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EQUALITY OF THE MIRKOVIĆ–VILONEN AND THETA BASES

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Abstract. We prove Kamnitzer's ICM 2022 conjecture that the Mirković–Vilonen and theta bases coincide.

AI use and scholarly credit

The initial draft was developed through seven ChatGPT-6-Pro conversations totaling approximately 23 hours and 10 minutes, interspersed with four hours of research using Pharos [25]. It was then reviewed and polished with Codex and its built-in mathematical paper-writing skill before being submitted to Rethlas-OSV [26, 27, 28] for review.

This was the first time Rethlas-OSV identified a nontrivial error in my work that invalidated the proof of the main theorem. Previously, manuscripts had either passed its review or yielded editorial errors, such as incorrect reference numbers or a missing plural ending, or readily repairable mathematical errors, such as an omitted hypothesis. I submitted the review report to ChatGPT-6-Pro and asked it to spend six hours repairing the proof. After it reported completing the repair, I used a separate ChatGPT conversation for approximately three rounds of review totaling about 88 minutes. These reviews found no further errors. Codex then reviewed and polished the manuscript, expanding it from 21 to 52 pages. The revised manuscript was then reviewed using Rethlas-OSV.

The second Rethlas-OSV review identified further errors in the manuscript, including one that again blocked the proof as written. They were classified as minor because the report also supplied corrections. After these errors were corrected and the exposition was polished again, the manuscript passed Rethlas-OSV review.

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1. Introduction

Let G be a connected complex semisimple group and N a maximal unipotent subgroup. Distinguished bases of $\mathbb{C}[N]$ encode bases of highest-weight representations, crystal operators and tensor-product multiplicities. Three representation-theoretic constructions are the dual canonical basis, the dual semicanonical basis in simply-laced type, and the Mirković–Vilonen (MV) basis. A fourth construction, the theta basis, comes from the cluster structure on $\mathbb{C}[N]$. Kamnitzer's Conjecture 6.9 in his ICM 2022 report [12] predicts that the MV and theta bases coincide. We prove this equality for every G.

1.1. Geometric bases and cluster algebras.

The canonical-basis theory of Lusztig and the crystal-basis theory of Kashiwara provide compatible bases of quantum groups and their integrable highest-weight representations [23, 24]. At the classical parameter, dualizing the positive enveloping algebra gives a distinguished basis of the unipotent coordinate ring. The crystal records the leading behavior of the simple-root operators. Lusztig's semicanonical

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construction instead uses constructible functions on varieties of representations of preprojective algebras [44]. Its irreducible components carry the same crystal combinatorics, while the resulting basis vectors can differ from the canonical ones.

The MV construction arises from geometric Satake. Mirković and Vilonen identify representations of G with suitable perverse sheaves on the affine Grassmannian of the Langlands dual group [14]. The irreducible components of the relevant intersections with semi-infinite orbits provide fundamental classes representing weight vectors. Baumann and Gaussent describe the crystal operations geometrically [4]; Kamnitzer describes the associated MV polytopes by tropical Plücker relations and relates their parameters to Lusztig data [22]. These results give a concrete common indexing of the geometric and canonical constructions.

Passing from representations to $\mathbb{C}[N]$ requires compatibility as the highest weight increases. Baumann, Kamnitzer and Knutson establish this compatibility and obtain the normalized MV basis of $\mathbb{C}[N]$ [1]. It is biperfect: for both left and right simple-root derivations, the highest nonzero term determines the corresponding crystal operator, with a prescribed leading coefficient. They also express multiplication of MV functions through intersection multiplicities in a spatial fusion family in the Beilinson–Drinfeld Grassmannian. This multiplication formula is the geometric input for the present comparison.

Biperfectness alone allows several bases. The examples in [1] distinguish the MV, dual canonical and dual semicanonical constructions; their SL_6 calculation separates MV and dual semicanonical vectors despite the shared crystal labels. Their Duistermaat–Heckman measures detect information beyond the moment polytope. More recently, Baumann has recovered MV polytopes from the coproduct on the unipotent coordinate ring and related their extremal operations to stability decompositions [48]. Thus the common combinatorics is well understood, while identification of the actual functions requires finer information.

Fomin and Zelevinsky introduced cluster algebras in connection with total positivity and canonical bases [15]. Berenstein, Fomin and Zelevinsky constructed upper cluster structures on double Bruhat cells using generalized minors [16]. For $\mathbb{C}[N]$, a reduced word for the longest Weyl-group element gives an initial cluster. Its mutable variables change under exchange relations; its frozen variables remain coefficient functions. We use the original coordinate ring, in which the frozen minors are polynomial variables. Localizing at these minors produces the open cluster variety and is a separate operation in the argument.

Categorification established several links between cluster monomials and representation-theoretic bases. Geiss, Leclerc and Schröer use subcategories of modules over preprojective algebras to construct cluster structures on unipotent coordinate rings in the symmetric setting, with cluster monomials in the dual semicanonical basis [45]. Kang, Kashiwara, Kim and Oh use symmetric quiver Hecke algebras to obtain monoidal categorifications and the corresponding inclusion in dual canonical bases [46]. Qin's common triangular bases and injective-reachability results extend the cluster-basis comparison to the symmetrizable Lie-theoretic setting used here [6, 7, 20]. These results identify particular cluster families and supply the existence and triangularity inputs needed to compare complete bases.

Gross, Hacking, Keel and Kontsevich construct theta functions from consistent scattering diagrams [10]. A broken line begins with an integral tropical label and changes its attached monomial when it crosses a wall; summing its final monomials gives the theta function. In a fixed cluster, the function has a distinguished Laurent monomial with coefficient one, and all other exponents differ from its exponent by nonnegative combinations of exchange columns. This is pointedness. In the unipotent setting, the theta functions regular on N form a basis of $\mathbb{C}[N]$ and include the cluster monomials. Restriction from the localized algebra is controlled by valuations at the frozen divisors. Cheung, Magee, Mandel and Muller prove general valuative independence and theta reciprocity, extending the optimized-seed arguments of [10] [47]. We give the short frozen-divisor argument needed for this ring in Section 6.

1.2. **Kamnitzer's conjecture and subsequent work.** Kamnitzer's ICM 2022 report places the dual canonical, dual semicanonical and MV bases in a common framework of

perfect bases, crystals, polytopes and measures [12]. Its cluster-algebra section separates three assertions. Question 6.5 asks whether the theta basis is biperfect. Conjecture 6.6 asks whether the MV basis contains every cluster monomial and whether every MV element is pointed in every cluster. Conjecture 6.9 asserts equality of the MV and theta bases as functions. The last assertion determines the lower Laurent terms as well as the leading labels, and consequently identifies the entire multiplication table.

Several calculations support this comparison. Baumann, Gaussent and Littelmann prove cluster-monomial compatibility for reduced-word clusters satisfying the condition in their Proposition 7.2 [17]. Bai, Dranowski and Kamnitzer use the Mirković–Vybornov correspondence to compute fusion in type A; for the upper-unitriangular group of size four they verify the cluster exchange relations through MV products [18]. In rank-two cluster algebras, Cheung, Gross, Muller, Musiker, Rupel, Stella and Williams identify the greedy and theta bases [19]. That comparison is part of the evidence for the parallel between representation-theoretic and cluster bases discussed in the ICM report.

Fei develops a complementary approach through crystal structures on upper cluster algebras [21]. Reachable frozen vertices determine crystal operators on tropical labels; under the stated hypotheses these lift to generic bases. His classification of bases compatible with both crystal actions describes the freedom remaining after the two crystal structures have been fixed. The June 2026 revision still states the corresponding theta and triangular assertions as Conjecture 9.25. In a different geometric realization, his work on universal Schubert polynomials compares generic, canonical and MV elements attached to Schubert labels [42]. His ramified-quiver construction treats skew-symmetrizable exchange matrices admitting good realizations: under reachability, monodromy cluster characters give bases of the middle cluster algebra with frozen variables inverted, and of the upper cluster algebra when these two algebras coincide [43]. These hypotheses and localizations are relevant when transporting categorical constructions to the ring considered here.

Another recent direction uses characteristic cycles. Leroux-Lapierre relates asymptotic characters in category $\mathcal{O}$ for truncated shifted Yangians to equivariant multiplicities of characteristic cycles [49]. The proposed general change of basis from dual canonical to MV vectors appears there as Conjecture 6.3. Kamnitzer, Labelle, Leroux-Lapierre, Pinet and Weekes identify the complexified Grothendieck ring for simply-laced shifted Yangians with the Cox ring of an open bi-infinite Bott–Samelson variety and establish its cluster-algebra realization [50]. This supplies a categorical setting linking affine-Grassmannian geometry, canonical bases and cluster structures. The present comparison uses the spatial-fusion multiplication of MV cycles and the curve-counting multiplication of theta functions.

The enumerative construction of theta multiplication comes from logarithmic mirror symmetry. Gross and Siebert develop the logarithmic and intrinsic mirror constructions [11, 39]; Abramovich, Chen, Gross and Siebert construct punctured logarithmic maps and their obstruction theories [32]. Keel and Yu construct the mirror algebra of an affine log Calabi–Yau variety containing a torus by counts of non-archimedean curves [29]. Johnston compares these counts with punctured logarithmic invariants [30]. In our setting the relevant interior is an affine Richardson variety for the Langlands dual group. Its full torus, including the frozen directions, relates these curve counts to the cluster scattering diagram.

1.3. **The coefficient comparison.** Write $\mathcal{B}_{\mathrm{MV}}(\mathbb{C}[N])$ for the normalized MV basis and $\mathcal{B}_{\theta}(\mathbb{C}[N])$ for the regular theta basis. Our result is

$$\mathcal{B}_{\mathrm{MV}}(\mathbb{C}[N]) = \mathcal{B}_{\theta}(\mathbb{C}[N]). \tag{1}$$

Theorem 17.1 states the equality in the original coordinate ring. We first pair the bases by their values under a reduced-word valuation. Both are homogeneous, each weight space is finite-dimensional, and the values distinguish the elements within either basis. The pairing fixes the unit and every initial minor. It remains to compare multiplication coefficients:

multiplicativity then forces this pairing to fix every function, since the initial minors generate the function field.

For the geometric comparison, an actual MV cycle is represented by based polynomial loops. Keeping every coordinate of its fundamental representation vectors gives a flag map with controlled degrees. The auxiliary zeros of the frozen minors define a finite coefficient map. A double Bott–Samelson compactification of the Richardson variety has a nef divisor supported on its entire boundary. On this model, a general marked-curve and frame constraint has an entire incidence fiber consisting of smooth free maps. A pole-transfer argument recovers the original MV loop from such a map.

With two inputs, the Beilinson–Drinfeld fusion family and the logarithmic mapping space have the same generic incidence. Cartier specialization compares their classes over the collision. To extract an individual coefficient, both classes are pushed to the same finite set of actual central flag maps. The logarithmic source divisor is expanded over all separating cuts. Positive-dimensional central tropical types vanish as supported bivariant operations; the surviving central maps have weight one. The tail contribution retains the puncturing thickness, the finite logarithmic gluing degree and all admissible curve classes. Their product gives the punctured three-point coefficient at that same central map.

Finally, the full Richardson torus identifies the mirror labels with the valuation labels. We prove the weighted extension of the relevant Keel–Yu scattering comparison, using primitive center characters in every finite type. Restriction to the regular theta functions then identifies their multiplication with the coefficients obtained from fusion. The divisorial point-constraint calculation in Appendix A supplies the normalization for nonprimitive outputs.

2. An algebraic criterion

Lemma 2.1. *Let A be an integral algebra over a field k. Suppose that $(b_i)_{i\in I}$ and $(t_i)_{i\in I}$ are two k-bases, with corresponding unit elements, and that their finite multiplication expansions have the same coefficients:*

$$b_i b_j = \sum_{l\in I} c_{ij}^{l} b_l, \qquad t_i t_j = \sum_{l\in I} c_{ij}^{l} t_l.$$

Suppose that $x_1, \dots, x_m$ generate $\mathrm{Frac}(A)$ *as a field over k, and that each x_a is a common corresponding basis element. Then $b_i = t_i$ for every i.*

Proof. The linear bijection $P(b_i) = t_i$ preserves multiplication and the unit. It is therefore an algebra automorphism of A, and extends to an automorphism of $\mathrm{Frac}(A)$. It fixes every x_a, so its extension fixes $k(x_1, \dots, x_m) = \mathrm{Frac}(A)$. Its restriction to A is the identity. □

For $A = \mathbb{C}[N]$, one full initial cluster, including its frozen variables, generates the function field. Thus the criterion requires the identities of *all* multiplication coefficients, including the vanishing coefficients. It requires only the individual initial variables to be common basis elements.

3. Root data and prefix coordinates

We write $\mathbb{N} = \mathbb{Z}_{\geq 0}$. Fix a pinning of G, with simple roots α_i, simple coroots h_i, fundamental weights ω_i, and simple reflections s_i, indexed by a finite set I. Write e_i for the pinned positive Chevalley generator and $x_i(t) = \exp(te_i)$ for its root subgroup. Set $a_{ij} = \langle h_i, \alpha_j \rangle$. Let w_0 be the longest Weyl group element, and write i^* for the index satisfying $w_0\alpha_i = -\alpha_{i^*}$. Fundamental weights and their representations may be taken on the simply connected cover; its restriction to N is a pinned isomorphism.

Fix a reduced word $\mathbf{i} = (i_1, \dots, i_m)$ for w_0, and put $w_k = s_{i_1} \cdots s_{i_k}$. For a dominant weight λ, define $\Delta_{\lambda, w\lambda}(n) = v_\lambda^*(n v_{w\lambda})$, where $v_{w\lambda}$ is the pinned extremal vector and v_λ^* is the highest coefficient functional normalized by $v_\lambda^*(v_\lambda) = 1$. The prefix minors are

$$X_k = \Delta_{\omega_{i_k}, w_k\omega_{i_k}}\big|_N, \qquad 1 \leq k \leq m.$$

Let k^- and k^+ be the preceding and following occurrences of i_k, with absent indices 0 and $m+1$. The mutable set is $J = \{k : k^+ \le m\}$; the other positions are frozen.

We use the full $m \times |J|$ column matrix B given by

$$B_{ak} = \begin{cases} 1 & a = k^-, \\ -1 & a = k^+, \\ a_{i_a i_k} & a < k < a^+ < k^+, \\ -a_{i_a i_k} & k < a < k^+ < a^+, \\ 0 & \text{otherwise.} \end{cases}$$

The source convention uses the opposite exchange matrix $C = -B$, required by the inverse successor coordinates on the dual variety. The exponent lattice is the full lattice $M = \mathbb{Z}^m$ with basis given by the X_k. The successor-row minor makes $C : \mathbb{Z}^J \to M$ injective with saturated image: order the mutable columns by increasing successor k^+ and select the successor rows in that order. Since $C_{ak} = 0$ for $a > k^+$ and $C_{k^+,k} = 1$, this auxiliary minor is upper unitriangular. Its integral inverse gives an integral left inverse to C. The ambient coordinate order used for valuations remains unchanged.

For $a \in \mathbb{Z}^m$, put $X^a = \prod_{k=1}^m X_k^{a_k}$. For $a, b \in \mathbb{Z}^m$, write $a < b$ in reverse lexicographic order when the last nonzero coordinate of $b-a$ is positive. For a nonzero Laurent polynomial $f = \sum_a f_a X^a$, define $v_X(f) = \min\{a : f_a \neq 0\}$ in this order. Set $v_X(f/g) = v_X(f) - v_X(g)$ for nonzero Laurent polynomials f, g. The valuation v_t is defined in the same way in the factorization coordinates below. The coefficient of X^a means the scalar f_a, taken as zero when that monomial is absent.

For an integral exponent vector g, a theta function with label g in this convention has an expansion with scalar coefficients c_v of the form

$$\theta_g = X^g \left(1 + \sum_{0 \neq v \in \mathbb{N}^J} c_v X^{Cv}\right).$$

Only labels giving regular functions in $\mathbb{C}[N]$ are used as basis labels below.

The right Chevalley derivation is

$$D_i f(n) = \left.\frac{d}{dt}\right|_{t=0} f(n x_i(t)).$$

Its string coordinates use the *reverse product* chart

$$(t_1, \dots, t_m) \longmapsto x_{i_m}(t_m) \cdots x_{i_1}(t_1). \tag{2}$$

This order matters: right multiplication by $x_{i_1}(u)$ sends t_1 to $t_1 + u$ in this chart. Pinned transpose is an anti-isomorphism and reverses the factor order.

Define the lower unitriangular integral matrix

$$\mathsf{M}_{ak} = \begin{cases} \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_a} \omega_{i_a} \rangle & k \le a, \\ 0 & k > a. \end{cases}$$

The prefix-minor valuation formula of Fujita–Oya [2, Proposition 6.3 and Theorem 6.5], after transpose and the order convention (2), uses the rows of this matrix. The least reverse-lexicographic monomial of X_a has exponent $\mathsf{M}_{a,*}$ and coefficient one. For row vectors, multiplication by a lower unitriangular matrix preserves reverse lexicographic order: if the last nonzero coordinate of $b - a$ is j, then the coordinates of $(b-a)\mathsf{M}$ after j vanish and its j-th coordinate is $b_j - a_j$. Consequently different monomials X^a have different leading t-exponents $a\mathsf{M}$. A finite Laurent sum has the leading exponent of its least summand, since those exponents are distinct. Taking quotients gives

$$v_t(f) = v_X(f)\mathsf{M} \qquad (f \in \mathbb{C}(N)^\times).$$

Remark 3.1. For $G = SL_3$ and $\mathbf{i} = 121$, write

$$n = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(X_1, X_2, X_3) = (x, xy - z, z)$. Chart (2) gives $x = t_1 + t_3$, $y = t_2$, $z = t_3 t_2$, and the rows of M are $(1,0,0), (1,1,0), (0,1,1)$.

4. Polynomial-loop representatives

Let H be the Langlands dual group, with its dual pinning and opposite unipotent groups U_H^+ and U_H^-. The roots α_i of G are identified with the simple coroots of H. Write $B_\pm = T_H U_H^\pm$ for its pinned Borel subgroups and H_{ad} for its adjoint form. Use the pinned representative $\dot{w}_0 \in H$ of its longest Weyl element. Put $K = \mathbb{C}((z))$, $O = \mathbb{C}[[z]]$, and $\mathrm{Gr}_H = H(K)/H(O)$. For a coweight η, set $L_\eta = z^\eta H(O)$ and $S_\eta^\pm = U_H^\pm(K) L_\eta$. A stable upper MV cycle of charge $\nu = \sum_i \nu_i \alpha_i$, with $\nu_i \in \mathbb{N}$, is an irreducible component of the reduced closure of $S_0^+ \cap S_{-\nu}^-$.

The normalized function b_Z is defined by geometric Satake: for a sufficiently dominant allowed weight λ, it is the matrix coefficient $b_Z(n) = v_\lambda^*(n[z^\lambda Z])$, where the bracket denotes the fundamental cycle class in $V_G(\lambda)$ and $v_\lambda^*(v_\lambda) = 1$. Stabilization makes it independent of the sufficiently large choice [1, Section 6.1].

Its affine polynomial-loop part consists of points

$$n(z^{-1})^{-1} L_0, \qquad n(t) \in U_H^+(\mathbb{C}[t]), \quad n(0) = 1.$$

Ordered root coordinates give uniqueness: split off negative and nonnegative Laurent powers successively in root height; commutators change only higher roots. This construction is polynomial on each bounded pole chart and works over arbitrary coefficient algebras.

All charges used here belong to the coroot lattice of H. The simply connected covering group has the same unipotent subgroups and identifies its affine Grassmannian with the neutral component relevant to these charges. Fundamental representations below are taken on this cover when necessary. For each fundamental representation $V_H(\omega_i)^*$, let v_i^- be its lowest vector normalized by $\langle v_i^-, v_i^+ \rangle = 1$, where v_i^+ is the pinned highest vector of $V_H(\omega_i)$. Its lowest weight is $-\omega_i$. Fix a weight basis containing v_i^-. The degree of a polynomial vector is the maximum of the degrees of its nonzero coordinates; this is independent of the chosen constant basis. All coordinates of $n(t)v_i^-$ are retained.

Lemma 4.1. *Over a geometric field, the point $n(z^{-1})^{-1} L_0$ belongs to $S_{-\nu}^-$ if and only if*

$$\deg_t n(t) v_i^- = \nu_i \quad \textit{for every } i.$$

The inequalities with $\le \nu_i$ are polynomial coefficient equations on every bounded chart.

Proof. Write $n^{-1} = u_- z^\mu k$ with $u_- \in U_H^-(K)$ and $k \in H(O)$. Since u_- fixes the lowest vector,

$$n v_i^- = z^{\langle \omega_i, \mu \rangle} k^{-1} v_i^-.$$

The last vector is integral and primitive. Its minimum coordinate order is therefore $\langle \omega_i, \mu \rangle$, which is minus the degree of the polynomial vector. Thus $\mu = -\nu$ is equivalent to the displayed degree equalities. The Iwasawa orbit label is unique, and the fundamental weights detect it. The equations for an upper degree bound are precisely the vanishing of all higher coefficients. □

Index fundamental degrees by these lowest-line representations, with flag target H/B_-. The same vectors give an actual based flag quasimap through

$$P_i(z) = z^{\nu_i} n(z^{-1}) v_i^-.$$

They satisfy the homogeneous Plücker relations because they do so away from $z = 0$, where they come from a group element; multiplication by z is injective in the polynomial coefficient

ring. At a point where all vectors are nonzero they define the flag, and a common vector factor records the corresponding defect.

Write $V = U_H^-$. To define its fundamental extremal minors, take the representation $V_H(\omega_i)$, a pinned highest vector v_i^+ and the coefficient functional ϕ_i^- of its pinned lowest vector $\dot{w}_0 v_i^+$. Put

$$F_i(v) = \phi_i^-(v v_i^+) \qquad (v \in V).$$

These are regular functions on the unipotent group, also when the representation is taken on its simply connected cover. The dual matrix-coefficient calculation pairs the row of n^{-1} with $n v_i^-$; conjugation by $\dot{w}_0$ exchanges its highest and lowest extremal rows. This fixes the same index i as in Lemma 4.1. Let

$$Y = \{v \in V : F_i(v) \neq 0 \text{ for all } i\}.$$

Let $\mathscr{H} \subset V$ be a nonempty open of constant frames h with all $F_i(h^{-1}) \neq 0$; it will be shrunk when finitely many genericity conditions are imposed. Choose $h \in \mathscr{H}$ and define

$$g_{n,h}(z) = \dot{w}_0 n(z^{-1})^{-1} \dot{w}_0^{-1} h^{-1}. \tag{3}$$

The monic polynomials

$$Q_i(z) = z^{\nu_i} \frac{F_i(g_{n,h}(z))}{F_i(h^{-1})}$$

have degree ν_i. Indeed, the definitions and the contragredient action give

$$F_i(g_{n,h}(z)) = \langle n(z^{-1}) v_i^-, \dot{w}_0^{-1} h^{-1} v_i^+ \rangle.$$

Lemma 4.1 bounds its pole by ν_i, and its value at infinity is $F_i(h^{-1})$. It extends to the reduced affine closure by vanishing of polynomial coefficients.

5. Finite coefficient maps

For each fundamental representation the map

$$U_H^+ \longrightarrow \prod_i V_H(\omega_i)^*, \qquad u \longmapsto (u v_i^-)_i$$

is a closed immersion into the affine subspace where every lowest coordinate is one. This follows by restricting the fundamental Plücker embedding of H/B_- to its big cell and normalizing these coordinates. Every root coordinate is consequently a polynomial in the complete-vector coordinates.

Let $\mathcal{P}_\nu$ be the affine scheme of based polynomial loops with $\deg n(t) v_i^- \leq \nu_i$ for every i. It is defined in the finite space of bounded-vector coefficients by the Plücker equations and $n(0) = 1$. These coefficient equations represent the bounded functor over arbitrary coefficient algebras. The exact-charge criterion of Lemma 4.1 is used over geometric fields. Let Z^{aff} be the reduced closure of the corresponding irreducible exact-charge locus in $\mathcal{P}_\nu$. It is integral and has dimension $d = \mathrm{ht}(\nu) = \sum_i \nu_i$ [4, Proposition 7]. This explicit bounded construction also respects the finite-stage correction in [34].

Loop rotation preserves each component, since $\mathbb{G}_m$ is connected. It extends to $n(t) \mapsto n(ct)$ at $c = 0$ and contracts the source to $n = 1$. Its coordinate ring is positively graded with degree-zero part $\mathbb{C}$.

Lemma 5.1. *Suppose a smooth variety Y has a smooth projective compactification (W, E) with reduced simple normal crossings (SNC) boundary and $K_W + E \simeq \mathcal{O}_W$. No finite-type family of nonconstant morphisms $\mathbb{A}^1 \to Y$ covers Y.*

Proof. Such a family would give, after a transverse parameter slice, a dominant generically finite morphism $T \times \mathbb{A}^1 \to Y$ with $\dim T = \dim Y - 1$. After shrinking smooth T, properness extends it to $T \times \mathbb{P}^1 \to W$. Indeed the indeterminacy is in $T \times \{\infty\}$ and has codimension at least two, so its image can be removed from T. Pullback of a nonzero logarithmic volume

form is a nonzero section of $K_{T\times\mathbb{P}^1}(T\times\{\infty\})$. This line bundle has degree -1 on every $\mathbb{P}^1$ fiber and hence has no such section. □

For the open Richardson variety Y, the double Bott–Samelson compactification has these properties [13, Section 4].

Proposition 5.2. *For a sufficiently general frame h, the map*

$$\chi_h : Z^{\mathrm{aff}} \longrightarrow \prod_i \mathbb{A}^{\nu_i}$$

recording the coefficients of $z^0,\dots,z^{\nu_i-1}$ in each monic Q_i is finite and surjective. For $\nu_i=0$ this coefficient list is empty.

Proof. Use the equivalent coefficients of $F_i(g_{n,h}(t^{-1}))/F_i(h^{-1})$. Their coefficients of t^k are homogeneous of positive degree k. If their common zero locus contained a nonvertex point for general h, the resulting incidence family would give the nonconstant curves

$$t \longmapsto \dot{w}_0 n(t)^{-1}\dot{w}_0^{-1}h^{-1} \in Y.$$

They remain in Y because all extremal minors are constant and nonzero. The incidence is of finite type because the polynomial-loop degrees have a common bound. If it dominated the frame open, one of its finitely many irreducible components would dominate; the evaluation at $t=0$ on that component would be dominant onto Y. Its nonvertex points give nonconstant maps because $n(t)$ is based and nonconstant. This produces exactly the family forbidden by Lemma 5.1. Thus, on a nonempty frame open, the common zero fiber is supported at the vertex.

Let $R=\mathbb{C}[Z^{\mathrm{aff}}]$ and let J be the ideal generated by these positive-degree coefficients. Then R/J is finite-dimensional. Choose homogeneous lifts of a vector-space basis of R/J. For each homogeneous $f\in R$, subtract the linear combination of these lifts representing its quotient class. The remainder belongs to J. Express it as a sum of homogeneous coefficient generators times homogeneous elements of smaller degree and apply induction. These lifts generate R over the coefficient subalgebra, so χ_h is finite. Its image has dimension d, equal to the dimension of its affine target; since the source is integral the image is dense, and finiteness makes it surjective. □

In characteristic zero the finite map (χ,h) is generically étale. For finitely many sources, first choose a common dense open of the coefficient/frame base over which all these maps are étale, and then intersect it with the locus of distinct auxiliary roots avoiding the input and unit marks. On its full root-ordering cover, fixing the marked positions leaves $m=\dim N$ independent frame directions.

6. Valuations and the regular basis pairing

For a generic seed in the next argument, M° is its full exponent lattice, $N^\circ=\mathrm{Hom}(M^\circ,\mathbb{Z})$, and N is the seed lattice with basis (e_i) containing N° with finite index. These lattice symbols are local to the scattering argument; the algebraic group remains the maximal unipotent subgroup of G. The integers $d_i>0$ specify N° by its basis d_ie_i. The mutable sublattice is $N_{\mathrm{uf}}=\bigoplus_{i\in I_{\mathrm{uf}}}\mathbb{Z}e_i$, where I_{uf} indexes the mutable seed vectors. For the seed skew form $\{\ ,\ \}$, define $p^*:N_{\mathrm{uf}}\to M^\circ$ by $p^*(n)(n^\circ)=\{n,n^\circ\}$ for $n^\circ\in N^\circ$. Pairings with N use the rational extension of the natural pairing with N°. Write x^m for a formal Laurent monomial of exponent $m\in M^\circ$, and y^n for a positive graded Lie-algebra symbol, where n is a nonzero nonnegative combination of mutable seed vectors. Its action is denoted D_n below. The Lie bracket is $[y^n,y^{n'}]=\{n',n\}y^{n+n'}$, as in [10, equation (1.1)]; completion is by the sum of the mutable coefficients of n.

The string of a normalized MV function b along $\mathbf{i}$ is obtained by successively taking the largest nonzero powers of $D_{i_1},\dots,D_{i_m}$ and dividing each maximal derivative by its factorial.

Denote the resulting vector by $s_{\mathbf{i}}(b)$. More explicitly, start with $b^{(0)} = b$ and put

$$s_k = \max\{a \geq 0 : D_{i_k}^a b^{(k-1)} \neq 0\}, \qquad b^{(k)} = \frac{D_{i_k}^{s_k}}{s_k!} b^{(k-1)}.$$

Right multiplication on N induces this derivation; it is the left action on functions in [1, Section 2.1]. The maximum is finite because the derivations are locally nilpotent on $\mathbb{C}[N]$. The perfect-basis property and normalization in [1, Theorems 5.4 and 6.2] make the final endpoint equal to 1.

6.1. **Opposite theta functions in the same function field.**
Opposition for the full skew-symmetrizable scattering data. Although Appendix A.2 of [7] is introduced in the context of skew-symmetric quivers with potential, its opposition calculation is valid for arbitrary positive symmetrizers, as follows. Keep the lattices N, N°, M°, all frozen coordinates and every d_i fixed. Replacing the seed by its opposite changes the skew form to its negative and changes p^* to $-p^*$. For the positive graded Lie algebras write $\iota : \mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$ for the identity on the monomial symbols y^n. Then

$$[\iota u, \iota v]_{\mathrm{op}} = -\iota[u, v].$$

Consequently ι induces a group *anti*-isomorphism on the completed groups defined by the Baker–Campbell–Hausdorff formula. On the Laurent-series modules put $\kappa(x^m) = x^{-m}$. The two actions on monomials satisfy

$$D_n(x^m) = \langle m, n\rangle x^{m+p^*n},$$
$$D_n^{\mathrm{op}}(\kappa x^m) = -\langle m, n\rangle x^{-m-p^*n} = -\kappa D_n(x^m).$$

For a generic path γ, let P_γ be its ordered wall-crossing automorphism. Reflecting a path reverses its wall-crossing signs. Combining this sign reversal with this anti-isomorphism gives

$$P_{\kappa\gamma}^{\mathrm{op}}\,\kappa = \kappa P_\gamma.$$

The reflected diagram is consistent, and its incoming walls are exactly those of the opposite seed, with the unchanged multipliers d_i. Uniqueness of consistent completion identifies it with the opposite cluster scattering diagram. Write $\Theta(t)$ for the theta set of the seed t. Reflection of the defining broken lines then identifies its initial-chamber theta set with $\kappa P\Theta(t)$, where P transports the original positive chamber to the reachable negative chamber.

A seed is injective-reachable if a reachable seed $t[1]$ has mutable parts of its mutable g-vectors equal, up to permutation, to the negatives of the initial coordinate vectors.

Lemma 6.1. *Let a full-rank seed t be injective-reachable. Identify the initial variables of t and its opposite t^{op}, whose exchange matrix is $-B(t)$, in the same rational-function field. Then the two sets of Laurent theta functions are equal as sets of actual rational functions. All frozen variables remain in this identification.*

Proof. By [7, Theorem A.1.5], the full-rank injective-reachable seed has Laurent polynomial theta functions. The strict-cone hypothesis in the A-variety form of the GHKK result holds because the exchange columns are linearly independent: a rational linear functional can be chosen to take value one on every such column. Thus the Laurentness hypothesis in [8, Section 6.1 and Theorem 6.2] is satisfied before we apply the twist automorphism.

Set $t' = t[1]$, relabeling its mutable indices by the reachability permutation. Let $F(t)$ and $F(t')$ be their rational-function fields in all mutable and frozen variables, and let $\mu^* : F(t') \to F(t)$ be the mutation isomorphism. Write $\psi : M^\circ(t') \to M^\circ(t)$ for the linear map of the reachable chamber. This chamber is $\mathcal{C}^-$, whereas the initial one is $\mathcal{C}^+$. Let P denote the wall-crossing operation from $\mathcal{C}^+$ to $\mathcal{C}^-$. For each cluster monomial of t', the monomial expression in its own chamber gives

$$P \circ \mu^* = \psi_*, \qquad P = \psi_* \circ (\mu^*)^{-1}.$$

It suffices to check this on the cluster variables and invertible frozen variables, which generate the field. Here a subscript $*$ means the monomial map induced by the indicated lattice map. These identities are the mutation/wall-crossing compatibility recalled in [7, Appendix A.1].

Let $\kappa(x^m) = x^{-m}$. Reflection of broken lines, with their full exponents, gives

$$\Theta(t^{\mathrm{op}}) = \kappa P \Theta(t)$$

by [7, Proposition A.2.5 and Lemma A.2.6]. The variation of [8, Proposition 5.1] satisfies

$$\psi \, \mathrm{var}^M = -I, \qquad \mathrm{var}^M = \begin{pmatrix} I_{\mathrm{uf}} & 0 \\ -F_{\mathrm{f,uf}} & -I_{\mathrm{f}} \end{pmatrix}.$$

Here $F_{\mathrm{f,uf}}$ is the frozen-by-mutable block of the reachable-chamber map $\psi = \left(\begin{smallmatrix} -I_{\mathrm{uf}} & 0 \\ F_{\mathrm{f,uf}} & I_{\mathrm{f}} \end{smallmatrix}\right)$; the index sets uf and f denote mutable and frozen coordinates. Thus the Donaldson–Thomas twist $\mathrm{DT} = \mu^* \circ \mathrm{var}_*$ obeys

$$\mathrm{DT}^{-1} = (-\psi)_* \circ (\mu^*)^{-1} = \kappa P.$$

Theorems 5.2 and 6.2 of [8] say that this twist is an automorphism preserving the theta set. Therefore $\Theta(t^{\mathrm{op}}) = \mathrm{DT}^{-1}\,\Theta(t) = \Theta(t)$.

□

Frozen localization. In this cluster-algebra argument, F_i denotes the prefix minor X_k at the last occurrence of color i. The twist acts on these frozen variables by $\mathrm{DT}(F_i) = F_i^{-1}$. The preceding equality is an equality of localized theta sets in the common function field. Intersecting both sets with $\mathbb{C}[N]$ gives equality of their regular subsets. The valuation in the fixed initial variables will label these functions.

6.2. **The regular theta basis.** Put $A = \mathbb{C}[N]$ and $A^\circ = A[(\prod_i F_i)^{-1}]$. The all-type unipotent-cell cluster theorem [2, Theorem 5.3 and Appendix B] identifies A° with the full upper cluster algebra. The initial seeds are injective-reachable by [6, Theorem 8.2.2 and Lemma 9.1.3], and have full column rank. Thus [7, Theorem A.1.5] gives a Laurent theta basis of A°, indexed by the full lattice M. Lemma 6.1 permits the convention $C = -B$ with the initial variables fixed.

For the restriction to A, each F_i cuts out an irreducible reduced divisor in N: it is the restriction to the big cell of the corresponding fundamental Schubert divisor. For a fixed color i, choose a reduced word for w_0 ending in i. All such prefix seeds are mutation equivalent by [2, Theorem 5.4]. In this seed the frozen position $f = m$ has

$$C_{fk} = \begin{cases} 1 & k^+ = f, \\ 0 & \text{otherwise}, \end{cases} \qquad k \in J.$$

In particular every column has nonnegative f-coordinate.

The geometric-type Laurent phenomenon, together with the all-type identification of the nonlocalized cluster algebra with A [6, Theorem 9.2.3], gives

$$A \subset \mathbb{C}[X_f, X_j^{\pm 1} \ (j \neq f)].$$

The X_f-adic Gauss valuation on the common function field is therefore nonnegative on A. Its center contains $F_i = X_f$. The residue algebra contains the $m-1$ algebraically independent remaining initial variables, so its center has height one. Since F_i is prime, the center is (F_i). The DVR $A_{(F_i)}$ is dominated by the valuation ring; its units have valuation zero and its uniformizer F_i has valuation one. Thus the valuation is exactly ord_{F_i} on the common function field.

For every theta function in this seed, its finite pointed expansion has support in $g + C\mathbb{N}^J$ and leading coefficient one. It follows that $\mathrm{ord}_{F_i}(\theta_g) = g_f$. Moreover, for every finite linear

combination with nonzero coefficients,

$$\operatorname{ord}_{F_i}\Big(\sum_g a_g\theta_g\Big) = \min_{a_g\neq 0} \operatorname{ord}_{F_i}(\theta_g). \tag{4}$$

To see the last assertion, keep the terms of the least f-degree. On these terms, corrections have zero f-coordinate. Their nonzero exponents lie in the strictly convex cone generated by the corresponding independent columns of C. A total order positive on that cone makes the least remaining leading monomial unique, so it cannot cancel. This is the full-rank optimized-seed argument of [10, Proposition 9.7]; see [47] for general valuative independence.

The variety $N = \operatorname{Spec} A$ is normal, and $N \setminus \operatorname{Spec} A^\circ$ is the union of the divisors $F_i = 0$. Equivalently, inside A° a function lies in A exactly when all ord_{F_i} are nonnegative. Applying (4) for every i proves that

$$\{\theta_g : \theta_g \in \mathbb{C}[N]\}$$

is a basis of $\mathbb{C}[N]$.

Proposition 6.2. *The regular theta basis and the normalized MV basis admit a unique weight-preserving pairing with equal v_X-values. This pairing fixes* 1 *and every initial variable X_k.*

Proof. Let $\tau_{\mathrm{pin}} : U_G^- \to N_G$ be pinned transpose, and transport functions by $f \mapsto f \circ \tau_{\mathrm{pin}}$. Write f_i for the negative Chevalley generator and $y_i(t) = \exp(tf_i)$ for its root subgroup. The action called the left action in [1, Section 2.1] is $(n \cdot f)(x) = f(xn)$, so its infinitesimal action is precisely our D_i. In the conventions of [3, Section 3],

$$(-f_i) \cdot (f \circ \tau_{\mathrm{pin}}) = (D_i f) \circ \tau_{\mathrm{pin}},$$

because $\tau_{\mathrm{pin}}(y_i(t)v) = \tau_{\mathrm{pin}}(v)x_i(t)$. The normalized MV basis satisfies the two positivity conditions of [3, Section 4]: the Chevalley coefficients are nonnegative by [1, Theorem 5.4], and multiplication by the simple-root basis elements is nonnegative by [1, Theorem 7.11]. Its stability under the involution fixing the simple enveloping generators, and the exchange of the two actions, are [1, Proposition 6.6 and Remark 6.7]. The root-length factor in Theorem 5.4 cancels against the identical factor in [1, Proposition 5.5]; the normalized leading Chevalley coefficient is therefore ε_i in every finite type.

Pinned transpose changes the forward negative product chart of [3] into (2). [3, Proposition 3.28 and Theorem 5.1] identify the highest lexicographic exponent with the least reverse-lexicographic exponent, after undoing the reversal in the reported vector for the latter valuation. In the coordinate order $(t_1, \dots, t_m)$ both exponents are $s_{\mathbf{i}}(b)$. Their coefficient is one: successive maximal divided derivatives $D_{i_k}^{s_k}/s_k!$ reach the normalized basis element 1, so the highest lexicographic monomial has coefficient one; the lowest reverse-lexicographic monomial is the same monomial. Consequently

$$v_X(b) = s_{\mathbf{i}}(b)\mathsf{M}^{-1}. \tag{5}$$

These values distinguish MV basis elements.

The full-rank and injective-reachability hypotheses used above hold for the prefix seeds in every finite symmetrizable type by [6, Theorems 8.1.2, 8.2.2, 9.2.3 and Lemma 9.1.3]. Lemma 6.1 identifies the two theta sets as actual functions in the specified common field, with the frozen variables retained. The preceding regularity argument selects the basis of the original ring $\mathbb{C}[N]$.

Each column of $C = -B$ is positive in reverse lexicographic order: its last nonzero entry is 1 in its successor row. The same is true for a nonzero nonnegative combination. Thus $v_X(\theta_g) = g$, with leading coefficient one. Both bases are homogeneous for the root grading, and each homogeneous part of $\mathbb{C}[N]$ is finite-dimensional. If a basis of a finite-dimensional space has distinct values under a valuation, the values of its nonzero linear combinations are

exactly its basis values: the least term cannot cancel. Applying this observation to the two bases gives a unique weight- and unit-preserving pairing

$$P(b_Z) = \theta_{g_Z}, \qquad g_Z = v_X(b_Z). \tag{6}$$

Each X_k is a normalized extremal matrix coefficient and hence is an MV basis element. It is also the initial theta monomial. Its valuation is the k-th coordinate vector, and its extremal divided-power normalization is one. Therefore $P(X_k) = X_k$. □

6.3. **The actual source contact.** For a reduced word $\mathbf{r} = (r_1, \dots, r_m)$ in the geometric group H, write

$$u = y_{r_1}(u_1) \cdots y_{r_m}(u_m)$$

for the ordinary negative factorization coordinates on V, where $y_i(u) = \exp(u f_i^H)$ and f_i^H is the pinned negative Chevalley generator of H. Define the full successor ratios

$$q_k = u_{k^+}/u_k \quad (k^+ \le m), \qquad q_k = 1/u_k \quad (k^+ = m+1).$$

Here $\mathbf{i} = \mathbf{r}^*$, with color change and no word reversal. Put $\nu^* = -w_0\nu$. Opposition sends $S_0^+ \cap S_{-\nu}^-$ to $S_{\nu^*}^+ \cap S_0^-$, so $\dot{w}_0 Z$ has the $\mu_- = 0$ normalization of [4]. Let $\varepsilon_i(Z)$ be the first right-string coordinate of b_Z in a word starting with i. The operators $\widetilde{e}_i$ and $\widetilde{f}_i$ denote the corresponding crystal raising and lowering maps; superscript BG denotes the normalization of [4]. The stable endpoint formula in [1, Section 5.2] and the rank-one definition of φ_i^{BG} in [4, Section 4.1] give

$$\varepsilon_i(Z) = \max\{a \ge 0 : L_{-\nu + a\alpha_i} \in Z\} = \varphi_{i^*}^{\mathrm{BG}}(\dot{w}_0 Z).$$

The containment and weight characterizations of the crystal operators then imply

$$\dot{w}_0(\widetilde{e}_i Z) = \widetilde{f}_{i^*}^{\mathrm{BG}}(\dot{w}_0 Z).$$

Thus the BG shrinking string of $\dot{w}_0 Z$ in the word $\mathbf{r}$ is exactly our right string $s_{\mathbf{i}}(b_Z)$, with the same order of successive operations.

The coordinate valuation statement uses the full Laurent cylinder in [4, Section 4.5], which supplies a nonempty open of independent nonzero leading coefficients for Laurent parameters of the modified string valuations, whose Grassmannian image is dense in the cycle. For the fixed valuations used here, a faithful representation gives a uniform pole bound. Root-height Laurent splitting then shows that the normalized polynomial-loop representative depends on only finitely many jets of those parameters. Hence this cylinder argument takes place in the explicit bounded chart of Section 5; see also [34].

The Cartan matrix of H is the transpose of that of G. In the formula of [4, Section 4.5], the coefficient is the root–coroot pairing $\langle \alpha_{r_k}^H, h_{r_l}^H \rangle = a_{r_k r_l}$. Write s_k for the kth entry of $s_{\mathbf{i}}(b_Z)$. The ordinary-factorization cylinder has coordinate valuations

$$c_k = -s_k - \sum_{l>k} a_{r_k r_l} s_l.$$

In the following matrix calculation, $s = (s_k)$ and $c = (c_k)$ are column vectors. Let D be the full successor-difference matrix, $(\mathsf{D}c)_k = c_k - c_{k^+}$, with the absent successor term equal to zero. Taking successor ratios gives $-\mathsf{D}c$. If K is upper triangular with $\mathsf{K}_{kk} = 1$ and $\mathsf{K}_{kl} = a_{r_k r_l}$ for $l > k$, then $c = -\mathsf{K}s$ and

$$\mathsf{D}\mathsf{K}\mathsf{M}^t = I.$$

Indeed the k-th row of $\mathsf{D}\mathsf{K}$ has entries 1 at k and k^+, the Cartan entries between them, and zero afterward. Opposition preserves the Cartan entries, so they can be written $a_{i_k i_l}$. Fix a row a of M. If $a < k$, the product is zero, while for $a = k$ it is one. If $k < a < k^+$, the reflection formula gives

$$\mathsf{M}_{ak} + \sum_{l=k+1}^{a} a_{i_k i_l} \mathsf{M}_{al} = \langle h_{i_k}, \omega_{i_a} \rangle = 0.$$

If $k^+ \le a$, the same telescoping stopped at the next occurrence gives

$$\begin{aligned}\mathsf{M}_{ak} + \sum_{l=k+1}^{k^+-1} a_{i_k i_l}\mathsf{M}_{al} &= \langle h_{i_k}, s_{i_{k^+}} \cdots s_{i_a}\omega_{i_a}\rangle \\ &= -\mathsf{M}_{a,k^+},\end{aligned}$$

since $i_{k^+} = i_k$. At a last occurrence only the preceding case occurs. These are all entries of DKM^t. Hence $-\mathsf{D}c = \mathsf{M}^{-t}s$, exactly the column form of (5).

Lemma 6.3. *For any finite collection of generic MV sources and reduced-word tests, a single nonempty open of constant frames makes the orders of the actual successor-ratio functions in (3) equal to g_Z. The assertion includes frozen positions and remains true for regular power-series frames with the same sufficiently general constant term.*

Proof. Changes between ordinary reduced-word coordinates are positive birational maps in both directions [5, Theorem 5.2 and Section 7]. At a fixed tropical value their initial rational maps are inverse birational maps: the positive expressions do not cancel at a generic leading coefficient. A generic right multiplication by a simple root subgroup can be computed in a word ending in that root. It adds a constant to the last parameter, whose order is the negative of a string length and is nonpositive. Its leading coefficients remain generic; returning to the original word preserves the tropical value. Iterating gives the assertion for a generic constant right frame on the generic cylinder.

For a field extension $L/\mathbb{C}$ and a fixed representative $g \in V(L((z)))$, use polynomial root coordinates $(h_1, \dots, h_m)$ on V. For a nonzero polynomial $P = \sum_\alpha a_\alpha(z)h^\alpha$, define $v_{\mathrm{G}}(P) = \min_{a_\alpha \neq 0} \operatorname{ord}_z a_\alpha$, and extend to nonzero rational functions by $v_{\mathrm{G}}(P/Q) = v_{\mathrm{G}}(P) - v_{\mathrm{G}}(Q)$. For $f \in \mathbb{C}(V)^\times$, integral right multiplication of g leaves $v_{\mathrm{G}}(h \mapsto f(gh))$ unchanged, because left translation by an integral element of V preserves the generic special-fiber valuation of $V_{L[[z]]}$. The generic cylinder dominates the actual cycle. After a common field extension, its representative and any other generic representative differ by an element of $V(L[[z]])$. Thus the same Gauss values apply to the actual representative.

Only finitely many numerator and denominator leading polynomials are involved. Their nonvanishing can be imposed by one nonempty open of complex constant frames. For coefficients in a field extension, expand the finitely many coefficients in a complex vector-space basis; nonvanishing of one resulting complex polynomial suffices. Substituting a regular power-series frame with that constant term does not change these leading polynomials. This proves the statement. □

The frame open can also impose the finite tests $1 + q_k$ when $\operatorname{ord} q_k = 0$. Generic cylinder residues avoid their vanishing. Thus the generic input avoids the old binomial centers appearing in the next construction. Write p_Z for the integral tropical point represented by g_Z in this full chart. On a specified simple normal crossings model, $c_\rho(p_Z)$ denotes its contact order with D_ρ; the stratum component is the one attained by the actual generic map. These are the boundary data used below.

Fix a smooth SNC model before specializing the frame. On the generic Laurent cylinder, the independent leading coefficients give the monomial valuation of the full torus function field; the Gauss argument in Lemma 6.3 transports that valuation to the actual source. Near its center on the chosen model, take finitely many affine algebra generators, local boundary equations, denominators and functions whose nonvanishing selects the required stratum component. Impose the nonvanishing of all their leading numerator and denominator coefficients. This gives one dense frame open for these tests. Their orders determine the complete boundary contact and its stratum on this model. Repeating for the finite collection of sources and models used below amounts to intersecting finitely many dense opens.

7. The full torus in the Richardson variety

Proposition 7.1 (The full quotient torus and its toric model)**.** *Let H be the pinned Langlands dual group, let H_{ad} be its adjoint form, and let T_{ad} be the maximal torus of that adjoint form. Set*

$$Y = U_H^- \cap B_+ \dot{w}_0 B_+.$$

For the reduced word $\mathbf{r} = (r_1, \dots, r_m)$, write $u = y_{r_1}(u_1) \cdots y_{r_m}(u_m)$ on its factorization torus and put

$$q_k = \begin{cases} u_{k^+}/u_k & k^+ \le m, \\ 1/u_k & k^+ = m+1. \end{cases}$$

With $\mathbf{i} = \mathbf{r}^*$ and the full columns $C = -B$ of Section 3, the initial q-torus and its singly mutated tori embed openly in Y and cover it in codimension one. The partial toric blowup with rays C_k, centers $1+q_k = 0$, and strict toric boundary removed admits an open immersion into Y with complement of codimension at least two. This immersion preserves the specified full torus and its logarithmic volume form.

Proof. We descend the decorated spaces of [9] while retaining the frozen variables. Central isogenies identify pinned unipotent groups, so we may use H_{ad} to describe the target. The factorization $B_- = T_{\mathrm{ad}} U_H^-$ restricts to an isomorphism

$$T_{\mathrm{ad}} \times Y \longrightarrow H_{\mathrm{ad}}^{w_0,e}, \qquad (t, v) \longmapsto tv.$$

In particular, the left-torus quotient is a globally trivial principal bundle. The quotient uses the free T_{ad}-action; the map $T_H \to T_{\mathrm{ad}}$ may have a finite central kernel.

For reduced words, the identification in [9, Section 1.4] gives

$$\mathrm{Conf}_e^{w_0}(\mathcal{A}_{\mathrm{ad}}) \simeq H_{\mathrm{ad}}^{w_0,e},$$

and similarly for the simply connected group. Under these identifications the map from the simply connected decorated space to the adjoint one is the finite etale central isogeny. The initial A torus and its singly mutated A tori are open by [9, Lemma 3.37]; their union has complement of codimension at least two by [9, Proposition 3.42]. We use the all-top-word triangulation, which is one of the triangulations used in that proof.

Propositions 3.43–3.44 of [9] identify the restriction of the central map to the initial full seed torus as a finite monomial isogeny. Both propositions are needed, since the latter supplies the open-string coordinates. By [9, Propositions A.28–A.30], this map transports through mutation as a monomial map with the same finite lattice index. On each singly mutated A torus its kernel translations agree on the function field with the deck transformations of the central isogeny. Thus that torus is invariant under the deck group. Its finite quotient is the corresponding X torus and is open in the adjoint decorated space. Finite etale descent preserves codimension, so the initial and singly mutated X tori cover $H_{\mathrm{ad}}^{w_0,e}$ in codimension one.

We next compute the left-torus quotient on the full coordinate set. On the SW factorization chart write

$$g = y_{r_1}(p_1) \cdots y_{r_m}(p_m) t, \qquad t_i = \alpha_i(t).$$

Write $\mathrm{first}(i)$ for the first occurrence of color i. Let L_i be the leftmost open string on level i, and let S_k be the string immediately to the right of node k. The latter is closed when $k^+ \le m$ and is a rightmost open string otherwise. Definitions 3.12–3.13 of [9] give

$$L_i = p_{\mathrm{first}(i)}^{-1}, \qquad S_k = \begin{cases} p_k / p_{k^+} & k^+ \le m, \\ p_k t_{r_k} & k^+ = m+1. \end{cases}$$

Left multiplication by $s \in T_{\mathrm{ad}}$ sends p_k to $\alpha_{r_k}(s)^{-1} p_k$ and t_i to $\alpha_i(s) t_i$. It therefore multiplies L_i by $\alpha_i(s)$ and fixes every S_k. The simple roots form an integral character basis of T_{ad}. Consequently the quotient deletes precisely the leftmost open strings, while retaining every rightmost frozen string.

Moving all Cartan factors to the left gives

$$u_k = p_k t_{r_k} = \prod_{\substack{l \geq k \\ r_l = r_k}} S_l.$$

The exponent matrix of this transformation is upper unitriangular. Telescoping now gives $q_k = S_k^{-1}$, including the last occurrence of each color. Thus this is an integral identification of the full m-dimensional torus.

Every mutable X coordinate is a closed-string coordinate and is invariant under the left torus. A mutation multiplies every other coordinate by a rational expression in that invariant coordinate. Hence the left open strings keep their simple-root weights, and all retained coordinates remain invariant, in each singly mutated chart. These invariant open charts descend through the principal torus bundle to open tori in Y. The complement has codimension at least two because its inverse image has codimension at least two upstairs.

For the exchange columns, the Cartan matrix of H is the transpose of (a_{ij}). At a negative SW node of color i, with left and right strings a, b, its contributions are $\epsilon_{ab} = 1$ and $\epsilon_{ba} = -1$. If a string c on level j crosses its triangle, its contributions are

$$\epsilon_{ac} = \tfrac{1}{2}a_{ij}, \quad \epsilon_{bc} = -\tfrac{1}{2}a_{ij}, \quad \epsilon_{ca} = -\tfrac{1}{2}a_{ji}, \quad \epsilon_{cb} = \tfrac{1}{2}a_{ji}.$$

Sum these over the endpoints of the closed string S_k. The predecessor and successor entries are respectively 1 and -1. For $a < k < a^+ < k^+$, the two half-contributions sum to $a_{r_a r_k} = a_{i_a i_k}$; for $k < a < k^+ < a^+$ they sum to its negative. All other contributions cancel or vanish. This calculation includes the rightmost frozen strings, with missing successors written as $m + 1$. It gives precisely the full columns B. Opposition changes colors but preserves Cartan entries; no word reversal is used.

For an integer b, put $[b]_+ = \max(b, 0)$. SW's mutation formula is

$$S'_a = S_a S_k^{[B_{ak}]_+}(1 + S_k)^{-B_{ak}} \quad (a \neq k), \qquad S'_k = S_k^{-1}.$$

Since $q = S^{-1}$, it becomes

$$q'_a = q_a q_k^{[C_{ak}]_+}(1 + q_k)^{-C_{ak}}, \qquad q'_k = q_k^{-1}, \qquad C = -B.$$

In particular the dual Cartan transpose and the full frozen rows are both retained.

Let Y° be the indicated partial toric blowup. The successor minor makes each ray C_k primitive. The center character q_k is primitive and annihilates C_k, so its center is smooth, connected, and reduced. Distinct centers lie in disjoint boundary divisors of the partial fan. Thus Y° is smooth. The inverse mutation formula shows that an adjacent torus lifts to this blowup: along $1 + q'_k = 0$ the valuation vector of q is C_k, and the pullback of the center ideal is invertible. The adjacent torus meets the exceptional divisor densely. In the standard two-dimensional blowup factor, these tori omit the codimension-two locus $\rho = t = 0$ in $\operatorname{Spec} \mathbb{C}[\rho, t, (\rho t - 1)^{-1}]$.

For $f \in \mathcal{O}(Y)$, its restrictions to the initial and adjacent tori are regular. Every prime divisor of Y° meets one of these tori: the only divisors missing the initial torus are the exceptional divisors. Normality therefore extends f over Y°. Since Y is affine, these extensions define a morphism $Y^\circ \to Y$.

The rational form $\bigwedge_a d \log q_a$ is nonvanishing on every quotient seed torus, since a mutation has logarithmic Jacobian determinant -1. Codimension-one coverage extends it as a nowhere-zero top form on smooth Y. It is likewise nonvanishing on Y°: blowing up a smooth codimension-two center in one boundary component preserves this logarithmic form after removing the strict boundary. The morphism pulls the target form back to the source form, since they agree on the common torus. Its Jacobian is therefore invertible, so it is etale. Being separated and birational over a normal target, it is an open immersion by Zariski's main theorem. Its image contains the initial and adjacent quotient tori, and its complement has codimension at least two. □

8. A nef Richardson compactification

The open embedding

$$j : Y \longrightarrow H/B_-, \qquad j(g) = \dot{w}_0^{-1} g^{-1} \dot{w}_0 B_-$$

identifies Y with an open Richardson variety. Inversion permutes its extremal minors, up to nonzero pinned constants. The double Bott–Samelson construction of [13, Sections 4.2 and 4.5, equation (11)] gives a smooth projective compactification X and a projective map $\pi : X \to H/B_-$ extending j, with reduced SNC boundary $D = (X \setminus Y)_{\mathrm{red}} = \sum_i D_i$ and $K_X + D \sim 0$. Indices on D_i run through all boundary components, independently of the simple-root indices on fundamental line bundles. Write $\mathcal{L}_i$ for the fundamental globally generated line bundle on H/B_-, normalized to have degree δ_{ij} on the jth simple Schubert curve. Here 2ρ is twice the sum of the fundamental weights, and $\mathcal{L}(2\rho) = \bigotimes_i \mathcal{L}_i^{\otimes 2}$. The divisor of the pulled-back boundary section of $\mathcal{L}(2\rho)$ is nef and has the form

$$A = \sum_i a_i D_i, \qquad a_i > 0.$$

Indeed, the flag boundary is an effective Cartier divisor of class 2ρ, and $\pi^{-1}(Y) = Y$. Its local equation therefore has positive order at the generic point of every D_i. These properties persist under blowups of logarithmic strata. We use a common logarithmic modification as X, chosen below.

There is a zero-dimensional boundary stratum. To see this in the double Bott–Samelson construction, take the intersection of all standard boundary divisors in the ordinary Bott–Samelson factor. It is the all-identity point, whose image is the base flag. That flag belongs to the open opposite Bruhat cell, over which the opposite factor is an isomorphism. Near the resulting point the fiber product is consequently the ordinary factor, and its m standard boundary divisors intersect transversely in that point. A subdivision of a maximal cone under a logarithmic modification again has maximal cones. Thus the final model still has a zero-dimensional boundary stratum.

Since $\mathcal{O}(Y)$ is a localization of a polynomial ring, $\mathrm{Pic}(Y) = 0$. Divisor localization gives

$$\bigoplus_i \mathbb{Z}[D_i] \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y) \longrightarrow 0.$$

In particular, the complete boundary intersection vector determines the numerical curve class. All lifted effective classes must still be retained when a comparison theorem is applied.

Lemma 8.1. *Let $\omega = \bigwedge_{a=1}^{m} d\log q_a$ be the volume form on the specified full factorization torus. On the full double Bott–Samelson compactification, and on every smooth logarithmic modification used here, one has*

$$\mathrm{div}_X(\omega) = -D.$$

Thus the cluster volume form and the log Calabi–Yau volume form of the compactification differ by a nonzero constant. In particular, every boundary component is essential for this specified volume form.

Proof. Proposition 7.1 shows that ω extends to a nowhere-zero regular top form on Y. At the generic point of any prime divisor of a smooth compactification, choose a uniformizer t and write $q_a = t^{v_a} u_a$, with u_a a unit in the local discrete valuation ring. Then

$$d\log q_a = v_a \frac{dt}{t} + \frac{du_a}{u_a}.$$

In the top exterior product, at most one factor dt/t occurs. Consequently ω has at worst a simple pole along every prime divisor. Since it has no zero or pole in Y, the divisor $\mathrm{div}_X(\omega) + D$ is effective. It is linearly equivalent to $K_X + D \sim 0$. An effective linearly trivial divisor on an integral projective variety is zero, proving the displayed identity. The same

argument applies after every smooth logarithmic modification, which preserves $K_X + D \sim 0$ and the interior Y. □

Fix an interior frame h and define

$$\pi_h = (\dot{w}_0^{-1} h^{-1} \dot{w}_0) \cdot \pi, \qquad Y_h = \pi_h(Y) \subset H/B_-,$$

where the group acts by left translation on the flag variety. Conditions on the values of a normalized flag map below use Y_h and the corresponding translated auxiliary isomorphism loci. For the actual MV-source map

$$g(z) = \dot{w}_0 n(z^{-1})^{-1} \dot{w}_0^{-1} h^{-1}$$

one has the identity

$$\pi_h(g(z)) = n(z^{-1}) B_-. \tag{7}$$

Thus this flag map retains every complete vector $n(z^{-1}) v_i^-$. Constant left translation preserves the fundamental line bundles and their degrees. For $v \in V_H(\omega_i)^*$, define

$$\ell_{i,h}(v) = \langle v, \dot{w}_0^{-1} h^{-1} v_i^+ \rangle.$$

For $u \in V_H(\omega_i)^*$, invariance of the representation pairing gives

$$\ell_{i,h}(\dot{w}_0^{-1} h^{-1} \dot{w}_0 u) = \langle u, \dot{w}_0^{-1} v_i^+ \rangle. \tag{8}$$

The pullback divisor of this section is consequently independent of h. On the source, its normalized homogeneous section is

$$Q_i(z) = z^{\nu_i} F_i(g(z)) / F_i(h^{-1}).$$

The boundary contact of a generic auxiliary zero of color i is universal: it is the divisorial valuation of the reduced irreducible divisor $F_i = 0$ in V. The fundamental extremal section has degree δ_{ij} on the simple Schubert curves, so the corresponding Schubert divisor occurs with multiplicity one. Properness gives this valuation its unique center on X. The birational map $X \to H/B_-$ is an isomorphism over a dense open of this divisor. We retain its color and connected stratum component as part of the contact data.

For a generic source Z, let β_Z be its curve class on this target. For contacts $(p_Z, p_W; -p_T)$, with $\nu_T = \nu_Z + \nu_W$, the universal auxiliary contacts cancel. Boundary localization and the contact-degree identity consequently force the tail class

$$\beta_Z + \beta_W - \beta_T. \tag{9}$$

This uniqueness concerns numerical classes on the specified model. When a comparison changes models, every admissible lifted class is retained until the proper pushforward has been taken.

8.1. **A common divisorial model.** For a projective target, $NE(X)$ denotes the monoid of effective integral numerical curve classes, including zero. Homological classes will be retained separately in mapping spaces. A contact is *divisorial* when it is an integral multiple of one ray of the boundary cone complex. The coefficient $c_i(p)$ is its value on a local equation for D_i, and its connected stratum is part of the data.

Lemma 8.2. *Fix two source cycles, their total numerical curve class and their complete external and auxiliary contacts. There is a projective logarithmic modification on which these contacts and every nonzero nonnegative central contact arising from an effective splitting are divisorial.*

Proof. Choose an ample divisor H and integral divisor generators L_j. For sufficiently large integers M_j, both $M_j H + L_j$ and $M_j H - L_j$ are ample. An effective splitting $\beta = \beta_C + \beta_T$ satisfies

$$|L_j \cdot \beta_C| \le M_j H \cdot \beta.$$

There are therefore finitely many possible numerical central classes. For each one, its boundary contact is forced by

$$c_i(p_C) = D_i \cdot \beta_C - \sum_{a \text{ auxiliary}} c_i(p_a).$$

Keep the nonnegative vectors and all compatible connected strata. Insert their rays and the prescribed external rays by a common projective subdivision, followed by a smooth subdivision.

If a nonnegative central contact appears on the new model, its pushforward is nonnegative because the pullbacks of the old boundary divisors have nonnegative coefficients. It satisfies the same degree equations and belongs to the finite list. A subdivision preserves the underlying cone-complex point, and the inserted ray persists. Thus this one model has the asserted property. □

All subsequent classes and contacts refer to this fixed model. The boundary still generates its Picard group and supports a nef divisor with positive coefficients. Fix a very ample divisor H. Genus-zero stable maps with a fixed number of markings and fixed H-degree form a finite-type stack. Their integral homology class is locally constant, so only finitely many homological classes occur. This applies also to effective subcurves of bounded degree. Every mapping space below includes all these homological decorations.

For a tail with fixed complete contacts, all its boundary intersections are fixed. Expressing H as an integral linear combination of the boundary divisors fixes its H-degree and proves finiteness of the possible homological tail classes. The same reasoning applies on each refinement, using the new boundary and lifted contacts. The all-lifts identity of [31, Corollary 1.4 and Section 10] is applied first to each downstairs homological class, and only then summed over these finite collections.

Logarithmic mapping spaces. We write commutative monoids additively. For a monoid Q, its groupification is Q^{gp}, its real dual cone is $Q^{\vee}_{\mathbb{R}} = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0})$, and $K[Q]$ is its monoid algebra over a field K, with basis χ^q and multiplication $\chi^q \chi^{q'} = \chi^{q+q'}$. A fine saturated monoid is finitely generated, embeds in its groupification, and contains every element of that groupification whose positive integral multiple lies in the monoid. It is sharp if its only unit is zero. A homomorphism is local if the inverse image of the units is the set of units.

For a boundary component D_i, put $D_i^{\circ} = D_i \setminus \bigcup_{j \neq i} D_j$. The boundary cone complex $\Sigma(X)$ has a cone $\mathbb{R}^r_{\geq 0}$ for each connected stratum lying in exactly r boundary components, with face maps induced by inclusions of strata; its integral coordinates record boundary orders. Write $T_X^{\log} = T_X(-\log D)$ for the vector bundle of derivations preserving the ideals of the boundary components. We use rational Chow groups. For a morphism $F \to B$, $A^e(F \to B)$ denotes bivariant operations lowering cycle dimension by e after every allowed base change to B, with support in the corresponding base change of F. A superscript ! denotes refined Gysin pullback for a regular embedding, or the virtual pullback specified by a relative perfect obstruction theory. Proper pushforward includes residue-field degrees and stack stabilizers. The class $[\mathcal{L}]^{\mathrm{vir}}$ is the virtual fundamental class defined by that obstruction theory. A basic logarithmic map has the minimal fine saturated (fs) base monoid determined by its contact and smoothing data. For a tropical type τ, its cone parametrizes vertex positions and nonnegative edge lengths satisfying the contact relations. A decoration assigns an actual curve class to each vertex. A τ-marked space includes the degenerations of this type. The Artin fan $\mathcal{X}$ of (X, D) retains these boundary monoids and forgets the geometry of their strata. Evaluation enhancement at a marking is base change from its Artin-fan evaluation to the corresponding target stratum. Punctured maps allow a negative contact at a specified leg; their base retains the puncturing ideal. We use the constructions of [32, Sections 2–5] with these conventions.

The target is projective with Zariski logarithmic structure, and the labelled boundary divisors generate its characteristic group sheaf. Theorems 3.12 and 3.18 and Corollary 3.19 of [32] therefore give properness of the stable mapping spaces with fixed global contacts

and homological class. Forgetting logarithmic structures gives a proper map to the ordinary stable-map space. For a divisorial negative output, its fiber over a point of the open boundary stratum is proper. The logarithmic point constraint is finite over that fiber by Lemma A.1.

9. Generic smooth incidence

For $r \geq 3$, $\overline{\mathcal{M}}_{0,r}$ is the moduli space of stable genus-zero curves with r labelled markings, and $\mathcal{M}_{0,r}$ is its smooth-domain locus. The map st forgets the target map and stabilizes the marked curve; ev_j evaluates it at the jth marking.

A component of a mapping space is horizontal when its curve/frame map dominates the product of the smooth marked-curve space and Y. A nondegenerate map has no component contained in the boundary. A map with smooth rational domain is free here when $f^*T_X(-\log D)$ is trivial. We first establish that all points of a general incidence have these properties.

9.1. **The support of the incidence.** Throughout this section, $m = \dim X$, as above. For the compactifications used here, the integral affine structure needed below is the one of [30, Section 3, Assumption 3.1 and Lemma 3.2]. Lemma 8.1 identifies the specified torus volume form with a generator having divisor $-D$. Hence every cone is good, $B = |\Sigma(X)|$, and the union B_0 of the relative interiors of the cones of dimensions m and $m-1$ is an open integral affine manifold. A point of B_0 has an affine neighborhood in the whole support B. Here a cone is good when its boundary components are essential for the chosen logarithmic volume form.

For a tropical type α, write G_α for its graph and $\sigma(v)$ for the target cone of a vertex v. At a parameter $t \in \alpha$, $h_t(v) \in \sigma(v)$ is its position; $\mathrm{ev}_v : \alpha \to \sigma(v)$ is the resulting evaluation map. An oriented edge $e : v \to w$ has integral slope $u(e)$ and length function ℓ_e, with $h_t(w) - h_t(v) = \ell_e(t)u(e)$ in a common cone chart. We write relint for relative interior. A type is realizable when these positions and lengths realize its prescribed cones and graph on the relative interior of its parameter cone. It is balanced when the outgoing slopes sum to zero at each vertex in an affine chart of B_0.

A nontrivial broken-line type is a realizable balanced genus-zero type with two legs, a nonzero nonnegative incoming contact and a nonzero outgoing contact in a good cone. Its parameter cone has dimension $m-1$, and the outgoing leg sweeps an m-dimensional set. Its spine is the path joining the two legs, including those legs; see [30, Definitions 4.1, 4.4 and 4.6].

A monoid is integral if it embeds in its groupification. For integral monoids P, Q, a homomorphism $f : P \to Q$ is integral if every relation $f(p_1) + q_1 = f(p_2) + q_2$ admits $a_1, a_2 \in P$ and $q \in Q$ such that

$$p_1 + a_1 = p_2 + a_2, \qquad q_1 = f(a_1) + q, \qquad q_2 = f(a_2) + q.$$

It is exact if $(f^{\mathrm{gp}})^{-1}(Q) = P$ inside P^{gp}, where f^{gp} is the induced homomorphism of groupifications.

Lemma 9.1. *Let $P \to Q$ be a local integral homomorphism of sharp fine saturated monoids. It is injective and exact, and the map of real dual cones $Q^\vee_{\mathbb{R}} \to P^\vee_{\mathbb{R}}$ is surjective.*

Proof. Write $f : P \to Q$. Applying the integrality criterion with $q_1 = q_2 = 0$, sharpness and locality give $a_1 = a_2 = q = 0$, proving injectivity. If $f^{\mathrm{gp}}(p_1 - p_2) = q_2 \in Q$, apply the criterion with $q_1 = 0$. It gives $a_1 = q = 0$ and $p_1 - p_2 = a_2 \in P$, which is exactness.

Identify $P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$ with its image in $Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathrm{cone}(P)$ and $\mathrm{cone}(Q)$ be the nonnegative real spans of the respective monoids in these spaces. Exactness gives

$$\mathrm{cone}(Q) \cap (P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}) = \mathrm{cone}(P).$$

Indeed, clear denominators to prove this on rational points and use the rational polyhedrality of the intersection. Restriction of nonnegative linear functionals is consequently surjective

by the polyhedral extension theorem: the dual of its image is the displayed intersection, and the image is closed because it is polyhedral. □

Lemma 9.2. *Let α be the tropical type of an actual genus-zero stable, possibly punctured, logarithmic map. Suppose v maps to a good zero stratum of a target of dimension m, and $\dim \mathrm{ev}_v(\alpha) = m$. Then every branch of $G_\alpha \setminus \{v\}$ contains a marked leg.*

Proof. Suppose a branch contains none. If its first edge has nonzero slope, cut it and retain the component away from v. Replacing the node by a puncture gives an actual stable punctured map. Its type is realizable and balanced by [30, Lemma 4.3], has exactly one leg of nonzero slope, and that leg lies in the good maximal cone $\sigma(v)$. Its leg image contains $\mathrm{ev}_v(\alpha)$, contradicting [30, Lemma 4.7(1)].

If the first slope is zero, follow the maximal connected subtree of zero-slope edges within this branch. All its vertices have the same tropical position as v and map to the same zero stratum. A first nonzero edge leaving this subtree gives the preceding contradiction. If none exists, the whole branch is a constant unmarked tree with one attachment. For N vertices it has $2(N-1)+1$ flags in total, whereas stability would require at least $3N$. This is impossible. □

Lemma 9.3. *Suppose $K_X + D \sim 0$. A face contraction of the tropical type of an actual stable logarithmic map is realizable and balanced in the sense of* [30, Definitions 2.2 and 4.1].

Proof. Restrict the universal tropical family to the face and contract its zero-length edges. This gives the tropical realizability required by the stated definitions.

Write $\tau \subset \alpha$ for the face, choose $q \in \operatorname{relint}\tau$ and $q_0 \in \operatorname{relint}\alpha$, and consider $q_\epsilon = q + \epsilon q_0$. Let v be a vertex of the contracted type whose image $h_q(v)$ lies in B_0. Choose an integral affine neighborhood $W \subset B_0$ of that point. The finitely many vertices and internal edges of the cluster contracted to v converge to $h_q(v)$ as $q_\epsilon = q + \epsilon q_0$ tends to q. Shrinking W within an affine chart and taking $\epsilon > 0$ sufficiently small puts the whole cluster in this one chart. Each of its vertices satisfies the balancing equation by [30, Lemma 4.3]. Summing these equations in the common tangent lattice cancels each internal edge with its opposite orientation, and gives precisely the balancing equation at v.

□

Lemma 9.4. *Suppose $K_X + D \sim 0$, so that all cones of $\Sigma(X)$ are good. Let γ be the type of an actual stable punctured map marked by a nontrivial broken-line type δ. Every spine vertex of γ has evaluation image of dimension $m-1$. The slope of every spine edge, and of the outgoing spine leg, is transverse to the span of the evaluation image at its incident spine vertices. If one spine vertex maps to the origin at a parameter value, all spine vertices do.*

Proof. This is [30, Corollary 4.9], with the following details also accounting for zero-slope side branches and the outgoing leg. Let $\phi : G_\gamma \to G_\delta$ be the contraction. The restriction to δ and [30, Lemma 4.7(2)] give evaluation dimension at least $m-1$ at every spine vertex v. Its target cone therefore has dimension $m-1$ or m. In the former case the upper bound is automatic. In the latter case the actual component is constant and stable, so there is an unmarked branch off the spine. Lemma 9.2 excludes evaluation dimension m. Thus the span H_v has dimension $m-1$ and equals the span obtained from the broken-line face.

Every off-spine slope is tangent to H_v. Otherwise cutting that edge would give a balanced one-leg type whose leg sweeps a m-dimensional cone, contrary to [30, Lemma 4.7(1)]. A zero slope is already tangent. The outgoing spine leg is transverse to H_v, since its image on the broken-line face has dimension m. Balancing at v consequently forces the next spine direction to be transverse as well. Across an edge contracted by ϕ, the spans at its two endpoints agree. Across an edge not contracted by ϕ, transversality at both ends is the transversality proved in [33, proof of Lemma 2.5] for the broken-line face. Induction along the spine gives the assertion for every edge.

For consecutive spine vertices v, w and any parameter t, write $h_t(w) = h_t(v) + \ell_e(t)u(e)$ in the cone containing the edge. If $h_t(v) = 0$, then $\ell_e(t)u(e) \in H_w$. Transversality implies $\ell_e(t) = 0$, hence $h_t(w) = 0$. Induction proves the last assertion. □

Lemma 9.5. *Let (X, D) be a smooth projective pair over an algebraically closed field of characteristic zero, with reduced simple normal crossings boundary, $K_X + D \sim 0$, and at least one zero-dimensional boundary stratum. Suppose $U = X \setminus D$ is affine, contains a dense torus, and $\sum_i a_i D_i$ is nef with every $a_i > 0$. Fix genus-zero discrete data with $r \geq 3$ markings, at least one of contact zero, denoted x_∞. All other contacts are nonnegative, and every nonzero contact is divisorial on this model X. Let $\mathcal{L}$ be the proper stack of basic stable logarithmic maps with these data, and set*

$$e = (\mathrm{st}, \mathrm{ev}_\infty) : \mathcal{L} \longrightarrow \overline{\mathcal{M}}_{0,r} \times X.$$

There is a nonempty open $B^\circ \subset \mathcal{M}_{0,r} \times U$ whose entire inverse image is finite étale *over B°. Its actual domains are smooth. Write the nonzero contact at x_j as m_j times the primitive vector of the ray corresponding to $D_{i(j)}$. Each map f in this inverse image satisfies*

$$f^{-1}D = \sum_{j:\, m_j > 0} m_j x_j, \qquad f(x_j) \in D^\circ_{i(j)}.$$

Their logarithmic tangent pullbacks are trivial.

For every geometric point $b \in B^\circ$,

$$b^![\mathcal{L}]^{\mathrm{vir}} = [e^{-1}(b)] \quad \text{in } A_0(e^{-1}(b))_{\mathbb{Q}}. \tag{SGI}$$

One open suffices for finitely many fixed discrete data with the same number of markings.

Proof. Write β for the fixed discrete data and put $\mathfrak{A} = \mathcal{M}^{\mathrm{ev}}(\mathcal{X}, \beta)$, with the finite-type restriction of [30, Remark 2.3]. Choose a zero stratum s and the degeneration $T \to X$ of [30, §5, Step 3]. The closed log point has characteristic $P = \mathbb{N}^m$, although the underlying scheme T is a DVR. The dual cone map has full-dimensional image $C_0 \subset \sigma_s$. It can be chosen unimodular, by choosing a smooth full-dimensional cone in a sufficiently fine subdivision, and transverse to the finitely many evaluation cones. By [30, Lemmas 5.5–5.6], $\mathfrak{A}_T \to T$ and $\mathfrak{A}_s \to s$ are integral, and log and ordinary fiber dimensions agree.

The map $\mathcal{L} \to \mathfrak{A}$ is strict; see [32, Definition 5.14 and the following paragraph, p. 97]. Strictness persists under the fs base changes used here. At an actual point $z \in \mathcal{L}_s$, let α be its unconstrained basic tropical type and Q its characteristic after the point-constraint base change. It is also the characteristic at its image in $\mathfrak{A}_s$. Thus $P \to Q$ is local and integral, and Lemma 9.1 gives a surjection

$$Q^\vee_{\mathbb{R}} = \alpha \times_{\sigma_s} C_0 \longrightarrow C_0.$$

Consequently the output evaluation of the *actual* type α has dimension m.

Fix a smooth marked curve $c \in \mathcal{M}_{0,r}$ and consider an actual incidence point over the generic point of T. Properness, after a finite field extension, extends it over a DVR dominating T; separatedness keeps its stabilization equal to c. The output component in its special fiber maps to s and is constant. Lemma 9.2 shows that it has no unmarked branches. It therefore survives stabilization, and each branch contains exactly one of the other markings, since c is smooth. Its actual valence is exactly r.

Choose a generic point ξ of a component of $\mathfrak{A}_s$ specializing to the image of z, and let $\tau \subset \alpha$ be the resulting face type before the point-constraint base change. Equality of log and ordinary fiber dimension gives $\operatorname{rank} Q^{\mathrm{gp}}_\xi = m$. The cone $Q^\vee_\xi = \tau \times_{\sigma_s} C_0$ has a point interior in both factors, because ghost morphisms at points are local. Since $C_0^{\mathrm{gp}} \to \sigma_s^{\mathrm{gp}}$ is an isomorphism,

$$\dim \tau = \dim \mathrm{ev}_{\mathrm{out}}(\tau) = m.$$

Lemma 9.3 proves realizability and balancing of τ from the actual special type α.

Let K be the connected cluster of special vertices contracted to the generic output. For every $v \in K$, its evaluation restricted to τ equals the output evaluation. Thus $\sigma(v) = \sigma_s$, its actual component is constant and stable, and its full evaluation image has dimension m. Lemma 9.2 applies at each v. An unmarked generic output branch would pull back to an unmarked branch attached to one such v, so the generic output has at most r flags. But contraction gives

$$\operatorname{val}(\mathrm{out}_\tau) = \sum_{v \in K} \operatorname{val}(v) - 2(|K| - 1) \geq r + |K| - 1.$$

Hence K is the singleton consisting of the actual output, and the generic output has exactly r flags. No adjacent edge is contracted.

Cut each nontrivial generic branch at its edge adjacent to the output, and call its type τ_i. Let v_i be the vertex adjacent to the new outgoing leg, and orient its slope u_i towards the omitted output. In compatible framed cone charts, the basic-cone description in [32, Proposition 2.64, equation (2.31)] gives the endpoint space

$$E_i = \{(q_i, \lambda_i) \in \tau_i \times \mathbb{R}_{\geq 0} : \mathrm{ev}_{v_i}(q_i) + \lambda_i u_i \in \sigma_s\}, \qquad \epsilon_i(q_i, \lambda_i) = \mathrm{ev}_{v_i}(q_i) + \lambda_i u_i.$$

At a point induced by $\operatorname{relint}\tau$, the cut length is positive, all retained vertices and edges have their prescribed types, and the endpoint lies in $\operatorname{int}\sigma_s$. Thus $q_i \in \operatorname{relint}\tau_i$, the endpoint condition imposes only an open inequality there, and $\dim E_i = \dim \tau_i + 1$. Restoring the root and the cut edges in the same basic-cone coordinates identifies the whole cone, locally at this point, with the fiber product of the E_i over σ_s; a direct marked leg adds no coordinate. Each ϵ_i has rank m, since the full-dimensional output evaluation of τ factors through it. Consequently

$$m = \dim \tau = m + \sum_i (\dim E_i - m).$$

Every summand is nonnegative. Hence every nontrivial branch has $\dim \tau_i = m - 1$ and outgoing-leg image of dimension m. Its outgoing slope is necessarily nonzero.

If its input marking had zero contact, deleting that leg would leave the same cone and balancing equations, producing a one-leg type contradicting [30, Lemma 4.7(1)]. Here we delete the zero leg without stabilizing the graph. The tropical types in [30, Definitions 2.1 and 4.1] have no stable-valence requirement, and deletion of a zero-slope leg removes neither a basic-cone coordinate nor a balancing term. Thus two-valent vertices cause no change in the cone, and [30, Lemma 4.7(1)] applies to the resulting one-leg type. Thus all extra zero markings are directly on the generic output, and also on the actual special output because K is a singleton. Every nontrivial remaining branch is a broken-line type: its input is positive, its outgoing slope is nonzero, it is realizable and balanced, and it has the two required dimensions. Both leg cones are good because $K_X + D \sim 0$.

Splitting the actual special curve at the corresponding surviving node, followed by prestabilization and basicification, gives a stable punctured map of type γ_i marked by τ_i; compatibility of splitting with markings is [32, Propositions 5.2 and 5.4]. Replacing a node by a puncture preserves the special-point counts, so the retained actual components remain stable.

Let $\rho \subset \alpha$ describe the original generic fiber over the DVR. Its output lies at the origin because the generic point constraint lies in U. The cut outgoing slope is transverse to the span at the adjacent vertex by Lemma 9.4, so that vertex is also at the origin. The same lemma propagates zero along each branch spine. Direct legs have no additional vertices. Every vertex of the generic spine therefore lies in the zero cone.

If a generic component lay in D, choose the first boundary vertex on a path from the spine to it. Its predecessor lies in the zero cone, and the branch beyond the connecting edge has no markings. The nef full-boundary divisor excludes this tree by [30, §5, Step 1]. Thus every actual generic incidence point is nondegenerate.

The required smoothness and freeness follow from the following arguments in [29, Proposition 3.13]. On the present minimal log Calabi–Yau compactification, every boundary component is essential and the prescribed nonzero contacts are divisorial. On the nondegenerate locus, [30, Lemma 5.2] identifies logarithmic maps with ordinary maps having exactly these boundary contacts and a unique basic logarithmic lift.

First fix a smooth marked rational curve. At a map with the prescribed contacts, the tangent space with the marked domain fixed is $H^0(\mathbb{P}^1, E)$, where $E = f^*T_X(-\log D)$ has degree zero. If evaluation at a zero-contact marking dominates U on any irreducible locally closed locus, generic smoothness supplies m such sections which span the fiber at that marking. Their determinant is a section of a degree-zero line bundle and is nonzero at one point, so it is nowhere zero. Hence $E \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus m}$. Applying the same argument to the nonfree locus shows that its evaluation image is not dense. On the free locus, evaluation at every zero-contact marking has full rank, so finitely many such evaluations can simultaneously avoid any prescribed proper closed subset. This proves the evaluation assertion of [29, Lemma 3.9(1)].

Next, bounded-degree nonconstant $\mathbb{A}^1$-curves have images in a proper closed subset $Z \subset U$. Indeed their moduli are of finite type, and a component whose evaluation image was dense would give a finite-type covering family of $\mathbb{A}^1$-curves, excluded by the logarithmic volume-form argument above. For a nondegenerate stable map whose stabilized marked domain is smooth, a component which is unstable as a marked curve must be nonconstant. Affineness of U forces such a component to meet D. The exact contact condition and the bound of two special points then force it to have one boundary marking and one attaching node. Thus it is an $\mathbb{A}^1$-leaf. Contracting all these leaves gives the same smooth marked stabilization, with the former attachments now assigned contact zero. Only finitely many subsets of the positive markings and finitely many effective residual curve classes occur. Apply the preceding free-map argument to all these data and require the attaching evaluations to avoid Z. An $\mathbb{A}^1$-leaf would put one of them in Z, a contradiction. This is the argument of [29, Lemmas 3.10–3.11].

The argument applies to every zero-contact set I in [29, equation (3.1)]. Varying the marked smooth curve gives the nonempty open of $\mathcal{M}_{0,r} \times U$ used in [29, Proposition 3.13]. Thus, when there are at least two positive inputs, we can choose c and the generic point constraint so that every nondegenerate incidence point has stable, hence smooth, actual domain and trivial logarithmic tangent pullback.

The cases with fewer positive inputs are immediate from the same generic type: with one positive input, balancing at the output is impossible; with none, all legs are at the output and there are no other branches, leaving only the constant class. For that class the asserted incidence is the family of constant maps.

A nonempty bad generic fiber of finite type would have a closed point over a finite extension. The preceding argument therefore controls the entire actual generic incidence.

Let $G \subset \mathcal{L}$ be the locus of nondegenerate maps with smooth marked domains and trivial logarithmic tangent pullback. This is open: in the smooth genus-zero family the pullback has degree zero, and its triviality is the open condition that both cohomology groups after twisting by $-x_\infty$ vanish. Properness of e makes $e(\mathcal{L} \setminus G)$ closed. The generic-constraint argument shows that its complement in $\mathcal{M}_{0,r} \times U$ is nonempty; choose B° in this complement. Every geometric point above B° has trivial stabilizer: the smooth rational domain has $r \geq 3$ labelled markings, and the nondegenerate map has a unique basic logarithmic lift. With its marked domain and output evaluation fixed, the relative deformation-obstruction complex is

$$R\Gamma\big(\mathbb{P}^1, f^*T_X(-\log D)(-x_\infty)\big) = R\Gamma\big(\mathbb{P}^1, \mathcal{O}(-1)^m\big) = 0.$$

Thus e is representable and étale above B°. Being also proper, it is finite étale. On this open the logarithmic obstruction theory is unobstructed of the expected dimension, so the virtual class restricts to the fundamental class.

Its refined fiber is consequently the fundamental zero-cycle of the finite étale fiber. Since the entire fiber lies in this open, the equality holds in $A_0(e^{-1}(b))_{\mathbb{Q}}$ itself. Intersecting finitely many dense opens proves the final assertion. □

Lemma 9.6. *Let a family of smooth-domain maps with fixed marked positions and contact data dominate the interior frame parameter. Then the logarithmic tangent bundle of its general map is trivial. At every marked contact, the leading units and the coordinates of the boundary stratum vary with full rank as the frame varies.*

Proof. At a general point choose m parameter directions whose evaluations at the frame form a basis of the target tangent space. They give m sections of $E = f^*T_X(-\log D)$ on the already smooth $\mathbb{P}^1$. Their determinant is nonzero at the frame. Since $\det E \simeq \mathcal{O}_{\mathbb{P}^1}$, it is nowhere zero and $E \simeq \mathcal{O}_{\mathbb{P}^1}^m$. If locally $x_j = z^{c_j}u_j(z)$ are the boundary coordinates and y_l are stratum coordinates, evaluation of these logarithmic sections at $z = 0$ is exactly the differential of $(u_j(0), y_l(0))$. Its rank is m. □

Triviality of E gives

$$H^0(\mathbb{P}^1, E(-\infty)) = H^1(\mathbb{P}^1, E(-\infty)) = 0. \tag{10}$$

For a rational tree C and a rank-m vector bundle E trivial on each component, the same vanishing holds after twisting by a smooth frame point x_∞. Indeed, remove a leaf L not carrying x_∞, attached to the remaining tree C' at p. The normalization sequence is

$$0 \longrightarrow E(-x_\infty) \longrightarrow E|_{C'}(-x_\infty) \oplus E|_L \longrightarrow E_p \longrightarrow 0.$$

Evaluation $H^0(L, E|_L) \to E_p$ is an isomorphism and $H^1(L, E|_L) = 0$. The cohomology sequence reduces both vanishings to C'. Induction ends with $\mathcal{O}_{\mathbb{P}^1}(-1)^m$. We use this for free maps glued at interior points.

10. Recovering the MV source

For a source of charge ν, set $d = \mathrm{ht}(\nu)$. Use one input at 0, the old unit at 1, the frame at ∞, and d auxiliary markings. Initially work with all roots ordered. The full product of permutation groups on identical auxiliary types acts on both source and receiver; its quotient is used when the roots are unordered.

Proposition 5.2 and Lemma 9.6 make the contact leading units and stratum coordinates jointly generic in the source family. The same lemma applies to every smooth-domain horizontal receiver component.

Lemma 10.1. *Let F be a rational function on X regular on Y. A pole bound for F on the generic complete leading family at a fixed contact stratum holds for every contact arc over a sufficiently small dense open of that stratum, including arcs over Artin coefficient rings.*

Proof. Let Σ be the specified stratum, locally cut out by boundary equations $x_1, \dots, x_k$, and let $c_j > 0$ be the prescribed order of x_j along an arc centered there. Near its generic point write $F = A/(x^a v)$, where A is regular, v is a unit, $a \in \mathbb{N}^k$ and $x^a = \prod_j x_j^{a_j}$. For $b \in \mathbb{Z}$, let J_b in $\mathbb{C}(\Sigma)[[x_1, \dots, x_k]]$ be generated by the monomials x^α, $\alpha \in \mathbb{N}^k$, with $\sum_j c_j\alpha_j \geq b$; thus J_b is the whole ring for $b \leq 0$. If $l \geq 0$ is an integer, the bound $\mathrm{ord}_t F \geq -l$ on the generic leading family forces

$$A \in J_{\sum a_j c_j - l}.$$

Otherwise the first lower-weight coefficient is a nonzero polynomial in the independent leading units and cannot vanish generically. Faithful flatness of completion gives the ideal membership in the local ring; it spreads to a dense open of the stratum. Substitution $x_j = t^{c_j}u_j$ sends this ideal into the required power of t over every coefficient ring. Equality of generic orders follows by applying the argument to numerator and denominator initial terms separately. □

10.1. **Reconstruction from the prescribed poles.**

Lemma 10.2. *In the smooth-domain inverse, the complete-vector bounds may be transferred using rational functions on the target that do not depend on the frame.*

Proof. Use the coordinates of the vector

$$v \longmapsto \dot{w}_0^{-1} v^{-1} \dot{w}_0 v_i^-$$

as rational functions on X, regular on $V = U_H^-$ and hence on Y. For a source map $g = \dot{w}_0 n^{-1} \dot{w}_0^{-1} h^{-1}$, this vector is

$$(\dot{w}_0^{-1} h \dot{w}_0) n v_i^-.$$

The left factor is a constant invertible matrix. Consequently every coordinate has pole order at most ν_i at the input. The smooth-domain frame determinant makes the leading units and the stratum coordinates jointly general, so the monomial-ideal pole-transfer argument applies to these frame-independent functions. On a receiver map, multiply afterward by the constant inverse matrix. This gives the same upper bound for all coordinates of the reconstructed nv_i^-. The exact generic orders of the frame-independent extremal minors F_i force the opposite inequality and hence the exact degree ν_i. The same argument at each of two separate inputs is unchanged by an invertible integral left factor there. □

Proposition 10.3. *For the complete contact data of Z, the smooth-domain horizontal ordinary receiver, modulo the full auxiliary relabeling group, is birational to $Z^{\mathrm{aff}} \times \mathscr{H}$. The unordered receiver has a unique smooth-domain horizontal component for these data, and its class is β_Z. Before taking the full auxiliary relabeling quotient, the union of all such ordered components is birational to the full ordered-root cover of the source. This cover may have several components.*

Proof. On an arbitrary such component, compose the map on its dense interior with $Y \subset V$ to recover a rational V-valued map $g(z)$. Polynomial generators of $\mathcal{O}(V)$ have no poles at the auxiliary points in the actual source family. Lemma 10.1 transfers these bounds to the receiver. There are no other unmarked boundary preimages. Thus g is regular away from 0.

Set $h = g(\infty)^{-1}$ and

$$n(t) = \dot{w}_0^{-1} h^{-1} g(t^{-1})^{-1} \dot{w}_0.$$

It is a based polynomial U_H^+-loop. Apply Lemma 10.2 to the coordinates of $\dot{w}_0^{-1} g^{-1} \dot{w}_0 v_i^-$. The constant matrix of h preserves these pole bounds, giving

$$\deg(n(t) v_i^-) \le \nu_i.$$

The exact frozen orders, detected by the generic leading family, force equality. Lemma 4.1 places the loop in an actual MV source of the required charge.

The reconstruction retains n, h and, on ordered covers, the root ordering, and determines the original X-map by separatedness. First take the quotient by the full auxiliary relabeling group. A horizontal component still has dimension $m + d$ by (10). Its generically injective reconstruction is therefore dense in an actual $Z'^{\mathrm{aff}} \times \mathscr{H}$. Lemma 6.3 and the injectivity of (6) give $Z' = Z$. The forward construction and separatedness identify the unordered component uniquely with the source birationally. On the distinct-root open the ordered space is the full root-ordering torsor over this unordered space. Thus its components correspond bijectively to the components of the full ordered-root cover of the source; all components are retained. The full relabeling group acts transitively on them and does not change the underlying map, so each has class β_Z.

□

Specialization to the collision divisor will use Lemma 13.1.

11. Two inputs and their Cartier fusion

Let the input sources be Z, W, with total auxiliary count $d = \mathrm{ht}(\nu_Z) + \mathrm{ht}(\nu_W)$. Retain inputs at $0, a$, the old unit at 1 and the frame at ∞. The cross ratio a varies.

Lemma 11.1. *Let R be a commutative $\mathbb{C}$-algebra. Let $a_1, \dots, a_r \in R$ with every $a_i - a_j$ invertible for $i \neq j$, and set $D = \bigcup_i \{z = a_i\} \subset \mathbb{P}^1_R$ and $t_i = z - a_i$. For $q_i \in U_H^+(R((t_i)))$, there is a unique morphism*

$$q : \mathbb{P}^1_R \setminus D \longrightarrow U_H^+$$

such that $q(\infty) = 1$ and $qq_i^{-1} \in U_H^+(R[[t_i]])$ for every i. The construction is functorial in R and is algebraic on charts with fixed pole bounds and the finitely many required jet bounds.

Proof. Choose a central filtration of the split unipotent group whose successive quotients are vector groups. For a vector-group quotient, sum the finite principal parts at the sections and normalize the value at infinity to zero. The differences of the sections are units, so each summand has a formal expansion regular at the other sections. At each later central step, the already constructed lower coordinates make the remaining discrepancy additive. Sum its principal parts by the same procedure. Commutators only involve higher filtration levels, so this finite induction constructs q. All corrections are regular away from D, and every correction is zero at infinity. Only finitely many coefficients of each prescribed local series occur at a fixed bounded chart; all denominators are powers of the units $a_i - a_j$. This also proves functoriality and the bounded algebraicity claim.

For uniqueness, if q and q' are two such morphisms, then at each section

$$q'q^{-1} = (q'q_i^{-1})(qq_i^{-1})^{-1}$$

is regular. It is already regular away from D. Entrywise partial-fraction expansion, valid over R because the section differences are units, extends this quotient to a morphism $\mathbb{P}^1_R \to U_H^+$. Since U_H^+ is affine and $\Gamma(\mathbb{P}^1_R, \mathcal{O}) = R$, this morphism is constant. Its value at infinity is one, hence $q' = q$. □

Use $g = \dot{w}_0 q^{-1} \dot{w}_0^{-1} h^{-1}$ and write $k_i = qq_i^{-1} \in U_H^+(R[[t_i]])$. At each input the left integral correction to q_i becomes a right integral frame correction to g. Lemma 6.3 gives exactly p_Z, p_W. It also preserves complete vector bounds, since each k_i is an invertible integral matrix in every fundamental representation.

Proposition 11.2 (Generic two-input classification)**.** *On the common target constructed above, for prescribed input contacts p_{Z_1}, p_{Z_2} and the universal auxiliary list of total charge $\nu = \nu_1 + \nu_2$, the generic smooth ordinary receiver, after quotienting by the full auxiliary relabeling group, has a unique horizontal component, birational to*

$$Z_1^{\mathrm{aff}} \times Z_2^{\mathrm{aff}} \times \mathscr{H} \times \{a \neq 0\}.$$

Before quotienting, the union of all generic smooth horizontal components is birational to the full ordered-root cover of this source, which need not be irreducible. Every such component has ordinary curve class $\beta_{Z_1} + \beta_{Z_2}$.

The entire generic curve/frame incidence consists of the source fiber, and its virtual class is the fundamental zero-cycle.

Proof. For a smooth receiver, the smooth-domain frame calculation supplies jointly general contact units and stratum evaluations. The target map recovers a rational V-map g from its dense Y-valued locus. Pole transfer for polynomial generators of $\mathcal{O}(V)$ proves regularity at the auxiliary markings. The supported generic-incidence theorem excludes every other unmarked boundary preimage. Hence g is regular away from $0, a$, and

$$q(z) = \dot{w}_0^{-1} h^{-1} g(z)^{-1} \dot{w}_0$$

is a based rational U_H^+-map with precisely those allowed poles. At the input $z = a_j$, apply Lemma 11.1 with one pole to the local Laurent series q, using $t_j = z - a_j$. It gives a

unique based polynomial loop $q_j(t_j^{-1})$ for which $q_j q^{-1}$ is integral. Its inverse $k_j = (q_j q^{-1})^{-1}$ is integral as well, and $q = k_j q_j$. Lemma 10.2 and the exact frozen orders give the full fundamental degrees ν_j. Thus q_j belongs to an actual MV source of that charge.

Conversely, the principal-parts lemma reconstructs one and only one global q from the two local representatives, with regularity away from $0, a$ and $q(\infty) = 1$. This construction is algebraic on bounded charts, including over nonreduced coefficient rings. Separatedness of X determines the extension of g from its dense interior. The inverse therefore retains the actual loop representatives, the frame, the input positions, and the complete root ordering.

First work after the full auxiliary relabeling quotient. A horizontal smooth component has dimension $m + d + 1$, where $d = \mathrm{ht}(\nu_1) + \mathrm{ht}(\nu_2)$, because the curve/frame morphism is etale; taking a finite quotient does not change this dimension. Its generically injective reconstruction has image in the finite union of products of actual charge-ν_1 and charge-ν_2 MV sources, times the frame and input parameter. Every such product has the same dimension, so the image is dense in one product. The full source-contact calculation and injectivity of the MV/contact pairing identify its two factors with Z_1, Z_2. The unique principal-parts reconstruction and separatedness show that there is at most one such unordered component. On the distinct-root open the ordered space is the full root-ordering torsor; its components therefore correspond to all components of the full ordered-root cover of that product. The full relabeling group acts transitively on these components and preserves the underlying map and its curve class.

For existence, glue general free one-input source maps at their interior frame values, keeping their former unit markings as the new unit and frame. Their logarithmic tangent bundles are trivial on both components. On the resulting rational tree the tangent bundle is trivial and, after fixing the new frame, has vanishing H^0 and H^1. The usual logarithmic map deformation theory therefore gives actual smoothings dominating the curve/frame base. The glued node is interior. The class is the sum of the two source classes. The component correspondence gives that same class on every ordered component of the generic smooth source. Lemma 9.5 identifies the entire general incidence with this free source fiber. Its supported virtual class is therefore the source fundamental zero-cycle. □

11.1. **A proper graph with complete vectors.** Set $\nu_i = \nu_{1,i} + \nu_{2,i}$ and

$$\Delta_i(z, a) = z^{\nu_{1,i}}(z - a)^{\nu_{2,i}}.$$

The complete vector on the separated source is

$$P_i(z) = \Delta_i(z, a) q(z) v_i^- \in H^0(\mathbb{P}^1, \mathcal{O}(\nu_i)) \otimes V_H(\omega_i)^*. \tag{PV}$$

Its lowest coordinate is exactly Δ_i, and its value at infinity in the standard trivialization is v_i^-.

Lemma 11.3. *Let B be the reduced raw BD closure of the separated source, before adjoining complete-vector records. Adjoin the independent frame parameter. There is an open neighbourhood U of every actual central polynomial-loop point of full fundamental degrees ν_i on which the complete vectors*

$$P_i(z, a) = z^{\nu_{1,i}}(z - a)^{\nu_{2,i}} q(z) v_i^-$$

are canonical regular coefficient vectors of degree at most ν_i. Their lowest coordinates are exactly $z^{\nu_{1,i}}(z - a)^{\nu_{2,i}}$. On this open, adjoining their projective records gives the graph of a regular morphism; its projection to U is an isomorphism.

Shrink U so that the auxiliary polynomials have simple roots and the required noncollision and interior-frame conditions hold. The full ordered-root cover $R \to U$ is finite étale. The enriched closure with all root orderings, restricted over U, is isomorphic to R. These are isomorphisms before taking the collision Cartier fibre or the retained curve/frame incidence fibre.

Proof. Let $\mathfrak{h}$ be the Lie algebra of H. At the trivial bundle with its infinity frame, framed deformations and automorphisms are

$$H^1(\mathbb{P}^1, \mathfrak{h} \otimes \mathcal{O}(-\infty)) = 0, \qquad H^0(\mathbb{P}^1, \mathfrak{h} \otimes \mathcal{O}(-\infty)) = 0.$$

The framed bundle functor is locally of finite presentation. Its trivial point is therefore an etale monomorphism, hence an open immersion. The same vanishings give a unique normalized trivialization over square-zero extensions and thus over Artin thickenings. Work on an affine chart $\operatorname{Spec} A$ of this open in the reduced raw closure. The normalized trivialization gives a regular H-valued map on the complement of $z(z-a) = 0$. Its U_H^+ equations hold on the dense separated source, and therefore hold on this complement: the latter is smooth over the reduced base and is reduced. In particular the lowest coordinate of $q(z)v_i^-$ is one.

After shrinking the chart, choose a common integer N at least as large as the prescribed two pole orders and write each coordinate as

$$q(z)v_i^- = \frac{R_i(z)}{z^N(z-a)^N}, \qquad R_i(z) \in A[z] \otimes V_H(\omega_i)^*.$$

Divide R_i by the monic polynomial

$$z^{N-\nu_{1,i}}(z-a)^{N-\nu_{2,i}}.$$

The remainder vanishes on the dense separated source, where the prescribed pole bounds hold. Reducedness of A makes every coefficient of the remainder zero. Thus P_i has regular coefficients over A, including at $a = 0$, since the divisor is monic over A. Its coefficients of degree greater than ν_i vanish by the same argument, or by regularity of q at infinity. Its lowest coordinate is the stated monic polynomial, so its coefficient vector is nowhere zero and determines a regular map to the projective record space.

The graph of this regular map is closed over U. The separated source is dense in every component of the reduced raw closure. Consequently the reduced closure of its graph, restricted over U, is exactly this graph, scheme-theoretically.

On the interior-frame open, applying the auxiliary linear functional to P_i and dividing by its nonzero frame value gives a monic polynomial with regular coefficients. On its discriminant open the scheme of all orderings of its roots is finite étale. Include the open conditions that the roots avoid the input and unit markings. Finite étale maps are open, so the inverse image of the dense separated source is dense in every component of this full ordering cover. Since the construction retains every ordering, its reduced closure is the whole cover R. The preceding canonical-record argument then identifies the enriched closure over U with R. □

Let S be a smooth chart for the ordered marked positions and the cross-ratio a. Write c for the d local coordinates of the retained $(d+3)$-pointed curve obtained by forgetting the second input. We choose the chart with local parameters (c, a) so that $a = 0$ has partition

$$\{\text{input}_1, \text{input}_2\} \mid \{\text{unit}, \text{frame}, \text{auxiliary markings}\}.$$

The auxiliary markings remain distinct. Retain the frame as a base parameter and put $S_{\mathrm{fr}} = S \times \mathscr{H}$.

Let $\mathcal{M}_S$ be the ordinary stable-map space for the fixed markings and the finite collection of curve classes, pulled back to S by stabilization; it is proper over S. Define the framed ordinary space by

$$\mathcal{M}_{\mathrm{fr}} = \{(f, h) \in \mathcal{M}_S \times \mathscr{H} : \mathrm{ev}_\infty(f) = h^{-1} \text{ in } X\}. \tag{11}$$

Inversion identifies $\mathscr{H}$ with the open subset $\mathscr{H}^{-1} = \{h^{-1} : h \in \mathscr{H}\}$ of $Y \subset X$. The displayed condition is the pullback of the closed diagonal of X, so $\mathcal{M}_{\mathrm{fr}}$ is proper over S_{fr}. Its projection to $\mathcal{M}_S$ identifies it with the open locus whose frame evaluation lies in $\mathscr{H}^{-1}$. Pull every positive logarithmic mapping space over this chart back to $\mathcal{M}_{\mathrm{fr}}$. These are open restrictions of the actual mapping spaces, with their restricted virtual classes.

Choose a bounded projective BD ambient space containing the separated sources. Uniform polynomial-loop bounds, including inverse-matrix bounds in a faithful representation, supply such a space. Base change it to S_{fr} and add the projective complete-vector records

$$\prod_i \mathbb{P}\big(H^0(\mathbb{P}^1, \mathcal{O}(\nu_i)) \otimes V_H(\omega_i)^*\big).$$

Let $\widetilde{\mathcal{B}}$ be the reduced closure of the full ordered-root source in this product, retaining all components of the root-ordering cover. It is projective over S_{fr}. The source identity $q(\infty) = 1$ gives $\mathrm{ev}_\infty(g) = h^{-1}$. Thus its ordinary map takes values in $\mathcal{M}_{\mathrm{fr}}$. Let Γ be the reduced closure of its graph in

$$\widetilde{\mathcal{B}} \times_{S_{\mathrm{fr}}} \mathcal{M}_{\mathrm{fr}}.$$

A subscript 0 denotes the fiber at $a = 0$. Write p_{BD} and p_{ord} for the two projections. Each ambient projection is a base change of a proper morphism; its restriction to the closed graph is therefore proper. The smooth-source inverse gives generic degree one on every component, so the fundamental-class pushforwards are the fundamental classes of the corresponding source closures. These proper morphisms remain proper after fixing (c, h) and after collision base change. Stabilizing the first input, unit, and frame gives the actual domain a degree-one projection to $\mathbb{P}^1$. Composition with $\pi_h : X \to H/B_-$ gives the common stable flag graph.

Lemma 11.4. *The ordinary projection of Γ factors through the scheme-theoretic proper image in $\mathcal{M}_{\mathrm{fr}}$ of the positive logarithmic mapping space with the fixed external data and frame condition* (11).

Proof. On the separated smooth source, the prescribed boundary intersections provide its canonical basic logarithmic enhancement. All inputs are positive and divisorial on the common model. Thus its ordinary map factors through the framed positive logarithmic space. Its forgetful map to $\mathcal{M}_{\mathrm{fr}}$ is proper by base change from the ordinary forgetful map, so its scheme-theoretic image is closed. Pull the ideal of this image to the reduced graph Γ. It vanishes on the dense separated part of every irreducible component, hence vanishes on all of Γ. This proves the factorization, which persists under incidence base change, including over nonreduced bases. □

Lemma 11.5. *Let q_0 be one of the finite genuine central flag maps of total fundamental degrees ν_i. Every point of Γ_0 above q_0 lies in the actual framed polynomial-loop open of $\widetilde{\mathcal{B}}_0$, and its BD point is the loop reconstructed from q_0. Conversely, every point above that actual central BD point has retained flag map q_0. These assertions hold on opens containing the entire corresponding fibres.*

Proof. The generic source identities between its flag lines and (PV) are closed incidence equations on the universal degree-one graph. They therefore hold on Γ. On a special fibre over q_0, the projective record $[P_i]$ is a nonzero polynomial vector. On the degree-one core, away from its finitely many zeros, its line is the line of q_0. The lowest coordinate of P_i is a scalar multiple $\lambda_i z^{\nu_i}$. If $\lambda_i = 0$, the lowest coordinate of the rational flag line would vanish identically. This contradicts the fact that the core meets the big cell and that its normalized frame flag is the lowest flag. Hence $\lambda_i \neq 0$; normalize it to one.

Let B_i be the common factor of the coordinates of P_i as a section of $\mathcal{O}(\nu_i)$. The associated rational flag line has degree $\nu_i - \deg B_i$. The retained map has degree ν_i, so B_i is constant. Thus no fundamental vector has a base point. Nonvanishing of λ_i and absence of a common vector zero are open conditions in the projective coefficient records. On this open the full Plucker relations give a genuine flag morphism on $\mathbb{P}^1$, and its normalized lowest coordinates are $\Delta_i(z, a)$.

Away from the input divisor all these coordinates are invertible. The flag therefore lies in $U_H^+ B_-/B_-$ and has its unique U_H^+ lift $q(z)$, with $q(\infty) = 1$. This defines an actual BD modification of the trivial bundle, functorially on that open. It agrees with the original BD projection on the dense separated part of the reduced graph. Separatedness of the bounded

BD space makes the two morphisms equal there. In particular every point over q_0 has the stated actual BD point, including graph boundary points.

Conversely, over that actual BD point the normalized complete polynomial vectors are those of q_0 and have no base points. They determine the flag map on the degree-one component. Its fundamental degrees already exhaust the total degrees ν_i. Nefness of every fundamental flag bundle makes every other flag component constant. The generic auxiliary isomorphism conditions and the prescribed two-input boundary partition then leave only the input subtree, as in the finite-receiver construction. After forgetting the second input it contracts, and the retained flag map is q_0. This proves the assertions on the full fibers. □

11.2. **The BD coefficient in the supported finite fibre.** Define the MV multiplication coefficients by

$$b_{Z_1} b_{Z_2} = \sum_T m^T_{Z_1 Z_2} b_T.$$

The sum ranges over MV cycles T of charge $\nu_1 + \nu_2$; it is finite by homogeneity. For fixed ordered auxiliary positions and frame, write Q_T for the coefficient-map fiber of T, with that root ordering.

Proposition 11.6 (Cartier coefficient at an actual central point)**.** *Choose general ordered auxiliary positions c and frame h for which all the charge-ν coefficient fibres are finite etale, disjoint from intersections of distinct MV source components, and satisfy the genuine flag conditions. Let Q be their finite union. Form all supported incidence classes before reducing any special fibre. Then the pushforward to $A_0(Q)$ of the BD graph's collision class is*

$$\sum_T \sum_{q_0 \in Q_T} m^T_{Z_1 Z_2} [q_0]. \tag{BQ}$$

A point with empty graph fibre has coefficient zero.

Proof. By [1, Theorem 7.11], the collision Cartier cycle on the raw actual central MV open is

$$0^![B] = \sum_T m^T_{Z_1 Z_2} [T].$$

Pull back along the smooth frame factor $\mathscr{H}$ and restrict to the open of Lemma 11.3 and then take its full ordered-root étale cover. On this neighbourhood the enriched space is that cover itself, so flat pullback of the raw BKK formula is the Cartier formula for the enriched space there.

Lemma 11.5 puts the entire graph inverse image of the actual finite set Q inside this neighbourhood. Near Q the coefficient map and the root-ordering map are étale. The supported pullback fixing (c, h) therefore takes each $[T]$ to $\sum_{q_0 \in Q_T} [q_0]$, with coefficient one at every point. Proper pushforward of the graph class and its two refined pullbacks is taken on this full inverse image. Its ordinary and BD maps to Q agree, so this gives (BQ) in $A_0(Q)$.

On S_{fr}, fixing (c, h) is a regular embedding of codimension $d + m$, leaving the cross-ratio parameter free. Form the one-cycle

$$\widetilde{\gamma} = (c, h)^![\Gamma].$$

For either proper projection $p = p_{\mathrm{BD}}$ or $p = p_{\mathrm{ord}}$, let $p_{c,h}$ be its incidence base change. Proper pushforward gives

$$(p_{c,h})_* \widetilde{\gamma} = (c, h)^! p_* [\Gamma]$$

in the Chow group of the corresponding incidence fiber. The regular pullbacks fixing (c, h) and setting $a = 0$ commute on this common smooth base, and commute with these proper pushforwards. The BD projection consequently gives (BQ) for the same supported collision class.

Lemma 11.5 says that the preimage of the good ordinary receiver and the preimage of the actual finite BD fibre Q are the same open in Γ_0. On this open their maps to Q agree. Their

proper pushforwards therefore identify the left hand side of (BQ) with the class used in the ordinary receiver comparison. All Cartier lengths, normalization valuations and residue-field degrees are already retained by the refined pullback and proper pushforward. One may use reduction-invariance of Chow groups only after this class has been formed.

The source intersection locus for two distinct charge-ν MV components has dimension less than $\mathrm{ht}(\nu)$. Finiteness of χ_h gives it proper image in the coefficient base. A common general choice avoids all those images. Thus every q_0 is attached to one actual T. Over a nonclosed base field one makes a common finite separable extension splitting Q; the same base change is made on both sides. The equality therefore compares coefficients at individual geometric points. □

12. Central flag maps and a finite receiver

Over S_{fr}, let $O^{\mathrm{pre}} \subset \mathcal{M}_{\mathrm{fr}}$ be the scheme-theoretic union of the proper images of the framed positive logarithmic mapping spaces and the BD graph. Include all homological decorations of the total numerical datum. The prescribed auxiliary contacts factor evaluations through their closed divisors. This is a closed receiver proper over S_{fr}, and p_{ord} factors as a proper map $p_{O^{\mathrm{pre}}} : \Gamma \to O^{\mathrm{pre}}$. Define O by fixing the retained curve and frame (c, h) and pulling back to the cross-ratio trait at $a = 0$. Then O is proper over that trait. All refined classes use this same base change. Under the identification of $\mathcal{M}_{\mathrm{fr}}$ with its open image in $\mathcal{M}_S$, fixing h is exactly the original refined frame insertion $\mathrm{ev}_\infty = h^{-1}$, of Gysin degree m.

Lemma 12.1. *Every geometric map in this receiver has a positive logarithmic lift after a field extension. If a nondegenerate component C_0 has an unmarked attached tree T, and c_i is the contact of C_0 with D_i at the attachment, then*

$$D_i \cdot \beta_T = -c_i \qquad \text{for every } i.$$

The same identity holds in a punctured central space for a tree carrying none of its markings.

Proof. The logarithmic image is closed by properness, and every geometric point of it lifts after a field extension. Lemma 11.4 puts the graph image in that same closed image, scheme-theoretically. These facts persist after incidence base change. Sum the contact-degree identity for D_i over the vertices of T. The internal edge terms cancel. There are no marked legs on T, and its only external edge contributes $-c_i$. This proves the identity, also when a puncture lies on the complementary component. □

Lemma 12.2. *Let $f : C \to X$ be an ordinary coarse map with the fixed external data of total charge ν. Suppose the unit, frame, and all auxiliary markings lie on one smooth component C_0, and the complement is a connected subtree containing the two input markings. Suppose the frame is interior and the color-i auxiliary markings are the ν_i distinct prescribed zeros of the section in (8). Then*

$$\deg_{C_0}(\pi_h f)^*\mathcal{L}_i = \nu_i, \qquad \deg_{C'}(\pi_h f)^*\mathcal{L}_i = 0 \quad (C' \neq C_0).$$

Every component off C_0 is constant as a map to H/B_-.

Proof. The section on C_0 is nonzero at the frame and vanishes at the ν_i distinct auxiliary points. Its degree is therefore at least ν_i. The total degree is ν_i and $\mathcal{L}_i$ is nef. Equality follows, as does degree zero on every other component. A positive sum of the fundamental bundles is ample; hence all these other components have constant flag image. They may still be nonconstant in an exceptional fiber of $X \to H/B_-$. □

Compose a stable X-map with π_h, forget the second input marking, and stabilize as a stable map to H/B_-. This defines a proper morphism with $d + 3$ retained markings (the first input, unit, frame, and d auxiliary markings),

$$\Phi : O_0 \longrightarrow \overline{\mathcal{M}}_{0,d+3}(H/B_-, \nu). \tag{12}$$

On the locus of Lemma 12.2, its value is the central flag map. Here ν specifies the flag curve class by its fundamental degrees ν_i. Stabilization defines this morphism functorially in families.

Lemma 12.3. *Let q be a smooth central flag map with the fixed frame and auxiliary data. Suppose its rational lift g is regular in V away from the input, has its stated generic auxiliary contacts, and has complete flag degrees ν_i. Then*

$$n(t) = \dot{w}_0^{-1} h^{-1} g(t^{-1})^{-1} \dot{w}_0$$

is a based polynomial loop in U_H^+ of semi-infinite charge ν. It belongs to an actual MV source, and its coefficient map has exactly the prescribed ordered auxiliary roots.

Proof. The displayed expression is regular for finite t and equals one at $t = 0$. Each complete vector $n(t)v_i^-$ has its lowest coordinate identically one. Consequently its coordinates have no common polynomial factor, and their maximum degree is the degree of the associated flag line bundle, namely ν_i. The complete-vector characterization of the semi-infinite orbit now gives the charge ν. The irreducible-component description of the resulting finite-dimensional orbit intersection supplies an actual MV source. The section identity above identifies its auxiliary roots with the prescribed ones. □

Lemma 12.4. *On the fixed separating-curve fiber, let G be the relative open locus where the unit, frame, and all auxiliary markings lie on one smooth component C_0, every auxiliary evaluation is in the prescribed generic V-divisor locus where $X \to H/B_-$ is an isomorphism, and unit/frame evaluations lie in Y. These opens are taken after imposing the closed incidence. On the common ordinary receiver, $C \setminus C_0$ is one connected subtree containing both inputs.*

Proof. The property defining G survives smoothing edges of the domain graph, so it is open in the finite collection of domain strata. The fixed stable curve has the partition {input 1,input 2} | {all other marks}. Hence the inputs lie in the same connected component of $C \setminus C_0$; every other such component is an unmarked tree T. The frame makes C_0 nondegenerate. Write $c_i \geq 0$ for its ordinary contact with D_i at the attaching point of T.

Lemma 12.1 gives $D_i \cdot \beta_T = -c_i$ for every boundary divisor.

If some $c_i > 0$, nefness contradicts $A \cdot \beta_T = -\sum_i a_i c_i < 0$. If every $c_i = 0$, the attachment lies in Y. The first component of T has degree zero against the effective full-support divisor A, so it stays in Y and is constant by affineness. Repeat along the connected tree. An unmarked constant tree with one attachment cannot be stable. Thus T does not exist. □

12.1. **A finite proper receiver for individual coefficients.** Let $G \subset O_0$ be the open from Lemma 12.4. Its complement to the central component is exactly one connected input subtree. Let Q be the finite reduced union of the general coefficient-map fibers over all actual MV sources of charge ν, regarded as actual normalized smooth central flag maps. The finite-map theorem and generic etaleness give the asserted finiteness and reducedness. For every point of G, the divisorwise restrictions exclude unmarked boundary intersections, and the nef cut leaves only the input subtree. The auxiliary evaluation conditions put its other boundary values in V. Its rational V-map is therefore regular away from the input. Fundamental degree exhaustion and Lemma 12.3 reconstruct an actual source of charge ν with the prescribed coefficient fibre. Consequently $\Phi(G) \subset Q$.

Choose the general incidence so that every $q \in Q$ takes values in Y_h off the input and auxiliary markings, and its auxiliary values lie in the π_h-images of the prescribed generic V-divisor loci where π_h is an isomorphism. These are simultaneous generic conditions on the finitely many source families; the smooth-domain frame argument supplies the auxiliary genericity.

Lemma 12.5. *Under the preceding incidence choices, every geometric fiber $\Phi^{-1}(q)$, $q \in Q$, lies in G. There is an open subspace $O_{\mathrm{good}} \subset O_0$ containing these fibers and a proper map*

$$(O_{\mathrm{good}})_{\mathrm{red}} \longrightarrow Q.$$

It induces a proper pushforward $A_0(O_{\mathrm{good}}) \to A_0(Q)$, where $A_0(Q)$ is freely generated by the actual points q.

Proof. A map over q has exactly one component surviving as the smooth marked domain of q; stable-map stabilization never contracts a nonconstant flag component. This formulation includes charge zero. Every other component is mapped into a fiber of π_h. Away from the input, that fiber is a point: q takes values in Y_h or in the translated auxiliary isomorphism loci. The fixed separating marked curve puts the two inputs over its input point. Thus an off-input subtree has no input marking and at most one non-input marking. A tree of constant components with at most one marking and one attaching point cannot be stable. Hence every non-input marking lies on the main component and every remaining component is attached at the input. The prescribed separating marked curve puts both inputs in the same connected subtree there. A nodal curve cannot attach a second subtree at the same smooth point of its main component. This proves $\Phi^{-1}(q) \subset G$.

Properness of Φ makes $\Phi(O_0 \setminus G)$ closed and disjoint from Q. Put

$$O_{\mathrm{good}} = \Phi^{-1}\big(\overline{\mathcal{M}}_{0,d+3}(H/B_-, \nu) \setminus \Phi(O_0 \setminus G)\big)\,.$$

Then $O_{\mathrm{good}} \subset G$ and its image is contained in Q. Its reduction factors through the finite reduced Q and is proper over Q, by properness of the corresponding base change of Φ. Chow groups are unchanged by nilpotent thickening, which gives the displayed pushforward. The cycle to be pushed is formed before invoking this nilpotent-invariance statement: Cartier-fiber multiplicities are retained. □

The same open can be chosen for the finitely many central punctured spaces arising from cuts with zero tail flag degree. Their ordinary images are proper. In each of them, impose the same auxiliary isomorphism loci and interior conditions to define G_C. The preceding tree argument applies to branches without the input marking, and the degree argument gives $\Phi_C(G_C) \subset Q$. Every fiber over Q lies in G_C, because off the input the fibers of π_h are points and a constant tree with at most one marking and one attachment is unstable. Delete the finite union of the proper closed bad images from the flag space. Its complement contains Q and works for the total and every central receiver. For zero tail flag degree, flag stabilization after gluing factors through the central map; hence this common-open restriction is pulled back from the central factor.

13. Supported Cartier specialization

Lemma 13.1. *Let O be proper over a discrete valuation ring R, with closed point 0, and let α, β be relative one-cycle classes whose generic restrictions agree. Then*

$$0^!\alpha = 0^!\beta \quad \text{in } A_0(O_0)_{\mathbb{Q}}.$$

Suppose $\Phi : O_0 \to Z$ is proper, $V \subset Z$ is open, and $\Phi(O_0) \cap V$ is contained in a finite reduced closed scheme $Q \subset Z$ which is contained in V. The equality restricts to $O_V = \Phi^{-1}(V)$ and then pushes to an equality in $A_0(Q)_{\mathbb{Q}}$.

Proof. Localization writes $\alpha - \beta = i_*\gamma$ for a class supported on O_0. The refined pullback is the base change of the regular embedding $0 \to \operatorname{Spec} R$. The self-intersection rule gives $0^! i_*\gamma = c_1(N_{0/R}) \cap \gamma = 0$, because the normal line is trivial.

By properness the image inside V is closed. Its support is finite, so O_V is also closed in O_0. After the Cartier pullback has been formed, reduction preserves Chow groups and its already-computed coefficients. The reduced map factors through Q, and is proper: it is the restriction of a proper map to a closed finite receiver. Restrict and push the displayed equality. It is enough that the image be *contained* in Q. A candidate point with empty

inverse image has coefficient zero. Over a nonclosed residue field one may make one finite separable extension splitting the finite *étale* candidate set; all stack stabilizers and base-change degrees are retained on both sides. The equality then compares the coefficients of each geometric point. □

The auxiliary-divisor conditions are part of the prescribed contact data. The auxiliary isomorphism loci define opens within that closed incidence. The actual insertions fix the retained curve and frame, of codimensions d and m. They reduce the two-input virtual dimension $m+d+1$ to one; collision then gives a zero-cycle.

14. Logarithmic collision

We compare the coefficients directly using logarithmic virtual classes. The evaluation-enhanced gluing theories are those of [32, Sections 4.2 and 5.3] and [40, Theorems 5.1, 5.3, 5.5], always on the original scheme target. The Artin-stack Chow operations and their rational proper descent are those of [36, Appendices B–C].

Lemma 14.1. *Work with rational Chow groups over a characteristic-zero field. Let B be a finite-type algebraic stack stratified by global quotient stacks, with $\dim B \le b$. Let $F \to B$ be a morphism with F a finite-type Deligne–Mumford stack, and let*

$$c \in A^e(F \longrightarrow B)$$

be a bivariant class on finite-type quotient-stratified test stacks: its operations commute with representable projective pushforward, flat pullback, and refined pullback for representable lci morphisms. If $e > b$, then $c = 0$.

Proof. Take the reduction of B and apply functorial resolution of singularities to it. More explicitly, [41, Theorem 5.1.1] extends the smooth-functorial sequence of ordinary blowups from a smooth atlas and its groupoid to a sequence of blowups of the reduced Artin stack. Thus there is a projective representable surjection

$$r : \widetilde{B} \longrightarrow B$$

with $\widetilde{B}$ smooth, whose connected components have dimension at most b. Compose the resolution of the reduction with its closed immersion in B. Properness, representability, and surjectivity hold after every base change. Since r is representable, $\widetilde{F} = F \times_B \widetilde{B}$ is still Deligne–Mumford. The pullback of c to any smooth pure-dimensional component B' of $\widetilde{B}$, of dimension $b' \le b$, satisfies

$$c_{B'}[B'] \in A_{b'-e}(F \times_B B')_{\mathbb{Q}} = 0.$$

Here negative-dimensional rational Chow groups vanish because the support is Deligne–Mumford.

The supported smooth-base duality formula is as follows. If Y is smooth of pure dimension b' and $a \in A^e(H \to Y)$, then, for every $g : U \to Y$ and $\eta \in A_*(U)$,

$$a_g(\eta) = \Gamma_g^!\big(\eta \times a_{\mathrm{id}}[Y]\big) \quad \text{in } A_*(H \times_Y U).$$

Here $\Gamma_g : U \to U \times Y$ is representable lci. It need not be a regular closed immersion for an Artin target; its virtual codimension is b', which may be negative. The formula follows by flat exterior product and compatibility with refined lci pullback: first prove it for a fundamental cycle, then use representable projective pushforward and homotopy invariance to pass to Kresch Chow classes. These are the operations in the proof of [36, Theorem C.6], with support in H retained at every step. Consequently $c_{B'}[B'] = 0$ implies $c_{B'} = 0$, and hence $r^*c = 0$.

For an arbitrary test stack $U \to B$, set $\widetilde{U} = U \times_B \widetilde{B}$. The map $\widetilde{U} \to U$ is projective, surjective, and representable. [36, Proposition B.19] gives a surjection on rational Chow

groups in every dimension. Lift $\eta \in A_*(U)$ to $\widetilde{\eta} \in A_*(\widetilde{U})$. The projective-pushforward axiom then gives

$$c_U(\eta) = r_{F,U*}\big((r^*c)_{\widetilde{U}}(\widetilde{\eta})\big) = 0.$$

□

Lemma 14.2. *Let (X, D) be a smooth projective simple-normal-crossings log Calabi–Yau pair of dimension m, and let τ_C be a realizable genus-zero decorated type with $d+3$ labelled legs. Enhance its Artin-fan mapping stack at precisely one distinguished cut leg:*

$$B_C = \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau_C), \qquad \epsilon_C : L_C = \mathcal{M}(X, \tau_C) \longrightarrow B_C.$$

Here the Artin base uses the underlying undecorated type; the actual mapping stack retains the specified homological decorations, by [32, Proposition 5.21 and diagram (5.22)]. *Use a finite-type restriction containing the actual maps under consideration, and the compatible enhanced perfect obstruction theory of ACGS Sections 4.2 and 5.3. Let P be the smooth curve/frame parameter space of dimension $m+d$, let $b \in P$, and let $L_{C,b}$ be the corresponding actual incidence. Then*

$$c_C := b^! \epsilon_C^! \in A^{m+d}(L_{C,b} \longrightarrow B_C)$$

vanishes if $\dim \tau_C > 0$. In particular, after every base change in the stated test category it annihilates every gluing-base cycle, including cycles not expressible as products of vertex classes.

Proof. ACGS Proposition 3.30 and Gross's dimension calculation in the proof of Theorem 5.3 give

$$\dim B_C = (d+3) - 3 - \dim \tau_C + m = m + d - \dim \tau_C.$$

The enhancement is smooth of relative dimension m, by ACGS Definition 5.14. Write $\pi : \mathcal{C} \to L_C$ and $f : \mathcal{C} \to X$ for the universal curve and map, and p_{cut} for its cut section. Its compatible relative obstruction complex is

$$(R\pi_* f^* T_X^{\log}(-p_{\mathrm{cut}}))^\vee.$$

Its virtual rank is $\chi(f^*T_X^{\log}) - m = m - m = 0$: genus zero and $c_1(T_X^{\log}) = 0$ are used, and the sign of a puncture does not alter this Euler characteristic. The incidence pullback has degree $\dim P = m + d$. Thus c_C has the displayed degree. The actual stable punctured mapping stack and its incidence are Deligne–Mumford. Its virtual pullback, composed with the regular incidence pullback, has all the bivariant compatibilities in the preceding lemma [38, Theorems 4.3–4.4]. The genus-zero prestable-curve automorphism groups are affine. The kernel contributed by logarithmic maps to the Artin fan is a subgroup of the product of the finitely many torus automorphism groups of the ghost charts, with finite graph automorphisms; see [32, Remark 3.27 and diagrams (3.8)–(3.9)]. Thus B_C has affine geometric stabilizers; evaluation enhancement is representable and preserves this property. The finite-type restrictions are therefore stratified by global quotient stacks, by [37, Proposition 3.5.9]. Lemma 14.1 proves the claim.

For gluing, first choose the enhanced obstruction theories on the central and tail factors, and on their product choose the external direct sum, including its arrow to the relative cotangent complex. The proof of [32, Theorem 5.19(1)], in its final paragraph, then permits the dashed arrow for the glued enhanced obstruction theory to be chosen so that this theory is the pullback of that product theory. Indeed, the proof fixes the split arrow and adjusts the glued arrow through the ambiguity described in [32, Remark 4.4]. The resulting glued theory still belongs to the compatible triangle of Section 4.2, so its comparison with the unenhanced theory is preserved.

Write B_T for the tail Artin base enhanced at its cut marking, and $\epsilon_T : L_T \to B_T$ for its actual mapping space with the compatible obstruction theory. Write $G = B_{\mathrm{gl}}^{\mathrm{ev}}$ for the glued enhanced Artin base and $j : G \to B_C \times B_T$ for its splitting map. By [32, Proposition 5.17], the square defining the actual glued stack is Cartesian in the category of ordinary stacks.

Since the retained curve/frame incidence is pulled back from the central factor, it identifies the actual glued incidence with

$$L_{\mathrm{gl},b} \simeq G \times_{B_C \times B_T} (L_{C,b} \times L_T).$$

Let $\epsilon_{T,G} : U_T \to G$ be the base change of ϵ_T along $G \to B_T$. For any $\alpha \in A_*(G)$, set

$$U_T = G \times_{B_T} L_T, \qquad \eta = \epsilon_{T,G}^!(\alpha) \in A_*(U_T).$$

The stack U_T is an admissible test stack for c_C: it is finite type on the chosen restrictions, and $U_T \to G$ is of DM-type, so it inherits quotient stratification. Hence

$$(c_C)_{U_T}(\eta) = 0 \quad \text{in } A_*(L_{\mathrm{gl},b}).$$

The split compatible triangle and [38, Theorem 4.8] identify the virtual pullback for the product obstruction theory with the composite of the tail and central virtual pullbacks; [38, Theorem 4.3] permits commutation with the central incidence pullback. With the compatible glued arrow chosen above, the last displayed zero is therefore exactly $b^! \epsilon_{\mathrm{gl}}^!(\alpha)$.

Thus the glued virtual-incidence operation vanishes on its actual carrier for every class on the glued enhanced base. □

14.1. **The complete source Cartier cycle and its tails.**
The source boundary. There are $d+4$ marked points before the two distinguished inputs collide. Work over the curve chart on which stabilization forgets the second input to give the retained $(d+3)$-pointed curve, and the remaining coordinate is the cross-ratio a. At $a = 0$ the two input marks lie on the three-point tail of the stabilized curve; its other component carries the unit, frame, and all auxiliary markings. Forgetting the second input contracts that tail and transfers the first input label to its attachment on the retained curve. Let $\mathfrak{A}$ be the Artin-fan mapping stack for the full positive *undecorated* global datum: genus, labelled markings, and complete global contact vectors. Homology classes are imposed on the actual mapping spaces. At this stage the full global-contact stack precedes curve/frame incidence. Use a finite-type restriction containing every actual map occurring over the proper ordinary receiver under discussion. Let

$$\epsilon : \mathcal{L} \longrightarrow \mathfrak{A}$$

be the positive stable-log-map morphism with its usual relative obstruction theory, taking the disjoint union over the entire bounded collection of actual homological classes when needed.

The full positive Artin base is logarithmically smooth over the ground field. We compute its canonical idealized structure from the global datum. Write $\bar{\beta}$ for the undecorated class datum: it has one vertex, no edges, and all reference strata equal to the origin, as in ACGS Definition 2.44(4). For a geometric point w of the full mapping base, let τ_w be its actual type and $\tau_w \to \bar{\beta}$ its marking contraction. Every vertex is sent to the origin and every internal edge is contracted. Thus the face in ACGS equation (2.21) is

$$Q^\vee_{\bar{\beta}\tau_w} = \{(V_v, \ell_e) \in Q^\vee_{\tau_w} : V_v = 0 \text{ for all } v,\ \ell_e = 0 \text{ for all } e\} = \{0\}.$$

Consequently the target sharp monoid $Q_{\bar{\beta}\tau_w}$ of the localization in equation (2.22) is zero. All three sets of generators in ACGS Definition 3.20 are therefore empty: in (i), each reference cone is zero, so $P_x \setminus \chi_x^{-1}(0) = \varnothing$; in (ii), no node is preserved by the marking; and in (iii), $Q_{\bar{\beta}\tau_w} \setminus \{0\} = \varnothing$. Furthermore, the prescribed external contacts are all nonnegative. The prestable puncturing hence has no negative contact generator, and its puncturing log ideal is empty by ACGS Definitions 2.49 and 2.55 and the generator description in the proof of Proposition 2.57. The canonical idealized structure of Definition 3.23 is empty.

The curve base for this one-vertex marking has no marked nodal ideal either. ACGS Theorem 3.25 consequently gives an ordinary logarithmically etale morphism from the full positive Artin base to the prestable-curve stack with its nodal log structure. The latter is logarithmically smooth over the ground field, so the same is true of the former. In particular, toroidal smooth charts are normal and reduced and have dense locus of trivial characteristic.

The pulled-back stabilization parameter a is, on each such chart, a unit times a monomial and is a nonzerodivisor.

Lemma 14.3. *The Cartier cycle* $[a = 0]$ *on* $\mathfrak{A}$ *is the sum of its one-dimensional tropical-type cycles, with coefficient*

$$\lambda_\tau = \sum_{e \in E_{\mathrm{sep}}(\tau)} \ell_e(v_\tau).$$

Here v_τ *is the primitive generator of the lattice of the one-dimensional cone* τ*. The set* $E_{\mathrm{sep}}(\tau)$ *consists of the edges whose removal separates both input markings from the unit, frame, and all auxiliary markings. Consequently its virtual pullback, followed by the retained curve/frame insertion, is a sum of terms indexed by a type* τ *and one chosen separating edge* e*, with weight* $\ell_e(v_\tau)$*. Each term cuts exactly one edge.*

The equality holds on the actual special mapping stack and commutes with proper push-forward to the ordinary receiver.

Proof. The edges inducing this partition form the chain contracted to the separating node of the stabilized $(d+4)$-pointed curve. On a chart of the prestable-curve stack their smoothing parameters satisfy

$$a = u \prod_{e \in E_{\mathrm{sep}}} s_e,$$

where u is invertible. This follows by successively contracting the two-point components in the unique separating chain; for two adjacent nodes their product is the parameter of the resulting node. Nodes off this chain do not occur in the stabilization parameter. Pullback to the basic positive Artin-fan map stack writes each s_e as a unit times the monomial χ^{ℓ_e} in its basic monoid. Thus

$$a = u' \chi^{\sum_{e \in E_{\mathrm{sep}}} \ell_e}.$$

This is an equality of Cartier equations. It retains contacts of arbitrary integral multiplicity.

Every codimension-one component of this monomial divisor has generic sharp characteristic monoid $\mathbb{N}$; its generic tropical type has dimension one. The multiplicity on that component is the value of its exponent on the primitive valuation, which gives the asserted sum. Conversely every component of the divisor is detected this way, since the positive base is normal and the equation is a nonzero monomial on each toroidal chart.

First use undecorated tropical types τ and the full τ-marked mapping stacks, which contain the exact-type locus as a dense open; ACGS Proposition 3.30 (v3, pp. 71–72) gives reducedness and pure dimension for the full marked stack of a realizable type, and Remark 3.31 gives its finite unramified forgetful morphism to the full mapping base. Its image is the appropriate type-stratum closure. Generic multiplicities and degrees are computed on the dense exact-type open, and the cycles are pushed from these full marked stacks. One can either use their quotients by type automorphisms or divide their pushed cycles by the generic type-marking degree. The displayed cycle identity is on these quotients. At this Artin-base stage the denominator is the finite marking degree $|\operatorname{Aut}\tau|$ of the undecorated type, and the sum includes all separating edges. After virtual pullback to an actual mapping space of a specified total class, ACGS Proposition 5.21 and diagram (5.22) identify the base change with the disjoint union of all compatible curve-class decorations of the vertices, with compatible obstruction theories. Applying orbit–stabilizer to the action of $\operatorname{Aut}\tau$ on these decorations converts the coefficient into $1/|\operatorname{Aut}\boldsymbol{\tau}|$ for each decorated type $\boldsymbol{\tau}$. This is the orbit–stabilizer calculation in ACGS Theorem 5.25, equations (5.24)–(5.25), applied after Proposition 5.21. Equivalently, the one-cut sum is over isomorphism classes of pairs $(\boldsymbol{\tau}, e)$ with denominator $|\operatorname{Aut}(\boldsymbol{\tau}, e)|$. Gluing and the edge-length functions are equivariant for these automorphisms, so the same quotient factors enter the gluing calculation.

The definition of the logarithmic virtual class is $\epsilon^!|\mathfrak{A}|$. Refined Cartier pullback commutes with the virtual pullback and with the regular curve/frame insertion. The equality of Cartier

cycles therefore gives the equality stated on the actual special mapping stack. When switching to an evaluation-enhanced base for gluing, ACGS Definition 5.14 and the compatible obstruction-theory triangle of its Section 4.2 identify the resulting virtual pullback with the same class. Finally proper pushforward of these operations gives the equality on any common ordinary receiver. □

Restriction to the finite receiver. Use the good flag open and finite set Q of Lemma 12.5. A tail of positive flag degree has empty inverse image of this open. For a zero flag degree tail, the entire tail tree is flag constant, so flag stabilization of the glued map factors through the central factor. The retained curve/frame insertion is also pulled back from the central factor: after forgetting the second input, stabilization contracts the two-input tail and transfers the first input label to the cut mark. The retained curve is therefore the stabilization of the central curve with that label at its cut leg.

For each one-cut term, let τ_C denote its induced central punctured type. Its legs are the $d+2$ original non-input markings and the cut leg, giving $d+3$ labelled legs. The vanishing assertion of Lemma 14.2 applies with evaluation enhanced at that one cut leg. In conjunction with the ordinary Cartesian square of ACGS Proposition 5.17 and the compatible pullback obstruction theory of Theorem 5.19(1), it kills the gluing operation itself whenever $\dim \tau_C > 0$.

A realizable type with a positive edge length, or with a nonzero vertex degeneracy, has positive-dimensional cone: simultaneous positive scaling supplies a nonzero ray. Hence a surviving central type has one nondegenerate vertex and no edges. Its generic source is smooth, and all its contacts are nonnegative. The type-marked mapping space still includes more degenerate types. Apply Lemma 9.5 to this whole positive central mapping space; it identifies its entire actual incidence with the finite reduced free maps $q \in Q$. Its virtual class there is $[q]$, of weight one. Proposition 10.3 recovers the full source T, the complete contact p_T, and its class β_T. For the nonunit products in question $p_T \neq 0$. Let ρ be the primitive generator of its ray on the chosen model, let k be its positive integral multiplicity, and let p_{cut} be the cut marking. Then

$$p_T = k\rho, \qquad k \geq 1, \qquad z = q(p_{\mathrm{cut}}) \in D_\rho^\circ.$$

The vanishing also removes every other cut farther down a separating chain whose central side is still nodal.

Lemma 14.4. *Fix a surviving free central point q, with complete output $k\rho$, and a tail curve class. Let $\widehat{\mathfrak{B}}_T$ be the full one-negative tail Artin base enhanced at its output, and $\widehat{\mathcal{M}}_T$ the corresponding actual punctured mapping space. Write $[\widehat{\mathcal{M}}_T]^{\mathrm{ref}}$ for the virtual pullback of the puncturing base cycle with its monomial multiplicities, as in Appendix A. Restrict evaluation to D_ρ° for the local calculation. The sum of all surviving one-cut terms, with their source Cartier weights, pushes on the same ordinary glued-map receiver to*

$$[q] \times z^![\widehat{\mathcal{M}}_T]^{\mathrm{ref}}.$$

The equality includes every top-dimensional component of the full tail base with its puncturing multiplicity.

Proof. The full tail has three markings and exactly one negative divisor-marking pair. Its universal puncturing condition is one monomial Cartier equation. On the output open it is log smooth over the logarithmic evaluation stack, as in [39, Theorem 3.16]. Smooth toroidal charts of its enhanced base therefore have the form

$$\operatorname{Spec} K[Q]/(\chi^h), \qquad h \in Q \setminus \{0\},$$

with Q fs and $K[Q]$ normal, times smooth factors. Its refined base cycle is its fundamental cycle with these monomial multiplicities, as established in the scheme-target point-constraint argument.

This enhanced base is pure-dimensional of dimension $m-1$. For the full marked stack of a realizable tail type τ_T, its dimension is $m-\dim\tau_T$ by ACGS Proposition 3.30 and evaluation enhancement. The negative output forces a positive offset h, so $\dim\tau_T \geq 1$. Every top-dimensional component is thus an image of a component of a full type-marked stack with $\dim\tau_T = 1$, and every such component is retained. The finite maps of ACGS Remark 3.31 supply the proper cycle carriers; their dense exact-type opens are used only for the generic calculation. At such a generic point let t be the primitive lattice parameter and write

$$h = b\,t, \qquad b > 0.$$

Its coefficient in the full refined base cycle is b. This coefficient retains the nonreduced thickness of the full base.

For schematic gluing, the virtual diagonal formula applies as follows. Both enhanced evaluations, central and tail, factor through the same smooth stratum D_ρ°. On the Artin fan the corresponding open ray stratum is $B\mathbb{G}_m$, and the enhanced base is the pullback of $D_\rho^\circ \to B\mathbb{G}_m$. Every morphism to $B\mathbb{G}_m$ over the field is flat: after its smooth point atlas it is a $\mathbb{G}_m$-torsor over a stack over a field. Thus these enhanced evaluations to D_ρ° are flat. Equivalently, [40, Theorem 5.3] applies to the permitted open: every allowed contraction keeps the cut-leg cone equal to ρ, so the stratum-dimension increment in that theorem is zero, and $\dim\tau' \geq \dim\tau$. The calculation is on the evaluation open D_ρ°. The diagonal of D_ρ° is lci. [40, Theorem 5.1 and Remark 5.2] now give the schematic gluing class by the ordinary refined diagonal. After central incidence this is precisely the tail point pullback at z paired with the point q.

The remaining logarithmic gluing is finite and representable. The central tropical lattice is zero. At the one cut the gluing lattice homomorphism of [40, Definition 4.2] is

$$\Psi : \mathbb{Z}t \oplus \mathbb{Z}\ell \longrightarrow \mathbb{Z}\rho, \qquad (t,\ell) \longmapsto bt - k\ell.$$

It has finite cokernel of order $g = \gcd(b,k)$. Therefore the gluing is tropically transverse; the flatness just checked and [40, Theorem 5.5] show that the finite map is dominant, hence surjective, and that its degree is g. The primitive generator s of the glued one-dimensional cone satisfies

$$t = (k/g)s, \qquad \ell = (b/g)s.$$

Consequently the contribution of this cut to the source Cartier parameter has weight b/g. Its product with the logarithmic gluing degree is

$$\underbrace{b/g}_{\text{source Cartier weight}} \qquad \underbrace{g}_{\text{finite gluing degree}} \qquad = b.$$

After a separable residue extension there are g unit branches; without that extension their residue-field degrees sum to g. Existing stack stabilizers and graph automorphisms are already in the degree and the quotient type cycles. This calculation agrees with the one-negative point constraint's saturated chart $bt = k\ell$, whose branches have transverse length b/g. For the surviving cut, this statement also follows directly from

$$\mathrm{Aut}(\tau, e) \simeq \mathrm{Aut}(\tau_T).$$

Indeed the central vertex is the unique vertex carrying the unit, frame, and all auxiliary legs; its class and complete contact data are fixed, and the chosen edge is fixed. Deleting that vertex and edge identifies its automorphisms with those of the tail with its three labelled legs. Thus the source marked-cut denominator is exactly the tail type denominator. The fixed actual free central curve has no additional automorphisms because its markings are retained.

ACGS Theorem 5.19, with the compatible choices of enhanced obstruction theories, transports this finite-base-cycle equality to the actual mapping spaces. The finite logarithmic gluing and the schematic diagonal are over the same ordinary nodal map. The equality consequently holds after proper pushforward to the ordinary receiver, before taking its degree.

Summing over the reduced tail-type components gives exactly the multiplicities b of $[\widehat{\mathfrak{B}}_T]$ and hence the full refined tail class, with its output point insertion.

It remains to verify coverage of the full tail class. From any one-dimensional tail type with output in D_ρ°, attach the nondegenerate central vertex of q and replace the two matching legs by the edge of length $h/k > 0$. All other legs and decorations are unchanged. This is a realizable positive total type of dimension one. Its total class is $\beta_T + \beta_{\mathrm{tail}}$. The finite gluing is surjective by the preceding transversality and flatness, so every matching geometric point of each full marked tail carrier is included. The coefficient b retains its scheme thickness. Conversely, deleting the surviving central vertex and the single cut of any surviving total type gives exactly such a tail type. The relation $h = k\ell$ identifies their real cones; hence a surviving total Cartier ray has a one-dimensional tail cone, and no other top-dimensional tail component is missed.

This correspondence is applied separately to every curve class in the entire bounded comparison collection. The already proved degree and boundary-localization identities impose the required total class and zero tail flag degree. More explicitly, the complete boundary degree equations determine the numerical tail class $\beta_Z + \beta_W - \beta_T$, and each fundamental flag line bundle has degree $\nu_{Z,i} + \nu_{W,i} - \nu_{T,i} = 0$ on it. Therefore every admissible tail in the reverse construction is flag constant, component by component, and the whole proper tail fiber has flag stabilization equal to the fixed ordinary point q after gluing. Thus every tail component maps to the same finite receiver. □

15. Multiplication coefficients

Products containing the paired unit already agree. For this section fix nonunit Z, W and put $\nu = \nu_Z + \nu_W$, $d = \sum_i \nu_i$. Use the direct logarithmic formula of Section 14.

Choose the common model for the total positive data and all effective nonnegative central contacts as above. Choose the retained $(d+3)$-pointed curve c and frame h generally. The choices avoid the closed bad images for all finitely many central data and lie in the projection of the good two-input open. Thus the generic value of the remaining cross-ratio a belongs to that two-input open. Work over the trait at $a = 0$ with coefficient field generated by these general parameters. One finite separable extension may be made to split the finite ordered coefficient fibers.

Let O be the base change of $O^{\mathrm{pre}} \subset \mathcal{M}_{\mathrm{fr}}$ along this fixed (c,h) trait in S_{fr}. It is proper over the trait by Section 12. Write α for the pushed whole logarithmic virtual class after the supported (c,h) insertion, summed over all actual homological decorations of the fixed numerical total datum. This collection is finite by the bounded-degree argument above. Proposition 11.2 identifies every generic map with the actual two-input BD source; decorations with no such generic source have zero generic class. The vertical-DVR argument therefore also controls their specialization, and none is discarded from the gluing sum. Use the graph Γ over S_{fr} from Section 11.1, with its complete-vector records. Write $\Gamma_{c,h}$ for the fixed (c,h) graph over the same cross-ratio trait and $p_O : \Gamma_{c,h} \to O$ for the base change of the proper map $p_{O^{\mathrm{pre}}}$. Then

$$\gamma = (c,h)^!(p_{O^{\mathrm{pre}}})_*[\Gamma] = (p_O)_*\big((c,h)^![\Gamma]\big).$$

Both expressions are the same refined one-cycle on O. By the smooth inverse and Lemma 9.5, $\alpha_\eta = \gamma_\eta$ on the generic cross-ratio fiber. Here (SGI) is applied at a geometric generic point. The resulting rational equivalence descends to a finite extension of the generic coefficient field. Restriction followed by proper transfer multiplies rational Chow classes by that extension degree, so the equality holds over the coefficient field of the trait itself. Hence Lemma 13.1 gives

$$0^!\alpha = 0^!\gamma \quad \text{in } A_0(O_0)_{\mathbb{Q}}. \tag{13}$$

The flag morphism and the degree-exhaustion argument isolate the finite candidate set Q of actual central maps. Restrict (13) to the good open and push to $A_0(Q)_{\mathbb{Q}}$. Candidate

points with empty fibers are retained with coefficient zero. The group $A_0(Q)_{\mathbb{Q}}$ has the actual points as an independent basis.

For an actual homological class A, denote by $N^{p_T,A}_{p_Z,p_W}(X,D)$ the degree of the Gross–Siebert three-point punctured virtual class with inputs p_Z, p_W and negative output $-p_T$, with its logarithmic output point constraint. Appendix A compares this constraint with ordinary point insertion when p_T is divisorial.

Proposition 15.1 (Individual coefficient comparison)**.** *For every actual output source T of charge $\nu_Z+\nu_W$, one has*

$$m^T_{ZW}=\sum_{\beta^{\mathrm{tail}}_T} N^{p_T,\beta^{\mathrm{tail}}_T}_{p_Z,p_W}(X,D), \qquad \nu_T=\nu_Z+\nu_W, \tag{14}$$

where every admissible tail class on the chosen model is retained. Its numerical class is $\beta_Z+\beta_W-\beta_T$ on this model. If $\sigma: X\to X^{(0)}$ is a preceding comparison modification, its pushforward is $\sigma_*(\beta_Z+\beta_W-\beta_T)$; all admissible lifts are retained when identifying the coefficient on $X^{(0)}$. The same argument includes zero coefficients.

Proof. Apply Lemma 14.3 to the logarithmic side. For each distinguished cut, positive tail flag degree is excluded by the good ordinary receiver. In the remaining splittings, the receiver and the retained incidence factor through the whole central mapping space. Lemma 14.2 annihilates all positive-dimensional central tropical types on their actual gluing carriers. The zero-dimensional central type has positive contacts; Lemma 9.5 identifies its whole incidence with the finite smooth points q. The inverse construction recovers at each q its actual MV source T, complete contact p_T and class β_T.

If the numerical class $\beta_Z+\beta_W-\beta_T$ has no effective homological representative, the complete boundary-degree equations make every punctured tail space with output p_T empty. For such a candidate, apply Lemma 14.3 and Lemma 14.2 before invoking any divisorial tail formula: a surviving one-cut term over $q\in Q_T$ would have a smooth central map reconstructed as T and an effective tail of numerical class $\beta_Z+\beta_W-\beta_T$, a contradiction. Here Lemma 9.5 is applied only to central data of actual effective splittings, which Lemma 8.2 makes divisorial; no divisoriality assertion is needed for the empty candidate itself. Thus the logarithmic specialization has coefficient zero at q, and (13) together with Proposition 11.6 gives $m^T_{ZW}=0$. For every remaining candidate, an effective representative of $\beta_Z+\beta_W-\beta_T$ gives an effective splitting in Lemma 8.2; hence the chosen target makes $p_T=k\rho$ divisorial. This does not require the corresponding tail mapping space to be nonempty. Lemma 14.4 identifies the full contribution over that same q with the one-negative refined tail point class, including every tail-type component, homological decoration, monomial thickness and torsion-unit degree.

For each full tail point fiber let ξ be its refined zero-cycle and ρ_Q its proper map to Q. Its flag image is the fixed point q, so $(\rho_Q)_*\xi=\deg(\xi)[q]$. By Proposition A.2, this degree is the Gross–Siebert point-constrained scalar. Thus the scalar appears after the proper pushforward of the entire tail point class.

Proposition 11.6 gives coefficient m^T_{ZW} for the BD side at the same actual point. Equality in the free zero-cycle group $A_0(Q)_{\mathbb{Q}}$ proves the asserted identity, including zero coefficients. The all-lifts identity of [31, Corollary 1.4 and §10] gives the stated sum on a preceding model. It is applied to each downstairs homological class and then summed over its finite collection of decorations. The auxiliary contacts cancel by color and charge, so boundary localization gives precisely the stated numerical tail class. □

The MV basis is homogeneous. The theta functions are homogeneous in the root grading: every exchange column has weight zero, so every broken-line correction to an initial exponent has the same weight. The pairing (6) preserves this grading. Consequently the two multiplication coefficients both vanish when $\nu_T\neq\nu_Z+\nu_W$. Charge zero gives the unit. This

is compatible with the geometric observation that a reconstructed unipotent map regular on all of $\mathbb{P}^1$ is constant.

16. Weighted scattering and the affine mirror

We use the full lattice, including the frozen coordinates. In nonsimply-laced types, the primitive normal of a binomial center and the covector supplied by the alternating form differ by a root-length factor. We incorporate this factor in the scattering comparison of [29, Section 21].

Let M be a free integral lattice of finite rank, put $M^* = \mathrm{Hom}(M, \mathbb{Z})$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, and choose a basis (n_k) of M^*. Let J be the mutable set. Let $C_k \in M$, $k \in J$, be independent vectors spanning a saturated sublattice, with $n_k(C_k) = 0$. Write $C_{jk} = n_j(C_k)$. If $\bar{d}_k > 0$ is a conventional row symmetrizer, then

$$\bar{d}_j C_{jk} = -\bar{d}_k C_{kj} \qquad (j, k \in J).$$

Choose positive integral row symmetrizers and put $d_k = \mathrm{lcm}_{j \in J}(\bar{d}_j)/\bar{d}_k$ for $k \in J$. Consequently

$$\frac{n_j(C_k)}{d_j} = -\frac{n_k(C_j)}{d_k}. \tag{15}$$

These d_k are the GHKK multipliers in the convention used below; they are reciprocal, up to common scale, to the row symmetrizers.

More precisely, in the notation of [10, Appendix A], our M is M°, the seed vector is $e_k \in \mathsf{N}$, and

$$n_k = d_k e_k \in \mathsf{N}^\circ = \mathrm{Hom}(M, \mathbb{Z}), \qquad C_k = p_1^*(e_k). \tag{16}$$

To complete the fixed data, choose $d_f = 1$ for each frozen coordinate index f, put $e_f = n_f$, and take

$$\mathsf{N} = \bigoplus_{i=1}^{\operatorname{rank} M} \mathbb{Z} e_i, \qquad \mathsf{N}^\circ = \bigoplus_{i=1}^{\operatorname{rank} M} \mathbb{Z} n_i, \qquad \mathsf{N}_{\mathrm{uf}} = \bigoplus_{k \in J} \mathbb{Z} e_k.$$

Define

$$\{e_k, e_j\} = \frac{C_{jk}}{d_j} \quad (k \in J),$$

extend by skew-symmetry, and set the frozen–frozen pairings equal to zero. Equation (15) makes this definition consistent when both indices are mutable. The relations

$$\{\mathsf{N}_{\mathrm{uf}}, \mathsf{N}^\circ\} \subset \mathbb{Z}, \qquad \{\mathsf{N}, \mathsf{N}_{\mathrm{uf}} \cap \mathsf{N}^\circ\} \subset \mathbb{Z}$$

follow from the integral columns C_k (for a frozen first index, $\{e_f, d_k e_k\} = -d_k C_{fk}$). These are the integrality conditions in [10, Appendix A]. Moreover $p_1^*(e_k) = C_k$, so the injectivity assumption follows from the independence of the full columns. This is an explicit realization of the scattering data; the lattices retain their specified finite index.

The initial wall normal primitive in N° is therefore n_k. These equations, or equivalently the primitive initial automorphisms below, fix the integral convention.

Let $T_M = \operatorname{Spec} \mathbb{C}[M^*]$. Its characters are denoted z^n for $n \in M^*$; the mirror monomials z^a, $a \in M$, belong instead to the group algebra of M. Form the toric variety whose fan has the zero cone and the rays $\mathbb{R}_{\geq 0} C_k$. In the open divisor with ray C_k, blow up

$$Z_k = \{1 + z^{n_k} = 0\},$$

and remove the strict transform of the toric boundary. Denote this open toric blowup by Y°.

We use the following tropical terminology. A spine is the marked metric tree obtained from a skeletal analytic curve by retaining the paths between its prescribed ends, with its piecewise integral affine map to the skeleton. A twig is an attached part omitted from this spine; its monomial is its outgoing integral direction at the attachment. A toric tail extends

a prescribed end in the toric model. A pointed monoid P here is an additive submonoid of the direction lattice whose real cone contains no line. For a spine S and effective curve class γ, $N(S,\gamma)$ denotes the degree of the corresponding finite analytic incidence fiber, with the toric-tail classes subtracted from its complete curve class, as in [29, Sections 9–11].

Lemma 16.1. *Use a compactification and toric model satisfying* [29, Assumption 2.4]. *Suppose every nonempty twig under consideration has monomial in $P \setminus \{0\}$, where P is strictly convex. Let S be a transverse spine without bending vertices. Assume also that its extension by the prescribed affine toric tails is fan-generic: its image avoids the codimension-two skeleton of the toric fan, and the inverse image of the codimension-one skeleton is a finite set of two-valent points. Auxiliary contracted evaluation legs are attached away from the codimension-one skeleton. These are the conditions in* [29, Lemma 7.2(1–2)]. *Then exactly one curve class γ_S has nonzero count, and*

$$N(S,\gamma_S) = 1.$$

The class γ_S need not be zero. The zero-class conclusion requires the additional fan-avoidance condition in [29, Lemma 12.3].

Proof. Append the prescribed toric tails using [29, Construction 9.3]. At a vertex of the original spine the sum of its outgoing vectors is zero. Balancing of the complete tropical curve therefore makes the sum of its attached twig monomials zero. Since every nonempty twig has a nonzero monomial in the pointed monoid P, there are no such twigs. The toric-tail condition, equivalently [29, Lemma 11.3(4)], excludes twigs on the appended portions. Thus the complete tropicalization is the extended spine.

As in [29, Lemma 12.3 and Proposition 20.33], every contributing complete curve has image in the common toric isomorphism locus W of [29, Lemma 2.6]. Conversely, the toric skeletal curve avoids the complement of W in the toric model: that complement is a closed subset of codimension at least two, and [29, Lemma 8.21(1)] applies to the toric target. Hence the finite moduli fibers on the two sides are identified through W, including their scheme structures.

First suppose the metric and evaluation data are rational. Their Berkovich point has a completed residue field with discrete value group; finite extensions used to split the finite etale fiber still have discrete value group. The fan conditions allow the application of [29, Lemma 9.4 and Proposition 7.11]. They give one map of multiplicity one, in the determined class after subtracting the prescribed tail classes. Specifically, γ_S is the class assigned to the original spine map h by [29, Definition 7.9] on the fixed toric model. Write $\delta_{\widehat{h}}$ for the class of its extension by the prescribed tails, indexed by a finite set F, and δ_j for the class of the jth tail. Additivity gives

$$\delta_{\widehat{h}} - \sum_{j\in F} \delta_j = \gamma_S.$$

This is the unique class in the rational calculation; in particular it is effective. Fix the directions and the open fan cones containing the endpoints of each affine domain. By [29, Remark 7.10], the same class γ_S is constant throughout a sufficiently small neighborhood of these data.

For real metric or evaluation data, perform this computation after an arbitrarily small rational perturbation within the same finite fan and wall combinatorial chamber. Such a perturbation exists: the affine constraints have rational coefficients, and the required transversality and fixed open-cell conditions are open. For each fixed class, choose a finite wall bound large enough for that class before perturbing. The count in that class is unchanged by [29, Theorem 11.9], while the class of each appended affine tail is unchanged by [29, Remark 7.10]. Every such perturbation therefore computes the same class γ_S, independently of which class was fixed for the deformation argument. Consequently the count for γ_S is one and the count for each other class is zero at the original real data. Equivalently, one can use the constant degree of the finite etale fibers in [29, Lemma 9.11]. This use of

deformation is necessary because the curve-class computation in [29, Section 7] is first made over discretely valued fields.

□

Lemma 16.2. *Let Σ be a finite rational fan in $M_{\mathbb{R}}$, and consider a finite collection of transverse piecewise affine paths with nonzero rational directions, rational wall hyperplanes, and a common endpoint x. At each wall $H = \ker n$, assume the directions v, w on the two sides satisfy*

$$n(v) = n(w) \neq 0.$$

For each fixed wall itinerary there is a finite union of proper rational linear subspaces of $M_{\mathbb{R}}$ outside which every affine line supporting a segment avoids the codimension-two skeleton of Σ and is not contained in the span of a codimension-one cone. The same assertion holds simultaneously for finitely many itineraries.

In particular, an endpoint outside every proper rational hyperplane satisfies these assertions for every finite collection. Such endpoints exist in the interior of the positive cluster chamber. We use them as real skeletal points.

Proof. For a nonzero rational direction v, write $L_v = M_{\mathbb{R}}/\mathbb{R}v$ for the space of affine lines parallel to v. If $H = \ker n$ is transverse to v, the restriction $H \to L_v$ is a rational linear isomorphism. At a crossing where both v and w are transverse to H, the common crossing point therefore identifies L_v and L_w by a rational linear isomorphism. Composing these identifications backwards along the path expresses the offset of every supporting line as a rational linear isomorphism applied to the offset of the final line through x.

For a cone σ of codimension at least two, the offsets of lines parallel to v that meet σ lie in the proper rational subspace

$$\operatorname{im}(\operatorname{span}_{\mathbb{R}} \sigma \longrightarrow L_v),$$

whose dimension is at most $\dim M_{\mathbb{R}} - 2$. If a codimension-one span H_σ contains v, the offsets of lines contained in H_σ similarly form the proper subspace $H_\sigma/\mathbb{R}v$ of L_v. Their inverse images in the endpoint space, under the preceding rational isomorphisms and $M_{\mathbb{R}} \to L_{v_{\mathrm{final}}}$, are proper rational linear subspaces. There are finitely many cones, segments and chosen itineraries. This proves the first assertion. The complement of all rational hyperplanes meets every nonempty real open set, which proves the last assertion.

A supporting line satisfying these conditions meets each fan facet transversely in at most one point. Cut points and contracted evaluation legs can therefore be chosen away from the finite collection of such intersections. Extending a resulting straight piece by affine tails keeps it on the same supporting line, so it satisfies precisely the fan conditions of Lemma 16.1. □

Proposition 16.3 (Weighted affine mirror comparison)**.** *Suppose Y is smooth affine and $Y^\circ \hookrightarrow Y$ is a big open immersion preserving the specified torus and its logarithmic volume form. Then the Keel–Yu scattering diagram of Y, after sending every curve-class monomial to 1, is equivalent to the GHKK diagram with initial walls*

$$\big(n_k^\perp, 1 + z^{C_k}\big), \qquad k \in J,$$

and positive wall crossing

$$z^a \longmapsto z^a (1 + z^{C_k})^{\langle n_k, a \rangle}. \tag{17}$$

The basis identification indexed by the full lattice $M = \mathrm{Sk}(Y, \mathbb{Z})$ identifies the specialized Keel–Yu mirror algebra with the formal broken-line theta algebra for the GHKK diagram. Regular functions will be selected in Lemma 16.5. In particular, every GHKK multiplication coefficient is the finite sum of the corresponding analytic coefficients over curve classes.

Proof. We extend the skew-symmetric comparison of [29, Section 21] using the specified primitive normals.

The empty mutable set. If $J = \varnothing$, the partial model is the torus T_M itself. The big-open hypothesis and normality give $\mathcal{O}(Y) = \mathcal{O}(T_M) = \mathbb{C}[M^*]$; since Y is affine, its specified open torus is all of Y. The comparison is then the toric calculation of [29, Lemma 9.4], with empty scattering diagram and Laurent monomial basis. We may therefore assume $J \neq \varnothing$ below.

A nondegenerate alternating form. Put $E = \operatorname{span}_{\mathbb{Q}}\{C_k : k \in J\}$. For $v \in M_{\mathbb{Q}}$, prescribe $\omega_0(C_k, v) = n_k(v)/d_k$. Equation (15) makes this prescription skew-symmetric on $E \times E$. Choose $M_{\mathbb{Q}} = E \oplus F$ and extend it to an alternating form by putting $\omega_0|_{F\times F} = 0$. If $R = \operatorname{rad}(\omega_0)$, then $R \cap E = 0$: an element $\sum a_k C_k$ in this intersection would give $\sum (a_k/d_k) n_k = 0$, and the coordinate normals are independent.

Choose a rational vector space K of dimension $\dim R$ and a map $j : K \to M^*_{\mathbb{Q}}$ whose image annihilates E and pairs perfectly with R. Such a map exists because $R \cap E = 0$. On $M_{\mathbb{Q}} \oplus K$ put

$$\omega((v,k),(w,l)) = \omega_0(v,w) + j(l)(v) - j(k)(w).$$

This form is nondegenerate. Indeed, a radical vector (v,k) pairs with $R \oplus 0$ to force $k = 0$, then with $M_{\mathbb{Q}} \oplus 0$ to force $v \in R$, and finally with $0 \oplus K$ to force $v = 0$. It satisfies

$$\omega(C_k, (v,l)) = \frac{n_k(v)}{d_k}. \tag{18}$$

Choose a lattice $K_{\mathbb{Z}}$ in K and set $\widetilde{M} = M \oplus K_{\mathbb{Z}}$.

For a vector a, write $a^{\perp_\omega} = \{v : \omega(a,v) = 0\}$; for several vectors this denotes the intersection of their orthogonal subspaces. A rational hyperplane can be written as $a^{\perp_\omega}$ for an integral vector $a \in \widetilde{M}$, after multiplying the rational inverse image by a positive denominator. Thus the genericity argument in [29, Lemma 21.8] applies. The primitive center characters and wall normals remain those of the fixed integral lattice.

Temporarily replace Y by $\widetilde{Y} = Y \times T_{K_{\mathbb{Z}}}$. The primitive ray and center data remain $(C_k, 0)$ and $(n_k, 0)$. The center remains the reduced binomial $1 + z^{n_k} = 0$. The following twig and wall arguments take place in $\widetilde{M}$ and $\widetilde{Y}$; the original C_k, n_k denote their inclusions with zero auxiliary coordinates.

The twig argument. The monoid $P = \sum_{k\in J} \mathbb{N} C_k$ is free, saturated and strictly convex, since the C_k extend to an integral basis. Its degree is $\deg(\sum a_k C_k) = \sum a_k$.

The common-model argument of [29, Lemma 21.16] applies unchanged to the present centers. After toric refinements, the birational comparison with the partial toric blowup is an isomorphism away from a codimension-two subset of the interior and essential boundary. Skeletal curves avoid that subset by [29, Lemma 8.21(1)]. An unmarked toric contact therefore lies over some Z_k. Since $z^{n_k} = -1$ there, its tropical end lies in $n_k^\perp$, and its outgoing vector is a positive integral multiple of the primitive ray C_k. By (18),

$$n_k^\perp = C_k^{\perp_\omega}.$$

These are exactly the end conditions in [29, Definition 21.5].

Balancing gives every edge vector as a nonzero element of P. Induction from the ends places that edge in its ω-orthogonal hyperplane: at a vertex the incident vectors add, and $\omega(v,v) = 0$. This is [29, Lemma 21.6]. The genericity propagation in [29, Lemmas 21.7–21.8] uses nondegeneracy to identify ranks with codimensions of rational orthogonals; the form constructed above has exactly these properties.

For two walls with monomials v_1, v_2 and $\omega(v_1, v_2) \neq 0$, a sum wall is supported on

$$\mathfrak{d}_1 \cap \mathfrak{d}_2 - \mathbb{R}_{\geq 0}(v_1 + v_2).$$

If it contained $v_1 + v_2$, a positive multiple of this vector would lie in $(v_1, v_2)^{\perp_\omega}$, contradicting the nonzero pairing. For parallel monomials, an incoming sum requires both summands incoming. Thus the induction of [29, Construction 21.13 and Lemmas 21.14–21.15] produces no new incoming wall. Every bend belongs to P, so [29, Assumption 20.27] holds. The finiteness statements [29, Lemmas 20.29–20.31] permit forgetting curve classes at every finite

P-adic order. The generic-wall argument of [29, Lemmas 21.17–21.19] then shows that a wall function contains only positive multiples of its single monomial direction. The resulting diagram is Kontsevich–Soibelman consistent by [29, Proposition 20.36], which follows there from the theta-function consistency of [29, Proposition 20.35].

The primitive incoming wall count. At a generic point of an incoming wall of direction C_k, every contributing twig has total monomial a positive multiple of C_k. Since the C_j are part of a basis, every end of such a twig has type k. The reduction in [29, Claim 21.26 and Lemma 21.25] therefore takes place in the single blowup at Z_k.

Choose $v \in M$ with $n_k(v) = 1$. Because C_k is primitive in $\ker n_k$, the pair C_k, v extends to an integral basis with all remaining vectors in $\ker n_k$, and adjoining a basis of $K_{\mathbb{Z}}$ gives a basis of $\widetilde{M}$. The local target is consequently

$$\mathrm{Bl}_{(0,-1)}(\mathbb{A}^1 \times \mathbb{G}_m) \times \mathbb{G}_m^{\operatorname{rank}\widetilde{M}-2}.$$

For a cylinder used to compute the wall function, the input exponent can be taken to be v, by [29, Lemma 20.12]; its output differs by a multiple of C_k. The other torus projections are constant. The local calculation is as follows.

The primitive local count and higher covers. The rank-two calculation is uniform in the root system. In the basis beginning with C_k, v chosen above, let x be the torus character dual to C_k, and put $y = z^{n_k}$, the character dual to v. Thus x vanishes on the toric divisor and y is a unit there. The center is $(x, y) = (0, -1)$. For the cylinder with input v and output $v + rC_k$, $r \geq 0$, its complete three-pointed skeletal curve has one simple zero and one simple pole of y. Thus y has degree one and is a coordinate on the source. The other torus factors are constant. The boundary condition forces every remaining zero of x to lie at $y = -1$, and forces its pole to be at $y = \infty$. Consequently

$$x = c(1 + y)^r \qquad (c \neq 0).$$

For $r > 0$, on the blowup chart with coordinates $t = 1 + y$ and $\rho = x/(1 + y)$, this curve is $\rho = ct^{r-1}$. The strict transform of the old divisor $x = 0$ is $\rho = 0$. It has been removed, so $r \geq 2$ is impossible, whereas $r = 1$ is allowed and meets the exceptional divisor off the removed boundary. The case $r = 0$ is the straight toric curve. Evaluation at a general framed point determines c uniquely in each allowed case. KY's finite etale fiber property gives length one; the degree-one y-map also rules out nontrivial multiple covers. Thus the wall function is exactly $1 + z^{C_k}$.

The scattering Lie algebra. For $a = (a_k) \in \mathbb{N}^J \setminus \{0\}$ write

$$m_a = \sum a_k C_k, \qquad u_a = \sum \frac{a_k}{d_k} n_k = \sum a_k e_k.$$

For $u \in M^* \otimes_{\mathbb{Z}} \mathbb{Q}$, define the logarithmic derivation on the group algebra by $\partial_u(z^m) = \langle u, m\rangle z^m$ for $m \in M$; it acts trivially on the auxiliary characters. Then $\omega(m_a, -) = u_a$. The homogeneous log derivation space is $\mathfrak{g}_a = \mathbb{Q} z^{m_a} \partial_{u_a}$. Its bracket is

$$[z^{m_a}\partial_{u_a}, z^{m_b}\partial_{u_b}] = \omega(m_a, m_b) z^{m_{a+b}} \partial_{u_{a+b}}.$$

These are exactly the homogeneous spaces and bracket of [10, Section 1], since

$$m_a = p_1^* \left(\sum a_k e_k \right).$$

For any noninitial wall, its primitive geometric normal is the primitive positive vector on $\mathbb{R}_{\geq 0} u_a$ in $M^* = \mathsf{N}^\circ$. This is the vector n_0' of [10, Definition 1.2]; all nonprimitive root factors are therefore retained on both sides. For an explicit integral normalization at every wall, put

$$D = \operatorname{lcm}_{k \in J}(d_k), \qquad h_a = \gcd_{k \in J} \left(\frac{D a_k}{d_k} \right) \quad (a \in \mathbb{N}^J \setminus \{0\}),$$

where zero entries do not affect the gcd. The primitive positive normal is

$$n'_a = \sum_{k\in J} \frac{Da_k}{h_a d_k}\, n_k = \frac{D}{h_a} u_a \in M^* = \mathsf{N}^\circ.$$

Its integral coordinates have gcd one, so it is precisely the generator of $\mathbb{R}_{\geq 0}u_a \cap \mathsf{N}^\circ$ used in [10, Definition 1.2]. This remains the same primitive vector when a is replaced by a positive integral multiple. Taking a to be the k-th coordinate vector gives $n'_a = n_k$, so the initial automorphism contains its root-length multiplier exactly once. For a general wall with attached function f, $\log(f)\partial_{n'_a}$ has every homogeneous part in the previously identified GHKK graded Lie algebra.

Every geometric wall belongs to this same completed group. The incoming walls and their automorphisms agree, all other walls are outgoing, and the diagram is consistent. Hence [10, Theorem 1.12] identifies the diagrams up to equivalence at every finite degree.

Wall and curve-class bounds. The geometric walls and the toric fan serve different purposes. Choose a real point $x \in \widetilde{M}_{\mathbb{R}} = \widetilde{M} \otimes_{\mathbb{Z}} \mathbb{R}$ in the inverse image of the positive cluster chamber under $\widetilde{M}_{\mathbb{R}} \to M_{\mathbb{R}}$, outside every proper rational hyperplane of this enlarged space. This is allowed in [29, Definition 20.20]: by [29, Lemma 2.2], the whole space $\widetilde{M}_{\mathbb{R}}$ is the Berkovich skeleton of $\widetilde{Y}$, and [29, Constructions 9.2–9.3 and Lemma 9.11] take finite etale fibers over the completed residue fields of its points.

Fix an input $m \in M \subset \widetilde{M}$ and an order $\ell \in \mathbb{N}$. Every retained intermediate exponent belongs to the finite set

$$\{m + Ca : a \in \mathbb{N}^J,\ \sum_{k\in J} a_k \leq \ell\}.$$

The case $m = 0$ is the unit and is treated separately. Starting with $m \neq 0$, a transverse bend cannot produce a zero exponent: its incoming and outgoing exponents have the same nonzero pairing with the crossing normal. Thus all directions used in the supporting-line lemma are nonzero. Every nonzero bend consumes positive P-degree, so a retained path has at most ℓ bends. The finite-degree twig-wall construction [29, Construction 21.13 and Lemma 21.15] gives finitely many possible wall cones and directions through this degree. Hence there are only finitely many candidate itineraries. Each itinerary, together with the endpoint, determines its supporting lines by the rational crossing maps in Lemma 16.2.

For the geometric counts, [29, Lemmas 20.29–20.31] give finitely many cylinder and local-theta classes at these orders, and the proof of [29, Proposition 20.21] gives finitely many contributing spines. Include the classes in the products of the retained local factors on the finitely many candidate itineraries as well. This is still a finite set: cylinder classes are finite by Lemma 20.29, and a straight-piece class depends only on its direction and the open fan cones of its endpoints, by [29, Definition 7.9 and Remark 7.10]. These choices are finite. Include every effective numerical decomposition of these total classes. There are only finitely many: intersection with an ample divisor bounds the degree of every summand, and effective numerical classes of bounded ample degree form a finite set in the numerical lattice. On a common compactification, write π_{tor} for its map to the toric model, whose fan lies in $\widetilde{M}_{\mathbb{R}}$. Write D_{tor} for the reduced toric boundary, and D_{cmp} for its own reduced boundary. For a contributing curve of class γ, its total twig degree is $\gamma \cdot (\pi_{\mathrm{tor}}^* D_{\mathrm{tor}} - D_{\mathrm{cmp}})$ by [29, Proposition 4.24]. Choose an integer $A \geq 0$ bounding this degree for every retained class, as in [29, Definition 4.28]. Here A is a degree bound, local to this scattering calculation. Write Wall_A for the finite polyhedral wall set constructed from twigs of degree at most A in [29, Construction 4.16]. Appending the toric tails does not require an unbounded further enlargement, by [29, Lemma 9.12].

Use Wall_A to cut paths into short crossing pieces and pieces avoiding these walls. Separately, use the finite intersections with the codimension-one skeleton of this toric fan to choose all cut points and evaluation legs away from that skeleton. The supporting-line lemma already excludes codimension-two fan intersections and lines contained in fan facets. Thus

every straight piece to which the count-one lemma is applied satisfies the full hypotheses of KY Lemma 7.2, including its affine tails. Any additional subdivision caused by enlarging A only splits the already retained total classes into effective summands, which were included when choosing A; the chosen bound remains valid.

At a crossing, the bend is tangent to its wall, so the equality $n(v) = n(w) \neq 0$ needed above follows from [29, Lemma 20.3] (and from the same wall-crossing formula on the GHKK side). For a retained output with a nonzero P-increment, a nonempty twig is necessary, and its image meets Wall_A. Hence a wall-free piece has only its unchanged exponent through order ℓ. That unchanged-exponent spine is straight, and its coefficient is one after summing classes by Lemma 16.1. This statement is taken modulo terms of P-degree greater than ℓ.

The gluing formula [29, Theorem 12.4] applies with this single A to every retained class decomposition. Crossing pieces give the previously identified wall automorphisms, and straight pieces give identity transmission. Conversely, each retained broken-line choice uses one of the included itineraries and these same local factors, so the gluing formula reconstructs exactly its geometric coefficient. This proves equality of the local expansions through order ℓ. If rational data are used to calculate any straight factor, the small perturbations remain in this same finite wall and fan chamber; [29, Theorem 11.9 and Remark 7.10] preserve its classwise count and tail classes. The coefficient is therefore the coefficient at the original real point x.

The endpoint x can be fixed for all ℓ, and truncations merely discard positive P-degree. The identities are therefore compatible as ℓ grows. The structure-coefficient comparison is then the gluing argument in the proof of [29, Theorem 21.28], with the weighted diagram already identified. For a fixed product the sum over output labels and curve classes is finite by [29, Theorem 1.6].

Removing the auxiliary torus. All counts used above have a definition by complete rational curves. For a disk count this is [29, Definition 1.5], which appends the opposite output contact and a toric tail; for finite-ended spines it is [29, Construction 9.3]. Use a product compactification with $(\mathbb{P}^1)^{\operatorname{rank} K}$ in the new factors. For labels $(p, 0)$, every original and every appended contact has zero new coordinate. The complete curve has no other boundary preimages. Each new torus character therefore pulls back to a regular invertible function on a complete connected rational curve, so is constant, including on nodal domains. For a family $\pi_C : C \to S$ of connected genus-zero curves, $(\pi_C)_*\mathcal{O}_C = \mathcal{O}_S$, so the auxiliary torus morphism factors through S, also over nonreduced bases. The frame evaluation fixes that constant. This proves an identification of the finite moduli fibers with those for Y, with degree-one torus factor and zero new curve class.

On the scattering side, the initial walls are pulled back from $M_{\mathbb{R}}$ and all functions and normals have zero new coordinate. Uniqueness of the consistent completion gives the pulled-back diagram. Broken lines of label $(p, 0)$ have constant extra coordinates and zero extra monomial exponents, hence give the original θ_p. The comparison therefore descends to the original full lattice M. □

Corollary 16.4. *In the situation of Proposition 16.3, let (W, E) be a smooth projective SNC log Calabi–Yau compactification of the same affine Y. Suppose a nef divisor with strictly positive coefficient on each component of E is supported on E. Then, for full lattice labels p, q, r, the GHKK structure coefficient with inputs p, q and output r equals*

$$\sum_{\beta} N_{p,q}^{r,\beta}(W, E),$$

where $N_{p,q}^{r,\beta}$ is the point-constrained punctured invariant defined in Section 15. The sum is finite. If the complete contact-degree constraints admit only one numerical curve class β_0, all other numerical terms are empty. The surviving coefficient is still the finite aggregation of every actual homological class with numerical class β_0. For our Richardson target, the boundary-generated Picard group and the displayed tail-class calculation establish precisely this condition.

Proof. [30, Corollary 5.10] compares the punctured logarithmic coefficients on (W, E) with the analytic disk coefficients of Y, including the toric tail condition. Apply Proposition 16.3 and the finiteness in [29, Theorem 1.6]. The labels are the same valuations of the fixed torus; Lemma 8.1 identifies its volume form and essential skeleton with those of the chosen Richardson target.

□

Lemma 16.5. *Let $P = \sum_{k\in J} \mathbb{N}C_k \subset M$. Write $\mathbb{C}[[P]]$ for the completion of $\mathbb{C}[P]$ at the ideal generated by z^{C_k}, $k \in J$, and put*

$$\widehat{L} = \mathbb{C}[[P]] \otimes_{\mathbb{C}[P]} \mathbb{C}[M].$$

Suppose the weighted mirror comparison identifies the specialization of the Keel–Yu algebra with the formal broken-line theta algebra in $\widehat{L}$, with basis indexed by M. Suppose also that the regular theta functions

$$\{\theta_g : g \in R\}, \qquad R = \{g \in M : \theta_g \in \mathbb{C}[N]\},$$

form a basis of $\mathbb{C}[N]$. Then, for $p, q \in R$, the specialized Keel–Yu coefficient with output r equals the multiplication coefficient in this regular basis when $r \in R$, and is zero when $r \notin R$.

Proof. The successor minor makes the columns C_k a basis of a saturated direct summand of M. Hence $\mathbb{C}[M]$ embeds in $\widehat{L}$; after choosing a complementary lattice, the latter is a power-series ring in $|J|$ variables with those variables inverted, tensored with the group algebra of the complement.

The local theta expansion, equivalently [29, Lemma 20.34], is

$$\theta_g = z^g \left(1 + \sum_{0\neq a\in\mathbb{N}^J} c_{g,a} z^{Ca}\right).$$

Every nonzero Ca is positive in the fixed reverse lexicographic order. For a nonzero finite linear combination of these series, the least label with a nonzero coefficient contributes its monomial z^g with coefficient one; no series with a greater label can contribute that monomial. Thus the formal theta functions indexed by all of M are linearly independent for finite linear combinations.

For $p, q \in R$, write their finite expansion in the regular basis as

$$\theta_p\theta_q = \sum_{r\in R} a_r\theta_r.$$

The initial-torus restriction embeds $\mathbb{C}[N]$ into $\mathbb{C}[M]$, so this is an equality in the same ring $\widehat{L}$. By the weighted mirror comparison the Keel–Yu multiplication gives another equality there,

$$\theta_p\theta_q = \sum_{r\in M} \kappa_r\theta_r.$$

Here κ_r is the sum of the analytic three-point coefficients with output r over their effective curve classes β. The sum over pairs (r, β) is finite by [29, Theorem 1.6]. Subtracting the two finite expansions and applying the linear independence just proved gives $\kappa_r = a_r$ for $r \in R$ and $\kappa_r = 0$ otherwise. □

17. Equality of the bases

Theorem 17.1. *The normalized MV and regular theta functions agree in the original coordinate ring $\mathbb{C}[N]$ for every connected complex semisimple group G.*

Proof. The pairing (6) is a weight- and unit-preserving bijection. Proposition 15.1, Corollary 16.4 and Lemma 16.5 identify its multiplication coefficients, including the unit and unequal-weight cases. Its linear extension is therefore multiplicative. Proposition 6.2 shows that it fixes the initial variables, which generate $\mathbb{C}(N)$, so Lemma 2.1 makes it the identity

on $\mathbb{C}[N]$. Central covers do not change the pinned unipotent subgroup, and products give tensor products of all the constructions. □

Appendix A. Divisorial point constraints

Let (X, D) be a smooth simple-normal-crossings pair over an algebraically closed field of characteristic zero. Write $D = \sum_i D_i$ and fix genus-zero discrete data β with precisely one negative divisor-marking pair: the output has contact $-k\rho$, where $k \geq 1$ and ρ is the primitive ray of D_1, and all other contacts are nonnegative. Set

$$D_1^\circ = D_1 \setminus \bigcup_{i\neq 1} D_i \quad \text{and choose } z \in D_1^\circ.$$

All finite-type restrictions below are open neighborhoods of the relevant actual-map images.

We distinguish full mapping spaces from the open spaces used in the local calculation. Let $\widehat{\mathcal{M}}$ be the full actual punctured mapping stack, $\widehat{\mathfrak{A}}$ the corresponding Artin-fan mapping stack, and $\widehat{\mathfrak{B}}$ its evaluation enhancement at the output. The underlying output evaluation on $\widehat{\mathcal{M}}$ and $\widehat{\mathfrak{B}}$ factors through D_1. Let $\mathcal{M}$ and $\mathfrak{B}$ be their restrictions to evaluation in D_1°, and let $\mathfrak{A}$ be the corresponding restriction of the Artin-fan mapping stack to the open ray stratum. The morphisms

$$\widehat{\epsilon} : \widehat{\mathcal{M}} \longrightarrow \widehat{\mathfrak{B}}, \qquad \epsilon : \mathcal{M} \longrightarrow \mathfrak{B}$$

are strict.

Choose an evaluation-enhanced relative perfect obstruction theory compatible with the unenhanced theory and the smooth morphism $\widehat{\mathfrak{B}} \to \widehat{\mathfrak{A}}$, as in [32, Section 4.2]. The isomorphism class of the enhanced complex and the resulting virtual classes do not depend on the compatible arrow chosen as in its Remark 4.4. All obstruction theories below are restrictions or base changes of this one chosen compatible construction. If $\pi : C \to \mathcal{M}$ and $f : C \to X$ are the universal curve and universal map, and $p = x_{\mathrm{out}}$ is the output section, its relative complex is

$$(R\pi_* f^* T_X^{\log}(-p))^\vee.$$

On the output open stratum, the logarithmic evaluation stack has underlying stack

$$\underline{\mathcal{P}(X, k\rho)} = D_1^\circ \times B\mathbb{G}_m.$$

Here and below this notation denotes its restriction to D_1°. Let

$$Z_k := \mathcal{P}(X, k\rho)|_{\{z\}\times B\mathbb{G}_m}$$

with the induced log structure. Thus $Z_k \hookrightarrow \mathcal{P}(X, k\rho)$ is strict. In the universal-torsor convention of [39, Proposition 3.8], let $\mathcal{U}$ be the universal line bundle on $B\mathbb{G}_m$. The log generator on Z_k has torsor and zero section $(\mathcal{U}^{\otimes k}, 0)$. Let $W = B\mathbb{G}_m^\dagger$ have generator $(\mathcal{U}, 0)$. The point-constraint morphism

$$W \longrightarrow Z_k \longrightarrow \mathcal{P}(X, k\rho)$$

has underlying first map the identity of $B\mathbb{G}_m$ and characteristic map $\mathbb{N} \to \mathbb{N}$, $1 \mapsto k$. This specifies the actual logarithmic map, after the allowed choice of units and of the boundary-line trivialization at z.

Define

$$\begin{aligned}
\mathfrak{B}_0 &= \mathfrak{B} \times_{D_1^\circ} \{z\} = \widehat{\mathfrak{B}} \times_{D_1} \{z\},\\
\mathcal{M}_0 &= \mathcal{M} \times_{D_1^\circ} \{z\} = \widehat{\mathcal{M}} \times_{D_1} \{z\},\\
\mathfrak{B}_{\log} &= \mathfrak{B} \times^{\mathrm{fs}}_{\mathcal{P}(X,k\rho)} W, \qquad \mathcal{M}_{\log} = \mathcal{M} \times^{\mathrm{fs}}_{\mathcal{P}(X,k\rho)} W.
\end{aligned}$$

The strict inclusion defining Z_k identifies $\mathfrak{B}_0$, with its pulled-back basic log structure, with $\mathfrak{B} \times^{\mathrm{fs}}_{\mathcal{P}(X,k\rho)} Z_k$. The ordinary fiber product with W has the same underlying stack $\mathfrak{B}_0$: the $B\mathbb{G}_m$ factor is retained on both sides, and its identity map cancels in the fiber product.

Lemma A.1. *The natural morphism $\nu : \mathfrak{B}_{\log} \to \mathfrak{B}_0$ is representable and finite. Its pushforward on fundamental cycles is*

$$\nu_*[\mathfrak{B}_{\log}] = [\mathfrak{B}_0]. \tag{19}$$

Moreover, in the Chow group of $\mathfrak{B}_0$,

$$[\mathfrak{B}_0] = z^![\mathfrak{B}] = z^![\widehat{\mathfrak{B}}],$$

where the last equality uses the identification of the two point fibers and locality of refined Gysin pullback around z.

Proof. Because the pair is simple normal crossings and D_1 is a labelled irreducible component, the open ray stratum of its Artin fan is $\mathfrak{D}_1^\circ \simeq B\mathbb{G}_m$. There is no characteristic-monoid monodromy on this stratum: the sharp monoid is $\mathbb{N}$ and its automorphism group is trivial. The morphism $D_1^\circ \to B\mathbb{G}_m$ classifies the normal line bundle, including its twisting. Evaluation enhancement gives

$$\mathfrak{B} = \mathfrak{A} \times_{B\mathbb{G}_m} D_1^\circ \tag{20}$$

as underlying stacks, with the basic log structure pulled back strictly from $\mathfrak{A}$.

Every morphism of algebraic stacks over the ground field to $B\mathbb{G}_m$ is flat: after the smooth surjective atlas $\operatorname{Spec} k \to B\mathbb{G}_m$ its source is a $\mathbb{G}_m$-torsor and its target is $\operatorname{Spec} k$. Consequently $\mathfrak{B} \to D_1^\circ$ is flat. Regular point Gysin pullback therefore gives $z^![\mathfrak{B}] = [\mathfrak{B}_0]$. The full-space statement follows since the point fiber is contained in this open restriction.

The map $W \to Z_k$ is integral, as its characteristic map is $\mathbb{N} \xrightarrow{\times k} \mathbb{N}$. Thus the fine logarithmic fiber product has underlying stack $\mathfrak{B}_0$. Saturation is finite and representable, including for logarithmic algebraic stacks; see [39, Section 2.1.4 and Remark 2.1]. This proves the corresponding assertions for ν. Saturation is applied to the base log structure.

Since every other contact is nonnegative, [39, Theorem 3.16] says that $\mathfrak{B} \to \mathcal{P}(X, k\rho)$ is log smooth. Its strict base change $\mathfrak{B}_0 \to Z_k$ is therefore log smooth. Choose strict smooth atlases and frames for the log lines. The logarithmic chart criterion then gives, locally, a smooth map to a toric hypersurface

$$S = \operatorname{Spec} k[Q]/(\chi^h), \qquad h \in Q \setminus \{0\},$$

where Q is a sharp fs monoid and the map from the target characteristic monoid sends 1 to h. The case that h is a unit gives an empty chart. For the saturated fiber the corresponding chart is

$$S' = \operatorname{Spec} k[P^{\mathrm{sat}}]/(\chi^w), \qquad P = Q \oplus_{\mathbb{N}} \mathbb{N}w, \qquad h = kw.$$

Indeed, before saturation the equation $\chi^w = 0$ also implies $\chi^h = (\chi^w)^k = 0$; after saturation it is still the equation $\chi^w = 0$ that is imposed. These chart constructions commute with the strict smooth base changes just used.

After splitting the finite torsion units, $k[P^{\mathrm{sat}}]$ is a product of normal toric domains. Their real cones have the same dimension and face structure as the cone of Q: the additional real coordinate is uniquely h/k. On each such domain χ^w is a nonzero, nonunit monomial. Its zero locus is a Cartier divisor of pure codimension one. Its irreducible components correspond to exactly the same rays on which h is positive as do the components of $(\chi^h = 0)$. Thus every irreducible component of S' dominates a component of S, and the same assertion holds after the indicated smooth base changes. In particular, ν has no additional irreducible components supported over proper closed subsets of components of $\mathfrak{B}_0$.

It now suffices to compare the multiplicities at each generic component of $\mathfrak{B}_0$. Invert its toric face. The resulting sharp characteristic monoid is $\mathbb{N}q$, and $h = aq$ for some $a > 0$. After passing to the generic residue field and choosing frames, the transverse ring is $K[x]/(x^a)$. The saturated pushout is obtained from

$$\mathbb{N}q \oplus_{\mathbb{N}p} \mathbb{N}w, \qquad p \longmapsto aq, \quad p \longmapsto kw.$$

Put $d = \gcd(a, k)$. The groupification of this pushout is $\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$. Its saturation has sharp quotient $\mathbb{N}u$, with

$$q = (k/d)u, \qquad w = (a/d)u$$

up to torsion units. After an etale extension splitting the units, its underlying finite algebra has d branches. Since $\chi^w = 0$ on the point constraint, each branch has transverse ring

$$K[u]/(u^{a/d}).$$

The old equation $\chi^{aq} = 0$ pulls back to $u^{ka/d} = 0$ and is redundant. The total generic cycle coefficient is consequently

$$d \cdot (a/d) = a,$$

the coefficient of the original component. Without splitting the units, the sum of residue-field degrees gives the same factor d.

This computation takes place on one strict smooth atlas and its base change. Representability of ν means that these torsion units describe branches of a finite algebra. Existing stabilizers and type automorphisms descend with the atlas calculation. Together with the absence of vertical components proved above, this proves the equality of fundamental cycles (19). □

Proposition A.2 (Ordinary divisor-point insertion and the log point constraint)**.** *Let $\nu_{\mathcal{M}} : \mathcal{M}_{\log} \to \mathcal{M}_0$ and $\widehat{j} : \mathcal{M}_0 \to \widehat{\mathcal{M}}$ be the natural maps, and write $\widehat{\mathrm{ev}}_{D_1} : \widehat{\mathcal{M}} \to D_1$ for output evaluation. For the refined class on the full punctured mapping stack and the Gross–Siebert point-constrained punctured class, one has*

$$(\nu_{\mathcal{M}})_*[\mathcal{M}_{\log}]^{\mathrm{vir}} = z^![\widehat{\mathcal{M}}]^{\mathrm{ref}} \quad \text{in } A_*(\mathcal{M}_0)_{\mathbb{Q}}. \tag{21}$$

Consequently

$$(\widehat{j} \circ \nu_{\mathcal{M}})_*[\mathcal{M}_{\log}]^{\mathrm{vir}} = \widehat{\mathrm{ev}}_{D_1}^*[z] \cap [\widehat{\mathcal{M}}]^{\mathrm{ref}}. \tag{22}$$

These identities commute with capping by classes pulled back from the full mapping stack, including ordinary descendants: if x_j is the jth marked section and ω_π is the relative dualizing line bundle of the universal curve, these are powers of $\psi_j = c_1(x_j^*\omega_\pi)$. Equation (21) may be pushed to any common receiver to which the morphism from $\mathcal{M}_0$ is proper and preserves the underlying ordinary stable map.

Proof. The total puncturing rank is one. The universal puncturing substack is therefore a Cartier divisor in its normal toroidal Artin fan, and its refined class equals its fundamental class, with its nonreduced multiplicities. Concretely its local equation is a nonzero monomial in a normal toric domain, hence a nonzerodivisor; where the offset is zero in the characteristic monoid the zero locus is empty. This is the rank-one case of [35, Section 1.5.1, equation (8), and Definitions 1.11–1.13]. Smooth pullback to the mapping base and the chosen compatible obstruction theories consequently give

$$[\widehat{\mathcal{M}}]^{\mathrm{ref}} = \widehat{\epsilon}^![\widehat{\mathfrak{B}}], \qquad [\mathcal{M}]^{\mathrm{ref}} = \epsilon^![\mathfrak{B}].$$

The comparison of the two obstruction theories includes the morphisms to their relative cotangent complexes. The smooth morphism $\mathfrak{B} \to \mathfrak{A}$ is obtained from $X \to \mathcal{X}$, where $\mathcal{X}$ is the Artin fan. Its relative tangent bundle pulls back to $p^*f^*T_X^{\log}$. Over D_1° it is the Atiyah extension

$$0 \longrightarrow \mathcal{O}_{D_1^\circ} \longrightarrow T_X^{\log}|_{D_1^\circ} \longrightarrow T_{D_1^\circ} \longrightarrow 0,$$

with the line convention determined by $X \to \mathcal{X}$. Its rank is $\dim X$. The evaluation exact sequence gives the compatible triangle

$$(p^*f^*T_X^{\log})^\vee \longrightarrow (R\pi_*f^*T_X^{\log})^\vee \longrightarrow (R\pi_*f^*T_X^{\log}(-p))^\vee \longrightarrow .$$

[32, Section 4.2, diagram (4.15), Remark 4.4, and Proposition 4.5] construct a morphism from this triangle to the cotangent triangle for $\mathcal{M} \to \mathfrak{B} \to \mathfrak{A}$. One chooses a compatible final arrow in that construction; its resulting virtual class is independent of that choice.

Strictness of ϵ makes

$$\begin{array}{ccc} \mathcal{M}_{\log} & \longrightarrow & \mathcal{M}_0 \\ \downarrow & & \downarrow \\ \mathfrak{B}_{\log} & \xrightarrow{\nu} & \mathfrak{B}_0 \end{array}$$

cartesian on underlying algebraic stacks. Write ϵ_0 and $\epsilon_{\log}$ for the base changes of ϵ to $\mathfrak{B}_0$ and $\mathfrak{B}_{\log}$. Both vertical arrows carry the pullback of the chosen enhanced perfect obstruction theory. By [39, Definition 3.10 and Proposition 3.12], the class on the upper left is $\epsilon_{\log}^{!}[\mathfrak{B}_{\log}]$. The log smoothness used in the lemma also gives the required pure-dimensional base; compare their Proposition 3.19. The proper-pushforward property of the virtual pullback class, the lemma, and its compatibility with the regular point Gysin map now give

$$\begin{aligned} (\nu_{\mathcal{M}})_*[\mathcal{M}_{\log}]^{\mathrm{vir}} &= \epsilon_0^{!}\nu_*[\mathfrak{B}_{\log}] \\ &= \epsilon_0^{!}[\mathfrak{B}_0] \\ &= z^{!}\epsilon^{!}[\mathfrak{B}] \\ &= z^{!}[\widehat{\mathcal{M}}]^{\mathrm{ref}}. \end{aligned}$$

The last equality uses locality near the point fiber. Pushing by $\widehat{j}$ proves (22).

Fine pullback of a puncturing preserves the underlying curve and prestability; see [32, Proposition 2.8 and Corollary 2.10]. The source puncturing is preserved. Hence the ordinary stable map and its cotangent lines are the same after the indicated base changes. The descendant statement and any proper push to an ordinary receiver follow from the projection formula. □

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