_j - \arg x_j) + \sum_{j \in A \cap Z_x} c_{j,A} \varepsilon^{\pi(j)-1}$$
$$\le \varepsilon^{\pi(j_0)-1} \left(-s c_- + c_+ \frac{\varepsilon}{1-\varepsilon} \right) < 0.$$

The final inequality follows from $c_+ \varepsilon/(1-\varepsilon) = s c_-/2$. All tight inequalities become strict for sufficiently small positive displacement. The other inequalities remain strict by continuity, and finiteness gives a common positive step into the interior. This includes zero coordinates and nonsmooth boundary points.

The order and angular events are independent. Thus the same distribution illuminates every boundary point with probability at least

$$2^{-|Z_x|} \left(\frac{\arccos s}{\pi} \right)^{|S|} \ge 2^{-n} \left(\frac{2 \arccos s}{\pi} \right)^n.$$

Scaling by the reciprocal and taking the infimum as $s \downarrow 0$ proves the fractional bound. This uses a different measure for each s and does not claim illumination by a limiting measure.

For the ordinary bound, fix $\varepsilon = c_-/(32 n c_+)$, and again choose random orders and angles. We find a finite family serving every nonempty support S, every assignment $i : [n] \setminus S \to S$, and every vector of grid angles

$$(\alpha_j)_{j \in S} \in \left\{ \frac{2\pi k}{m_n} : k = 0, \dots, m_n - 1 \right\}^S.$$

There are at most $(n+1)^n$ choices of support and assignment: encode a member of S by zero and every other coordinate by its assigned member of S. There are at most m_n^n grid vectors per choice. For one such choice, a random direction meets the order event and $\cos(\theta_j - \alpha_j) \le -1/(4n)$ on S with probability at least

$$2^{-(n-|S|)} \left(\frac{\arccos(1/(4n))}{\pi} \right)^{|S|} \ge 2^{-n} \left(1 - \frac{1}{4n} \right)^n \ge \frac{3}{4} 2^{-n}.$$

Here $\arcsin u \le \pi u/2$ on $[0,1]$, and the last inequality is Bernoulli's inequality. Let M be the right side of (20), and choose M independent directions. The probability that at least one assignment-grid pair is unserved is at most

$$((n+1) m_n)^n \exp\left(-\frac{3}{4} 2^{-n} M \right) \le e^{-1} < 1.$$

Fix a successful family before choosing $x \in \partial K$. For that point choose its assignments as above and the nearest grid angles. Their circular distance from $\arg x_j$ is at most $\pi/m_n \le 1/(16n)$. Since cosine is 1-Lipschitz, a serving direction satisfies

$$\cos(\theta_j - \arg x_j) \le -\frac{1}{4n} + \frac{1}{16n} < -\frac{1}{8n} \qquad (j \in S).$$

Also, since $c_- \le c_+$,

$$c_+ \frac{\varepsilon}{1-\varepsilon} \le \frac{c_-}{32n - 1} < \frac{c_-}{8n}.$$

The derivative estimate therefore makes this direction illuminate x, proving (20).

For the sharper asymptotic estimate, put $c_n = \log(n+1)$ and $\widetilde{m}_n = \lceil 4\pi c_n n \rceil$. For sufficiently large n, use grid threshold $1/(c_n n)$, grid size $\widetilde{m}_n$, and ratio $\varepsilon = c_-/(4c_n n c_+)$. The same counting argument has at most $((n+1)\widetilde{m}_n)^n$ pairs and success probability at least $(1 - 1/c_n)2^{-n}$ for each pair. It gives a family of size

$$\left\lceil \frac{2^n}{1 - 1/c_n} \left[n \log\bigl((n+1)\widetilde{m}_n\bigr) + 1 \right] \right\rceil = (2 + o(1)) n \log n \, 2^n.$$

The grid error is at most $1/(4c_n n)$, leaving an inward cosine of magnitude at least $3/(4c_n n)$. The weighted suffix satisfies

$$c_+ \frac{\varepsilon}{1-\varepsilon} \le \frac{c_-}{4c_n n - 1} < \frac{c_-}{2c_n n}$$

for sufficiently large n. Strict negativity of the derivative again proves illumination. The counts are independent of the coefficients, but the direction magnitudes may depend on them.

Finally suppose that $\mathcal{F}$ is laminar. Fix a boundary point and a zero coordinate j belonging to a tight set. Choose the smallest tight set A_j containing j. All sets containing j form a chain, so A_j is contained in every tight set containing j. Since $b_{A_j} > 0$, tightness implies $A_j \cap S \neq \varnothing$. Any $i(j) \in A_j \cap S$ satisfies (19), completing the proof. □

7 A proposed fractional-equality class

The equality cases in Corollaries 5 and 7 are products of discs and copies of $B = \{(z,w) : |z| + |w| \le 1\}$. For general normalized unconditional bodies, the planar equality criterion in the companion manuscript [13] motivates a larger proposed class. That source is an unpublished working draft, and its planar theorem is used here only as motivation. For $0 < a, b \le 1$, define

$$L(a,b) = \{(z,w) \in \mathbb{C}^2 : |z| + a|w| \le 1,\ b|z| + |w| \le 1\}.$$

Conjecture 10. *Let $K \subset \mathbb{C}^n$ be a normalized complex 1-unconditional convex body. Then $\mathrm{ill}^*(K) = 2^n$ if and only if, after a coordinate permutation, there are integers $p, r \ge 0$ with $p + 2r = n$ and $a_j, b_j \in (0,1]$ satisfying $a_j b_j \ge 1/2$ for $j \in [r]$, such that*

$$\mathbb{D}^p \times B^r \ \subseteq K \ \subseteq \mathbb{D}^p \times \prod_{j \in [r]} L(a_j, b_j). \tag{21}$$

Factors with exponent zero are omitted.

For $r = 0$ the two bounds coincide with $\mathbb{D}^n$. Taking $a_j = b_j = 1$ makes $L(a_j, b_j) = B$, so the Hanner equality cases are included. The planar classification is proved in [13, Section 7]; the conjecture above in higher dimensions is not a consequence of the results of this paper. For arbitrary bodies between the two displayed bounds, the upper estimate $\mathrm{ill}^*(K) \le 2^n$ remains unproved. The converse assertion that every equality body has this form is also open. Neither assertion is used in the preceding proofs.

References

[1] E. D. Baladze and V. G. Boltyanski, *Illumination of direct sums of two convex figures*, Beitr. Algebra Geom. **47** (2006), no. 2, 345–350. Full text.

[2] V. Boltyanski and H. Martini, *Illumination of direct vector sums of convex bodies*, Studia Sci. Math. Hungar. **44** (2007), no. 3, 367–376. doi:10.1556/SScMath.44.2007.3.6.

[3] V. Chvátal, *On certain polytopes associated with graphs*, J. Combin. Theory Ser. B **18** (1975), no. 2, 138–154. doi:10.1016/0095-8956(75)90041-6.

[4] L. Danzer and B. Grünbaum, *Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee*, Math. Z. **79** (1962), 95–99. doi:10.1007/BF01193107.

[5] J. Edmonds, *Submodular functions, matroids, and certain polyhedra*, in *Combinatorial Structures and Their Applications* (Proc. Calgary Internat. Conf., 1969), Gordon and Breach, New York, 1970, 69–87. Reprinted in *Combinatorial Optimization—Eureka, You Shrink!*, Lecture Notes in Comput. Sci. **2570**, Springer, 2003, 11–26. doi:10.1007/3-540-36478-1_2 (reprint).

[6] O. Hanner, *Intersections of translates of convex bodies*, Math. Scand. **4** (1956), 65–87. doi:10.7146/math.scand.a-10456.

[7] J. Kim, *Minimal volume product near Hanner polytopes*, J. Funct. Anal. **266** (2014), no. 4, 2360–2402. doi:10.1016/j.jfa.2013.08.008. arXiv:1212.2544.

[8] F. Kohl, M. Olsen, and R. Sanyal, *Unconditional reflexive polytopes*, Discrete Comput. Geom. **64** (2020), 427–452. doi:10.1007/s00454-020-00199-8.

[9] L. Lovász, *On the ratio of optimal integral and fractional covers*, Discrete Math. **13** (1975), no. 4, 383–390. doi:10.1016/0012-365X(75)90058-8.

[10] M. Naszódi, *Fractional illumination of convex bodies*, Contrib. Discrete Math. **4** (2009), no. 2, 83–88. doi:10.55016/ojs/cdm.v4i2.62022.

[11] S. Reisner, *Certain Banach spaces associated with graphs and CL-spaces with 1-unconditional bases*, J. London Math. Soc. (2) **43** (1991), no. 1, 137–148. doi:10.1112/jlms/s2-43.1.137.

[12] L. Rotem, A. Schejter, and B. A. Slomka, *The complex Illumination problem*, Combinatorica **46**, article 3 (2026). doi:10.1007/s00493-025-00195-7. arXiv:2410.12021.

[13] *Illumination of complex* 1*-unconditional convex bodies*, unpublished companion working manuscript, version of 25 October 2026, Section 7, "Fractional illumination in two complex dimensions."