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Coincidences of q-analogues of Markoff numbers outside the known families: an answer to a question of Labbé, Lapointe and Steiner

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Abstract

Labbé, Lapointe and Steiner proved that $w \mapsto \mu_q(w)_{12}$ is injective on Christoffel words, gave two families of identities showing that it is not injective on $\{0,1\}^*$, and asked whether every coincidence $\mu_q(x)_{12} = \mu_q(y)_{12}$ comes from these two families. We show that it does not. The smallest counterexamples have length 24; by an exhaustive computation, among words of length at most 32 there are exactly 50 classes of coincidences not explained by the two families, 10 of length 24 and 40 of length 30. Each of them contains an instance of one of a few new identities, which we prove for a generic 2×2 matrix in place of the matrix of a factor of the word. One of these identities belongs to an infinite family, indexed by $k \equiv 1 \pmod 4$, which we prove for every such k.

1 Introduction

Let $L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}$, $R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$, let $M_q : \{0,1\}^* \to \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}])$ be the morphism with $M_q(0) = L_q$, $M_q(1) = R_q$, and let μ_q be the morphism with

$$\mu_q(0) = R_q L_q = M_q(10), \qquad \mu_q(1) = R_q R_q L_q L_q = M_q(1100),$$

the q-analogue of the Markoff matrices introduced by Leclere and Morier-Genoud [2]. For a Christoffel word w, $\mu_q(w)_{12}$ is a q-analogue of the Markoff number of w. Labbé, Lapointe and Steiner proved that $w \mapsto \mu_q(w)_{12}$ is injective on Christoffel words [1], and showed that it is not injective on $\{0,1\}^*$ by two families of identities, their equations (4) and (7):

$$\mu_q(0\,w\,1)_{12} = \mu_q(0\,\widetilde{w}\,1)_{12}, \tag{4}$$

$$\mu_q(0\,\psi_w(v)\,w\,1)_{12} = \mu_q(0\,\psi_w(\widehat{v})\,w\,1)_{12}, \tag{7}$$

for $w \in \{0,1\}^*$ and $v \in \{0,1,2,3\}^*$. Here $\widetilde{w}$ is the mirror image of w, ψ_w is the morphism $0 \mapsto w\,01\,\widetilde{w}\,01$, $1 \mapsto w\,10\,\widetilde{w}\,10$, $2 \mapsto w\,01\,\widetilde{w}\,10$, $3 \mapsto w\,10\,\widetilde{w}\,01$, and $\widehat{v}$ is the mirror image of v with $0 \leftrightarrow 1$ exchanged and $2,3$ fixed. (The published version of [1] has $\overline{v}$ in place of $\widehat{v}$ in both identities of its Theorem 3.2; the correction is made in the latest arXiv version, and we use the corrected form.) They asked [1, Question 3.5]:

Do there exist $x, y \in \{0,1\}^*$ satisfying $\mu_q(x)_{12} = \mu_q(y)_{12}$ which are not given by Equation (4) or (7)?

They state that they believe (4) and (7) account for all coincidences. The answer to the question is yes.

Theorem 1. *(a) The words*

$$A = \texttt{00101100101110100110010 11},$$
$$B = \texttt{00110100101011001101001 1}$$

satisfy $\mu_q(A)_{12} = \mu_q(B)_{12}$*, a polynomial of degree* 69 *whose value at* $q = 1$ *is* 43 633 212 873 750*, and* B *cannot be obtained from* A *by any finite chain of identities (4) and (7).*

(b) Among the words of length at most 32*, the coincidences of* $\mu_q(\cdot)_{12}$ *that are not explained by (4) and (7) form exactly* 50 *classes:* 10 *of length* 24 *and* 40 *of length* 30 *(Table 2). In particular* 24 *is the smallest length at which a counterexample occurs.*

(c) Each of these 50 *classes contains an instance of Theorem 2 (*30 *classes) or of Theorem 3 (*20 *classes).*

(d) Theorem 2(i) is the first member of an infinite family of identities (Theorem 4). Its second member gives, for instance, a pair of words of length 50*, listed in Section 7, which is again not explained by (4) and (7).*

Parts (a) and (c), the closing assertion of (d), and the identities of Theorems 2 and 3 are checked in exact arithmetic by the script attached to this note (Section 7). Part (b) rests on an exhaustive computation which is not part of the deposit; see Section 6.

2 Four facts

Write $\overline{z}$ for the mirror image of $z \in \{\texttt{0},\texttt{1}\}^*$ with $\texttt{0} \leftrightarrow \texttt{1}$ exchanged. Let $Q = \mathrm{diag}(q, 1)$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $J = SQ = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$, and for a 2×2 matrix X put $\varphi(X) = (QXQ^{-1})^{\mathsf{T}} = Q^{-1}X^{\mathsf{T}}Q$. Let $r = (q, 1)$ and $s = (1, 1)^{\mathsf{T}}$.

Lemma 1. *(F1)* $M_q(\texttt{0110}) = M_q(\texttt{1001}) + (q^3 + 1)J$.

(F2) $XJ\varphi(X) = \det(X)\, J$ *for every* 2×2 *matrix* X.

(F3) φ *is an involutive anti-automorphism, and* $\varphi(M_q(z)) = M_q(\overline{z})$.

(F4) $M_q(\texttt{1}z\texttt{1})_{12} = r\, M_q(z)\, s$*; and* $\mu_q(w)_{12} = M_q(\sigma(w))_{12}$*, where* σ *is the morphism* $\texttt{0} \mapsto \texttt{10}$*,* $\texttt{1} \mapsto \texttt{1100}$*, and this entry does not change if the final letters* 0 *of* $\sigma(w)$ *are deleted.*

Proof. (F1) is a direct computation. (F2): $XJ\varphi(X) = XSQQ^{-1}X^{\mathsf{T}}Q = XSX^{\mathsf{T}}Q = \det(X)SQ$. (F3): φ reverses products, and $\varphi(L_q) = R_q$, $\varphi(R_q) = L_q$. (F4): the first row of R_q is r and its second column is s; the second column of L_q is $(0, 1)^{\mathsf{T}}$, so $(XL_q)_{12} = X_{12}$. □

Fact (F1), and (F2) for $X = M_q(z)$ in the form $M_q(z)\, SQ\, M_q(\overline{z}) = q^{|z|}SQ$, appear in the proof of [1, Lemma 3.3]; (F3) is equivalent to their equation (5). By (F4), for $w \in \{\texttt{0},\texttt{1}\}^*$ the word $\sigma(w)$ with its final 0's deleted begins and ends with 1, and coincidences of $\mu_q(\cdot)_{12}$ are coincidences of $z \mapsto r\, M_q(z)\, s$ on the words obtained by deleting these two letters.

3 Identities with five windows

The two words 0110 and 1001 are called *windows*; note $\overline{\texttt{0110}} = \texttt{1001}$. For a word T, put $T^+ = T$ and $T^- = \overline{T}$. For $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k) \in \{\pm\}^{k+1}$ and windows $a_1, \dots, a_k$, put

$$Z_\varepsilon(T; a_1, \dots, a_k) = \texttt{1}\, T^{\varepsilon_0} a_1 T^{\varepsilon_1} a_2 \cdots a_k T^{\varepsilon_k}\, \texttt{1}.$$

For a set F of positions, a^F denotes the sequence obtained from a by exchanging 0110 and 1001 at the positions of F.

Theorem 2. *Let $T \in \{\texttt{0},\texttt{1}\}^*$ and let $a_1, \dots, a_5$ be windows. Then $M_q(Z_\varepsilon(T; a))_{12} = M_q(Z_\varepsilon(T; a^F))_{12}$ in each of the following cases:*

(i) $\varepsilon = (+,-,+,-,-,+)$, $F = \{1,3,5\}$, *and* $a_1 = a_5 \neq a_3$;

(ii) $\varepsilon = (+,-,-,+,-,+)$, $F = \{1,3,5\}$, *and* $a_1 = a_5 \neq a_3$;

(iii) $\varepsilon = (+,-,-,+,+,-)$, $F = \{2,4\}$, $a_1 = a_5 \neq a_3$ *and* $a_2 = a_4$.

Proof. By (F3) and (F4), $M_q(Z_\varepsilon(T; a))_{12} = r\, G^{\varepsilon_0} A_1 G^{\varepsilon_1} \cdots A_5 G^{\varepsilon_5}\, s$ with $G = M_q(T)$, $G^+ = G$, $G^- = \varphi(G)$ and $A_i = M_q(a_i)$. It therefore suffices to prove the corresponding identities with $G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ generic, that is, in $\mathbb{Z}[\alpha, \beta, \gamma, \delta, q]$ after clearing the common power of q coming from $\varphi(G)$. These are finitely many polynomial identities (4, 4 and 2 in the three cases, up to exchanging the two sides), and they are verified by exact expansion; see Section 7. □

Remark 1. For the generic matrix the conditions on a are exact: for each of the three pairs (ε, F), the identity fails for every other choice of the five windows. In case (i) the factor 0110 or 1001 that is exchanged produces, by (F1), a term in J, which (F2) absorbs when the two neighbouring factors are G and $\varphi(G)$ in this order or the reverse; this is the mechanism behind the infinite family of Section 5.

4 Self-conjugate words

If $\overline{T} = T$ (for instance $T = \texttt{01}$), then all exponents ε_i are irrelevant, and we write $Z(T; a)$. Table 1 lists 20 pairs (a, F) with $k = 14$ windows, written with + for 0110 and − for 1001.

Theorem 3. *Let $T \in \{\texttt{0},\texttt{1}\}^*$ with $\overline{T} = T$. For each of the 20 pairs (a, F) of Table 1, $M_q(Z(T; a))_{12} = M_q(Z(T; a^F))_{12}$.*

Proof. By (F3), $G = M_q(T)$ satisfies $\varphi(G) = G$, that is, $G = \begin{pmatrix} \alpha & \beta \\ q\beta & \delta \end{pmatrix}$. As in the proof of Theorem 2, it suffices to verify 20 identities in $\mathbb{Z}[\alpha, \beta, \delta, q]$ for this generic G, which is done by exact expansion. □

5 An infinite family

For $m \geq 1$ put $k = 4m + 1$ and $\varepsilon = (+,-)^{2m}(-,+) \in \{\pm\}^{k+1}$. Call a sequence of windows $a_1, \dots, a_k$ *admissible* if its odd-indexed terms alternate, starting with 0110: $a_{2t+1} = \texttt{0110}$ for t even and $a_{2t+1} = \texttt{1001}$ for t odd; the even-indexed terms are arbitrary. Let $F_{\mathrm{odd}} = \{1, 3, \dots, k\}$.

a	F	a	F
+++-++---+-+++	$\{2,3,7,8,12,13\}$	+++-+---+++++-	$\{5,10\}$
+++-+---++-+++	$\{2,3,7,8,12,13\}$	++-++--+-+++-+	$\{5,10\}$
+++--+-----+++	$\{2,3,7,8,12,13\}$	+-+++-+--++-++	$\{5,10\}$
+++-----+--+++	$\{2,3,7,8,12,13\}$	-+++++---+-+++	$\{5,10\}$
+++-++-----+++	$\{1,4,11,14\}$	++-++-+--++-++	$\{2,3,5,10,12,13\}$
+++-+---+--+++	$\{1,4,11,14\}$	++-++--+-++-++	$\{2,3,5,10,12,13\}$
+++--+---+-+++	$\{1,4,11,14\}$	++-+--+---+-++	$\{2,3,5,10,12,13\}$
+++-----++-+++	$\{1,4,11,14\}$	++-+---+--+-++	$\{2,3,5,10,12,13\}$
++-++-+---+-++	$\{2,3,12,13\}$	++-+--+--++-++	$\{2,3,12,13\}$
++-++--+--+-++	$\{2,3,12,13\}$	++-+---+-++-++	$\{2,3,12,13\}$

Table 1: The 20 identities of Theorem 3 ($k = 14$; + stands for `0110`, − for `1001`). Each set F is invariant under $i \mapsto 15 - i$.

Theorem 4. *For every $m \geq 1$, every word T and every admissible a,*

$$M_q\big(Z_\varepsilon(T;a)\big)_{12} = M_q\big(Z_\varepsilon(T;a^{F_{\mathrm{odd}}})\big)_{12}.$$

For $m = 1$ this is Theorem 2(i) with $a_1 = \mathtt{0110}$.

Proof. As before it suffices to treat a generic matrix $G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ over the field $K = \mathbb{Q}(\alpha, \beta, \gamma, \delta, q)$, with $G' = \varphi(G)$, $P = M_q(\mathtt{0110})$, $N = M_q(\mathtt{1001})$; the identity to be proved is a polynomial identity, so it holds after any specialisation of G. Put

$$U = GNG', \quad V = G'NG, \quad K_0 = (q^3 + 1)\det(G)\, J.$$

By (F1) and (F2), $GPG' = U + K_0$ and $G'PG = V + K_0$ (for the second, apply (F2) to $X = G'$, noting $\varphi(G') = G$ and $\det G' = \det G$). Writing a_i also for $M_q(a_i)$, group the factors of the product attached to $Z_\varepsilon(T;a)$ as

$$(G\, a_1\, G')\, a_2\, (G\, a_3\, G')\, a_4 \cdots (G\, a_{4m-1}\, G')\, a_{4m}\, (G'\, a_{4m+1}\, G).$$

For admissible a the bracketed factors are, in order, $U + K_0, U, U + K_0, \ldots, U$ ($2m$ factors) followed by $V + K_0$; after the exchange they are $U, U + K_0, \ldots, U + K_0$ followed by V. With

$$C_{XY} = (U + K_0)\, X\, U\, Y, \qquad C'_{XY} = U\, X\, (U + K_0)\, Y \qquad (X, Y \in \{P, N\}),$$

four matrices each, the statement becomes: for all $X_1, \ldots, X_{2m} \in \{P, N\}$,

$$r\, C_{X_1X_2} \cdots C_{X_{2m-1}X_{2m}}\, (V + K_0)\, s = r\, C'_{X_1X_2} \cdots C'_{X_{2m-1}X_{2m}}\, V\, s. \tag{1}$$

Write $C_1, \ldots, C_4$ and $C'_1, \ldots, C'_4$ for the two quadruples. The following are verified by exact computation in $\mathbb{Z}[\alpha, \beta, \gamma, \delta, q]$ (Section 7): (a) $\operatorname{tr} C_i = \operatorname{tr} C'_i$, $\operatorname{tr}(C_iC_j) = \operatorname{tr}(C'_iC'_j)$ for $i \leq j$, and $\operatorname{tr}(C_iC_jC_l) = \operatorname{tr}(C'_iC'_jC'_l)$ for $i < j < l$; (b) $C_1, \ldots, C_4$ are linearly independent over K; (c) $(V + K_0)s \neq 0$; (d) (1) for $m = 1$, which is Theorem 2(i).

Step 1: equal traces on all words. For 2×2 matrices over a field of characteristic 0, the trace of any product of $X_1, \ldots, X_r$ is a polynomial in $\operatorname{tr} X_i$, $\operatorname{tr}(X_iX_j)$ ($i \leq j$) and $\operatorname{tr}(X_iX_jX_l)$ ($i < j < l$): this follows from the Cayley–Hamilton identity $XY + YX = \operatorname{tr}(X)Y + \operatorname{tr}(Y)X +$

$(\operatorname{tr}(XY) - \operatorname{tr} X \operatorname{tr} Y)I$, see [3, 4, 5]. Hence by (a), $\operatorname{tr} w(C) = \operatorname{tr} w(C')$ for every word w in four letters, and by linearity $\operatorname{tr} f(C) = \operatorname{tr} f(C')$ for every f in the free algebra $K\langle x_1, \dots, x_4\rangle$.

Step 2: conjugacy. By (b) the C_i span $M_2(K)$, and the trace form $(X, Y) \mapsto \operatorname{tr}(XY)$ is nondegenerate on $M_2(K)$, so the Gram matrix $(\operatorname{tr} C_i C_j)$ is invertible; it equals $(\operatorname{tr} C'_i C'_j)$, hence the C'_i span $M_2(K)$ too. If $f(C) = 0$ then $\operatorname{tr}(f(C')\, g(C')) = \operatorname{tr}(f(C)\, g(C)) = 0$ for all g, so $f(C') = 0$. Thus $C_i \mapsto C'_i$ defines a K-algebra endomorphism of $M_2(K)$, injective since $M_2(K)$ is simple, hence an automorphism, and by the Skolem–Noether theorem there is $W \in \mathrm{GL}_2(K)$ with $C'_i = W^{-1} C_i W$ for all i.

Step 3: the two ends. By (d), $r\, C_i\, (V + K_0)s = r\, C'_i\, Vs = (rW^{-1})\, C_i\, (WVs)$ for $i = 1, \dots, 4$. The linear forms $X \mapsto r\, X\, (V + K_0)s$ and $X \mapsto (rW^{-1})\, X\, (WVs)$ on $M_2(K)$ agree on the basis $C_1, \dots, C_4$, hence everywhere; that is, the rank-one matrices $(V+K_0)s\, r$ and $WVs\, rW^{-1}$ are equal. By (c) and $r \neq 0$ they are nonzero, so there is $\lambda \in K^\times$ with $rW^{-1} = \lambda^{-1} r$ and $WVs = \lambda(V+K_0)s$.

Conclusion. For any $i_1, \dots, i_m$, $r\, C'_{i_1} \cdots C'_{i_m} Vs = rW^{-1} C_{i_1} \cdots C_{i_m} WVs = r\, C_{i_1} \cdots C_{i_m} (V + K_0)s$. □

Remark 2. The congruence $k \equiv 1 \pmod 4$ is where the last odd window falls on the factor $G'\, a_k\, G$, as the grouping requires. The analogous statement with $k = 3$, $\varepsilon = (+, -, -, +)$ and $a_1 = \texttt{0110}$, $a_3 = \texttt{1001}$ is false (checked on several words T with $\overline{T} \neq T$).

6 The census

For each $n \leq 32$, all 2^n words of length n were sorted by the value of $\mu_q(\cdot)_{12}$, and the classes of equal values were compared with the classes generated by (4) and (7) on words of length n; both identities preserve length. Since a coincidence preserves the numbers of letters `0` and `1` ([1, Proposition 2.3], numbering of the arXiv version), the computation can be split by content. Every class found was confirmed in exact polynomial arithmetic. Table 2 gives the number of classes of coincidence that are not reduced to a pair $\{\texttt{0}w\texttt{1}, \texttt{0}\widetilde{w}\texttt{1}\}$ given by (4), and among them the number not explained by (4) and (7) together. No other length $n \leq 32$ carries any class of either kind. The computation was carried out with three independent implementations, and the counts up to length 24 were reproduced by a fourth, written separately.

n	12	15	18	20	21	24	25	27	28	30	total
classes beyond (4)	1	2	6	6	12	38	12	56	28	196	357
not explained by (4) and (7)	0	0	0	0	0	10	0	0	0	40	50

Table 2: Classes of coincidences of $\mu_q(\cdot)_{12}$ among words of length $n \leq 32$.

Each of the 50 classes consists of exactly four words $\{A, A', B, B'\}$, where A' and B' are the partners of A and B under (4); identities (4) and (7) relate A to A' and B to B', and nothing else. The census answers Question 3.5 only for $n \leq 32$; the pair of length 50 of Section 7, an instance of Theorem 4, shows that the phenomenon is not confined to these lengths.

7 Data and verification

The 50 pairs (A, B) of Theorem 1(b), the pair of length 50 of Theorem 1(d), namely

`00010110001001101000100101100010011010001100010011`,

`00011010001001011000100110100010010110001100100011`,

and all the identities of this note are checked by the script `anc/verify_markoff_q35.py` (Python 3, no package, no argument, about one minute). It checks, in exact arithmetic over $\mathbb{Z}$ and $\mathbb{Z}[\alpha, \beta, \gamma, \delta, q]$: the values of [1] used for calibration; $\mu_q(A)_{12} = \mu_q(B)_{12}$ for the 51 pairs; that for each of them B is not reached from A by any chain of instances of (4) and (7), the set reached from A being $\{A, A'\}$ and the set reached from B being $\{B, B'\}$; facts (F1) and (F2), and (F3)–(F4) on all words of length at most 8; the identities of Theorems 2 and 3, and the exactness of the conditions in Theorem 2; that each of the 50 classes contains an instance of one of them; the inputs (a)–(c) of the proof of Theorem 4 (input (d) is Theorem 2(i)), direct instances of it for $m = 1, 2, 3$, and the fact that the pair of length 50 is an instance with $m = 2$ and $T = \texttt{0101011001}$.

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