Rational limits in Markov–Euclid asymptotics

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Abstract

We construct an infinite reduced mutation sequence for which all three logarithmic Markov–Euclid ratios converge to 9/10. This disproves the irrationality conjecture of Zhichao Chen and Zelin Jia (Conjecture 6.12).

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The main results and their proofs were developed in a ChatGPT-6-Pro conversation lasting approximately 34 minutes. Codex subsequently carried out an adversarial review of the arguments, checked the supporting computations and literature, and prepared the manuscript using the mathematical writing skill.

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

The classical Markov equation

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3 \tag{1}$$

arose in Markov's study of indefinite binary quadratic forms and Diophantine approximation [1]. Its positive integer solutions are called Markov triples. Replacing one coordinate by the other root of the corresponding quadratic equation gives the Vieta mutation

$$x_i \longmapsto 3x_jx_k - x_i, \qquad \{i,j,k\} = \{1,2,3\},$$

with the other two coordinates fixed. These operations generate the positive solutions from $(1,1,1)$ and organize them into a tree.

Zagier's counting argument related the Markov tree to a lattice-point problem; Baragar [2] developed logarithmic comparisons with Euclid trees for higher-dimensional Markoff–Hurwitz equations. In these comparisons, the product appearing in a Vieta mutation becomes a sum after taking logarithms. Spalding and Veselov [8] subsequently studied Lyapunov exponents of Markov and Euclid dynamics along paths. Their work also connects the Farey parametrization of the Markov tree with hyperbolic lengths and Fock's function.

Cluster algebras, introduced by Fomin and Zelevinsky [3], provide another setting for these recursions. On a Markov triple, the equation gives

$$3x_jx_k - x_i = \frac{x_j^2 + x_k^2}{x_i},$$

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which is the exchange relation of the Markov cluster pattern. Gyoda and Matsushita [4] studied the generalized equations

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + \lambda_3 x_1 x_2 + \lambda_1 x_2 x_3 + \lambda_2 x_3 x_1 \\ = (3 + \lambda_1 + \lambda_2 + \lambda_3) x_1 x_2 x_3, \end{aligned} \tag{2}$$

where $\lambda_1, \lambda_2, \lambda_3$ are nonnegative integers. They gave recursive descriptions of the positive integer solutions and related these recursions to generalized cluster algebras. The choice $\lambda_1 = \lambda_2 = \lambda_3 = 0$ recovers (1).

Chen and Jia [6] investigated the tropicalization and asymptotic behavior of (2). They constructed a deformed Fock–Goncharov tropicalization, compared generalized Euclid trees with classical Euclid trees, and established logarithmic comparison limits along infinite reduced mutation sequences. Their Theorem 6.10 distinguishes paths in which all three directions recur infinitely often from paths that eventually use only two directions. In the first case, the three coordinate ratios have a common limit; in the second, the comparison concerns the two changing coordinates. Their Fibonacci example, Example 6.11, motivates Conjecture 6.12, also stated as Conjecture 1.5: every such limit is a positive irrational number. Chen, Jia and Wu [5] extended the logarithmic asymptotic comparison to a generalized Markov–Hurwitz family in arbitrarily many variables.

For the classical equation, we use the following normalization. For a positive triple (e_1, e_2, e_3), the Euclid operation in direction i replaces e_i by $e_j + e_k$ and fixes the other coordinates. A word $\mathbf{w} = (w_n)_{n \geq 1}$ with $w_n \in \{1, 2, 3\}$ is reduced if $w_{n+1} \neq w_n$ for every $n \geq 1$. Starting both triples at $(1, 1, 1)$, apply the first n letters of the same word by Markov mutations and Euclid operations, respectively. Denote the resulting coordinate i by $M_i(n)$ and $E_i(n)$, for $i \in \{1, 2, 3\}$ and integers $n \geq 0$. The limit in Chen and Jia's conjecture, for a path with all three directions recurring, is

$$\lim_{n \to \infty} \frac{\log M_i(n)}{E_i(n)}, \qquad i = 1, 2, 3.$$

Here log is the natural logarithm. We write arcosh : $[1, \infty) \to [0, \infty)$ for the inverse of cosh restricted to $[0, \infty)$.

Theorem 1.1. *Let*

$$I = \left(\frac{1}{2} \operatorname{arcosh} 3, \ \operatorname{arcosh} \frac{3}{2} \right).$$

For every real algebraic $c \in I$, there is a computable infinite reduced word $\mathbf{w}$ in which each of $1, 2, 3$ occurs infinitely often, such that

$$\lim_{n \to \infty} \frac{\log M_i(n)}{E_i(n)} = c \qquad (i = 1, 2, 3). \tag{3}$$

In particular, $c = 9/10$ is a counterexample to Chen–Jia Conjecture 6.12.

The arithmetic input is the Hermite–Lindemann theorem: the exponential of a nonzero algebraic number is transcendental. It implies that the normalized lengths at rational Farey vertices are transcendental. Their continuous interpolation takes every intermediate value, so an algebraic target in the displayed interval must occur at an irrational Farey parameter. We construct its path by a strict sign comparison and identify every vertex denominator with the corresponding Euclid coordinate.

The interpolation used here is Fock's function. Its continuous convex extension is recorded in Sorrentino–Veselov [7, Theorem 1] and Spalding–Veselov [8, Theorem 10]. We prove the required continuity directly from the integer recursion, with a uniform error bound. The proof of Theorem 1.1 concludes in Section 6.

2 An integer-labeled Farey tree

An active interval has reduced rational endpoints $a/b < c/d$ in $[0,1]$, with integer numerators a, c and positive integer denominators b, d, satisfying

$$bc - ad = 1, \qquad b, d \geq 1. \tag{4}$$

Attach positive integer labels x, y to its left and right endpoints and a positive integer label u to the vertex opposite that interval in the preceding subdivision. At the root the opposite label is an auxiliary initial value. Start from $[0/1, 1/1]$ with $(x, y; u) = (1, 1; 1)$. Insert the mediant and its new label

$$\frac{a+c}{b+d}, \qquad z = 3xy - u. \tag{5}$$

The left and right children respectively have data

$$([a/b, (a+c)/(b+d)], x, z; y), \qquad ([(a+c)/(b+d), c/d], z, y; x).$$

Existing endpoint labels are inherited by both children. The root interval has depth zero, and a child of a depth-n interval has depth $n+1$.

Lemma 2.1. *At every interval, u, x, y are positive integers, $u^2 + x^2 + y^2 = 3uxy$, and $u \leq \max(x, y)$. The new label satisfies*

$$z \geq 2xy \geq 2\max(x, y) > u. \tag{6}$$

At depth n of the interval tree, $xy \geq 2^n$.

Proof. The invariant assertions hold at the root. Since $\max(x, y) \leq xy$, $z = 3xy - u \geq 2xy$. The identity

$$x^2 + y^2 + (3xy - u)^2 - 3xy(3xy - u) = x^2 + y^2 + u^2 - 3xyu = 0$$

shows that the Markov equation is preserved. In either child, the opposite label is one of x, y, whereas the new endpoint z is larger than both. This proves the induction. The endpoint maximum at least doubles on each descent; the root maximum is one. Finally $xy \geq \max(x, y) \geq 2^n$. □

Lemma 2.2. *Each refinement is a partition into intervals satisfying (4). Every rational in $[0,1]$ appears at a finite level and has a well-defined inherited label.*

Proof. The two child determinants are

$$b(a+c) - a(b+d) = bc - ad = 1, \qquad (b+d)c - (a+c)d = bc - ad = 1.$$

A common divisor of $a+c$ and $b+d$ would divide the first determinant, so the mediant is reduced. Its distances from the two endpoints are

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{1}{b(b+d)}, \qquad \frac{c}{d} - \frac{a+c}{b+d} = \frac{1}{d(b+d)}.$$

They are positive. Thus each level is a partition, and every new vertex lies in the interior of exactly one parent interval. Its label is assigned once and is inherited thereafter.

Every reduced rational $p/q \in (0,1)$ occurs at a finite level. To prove this, suppose it is interior to $[a/b, c/d]$ and define the positive integers

$$\mu = cq - dp, \qquad \nu = bp - aq.$$

The determinant identity yields

$$(p, q) = \mu(a, b) + \nu(c, d).$$

Since

$$\mu - \nu = q(a+c) - p(b+d),$$

if $p/q < (a+c)/(b+d)$, then $\mu > \nu$ and the coefficients relative to the left child are $(\mu - \nu, \nu)$. If $p/q > (a+c)/(b+d)$, the coefficients relative to the right child are $(\mu, \nu - \mu)$. In either case their positive integer sum decreases. Eventually $\mu = \nu$; then $(p, q) = \mu(a+c, b+d)$, and reducedness forces $\mu = 1$. Hence p/q is the inserted mediant. The endpoints $0, 1$ occur at level zero. □

For each rational $r \in [0, 1]$, let $m(r)$ denote its inherited positive integer label.

3 A quantitative length defect

For an integer $m \geq 1$, define

$$h(m) = \operatorname{arcosh} \frac{3m}{2} = \log \frac{3m + \sqrt{9m^2 - 4}}{2}. \tag{7}$$

Lemma 3.1. *For every insertion (5),*

$$0 < h(x) + h(y) - h(z) \leq \frac{4}{15x^2y^2}. \tag{8}$$

Proof. Both u and z solve $t^2 - 3xyt + x^2 + y^2 = 0$. Its discriminant is

$$D = 9x^2y^2 - 4x^2 - 4y^2.$$

Since $x^2 + y^2 \leq 2x^2y^2$, we have $D \geq x^2y^2 > 0$. As $z > u$ and $z + u = 3xy$, we have $z = (3xy + \sqrt{D})/2$. Since $\sinh h(x) = \sqrt{9x^2 - 4}/2$, the addition formula gives

$$\cosh(h(x) + h(y)) = \frac{9xy + \sqrt{(9x^2 - 4)(9y^2 - 4)}}{4}.$$

Here $(9x^2 - 4)(9y^2 - 4) = 9D + 16$ and $\cosh h(z) = 3z/2 = (9xy + 3\sqrt{D})/4$. Subtracting and rationalizing,

$$\cosh(h(x) + h(y)) - \cosh h(z) = \frac{\sqrt{9D + 16} - 3\sqrt{D}}{4} = \frac{4}{\sqrt{9D + 16} + 3\sqrt{D}} > 0.$$

The positive difference and strict increase of cosh on positive reals imply $h(x) + h(y) > h(z)$. Moreover,

$$\cosh(h(x) + h(y)) - \cosh h(z) = \int_{h(z)}^{h(x)+h(y)} \sinh v \, dv \geq \big(h(x) + h(y) - h(z)\big) \frac{\sqrt{9z^2 - 4}}{2}.$$

Consequently

$$0 < \big(h(x) + h(y) - h(z)\big) \leq \frac{8}{\sqrt{9z^2 - 4}(\sqrt{9D + 16} + 3\sqrt{D})}.$$

Also

$$9z^2 - 4 \geq 36x^2y^2 - 4 \geq 25x^2y^2,$$

since $x^2y^2 \geq 1$. Thus $\sqrt{9z^2 - 4} \geq 5xy$, while $\sqrt{9D + 16} + 3\sqrt{D} \geq 6xy$. These inequalities prove (8). □

4 Continuous Farey interpolation

For each integer $n \geq 0$, let P_n be the ordered set of vertices after n complete refinement levels. Thus $P_0 = \{0, 1\}$, and the gaps between consecutive elements of P_n are exactly the depth-n intervals. Define $F_n : [0, 1] \to \mathbb{R}$ to be affine on each gap and to take, at every reduced vertex $p/d \in P_n$, the value

$$F_n(p/d) = \frac{h(m(p/d))}{d}. \tag{9}$$

For a depth-n interval $[a/b, c/d]$ with endpoint labels x, y and mediant label $z = 3xy - u$, the identity

$$\frac{a+c}{b+d} = \frac{b}{b+d}\frac{a}{b} + \frac{d}{b+d}\frac{c}{d}$$

gives

$$F_n\left(\frac{a+c}{b+d}\right) = \frac{h(x)+h(y)}{b+d}, \qquad F_{n+1}\left(\frac{a+c}{b+d}\right) = \frac{h(z)}{b+d}.$$

On each parent interval, $F_n - F_{n+1}$ vanishes at both endpoints and is affine on each child. Its value at the mediant is

$$\frac{h(x)+h(y)-h(z)}{b+d} > 0,$$

which is therefore its maximum on that interval. Lemma 2.1, Lemma 3.1, and $b + d \geq 2$ give

$$0 \leq F_n(t) - F_{n+1}(t) \leq \frac{2}{15}4^{-n} \qquad (0 \leq t \leq 1). \tag{10}$$

Proposition 4.1. *The functions F_n converge uniformly to a continuous function F on $[0, 1]$. For every integer $n \geq 0$ and $t \in [0, 1]$,*

$$0 \leq F_n(t) - F(t) \leq \frac{8}{45}4^{-n}. \tag{11}$$

At every reduced rational $p/d \in [0, 1]$,

$$F(p/d) = \frac{\operatorname{arcosh}(3m(p/d)/2)}{d}. \tag{12}$$

Proof. For integers $N > n \geq 0$, telescoping (10) gives

$$0 \leq F_n(t) - F_N(t) \leq \frac{2}{15}\sum_{k=n}^{N-1} 4^{-k} \leq \frac{8}{45}4^{-n} \qquad (t \in [0, 1]).$$

Thus (F_n) is uniformly Cauchy and has a uniform limit F; letting $N \to \infty$ gives (11). To check continuity at $t_0 \in [0, 1]$, fix $\varepsilon > 0$ and choose n with $\sup_t |F_n(t) - F(t)| < \varepsilon/3$. Continuity of F_n gives a neighborhood of t_0 on which $|F_n(t) - F_n(t_0)| < \varepsilon/3$. The triangle inequality then gives $|F(t) - F(t_0)| < \varepsilon$. Every rational vertex appears at a finite level and keeps the same assigned value at all subsequent levels. Taking its limit proves (12). □

5 Algebraic values along infinite Farey paths

At the two endpoints and the first mediant,

$$F(0) = F(1) = \operatorname{arcosh}(3/2), \qquad F(1/2) = \tfrac{1}{2}\operatorname{arcosh} 3. \tag{13}$$

The power series for cosh gives

$$\cosh(9/5) > 1 + \frac{(9/5)^2}{2} + \frac{(9/5)^4}{24} = \frac{15287}{5000} > 3.$$

For every integer $k \geq 2$,

$$\frac{(9/10)^{2k+2}/(2k+2)!}{(9/10)^{2k}/(2k)!} = \frac{81}{100(2k+2)(2k+1)} \leq \frac{27}{1000}.$$

Bounding the tail beginning with $k = 2$ by a geometric series gives

$$\cosh(9/10) \leq 1 + \frac{81}{200} + \frac{(9/10)^4}{24(1 - 27/1000)} = \frac{557761}{389200} < \frac{3}{2}.$$

As cosh increases on positive reals, these prove

$$\frac{1}{2}\operatorname{arcosh} 3 < \frac{9}{10} < \operatorname{arcosh}\frac{3}{2}. \tag{14}$$

Lemma 5.1. *For every rational $r \in [0, 1]$, $F(r)$ is transcendental.*

Proof. Write $r = p/d$ in reduced form with integers $0 \leq p \leq d$ and $d \geq 1$. If $F(r)$ were algebraic, then $dF(r) = h(m(r))$ would be a nonzero algebraic number, whereas

$$\exp(dF(r)) = \frac{3m(r) + \sqrt{9m(r)^2 - 4}}{2}$$

is algebraic. This contradicts the Hermite–Lindemann theorem [9]. □

Proposition 5.2. *For each real algebraic $c \in I$, the following rule defines an infinite nested sequence of Farey intervals J_k ($k \geq 0$), with $J_0 = [1/2, 1]$: insert the mediant in J_k and let J_{k+1} be the child whose endpoint values lie on opposite sides of c. Their intersection is a single irrational number α satisfying $F(\alpha) = c$. Each child can be determined by a terminating comparison of rational bounds.*

Proof. The endpoint values of J_0 satisfy $F(1/2) < c < F(1)$. Suppose the endpoints of J_k have values below and above c, respectively. By Lemma 5.1, the mediant value differs from c. If it is below c, keep the right child; if it is above c, keep the left child. In either case the new left value is below c and the new right value is above c. This proves that the rule continues uniquely for all k.

If J_k has endpoint denominators b, d, its length is $1/(bd)$ by (4). The denominator product in the next interval is either $b(b+d) = bd + b^2$ or $d(b+d) = bd + d^2$, and hence increases by at least one. Since the initial product is two, the length of J_k is at most $1/(k+2)$. The left endpoints are nondecreasing and bounded above, while the right endpoints are nonincreasing and bounded below. Their limits coincide because the lengths tend to zero. Denote this common limit by α. Nestedness gives $\bigcap_{k\geq 0} J_k = \{\alpha\}$. Continuity and the strict endpoint inequalities imply $F(\alpha) \leq c$ and $F(\alpha) \geq c$, respectively, so

$$F(\alpha) = c. \tag{15}$$

If α were rational, Lemma 5.1 would contradict the algebraicity of c.

Each decision in this rule can be determined by rational bounds. Indeed, for a reduced mediant p/d with label m, comparing $F(p/d)$ with c is equivalent to comparing

$$\frac{3m + \sqrt{9m^2 - 4}}{2} \quad \text{with} \quad e^{dc}.$$

Bisection for the square root gives rational bounds of arbitrarily small width for the first number. Specify c by a squarefree integer polynomial and a positive rational interval containing exactly that real root, with neither endpoint a root. The root is simple, so the polynomial has opposite signs at the endpoints. Evaluating it at the rational midpoint either determines c exactly or selects a half-interval with the same sign-change property. Repeating this step refines the bounds

for c. For rational $v \geq 0$ and a nonnegative integer N with $v < N + 2$, the exponential series has the bound

$$0 \leq e^v - \sum_{j=0}^{N} \frac{v^j}{j!} \leq \frac{v^{N+1}}{(N+1)!} \frac{1}{1 - v/(N+2)}.$$

This follows because the ratios of consecutive terms after the first omitted term are at most $v/(N+2)$. Apply these bounds at rational lower and upper bounds for dc, refining those bounds and increasing N. All these rational arguments lie below a fixed upper bound. For N larger than twice that bound, the displayed remainder is at most twice its first omitted term and tends to zero as $N \to \infty$. Together with continuity of the exponential, this shows that the enclosure widths tend to zero. Lemma 5.1 excludes equality in the comparison, so two disjoint enclosures are reached after finitely many refinements. □

6 Markov–Euclid limits

Proof of Theorem 1.1. Fix $c \in I$ algebraic and take the intervals J_k and α from Proposition 5.2. Set $w_1 = 1$ and apply this mutation to both roots. The triples become $(2, 1, 1)$. For each integer $n \geq 1$, let $r_i(n)$ be the rational label attached to coordinate i after n mutations. At $n = 1$ set

$$(r_1(1), r_2(1), r_3(1)) = (1/2, 1/1, 0/1), \qquad (M_i(1))_{i=1}^3 = (E_i(1))_{i=1}^3 = (2, 1, 1). \tag{16}$$

The active interval is $J_0 = [1/2, 1]$, with opposite vertex 0 in coordinate 3. Its endpoint labels are $2, 1$ and its opposite label is 1, as in the right child of the root Farey subdivision. The next mutation therefore has direction 3.

Inductively, after n mutations the active interval is J_{n-1}, and the three fractions consist of its endpoints and its opposite vertex. Suppose its endpoints occupy coordinates j, k, with labels x, y and reduced denominators b, d, while the opposite coordinate i has label u. Set $w_{n+1} = i$ and mutate coordinate i, replacing its rational label by the mediant. Its Markov label becomes $3xy - u$ and its Euclid coordinate becomes $b + d$. Select J_n by Proposition 5.2. Its opposite vertex is the discarded endpoint of J_{n-1}, giving the next induction state.

The initial denominators equal $(2, 1, 1)$. The preceding update and the determinant-one property prove by induction, for every $n \geq 1$, that $r_i(n)$ is reduced, its denominator is $E_i(n)$, and its Markov label is $M_i(n) = m(r_i(n))$.

The newly inserted mediant is retained by either child. Thus the next opposite coordinate is one of the two other coordinates. Successive directions after the first insertion differ. The initial pair is $1, 3$, so the whole word is reduced.

Write $|J|$ for the length of an interval J. For $n \geq 2$, all three fractions after mutation n are the endpoints and mediant of J_{n-2}. Consequently

$$|r_i(n) - \alpha| \leq |J_{n-2}| \leq \frac{1}{n} \qquad (i = 1, 2, 3),$$

so each $r_i(n)$ tends to α. Fix a coordinate i. If $E_i(n)$ failed to tend to infinity, there would be a positive integer B and infinitely many n with $E_i(n) \leq B$. There are only finitely many reduced fractions in $[0, 1]$ with denominator at most B. One of them would occur along an infinite subsequence of $r_i(n)$ and hence equal its limit α, contradicting irrationality. Likewise, no coordinate direction can stop appearing, because its fraction would then remain fixed. Therefore all three directions occur infinitely often.

Proposition 4.1 and (15) give

$$\frac{h(M_i(n))}{E_i(n)} = F(r_i(n)) \longrightarrow c.$$

Finally, for all $m \geq 1$,

$$0 < h(m) - \log m = \log \frac{3 + \sqrt{9 - 4/m^2}}{2} < \log 3. \tag{17}$$

Divide by $E_i(n) \to \infty$. This proves (3) for all three coordinates. The choice $c = 9/10$ is valid by (14). □

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