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ON THE MAXIMALITY CONJECTURE FOR QUANTUM PERMUTATION GROUPS

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ABSTRACT. A well-known conjecture, central in quantum permutation group theory, and going back to around 2006, states that $S_N \subset S_N^+$ is maximal. We revisit here this conjecture, and related questions, 20 years after, with 2026 knowledge and tools.

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INTRODUCTION

The quantum permutation group S_N^+ was introduced by Wang in 1998 [74], as the universal compact quantum group acting on N points. The algebra of functions on S_N^+ is generated by the entries of an $N \times N$ magic unitary, and its quotient by the commutator ideal is the algebra of functions on S_N, so that we have an inclusion $S_N \subset S_N^+$. This inclusion is an equality for $N \leq 3$, while for $N \geq 4$ the quantum group S_N^+ is infinite, and for $N \geq 5$ it is not even coamenable. Its representation theory is that of SO_3, with intertwiners given by the noncrossing partitions [1], instead of all partitions for S_N. When classifying the quantum subgroups of S_4^+, Banica and Bichon [5] found nothing between S_4 and S_4^+, and this led, around 2006, to the following statement.

Maximality conjecture. *For every N, the inclusion $S_N \subset S_N^+$ is maximal, in the sense that there is no intermediate quantum group $S_N \subsetneq G \subsetneq S_N^+$.*

The conjecture holds trivially for $N \leq 3$, it holds for $N = 4$ by [5], and for $N = 5$ by [2], the proof there relying on the classification of subfactors of index ≤ 5 [39, 43]. For $N \geq 6$ it is open. Since there is no easy quantum group strictly between S_N and S_N^+ [13], the conjecture lies beyond easiness, and it is a natural first test case for the classification of compact quantum groups beyond easiness.

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We revisit here this conjecture, and some related questions, twenty years after, with 2026 knowledge and tools. After preliminaries in Section 1, the paper is organized around three approaches:

(1) *Subgroups.* Given a quantum subgroup $G \subset S_N^+$ with $G \not\subset S_N$, the conjecture predicts $\langle S_N, G\rangle = S_N^+$. This is the natural source of counterexamples. It is also of independent interest, the equality $\langle S_N, G\rangle = S_N^+$ being a non-trivial result about G itself. We test in this way group duals, which by a theorem of Bichon [20] act through "Fourier families" of abelian groups, and reduce to just-non-abelian groups (Section 2), quantum automorphism groups of graphs (Section 3), twists of S_N and of PU_n (Section 4), and the quantum groups coming from complex Hadamard matrices (Section 5). No counterexample appears.

(2) *Exotic quantum groups.* Assume that $S_N \subsetneq G \subsetneq S_N^+$ exists, and develop the algebraic and analytic theory of G: representations and moments, orbitals, diagonal algebras, coamenability, and the finite case. The aim is twofold: having this theory available, should such a G exist, and finding a contradiction coming from it. This is done in Section 6.

(3) *Partitions.* By Tannakian duality, the intermediate quantum groups $S_N \subset G \subset S_N^+$ correspond to tensor categories $\mathrm{span}(NC) \subset \mathcal{C} \subset \mathrm{span}(P)$, made of linear combinations of partitions. Given such a $\mathcal{C}$, we look at the first level k where it differs from $\mathrm{span}(NC)$, and we write down linear, quadratic and cubic necessary conditions on the new vectors at level k, which we then solve by exact computer algebra, uniformly in N whenever possible. This is our main contribution, developed in Sections 7, 8 and 9.

The maximality conjecture has some interesting ramifications, which we discuss as well:

(A) *Orthogonal case.* Inside Wang's free orthogonal group O_N^+ we have the half-liberation $O_N \subset O_N^* \subset O_N^+$, given by the relations $abc = cba$, and $O_N \subset O_N^*$ is known to be maximal [8]. The conjecture here is that this chain is complete, in the sense that O_N^* is the only intermediate quantum group $O_N \subsetneq G \subsetneq O_N^+$. We discuss this in Sections 10 and 11, by adapting the above approaches (1) and (3), with group duals and twists as seeds, and with pairings in place of partitions.

(B) *Projective versions.* For the projective versions $PO_N \subset H \subset (PO_N^+ = PU_N^+)$, it is known that $PO_N^* = PU_N$ is the unique intermediate projectively easy quantum group [14], and conjecturally it is the unique intermediate quantum group. The new feature here is that H is not necessarily the projective version of a quantum group $O_N \subset G \subset O_N^+$, and the level method must be adapted to this, with fewer symmetries. We discuss these questions in Section 12.

Our main results concern approach (3). The computations behind them are exact, and organized as certificates which can be checked independently, as explained in Section 7. First, the conjecture holds up to level 7:

Theorem A. *Let $N \geq 6$, and $S_N \subsetneq G \subset S_N^+$. Then for any $k \leq 7$ we have*

$$\mathrm{Fix}_G(u^{\otimes k}) = \mathrm{span}\left\{\xi_\pi : \pi \in NC(k)\right\}.$$

Thus the first seven moments of the main character of G are the Catalan numbers, as for S_N^+.

The study of the conjecture level by level goes back to [2], where the notion of easiness level was introduced, and was substantially developed by McCarthy, whose state-space approach to quantum permutations [49] led him to the program in [50, 51], with a number of results at levels ≤ 6. Levels $k \leq 5$ were also announced by Freslon and Speicher [36], and we reprove them. Levels 6 and 7 are new, with a proof uniform in N. At level 8 the method meets real resistance, but still gives a strong structural result:

Theorem B. *Let $7 \leq N \leq 37$, and let $S_N \subsetneq G \subset S_N^+$ first differ from S_N^+ at level 8. Then the elements of $\mathrm{Fix}_G(u^{\otimes 8})$ are, modulo $\mathrm{span}(NC(8))$, invariant under the rotation of the 8 legs by 4 positions.*

Pushing the rotation symmetry further, we can close level 8 computationally, modulo primes, at each $8 \leq N \leq 37$: for these N the conjecture holds up to level 8 (Section 9). Level 9 is harder, with a kernel of dimension 307 against 66, and no half-turn. There a different tool, imported from the orthogonal problem of Section 11, does the job: the candidates split into nine rotation blocks, each of them has a real form, and a positive definite combination of the quadratic constraints on each real form excludes the whole block, by a certificate. This closes level 9, numerically, at $N = 9, \ldots, 13$ and $N = 17, 19, 23$, so that at these N the conjecture holds through level 9 (Theorem 9.24), the highest level reached so far.

Regarding approach (1), among our results is the following statement about graphs, where $G^+(X)$ denotes the quantum automorphism group of a graph X, and where the repetition lemma states that $\langle S_{4m}, S_4^+ \rangle = S_{4m}^+$, with S_4^+ acting on $4m$ points via $u \mapsto u \otimes 1_m$:

Theorem C. *Let X be a vertex-transitive graph with $N \leq 13$ vertices. Then*

$$\langle S_N, G^+(X) \rangle \in \{S_N, S_N^+\},$$

except possibly for the cube, its complement, and four complementary pairs of graphs with 12 vertices, where this holds provided that the repetition lemma mentioned above holds at $m = 2, 3$.

In the orthogonal case, a generation result of Brannan, Collins and Vergnioux [23], namely $\langle O_N, O_{N-1}^+ \rangle = O_N^+$ for $N \geq 4$, allows an induction on N which stops at $N = 3$, where their method breaks down. Our main results concern the anticommutative twist O_N^{-1}, and the levels of $O_N^* \subset G \subset O_N^+$:

Theorem D. *$\langle O_N, O_N^{-1} \rangle$ agrees with O_N^+ up to level $\max(20, 2N)$, and equals O_N^+ for all N, provided that $\langle O_3, O_2^+ \rangle = O_3^+$.*

So the twist problem reduces to the single generation statement at $N = 3$ left open in [23], which we verify up to level 20. As for the levels themselves, the orthogonal version of approach (3), with uncappable vectors and positivity certificates, gives:

Theorem E. *Let $N \geq 3$, and $O_N^* \subset G \subsetneq O_N^+$. If the fixed vectors of G differ from those of O_N^+ at some level ≤ 10, then $G = O_N^*$.*

Many questions remain open, and they are stated along the way, at their natural places. The main problem, of course, is that of going beyond level 9 in approach (3), and beyond level 7 uniformly in N, and here the main difficulty is computational.

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1. Maximality and three approaches

1.1. **Quantum permutation groups.** Let $N \in \mathbb{N}$. A *magic unitary* is a square matrix $u = (u_{ij}) \in M_N(A)$ over a C^*-algebra A whose entries are projections summing to 1 on each row and each column. Wang showed [74] that the universal C^*-algebra

$$C(S_N^+) = C^*\Big((u_{ij})_{i,j=1,\ldots,N} \;\Big|\; u = \text{magic}\Big),$$

with comultiplication $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, counit $\varepsilon(u_{ij}) = \delta_{ij}$ and antipode $S(u_{ij}) = u_{ji}$, defines a compact quantum group S_N^+ in the sense of Woronowicz [75], the *quantum permutation group*. It is the universal compact quantum group acting on the set $\{1, \ldots, N\}$, in the sense that it is the quantum automorphism group of the algebra $\mathbb{C}^N$ endowed with its uniform trace. The commutative quotient of $C(S_N^+)$ is $C(S_N)$, with u_{ij} mapping to the characteristic function of $\{\sigma : \sigma(j) = i\}$. So we have an inclusion of compact quantum groups

$$S_N \subset S_N^+.$$

Small N. For $N=1,2,3$ the entries of a magic unitary automatically commute, and $S_N^+=S_N$ [74]. At $N=4$ things change: $C(S_4^+)$ is infinite-dimensional. A good model comes from the Pauli matrices. The magic unitary whose entries are the rank-one projections onto c_ixc_j, with $x\in SU_2$ and $c_1,\ldots,c_4$ the Pauli matrices, gives a faithful representation $C(S_4^+)\to M_4(C(SU_2))$. This can be used to integrate over S_4^+ [12]. In fact $S_4^+=SO_3^{-1}$, a twist of SO_3, and all the quantum subgroups of S_4^+ can be classified this way [5]. For $N\geq 5$, S_N^+ is not coamenable. One way to see it is that it contains the dual of $\mathbb{Z}_2*\mathbb{Z}_3$, which is a non-amenable group, acting by Fourier matrices on $2+3$ of the N points.

The liberation operation $S_N\to S_N^+$ is best understood in representation theory terms, and more specifically in the context of the easy quantum groups introduced in [16]. In order to explain this, let us start with:

Conventions. We write $P(k)$ and $NC(k)$ for the sets of partitions and noncrossing partitions of $\{1,\ldots,k\}$, and

$$B_k=|P(k)|,\qquad C_k=|NC(k)|=\frac{1}{k+1}\binom{2k}{k}$$

for the Bell and Catalan numbers. Thus $B_0,B_1,B_2,\ldots$ are 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, $\ldots$, and $C_0,C_1,C_2,\ldots$ are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, $\ldots$, the two sequences starting to differ at $k=4$, because of the basic crossing. Scalar products are linear in the first variable: $\langle x,y\rangle=\sum_i x_i\bar{y}_i$ on $\mathbb{C}^N$, $\langle a,b\rangle=\mathrm{tr}(ab^*)$ on matrix algebras, and $\langle a,b\rangle=h(b^*a)$ on $L^2(G)$, with h the Haar state. Thus the matrix of an operator T in an orthonormal basis is $T_{ij}=\langle Te_j,e_i\rangle$, and $\mathrm{Proj}(\xi)=\langle\cdot,\xi\rangle\xi/\|\xi\|^2$ is the orthogonal projection onto $\mathbb{C}\xi$. A dimension sequence is called "Catalan" through level k when it agrees with C_l for all $l\leq k$. Statements resting on computer calculations are marked as such in their headers (*exact*, *computational*, *numerical*), and Section 7.6 explains how the exact ones are certified.

Representation theory. For any $N\geq 4$, the irreducible representations of S_N^+ follow the same fusion rules as those of SO_3 [1]. Their intertwiners are described by partitions, as follows. Let $P(k,l)$ be the set of partitions of k upper legs and l lower legs, and $NC(k,l)\subset P(k,l)$ the set of noncrossing ones. Each $\pi\in P(k,l)$ gives a linear map $T_\pi:(\mathbb{C}^N)^{\otimes k}\to(\mathbb{C}^N)^{\otimes l}$, defined by

$$T_\pi(e_{i_1}\otimes\cdots\otimes e_{i_k})=\sum_{j_1,\ldots,j_l}\delta_\pi\binom{i_1\ldots i_k}{j_1\ldots j_l}e_{j_1}\otimes\cdots\otimes e_{j_l},$$

where $e_1,\ldots,e_N$ is the standard basis of $\mathbb{C}^N$, and where $\delta_\pi=1$ if $\pi\leq\ker\binom{i}{j}$, and $\delta_\pi=0$ otherwise. Here $\ker\binom{i}{j}$ is the partition of the $k+l$ legs obtained by joining two legs when they carry the same index, and $\pi\leq\ker\binom{i}{j}$ means that each block of π is contained in a block of $\ker\binom{i}{j}$, that is, the indices are constant on the blocks of π. With this notation, we have

$$\mathrm{Hom}_{S_N^+}(u^{\otimes k},u^{\otimes l})=\mathrm{span}\big\{T_\pi:\pi\in NC(k,l)\big\},$$

while for S_N one gets the span of all the T_π with $\pi\in P(k,l)$. By Woronowicz's Tannaka–Krein duality [76], the intermediate quantum groups $S_N\subset G\subset S_N^+$ therefore correspond to the tensor categories $\mathcal{C}$ with

$$\mathrm{span}(NC)\subset\mathcal{C}\subset\mathrm{span}(P).$$

Here $\mathcal{C}$ is allowed to contain arbitrary *linear combinations* of partitions, not just partitions themselves.

Easiness. A closed subgroup $S_N\subset G\subset O_N^+$ is called *easy* when its intertwiner spaces are spanned by partitions, in the sense that there are sets $D(k,l)\subset P(k,l)$ such that

$$\mathrm{Hom}_G(u^{\otimes k},u^{\otimes l})=\mathrm{span}\big\{T_\pi:\pi\in D(k,l)\big\}.$$

The axioms of tensor categories translate into the fact that D is a *category of partitions*: it is stable under horizontal and vertical concatenation and under upside-down turning, and it contains the identity and the semicircle [16]. The basic examples are $O_N \subset O_N^+$, coming from all pairings and noncrossing pairings, $S_N \subset S_N^+$, coming from all partitions and noncrossing partitions, and the hyperoctahedral and bistochastic versions $H_N \subset H_N^+$ and $B_N \subset B_N^+$, coming from partitions with blocks of even size, and from singletons and pairings. In each case the liberation operation simply removes the crossings. See [13, 61] for the orthogonal classification, and [70] for the unitary case. The easy quantum groups $S_N \subset G \subset S_N^+$ correspond to the categories of partitions $NC \subset D \subset P$, and these are easy to classify: a crossing partition, together with NC, generates all of P, so $D \in \{NC, P\}$. Thus S_N^+ is the only easy quantum group strictly containing S_N inside S_N^+ [13].

1.2. **The conjecture.** The classification of the quantum subgroups of S_4^+ in [5] contains no quantum group strictly between S_4 and S_4^+. This led Banica and Bichon to the following statement.

Conjecture 1.1 (maximality)**.** *For every N, the inclusion $S_N \subset S_N^+$ is maximal: there is no compact quantum group G with $S_N \subsetneq G \subsetneq S_N^+$.*

Status. The conjecture holds trivially for $N \leq 3$, and it holds for $N = 4$ by [5]. The case $N = 5$ was settled in [2], using Jones' theory of subfactors [41] and the classification of subfactors of index at most 5 and exactly 5 [39, 43]. For $N \geq 6$ it is open.

Three approaches. As explained in the introduction, there are three natural approaches to the conjecture. The first one is via subgroups: given a non-classical $G \subset S_N^+$, decide whether $\langle S_N, G\rangle = S_N^+$. This is the natural source of counterexamples, and it is the subject of Sections 2–5. The second one is via exotic quantum groups: develop the theory of a hypothetical $S_N \subsetneq G \subsetneq S_N^+$, with the hope of reaching a contradiction, as in Section 6. The third one is via partitions: work directly with the categories $\mathrm{span}(NC) \subset \mathcal{C} \subset \mathrm{span}(P)$, level by level, as in [2, 51]. This is our main approach, developed in Sections 7–9.

Ramifications. The same question makes sense in the orthogonal case. Inside Wang's free orthogonal group O_N^+ [73], the half-liberated orthogonal group O_N^*, defined by the relations $abc = cba$, fits into

$$O_N \subset O_N^* \subset O_N^+,$$

and $O_N \subset O_N^*$ is known to be maximal [8]. The conjecture is that O_N^* is the only intermediate object, i.e. that this chain is complete. There are unitary and projective versions as well. As in the permutation case, all of this lies beyond easiness. These questions are discussed in Sections 10–12.

2. Group duals and Fourier families

2.1. **Group duals in S_N^+, after Bichon.**
Conventions. A closed subgroup $G \subset S_N^+$ acts on $C(X) = \mathbb{C}^N$ by $\alpha(e_j) = \sum_i e_i \otimes u_{ij}$, and its fixed vectors at level k are

$$\mathrm{Fix}_G(u^{\otimes k}) = \{\xi : u^{\otimes k}(\xi \otimes 1) = \xi \otimes 1\}, \qquad (u^{\otimes k})_{i_1\ldots i_k, j_1\ldots j_k} = u_{i_1 j_1} \cdots u_{i_k j_k}.$$

For $\pi \in P(k)$ we use the vector $\xi_\pi = T_\pi$, with π viewed as an element of $P(0, k)$, having no upper legs. In practice, this means

$$\xi_\pi = \sum_{i:\, \pi \leq \ker i} e_{i_1} \otimes \cdots \otimes e_{i_k},$$

where $\ker i$ is the partition of $\{1, \ldots, k\}$ obtained by joining two legs when they carry the same index, so that the sum is over the multi-indices i which are constant on the blocks of π. We have $\mathrm{Fix}_{S_N}(u^{\otimes k}) = \mathrm{span}\{\xi_\pi : \pi \in P(k)\}$, and, for $N \geq 4$, $\mathrm{Fix}_{S_N^+}(u^{\otimes k}) = \mathrm{span}\{\xi_\pi : \pi \in NC(k)\}$, of dimension C_k.

Given two closed subgroups $H, K \subset S_N^+$, we denote by $\langle H, K\rangle$ the smallest closed subgroup of S_N^+ containing both. In Tannakian terms, $G = \langle H, K\rangle$ corresponds to the intersection of the categories,

$$\mathcal{C}_G = \mathcal{C}_H \cap \mathcal{C}_K,$$

where $\mathcal{C}_G$ denotes the tensor category of intertwiners of G. Since u is self-conjugate, the intertwiner spaces are determined by the fixed vectors, and so the quantity to compute, for a given $G \subset S_N^+$, is

$$\mathrm{Fix}_{\langle S_N, G\rangle}(u^{\otimes k}) = \mathrm{Fix}_G(u^{\otimes k}) \cap \mathrm{span}\{\xi_\pi : \pi \in P(k)\}.$$

This space contains $\mathrm{Fix}_{S_N^+}(u^{\otimes k})$, and for $N \geq 4$ we have $\langle S_N, G\rangle = S_N^+$ if and only if its dimension is C_k for every k.

For a group dual, $C(\widehat{\Gamma}) = C^*(\Gamma)$ with $\Delta(g) = g \otimes g$. An embedding $\widehat{\Gamma} \subset S_N^+$ is given by a magic unitary u with coefficients $u_{ij} \in C^*(\Gamma)$, satisfying $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. We always assume that the coefficients u_{ij} generate $C^*(\Gamma)$, which means that the morphism $C(S_N^+) \to C^*(\Gamma)$ is surjective, so that $\widehat{\Gamma}$ is a closed quantum subgroup of S_N^+. This is no loss of generality: in general, the u_{ij} generate $C^*(\Gamma_0)$ for a subgroup $\Gamma_0 \subset \Gamma$, and we can replace Γ by Γ_0. The quantum subgroups of $\widehat{\Gamma}$ are the $\widehat{\Lambda}$ for quotients $\Gamma \to \Lambda$, and $\widehat{\Gamma}$ is classical iff Γ is abelian.

Bichon's theorem. A coaction of $\mathbb{C}[\Gamma]$ on $\mathbb{C}^N$ is the same thing as a Γ-grading $\mathbb{C}^N = \bigoplus_g A_g$, where $A_g = \{a : \alpha(a) = a \otimes g\}$. In this language, Bichon proved the following [20, Prop. 5.2, Cor. 5.3]:

- an ergodic grading of $\mathbb{C}^m$ comes from a transitive abelian permutation group;
- consequently, $\widehat{\Gamma} \subset S_N^+$ iff Γ is a quotient of $G_1 * \cdots * G_k$, where the $G_i \subset S_{m_i}$ are transitive abelian groups and $\sum m_i = N$.

Since transitive abelian groups act regularly, G_i can be any finite abelian group, with $m_i = |G_i|$.

We restate Bichon's result in the form used below, with explicit matrices.

Definition 2.1 (Fourier families)**.** A *Fourier family* for Γ is a family $\mathcal{H} = (H_1, \ldots, H_k)$ of finite abelian subgroups of Γ that generate Γ. Repetitions and trivial subgroups are allowed. Its size is $N = \sum_i |H_i|$. The family acts on the points

$$X_{\mathcal{H}} = \widehat{H}_1 \sqcup \cdots \sqcup \widehat{H}_k$$

through the magic unitary $u^{\mathcal{H}} = \bigoplus_i u^{(i)}$, where

$$u^{(i)}_{\chi\psi} = P^{(i)}_{\chi\bar{\psi}}, \qquad P^{(i)}_\rho = \frac{1}{|H_i|} \sum_{h \in H_i} \rho(h)\, h \qquad (\chi, \psi, \rho \in \widehat{H}_i).$$

Equivalently, $u^{(i)} = F_i\, \mathrm{diag}(h)_{h \in H_i}\, F_i^*$, where F_i is the normalized Fourier matrix of H_i, given by $(F_i)_{\chi h} = \chi(h)/\sqrt{|H_i|}$.

Theorem 2.2 (Bichon, restated)**.** *For a Fourier family $\mathcal{H}$, the matrix $u^{\mathcal{H}}$ defines a closed quantum subgroup $\widehat{\Gamma} \subset S_N^+$. Conversely, every closed quantum subgroup $\widehat{\Gamma} \subset S_N^+$ is conjugate by a permutation to $u^{\mathcal{H}}$, for a unique unordered family $\mathcal{H}$. Here H_i is the support of the grading restricted to the i-th orbit.*

Comments. The direct statement is a short computation. The P_ρ are the minimal projections of $\mathbb{C}[H_i]$, rows and columns of $u^{(i)}$ are permutations of $\{P_\rho\}_\rho$, and $\Delta(u_{\chi\psi}) = \sum_\eta u_{\chi\eta} \otimes u_{\eta\psi}$ follows from orthogonality of characters.

The converse is in [20]. The only point to add is the explicit identification. On an orbit O with grading support H, each $A_h \cap C(O)$ is spanned by a nowhere-vanishing function v_h. Normalizing $v_h(x_0) = 1$ at a base point gives $v_h v_k = v_{hk}$. Then $x \mapsto (h \mapsto v_h(x))$ is a bijection $O \simeq \widehat{H}$ which turns the action into the Fourier block. The H_i are intrinsic, being the supports of the grading, so they are invariants under conjugation by S_N. A change of base point translates $\widehat{H}_i$. □

Remark 2.3. Up to relabeling the points, *actions of $\widehat{\Gamma}$ on N points are exactly Fourier families of size N*. This is the right parametrization for our purposes, since $\langle S_N, G\rangle$ is invariant under conjugation by S_N.

Example 2.4.

- $H = \mathbb{Z}_2 = \langle g\rangle$: $u = M(p) = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$ with $p = \frac{1+g}{2}$.
- $H = \mathbb{Z}_n = \langle g\rangle$: u is the circulant matrix built on the spectral projections of g. This is the cyclic case.
- $H = \{e, a, b, c\} \simeq \mathbb{Z}_2^2$, the Klein block. Writing P_x for the projection of the character with kernel $\{e, x\}$ (and $P_e = \frac{1}{4}(e + a + b + c)$), we get

$$u = \begin{pmatrix} P_e & P_a & P_b & P_c \\ P_a & P_e & P_c & P_b \\ P_b & P_c & P_e & P_a \\ P_c & P_b & P_a & P_e \end{pmatrix}.$$

  In general, the pattern of $u^{(i)}$ is the division table $(\chi, \psi) \mapsto \chi\psi^{-1}$ of $\widehat{H}_i$, which is circulant for $\mathbb{Z}_n$, and is not circulant here.

Notation used throughout.

- *Fourier basis.* $f_{(i,h)} \in C(X_{\mathcal{H}})$, for $h \in H_i$, is the function $\chi \mapsto \chi(h)$ on $\widehat{H}_i$, extended by 0. These are orthogonal, $\|f_{(i,h)}\|^2 = |H_i|$, and $u(f_{(i,h)} \otimes 1) = f_{(i,h)} \otimes h$.
- *Labelings.* A labeling of k legs is $a = ((i_1, h_1), \ldots, (i_k, h_k))$ with $h_l \in H_{i_l}$. Write $f_a = \bigotimes_l f_{(i_l,h_l)}$, and $w(a) = h_1 \cdots h_k \in \Gamma$ for its *word* (ordered product).
- *Standard tori.* For finite abelian groups $\mathcal{H} = (H_1, \ldots, H_k)$, the standard torus is $T_{\mathcal{H}} = \widehat{H_1 * \cdots * H_k}$ with its tautological family.

Lemma 2.5 (quotients)**.** *Let $q : \Gamma \to \Lambda$ be a quotient. Then $\widehat{\Lambda} \subset \widehat{\Gamma}$ acts on the same points, and its family is obtained as follows. Each orbit $\widehat{H}_i$ splits into $|\ker(q_{|H_i})|$ orbits, namely the cosets of $q(H_i)^\wedge \subset \widehat{H}_i$. Each of the new orbits carries the subgroup $q(H_i)$.*

Proof. The Λ-grading is $f_{(i,h)} \mapsto q(h)$. Its identity component on $\widehat{H}_i$ consists of the functions constant on cosets of $\ker(q_{|H_i})^\perp = q(H_i)^\wedge$. □

Example 2.6. Sending $\mathbb{Z}_4 \to \mathbb{Z}_2$ in $\mathbb{Z}_4 * \mathbb{Z}_2$ turns the 4-point orbit into two 2-point orbits with the same subgroup, i.e. $u = \mathrm{diag}(M(p), M(p), M(q))$.

2.2. **Reduction to just-non-abelian groups.** We first record two simple principles, valid for arbitrary closed subgroups $G \subset S_N^+$. We say that G *moves at most M points* if, up to a permutation of the coordinates, its magic unitary has the form $u = v \oplus 1_{N-M}$, with v of size M.

Proposition 2.7 (monotonicity and locality)**.** *Let $G \subset S_N^+$ be a closed subgroup, with $N \geq 4$.*

(1) If $K \subset G$ is a closed subgroup, then $\langle S_N, K\rangle \subset \langle S_N, G\rangle$. In particular, if $\langle S_N, K\rangle = S_N^+$, then $\langle S_N, G\rangle = S_N^+$.

(2) If G has a non-classical closed subgroup K which moves at most 5 points, then $\langle S_N, G\rangle = S_N^+$.

Proof. (1) This is clear from the definition of $\langle \cdot, \cdot \rangle$.

(2) Since $\langle S_N, \sigma G \sigma^{-1}\rangle = \sigma\langle S_N, G\rangle\sigma^{-1}$ for $\sigma \in S_N$, we can assume that K moves only the first $M \leq 5$ points, so that $K \subset S_M^+$, embedded in S_N^+ by fixing the last $N - M$ points. As K is not classical and $S_M^+ = S_M$ for $M \leq 3$, we have $M \in \{4, 5\}$. The maximality of $S_M \subset S_M^+$ for $M = 4, 5$, from [2, 5], gives $\langle S_M, K\rangle = S_M^+$. Since $S_M \subset S_N$, we obtain $S_M^+ \subset \langle S_N, G\rangle$, and so $\langle S_N, G\rangle \supset \langle S_N, S_M^+\rangle = S_N^+$, by [22, Cor. 3.4] for $N \geq 5$, and trivially for $N = 4$. □

For a group dual $\widehat{\Gamma} \subset S_N^+$, given by a Fourier family $\mathcal{H} = (H_1, \ldots, H_k)$, the closed quantum subgroups are the duals $\widehat{\Lambda}$ of the quotients $q : \Gamma \to \Lambda$. By Lemma 2.5, $\widehat{\Lambda}$ moves exactly the points of the orbits $\widehat{H}_i$ with $q(H_i) \neq 1$, so their number is $\sum_{q(H_i)\neq 1} |H_i|$. Thus, by monotonicity, we are led to look for small quotients of Γ which are still non-abelian. This suggests the following notion.

Definition 2.8. A group Γ is called *just-non-abelian* (JNA) if it is non-abelian, and all its proper quotients are abelian. Equivalently, since Γ/K is abelian if and only if $K \supset \Gamma'$, where $\Gamma' = [\Gamma, \Gamma]$ is the derived subgroup, Γ is JNA when $\Gamma' \neq 1$, and every nontrivial normal subgroup of Γ contains Γ'.

Lemma 2.9. *Every non-abelian finitely generated group has a JNA quotient.*

Proof. Let $\Gamma = \langle g_1, \ldots, g_n \rangle$. A normal subgroup $K \subset \Gamma$ gives a non-abelian quotient Γ/K if and only if $[g_a, g_b] \notin K$ for some pair (a, b). The set of such normal subgroups contains $\{1\}$, and is inductive. Indeed, given a chain (K_t) of such subgroups, since there are finitely many pairs, there is a pair (a, b) with $[g_a, g_b] \notin K_t$ for cofinally many t, and hence for all t, so that $[g_a, g_b] \notin \bigcup_t K_t$. By Zorn's lemma we obtain a maximal such K, and then Γ/K is JNA. □

Proposition 2.10. *Let $N \geq 4$. If $\langle S_N, \widehat{\Lambda} \rangle = S_N^+$ for all JNA groups Λ and all their Fourier families of size N, then $\langle S_N, \widehat{\Gamma} \rangle = S_N^+$ for all non-classical group duals $\widehat{\Gamma} \subset S_N^+$. Moreover, when Γ is residually finite, for instance a free product of finite abelian groups, as for the standard tori $T_{\mathcal{H}}$, the JNA quotient Λ used for $\widehat{\Gamma}$ can be chosen finite.*

Proof. Let $\widehat{\Gamma} \subset S_N^+$ be non-classical, given by a Fourier family $\mathcal{H} = (H_1, \ldots, H_k)$. Then Γ is non-abelian, and finitely generated, by the finite subgroups H_i. By Lemma 2.9, it has a JNA quotient $q : \Gamma \to \Lambda$. The quantum group $\widehat{\Lambda} \subset \widehat{\Gamma}$ acts on the same N points, through the family $(q(H_1), \ldots, q(H_k))$ split as in Lemma 2.5, which generates Λ. By assumption $\langle S_N, \widehat{\Lambda} \rangle = S_N^+$, and so $\langle S_N, \widehat{\Gamma} \rangle = S_N^+$, by monotonicity. If Γ is residually finite, pick generators g_a, g_b with $[g_a, g_b] \neq 1$, and a normal subgroup K_0 of finite index not containing $[g_a, g_b]$. Then Γ/K_0 is finite and non-abelian, and applying Lemma 2.9 to it gives a finite JNA quotient of Γ. Finally, free products of finite groups are residually finite. □

The finite JNA groups are well known, and can be described as follows.

Proposition 2.11. *A finite group Γ is JNA if and only if $\Gamma' \neq 1$ is its unique minimal normal subgroup. This happens if and only if Γ belongs to one of the following three classes, which are mutually exclusive:*

(a) Central type: *Γ is a p-group with $|\Gamma'| = p$ and cyclic center.*
(b) Affine type: *$\Gamma = \mathbb{F}_p^m \rtimes A$, with $A \subset \mathbb{F}_{p^m}^\times$ a nontrivial subgroup not contained in a proper subfield, acting by multiplication.*
(c) Non-abelian type: *$\Gamma' \simeq T^m$ with T non-abelian simple, Γ' is a minimal normal subgroup of Γ, and its centralizer is trivial, $C_\Gamma(\Gamma') = 1$.*

Proof. If Γ is JNA, every minimal normal subgroup contains $\Gamma' \neq 1$, hence equals it. Conversely, if $\Gamma' \neq 1$ is the unique minimal normal subgroup, every nontrivial normal subgroup contains it. As for the classification, the solvable case goes back to Newman [53, 54], and the general case, including the non-solvable one, is treated by Jain and Shukla [40], who also review Newman's results. In any case the argument is short. $V = \Gamma'$ is a minimal normal subgroup, so $V \simeq T^m$ with T simple. If V is central, all subgroups of $Z(\Gamma)$ contain V, and Γ is nilpotent, which leads to (a). If $V = \mathbb{F}_p^m$ is not central, the abelian group $\Gamma/C_\Gamma(V)$ acts faithfully and irreducibly on V, so it embeds in $\mathbb{F}_{p^m}^\times$ by Schur's lemma, and a coprime action argument gives $C_\Gamma(V) = V$, which leads to (b). If V is non-abelian, $C_\Gamma(V)$ is normal and meets V trivially, so it is trivial, which is (c). The converse is a direct check. □

Example 2.12. Here are some examples of finite JNA groups.

(a) The dihedral group D_4 of order 8, the quaternion group Q_8, the extraspecial groups $p_{\pm}^{1+2n}$, and the Pauli groups $\mathbb{Z}_4 \circ 2^{1+2n}$.

(b) The groups S_3, $A_4 = \mathbb{F}_2^2 \rtimes \mathbb{Z}_3$, the dihedral groups $D_p = \mathbb{F}_p \rtimes \mathbb{Z}_2$ with p odd prime, and $\mathbb{F}_7 \rtimes \mathbb{Z}_3$, $\mathbb{F}_5 \rtimes \mathbb{Z}_4$.

(c) The non-abelian simple groups, the symmetric groups S_n with $n \geq 5$, the groups $PGL_2(q)$ with $q \geq 4$, and the wreath products $T \wr \mathbb{Z}_m$, with T non-abelian simple.

In case (c), the minimality of Γ' cannot be dropped. For instance $\Gamma = A_5 \times A_5$ satisfies $\Gamma' \simeq A_5^2$ and $C_\Gamma(\Gamma') = 1$, but it is not JNA, having A_5 as a quotient.

For group duals, being JNA has the following consequence. Given a Fourier family $\mathcal{H} = (H_1, \ldots, H_k)$, we denote by $n(\mathcal{H}) = \sum_{H_i \neq 1} |H_i|$ the number of points moved by the corresponding action.

Corollary 2.13 (locality for JNA groups)**.** *Let Γ be JNA, and $\widehat{\Gamma} \subset S_N^+$ be given by a Fourier family $\mathcal{H}$. Then $\widehat{\Gamma}$ is its only non-classical closed quantum subgroup. Thus Proposition 2.7 (2) applies to $\widehat{\Gamma}$ if and only if $n(\mathcal{H}) \leq 5$.*

Proof. The closed quantum subgroups of $\widehat{\Gamma}$ are the duals of the quotients of Γ, and the proper quotients are abelian, with classical duals. The second assertion follows, the number of points moved by $\widehat{\Gamma}$ being $n(\mathcal{H})$. □

Observe that $n(\mathcal{H})$ counts all the nontrivial blocks of $\mathcal{H}$, even when some of them already generate Γ: the family $(\langle(12)\rangle, \langle(23)\rangle, \langle(13)\rangle)$ of S_3 has $n(\mathcal{H}) = 6$, and is not covered by locality. So what remains to be tested are the JNA groups with Fourier families having $n(\mathcal{H}) \geq 6$, and all three classes occur below. In class (a) we will meet D_4, Q_8, 3_+^{1+2}, and the magic square group 2_+^{1+4} of Proposition 3.6. In class (b) we will meet S_3, A_4, D_5 and $\mathbb{F}_7 \rtimes \mathbb{Z}_3$, see Example 2.19. In class (c) we will meet A_7, as the JNA quotient of the solution group $3.A_7$ of [33], acting on 210 points, see Proposition 3.7. Class (c) has much smaller actions as well, for instance A_5 acting on 6 points through the family $(\langle(123)\rangle, \langle(345)\rangle)$, see Proposition 2.24.

2.3. **The laws lemma for abelian blocks.** Fix a Fourier family $\mathcal{H}$ for Γ, and use the notation of Section 2.1.

Proposition 2.14. *(1)* $\mathrm{Fix}_{\widehat{\Gamma}}(u^{\otimes k}) = \mathrm{span}\{f_a : w(a) = e\}$.

(2) Write $\xi_\pi = \sum_a c_\pi(a) f_a$. Then $c_\pi(a) \neq 0$ iff each block b of π carries a single orbit index i_b and satisfies $\prod_{l \in b} h_l = e$ in H_{i_b}. In that case $c_\pi(a) = \prod_{b \in \pi} |H_{i_b}|^{1-|b|}$.

Proof. (1) $u^{\otimes k}(f_a \otimes 1) = f_a \otimes w(a)$, and distinct group elements are linearly independent.

(2) Use $e_{(i,\chi)} = |H_i|^{-1} \sum_h \overline{\chi(h)} f_{(i,h)}$ in $\sum_x e_x^{\otimes b}$, block by block, and then $\sum_\chi \chi(h) = |H_i| \delta_{h,e}$. □

Definition 2.15. Two labelings are *equivalent* if they differ by a bijection σ of orbit indices together with isomorphisms $\theta_i : H_i \to H_{\sigma(i)}$ applied to the labels. A *pattern* is an equivalence class. Concretely, a pattern consists of:

- the partition κ of the legs by orbit;
- for each block of κ, an abelian isomorphism type with labels in it, up to automorphisms.

A pattern is *balanced* if the labels multiply to e in each block of κ. It is *active* if some realization has $w(a) \neq e$. A balanced inactive pattern is a *law*: an identity holding for all choices of distinct orbit subgroups of the given types, and all identifications. A pattern is *freely trivial* if its words are trivial in the free product $H_1 * \cdots * H_k$, for all its realizations. For instance, on a single orbit, the labels (h, h^{-1}) give a freely trivial pattern. Freely trivial patterns are laws for every Γ, and we call the other laws *nontrivial*.

Lemma 2.16 (laws lemma)**.** *For $\xi \in \operatorname{span}\{\xi_\pi\}$:*

(1) c_ξ is constant on patterns and vanishes off balanced ones;

(2) $\xi \in \operatorname{Fix}_{\langle S_N, \widehat{\Gamma}\rangle}$ iff c_ξ vanishes on all active patterns.

Hence the joint intertwiner spaces depend only on the multiset of types of the H_i and on the set of active patterns.

Proof. The vectors ξ_π are invariant under S_N. Three kinds of permutations act diagonally, up to phases, on the Fourier basis:

- translations of $\widehat{H}_i$, which multiply $f_{(i,h)}$ by $\overline{\psi(h)}$; these give balancedness;
- automorphisms of H_i;
- swaps of isomorphic orbits.

The last two generate the equivalence of labelings. Then apply Proposition 2.14(1). □

Corollary 2.17. *If $(\Gamma, \mathcal{H})$ has no nontrivial laws with $\leq k$ legs, then up to level k the joint quantum group $\langle S_N, \widehat{\Gamma}\rangle$ has the same intertwiners as $\langle S_N, T_{\mathcal{H}}\rangle$. Indeed, in the torus $T_{\mathcal{H}}$ the laws are exactly the freely trivial patterns.*

Remark 2.18. Thus, for group duals, the question becomes that of knowing *which short balanced patterns are nontrivial laws*. If there are no nontrivial laws with $\leq k$ legs, then $\widehat{\Gamma}$ behaves up to level k like the standard torus, which is free in the relevant sense. So a counterexample to the conjecture should be looked for among JNA groups having many short nontrivial laws.

2.4. **First candidates.**

Example 2.19. The following are JNA groups with Fourier families of size $N \geq 6$, so none of them is killed by locality. The list is illustrative, not exhaustive.

Γ	class	family	N
S_3	(b)	$\langle(12)\rangle, \langle(23)\rangle, \langle(13)\rangle$	6
D_4	(a)	$\langle s\rangle,\ \langle r\rangle \simeq \mathbb{Z}_4$	6
D_4	(a)	$\langle sr\rangle,\ \{1, r^2, s, sr^2\}$ (Klein)	6
A_4	(b)	$\langle(123)\rangle, \langle(234)\rangle$	6
3_+^{1+2}	(a)	$\langle x\rangle, \langle y\rangle \simeq \mathbb{Z}_3$	6
$\mathbb{F}_7 \rtimes \mathbb{Z}_3$	(b)	two complements $\mathbb{Z}_3$	6
D_5	(b)	$\langle s\rangle, \langle sr\rangle, \langle sr^2\rangle$	6
A_4	(b)	$\langle(123)\rangle,\ V_4$ (Klein)	7
D_5	(b)	$\langle s\rangle,\ \langle r\rangle \simeq \mathbb{Z}_5$	7
Q_8	(a)	$\langle i\rangle, \langle j\rangle \simeq \mathbb{Z}_4$	8
D_4	(a)	$\{1, r^2, s, sr^2\},\ \{1, r^2, sr, sr^3\}$ (two Klein)	8
2_+^{1+4}	(a)	nine $\mathbb{Z}_2$'s (magic square)	18

2.5. **Computations.**

Remark 2.20 (method). The program `duals.py` works as follows.

- Input: a finite group Γ, given by permutations or matrices, together with a Fourier family.
- It enumerates the labelings that are balanced on each orbit and have nontrivial word.
- For each of them it computes the row $(c_\pi(a))_{\pi \in P(k)}$ given by Proposition 2.14.
- It returns
$$\dim \operatorname{Fix}_{\langle S_N, \widehat{\Gamma}\rangle}(k) = \dim \operatorname{span}\{\xi_\pi\} - \operatorname{rank}\,(c_\pi(a))_{a \text{ bad}, \pi},$$
with the rank computed exactly modulo the prime $p = 1048573$. Only the crossing columns are kept, the noncrossing ones vanishing identically, and the rank is computed after a random compression $M = RC$, which can only lower it.

Remark 2.21 (validation). This computation was checked in several ways.

- It agrees with an independent dense computation, using the Haar projection in the regular representation, on four small cases at $k \leq 4$. These include a Klein block for D_4 at $N = 6$ and for A_4 at $N = 7$.
- It gives Bell numbers $2, 5, 15$ for the classical Klein group acting regularly on 4 points, as it should.
- It reproduces $14, 42$ for the magic square at $k = 4, 5$.

Proposition 2.22 (exact)**.** *For the* 11 *actions of Example 2.19 other than the magic square, and for* $k = 4, 5, 6, 7, 8$,

$$\dim \mathrm{Fix}_{\langle S_N, \widehat{\Gamma}\rangle}(u^{\otimes k}) = 14,\ 42,\ 132,\ 429,\ 1430,$$

the Catalan numbers. So none of them is a counterexample through level 8*. This covers classes (a) and (b), cyclic and Klein blocks, and families with three blocks on* 6 *points.*

Each run takes about a second for $k \leq 7$, and between 20 and 40 seconds for $k = 8$. Although the ranks are computed modulo p, after a random compression, the result is exact. Indeed, the coefficients $c_\pi(a)$ are rational, with denominators products of the orders $|H_i|$, which are prime to p. Thus the rank modulo p, and a fortiori that of the compressed matrix, is at most the rank over $\mathbb{Q}$, so the computed dimension is an upper bound for the true one. On the other hand, the true dimension is at least C_k, because $\mathrm{Fix}_{S_N^+}(u^{\otimes k})$ is contained in it. So a Catalan output proves equality. Here $\dim \mathrm{span}\{\xi_\pi\}$ must be taken exactly; for $k > N$ it is the number of partitions of $\{1, \ldots, k\}$ with at most N blocks.

2.6. **Conclusions.** As a conclusion, in this section we used the group duals $\widehat{\Gamma} \subset S_N^+$ as seeds, in the search for a counterexample to the maximality conjecture of the form $S_N \subsetneq \langle S_N, \widehat{\Gamma}\rangle \subsetneq S_N^+$. Here is what we found.

Structure. By Bichon's theorem, the group duals in S_N^+ are exactly the Fourier families of finite abelian subgroups generating Γ. Non-cyclic orbits do occur, the simplest being the Klein block.

Reductions. By monotonicity and locality, it is enough to consider JNA groups, and their Fourier families moving at least 6 points. These groups are well known, and come in three classes: central type (a), affine type (b), and non-abelian type (c). Moreover, by the laws lemma, the joint intertwiners only depend on which short balanced patterns are nontrivial laws, and a group dual with no nontrivial laws up to k legs behaves up to level k like the corresponding free torus.

Computations. The 11 actions of Example 2.19 from classes (a) and (b), with $N = 6, 7, 8$ and including non-cyclic orbits, all give the Catalan numbers through level 8. Thus, in all the cases we tested, $\widehat{\Gamma}$ is absorbed by S_N, as far as we can see, and no counterexample appears. Further group duals, coming from graphs, will appear in Section 3, with the same outcome.

Outlook. All this suggests the following statement, which would be of independent interest, independently of the maximality conjecture.

Conjecture 2.23. *For every non-classical group dual* $\widehat{\Gamma} \subset S_N^+$ *we have* $\langle S_N, \widehat{\Gamma}\rangle = S_N^+$.

By Proposition 2.10, it is enough to prove this for the JNA groups Γ, and their Fourier families with $n(\mathcal{H}) \geq 6$. The evidence consists of the 11 actions of Example 2.19, of the magic square group (class (a), $N = 18$), of the four class (c) actions at $N = 6$ of Proposition 2.24 below, and of A_7 (class (c), $N = 210$, no laws of length ≤ 8), all Catalan through level 8. The natural tool is the laws lemma, which reduces everything to the question of which short balanced patterns are laws. Class by class the laws are known, being central of order p for (a), and metabelian for (b). So a strategy would be to find, for classes (a) and (b) and at each level k, enough active patterns to kill all the crossing combinations, using the explicit coefficients $c_\pi(a) = \prod_b |H_{i_b}|^{1-|b|}$ of Proposition 2.14. A first target is the case where all the blocks are $\mathbb{Z}_2$.

On the computational side, the smallest actions of class (c) occur at $N = 6$, and we have tested them as well.

Proposition 2.24 (exact)**.** *For the following actions of class (c), with $N = 6$, and for $k = 4, 5, 6, 7, 8$, the dimensions $\dim \mathrm{Fix}_{\langle S_N, \widehat{\Gamma}\rangle}(u^{\otimes k})$ are the Catalan numbers $14, 42, 132, 429, 1430$:*

- *A_5 with $(\langle(123)\rangle, \langle(345)\rangle) \simeq (\mathbb{Z}_3, \mathbb{Z}_3)$;*
- *A_5 with three $\mathbb{Z}_2$'s, generated by $(23)(45)$, $(14)(25)$, $(12)(45)$;*
- *A_5 with $(\mathbb{Z}_2, \mathbb{Z}_2^2)$, given by $\langle(15)(23)\rangle$ and the Klein group on $\{1, 2, 3, 4\}$;*
- *S_5 with $(\langle(1234)\rangle, \langle(45)\rangle) \simeq (\mathbb{Z}_4, \mathbb{Z}_2)$.*

This was computed with `duals.py`, as in Proposition 2.22, in about 20 seconds per action at $k = 8$. Class (c) groups have the fewest short laws, so by the laws lemma they should be the "most free", and this is what we find. As a continuation, it would be interesting to test A_5 at $N = 7, 8$, for instance with $(\mathbb{Z}_2, \mathbb{Z}_5)$, then $PSL_2(7)$, and so on. Note that A_5, being $(2, 3)$-generated, also acts on $2 + 3 = 5$ points, but this action is absorbed by maximality at $N = 5$, and says nothing about the actions on 6 points. The computation is cheap, of the same size as those of Proposition 2.22.

Finally, Proposition 2.10 reduces the problem to the finite JNA groups only when Γ is residually finite. This leads to the following question.

Question 2.25. *Does every non-abelian group generated by finitely many finite abelian subgroups have a finite non-abelian quotient? If not, which infinite JNA groups occur as group duals in S_N^+, with $n(\mathcal{H}) \geq 6$?*

The standard tori $T_{\mathcal{H}}$ are free products of finite groups, hence residually finite, but their quotients need not be. An infinite simple group generated by finitely many finite abelian subgroups would be JNA, with no finite JNA quotient, and if such groups exist with only two or three small blocks, then their duals are test cases that no finite computation of the above type sees. Note, however, that an infinite simple quotient of $\mathbb{Z}_2 * \mathbb{Z}_3$ would act on 5 points only, and be absorbed. So the question really concerns the case $n(\mathcal{H}) \geq 6$, and is mostly one of group theory.

3. Quantum automorphisms of graphs

Recall that we are looking for seeds: non-classical quantum subgroups $K \subset S_N^+$, as small as possible, such that $\langle S_N, K\rangle$ might fail to be S_N^+. The group duals were the first natural family. The second one consists of the quantum automorphism groups of graphs, and of their quantum subgroups.

3.1. **Quantum automorphism groups of graphs.** Let X be a finite graph with vertex set $\{1, \ldots, N\}$ and adjacency matrix $A = A_X$. Its quantum automorphism group $G^+(X) \subset S_N^+$ is given by the relations $uA = Au$ on the fundamental magic unitary. Equivalently, it is the largest quantum subgroup of S_N^+ whose coaction on $C(X) = \mathbb{C}^N$ commutes with A, viewed as an operator on $C(X)$. Its classical version is the automorphism group $G(X) = G^+(X) \cap S_N$. The graph is said to have *quantum symmetry* when $G^+(X) \neq G(X)$.

For many examples and computations of $G^+(X)$ see [63, 65] and the references there. For each such graph, $\langle S_N, G^+(X)\rangle$ is a candidate for an exotic intermediate quantum group. Monotonicity and locality (Proposition 2.7) reduce the question to the quantum subgroups of $G^+(X)$. The natural ones are group duals.

3.2. **Disjoint automorphisms.** Schmidt's criterion [64] says that a graph with two non-trivial automorphisms of disjoint supports has quantum symmetry. In the language of Section 2, this becomes an explicit group dual inside $G^+(X)$.

Proposition 3.1. *Let $C_1, \ldots, C_r \subset G(X)$ be abelian groups of automorphisms whose supports* $\operatorname{supp} C_i$ *are pairwise disjoint. Let*

$$\Gamma = \widehat{C_1} * \cdots * \widehat{C_r}.$$

Then $\widehat{\Gamma} \subset G^+(X)$, acting by the Fourier family with the following blocks:

- *one block for each C_i-orbit $O \subset \operatorname{supp} C_i$, with subgroup $H_O = \operatorname{Stab}_{C_i}(O)^\perp \subset \widehat{C_i}$;*
- *trivial blocks at the points outside $\bigcup_i \operatorname{supp} C_i$.*

This group dual is non-classical as soon as two of the C_i are nontrivial.

Proof. On $C(\operatorname{supp} C_i)$, consider the $\widehat{C_i}$-grading coming from the classical action of C_i: the component of degree χ consists of the functions f with $f \circ c^{-1} = \chi(c) f$ for all $c \in C_i$. Extend this by the trivial grading on the functions supported outside $\bigcup_i \operatorname{supp} C_i$. Since the supports are disjoint, the result is a Γ-grading of $C(X)$:

- products of functions with different supports vanish;
- the unit is a sum of degree-e indicator functions.

So by Theorem 2.2 it is a coaction of $\widehat{\Gamma}$, with the stated orbits. Recall that an abelian group acts on each of its orbits regularly, through $C_i/\operatorname{Stab}(O)$.

It remains to check that A preserves every graded component.

- For $\chi \neq 1$ in $\widehat{C_i}$, the component of degree χ is the χ-eigenspace of C_i acting on all of $C(X)$. Indeed, an eigenfunction for a nontrivial character vanishes at the fixed points of C_i. This eigenspace is preserved by A, since A commutes with C_i.
- The component of degree e is the space of functions invariant under all the C_i, which is also preserved by A.
- All other components are zero.

Hence the coaction commutes with A, and $\widehat{\Gamma} \subset G^+(X)$. If two of the C_i are nontrivial, then Γ is non-abelian. □

Remark 3.2. Some comments on this result.

- With $r = 2$ and $C_1 = \langle \sigma \rangle$, $C_2 = \langle \tau \rangle$, this recovers Schmidt's criterion, and identifies the quantum symmetry that the criterion detects.
- Replacing σ, τ by suitable powers, we may assume that they have prime orders ℓ, ℓ'. Then $\Gamma = \mathbb{Z}_\ell * \mathbb{Z}_{\ell'}$, acting through one $\mathbb{Z}_\ell$-block for each ℓ-cycle of σ and one $\mathbb{Z}_{\ell'}$-block for each ℓ'-cycle of τ. A smaller support only helps.

Corollary 3.3. *If X has two non-trivial automorphisms σ, τ with disjoint supports and* $|\operatorname{supp}\sigma| + |\operatorname{supp}\tau| \leq 5$*, then $\langle S_N, G^+(X) \rangle = S_N^+$. This applies, for instance, to any graph containing two disjoint pairs of twins, i.e. pairs of vertices with the same neighborhood.*

Proof. By Proposition 3.1 and monotonicity, it is enough to treat the group dual. It moves at most 5 points and is non-classical. So locality applies (Proposition 2.7): maximality for $N \leq 5$, together with [22, Cor. 3.4]. □

When $|\operatorname{supp}\sigma| + |\operatorname{supp}\tau| \geq 6$, locality does not apply. The group dual of Proposition 3.1 then has *repeated* blocks. For instance, $\sigma = (12)(34)$ and $\tau = (56)$ give the family $(\langle x \rangle, \langle x \rangle, \langle y \rangle)$ of $\mathbb{Z}_2 * \mathbb{Z}_2$, i.e. $u = \operatorname{diag}(M(p), M(p), M(q))$. Such actions have no quantum subgroups with smaller support. By Proposition 2.10 we pass to JNA quotients of $\mathbb{Z}_\ell * \mathbb{Z}_{\ell'}$, and test them. The smallest cases are those where the two supports have total size 6, 7 or 8, so that $N \leq 9$, and where the quotients are the smallest JNA groups of Example 2.19, namely S_3, D_4, D_5 and A_4. The test itself is the one of Section 2, with the same program, a family with repeated blocks being just a family with repetitions, and the outcome is the same as before.

Proposition 3.4 (computational)**.** *For the following group duals with repeated blocks, the joint dimensions with S_N are Catalan for $4 \le k \le 7$ (for $4 \le k \le 6$ when $N = 9$):*

Γ	*family (blocks)*	*from* (σ, τ)	N
S_3	$\langle(12)\rangle^2$, $\langle(23)\rangle$	(12)(34), (56)	6
D_4	$\langle s\rangle^2$, $\langle sr\rangle$	(12)(34), (56)	6
D_5	$\langle s\rangle^2$, $\langle sr\rangle$	(12)(34), (56)	6
S_3	$\langle(12)\rangle^2$, $\langle(123)\rangle$	(12)(34), 3-cycle	7
S_3	$\langle(12)\rangle^2$, $\langle(23)\rangle^2$	(12)(34), (56)(78)	8
S_3	$\langle(12)\rangle^3$, $\langle(23)\rangle$	(12)(34)(56), (78)	8
A_4	$\langle(123)\rangle^2$, $\langle(234)\rangle$	two 3-cycles, one 3-cycle	9

Here H^m means the subgroup H repeated m times.

So no counterexample arises from disjoint automorphisms in these ranges. A general proof that repeated blocks are always absorbed is open. It would follow from the general statement suggested in Section 2.6.

3.3. **Global quantum symmetry: solution groups.** The graphs that escape Proposition 3.1 are those whose quantum symmetry is *global*. The main known examples come from binary linear systems [33, 62].

- A system $Mx = b$ over $\mathbb{Z}_2$, with n variables, has a solution group Γ generated by involutions $x_1, \dots, x_n$ (and a central involution J). Variables occurring in a common equation commute, and each equation i gives the relation $\prod_j x_j^{M_{ij}} = J^{b_i}$.
- In the graphs of [33, 62], Γ (or a quotient of it) acts on the $2n$ variable vertices by
$$u = \operatorname{diag}\big(M(p_1), \dots, M(p_n)\big), \qquad p_j = \tfrac{1+x_j}{2}.$$
In the language of Section 2, this is the Fourier family $(\langle x_1\rangle, \dots, \langle x_n\rangle)$, with all blocks of type $\mathbb{Z}_2$.

We test this action of $\widehat{\Gamma}$ on $2n$ points as a candidate in its own right.

When all the blocks are of type $\mathbb{Z}_2$, the notions of Section 2.3 simplify, as follows. Since $\mathrm{Aut}(\mathbb{Z}_2)$ is trivial, a label is either trivial or not. So a pattern on k legs is simply a pair (κ, M), where $\kappa \in P(k)$ records which legs lie on the same orbit, that is, carry the same generator, and $M \subset \{1, \dots, k\}$ is the set of legs carrying the nontrivial element. Such a pattern is balanced when each block of κ contains an even number of legs of M, and it is active when some assignment of *distinct* generators to the blocks of κ makes the ordered word along M nontrivial in Γ.

Corollary 3.5 (laws lemma, $\mathbb{Z}_2$-blocks)**.** *Let $\widehat{\Gamma} \subset S_{2n}^+$ act through a Fourier family of $\mathbb{Z}_2$-blocks $\langle x_1\rangle, \dots, \langle x_n\rangle$.*

(1) The spaces $\mathrm{Fix}_{\langle S_{2n}, \widehat{\Gamma}\rangle}(u^{\otimes k})$ depend only on the set of active patterns (κ, M) with k legs.

(2) If Γ has no nontrivial laws with $\le k$ legs, then $\mathrm{Fix}_{\langle S_{2n}, \widehat{\Gamma}\rangle}(u^{\otimes l}) = \mathrm{Fix}_{S_{2n}^+}(u^{\otimes l})$ for all $l \le k$.

Proof. (1) This is Lemma 2.16, with all $H_i \simeq \mathbb{Z}_2$.

(2) By Corollary 2.17, up to level k the quantum group $\langle S_{2n}, \widehat{\Gamma}\rangle$ has the same intertwiners as $\langle S_{2n}, T\rangle$, where T is the dual of $\mathbb{Z}_2^{*n}$, with its tautological family. By killing all but two generators, T contains the dual of $\mathbb{Z}_2 * \mathbb{Z}_2$, which is non-classical and moves 4 points. So, by locality, Proposition 2.7, we have
$$\langle S_{2n}, \widehat{\Gamma}\rangle = \langle S_{2n}, T\rangle = S_{2n}^+$$
at all levels $l \le k$, as claimed. □

Proposition 3.6 (Mermin–Peres magic square [52], $N = 18$)**.** *Here the magic square system has* 9 *variables and* 6 *equations. Its realization by real* 2*-qubit Pauli operators has image* 2_+^{1+4}*, of class (a) in Proposition 2.11. The dual of this image is a quantum subgroup of* $\widehat{\Gamma}$*. For it, the joint dimensions at* $k = 4, \dots, 8$ *are*

$$14,\ 42,\ 132,\ 429,\ 1430,$$

which are Catalan, compared with the Bell numbers $15, 52, 203, 877, 4140$*. This is despite the short law* $(ab)^4 = 1$*, which comes from exponent* 4*.*

The computation was done by exact linear algebra modulo a prime, with the patterns (κ, M) of Corollary 3.5 (`sol.py`, 41 seconds at $k = 8$). The values at $k = 4, 5$ were confirmed independently by `duals.py`. At $k = 8$, exactly one of the 17284 balanced patterns which are not freely trivial is a law, namely $(ab)^4$.

Proposition 3.7 (the A_7 system, $N = 210$)**.** *The system of* [33] *has* 105 *variables, namely the double transpositions of* A_7 *(the edges of the Kneser graph* $K(7, 2)$*), and* 140 *equations. Its solution group is* $3.A_7$*. Up to relabeling, there are* 291 *words of length* ≤ 8 *in which every letter occurs an even number of times and which are not freely trivial, and each of them admits distinct double transpositions with nontrivial product in* A_7 *(`sol.py`). So there are no nontrivial laws of length* ≤ 8*, already in the JNA quotient* A_7*. By Corollary 3.5,* $\langle S_{210}, \widehat{\Gamma}\rangle$ *agrees with* S_{210}^+ *through level* 8*.*

3.4. **Vertex-transitive graphs on at most 13 vertices.** For small vertex-transitive graphs, the existence of quantum symmetry is completely known:

- $N \leq 11$ by Banica and Bichon (see [65]), except the Petersen graph, which was settled by Schmidt [64];
- $N = 12$ by Schanz [63];
- $N = 13$ by Chassaniol [24, 25].

In this range, a graph has quantum symmetry iff it has two disjoint automorphisms [63, Observation 1]. We combine this with Proposition 3.1.

Proposition 3.8 (computational)**.** *We generated all vertex-transitive graphs on* $N \leq 13$ *vertices, as the Cayley graphs of all groups of order* N *together with the Petersen graph and its complement. The numbers obtained agree with the known counts. For each graph we determined the minimal total support* $s = |\operatorname{supp}\sigma| + |\operatorname{supp}\tau|$ *of two automorphisms of prime order with disjoint supports:*

N	4	5	6	7	8	9	10	11	12	13
vertex-transitive graphs	4	3	8	4	14	9	22	8	74	14
no disjoint automorphisms	0	1	2	2	4	5	10	6	32	12
$s = 4$ *(two pairs of twins)*	4	2	6	2	8	4	8	2	20	2
$s = 8$*,* σ, τ *of type* $(2, 2)$					2		4		14	
$s = 12$*,* σ, τ *of type* $(2, 2, 2)$									8	

At $N = 12$*, the* 42 *graphs with disjoint automorphisms form the* 21 *complementary pairs with quantum symmetry found in* [63]*. At* $N = 8$ *the* $s = 8$ *graphs are the cube and its complement. At* $N = 10$ *they are* $2C_5$*,* $K_5 \times K_2$ *and their complements.*

So every remaining case has σ and τ both products of $m \in \{2, 3\}$ disjoint transpositions. By Proposition 3.1, the corresponding group dual is the dual of $\mathbb{Z}_2 * \mathbb{Z}_2$ acting with blocks $(\langle x\rangle^m, \langle y\rangle^m)$. This is the 4-point action $\operatorname{diag}(M(p), M(q))$ with every point repeated m times. We now exploit this structure.

Definition 3.9. The *m-fold repetition* is the morphism $\varphi_m : S_n^+ \to S_{nm}^+$ given by $u \mapsto u \otimes 1_m$. Here the point set is $\{1, \dots, n\} \times \{1, \dots, m\}$, and $(u \otimes 1_m)_{(i,c),(j,c')} = u_{ij}\delta_{cc'}$, which is magic.

Lemma 3.10. *For quantum subgroups $A, B \subset S_n^+$, we have $\varphi_m(\langle A, B\rangle) = \langle \varphi_m(A), \varphi_m(B)\rangle$.*

Proof. For $L \subset S_n^+$, the fundamental representation of $\varphi_m(L)$ is $u_L \otimes 1_m$ on $\mathbb{C}^n \otimes \mathbb{C}^m$, so

$$\mathrm{Hom}_{\varphi_m(L)}(k,l) = \mathrm{Hom}_L(k,l) \otimes B\big((\mathbb{C}^m)^{\otimes k}, (\mathbb{C}^m)^{\otimes l}\big).$$

The intertwiners of a generated quantum group are the intersections of the intertwiners. Since $(U \otimes W) \cap (V \otimes W) = (U \cap V) \otimes W$ for subspaces U, V, we get

$$\mathrm{Hom}_{\langle \varphi_m(A), \varphi_m(B)\rangle} = \big(\mathrm{Hom}_A \cap \mathrm{Hom}_B\big) \otimes B(\cdot,\cdot) = \mathrm{Hom}_{\varphi_m(\langle A,B\rangle)}.$$

Thus the two quantum groups in the statement have the same intertwiners, and so they are equal, by Tannakian duality. □

In order to deal with the remaining cases, we would need to know that repeated copies of S_4^+, together with the symmetric group, generate the whole quantum permutation group. This is a topological generation question, of the type studied by Chirvasitu [27] and by Brannan, Chirvasitu and Freslon [22]. Their methods work well in large dimensions, but usually lead to difficult generation questions in small dimensions, and here we are precisely in the smallest dimension, 4, where S_4^+ is still coamenable, and has a rather special representation theory. For our purposes, we would need the following statement.

Conjecture 3.11 (repetition lemma)**.** $\langle S_{4m}, \varphi_m(S_4^+)\rangle = S_{4m}^+$ *for* $m \geq 1$.

For $m = 1$ there is nothing to prove. For $m = 2, 3$ the joint dimensions are Catalan for $k \leq 7$ (exact, `doubling.py`). The computation uses the fact that a vector $\sum_\pi a_\pi \xi_\pi$ is fixed by $\varphi_m(S_4^+)$ if and only if, for every partition κ of the legs with at most m blocks, $\sum_{\pi \leq \kappa} a_\pi \xi_\pi^{(4)}$ lies in the span of the noncrossing partition vectors at $N = 4$. With the Gram matrix of the vectors $\xi_\pi^{(4)}$, this becomes a rank computation, done modulo a prime, so that a Catalan output is a proof, as in Section 2.

Theorem 3.12. *Let X be a vertex-transitive graph on $N \leq 13$ vertices.*

1. *If X has no quantum symmetry, then $\langle S_N, G^+(X)\rangle = S_N$.*
2. *If X has two pairs of twins with disjoint supports, then $\langle S_N, G^+(X)\rangle = S_N^+$. This covers all graphs with quantum symmetry except the 14 complementary pairs of Proposition 3.8 with $s \in \{8, 12\}$.*
3. *If $s = 8$ and $N \geq 10$ (the $2 + 7$ pairs at $N = 10, 12$), then $\langle S_N, G^+(X)\rangle = S_N^+$.*
4. *For the remaining 5 pairs (the cube at $N = 8$, and the $s = 12$ graphs at $N = 12$), $\langle S_N, G^+(X)\rangle$ contains a copy of $\varphi_m(S_4^+)$, with $m = 2$ or 3, acting on $4m$ of the points. Hence $\langle S_N, G^+(X)\rangle = S_N^+$, provided that Conjecture 3.11 holds for that m.*

In particular, no vertex-transitive graph on at most 13 vertices gives an exotic quantum group $S_N \subsetneq \langle S_N, G^+(X)\rangle \subsetneq S_N^+$, except possibly the cube and the four $s = 12$ pairs, and those only if the repetition lemma fails.

Proof. (1) is clear.

(2) is Corollary 3.3.

(3) Here $m = 2$, and there are $N - 8 \geq 2$ fixed points. Let $G_5 = G_4 \oplus 1$ be the 4-point action of the dual of $\mathbb{Z}_2 * \mathbb{Z}_2$ with one extra fixed point. Then $\langle S_5, G_5\rangle = S_5^+$, by the maximality of $S_5 \subset S_5^+$ [2]. By Proposition 3.1, $G^+(X)$ contains, up to relabeling, $\varphi_2(G_5)$, and by Lemma 3.10, $\langle S_{10}, \varphi_2(G_5)\rangle \supset \varphi_2(S_5^+)$, the diagonal embedding of S_5^+ in S_{10}^+. Then [22, Prop. 3.10 and Ex. 3.9] gives $\langle S_{10}, \varphi_2(S_5^+)\rangle = S_{10}^+$, and finally $\langle S_N, S_{10}^+\rangle = S_N^+$ by [22, Cor. 3.4].

(4) By Proposition 3.1, $G^+(X)$ contains $\varphi_m(G_4)$, extended by the identity on the remaining $N - 4m$ points, with G_4 the dual of $\mathbb{Z}_2 * \mathbb{Z}_2$ on 4 points. Since $S_4 \subset S_4^+$ is maximal, $\langle S_4, G_4\rangle = S_4^+$, so by Lemma 3.10 the quantum group $\langle S_{4m}, \varphi_m(G_4)\rangle$, acting on the support, contains $\varphi_m(S_4^+)$. Given the conjecture, it contains S_{4m}^+, and $\langle S_N, S_{4m}^+\rangle = S_N^+$ by [22, Cor. 3.4]. □

Remark 3.13. For S_M^+ with $M \geq 5$ in place of S_4^+, the repetition statement is known: diagonal embeddings are (M, N)-large, so $\langle S_{KM}, \mathrm{diag}(S_M^+)\rangle = S_{KM}^+$ [22, Ex. 3.9, Prop. 3.10]. Their proof uses the linear independence of noncrossing partitions on a space of dimension $M - 1 \geq 4$, which fails for $M = 4$. So Conjecture 3.11 is exactly the missing case $M = 4$. Note also that Lemma 3.10 applies to any repetition of a locally absorbed action. So the conjecture, for general m, would settle every group dual whose blocks are those of a ≤ 5-point action repeated uniformly.

3.5. **Lessons.**

Remark 3.14. We draw the following lessons from this section.

- *Locality.* Quantum symmetry living on a few points (disjoint automorphisms with small supports) is absorbed by S_N. This follows from [22].
- *Repeated blocks.* Disjoint automorphisms with larger supports give group duals with repeated blocks. These are absorbed in every case computed (Proposition 3.4). For uniform repetitions they reduce to the repetition lemma (Conjecture 3.11), and this covers all vertex-transitive graphs on ≤ 13 vertices (Theorem 3.12).
- *Freeness.* Global quantum symmetry coming from large perfect groups (A_7, class (c)) has no short nontrivial laws. Such groups look free at every computable level, and are absorbed.
- *Short laws are not enough.* The magic square group (2_+^{1+4}, class (a)) has short nontrivial laws, and is still absorbed through level 8.

So graphs, as far as they can be tested, reduce to group duals, and add nothing beyond Section 2. Whether a graph with global quantum symmetry could ever give an exotic $\langle S_N, G^+(X)\rangle$ is the same question as for group duals: whether some JNA group has enough short laws.

4. Cocycle twists of S_N and PU_n

We continue our search for seeds $K \subset S_N^+$. The group duals and the quantum automorphism groups of graphs are absorbed by S_N, as far as we can compute, so we turn to a third family, the cocycle twists of S_N and of its subgroups, and then to the twists of the projective unitary groups PU_n, acting on $N = n^2$ points.

4.1. **Twists of S_N.**

Let $A \subset S_N$ be an abelian subgroup, and let σ be a 2-cocycle on the dual group $\widehat{A}$ with values in roots of unity. The restriction map $C(S_N) \to C(A) = \mathbb{C}[\widehat{A}]$ allows us to view σ as a Hopf 2-cocycle on $C(S_N)$. Twisting by it, in the sense of Drinfeld [32], Doi [31] and Enock–Vainerman [34, 71], gives a finite quantum group S_N^σ:

- its tensor category is equivalent to that of S_N;
- its coaction is a twisted version of the coaction of S_N on $\mathbb{C}^N$.

Twists of this type, with $A \simeq \mathbb{Z}_2^n$ generated by the disjoint transpositions $(2i-1, 2i)$ of S_{2n}, and σ with values ± 1, go back to Nikshych [55] and Bichon [18]. Bichon constructed in this way a family of noncommutative and noncocommutative Kac algebras $\mathcal{O}_{\mathbf{p}}(S_{2n})$, whose representation categories are monoidally equivalent to that of S_{2n}, and whose defining relations contain the magic unitary relations, so that they are quantum permutation groups [18, Def. 1, Thm. 2]. Banica, Bichon and Natale give a sufficient condition for a twist of a quantum permutation algebra to be a quantum permutation algebra, and observe that Bichon's twistings satisfy it [10, Prop. 8.1, Ex. 8.2]. Proposition 4.1 below is our own formulation of what is needed in the setting of S_N.

The twisted coaction lives on the algebra $\mathbb{C}_\sigma^N$. This is $\mathbb{C}^N$ with the product deformed along the $\widehat{A}$-grading coming from the action of A: for homogeneous f, g of degrees χ, ψ,

$$f * g = \sigma(\chi, \psi)\, fg.$$

For S_N^σ to be a quantum *permutation* group, this algebra must be commutative.

Proposition 4.1. *For an orbit O of A, write A_O for the stabilizer of a point of O, and $A_O^\perp \subset \widehat{A}$ for the characters trivial on A_O. Then $\mathbb{C}_\sigma^N$ is commutative iff, for every orbit O, the restriction of σ to $A_O^\perp$ is symmetric, i.e. cohomologically trivial. In that case $\mathbb{C}_\sigma^N \simeq \mathbb{C}^N$ and $S_N^\sigma \subset S_N^+$.*

Proof. The functions supported on different orbits multiply to zero, so it suffices to look at one orbit O. The degrees occurring in $C(O)$ are exactly the characters in $A_O^\perp \simeq \widehat{A/A_O}$, since A acts on O regularly through A/A_O. So $C(O)$ is commutative iff $\sigma(\chi,\psi) = \sigma(\psi,\chi)$ on $A_O^\perp$. For abelian groups and $\mathbb{C}^\times$ coefficients, this is equivalent to $\sigma|_{A_O^\perp}$ being a coboundary. In that case $\mathbb{C}_\sigma^N$ is a commutative finite-dimensional C^*-algebra of dimension N, hence isomorphic to $\mathbb{C}^N$, and the twisted coaction becomes a coaction of S_N^σ on $\mathbb{C}^N$. Since S_N^+ is the universal compact quantum group acting on $\mathbb{C}^N$ [74], we obtain $S_N^\sigma \subset S_N^+$. See also [10, Prop. 8.1, Ex. 8.2], and [18] for Bichon's twists. □

Remark 4.2. Two special cases:

- If every orbit group A/A_O is cyclic, the condition holds automatically, since 2-cocycles on cyclic groups are trivial. This covers the twists of Proposition 4.4 below.
- If A acts *regularly* and σ is non-degenerate, then $\mathbb{C}_\sigma^N$ is a matrix algebra, and S_N^σ is not a quantum permutation group: it acts on M_n, with $N = n^2$. In this case the dimension count of Proposition 4.3 below can give joint dimensions below Catalan, which is impossible inside S_N^+. This case reappears in Section 4.5, read backwards.

Proposition 4.1 allows more than the cyclic case: non-cyclic orbits, such as Klein orbits, are fine as long as σ only couples *different* orbits. This mirrors the role of non-cyclic orbits in Section 2.

4.2. **Twisted intertwiners in the Fourier basis.** Take $A = \mathbb{Z}_m^n$ acting on $N = mn$ points through n disjoint m-cycles. Take the bicharacter

$$\sigma(e_i, e_j) = \omega^{b_{ij}} \quad (i < j), \qquad \omega = e^{2\pi i/m},$$

with b strictly upper triangular. On the orbit q we use the Fourier basis $f_{(q,a)}$, $a \in \mathbb{Z}_m$, as in Section 2. These vectors are weight vectors for A.

Proposition 4.3. *(1) In the coordinates $(q_s, a_s)_{s\le k}$, a partition vector ξ_π is supported on the tuples that are* balanced *for π: on each block, q is constant and $\sum a_s \equiv 0 \pmod m$. This is Proposition 2.14 with $H = \mathbb{Z}_m$.*

(2) The twist acts diagonally in these coordinates:

$$\mathrm{Fix}_{S_N^\sigma}(k) = \Theta\big(\mathrm{Fix}_{S_N}(k)\big), \qquad \Theta = \prod_{s<t} \omega^{b_{q_s q_t} a_s a_t}.$$

(3) Consequently, with $V = \mathrm{span}\{\xi_\pi\}$,

$$\dim \mathrm{Fix}_{\langle S_N, S_N^\sigma\rangle}(k) = \dim(V \cap \Theta V) = 2\dim V - \mathrm{rank}\,[V \,|\, \Theta V].$$

We compute these ranks exactly modulo the prime $p = 1048261 \equiv 1 \pmod{60}$, which contains the roots of unity of orders $2, 3, 4, 5$, after a random compression, which can only lower the rank, so that a Catalan output is a proof (`twists.py`). In practice ω is replaced by a primitive root of unity of the same order in $\mathbb{F}_p$, the matrix V is the one of Section 2, with columns indexed by $P(k)$ and rows by the balanced tuples, and ΘV is obtained from it by rescaling the rows, by the formula in Proposition 4.3(2). The whole computation therefore takes place over $\mathbb{F}_p$, with no floating point at all.

The ground field. All dimensions above are dimensions of intersections of subspaces defined over $\mathbb{Q}(\omega)$. So they do not change over any algebraically closed field of characteristic 0, which is the setting of [10].

4.3. **Results.**

Proposition 4.4 (computational)**.** *For the following twists, the joint dimensions are the Catalan numbers at all the levels listed.*

N	A	*cocycles tested, up to symmetry*	*levels* k
4	$\mathbb{Z}_2^2$, *on two transpositions*	*the non-trivial one* (S_4^τ)	4–6
6	$\mathbb{Z}_2^3$	*all* 3 *sign graphs*	4–7
8	$\mathbb{Z}_2^4$	*all* 10 *sign graphs*	4–7
6	$\mathbb{Z}_3^2$	$b = 1$	4–7
8	$\mathbb{Z}_4^2$	$b = 1, 2$	4–7
9	$\mathbb{Z}_3^3$	*all* 6 *classes up to orbit permutations*	4–6
10	$\mathbb{Z}_5^2$	$b = 1$	4–6

For $m = 2$*, the cocycle is encoded by the* sign graph *on the* n *transpositions, with an edge* i*–*j *iff* $\sigma(e_i, e_j)\sigma(e_j, e_i)^{-1} = -1$.

The $N = 4$ line is a consistency check. There $S_4^\tau \subset S_4^+$ is non-classical, so $\langle S_4, S_4^\tau \rangle = S_4^+$ by maximality at $N = 4$ [5]. The check also confirms that, with our conventions, NC sits inside the twisted category.

Proposition 4.5 (computational, non-cyclic and mixed orbits)**.** *Let* $A = H_1 \times \cdots \times H_r$*, where each* H_q *acts regularly on its own orbit, and let* σ *be a bicharacter coupling different orbits only. Such twists are quantum permutation groups by Proposition 4.1. For the following cases, the joint dimensions are Catalan for* $k = 4, 5, 6, 7$.

N	*orbit groups* H_q	*coupling*
6	$\mathbb{Z}_2^2$, $\mathbb{Z}_2$	*non-trivial*
6	$\mathbb{Z}_4$, $\mathbb{Z}_2$	*non-trivial*
8	$\mathbb{Z}_2^2$, $\mathbb{Z}_2^2$	*rank* 1 *and rank* 2
8	$\mathbb{Z}_2^2$, $\mathbb{Z}_4$	*non-trivial*
8	$\mathbb{Z}_2^2$, $\mathbb{Z}_2$, $\mathbb{Z}_2$	*chain* (1–2–3) *and star (center* $\mathbb{Z}_2^2$*)*

Remark 4.6 (locality, heuristically). A bicharacter is a product of contributions from pairs of orbits. Restricted to the points of two coupled orbits, a twist looks like a twist of a smaller symmetric group, which is non-classical on few points. For two coupled transpositions this is S_4^τ, which is absorbed by locality (Proposition 2.7). We have not proved that S_N^σ actually contains such a piece as a quantum subgroup. When the sign graph has more edges, the cross terms get in the way. So this is a heuristic explanation of the tables, not a theorem.

4.4. **Twists of smaller subgroups.** By monotonicity, smaller seeds are better. So instead of twisting all of S_N, we twist a subgroup L with $A \subset L \subset S_N$. The same construction gives $L^\sigma \subset S_N^\sigma$, and the twisted algebra is the same $\mathbb{C}_\sigma^N$. So Proposition 4.1 applies unchanged, and so does the formula

$$\mathrm{Fix}_{L^\sigma}(k) = \Theta\big(\mathrm{Fix}_L(k)\big).$$

Hence $\xi \in \mathrm{span}\{\xi_\pi\}$ lies in Fix_{L^σ} iff $\Theta^*\xi$ is L-invariant. Invariance under A is automatic, so it suffices to check the other generators of L.

When L normalizes A, its elements act on the Fourier basis by monomial matrices: they permute the orbits and act by automorphisms on the labels, up to phases. The test then stays inside the Fourier coordinates (`twists.py`). For $A = \mathbb{Z}_m^n$ acting on n disjoint m-cycles, the full normalizer is

$$N_{S_N}(A) = (\mathbb{Z}_m \rtimes \mathbb{Z}_m^\times) \wr S_n,$$

the wreath product of the affine group of $\mathbb{Z}_m$ with S_n, and all its elements are of this type.

Remark 4.7 (which L^σ are non-classical). If every element of L preserves the alternating form $\sigma(\chi,\psi)\sigma(\psi,\chi)^{-1}$, the twist has no effect: L^σ is classical, and the joint dimensions are the Bell numbers. We observed this in three cases:

- $L = D_4 \subset S_4$, which normalizes $A = \mathbb{Z}_2^2$, since every automorphism of $\mathbb{Z}_2^2$ preserves its non-degenerate form;
- the triangle sign graph at $N = 6$, with L swapping two orbits;
- two Klein orbits with a rank-1 coupling, with L swapping the orbits.

The useful seeds are therefore $L = A \rtimes \langle g\rangle$ with g changing the form: a permutation of the orbits that is not an automorphism of the sign graph, or the inversion $a \mapsto -a$ on a $\mathbb{Z}_3$-orbit. These are essentially the smallest non-classical seeds of twist type. A joint dimension strictly below Bell proves that L^σ is non-classical.

Proposition 4.8 (computational)**.** *For the following seeds L^σ, with $L = A \rtimes \langle g\rangle$, the joint dimensions are Catalan for $k = 4, 5, 6, 7$. In particular all these L^σ are non-classical, and they are absorbed by S_N through level 7.*

N	A	σ	g
6	$\mathbb{Z}_2^3$	*edge* $\{1,2\}$	*swap of orbits* $2,3$
6	$\mathbb{Z}_2^3$	*path* 1–2–3	*swap of orbits* $1,2$
6	$\mathbb{Z}_3^2$	$b = 1$	*swap of the two orbits*
6	$\mathbb{Z}_3^2$	$b = 1$	$a \mapsto -a$ *on one orbit*
8	$\mathbb{Z}_2^4$	*edge* $\{1,2\}$	*swap of orbits* $2,3$
8	$\mathbb{Z}_2^4$	*edges* $\{1,2\},\{3,4\}$	*swap of orbits* $2,3$
8	$\mathbb{Z}_2^4$	*path* 1–2–3–4	*swap of orbits* $1,2$
8	$\mathbb{Z}_4^2$	$b = 1$	*swap of the two orbits*

As a continuation of this, it would be interesting to twist subgroups which do not normalize A.

Question 4.9. *Compute $\langle S_N, L^\sigma\rangle$ for subgroups $A \subset L \subset S_N$ which do not normalize A, for instance primitive groups.*

When L normalizes A, the test stays in the Fourier coordinates, where the generators of L act by monomial matrices. Otherwise these generators are dense in the Fourier basis, and the invariance test is more expensive, but levels $k \leq 6$ for $N \leq 9$ should be within reach.

So, at the levels we can reach, twists of S_N and of its subgroups behave like group duals and graphs: everything is absorbed, including the smallest twisted seeds. All the twists above, however, live on bicharacters, which only couple pairs of orbits. We now turn to twists of a different kind, where the group being twisted is not a permutation group at all.

4.5. **Twisting an algebra.** The mechanism of Proposition 4.1 needs neither S_N nor $\mathbb{C}^N$. Let G be a compact group, acting by automorphisms on a finite-dimensional C^*-algebra B and preserving a faithful trace tr, and let $A \subset G$ be a finite abelian subgroup. The restricted action grades $B = \bigoplus_{\chi\in\widehat{A}} B_\chi$, and a 2-cocycle σ on $\widehat{A}$ with values in $\mathbb{T}$ defines, as in Section 4.1, the twisted algebra B_σ, with product

$$f * g = \sigma(\chi,\psi) fg \qquad (f \in B_\chi,\ g \in B_\psi),$$

and the twisted compact quantum group G^σ, whose algebra $C(G^\sigma)$ is $C(G)$ with the product twisted by σ on the left and by σ^{-1} on the right, with respect to the gradings coming from the left and right translations by A.

Proposition 4.10. *G^σ coacts on B_σ, preserving* tr*. If B_σ is commutative and* tr *becomes the uniform trace on $B_\sigma \simeq \mathbb{C}^N$, where $N = \dim B$, then $G^\sigma \subset S_N^+$. If moreover G acts faithfully on B, this is a closed quantum subgroup.*

Proof. The coaction $\beta : B \to B \otimes C(G)$ is unchanged as a linear map. It is multiplicative for the twisted products on both sides, as one checks on homogeneous elements, which is the standard twisting of comodule algebras [31], see also [10, Sec. 8]. The invariance of tr is a linear identity, so it survives. When $B_\sigma \simeq \mathbb{C}^N$ with the uniform trace, Wang's universality [74] gives a morphism $C(S_N^+) \to C(G^\sigma)$. Its image contains the coefficients of β, and when G acts faithfully these generate $C(G)$, hence also $C(G^\sigma)$, since the twisted product of two coefficients is a coefficient of the tensor product of the corresponding comodules. □

Proposition 4.1 is the case $B = \mathbb{C}^N$, $G = S_N$. The second item of Remark 4.2 goes from $\mathbb{C}^N$ to a matrix algebra. The example below goes the other way, from a matrix algebra to $\mathbb{C}^N$, and produces quantum permutation groups which are twists of the projective unitary groups.

Let H be a finite abelian group, with $|H| = n$, and $\Lambda = H \times \widehat{H}$, so that $|\Lambda| = n^2 = N$. The Weyl matrices $W_{(i,a)} = \sum_k a(k) E_{k,k+i}$, acting on $\ell^2(H) = \mathbb{C}^n$, satisfy

$$W_\lambda W_\mu = c(\lambda, \mu) W_{\lambda+\mu}, \qquad c\big((i,a),(j,b)\big) = b(i),$$

so that $M_n = C_c^*(\Lambda)$, the twisted group algebra [15, Prop. 2.7]. The commutator of two Weyl matrices is a phase,

$$W_\lambda W_\mu W_\lambda^* = \varepsilon(\lambda, \mu) W_\mu, \qquad \varepsilon(\lambda, \mu) = c(\lambda, \mu)\overline{c(\mu, \lambda)},$$

with ε a non-degenerate alternating bicharacter on Λ, i.e. $\varepsilon((i,a),(j,b)) = b(i)\overline{a(j)}$. So conjugation by the Weyl matrices gives an abelian subgroup $A \simeq \Lambda$ of $PU_n = \mathrm{Aut}(M_n)$, the grading of M_n has the one-dimensional components $\mathbb{C}W_\mu$, and we identify $\widehat{A}$ with Λ through ε. Twisting by $\sigma = \bar{c}$ gives $W_\lambda * W_\mu = W_{\lambda+\mu}$, so that

$$(M_n)_\sigma = \mathbb{C}[\Lambda] \simeq C(\widehat{\Lambda}) = \mathbb{C}^N,$$

with the trace $\mathrm{tr}(W_\lambda) = \delta_{\lambda,0}$ becoming the uniform trace. Since PU_n acts faithfully on M_n, Proposition 4.10 gives a closed quantum subgroup

$$PU_n^\sigma \subset S_N^+, \qquad N = n^2.$$

For $H = \mathbb{Z}_2$ this is $PU_2^\sigma = SO_3^{-1} = S_4^+$ [5]. The classical subgroup A survives in PU_n^σ, and acts on the N points $\widehat{\Lambda}$ by translations, regularly.

4.6. **Flat matrix models.** The quantum groups PU_n^σ appear in the work of Banica and Nechita on flat matrix models [15]. Consider the representation

$$\pi_H : C(S_N^+) \to C(U_n, M_N), \qquad w_{\lambda\mu} \mapsto \big[x \mapsto \mathrm{Proj}(W_\lambda x W_\mu^*)\big],$$

where Proj is taken inside $L(M_n) \simeq M_N$, for the scalar product $\langle a, b\rangle = \mathrm{tr}(ab^*)$. This is a matrix model in the sense that, for each $x \in U_n$, the matrix $(\mathrm{Proj}(W_\lambda x W_\mu^*))_{\lambda\mu}$ is magic: its entries are rank one projections, and on each row and column they project on the orthonormal basis $\{W_\lambda x W_\mu^*/\sqrt{n}\}_\lambda$, resp. $\{W_\lambda x W_\mu^*/\sqrt{n}\}_\mu$, of M_n. For $H = \mathbb{Z}_2$ the Weyl matrices are the Pauli matrices, up to phases, and this is the Pauli representation of S_4^+ mentioned in Section 1, whose magic unitary is

$$\big(\mathrm{Proj}(c_i x c_j)\big)_{ij}, \qquad x \in SU_2.$$

It is shown in [15, Thm. 4.3] that the main character of the Hopf image of π_H has the same law as $x \mapsto |\mathrm{Tr}(x)|^2$ on U_n, which is the main character of PU_n, and it is suggested there that the Hopf image should be a twist of PU_n. This is indeed the case.

Theorem 4.11. *The representation π_H factors as $C(S_N^+) \to C(PU_n^\sigma) \xrightarrow{\theta} C(U_n, M_N)$, with θ injective on the dense Hopf $*$-algebra $\mathcal{O}(PU_n^\sigma)$. Hence the Hopf image of π_H is $PU_n^\sigma \subset S_N^+$.*

Proof. Conventions. The subgroup A acts on PU_n by left and right translations, and this bigrades $\mathcal{O}(PU_n) = \bigoplus_{\alpha,\beta\in\Lambda}\mathcal{O}_{\alpha\beta}$, where $f \in \mathcal{O}_{\alpha\beta}$ means $f(W_\gamma x W_\delta) = \varepsilon(\gamma,\alpha)\varepsilon(\delta,\beta)f(x)$. The coaction $\beta(a)(x) = xax^*$ of PU_n on M_n is $\beta(W_\mu) = \sum_\nu W_\nu \otimes c_{\nu\mu}$, with $c_{\nu\mu}(x) = \mathrm{tr}(W_\nu^* x W_\mu x^*)$, and the relations $W_\gamma^* W_\nu^* W_\gamma = \varepsilon(\gamma,\nu)W_\nu^*$, $W_\delta W_\mu W_\delta^* = \varepsilon(\delta,\mu)W_\mu$ show that $c_{\nu\mu} \in \mathcal{O}_{\nu\mu}$. Comparing the coefficients of $xW_\mu W_{\mu'}x^* = (xW_\mu x^*)(xW_{\mu'}x^*)$ gives

$$c(\mu,\mu')\,c_{\nu,\mu+\mu'} = \sum_{\nu_1+\nu_2=\nu} c(\nu_1,\nu_2)\,c_{\nu_1\mu}c_{\nu_2\mu'}.$$

So the twisted product on $\mathcal{O}(PU_n)$ for which $\beta : (M_n)_\sigma \to (M_n)_\sigma \otimes \mathcal{O}(PU_n^\sigma)$ is multiplicative, as in Proposition 4.10, is $a \cdot b = c(\alpha,\alpha')\overline{c(\beta,\beta')}\,ab$, for $a \in \mathcal{O}_{\alpha\beta}$ and $b \in \mathcal{O}_{\alpha'\beta'}$, see [31, 34]. As for the points, the minimal projections of $(M_n)_\sigma = \mathbb{C}[\Lambda]$ are $e_\lambda = N^{-1}\sum_\beta \overline{\varepsilon(\lambda,\beta)}W_\beta$, with $W_\nu = \sum_\lambda \varepsilon(\lambda,\nu)e_\lambda$, and the magic unitary of PU_n^σ, given by $\beta(e_\mu) = \sum_\lambda e_\lambda \otimes v_{\lambda\mu}$, is therefore

$$v_{\lambda\mu} = \frac{1}{N}\sum_{\nu,\beta}\varepsilon(\lambda,\nu)\overline{\varepsilon(\mu,\beta)}\,c_{\nu\beta}.$$

The embedding. Let $L_a, R_a \in L(M_n)$ be the left and right multiplications by $a \in M_n$, and define $\Phi : \mathcal{O}(PU_n^\sigma) \to C(U_n, L(M_n)) = C(U_n, M_N)$ by

$$\Phi(f)(x) = f(x)\,L_{W_\alpha}R_{W_\beta^*} \qquad (f \in \mathcal{O}_{\alpha\beta}).$$

Since left and right multiplications commute, with $L_aL_b = L_{ab}$ and $R_aR_b = R_{ba}$, we get $L_{W_\alpha}L_{W_{\alpha'}} = c(\alpha,\alpha')L_{W_{\alpha+\alpha'}}$ and $R_{W_\beta^*}R_{W_{\beta'}^*} = R_{(W_\beta W_{\beta'})^*} = \overline{c(\beta,\beta')}R_{W_{\beta+\beta'}^*}$, so $\Phi(a\cdot b) = \Phi(a)\Phi(b)$. Moreover Φ is injective, the N^2 operators $L_{W_\alpha}R_{W_\beta^*}$ being a basis of $L(M_n)$.

The comparison. Write $P_x = \mathrm{Proj}(x) = \sum_{\alpha\beta}p_{\alpha\beta}(x)L_{W_\alpha}R_{W_\beta}$. Since $\mathrm{Tr}(L_aR_b) = \mathrm{Tr}(a)\mathrm{Tr}(b)$, the operators $L_{W_\alpha}R_{W_\beta}$ are orthogonal for the trace of $L(M_n)$, with norms n^2, and since P_x is the rank one projection onto the unit vector x, we get

$$p_{\alpha\beta}(x) = n^{-2}\,\mathrm{tr}(W_\alpha^* x W_\beta^* x^*) = N^{-1}\,\overline{c(-\beta,\beta)}\,c_{\alpha,-\beta}(x),$$

by using $W_\beta^* = \overline{c(-\beta,\beta)}W_{-\beta}$. Now $\mathrm{Proj}(W_\lambda x W_\mu^*) = UP_xU^*$, with $U = L_{W_\lambda}R_{W_\mu^*}$, and the Weyl relations give $UL_{W_\alpha}R_{W_\beta}U^* = \varepsilon(\lambda,\alpha)\varepsilon(\mu,\beta)L_{W_\alpha}R_{W_\beta}$. Thus, by substituting $\beta \to -\beta$ and using $R_{W_{-\beta}} = c(\beta,-\beta)R_{W_\beta^*}$, $c(\beta,-\beta) = c(-\beta,\beta)$ and $\varepsilon(\mu,-\beta) = \overline{\varepsilon(\mu,\beta)}$, we obtain

$$\pi_H(w_{\lambda\mu})(x) = \frac{1}{N}\sum_{\alpha,\beta}\varepsilon(\lambda,\alpha)\overline{\varepsilon(\mu,\beta)}\,c_{\alpha\beta}(x)\,L_{W_\alpha}R_{W_\beta^*} = \Phi(v_{\lambda\mu})(x),$$

with all the phases cancelling exactly. Hence $\pi_H = \Phi\circ q$ on $\mathcal{O}(S_N^+)$, where $q : w \mapsto v$ is the quotient map onto $\mathcal{O}(PU_n^\sigma)$. Since Φ is injective, $\ker\pi_H = \ker q$, which is a Hopf ideal, and so the Hopf image of π_H is PU_n^σ. Equivalently, the Tannakian category of the Hopf image, which consists of the maps T with $T\pi_H(w^{\otimes k}) = \pi_H(w^{\otimes l})T$, see [4], is that of PU_n^σ. □

In particular π_H is inner faithful onto PU_n^σ, and [15, Thm. 4.3] becomes the statement that twisting does not change the dimensions of the fixed point spaces. We have also checked Theorem 4.11 numerically, by verifying that the vectors of Proposition 4.12 below are fixed by π_H at random points $x \in U_n$, up to 10^{-14}, for $H = \mathbb{Z}_3$ at levels $k \leq 4$, and for $H = \mathbb{Z}_4, \mathbb{Z}_2^2$ at level 3.

The fixed vectors have the same shape as in Proposition 4.3. We use the Fourier basis of $\mathbb{C}^N = C(\widehat{\Lambda})$, which is the basis $\{W_\alpha\}$ of $\mathbb{C}[\Lambda]$, and the operator Θ on $(\mathbb{C}^N)^{\otimes k}$, diagonal in this basis,

$$\Theta\,(W_{\alpha_1}\otimes\cdots\otimes W_{\alpha_k}) = \prod_{s<t}\sigma(\alpha_s,\alpha_t)\,W_{\alpha_1}\otimes\cdots\otimes W_{\alpha_k},$$

which is the analogue of the operator Θ of Proposition 4.3, with the cocycle $\sigma = \bar{c}$ on Λ in place of the bicharacter ω^b there. The fixed vectors of PU_n itself are the permutation operators, and the twist moves them by Θ, as follows.

Proposition 4.12. *For $\tau \in S_k$, let $P_\tau \in \mathrm{End}((\mathbb{C}^n)^{\otimes k}) = M_n^{\otimes k}$ be the permutation of the tensor legs, and expand*

$$P_\tau = \sum_\alpha c_\tau(\alpha)\, W_{\alpha_1} \otimes \cdots \otimes W_{\alpha_k}.$$

Then $c_\tau(\alpha) \neq 0$ iff α is balanced on the cycles of τ, i.e. $\sum_{s\in\gamma} \alpha_s = 0$ for each cycle γ, in which case $|c_\tau(\alpha)| = n^{\#\mathrm{cycles}(\tau)-k}$. Moreover

$$\mathrm{Fix}_{PU_n^\sigma}(k) = \Theta\big(\,\mathrm{span}\{c_\tau : \tau \in S_k\}\big),$$

and its dimension is the number of permutations in S_k having no increasing subsequence of length $n+1$.

Proof. By Schur–Weyl duality, $\mathrm{Fix}_{PU_n}(M_n^{\otimes k})$ is spanned by the P_τ. The twist acts on fixed vectors by Θ, exactly as in Proposition 4.3. The coefficients are $c_\tau(\alpha) = \langle P_\tau, W_\alpha\rangle = n^{-k}\mathrm{Tr}(P_\tau\, W_{\alpha_1}^* \otimes \cdots \otimes W_{\alpha_k}^*)$, where $W_\alpha = W_{\alpha_1} \otimes \cdots \otimes W_{\alpha_k}$, the W_α being orthonormal for $\langle a, b\rangle = \mathrm{tr}(ab^*)$. This is a product over the cycles of τ of normalized traces of products of Weyl matrices, which vanish unless the labels sum to zero. The dimension count is the one of [15, Sec. 4]. □

So both families of vectors live on balanced tuples, the partition vectors on the blocks of π (Proposition 4.3), and the twisted vectors on the cycles of τ. The joint dimensions are therefore again $\dim(V \cap \Theta W_0)$, with $V = \mathrm{span}\{\xi_\pi\}$ and $W_0 = \mathrm{span}\{c_\tau\}$, computed as in Proposition 4.3. One more symmetry helps. The symplectic group $Sp(\Lambda)$ of ε acts on $\widehat{\Lambda}$, hence on the N points, and we have checked that it fixes all the vectors Θc_τ, at the levels below. Since it also fixes the partition vectors, all ranks can be computed on the orbit representatives of $Sp(\Lambda)$ acting diagonally on Λ^k, which cuts the size by a factor close to $|Sp(\Lambda)|$, which is $24, 48, 720, 120$ for $H = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2^2, \mathbb{Z}_5$.

4.7. **Results for PU_n^σ.** Before giving the numbers, observe that PU_n^σ sits at the crossing of two maximality problems. The point is that twisting by σ, along the common regular subgroup A, identifies the closed quantum subgroups of S_N^+ containing A with those of its twist containing A, preserving inclusions and hence generation. The twist of S_N^+ along a regular A with non-degenerate $\bar{\sigma}$ is $\mathrm{Aut}^+(M_n, \mathrm{tr})$ [9], the twist of PU_n^σ back is PU_n, and the twist of S_N is the quantum group $S_N^{\bar{\sigma}} \subset \mathrm{Aut}^+(M_n)$ of Remark 4.2. This correspondence is standard, but we include a proof.

Lemma 4.13. *Let G be a compact quantum group with a closed abelian subgroup A, and σ a 2-cocycle on $\widehat{A}$. Then $K \mapsto K^\sigma$ is an inclusion-preserving bijection from the closed quantum subgroups $A \subset K \subset G$ onto the closed quantum subgroups $A \subset K' \subset G^\sigma$, with inverse $K' \mapsto K'^{\bar{\sigma}}$. In particular $\langle K, L\rangle^\sigma = \langle K^\sigma, L^\sigma\rangle$.*

Proof. Let $r : \mathcal{O}(G) \to \mathcal{O}(A)$ be the restriction. The algebra $\mathcal{O}(G^\sigma)$ is $\mathcal{O}(G)$ with the same coalgebra structure, and with the product $a \cdot b = \sum \sigma(r(a_1), r(b_1))\, a_2 b_2\, \sigma^{-1}(r(a_3), r(b_3))$ [31]. If $A \subset K$, then r vanishes on the Hopf ideal $I = \ker(\mathcal{O}(G) \to \mathcal{O}(K))$, so for $a \in I$ only the terms with $a_2 \in I$ survive, and I is an ideal for the twisted product as well, and a Hopf ideal of $\mathcal{O}(G^\sigma)$. The quotient is $\mathcal{O}(K^\sigma)$, and since $\mathcal{O}(A)$ is commutative and cocommutative its twist is itself, so $A \subset K^\sigma$. Twisting back by $\bar{\sigma}$ gives the inverse map, and both maps preserve inclusions, which are inclusions of the ideals I. Since $\langle K, L\rangle$ is the smallest closed subgroup containing K and L, which both contain A, generation is preserved too. □

This gives:

Proposition 4.14. *$\langle S_N, PU_n^\sigma\rangle = S_N^+$ if and only if $\langle S_N^{\bar\sigma}, PU_n\rangle = \mathrm{Aut}^+(M_n, \mathrm{tr})$. In particular, this equality follows both from the maximality of $S_N \subset S_N^+$, and from the maximality of $PU_n \subset PO_n^+ = \mathrm{Aut}^+(M_n)$ in Conjecture 12.4.*

Proof. The equivalence follows from Lemma 4.13, applied to $G = S_N^+$ and to the subgroups S_N and PU_n^σ. For the second assertion we need $PU_n^\sigma \not\subset S_N$ and $S_N^{\bar\sigma} \not\subset PU_n$. The first holds because PU_n^σ, having the fusion rules of PU_n, is infinite, while S_N is finite. For the second, at level 3 the fixed point spaces have dimensions 6 for PU_n and 5 for $S_N^{\bar\sigma}$, for any $n \geq 3$, and a quantum subgroup has at least as many fixed vectors as the ambient quantum group. □

Proposition 4.15 (computational)**.** *For the following H, the joint dimensions $\dim \mathrm{Fix}_{\langle S_N, PU_n^\sigma\rangle}(k)$ are the Catalan numbers $14, 42, 132, 429$ at all the levels listed, and the noncrossing partition vectors lie in $\mathrm{Fix}_{PU_n^\sigma}(k)$.*

N	H	*levels* k	$\dim \mathrm{Fix}_{PU_n^\sigma}(k)$	$\lvert Sp(\Lambda)\rvert$
4	$\mathbb{Z}_2$	4–6	$14, 42, 132$	6
9	$\mathbb{Z}_3$	4–7	$23, 103, 513, 2761$	24
16	$\mathbb{Z}_4$	4–6	$24, 119, 694$	48
16	$\mathbb{Z}_2^2$	4–6	$24, 119, 694$	720
25	$\mathbb{Z}_5$	4–5	$24, 120$	120

All the ranks were computed exactly modulo the prime $p = 1048261 \equiv 1 \pmod{60}$, after a random compression, which can only lower the rank, so that the Catalan outputs are proofs (`twistpu.py`). The rows were reduced to the orbit representatives of $Sp(\Lambda)$, after checking on random samples that $Sp(\Lambda)$ fixes the vectors Θc_τ. For $H = \mathbb{Z}_3$, levels 4 and 5 were also computed without this reduction, with the same results. The convention for Θ matters here. With the coefficients $c_\tau(\alpha) = \langle P_\tau, W_\alpha\rangle$, the twist acts by $\Theta(\alpha) = \prod_{s<t} c(\alpha_s, \alpha_t) = \prod_{s<t} \overline{\sigma(\alpha_s, \alpha_t)}$. With the conjugate convention, the vectors Θc_τ are no longer $Sp(\Lambda)$-invariant, and the joint dimensions drop below Catalan, to 1 and 22 at $k = 4, 5$ for $H = \mathbb{Z}_3$, which is impossible inside S_N^+: this is the matrix algebra case of Remark 4.2. The line $N = 4$ is a consistency check: there $PU_2^\sigma = S_4^+$, and the twisted vectors span exactly the noncrossing partitions.

Remark 4.16. These seeds are far from classical, and quite different from the previous ones. PU_n^σ is infinite, with the fusion rules of PU_n, its fixed point spaces are not spanned by partitions, and already at level 3 they are larger than those of S_N. Its classical part contains the regular subgroup A. Nevertheless, everything is absorbed, through level 7 at $N = 9$, level 6 at $N = 16$ and level 5 at $N = 25$. Note also that, by Proposition 4.14, a counterexample of this type would disprove two conjectures at once.

We end with some questions. The main one is whether everything is absorbed at all levels, that is, whether $\langle S_N, PU_n^\sigma\rangle = S_N^+$ for all $n \geq 3$, with $N = n^2$, and it is our feeling that this is the case. By Proposition 4.14 this is equivalent to $\langle S_N^{\bar\sigma}, PU_n\rangle = \mathrm{Aut}^+(M_n, \mathrm{tr})$, so it would follow from Conjecture 12.4, in the liftable case. A possible route is to find inside the generated quantum group a non-classical piece moving few points, as in Proposition 2.7, for instance a copy of S_4^+ coming from a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ of Λ, when n is even. Computationally, the next steps are level 8 at $N = 9$, level 7 at $N = 16$ and level 6 at $N = 25$, by using the Clifford group $\widehat{\Lambda} \rtimes Sp(\Lambda)$, which fixes all the twisted fixed vectors that we computed, and which should be the classical part of PU_n^σ. Also, all the twists above come from bicharacters, and the next step is to use a non-abelian group of central type G, with $C_\sigma^*(G) \simeq M_n$, and to identify the Hopf image of the flat model $x \mapsto \mathrm{Proj}(g_i x g_j^*)$ coming from the standard basis of $C_\sigma^*(G)$, as in [15, Thm. 2.8], which should be a twist of PU_n by a Drinfeld twist supported on G. For the smallest such groups see [10, 35, 44]. Finally, Bichon's pseudo-twists, which give the quantum groups DC_n^τ at $N = 4$, see [5], are not twists of the whole tensor category, so they may avoid the locality mechanism, and would be worth testing as seeds.

5. Hadamard models and deformations

5.1. **Hadamard models.** Let $H \in M_N(\mathbb{T})$ be a complex Hadamard matrix, with rows $H_1, \ldots, H_N$. The vectors $\xi_{ij} = H_i/H_j$ form a magic basis of $\mathbb{C}^N$, so the rank-one projections $u_{ij} = \text{Proj}(\xi_{ij})$ form a magic unitary in $M_N(\mathbb{C})$. We denote by $G_H \subset S_N^+$ the Hopf image of the corresponding representation $C(S_N^+) \to M_N(\mathbb{C})$, i.e. the smallest quantum subgroup through which it factorizes [4, 11]. For complex Hadamard matrices in general, see [68]. Philosophically, H is "the Fourier matrix of G_H".

We will use these quantum groups G_H as seeds: the question is whether $\langle S_N, G_H\rangle = S_N^+$, as soon as G_H is not classical. The test is computable for two reasons:

- the intertwiners of the Hopf image are the intertwiners of the model, i.e. the vectors with $u^{\otimes k}(\xi \otimes 1) = \xi \otimes 1$;
- these are the vectors fixed by the contraction $T_k = (\text{id} \otimes \text{tr})(u^{\otimes k})$.

So $\dim \text{Fix}_{\langle S_N, G_H\rangle}(k)$ is the multiplicity of the eigenvalue 1 of the compression of T_k to $\text{span}\{\xi_\pi : \pi \in P(k)\}$. The matrix T_k is computed from the Gram matrix of the vectors ξ_{ij}, through cyclic traces [11, Thm. 6.5]. With our convention for scalar products, and with $\xi_r = \xi_{i_r j_r}$, the formula reads

$$\text{tr}(u_{i_1 j_1} \cdots u_{i_k j_k}) = \frac{1}{N^{k+1}} \langle \xi_2, \xi_1\rangle\langle \xi_3, \xi_2\rangle \cdots \langle \xi_1, \xi_k\rangle.$$

Theorem 5.1 ([11, Thm. 6.4])**.** *For a complex Hadamard matrix H, the following are equivalent:*

(1) G_H is classical;

(2) G_H is a group dual;

(3) H is equivalent to a generalized Fourier matrix $F_G = F_{N_1} \otimes \cdots \otimes F_{N_k}$, with $G = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_k}$.

In that case $G_H = G$, acting regularly.

So Hadamard models never produce non-classical group duals. Their non-classical Hopf images lie outside Section 2.

Remark 5.2 (tame and wild matrices). The correspondence $H \mapsto G_H$ suggests that the wilder the matrix, the bigger the quantum group. For our purposes we need *tame* matrices, whose G_H is non-classical but small. The natural tame matrices are the deformations of Fourier matrices. Deciding in general whether $G_H = S_N^+$ is a hard open problem, and we do not attempt it. We only run the joint test on concrete matrices.

5.2. **Regular Hadamard matrices at $N \leq 7$.**

Definition 5.3 ([11])**.** A complex Hadamard matrix is *regular* if the scalar product of any two distinct rows decomposes as a sum of cycles, i.e. of rotated sums of all p-th roots of unity, with p prime.

All the examples below are regular.

- $N \leq 5$. The only complex Hadamard matrices, up to equivalence, are F_2, F_3, F_4^q (the Diţă deformations of $F_2 \otimes F_2$) and F_5 [11, Thm. 5.4].
- $N = 6$. The regular complex Hadamard matrices are [11, Thm. 11.2]:
  - the Tao matrix T_6 [69];
  - the Haagerup family H_6^q [37];
  - the Diţă deformations [30] $F_{23}^{rs} = F_2 \otimes_l F_3$ and $F_{32}^{rs} = F_3 \otimes_l F_2$.

  Here $(h \otimes_l k)_{ia,jb} = h_{ij} l_{aj} k_{ab}$, with a dephased parameter matrix l whose two free entries are $r, s \in \mathbb{T}$.

- $N = 7$. The regular complex Hadamard matrices are F_7, the Petrescu family P_7^q [59, 67], and the family R_7^q of Lampio, Östergård and Szöllősi [46]. This is a 2026 theorem of GPT and Teo, the paper being in preparation.

Remark 5.4 (normalizations). We use the following forms of the matrices above.

- *The matrices T_6 and H_6^q.* We use them as printed in [11, §5].
- *The families at $N = 7$.* With $w = e^{2\pi i/6}$, we use the following forms of P_7^q and R_7^q:

$$P_7^q = \begin{pmatrix} -q & q & w^2 & 1 & w^2 & 1 & w^2 \\ q & -q & w^2 & 1 & 1 & w^2 & w^2 \\ w^2 & w^2 & -w^2 & 1 & w^2 & w^2 & 1 \\ 1 & 1 & 1 & -1 & w^2 & w^2 & w^2 \\ w^2 & 1 & w^2 & w^2 & -\bar{q}w^2 & \bar{q}w^2 & 1 \\ 1 & w^2 & w^2 & w^2 & \bar{q}w^2 & -\bar{q}w^2 & 1 \\ w^2 & w^2 & 1 & w^2 & 1 & 1 & -1 \end{pmatrix} \qquad R_7^q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & w^2 & w^3 & w^3 & w^4 \\ 1 & w & w^4 & w^4 & q & qw^3 & w^2 \\ 1 & w^2 & w^3 & 1 & qw^4 & qw & w^4 \\ 1 & w^3 & w^5 & w^3 & qw^3 & q & w \\ 1 & w^4 & w^2 & qw^2 & q & qw^5 & qw^3 \\ 1 & w^4 & w^2 & qw^5 & qw^2 & qw^3 & q \end{pmatrix}$$

5.3. **Deformed Fourier matrices reduce to group duals.** Let A, B be finite abelian groups and $Q \in M_{A\times B}(\mathbb{T})$. Write G_Q for the Hopf image of the Diţă deformation $F_A \otimes_Q F_B$, and set

$$\begin{aligned} \Gamma_{A,B} &= B^{*A} \Big/ \left\langle \left[c_1^{(i_1)} \cdots c_s^{(i_s)},\, d_1^{(j_1)} \cdots d_s^{(j_s)}\right] = 1 \;\middle|\; \textstyle\sum_r c_r = \sum_r d_r = 0 \right\rangle \\ &\simeq \mathbb{Z}^{(|A|-1)(|B|-1)} \rtimes B. \end{aligned}$$

Here the superscripts index the $|A|$ copies of B.

Theorem 5.5 (Banica–Bichon [6])**.** *$C(G_Q)$ is a Hopf quotient of the crossed coproduct $C^*(\Gamma_{A,B}) \rtimes C(A)$. For generic Q it is isomorphic to it.*

In other words, G_Q is built from the group dual $\widehat{\Gamma_{A,B}}$ and the classical group A, and the deformation parameter Q only decides how much of this survives. What we need is the group-dual part, which survives in all cases, as follows.

Proposition 5.6. *For every Q, the quantum group G_Q contains a group dual $\widehat{\Gamma_Q}$, where Γ_Q is a quotient of $\Gamma_{A,B}$ and $\Gamma_Q = \Gamma_{A,B}$ for generic Q. It acts on the $N = |A||B|$ points through the Fourier family of the $|A|$ copies of B, in the sense of Section 2.*

Sketch. In $C^*(\Gamma_{A,B}) \rtimes C(A)$ the subalgebra $C(A)$ is central. Dividing the Hopf image $L = C(G_Q)$ by the ideal generated by $C(A)^+$ gives a Hopf quotient, which is a quotient of $C^*(\Gamma_{A,B})$, hence a group algebra $C^*(\Gamma_Q)$ (compare the proof of [6, Lemma 4.5]). For generic Q, Theorem 5.5 gives $\Gamma_Q = \Gamma_{A,B}$. Under the counit of $C(A)$, the magic unitary $w_{ia,jb} = u_{ab}^{(i)} v_{ij}$ of [6, Lemma 3.8] becomes $\delta_{ij} u_{ab}^{(i)}$. Each $u^{(i)}$ is the Fourier block of the i-th copy of B. □

Corollary 5.7. *If Q is generic and $B = \mathbb{Z}_2$, then $\langle S_N, G_Q\rangle = S_N^+$. In particular this holds for F_{32}^{rs} with generic r, s.*

Proof. Kill all the copies of B except two in $\Gamma_{A,\mathbb{Z}_2}$. The quotient is $\Gamma_{\mathbb{Z}_2,\mathbb{Z}_2} = \mathbb{Z} \rtimes \mathbb{Z}_2 = D_\infty$, acting through two $\mathbb{Z}_2$-blocks, i.e. on 4 points. It is non-abelian, so locality (Proposition 2.7, together with [22, Cor. 3.4]) and monotonicity give the result. □

Remark 5.8. Two comments on the cases not covered by Corollary 5.7.

- For F_{23}^{rs} with generic parameters, $\Gamma_{\mathbb{Z}_2,\mathbb{Z}_3} = \mathbb{Z}^2 \rtimes \mathbb{Z}_3$ acts through two $\mathbb{Z}_3$-blocks on 6 points, so locality does not apply. It has $A_4 = \mathbb{Z}_2^2 \rtimes \mathbb{Z}_3$ as a quotient. So $\langle S_6, G_Q\rangle$ contains the candidate of $\widehat{A_4}$ with two $\mathbb{Z}_3$-blocks, which is Catalan through level 8 (Proposition 2.22).
- For special Q, the same reduction works as long as Γ_Q remains non-abelian.

5.4. **Computations.**

Proposition 5.9 (computational)**.** *The joint dimensions* $\dim \mathrm{Fix}_{\langle S_N, G_H\rangle}(k)$ *are as follows.*

H	*parameters*	$\dim \mathrm{Fix}(G_H)(2),(3)$	*joint, levels*
T_6	–	$2, 5$	*Catalan,* $k \leq 5$
H_6^q	$q = 1, i, e^{2\pi i/3}, e^{i\pi/3}$*, generic*	$2, 5$	*Catalan,* $k \leq 5$
F_{23}^{rs}, F_{32}^{rs}	r, s *all* 12*th roots of unity*		$k = 4$*: see below*
F_{23}^{rs}, F_{32}^{rs}	*generic, and* 5 *root-of-unity pairs*	$4, 18$	*Catalan,* $k \leq 5$
P_7^q	9 *values, incl. roots of unity*	$2, 5$	*Catalan,* $k \leq 5$
R_7^q	$q = \pm 1, i, e^{2\pi i/3}, e^{i\pi/3}$*, generic*	$2, 5$	*Catalan,* $k \leq 5$

For the Diţă deformations at $k = 4$*,* 132 *of the* 144 *root-of-unity pairs give Catalan* (14)*, for both families. The remaining* 12 *give Bell* (15)*. At those points* $\dim \mathrm{Fix}(G_H)(2),(3) = 6, 36$*, which is the regular action of* $\mathbb{Z}_6$*: the deformation is Fourier-equivalent there, and* G_H *is classical, in accordance with Theorem 5.1. All these numbers were computed in floating point, from the closed-walk formula of Proposition 5.12 below (*`hadamard.py`*).*

Remark 5.10. The Hopf images split into two kinds:

- *Wild.* T_6, H_6^q, P_7^q and R_7^q have $\dim \mathrm{Fix}(G_H)$ equal to Catalan already at levels 2 and 3, and in fact at all the levels we could reach (Section 5.5). They look like S_N^+ themselves, as the tame/wild heuristic predicts, and for them the joint test is automatic.
- *Tame.* The non-Fourier Diţă points have small Hopf images, with $\dim \mathrm{Fix}(G_H)(2),(3) = 4, 18$. These are genuinely useful seeds, and they are still absorbed.

5.5. **Wildness and the Paley matrices.** The joint test has content only at the levels where G_H itself has more fixed vectors than S_N^+. Indeed, $\mathcal{C}_{\langle S_N, G_H\rangle} = \mathcal{C}_{S_N} \cap \mathcal{C}_{G_H}$, and the noncrossing partition vectors are fixed by both. So if $\dim \mathrm{Fix}(G_H)(k) = C_k$, then $\dim \mathrm{Fix}_{\langle S_N, G_H\rangle}(k) = C_k$ as well, whatever S_N does.

Definition 5.11. A complex Hadamard matrix $H \in M_N(\mathbb{T})$, with $N \geq 4$, is *wild at level* k if $\dim \mathrm{Fix}(G_H)(k) = C_k$, and *wild* if it is wild at all levels.

Since $u = \bar{u}$, the fixed spaces at all levels determine the category of G_H, so H is wild if and only if $G_H = S_N^+$. A wild matrix is useless as a seed: it is a matrix model of S_N^+ itself. Wildness at level k can be tested on G_H alone, without S_N, by computing the fixed space of the operator T_k of Subsection 5.1, namely

$$T_k = (\mathrm{id} \otimes \mathrm{tr})\pi_H(u^{\otimes k}) = \Big(\mathrm{tr}(u_{i_1 j_1} \cdots u_{i_k j_k})\Big)_{i,j} \in M_{N^k}(\mathbb{C}),$$

whose fixed vectors are those of G_H at level k. This matrix has N^{2k} entries, so it cannot be stored beyond small k, but there is no need to, thanks to the following formula.

Proposition 5.12. *Let* $H \in M_N(\mathbb{T})$ *be complex Hadamard, and let* $a_{yx} \in \mathbb{C}^N$ *be the vectors* $a_{yx}(i) = H_{iy}\bar{H}_{ix}$*. Then*

$$T_k = \frac{1}{N^{k+1}} \sum_{x_1, \ldots, x_k = 1}^{N} a_{x_k x_1} a_{x_k x_1}^* \otimes a_{x_1 x_2} a_{x_1 x_2}^* \otimes \cdots \otimes a_{x_{k-1} x_k} a_{x_{k-1} x_k}^*,$$

a sum over the closed walks $x_1 \to x_2 \to \cdots \to x_k \to x_1$ *on the columns of* H*. Consequently:*

(1) T_k *is a positive contraction;*

(2) the orthogonal complement W_k *of* $\mathrm{span}\{\xi_\pi : \pi \in NC(k)\}$ *is invariant under* T_k*, and* H *is wild at level* k *if and only if* $\lambda_k(H) = \max \sigma(T_k|_{W_k}) < 1$*;*

(3) $\lambda_k(H) \leq \lambda_{k+1}(H)$ *for all* k*.*

Proof. Since $u_{ij} = \mathrm{Proj}(\xi_{ij})$ with $\|\xi_{ij}\|^2 = N$, the trace of a product of rank-one projections gives

$$\mathrm{tr}(u_{i_1j_1}\cdots u_{i_kj_k}) = \frac{1}{N^{k+1}}\prod_{r=1}^{k}\sum_{x_r=1}^{N}\overline{\xi_{i_rj_r}(x_r)}\,\xi_{i_{r+1}j_{r+1}}(x_r),$$

with indices r taken modulo k. Expand $\xi_{ij}(x) = H_{ix}\bar{H}_{jx}$. The slot r appears in the factors $r-1$ and r, and contributes $H_{i_rx_{r-1}}\bar{H}_{i_rx_r}\cdot\overline{H_{j_rx_{r-1}}\bar{H}_{j_rx_r}} = a_{x_{r-1}x_r}(i_r)\overline{a_{x_{r-1}x_r}(j_r)}$, which is the formula. Each term is a tensor product of positive rank-one operators, so $T_k \geq 0$, and $\|T_k\| \leq 1$ because $T_k = (\mathrm{id}\otimes\mathrm{tr})(U)$ with U the image of the unitary $u^{\otimes k}$. The noncrossing vectors are fixed, because $G_H \subset S_N^+$, so their span is invariant, and so is W_k, since T_k is self-adjoint. For a positive contraction, the fixed vectors are the eigenvectors at the top eigenvalue 1, which gives (2). For (3), let $v \in W_k$ and $v' = v\otimes\sum_i e_i$. Since $\sum_j u_{ij} = 1$, we have $T_{k+1}v' = (T_kv)\otimes\sum_i e_i$. Also $v' \in W_{k+1}$, because summing ξ_π, $\pi \in NC(k+1)$, over its last index gives a multiple of $\xi_{\pi'}$, with $\pi' \in NC(k)$ the restriction of π. So the Rayleigh quotients of v and v' agree. □

Computationally, T_kv is obtained by a transfer along the walk, at a cost of order kN^{k+3}, and $\lambda_k(H)$ by the Lanczos method on W_k. The number $1-\lambda_k(H)$ is a spectral gap, measuring how wild H is at level k. For the Fourier-equivalent and the tame Diţă points of Proposition 5.9 we find $\lambda_2 = 1$, as we should.

Real Hadamard matrices are a natural place to look, and they tend to be forgotten in the complex theory. The basic constructions are those of Paley [58], see also [3]. Let q be an odd prime power, χ the quadratic character of $\mathbb{F}_q$, with $\chi(0) = 0$, and $Q = (\chi(b-a))_{a,b\in\mathbb{F}_q}$ the Jacobsthal matrix, which satisfies

$$QQ^t = q1_q - J, \qquad QJ = JQ = 0, \qquad Q^t = \chi(-1)Q,$$

with J the all-one matrix, these identities being what makes the matrices below Hadamard. With $\mathbf{1}$ the all-one column vector, the Paley I matrix, for $q \equiv 3 \pmod 4$, and the Paley II matrix, for $q \equiv 1 \pmod 4$, are

$$H = 1_{q+1} + \begin{pmatrix} 0 & \mathbf{1}^t \\ -\mathbf{1} & Q \end{pmatrix}, \qquad H = C\otimes\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + 1_{q+1}\otimes\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \mathbf{1}^t \\ \mathbf{1} & Q \end{pmatrix},$$

of orders $N = q+1$ and $N = 2(q+1)$ respectively. For $q \equiv 3 \pmod 4$ the matrix Q is antisymmetric and Paley I is skew-Hadamard, while for $q \equiv 1 \pmod 4$ the matrix C is a symmetric conference matrix. The small cases are as follows:

- $N = 4, 8$: Paley I with $q = 3, 7$ gives the Walsh matrices $F_2^{\otimes 2}, F_2^{\otimes 3}$, the only real Hadamard matrices of these orders, and G_H is classical.
- $N = 12$: Paley I with $q = 11$ and Paley II with $q = 5$ are equivalent, the real Hadamard matrix of order 12 being unique.
- $N = 20$: Paley I with $q = 19$ and Paley II with $q = 9$ are inequivalent, two of the three equivalence classes found by Hall [38].
- $N = 24$: Paley I with $q = 23$. $N = 28$: Paley I with $q = 27$ and Paley II with $q = 13$, inequivalent (their 4-row profiles differ). The next orders, $N = 32$ (Paley I, $q = 31$) and $N = 36$ (Paley II, $q = 17$), are beyond our reach.

For a real H the vectors $\xi_{ij} = H_iH_j$ are real and symmetric in i, j, so the model satisfies $u_{ij} = u_{ji}$, which means that the antipode fixes the generators, $S(u_{ij}) = u_{ij}$, a feature without analogue for genuinely complex matrices. Whether it says anything about G_H is not clear to us. We believe that the Paley matrices of order $N \geq 12$ are wild, i.e. $G_H = S_N^+$, and the evidence is as follows. Note that at every level where H is wild the joint test with S_N is empty, so for these matrices the numbers $\lambda_k(H)$ are the whole story, and the question is how far from 1 they stay when k grows. The cost of the walk transfer, of order kN^{k+3}, limits us to $k \leq 7$ at $N = 6, 7$, to $k \leq 6$ at $N = 12$, and to $k \leq 4$ at $N = 20, 24, 28$.

Proposition 5.13 (numerical)**.** *The values of $\lambda_k(H)$, rounded to three decimals, are as follows. All of them are < 1, so these matrices are wild at all the levels shown.*

H	*parameters*	$k=2$	3	4	5	6	7
T_6	–	0.250	0.428	0.619	0.619	0.651	0.651
H_6^q	$q=1,i$	0.370	0.370	0.667	0.667	0.667	0.667
	$q=e^{2\pi i/3}, e^{i\pi/3}$	0.499	0.499	0.665	0.665	0.694	
	generic	0.518	0.518	0.641	0.641	0.666	0.666
P_7^q	$q=1,e^{2\pi i/3}$	0.353	0.570	0.570	0.570	0.571	
	$q=i$	0.353	0.503	0.562	0.593	0.593	
	generic	0.353	0.551	0.551	0.570	0.591	0.591
R_7^q	$q=\pm1,e^{2\pi i/3}$	0.353	0.570	0.570	0.570	0.571	
	$q=i$	0.353	0.503	0.562	0.593	0.593	
	generic	0.353	0.555	0.555	0.565	0.588	0.588

Paley matrix	$k=2$	3	4	5	6
$N=12$, *I* ($q=11$) *and II* ($q=5$)	0.185	0.309	0.399	0.399	0.433
$N=20$, *I* ($q=19$)	0.292	0.292	0.292		
$N=20$, *II* ($q=9$)	0.392	0.462	0.508		
$N=24$, *I* ($q=23$)	0.207	0.207	0.240		
$N=28$, *I* ($q=27$)	0.200	0.200	0.200		
$N=28$, *II* ($q=13$)	0.528	0.570	0.570		

Here "generic" means $q=e^{2\pi it}$ with $t=0.1234$ for H_6^q and P_7^q, and $t=0.3711$ for R_7^q. The eigenvalues were computed by the Lanczos method on W_k, with relative tolerance 10^{-6}, applying T_k as a cyclic chain of 4-index tensors, $(T_k)_{ij}=N^{-k-1}\prod_s Q(i_s,j_s,i_{s+1},j_{s+1})$ with $Q(a,b,c,d)=\sum_x \bar H_{ax}H_{bx}H_{cx}\bar H_{dx}$ (`hadamard.py`), and the noncrossing vectors were checked to be fixed by T_k up to $2\cdot10^{-15}$. At $N=12$ the two Paley matrices give the same values, as they should.

Several features are visible. The values are nondecreasing in k, as they must be by Proposition 5.12, and they often stay constant for several levels. The Paley I matrices are much wilder than the complex matrices at $N=6,7$, and get wilder as N grows. The Paley II matrices are less wild than Paley I at the same N, and comparable to the matrices at $N=6,7$. None of these matrices is a useful seed: at all the levels shown, $\langle S_N, G_H\rangle$ is Catalan for free. As explained above, it is our feeling that the Paley matrices are wild, and this leads to the following question.

Question 5.14. *Prove that the Paley matrices of order $N\geq 12$ are wild, i.e. that $G_H=S_N^+$, or at least that $\lambda_k(H)<1$ for all k.*

The values above are often constant in k over several levels, for instance $\lambda_k=0.292$ for $k=2,3,4$ at $N=20$, Paley I, and at $N=12$ they seem to be $\lambda_2=5/27$, $\lambda_3=25/81$, $\lambda_4=\lambda_5=97/243$, as rational guesses from the decimals. So an exact formula for λ_k might be within reach, using the large automorphism groups of these matrices.

5.6. **Conclusion.**

Theorem 5.15. *Let H be a regular complex Hadamard matrix with $N\leq 6$, or one of F_7, P_7^q, R_7^q. Then exactly one of the following holds:*

(1) H is Fourier-equivalent, G_H is classical, and $\langle S_N, G_H\rangle = S_N$;

(2) $\langle S_N, G_H\rangle = S_N^+$.

The statement is proved for $N\leq 5$, by maximality at $N=4,5$, and for F_{32}^{rs} with generic parameters (Corollary 5.7). In the remaining cases it is verified through level 5, on the parameter values of Proposition 5.9, and through level 6 for T_6, H_6^q, P_7^q, R_7^q on the parameter values of Proposition 5.13, where these matrices are wild.

We end with a question about the joint test itself. Let $M_k(H)$ be the matrix with entries $\langle T_k\xi_\sigma, \xi_\pi\rangle$, for $\pi, \sigma \in P(k)$, that is, the compression of T_k to the span of the partition vectors. Its entries are the values of the model state on the S_N-bi-invariant sums $\sum u_{i_1j_1}\cdots u_{i_kj_k}$, with i, j constant on the blocks of π, σ. Since T_k is a positive contraction, $\dim \mathrm{Fix}_{\langle S_N, G_H\rangle}(k)$ is the corank of $G_k - M_k$, with G_k the Gram matrix of the partition vectors. So the joint test only sees M_k, and if $M_k(H^q)$ does not depend on q, then a whole family is a single test case.

Question 5.16. *For which families H^q of complex Hadamard matrices is $M_k(H^q)$ independent of q, and why?*

Here is what we know (computational). At $k \leq 3$, the matrix M_k is the same for all complex Hadamard matrices of a given size, as checked at $N = 6$ for T_6, H_6^q, Diţă and F_6. At $k = 4$, the matrices $M_4(P_7^q)$ and $M_4(R_7^q)$ are independent of q, and equal to each other, recall that P_7^q is equivalent to $\bar{R}_7^q$, but they differ from $M_4(F_7)$. On the other hand, $M_4(H_6^q)$ depends on q, and so does M_4 for the Diţă families, where the joint dimension at $k = 4$ jumps from 14 to 15 at the Fourier points. Also, the numbers $\lambda_k(P_7^q)$ do depend on q, so it is only the compression which is invisible. Is $M_k(P_7^q)$ independent of q at all levels k? What distinguishes P_7^q from H_6^q here, perhaps the absence of a Fourier point on the family? Is M_k determined by some finite set of invariants of H, such as the multisets $\{\sum_i H_{iy_1}\bar{H}_{ix_1}\cdots H_{iy_s}\bar{H}_{ix_s}\}$ along closed walks, as the closed-walk formula of Proposition 5.12 suggests?

6. Exotic quantum permutation groups

Throughout this section, G denotes a hypothetical *exotic* quantum permutation group, $S_N \subsetneq G \subsetneq S_N^+$. By maximality at $N \leq 5$, such a G can only exist for $N \geq 6$. We collect what can be said about it, in soft terms: representation theory, orbitals, subgroups, diagonal algebras, and analysis.

6.1. **Representation theory and moments.** By Tannakian duality, G corresponds to a tensor category $\mathcal{C}_G$ with

$$\mathrm{span}(NC) \subsetneq \mathcal{C}_G \subsetneq \mathrm{span}(P).$$

In particular $C_k \leq \dim \mathrm{Fix}_G(k) \leq \dim \mathrm{Fix}_{S_N}(k)$ for all k, where C_k is the Catalan number. Moreover $\mathcal{C}_G$ is not spanned by partitions: there is no easy quantum group strictly between S_N and S_N^+ [16]. So G is non-easy, and $\mathcal{C}_G$ must contain a linear combination of crossing partitions that is not itself a partition [51].

Proposition 6.1 (McCarthy [50])**.** *The Haar state of G is tracial and invariant under the antipode and under relabeling. All its integrals of degree ≤ 4 in the coordinates u_{ij} coincide with those of S_N^+. In particular the fourth moment of the character is* 14*, i.e.* $\dim \mathrm{Fix}_G(k) = C_k$ *for $k \leq 4$.*

So an exotic G is invisible to S_N^+ through level 4. The level computations of Sections 7–9 push this further. McCarthy also shows that no exotic category is generated by a linear combination of three crossing partitions [51].

6.2. **Orbitals.** Following Lupini, Mančinska and Roberson [47], write $(i_1, \ldots, i_m) \sim_m (j_1, \ldots, j_m)$ when $u_{i_1j_1}\cdots u_{i_mj_m} \neq 0$. Say that G has *free m-orbitals* if such a monomial vanishes only for the trivial reason, namely two consecutive factors in a common row or column.

Theorem 6.2 (McCarthy [50])**.** *An exotic G has free orbits, free* 2*-orbitals and free* 3*-orbitals.*

Remark 6.3. Two comments on this result.

- Freeness of orbits and 2-orbitals is inherited from S_N, through abelianization. Freeness of 3-orbitals is the main result of [50].
- McCarthy asks whether *any* proper quantum subgroup of S_N^+ can have free 3-orbitals. A negative answer would prove maximality.

6.3. **What G cannot contain.** The subgroup approach of Sections 2–5 turns into a list of properties of G, through the following trivial observation.

Lemma 6.4. *If $K \subset G$, then $\langle S_N, K\rangle \subset G$. In particular:*

- *if $\langle S_N, K\rangle = S_N^+$, then $K \not\subset G$;*
- *if $\langle S_N, K\rangle$ agrees with S_N^+ through level k, and $K \subset G$, then so does G.*

For $M \leq N$, view $S_M^+ \subset S_N^+$ as acting on M chosen points and fixing the others, i.e. $v \mapsto \mathrm{diag}(v, 1_{N-M})$. Define $G_M = G \cap S_M^+$, the intersection of quantum subgroups of S_N^+. It is the largest quantum subgroup of G that only moves these M points.

Proposition 6.5 (restriction)**.** *For $4 \leq M \leq N$, either $G_M = S_M$, or G_M is exotic at M, i.e. $S_M \subsetneq G_M \subsetneq S_M^+$. In particular:*

(1) if $S_M \subset S_M^+$ is maximal, then $G_M = S_M$;
(2) every quantum subgroup of G that moves at most 5 points is classical;
(3) more generally, if maximality holds at all $M \leq m$, then every quantum subgroup of G moving at most m points is classical.

Proof. We have $S_M \subset S_N \subset G$ and $S_M \subset S_M^+$, so that we have inclusions as follows:

$$S_M \subset G_M \subset S_M^+.$$

If $G_M = S_M^+$, then $S_M^+ \subset G$. Since $4 \leq M \leq N$, the result in [22, Cor. 3.4] gives

$$S_N^+ = \langle S_N, S_M^+\rangle \subset G,$$

contradicting $G \neq S_N^+$. This proves the dichotomy, and (1) follows.

For (2) and (3), let $K \subset G$ move at most m points. Conjugating by an element of $S_N \subset G$, we may assume these are the first m points. Then $K \subset G_m$, and $G_m = S_m$ by (1), with the cases $m \leq 3$ being trivial. Maximality at 4 and 5 is known [2, 5]. □

So exotic quantum permutations are *global*: they are invisible on every set of at most 5 points, and on every set of size M at which maximality holds. Inductively, the first exotic G, if any, restricts to S_M on every proper subset.

Corollary 6.6. *An exotic G contains none of the following:*

- *a non-classical quantum subgroup moving ≤ 5 points, for instance S_4^τ, or the dual of $\mathbb{Z}_2 * \mathbb{Z}_2$ or of $\mathbb{Z}_2 * \mathbb{Z}_3$ on 4 or 5 points;*
- *a diagonal copy of S_M^+ with $M \geq 5$ [22, Ex. 3.9, Prop. 3.10];*
- *$G^+(X)$ for a graph X with two disjoint pairs of twins, or more generally with two disjoint automorphisms of total support ≤ 5 (Corollary 3.3);*
- *G_H for a generic Diţă deformation $F_A \otimes_Q F_2$ (Corollary 5.7);*
- *a group-theoretical quantum group $H_M^{[\Gamma]} \subset H_M^+ \subset S_{2M}^+$ of Raum and Weber [60, 61], with $N = 2M$, as soon as the quotient of Γ by the normal subgroup generated by $g_3, \ldots, g_M$ is non-abelian, for instance $H_M^{[\infty]}$, where $\Gamma = \mathbb{Z}_2^{*M}$. Indeed, $H_M^{[\Gamma]}$ contains the dual of Γ, acting diagonally through $u_{ij} \mapsto \delta_{ij} g_i$, and hence the dual of this quotient, a non-abelian quotient of $\mathbb{Z}_2 * \mathbb{Z}_2$, which acts on 4 of the $2M$ points.*

If G contains any of the other seeds of Sections 2–5, then G agrees with S_N^+ through the level at which that seed was checked (Lemma 6.4).

6.4. **Relabeling and diagonal algebras.** Since $S_N \subset G$, every permutation $\sigma \in S_N$ gives a character of $C(G)$, namely $ev_\sigma(u_{ij}) = \delta_{i\sigma(j)}$, the evaluation at the permutation matrix of σ. These characters can be used to move the indices around, inside $C(G)$ itself.

Proposition 6.7 (relabeling)**.** *Assume $S_N \subset G \subset S_N^+$. For any $\sigma, \tau \in S_N$ there is a $*$-automorphism $\alpha_{\sigma\tau}$ of $C(G)$ such that*

$$\alpha_{\sigma\tau}(u_{ij}) = u_{\sigma(i)\tau(j)}.$$

Proof. Set $\alpha_{\sigma\tau} = (ev_{\sigma^{-1}} \otimes \mathrm{id} \otimes ev_\tau)\Delta^{(2)}$, where $\Delta^{(2)} = (\Delta \otimes \mathrm{id})\Delta$. This is a composition of $*$-homomorphisms, and

$$\alpha_{\sigma\tau}(u_{ij}) = \sum_{k,l} ev_{\sigma^{-1}}(u_{ik})\, u_{kl}\, ev_\tau(u_{lj}) = u_{\sigma(i)\tau(j)}.$$

Since $\alpha_{\sigma^{-1}\tau^{-1}}$ is an inverse, $\alpha_{\sigma\tau}$ is an automorphism. □

This is the mechanism behind McCarthy's invariance of the Haar state under relabeling (Proposition 6.1), and it has the following consequences, from [4, Props. 10.38, 10.39], where they are obtained through the Haar state and Hahn–Banach.

Corollary 6.8. *Assume $S_N \subset G \subset S_N^+$.*

1. *The algebra $\langle u_{i_1 j_1}, \ldots, u_{i_p j_p} \rangle \subset C(G)$ depends, up to isomorphism, only on $\ker i$ and $\ker j$, that is, on the relative position of the pairs (i_r, j_r) inside the square $\{1, \ldots, N\}^2$.*
2. *If G is not classical, then $[u_{ij}, u_{kl}] \neq 0$ for all $i \neq k$ and $j \neq l$.*
3. *The diagonal algebra $D(G) = \langle u_{11}, \ldots, u_{NN} \rangle$, regarded as a quotient of $C^*(\mathbb{Z}_2^{*N})$ through $g_i \mapsto 2u_{ii} - 1$, has an S_N-invariant kernel.*

Proof. (1) The automorphisms $\alpha_{\sigma\tau}$ act transitively on the pairs of multi-indices with given kernels. (2) If $i = k$ or $j = l$, then u_{ij} and u_{kl} commute, being equal or orthogonal projections. So, G being non-classical, some commutator with $i \neq k$, $j \neq l$ is nonzero, and then all of them are, by (1). (3) The automorphisms $\alpha_{\sigma\sigma}$ map $u_{ii} \mapsto u_{\sigma(i)\sigma(i)}$. □

At the level of diagonal algebras we have quotient maps $D(S_N^+) \to D(G) \to D(S_N)$, all of them S_N-equivariant. The last one is commutative and explicit: u_{ii} is the indicator of the permutations fixing i, so $D(S_N) = C(X_N)$, where X_N is the set of fixed-point sets of permutations, namely the subsets of $\{1, \ldots, N\}$ of size $\neq N - 1$. So $D(G)$ is an S_N-invariant quotient of $D(S_N^+)$ which still projects onto $C(X_N)$. Very little is known about $D(S_N^+)$ itself, and for this we refer to McCarthy [49, 50, 51].

As explained in [4, Chapter 8], a promising approach to the diagonal algebras of the various quantum permutation groups $S_N \subset G \subset S_N^+$ is expected to come from looking more generally at the diagonal algebras of quantum semigroups of partial permutations, in the spirit of the constructions of Sołtan [66]. Whether this idea can really lead to concrete results remains to be seen.

Question 6.9. *Understand the diagonal algebras $D(S_N^+)$ and $D(G)$, for $S_N \subset G \subset S_N^+$, and find out whether they can detect an exotic G.*

Some comments on this question.

- Conjecture 10.43 in [4] states that the u_{ii} are free in $C(S_N^+)$, so that $D(S_N^+) \simeq C^*(\mathbb{Z}_2^{*N})$. This seems to fail at $N = 4$: in the Pauli model one finds relations of type $p_a p_b p_c = p_a p_d p_c$, and polynomial growth, which is forced by subhomogeneity. This should be rechecked, and for $N \geq 5$ the question is open.
- The free wreath products $\mathbb{Z}_2 \wr_* G$ do not see G, their diagonal subgroup being always $\widehat{\mathbb{Z}_2^{*N}}$. The G-dependent wreath product $\langle G, \mathbb{Z}_2^N \rangle \subset H_N^+$ interpolates between H_N and H_N^+, but easy inputs give nothing here, by [13].

- As suggested in [4], $D(G)$ should come from an intermediate quotient $\mathbb{Z}_2^{*N} \to \Gamma \to \mathbb{Z}_2^N$. However, $D(S_N) = C(X_N)$ is a quotient of $C(\mathbb{Z}_2^N)$ which is not a group algebra, so the right intermediate objects are rather the S_N-invariant quotients of $C^*(\mathbb{Z}_2^{*N})$, not necessarily group algebras.

6.5. **Analysis, growth and Lie theory.** Beyond representation theory, one can ask about the analytic and geometric properties of an exotic G: coamenability, growth, and Lie-theoretic aspects. Regarding growth, the natural questions are those of polynomial growth of the dimensions of the irreducible representations, and of the related finiteness conditions, as in [28, 29], polynomial growth implying coamenability. Regarding Lie theory, this is only available, to some extent, in the half-classical case, see [17, 21], and its adaptation to exotic quantum permutations is an open question.

We focus in what follows on coamenability. For a compact quantum group of Kac type with self-adjoint fundamental representation, coamenability is equivalent to N lying in the support of the law μ_G of the character $\chi = \sum_i u_{ii}$ (a Kesten-type criterion). Since $0 \leq \chi \leq N$ and $\int_G \chi^k = \dim \mathrm{Fix}_G(k)$, this means

$$\limsup_{k\to\infty} (\dim \mathrm{Fix}_G(k))^{1/k} = N.$$

Lemma 6.10. *Let $H \subset G \subset S_N^+$ be closed quantum subgroups. If G is coamenable, then so is H.*

Proof. We have $\mathrm{Fix}_G(k) \subset \mathrm{Fix}_H(k)$, so the moments of μ_H dominate those of μ_G. Both growth rates are at most N, so if the rate of G is N, so is the rate of H. □

So coamenability goes down to quantum subgroups, and a non-coamenable quantum subgroup is an obstruction. Let us compare with the two extreme cases.

- S_N is coamenable, being a finite group.
- S_N^+ is not, for $N \geq 5$. The usual witness is the non-amenable group dual $\widehat{\mathbb{Z}_2 * \mathbb{Z}_3}$ acting on 5 points, together with Lemma 6.10.
- By Proposition 6.5, that witness cannot sit inside an exotic G. Nor can any non-classical subgroup on ≤ 5 points.

There is one case where coamenability is automatic, namely when G is finite, i.e. when $C(G)$ is finite dimensional. This case is detected by the law of the character.

Proposition 6.11. *Let $G \subset S_N^+$. If G is finite, then $\mu_G(\{N\}) = 1/\dim C(G)$. If G is infinite, then $\mu_G(\{N\}) = 0$.*

Proof. Let P be the spectral projection of χ at N, in the von Neumann algebra $L^\infty(G)$ of the Haar state h, so that $\mu_G(\{N\}) = h(P)$. From $\chi P = NP$ we get $\sum_i P(1 - u_{ii})P = 0$, a sum of positive elements, so $(1-u_{ii})P = 0$, that is, $u_{ii}P = P = Pu_{ii}$ for all i. For $i \neq j$ we get $u_{ij}P = u_{ij}u_{jj}P = 0$, and similarly $Pu_{ij} = 0$. Thus $xP = Px = \varepsilon(x)P$ for every x in the dense $*$-algebra $\mathcal{O}(G)$, where ε is the counit.

Now expand P in $L^2(G)$, in the orthonormal basis $\sqrt{d_\alpha}\, v_{ij}^\alpha$ of coefficients of irreducible representations, G being of Kac type. The coefficient of P on $\sqrt{d_\alpha}\, v_{ij}^\alpha$ is $\langle P, \sqrt{d_\alpha}\, v_{ij}^\alpha \rangle = \sqrt{d_\alpha}\, h((v_{ij}^\alpha)^* P) = \sqrt{d_\alpha}\, \delta_{ij}\, h(P)$, and Parseval gives

$$h(P) = h(P^*P) = \sum_\alpha d_\alpha^2\, h(P)^2.$$

So either $h(P) = 0$, or $\sum_\alpha d_\alpha^2 = 1/h(P)$ is finite, in which case G is finite, with $\dim C(G) = \sum_\alpha d_\alpha^2 = 1/h(P)$. Conversely, if G is finite, the counit is a character of the finite dimensional algebra $C(G)$, so it is supported by a nonzero central projection p, with $xp = \varepsilon(x)p$. Then $\chi p = Np$, so $p \leq P$, and $h(P) \geq h(p) > 0$, the Haar state being faithful on $C(G)$. So $h(P) = 1/\dim C(G)$. □

So an exotic G is either finite, and then coamenable, with $\dim C(G) > N!$ since $C(G)$ maps onto $C(S_N)$, or infinite, with no atom of μ_G at N. In the infinite case, coamenability means that N is an accumulation point of the support of μ_G.

Question 6.12. *Is an exotic G coamenable? Can an exotic G be finite?*

Our feeling is that an exotic G, if any, should rather not be coamenable. Indeed, the results of McCarthy on the Haar state and the orbitals (Proposition 6.1 and Theorem 6.2) suggest that such a G is "closer" to S_N^+ than to S_N, and the level computations of Sections 7–9 point in the same direction: for $8 \leq N \leq 37$, the first 8 moments of μ_G are the Catalan numbers, as for S_N^+. This does not decide the question, since the moment bounds $C_k \leq \dim \mathrm{Fix}_G(k) \leq \dim \mathrm{Fix}_{S_N}(k)$ say nothing about the growth rate, and a non-amenable witness inside G would have to be global, for instance a non-amenable group dual with all its non-trivial pieces on ≥ 6 points, such as $\widehat{\mathbb{Z}_3 * \mathbb{Z}_3}$ acting through two $\mathbb{Z}_3$-blocks.

6.6. **Finite exotic quantum permutations.** Let us try to exclude an exotic G which is finite, $\dim C(G) < \infty$, as in Question 6.12. Finiteness produces a Lagrange theorem, and lower bounds on the size of G growing with the number of settled levels, but no contradiction. We write $D = \dim C(G)$, $m_k = \dim \mathrm{Fix}_G(k)$ and $s_k = \dim \mathrm{Fix}_{S_N}(k)$.

Proposition 6.13. *Let G be a finite exotic quantum permutation group.*

1. *The characters of $C(G)$ are the evaluations ev_σ, $\sigma \in S_N$. So $C(G) = \mathbb{C}^{N!} \oplus A_{nc}$, where A_{nc} is a sum of matrix blocks M_{n_b} with $n_b \geq 2$.*
2. (Lagrange) *$D = q \cdot N!$, for an integer $q \geq 2$, the index of S_N in G.*
3. (Cosets) *The algebra $C(G/S_N)$ of right S_N-invariant elements has dimension q, and $C(G/S_N) \simeq \bigoplus_\alpha f_\alpha V_\alpha$ as a G-module, where $f_\alpha = \dim V_\alpha^{S_N}$. So*
$$q = \sum_\alpha d_\alpha f_\alpha.$$
4. *The law of χ splits as $\mu_G = \frac{1}{q}\,\mu_{S_N} + \left(1 - \frac{1}{q}\right)\nu$, where ν is the normalized law of χ on A_{nc}, a probability measure on $[0, N)$.*

Proof. (1) A character maps u to a permutation matrix, so the classical version of G is S_N, and the characters of a finite dimensional C*-algebra are its 1-dimensional blocks. (2) Dually, $\mathbb{C}[S_N]$ is a Hopf subalgebra of $C(G)^*$, and $N! \mid D$ by the Nichols–Zoeller theorem [57]. (3) $C(G)$ is free over its quotient $C(S_N)$, which gives the dimension, and by Peter–Weyl the right S_N-invariant part of $C(G) = \bigoplus_\alpha V_\alpha \otimes \bar{V}_\alpha$ is $\bigoplus_\alpha V_\alpha \otimes \bar{V}_\alpha^{S_N}$. (4) The Haar state is the normalized regular trace, $h = \frac{1}{D}\sum_b n_b \mathrm{Tr}_b$, so the classical part has weight $N!/D = 1/q$, and on it χ counts the fixed points. On a block of size ≥ 2, an eigenvector of χ at N would satisfy $u_{ij}\xi = \delta_{ij}\xi$, and give a 1-dimensional subrepresentation of an irreducible block. □

By (4), with $t = 1/(q-1)$, the moments of ν are $\nu_k = C_k + t\,(C_k - s_k)$ for all k at which G agrees with S_N^+. For $t \to 0$ these are the moments of the free Poisson law, but positivity fails quickly as t grows.

Theorem 6.14 (exact)**.** *A finite exotic G has index $q = \dim C(G)/N! \geq 31$, and $q \geq 32$ for $N \geq 7$.*

Proof. By Theorem 9.5, $m_k = C_k$ for $k \leq 7$. Since ν lives on $[0, \infty)$, the Hankel matrix $H = (\nu_{i+j+1})_{0 \leq i,j \leq 3}$ is positive semidefinite. For $N \geq 7$ we have $s_k = B_k$, the Bell numbers, for $k \leq 7$, and a direct computation gives $\det H = 1 - 30t - 14t^2 + 6t^3 + t^4$. With $x = q - 1 = 1/t$, this is $x^{-4}(x^4 - 30x^3 - 14x^2 + 6x + 1)$, which is negative for the integers $1 \leq x \leq 30$. So $x \geq 31$. For $N = 6$ we have $s_7 = B_7 - 1$, and the determinant is $1 - 29t - 16t^2 + 5t^3 + t^4$, which gives $x \geq 30$. □

With the level-8 result of Corollary 9.13, the Hankel matrix $(\nu_{i+j})_{0\le i,j\le 4}$ gives $q \ge 123$ for $8 \le N \le 37$. The thresholds for t obtained from the levels $4,\dots,8$ are 1, 0.414, 0.116, 0.0328, 0.0082, decreasing by a factor of about 4 per level. Representation theory does better.

Proposition 6.15 (computational, $8 \le N \le 37$)**.** *A finite exotic G has index $q \ge 1 + d_4$, where $d_0 = 1$, $d_1 = N - 1$ and $d_{k+1} = (N-2)d_k - d_{k-1}$. For instance $q \ge 1394$ at $N = 8$.*

Proof. By Corollary 9.13, G has the intertwiners of S_N^+ through level 8. So $u^{\otimes k}$, $k \le 4$, decomposes into irreducible representations $r_0,\dots,r_4$ of dimensions $d_0,\dots,d_4$, with the multiplicities m_{kj} of S_N^+, and $m_{44} = 1$. Counting S_N-fixed vectors gives $s_k = \sum_j m_{kj} f_j$. For $k \le 3$ we have $s_k = C_k = m_{k0}$, so $f_1 = f_2 = f_3 = 0$, and for $k = 4$ we get $15 = 14 + f_4$, so $f_4 = 1$. Now Proposition 6.13(3) gives $q \ge d_0 f_0 + d_4 f_4$. □

The same argument shows that the more levels are settled, the larger a finite exotic G must be. If G agrees with S_N^+ through level $2m$, then $q \ge \sum_{j\le m} d_j f_j$, where the multiplicities f_j are those of S_N^+, namely $1, 0, 0, 0, 1, 1, 6, 20, 86, 393, \dots$ for N large. So $q \ge d_m$ for $m \ge 4$, and d_m grows roughly like $(N-2)^m$.

All these are lower bounds. To exclude a finite exotic G, one needs an upper bound on q, or a contradiction between finiteness and the long initial agreement with S_N^+, and there are several possible sources for this: the coset algebra $C(G/S_N)$, a q-dimensional C*-algebra with an ergodic action of G, whose "double coset" algebra has dimension $\sum_\alpha f_\alpha^2$; the fusion graph of r_1, a finite connected graph with Perron–Frobenius eigenvalue $N-1$ and integer Perron–Frobenius vector (d_α), starting with the chain $r_0 - r_1 - r_2 - r_3$, and satisfying $\sum_\alpha d_\alpha^2 = N! \sum_\alpha d_\alpha f_\alpha$; the fixed-point subfactor of the action of G on $\mathbb{C}^N$, of index N and finite depth, whose planar algebra lies between Temperley–Lieb and the spin model planar algebra of S_N; and integrality, the spectrum of χ consisting of totally real algebraic integers in $[0, N]$, with $\sum_k m_k z^k$ a rational function with integer coefficients, by Proposition 6.11. None of these gives an upper bound, as far as we can see: each of them is satisfied by a "fake" finite G which looks like S_N^+ up to a high level, and then closes up. So the finite case does not seem to be easier than the maximality conjecture itself, or at least we do not know how to properly exploit the finiteness assumption.

7. Partitions, levels and the method

This section and the next two form the core of the paper. They attack the maximality conjecture head-on, through the Tannakian category of a hypothetical intermediate quantum group $S_N \subsetneq G \subsetneq S_N^+$. The idea is simple. The category of G sits between the noncrossing partitions and all partitions, and it must be closed under the usual categorical operations. So we look for the first "level" k at which it could contain a vector outside the noncrossing span, and we write down enough necessary conditions to show that no such vector exists.

At each level the conditions come in three layers: linear, quadratic and cubic. What makes the approach work in practice is that every layer can be reduced to finite, exact linear algebra, uniformly in N wherever possible. Such computations cannot be checked by hand, so we put some care into making them *certificates*: objects that can be verified independently, rather than numerical evidence.

This section sets up the framework and the tools. Section 8 settles level 6 for all $N \ge 6$. Section 9 settles level 7 for all $N \ge 6$, and proves a structural theorem at level 8 for $7 \le N \le 37$.

7.1. **Partition vectors and the two categories.** Let $N \ge 1$, and let $e_1,\dots,e_N$ be the standard basis of $\mathbb{C}^N$. For a partition π of $\{1,\dots,k\}$ we use the vector

$$\xi_\pi = \sum_{i_1,\dots,i_k} \delta_\pi(i)\, e_{i_1} \otimes \cdots \otimes e_{i_k} \ \in (\mathbb{C}^N)^{\otimes k},$$

where $\delta_\pi(i) = 1$ if the multi-index i is constant on each block of π, and $\delta_\pi(i) = 0$ otherwise. We write $\mathcal{P}(k)$ for the set of all partitions of k points, and $\mathcal{NC}(k) \subset \mathcal{P}(k)$ for the noncrossing ones. Recall that

$$\mathrm{Fix}_{S_N}(u^{\otimes k}) = \mathrm{span}\{\xi_\pi : \pi \in \mathcal{P}(k)\}, \quad \mathrm{Fix}_{S_N^+}(u^{\otimes k}) = \mathrm{span}\{\xi_\pi : \pi \in \mathcal{NC}(k)\}.$$

More generally, partitions with k upper and l lower points act as linear maps $T_\pi : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$, and composing or tensoring such maps corresponds to composing or juxtaposing partitions. Each closed loop created by a composition contributes a factor N, see [16].

Lemma 7.1. *The vectors ξ_π, $\pi \in \mathcal{P}(k)$, are linearly independent if and only if $k \leq N$. For $k = N+1$ there is exactly one linear relation, namely*

$$\sum_{\pi \in \mathcal{P}(k)} \mu(0_k, \pi)\, \xi_\pi = 0,$$

where μ is the Möbius function of the partition lattice, so that $\mu(0_k, \pi) = \prod_{B \in \pi} (-1)^{|B|-1}(|B|-1)!$.

Proof. By Möbius inversion, the vector $\sum_\pi \mu(0_k, \pi)\xi_\pi$ is the sum of all $e_{i_1} \otimes \cdots \otimes e_{i_k}$ with pairwise distinct indices. The vectors $\delta_{\ker = \pi}$, sums over the multi-indices whose kernel is exactly π, are related to the ξ_π by a unitriangular change of basis. They are nonzero, with disjoint supports, exactly when π has at most N blocks. For $k = N+1$, only the partition into singletons has more than N blocks. □

The noncrossing vectors behave better: for $N \geq 4$ the ξ_π, $\pi \in \mathcal{NC}(k)$, are linearly independent for every k. This is the nondegeneracy of the Temperley–Lieb Gram matrices at parameter $N \geq 4$.

7.2. **Levels.** Let $S_N \subset G \subset S_N^+$, and write $\mathcal{C}(k) = \mathrm{Fix}_G(u^{\otimes k})$. Then

$$\mathrm{span}\{\xi_\pi : \pi \in \mathcal{NC}(k)\} \;\subset\; \mathcal{C}(k) \;\subset\; \mathrm{span}\{\xi_\pi : \pi \in \mathcal{P}(k)\},$$

and the family $\mathcal{C} = (\mathcal{C}(k))_k$, together with the corresponding spaces of intertwiners, is a tensor category containing all the noncrossing partitions. We abbreviate $\mathcal{NC}(k)$ for the span of the noncrossing partition vectors, when no confusion can arise.

Definition 7.2. The category $\mathcal{C}$ *first differs at level k* if $\mathcal{C}(l) = \mathcal{NC}(l)$ for all $l < k$, and $\mathcal{C}(k) \neq \mathcal{NC}(k)$.

If $G \neq S_N^+$ then $\mathcal{C} \neq \mathcal{NC}$, so $\mathcal{C}$ first differs at some level. The maximality conjecture is therefore equivalent to the following: for $S_N \subsetneq G$, the category of G never first differs at any level. The work of this section and the next two is to exclude, one level at a time, that $\mathcal{C}$ first differs at level k.

When $k \leq N$, Lemma 7.1 shows that every element of $\mathrm{span}\{\xi_\pi : \pi \in \mathcal{P}(k)\}$ has a well-defined *crossing part*, namely its coordinates on the crossing partitions. Since $\mathcal{NC}(k) \subset \mathcal{C}(k)$, the crossing part of an element of $\mathcal{C}(k)$ is itself an element of $\mathcal{C}(k)$. So $\mathcal{C}(k) \neq \mathcal{NC}(k)$ iff $\mathcal{C}(k)$ contains a nonzero purely crossing combination.

Remark 7.3. The known results at low levels are as follows.

- There is no easy exotic G (Banica–Curran–Speicher [13]).
- Easiness holds at level 2 (Banica [2]) and at level 3 (McCarthy [50]).
- The Haar integrals of degree ≤ 4 agree with those of S_N^+ (McCarthy [50]).
- $\mathcal{C}(k) = \mathcal{NC}(k)$ for $k \leq 5$ was announced by Freslon and Speicher. We reprove it in Proposition 7.5.
- At order 6, no exotic category contains a combination of at most 31 crossing partitions (McCarthy [51]).

7.3. **The operations.** We use four kinds of operations, all of them legitimate in any category containing $\mathcal{NC}$. For a vector at level k, the legs are indexed cyclically by $\mathbb{Z}_k$.

- *Leg-sums:* $d_i\,\xi_\pi = N^{[\{i\}\in\pi]}\,\xi_{\pi\setminus i}$, which sums out the leg i. The factor N appears exactly when i is a singleton of π.
- *Merges:* $m_i\,\xi_\pi = \xi_{\pi/(i\sim i+1)}$, which identifies the legs i and $i+1$.
- *Ribbon contractions:* given ξ,η at level k, join an arc of j consecutive legs of ξ to an arc of j consecutive legs of η by j parallel strings, in the planar way. Each of the two outermost strings may carry one output leg. A ribbon is described by its *configuration* (s,t,j,o_1,o_2): the strings join the legs $s,s+1,\dots,s+j-1$ of ξ to the legs $t+j-1,\dots,t$ of η, and $o_1,o_2\in\{0,1\}$ say whether the first and the last strings carry an output leg. The output has level $2k-2j+o_1+o_2$, and every closed loop contributes a factor N.
- *Triple contractions:* the composite $x\mapsto \rho_2\big(\rho_1(x\otimes x)\otimes x\big)$ of a ribbon ρ_1 landing at level k and a ribbon ρ_2 landing at level $<k$.

Lemma 7.4. *Let $\mathcal{C}$ be the category of $S_N\subset G\subset S_N^+$.*

(1) If $\xi\in\mathcal{C}(k)$, then $d_i\xi, m_i\xi\in\mathcal{C}(k-1)$.
(2) If $\xi,\eta\in\mathcal{C}(k)$, then every ribbon contraction of $\xi\otimes\eta$ lies in $\mathcal{C}$.
(3) If $x\in\mathcal{C}(k)$, then every triple contraction of x lies in $\mathcal{C}$.

Proof. Each operation is obtained from ξ, from $\xi\otimes\eta\in\mathcal{C}(2k)$, or from $\rho_1(x\otimes x)\otimes x\in\mathcal{C}(2k)$, by applying a map T_σ with σ noncrossing. These are a singleton, a merging block, or a family of nested caps, possibly with a three-leg block at each end of the nest. Since $\mathcal{NC}\subset\mathcal{C}$ and $\mathcal{C}$ is a tensor category, the result stays in $\mathcal{C}$. The factors N are the usual loop factors of T_σ. □

We only ever use these operations to produce *necessary* conditions. Whether they generate all the relevant operations is therefore irrelevant for the non-existence statements below. It is reassuring, however, that a natural class of further linear conditions turns out to be redundant (Remark 7.7).

7.4. **The three-step method.** Fix k, and assume that $\mathcal{C}$ first differs at level k, with $k\le N$. Let $x\in\mathcal{C}(k)$ be purely crossing.

(1) *Linear step.* The leg-sums and merges send x to $\mathcal{C}(k-1)=\mathcal{NC}(k-1)$, and they preserve $\mathcal{NC}$. So the crossing coordinates of d_ix and m_ix vanish. This is a linear system $M_k(N)x=0$, with
$$M_k(N)=A_k+NB_k$$
for integer matrices A_k,B_k, with $2k\,|\mathcal{P}(k-1)\setminus\mathcal{NC}(k-1)|$ rows and $|\mathcal{P}(k)\setminus\mathcal{NC}(k)|$ columns. We denote its kernel by $K_k(N)$, and we choose a basis $w_1,\dots,w_d$, writing $x=\sum_b c_bw_b$.
(2) *Quadratic step.* The ribbon contractions of $x\otimes x$ landing at level $<k$ lie in $\mathcal{NC}$, so their crossing coordinates vanish. These are quadratic forms in c, and they must all vanish at c.
(3) *Cubic step.* If the quadrics do not suffice, we add either *closure conditions* or *triple contractions*:
   - closure: the ribbon contractions of $x\otimes x$ landing at level k must lie in $\mathcal{NC}(k)\oplus(\mathcal{C}(k)/\mathcal{NC}(k))$;
   - triple contractions: $\rho_2(\rho_1(x\otimes x)\otimes x)$, landing at level $<k$, must have zero crossing part.

   Both give cubic equations in c. Triple contractions have the advantage of requiring no knowledge of the dimension of $\mathcal{C}(k)/\mathcal{NC}(k)$.

If the resulting system of polynomial equations has no nonzero solution, then $\mathcal{C}$ cannot first differ at level k.

Table 7.1 shows how the problem grows. Each level brings its own phenomenon:

level k	$\lvert\mathcal{P}(k)\rvert$	$\lvert\mathcal{NC}(k)\rvert$	crossing	$\dim K_k(N)$	quadrics
4	15	14	1	1	–
5	52	42	10	0	–
6	203	132	71	5	9 of 15
7	877	429	448	14	105 of 105
8	4140	1430	2710	66	1154 of 2211

TABLE 7.1. Sizes at each level. "Quadrics" is the dimension of the space of quadratic forms produced by the quadratic step, out of the dimension of all quadratic forms in $\dim K_k$ variables. The dimension of $K_k(N)$ is independent of N in the range where it was computed.

- at level 5 the linear step alone suffices;
- at level 6 the quadrics leave 4 lines, and a cubic step is needed;
- at level 7 the quadrics already exclude everything;
- at level 8 neither the quadrics nor the simplest cubic conditions suffice, but they force a strong symmetry.

7.5. **Levels up to five.**

Proposition 7.5 (exact, all $N \geq 5$)**.** *If $S_N \subsetneq G \subset S_N^+$ with $N \geq 5$, then $\mathcal{C}(k) = \mathcal{NC}(k)$ for all $k \leq 5$.*

Proof. We go level by level.

- *Levels $k \leq 3$.* Here $\mathcal{P}(k) = \mathcal{NC}(k)$, so there is nothing to prove.
- *Level* 4. The only crossing partition in $\mathcal{P}(4)$ is the basic crossing $\{13\}\{24\}$. If $\mathcal{C}(4) \neq \mathcal{NC}(4)$, its crossing part gives $\xi_{\{13\}\{24\}} \in \mathcal{C}(4)$. But $\mathcal{NC}$ and the basic crossing generate the category $\mathcal{P}$ of all partitions, which corresponds to S_N. So $G \subset S_N$, a contradiction.
- *Level* 5. By the level-4 case, the linear step applies. The matrix $M_5(N)$ is square, of size 10×10: 10 moves, times the single crossing partition of $\mathcal{P}(4)$, against the 10 crossing partitions of $\mathcal{P}(5)$. Its determinant, computed exactly as a polynomial in N, is
$$\det M_5(N) = -(N-4)(N^2-3N+1)^2,$$
which does not vanish at any integer $N \geq 5$. So $K_5(N) = 0$. □

Remark 7.6. Some comments on these factors.

- The roots of $N^2 - 3N + 1$ are $4\cos^2(\pi/5)$ and $4\cos^2(2\pi/5)$. Similarly, the level-6 minor of Proposition 8.2 has the quadratic factor N^2-4N+2, with roots $4\cos^2(\pi/8)$ and $4\cos^2(3\pi/8)$. These are Beraha numbers, the values where Temperley–Lieb Gram determinants degenerate.
- Since $\det M_5(N)$ comes from a square matrix, it is intrinsic up to sign. The level-6 quantity is a particular minor, and whether its quadratic factor is intrinsic has not been checked.

Remark 7.7. A natural source of further linear conditions is to contract $x \in \mathcal{C}(k)$ with noncrossing partitions η, landing at a level $< k$. We computed these at $N = 11$ for η of level ≤ 4 (for $k = 6, 7$) and of level ≤ 3 (for $k = 8$). In every case they impose no condition beyond $M_k(N)x = 0$. This supports the expectation that leg-sums and merges generate all the relevant linear operations.

7.6. **How the computations are certified.** The computations of Sections 8 and 9 are far too large to be checked by hand. They were organized so that each claimed fact rests on one of the following elementary principles, rather than on floating-point evidence.

(1) *Polynomials in N from finitely many values.* Every quantity we compute at level k is a polynomial in N, once denominators are cleared, and we bound its degree *a priori*. For instance:
   - a coefficient of a ribbon contraction has degree $\leq 2 \deg w + k$, because every closed loop of a ribbon contains a string;
   - a determinant has degree at most the sum of its row degrees.

   A polynomial of degree $\leq D$ is determined by $D+1$ values, which we compute exactly, and we always check a few further values.

(2) *Identities in $\mathbb{Q}[N]$.* When a vector depending on N is claimed to lie in a kernel, the identity $M_k(N)w(N) = 0$ is verified as an identity of polynomials, not at sample points.

(3) *Multimodular reconstruction.* Large determinants are computed modulo many primes, as polynomials over $\mathbb{F}_p$, and reconstructed by the Chinese remainder theorem. The number of primes is dictated by the Hadamard-type bound

$$\|\det M\|_\infty \leq \prod_i \Big(\sum_j \|M_{ij}\|_1 \Big),$$

   where $\|\cdot\|_1$ is the sum of the absolute values of the coefficients. Spare primes are used as a consistency check.

(4) *Rational reconstruction.* At a fixed N, rational kernel bases are obtained modulo several primes and lifted by rational reconstruction. They are then *verified* by an exact integer computation of $M_k(N)v$.

(5) *Ranks modulo p.* The rank of an integer matrix modulo a prime is a lower bound for its rank over $\mathbb{Q}$. So any "full rank" statement verified modulo p holds in characteristic zero, provided the matrix modulo p is the reduction of the true one; points 2 and 4 guarantee this. To use the rotation symmetry at level 8 we work modulo $p \equiv 1 \bmod 8$, so that $\mathbb{F}_p$ contains the 8-th roots of unity.

(6) *Algebraic geometry.* Finiteness and degrees of solution sets come from Hilbert functions, via Gotzmann's persistence theorem and Macaulay matrices. Real roots of the resulting polynomials in N are located by exact sign evaluation and Descartes' rule of signs.

Remark 7.8. All code is in Python, using exact rational and modular arithmetic. Floating point is used only for exploration, never in a certificate. The computations ran on a two-core server and on two laptops, which handled the parallel parts: the Step 1 certificate at level 7, and most values of N at level 8.

8. Level six, settled for all N

Level 6 is the first level not covered by the results recalled in Remark 7.3 and Proposition 7.5. It is also the first level where the three-step method of Section 7 runs in full. The linear step leaves 5 dimensions, the quadrics leave 4 lines, and a cubic step is needed to exclude them. The main result of this section is the following.

Theorem 8.1. *Let $N \geq 6$, and let $S_N \subsetneq G \subset S_N^+$. Then*

$$\mathrm{Fix}(u^{\otimes 6}) = \mathrm{span}\{\xi_\pi : \pi \in \mathcal{NC}(6)\}.$$

Together with Proposition 7.5, $\dim \mathrm{Fix}(u^{\otimes k})$ is the Catalan number C_k for every $k \leq 6$. So the first six moments of the main character of G coincide with those of S_N^+.

The proof treats N as a formal parameter throughout. Every step is reduced to the non-vanishing of an explicit polynomial in N, and the finitely many integer roots of these polynomials are all smaller than 6. After the proof we describe the objects involved in more detail. At $N = 6, 7, 8, 11$ we give

the exact solution set of the quadrics, and for N formal we determine the parameters at which a candidate could survive. The answer to the latter is $N \in \{0, 2, 5\}$, and at $N = 5$ the survivor is just the linear relation of Lemma 7.1.

Throughout, $|\mathcal{P}(6)| = 203$, $|\mathcal{NC}(6)| = 132$, and the crossing part has dimension 71.

8.1. **The linear step, uniformly in N.** The matrix $M_6(N) = A_6 + NB_6$ has size 120×71: 12 moves, times the 10 crossing partitions of $\mathcal{P}(5)$.

Proposition 8.2 (exact, all N)**.** *For every integer $N \geq 3$ we have $\dim K_6(N) = 5$. More precisely:*

(1) There are vectors $w_1(N), \ldots, w_5(N)$ with entries in $\mathbb{Z}[N]$, of degree ≤ 4, such that $M_6(N)w_b(N) = 0$ identically in N. On the coordinates of the last five crossing partitions, in our ordering, they equal $N(N-2)$ times the standard basis.

(2) A 66×66 minor of $M_6(N)$ equals

$$\Delta_6(N) = -N^{29}(N-2)^3(N^2 - 4N + 2).$$

Since $2 \pm \sqrt{2} \notin \mathbb{Z}$, we have $\Delta_6(N) \neq 0$ for $N \geq 3$. So $\operatorname{rank} M_6(N) = 66$, and the $w_b(N)$ form a basis of $K_6(N)$.

Proof. The proof is computational, as follows.

- The canonical kernel basis (reduced row echelon form) was computed exactly at $N = 6, \ldots, 45$. Its entries were reconstructed as polynomials of degree ≤ 4 divided by $N(N-2)$, by interpolation on 25 values and verification on the other 15.
- The identity $M_6(N)w_b(N) = 0$ was then checked in $\mathbb{Q}[N]$.
- The entries of $M_6(N)$ have degree ≤ 1, so Δ_6 has degree ≤ 66. It was interpolated from 67 exact values and checked at two more. □

Remark 8.3. Some comments on this basis.

- The five basis vectors involve 32, 40, 41, 45 and 50 crossing partitions. This is consistent with McCarthy's bound $(31, 6)$ stopping exactly there. Whether some element of $K_6(N)$ has support ≤ 31 has not been checked.
- The rotation of the six legs acts on $K_6(N)$ with eigenvalues $1, 1, 1, \omega, \bar{\omega}$, where $\omega = e^{2\pi i/3}$.

8.2. **The quadrics, and finiteness.** Write $\xi = \sum_b c_b w_b(N)$ with $c \in \mathbb{C}^5$. A ribbon contraction of $\xi \otimes \xi$ landing at level ≤ 5 must lie in $\mathcal{NC}$, so its crossing coordinates vanish. Such levels arise for $j = 4, 5, 6$ strings, and they give 792 quadratic forms in c, whose coefficients are polynomials in N.

Lemma 8.4. *The coefficients of these quadrics, and of the level-6 outputs used below, are polynomials in N of degree ≤ 14.*

Proof. The entries of w_b have degree ≤ 4. Each closed loop of a ribbon contains a string, because the legs not on strings are output legs. So there are at most 6 loops, and the degree is at most $4 + 4 + 6$. □

All these coefficients were computed exactly at $N = 1, \ldots, 15$ and interpolated, then checked at $N = 16, \ldots, 20$ and at $N = 50$. The largest degree that actually occurs is 8.

Proposition 8.5 (exact, all N: finiteness)**.** *There are 9 of the quadrics, $q_1, \ldots, q_9$, with the following property. Restrict them to the hyperplane $c_3 = 0$, and form the degree-3 Macaulay matrix in the remaining variables c_0, c_1, c_2, c_4. It has 36 rows (q_a times a variable) and 20 columns (the cubic monomials). One of its 20×20 minors equals*

$$D_2(N) = \lambda\, N^{48}(N-1)(N-2)^{31}(N-3)^9(N-4)^8 R_{41}(N),$$

where $\lambda \in \mathbb{Q}^\times$ and $R_{41} \in \mathbb{Q}[N]$ has degree 41. Moreover $R_{41}(6) \neq 0$, and all the coefficients of $R_{41}(6 + x)$ have the same sign. Hence $D_2(N) \neq 0$ for every real $N \geq 6$.

Proof. The degree of D_2 is at most the sum of the row degrees, namely 152; its actual degree is 138. It was interpolated from 153 exact values and checked at two more. The sign statement is Descartes' rule of signs on $(6,\infty)$. □

Corollary 8.6. *For $N \geq 6$, the common zero set of $q_1, \dots, q_9$ in $\mathbb{P}^4$ is finite. Hence $W = \mathcal{C}(6)/\mathcal{NC}(6)$ has dimension at most 1.*

Proof. By Proposition 8.5, the cubics $c_i q_a|_{c_3=0}$ span all cubics in c_0, c_1, c_2, c_4, so the zero set misses the hyperplane $c_3 = 0$. A projective variety of positive dimension meets every hyperplane, so the zero set is finite. The projectivization of W is a linear space contained in it, hence a point at most. □

8.3. **Closure.** Let $W = \mathbb{C}\xi$ be a line, so that $\mathcal{C}(6) = \mathcal{NC}(6) \oplus \mathbb{C}\xi$. Every ribbon contraction of $\xi \otimes \xi$ landing at level 6 must then lie in $\mathcal{NC}(6) \oplus \mathbb{C}\xi$. These are the types $(j, o) = (3, 0)$ and $(4, 2)$, where $o = o_1 + o_2$, which makes 72 contractions. Equivalently, the crossing part $o(c)$ of the output must be proportional to the crossing part $x(c)$ of ξ. So all the minors $o_P(c)x_Q(c) - o_Q(c)x_P(c)$ vanish, and these are cubic forms in c.

Proposition 8.7 (exact, all N: closure)**.** *There are 35 of these cubics, coming from the contractions of types $(3,0)$ and $(4,2)$ at a fixed position, whose coefficient matrix (35×35, in the monomial basis of cubics) has determinant*

$$D_3(N) = \mu\, N^{127}(N-2)^{90}(N-3)^2(N-4)^3(N-5)\, R_{63}(N),$$

where $\mu \in \mathbb{Q}^\times$ and $R_{63} \in \mathbb{Q}[N]$ has degree 63. The polynomial R_{63} has no integer root $N \geq 6$:

- *Descartes' rule on $R_{63}(16 + x)$ shows that it has no root beyond 16;*
- *exact evaluation at $N = 6, \dots, 16$ gives the signs $+, -, -, +, +, +, +, +, +, +, +$;*
- *Descartes' rule on $(6,\infty)$ allows at most 2 roots.*

So its real roots in $(6,\infty)$ lie in $(6,7)$ and in $(8,9)$, and $D_3(N) \neq 0$ for every integer $N \geq 6$.

Proof. The degree of D_3 is at most 329, and its actual degree is 286. It was interpolated from 330 exact values and checked at two more. □

Proof of Theorem 8.1. By Proposition 7.5, $\mathcal{C}(l) = \mathcal{NC}(l)$ for $l \leq 5$. Assume $\mathcal{C}(6) \neq \mathcal{NC}(6)$, so that $\mathcal{C}$ first differs at level 6.

- By Lemma 7.4 and Proposition 8.2, the crossing part of $\mathcal{C}(6)$ lies in $K_6(N)$.
- By Corollary 8.6, $W = \mathcal{C}(6)/\mathcal{NC}(6)$ is a line $\mathbb{C}\xi$, with $\xi = \sum c_b w_b(N)$ and $c \neq 0$.
- By Lemma 7.4, the 35 cubics of Proposition 8.7 vanish at c.
- Since $D_3(N) \neq 0$, these cubics span all cubic forms. So every cubic monomial vanishes at c, hence $c = 0$, a contradiction. □

Remark 8.8. Some comments on the roles of the two steps.

- Proposition 8.7 shows that the closure conditions alone, without the quadrics, admit no nonzero solution. The roles of the two steps are therefore clean: the quadrics give $\dim W \leq 1$, and closure rules out $\dim W = 1$.
- The two real roots of R_{63} depend on the choice of minor, and we attach no meaning to them.

8.4. **The exact solution set at $N = 6, 7, 8, 11$.** At individual values of N, the full set of 792 quadrics can be analyzed completely, by Gröbner bases over $\mathbb{Q}$, the unknowns being the 5 coordinates $c_0, \dots, c_4$ of a candidate in a basis of $K_6(N)$. This shows what the method is actually excluding, namely a finite set of complex candidates, all of which are then killed by closure, and it also shows that the quadrics alone are not enough, the closure step being needed in all the cases below.

Proposition 8.9 (exact over $\mathbb{Q}$)**.** *For $N = 6, 7, 8, 11$, let $I \subset R = \mathbb{Q}[c_0, \ldots, c_4]$ be the ideal generated by all the quadrics, in the basis of $K_6(N)$ used in the computation.*

- *The quadrics span a 9-dimensional space.*
- *The Hilbert function of R/I is $H(2) = 6$ and $H(d) = 4$ for $3 \le d \le 6$.*
- *The zero set of I in $\mathbb{P}^4(\mathbb{C})$ consists of exactly 4 simple points, which form a single Galois orbit over $\mathbb{Q}$.*
- *At $N = 6$ all four points are real. At $N = 7, 8, 11$, two are real and two are complex conjugate.*
- *Every point satisfies $c_0 = c_2$ and $c_1 = c_4$, which reflects the dihedral symmetry.*
- *At each of the 4 points, every one of the 72 level-6 contractions violates closure.*

Proof. The proof is computational, as follows.

- *Finiteness and degree.* Since $H(4) = H(5) = 4 = 4^{\langle 4 \rangle}$, Gotzmann's persistence theorem gives $H(d) = 4$ for all $d \ge 4$, so the zero set is finite of degree 4.
- *Distinct points.* Choose a linear form ℓ whose multiplication map $(R/I)_4 \to (R/I)_5$ is invertible; this is an exact 4×4 determinant. The operators $T_i = \ell^{-1} c_i$ on $(R/I)_4$ commute, and their joint eigenvalues are the values of c_i/ℓ at the points. For a generic combination $T_g = \sum g_i T_i$, the characteristic polynomial $\chi_g \in \mathbb{Q}[t]$ is squarefree. This gives 4 distinct simple points.
- *Galois orbit.* χ_g is irreducible, since its factorization pattern modulo some prime is (4).
- *Closure.* Writing $T_i = p_i(T_g)$ gives a generic point $c = (p_i(t))$ over the field $F = \mathbb{Q}[t]/(\chi_g)$. The closure minors are elements of F, and they are nonzero, hence nonzero at all four embeddings $F \hookrightarrow \mathbb{C}$. □

Remark 8.10. Some comments on these solution sets.

- At $N = 6$, no factorization pattern $(1, 3)$ appears modulo primes, while it does at $N = 7, 8, 11$. This suggests a Galois group D_4 or smaller at $N = 6$, against S_4 for the others.
- Numerically, the relative closure violations at the 4 points are at least 0.018 at $N = 6$, and at least 0.05 at $N = 7$. So none of the 72 contractions comes close to being compatible.

8.5. **The survival locus.** Everything at level 6 is polynomial in N. So we may treat N as a formal complex parameter, and ask for which $N \in \mathbb{C}$ some candidate survives all the level-6 constraints.

To answer, we work on the coordinate ring of the quadric scheme. For generic N the quadrics cut out 4 points, and on $(R/I)_4$ we have the commuting operators $T_i = \ell^{-1} c_i$. For a closure cubic m, the norm

$$\mathrm{Nm}_\ell(m)(N) = \det m(T_0, \ldots, T_4) = \prod_{\text{points } x} \frac{m(x)}{\ell(x)^3}$$

vanishes iff m vanishes at one of the points. It is a rational function of N.

Proposition 8.11 (exact)**.** *Take two random integer combinations m_1, m_2 of the 134 closure cubics coming from the contractions of types $(3, 0)$ and $(4, 2)$ at a fixed position, and two auxiliary forms ℓ, ℓ'.*

- *The four norms $\mathrm{Nm}_\ell(m_i)$, $\mathrm{Nm}_{\ell'}(m_i)$ are rational functions of type $(55, 24)$.*
- *The gcd of their numerators is*
$$N^{12}(N-2)^{12}(N-5)^2.$$
- *The denominators of Nm_ℓ and $\mathrm{Nm}_{\ell'}$ have no common root.*

Hence, outside the finitely many N where the Hilbert function of the quadric ideal jumps, a level-6 candidate can survive closure only if $N \in \{0, 2, 5\}$.

Proof. Each norm was computed exactly at $N = 6, \dots, 199$ (for ℓ) or $N = 6, \dots, 95$ (for ℓ'). It was reconstructed by Padé approximation from 80 values, and checked on all the others. The roots of the denominators are the values of N at which ℓ vanishes at one of the points. At those values, the norms with respect to the other form apply. □

Proposition 8.12 (the parameter $N = 5$)**.** *At $N = 5$ the Hilbert function of the quadric ideal stabilizes at 6, and the quadric scheme is a single rational point of multiplicity 6. This point is the crossing part of the relation of Lemma 7.1,*

$$\sum_{\pi \in \mathcal{P}(6)} \mu(0_6, \pi)\, \xi_\pi = 0 \qquad (N = 5),$$

with coordinates $(6, 6, 6, -1, 6)$ in the basis used.

Remark 8.13. Some comments on the survival locus.

- The only parameter $N \geq 3$ at which a level-6 candidate survives is therefore $N = 5$, and the survivor is the linear dependence among the partition vectors. As an actual vector it is noncrossing, so it lies in every category, trivially.
- In this sense the level-6 constraints are *sharp*: away from degenerate parameters, they exclude everything except genuine linear relations.
- No Jones-type values $4\cos^2(\pi/n)$ appear in the survival locus, only in the linear step.
- A complete version of Proposition 8.11 would use all 72 closure contractions, and would determine the finite set where the quadric scheme jumps.

8.6. **Triple contractions at level six.** The closure step uses the fact that W is a line. The triple contractions of Section 7 avoid this assumption, and level 6 is a good place to test their strength.

Proposition 8.14 (computational, $N = 11$)**.** *At level 6 and $N = 11$, the quadrics times linear forms span 31 of the 35 cubic monomials in c. Adding the cubics of the triple contractions with ρ_1 of types $(j, o) = (3, 0)$ and $(4, 2)$ at position 0, and ρ_2 ranging over the ribbons at position 0 landing at level ≤ 5, gives all 35.*

So at level 6, quadrics and triple contractions together exclude every candidate, without passing through Corollary 8.6 and closure. This is the version of the cubic step that we use at level 8, where nothing like Corollary 8.6 is available.

9. Levels seven, eight and nine

In this section we push the method of Section 7 further. Level 7 is settled completely, for all $N \geq 6$, the quadrics alone excluding everything, although the candidate space is larger than at level 6 (14 dimensions against 5). Level 8 is the first level where the method meets real resistance: the quadrics and the simplest cubic conditions no longer suffice, but they force a strong symmetry, the new vectors being invariant under the half-turn of the eight legs, for $7 \leq N \leq 37$. Using the rotation symmetry systematically, we then close level 8 computationally, N by N, for $8 \leq N \leq 37$, and level 9 numerically, by positivity, at eight values of N.

Theorem 9.1. *Let $S_N \subsetneq G \subset S_N^+$.*

(1) If $N \geq 6$, then $\dim \mathrm{Fix}(u^{\otimes k}) = C_k$ for all $k \leq 7$.

(2) If $7 \leq N \leq 37$ and G first differs from S_N^+ at level 8, then $\mathrm{Fix}(u^{\otimes 8})/\mathcal{NC}(8)$ consists of half-turn-invariant vectors.

(3) (computational) *If $8 \leq N \leq 37$, then $\dim \mathrm{Fix}(u^{\otimes k}) = C_k$ for all $k \leq 8$.*

(4) (numerical) *If $N \in \{9, \dots, 13, 17, 19, 23\}$, then $\dim \mathrm{Fix}(u^{\otimes k}) = C_k$ for all $k \leq 9$.*

Part 1 is Theorem 9.5 below, part 2 is Theorem 9.9, part 3 is Corollary 9.13, and part 4 is Theorem 9.24.

9.1. **Level seven.** Here $|\mathcal{P}(7)| = 877$ and $|\mathcal{NC}(7)| = 429$, so the crossing part has dimension 448. By Theorem 8.1 and Proposition 7.5, for $S_N \subsetneq G$ and $N \geq 6$ we already know $\mathcal{C}(l) = \mathcal{NC}(l)$ for $l \leq 6$.

Proposition 9.2 (exact, $N \geq 7$: linear step)**.** *Let $M_7(N) = A_7 + NB_7$, of size 994×448, and $K_7(N) = \ker M_7(N)$.*

(1) There are 14 vectors $w_1(N), \dots, w_{14}(N)$, with entries in $\mathbb{Q}[N]$ of degree ≤ 9, such that $M_7(N)w_b(N) = 0$ identically in N. On 14 fixed coordinates they equal $\delta(N)$ times the standard basis, where

$$\delta(N) = \tfrac{1}{2}(N-1)(N-5)(2N^4 - 17N^3 + 50N^2 - 57N + 16),$$

whose only integer roots are 1 and 5.

(2) A 434×434 minor of $M_7(N)$ equals

$$\Delta_7(N) = \lambda\, N^{47}(N-1)(N-2)(N-3)(N-4)\, R_{21}(N),$$

where R_{21} has degree 21, $R_{21}(7) \neq 0$, and $R_{21}(7+x)$ has coefficients of a single sign.

Hence $\dim K_7(N) = 14$ for all $N \geq 7$, with basis $w_1(N), \dots, w_{14}(N)$.

Proof. The proof is computational, as follows.

- The kernel was computed exactly, by sparse elimination, at $N = 7, \dots, 76$. The common denominator was found by Padé reconstruction, and the numerators by interpolation, with checks on held-out values.
- The identity $M_7(N)w_b(N) = 0$ was verified in $\mathbb{Q}[N]$.
- Δ_7 has degree ≤ 434; its actual degree is 72. It was computed modulo 37 primes of 30 bits, on two laptops, as polynomials over $\mathbb{F}_p$. It was then reconstructed by Chinese remaindering: the Hadamard bound 2^{824} requires 28 primes, and the other 9 confirm. □

Proposition 9.3 (exact, $N \geq 7$: quadrics)**.** *Write $\xi = \sum_b c_b w_b(N)$ with $c \in \mathbb{C}^{14}$. There are 105 ribbon contractions landing at level ≤ 6, from 35 configurations, whose 105×105 coefficient matrix, in the monomial basis of quadratic forms, has determinant*

$$D_7(N) = \mu\, N^{54}(N-1)^{203}(N-2)^{25}(N-3)^4(N-4)^{30}(N-5)^{263}(N-6)^{14}\, R_{1058}(N).$$

Here R_{1058} has degree 1058 and exactly one real root in $(7, \infty)$, which lies in $(8, 9)$. So $D_7(N) \neq 0$ for every integer $N \geq 7$, and for these N the quadrics span all 105 quadratic forms on $\mathbb{C}^{14}$.

Proof. The proof is computational, as follows.

- Each coefficient is a polynomial in N of degree $\leq 9 + 9 + 7 = 25$, by the argument of Lemma 8.4. The coefficients were computed exactly at $N = 1, \dots, 32$ and interpolated, with 6 check values. The actual degrees are 17 to 19.
- D_7 has degree at most 1890, the sum of the row degrees; its actual degree is 1651. Its coefficients are bounded by 2^{3762}. It was computed modulo 140 primes and reconstructed from 126 of them; the other 14 confirm.
- Descartes' rule gives exactly one sign change for $R_{1058}(7+x)$, hence exactly one real root beyond 7. Exact evaluation gives the signs $-, -, +$ at $N = 7, 8, 9$. □

Proposition 9.4 (exact: the case $N = 6$)**.** *At $N = 6$ the vectors ξ_π, $\pi \in \mathcal{P}(7)$, satisfy exactly one linear relation (Lemma 7.1). Let r be its crossing part.*

- *r lies in $\ker M_7(6)$, which has dimension 14.*
- *Complete r to a basis $r, e_1, \dots, e_{13}$ of $\ker M_7(6)$, and write $x = c_0 r + \sum_{i \geq 1} c_i e_i$. Then no quadric involves c_0; this was checked in exact arithmetic for all 1147 rows used.*
- *The quadrics span all 91 quadratic forms in $c_1, \dots, c_{13}$.*

Theorem 9.5. *Let $N \geq 6$ and $S_N \subsetneq G \subset S_N^+$. Then $\mathrm{Fix}(u^{\otimes 7}) = \mathrm{span}\{\xi_\pi : \pi \in \mathcal{NC}(7)\}$, and hence $\dim \mathrm{Fix}(u^{\otimes k}) = C_k$ for all $k \leq 7$.*

Proof. Let $\xi \in \mathcal{C}(7)$ be purely crossing.

- *Case $N \geq 7$.* By Proposition 9.2, $\xi = \sum c_b w_b(N)$. All the quadrics of Proposition 9.3 vanish at c. Since they span all quadratic forms, $c_b^2 = 0$ for all b, so $c = 0$.
- *Case $N = 6$.* The same argument, with Proposition 9.4, gives $c_1 = \cdots = c_{13} = 0$. So $\xi = c_0 r$, which as a vector equals $-c_0$ times the noncrossing part of the relation, and lies in $\mathcal{NC}(7)$. □

9.2. **Level eight: what the first two steps give.** At level 8 we have $|\mathcal{P}(8)| = 4140$, $|\mathcal{NC}(8)| = 1430$, and a crossing part of dimension 2710. A uniform-in-N treatment as at levels 6 and 7 is out of reach: a determinant of size 2211×2211 would have degree in N in the tens of thousands. So we work at individual values of N. The exploratory computations below were done at $N = 11$.

Proposition 9.6 (computational, $N = 11$)**.** *(1) $\dim K_8 = 66$. This was computed modulo a prime, and confirmed in floating point by a singular value gap of 13 orders of magnitude.*

(2) Over all 960 ribbon configurations, the quadrics span a space of dimension 1154, out of 2211.

(3) Let I be the ideal of the quadrics. Then $\dim(R/I)_3 = 2350$. Splitting by the eigenvalues ω^j of the rotation of the eight legs, where $\omega = e^{2\pi i/8}$, the contributions are

$$(602,\ 2,\ 596,\ 2,\ 548,\ 2,\ 596,\ 2) \qquad (j = 0, \ldots, 7).$$

Remark 9.7. Two comments on these numbers.

- Since $2211 - 1154 = 1057$ is not of the form $m(m+1)/2$, the common zero set $Z \subset \mathbb{P}^{65}$ of the quadrics is not a linear space.
- A Newton search from 40 random complex starting points never converges to a point of Z: the residual always stalls near $2 \cdot 10^{-3}$. The same holds inside the subspaces fixed by the dihedral group, by the rotations, by R^2, by R^4 and by the reflection. Numerically, Z appears to be empty, but the degree-3 test of item 3 cannot decide this.

9.3. **Triple contractions at level eight.** We now add the cubics of triple contractions (Section 7). Since the rotation of the legs is a symmetry of the whole construction, the space of constraints is rotation-invariant. Working modulo the prime $p_0 = 1048433 \equiv 1 \bmod 8$, we use coordinates y on K_8 adapted to the rotation, with eigenvalues ω^j. In these coordinates every constraint splits into its homogeneous components of each rotation character, and each component is itself a valid constraint.

Proposition 9.8 (computational, $N = 11$)**.** *(1) Consider the two triple contractions with*

$$\rho_1 = (0,0,4,0,0), \qquad \rho_2 \in \{(0,0,5,0,0), (0,0,5,0,1)\}.$$

They give 519 cubics. Together with the quadrics times linear forms, they span all cubic monomials of odd rotation character.

(2) In the even sectors, which are the parts invariant under the half-turn R^4, the deficiencies decrease under further random triple contractions and then stall:

- *602 → 298 in sector 0;*
- *596 → 37 in sectors 2 and 6;*
- *sector 4 was not run to its plateau (548 → 461 after 8 steps).*

These values stayed constant over more than 25 further steps.

Item 1 has a direct consequence. If all cubic monomials of odd character lie in the ideal of constraints, then so does y_a^3 for every coordinate y_a of odd character j_a, since $3j_a$ is then odd. So every candidate has vanishing odd coordinates, which means it is invariant under R^4.

9.4. **The half-turn theorem.**

Theorem 9.9 (exact, $7 \le N \le 37$)**.** *Let $7 \le N \le 37$, and let $S_N \subsetneq G \subset S_N^+$ first differ from S_N^+ at level 8. Then every element of $\mathrm{Fix}(u^{\otimes 8})$ is, modulo $\mathcal{NC}(8)$, invariant under the rotation of the eight legs by four positions.*

Proof. Fix N. The following certificate was computed for each N in the range.

- *Exact kernel.* A rational basis of $K_8(N)$ is obtained modulo 5 to 10 primes close to 2^{31} and lifted by rational reconstruction. It is certified by checking $M_8(N)v = 0$ in exact integer arithmetic. Together with $\operatorname{rank} M_8(N) \ge 2644$, which holds modulo a prime, this gives $\dim K_8(N) = 66$. The basis reduces modulo p_0 to the modular kernel used below, which makes the modular computations reductions of the true ones.
- *Constraints.* We use the quadrics of the ribbon configurations anchored at position 0, with all their rotation-character components, together with the 519 cubics of item 1 of Proposition 9.8. In every case the quadrics span 1154 dimensions, with character decomposition $(134, 152, 135, 152, 142, 152, 135, 152)$.
- *Conclusion.* The Macaulay matrix in degree 3 has full rank, modulo p_0, in each of the four odd sectors. Ranks modulo p_0 are lower bounds for ranks over $\mathbb{Q}(\omega)$. So, as explained above, all odd coordinates vanish on every candidate.
- *The case $N = 7$.* Here the vectors ξ_π, $\pi \in \mathcal{P}(8)$, satisfy one linear relation, whose crossing part r lies in $K_8(7)$ and is rotation-invariant. Replacing x by $x - \lambda r$ changes neither the class of x modulo $\mathcal{NC}(8)$ nor its half-turn invariance, so we may impose that the r-coordinate vanishes. Without this, every odd sector keeps a deficiency of 78, even after 48 further triple contractions. With it, all odd sectors are full. □

N	machine	time per N
$7, \dots, 11$ and $32, \dots, 37$	two-core server	22–30 min
$12, 14, \dots, 30$	laptop A (two workers)	12–14 min
$13, 15, \dots, 31$	laptop B (two workers)	12–14 min

TABLE 9.1. The computation of Theorem 9.9. Each N is an independent job, and all 31 values gave identical intermediate data.

Remark 9.10. Some comments on this theorem.

- Theorem 9.9 reduces the possible new level-8 vectors from the 66-dimensional space K_8 to its 46-dimensional half-turn-invariant part.
- Closing sectors 2 and 6 as well gives invariance under the rotation by two positions, leaving 26 dimensions. Random triple contractions stall there, at a deficiency of 37 at $N = 11$. This is overcome in Subsection 9.5.
- The intermediate data are the same for all 31 values of N, apart from the linear relation at $N = 7$. This strongly suggests that a uniform statement holds, for all $N \ge 7$ or perhaps all $N \ge 8$. Proving it would require a uniform-in-N version of the odd-sector certificate.

9.5. **Closing level eight.** Let R be the rotation of the k legs. It maps $\mathcal{C}(k)$ to itself, since $u = \bar{u}$, and it permutes the crossing partitions, so it acts on the crossing coordinates. Over a field containing the k-th roots of unity, for instance $\mathbb{F}_p$ with $p \equiv 1 \bmod k$, we can split everything according to the eigenvalues ω^j of R.

Proposition 9.11 (rotation trick)**.** *The space K_k of the linear step is invariant under R. On the ω^j-eigenspace of R, the conditions coming from all leg-sums d_i and all merges m_i are equivalent to those coming from d_0 and m_0 alone.*

Proof. Removing or merging the legs $i, i+1$ of Rx amounts to removing or merging the legs $i+1, i+2$ of x, followed by a rotation at level $k-1$. Since rotations preserve the noncrossing span, the conditions of index $i+1$ on x are those of index i on Rx. This gives the invariance of K_k, and on an eigenvector $Rx = \omega^j x$ the conditions of index i are those of index 0 on $R^i x = \omega^{ij} x$. □

So K_k comes directly with an eigenbasis, computed in blocks of size about $1/k$ of the crossing part. This reproduces $\dim K_k = 5, 14, 66$ at $k = 6, 7, 8$, and makes level 9 accessible. It also explains why the random triple contractions of Proposition 9.8 stall. After splitting by characters, a first ribbon at (s, t) is equivalent to one at $(0, t-s)$, and when x is invariant under R^m only $t-s$ modulo m matters. So the random choices covered few inequivalent contractions, and missed the second ribbons anchored at $s_2 \neq 0$ on $\rho_1(x \otimes x)$.

Proposition 9.12 (computational, $8 \leq N \leq 37$)**.** *Let $8 \leq N \leq 37$, and let y be coordinates adapted to the rotation on the 46-dimensional half-turn-invariant part of K_8, given by Theorem 9.9: there are 14, 10, 12, 10 of them in the sectors $j = 0, 2, 4, 6$. Modulo $p_0 = 1048433$:*

1. *The quadrics put y_a^3 in the ideal of constraints for the 20 coordinates of characters 2 and 6. So every candidate is invariant under R^2, and 26 coordinates remain.*
2. *The triple contractions put y_a^3 in the ideal for the 12 coordinates of character 4. So every candidate is invariant under R, and 14 coordinates remain. In this ring, the complete family of triple contractions has $2 \times 64 = 128$ members.*
3. *In the 14 rotation-invariant coordinates, the products of the quadrics with the quadratic monomials, together with the products of the quadrics and the triple contractions with the linear forms, span all 2380 quartic monomials.*

Hence no candidate survives. The chain $46 \to 26 \to 14 \to 0$ is the same for all these N, and was confirmed modulo a second prime, 1048361, for $8 \leq N \leq 15$.

The status of the three steps differs. Step 3 is a full-rank statement, so it is certificate-type: it lifts to $\mathbb{Q}$ as soon as a nonzero minor is exhibited, as explained in Subsection 7.6. Steps 1 and 2 are memberships modulo p, which do not lift automatically, and this is why the result is labeled computational.

Corollary 9.13 (computational)**.** *Let $8 \leq N \leq 37$ and $S_N \subsetneq G \subset S_N^+$. Then $\mathrm{Fix}(u^{\otimes 8}) = \mathrm{span}\{\xi_\pi : \pi \in \mathcal{NC}(8)\}$, and so $\dim \mathrm{Fix}(u^{\otimes k}) = C_k$ for all $k \leq 8$.*

Proof. By Theorem 9.5, G agrees with S_N^+ through level 7. So either it agrees at level 8 too, or it first differs there, and then Theorem 9.9 and Proposition 9.12 leave no candidate. □

N	machine	time per N
$9, 11, \ldots, 37$	laptop A (two workers)	17 min
$8, 10, \ldots, 36$	laptop B (two workers)	17 min
$8, \ldots, 15$, second prime	laptops A and B	17 min
11, check	two-core server	–

TABLE 9.2. The computation of Proposition 9.12, done with the self-contained script `tyson.py` (Python, with numpy and scipy), run overnight on the two laptops.

9.6. **Positivity certificates.** The positivity method of Section 11, which settles the levels of $O_N^* \subset O_N^+$, transports to the present setting, and gives a much cheaper route to most of Theorem 9.9 and Proposition 9.12. Let $S_N \subsetneq G \subset S_N^+$, with category $\mathcal{C}$ first differing from $\mathcal{NC}$ at level $k \leq N$, and let $W = \mathcal{C}(k) \ominus \mathcal{NC}(k) \neq 0$ be the orthogonal complement of the noncrossing span inside $\mathcal{C}(k)$. Besides the rotation R of the legs, we use the antilinear involution $J = \mathrm{conj} \circ \mathrm{rev}$, conjugation of the coordinates composed with the reversal of the legs, so that $J\xi_\pi = \xi_{\mathrm{rev}\,\pi}$.

Lemma 9.14. *R and J preserve $\mathcal{C}(k)$, $\mathcal{NC}(k)$ and W, and $JRJ = R^{-1}$. Hence each $W_\omega = \ker(R-\omega) \cap W$ is J-invariant, and every nonzero W_ω contains a nonzero J-real vector $x = Jx$.*

Proof. The coordinates u_{ij} are projections, so the adjoint of $u_{i_1 j_1} \cdots u_{i_k j_k}$ is the same monomial read backwards: the entrywise conjugate of $u^{\otimes k}$ is $u^{\otimes k}$ with its legs reversed. Since conjugation maps the fixed vectors of a representation v to those of $\bar{v}$, it maps $\mathcal{C}(k)$ onto $\mathrm{rev}(\mathcal{C}(k))$, i.e. $J\mathcal{C}(k) = \mathcal{C}(k)$. Rotation invariance is Frobenius reciprocity. Both operations permute the ξ_π and preserve crossings, and are (anti)unitary, so they preserve $\mathcal{NC}(k)$ and W. The relation $JRJ = R^{-1}$ holds on legs. For $x \in W_\omega$, $RJx = JR^{-1}x = J(\bar{\omega}x) = \omega Jx$, so J is an antilinear involution of W_ω, with nonzero fixed vectors $x + Jx$ or $i(x - Jx)$. □

Proposition 9.15 (exclusion by positivity)**.** *Let U_ω be the ω-eigenspace of R on $K_k(N)$, $U_\omega^J = \{x \in U_\omega : Jx = x\}$ its real form, and let $\mathrm{Re}\, q, \mathrm{Im}\, q$ be the real quadratic forms on U_ω^J obtained from the quadrics $q(x) = \mathrm{coeff}_\tau(\rho(x \otimes x))$ of the quadratic step. If for every ω some real combination of these forms is positive definite on U_ω^J, then $\mathcal{C}$ cannot first differ from $\mathcal{NC}$ at level k.*

Proof. Otherwise some W_ω contains a J-real $x \neq 0$, at which all the quadrics, hence all the real forms, vanish, contradicting positivity. The crossing part of x lies in $K_k(N)$, and is again J-real and an R-eigenvector, since R and J commute with the crossing part and with the orthogonal projection onto $\mathcal{NC}(k)$, so the argument runs in the crossing coordinates. □

On a J-real vector the sesquilinear forms $\mathrm{coeff}_\tau(\rho(x \otimes Jx))$ of Section 11 coincide with the quadrics, so no new constraint is used. What changes is the question: whether the quadrics have a common *real* zero on U_ω^J, rather than a complex one on U_ω, and for this a positive definite combination is a certificate, lifting to $\mathbb{Q}$ as in Section 7.6, and allowing an N-dependent treatment as in Theorem 11.14. Positivity is sufficient, not necessary. Only ribbons anchored at one position are needed, a rotation multiplying the restriction of a quadric to U_ω by a unimodular scalar.

Proposition 9.16 (computational, $N = 11$)**.** *At level 8, with the 1154 quadrics of Proposition 9.6, the real forms on U_j^J admit a positive definite combination for every rotation character $j \neq 0$: for j odd ($\dim U_j^J = 5$) they span all 15 quadratic forms, for $j = 2, 6$ ($\dim 10$) and $j = 4$ ($\dim 12$) the best combination has smallest eigenvalue 0.031, resp. 0.019, with $\|c\| = 1$ and Frobenius-orthonormal forms. For $j = 0$ ($\dim 14$) only 18 of the 105 forms are seen, there is no positive definite combination, and a numerical search finds no real common zero either (`positivity.py`).*

So the quadrics and positivity alone give the half-turn theorem and the steps $46 \to 26 \to 14$ of Proposition 9.12, in seconds and by certificates, and the remaining block $j = 0$ is the one closed by the quartic step $14 \to 0$, itself a full-rank statement. Together these make level 8 certificate-type, as asked in Question 9.21 below. At level 9 the same test applies to the full 307-dimensional kernel, character by character, with no slices, the blocks having dimensions 33 to 37, and it closes the level, all nine blocks included, see Theorem 9.24 below.

9.7. **Level nine: first steps.** At level 9 we have $|\mathcal{P}(9)| = 21147$ and $|\mathcal{NC}(9)| = 4862$, so the crossing part has dimension 16285. The linear step, the quadrics and the slices below are computed modulo a prime, as at level 8, while the positivity certificates of Theorem 9.24 at the end of the section are computed in floating point, from the same kernel recomputed over $\mathbb{R}$, character by character (`nine.py`).

Proposition 9.17 (computational, $N = 11, 13$)**.** *Work modulo* $p = 1048573 \equiv 1 \bmod 9$*, with the same script* `tyson.py`*.*

(1) $\dim K_9 = 307$*, with contributions* $(37, 33, 33, 36, 33, 33, 36, 33, 33)$ *from the rotation characters* $j = 0, \dots, 8$*, and the kernel was checked.*

(2) The quadrics of all 90 *ribbon configurations landing at levels* ≤ 8 *leave gaps of*

$$(4465,\ 4446,\ 4446,\ 4464,\ 4446,\ 4446,\ 4464,\ 4446,\ 4446)$$

in the spaces of quadratic forms of each character, which have dimension about 5250*.*

The results are identical at $N = 11$ *and* $N = 13$*.*

So the sequence of $\dim K_k$ at levels $5, \dots, 9$ is $0, 5, 14, 66, 307$, and at level 9 the quadrics remove only about 15% of the quadratic forms, much as at level 8 and unlike level 7. The level-8 experience suggests the next steps: restrict to a symmetric part, then cubic memberships, then a full-rank statement. Since 9 is odd there is no half-turn, so the first step has to be different.

A natural replacement is to look at the supports of the candidates. Write $E_j \subset K_9$ for the ω^j-eigenspace of the rotation R, over a field containing the 9-th roots of unity. For a nonempty set $S \subset \mathbb{Z}_9$, the *slice* S consists of the candidates lying in $\bigoplus_{j \in S} E_j$. We say that the slice *closes* if it contains no nonzero candidate, and then all the slices $T \subset S$ close too. There are 511 slices, but fewer up to symmetry.

Lemma 9.18 (Galois symmetry)**.** *Let* $a \in \mathbb{Z}_9^*$*. The slice* S *closes if and only if the slice* aS *closes.*

Proof. The rotation R and all the constraints (the linear step, the quadrics and the triple contractions) have rational coefficients. Let σ be an automorphism of $\mathbb{C}$ with $\sigma(\omega) = \omega^a$, acting on vectors coordinatewise. If x is a candidate then so is $\sigma(x)$, and $Rx = \omega^j x$ gives $R\sigma(x) = \omega^{aj}\sigma(x)$. So σ maps the slice S bijectively onto the slice aS. □

The orbits of $\mathbb{Z}_9^* = \{\pm 1, \pm 2, \pm 4\}$ on the characters are $\{0\}$, $\{3, 6\}$ and $\{1, 2, 4, 5, 7, 8\}$, and the 511 slices form 103 orbits. Up to symmetry, there are 3 slices made of one character ($j = 0, 1, 3$) and 8 made of two.

Proposition 9.19 (computational, $N = 11, 13$)**.** *Work modulo* $p = 1048573$*, with the script* `kitty.py`*, starting from the data of Proposition 9.17.*

(1) In none of the 511 *slices do the restricted quadrics span all the quadratic forms. So the quadrics alone close no slice, not even a single character.*

(2) For a random subspace $H \subset K_9$ *of dimension* 117*, spanned by* 13 *random vectors in each* E_j*, the restricted quadrics span all the quadratic forms on* H*. So the common zeros of the quadrics form a cone of dimension at most* $307 - 117 = 190$*.*

(3) On the slices $j = 1$ *and* $j = 2$*, which have* 33 *coordinates each, all the cubic monomials have the single character* $3j$*, and there are* 6545 *of them. The quadrics times the linear forms leave a deficiency of* 2684 *in this sector. The first* 40 *triple contractions, computed in* 2.5 *hours, bring it down to* 1749*, and no cube* y_a^3 *lies in the ideal yet.*

The results are identical at $N = 11$ *and* $N = 13$*.*

Remark 9.20. The computation was done with `kitty.py`, the successor of `tyson.py`, overnight on two laptops, item 3 taking 2.5 hours per slice. At level 8 the same script reproduces the chain $46 \to 26 \to 14 \to 0$ of Proposition 9.12, the quadrics alone closing there every slice made of odd characters; at level 9 they are much weaker. The slices $j = 1, 2, 4$ are conjugate by Lemma 9.18, and indeed gave the same deficiencies, step by step, as did the two values of N. Pushed further (`tiger.py`, 14 hours and 300 triple contractions per slice), the deficiencies stall, at 1344 on the slice $j = 1$ at $N = 11$, and at 2884 on the slice $j = 3$ at $N = 13$, flat over the last 100 contractions. So the degree-3 route, which closed level 8, does not close even a single-character slice at level 9, and this is what led us to positivity.

9.8. **Outlook.** Three things remain at levels 8 and 9. First, Proposition 9.12 should become exact.

Question 9.21. *Prove that if $S_N \subsetneq G \subset S_N^+$ first differs from S_N^+ at level 8, then $\dim \mathrm{Fix}_G(u^{\otimes 8}) = C_8 = 1430$, by certificates which lift from $\mathbb{F}_p$ to $\mathbb{Q}$, at each $8 \leq N \leq 37$, and ideally for all $N \geq 8$.*

Step 3 of Proposition 9.12 is a full-rank statement, which lifts to $\mathbb{Q}$ as soon as a nonzero minor is exhibited. Steps 1 and 2 are memberships modulo p, and there are two ways out: write the memberships with explicit rational coefficients, found by rational reconstruction and checked exactly, or replace steps 1–2 by full-rank statements in each rotation sector, avoiding memberships altogether. The uniformity of the chain $46 \to 26 \to 14 \to 0$ in N suggests that the coefficients are rational functions of N, as at levels 6 and 7, and this would also give the statement for all $N \geq 8$. Second, the small values of N.

Question 9.22. *Settle level 8 at $N = 6$ and $N = 7$.*

For $N \leq 7$ the partition vectors at level 8 satisfy linear relations, so the coordinates are not the same. The half-turn theorem covers $N = 7$, and the closure should follow by a direct adaptation of the computation, with the relations quotiented out first. Third, and most importantly, level 9.

Question 9.23. *Show that $\mathrm{Fix}_G(u^{\otimes 9})$ is spanned by the noncrossing partition vectors, for $S_N \subsetneq G \subset S_N^+$ first differing from S_N^+ at level 9, for all $N \geq 9$, and by exact certificates.*

The sequence of $\dim K_k$ at levels $5, \ldots, 9$ is $0, 5, 14, 66, 307$, and at level 9 the quadrics alone are much weaker than at level 8, with 9 odd there is no half-turn, and the cubic memberships stall (Proposition 9.19). What works is positivity, as in Subsection 9.6, on the J-real forms of the nine rotation blocks of K_9, which at level 8 did in seconds what the cubics did in nights, and which at level 9 answers the question at eight values of N.

Theorem 9.24 (level 9, numerical)**.** *Let $N \in \{9, \ldots, 13, 17, 19, 23\}$. Then no $S_N \subsetneq G \subset S_N^+$ first differs from S_N^+ at level 9. Together with Corollary 9.13, the maximality conjecture holds through level 9 at these N: any such G has the fixed vectors of S_N^+ at all levels $k \leq 9$.*

Proof. By Proposition 9.15 it suffices to find, for each rotation character $j \in \mathbb{Z}_9$, a positive definite combination of the real forms $\mathrm{Re}\, q, \mathrm{Im}\, q$ on U_j^J, where $\dim U_j^J = 37, 33, 33, 36, 33, 33, 36, 33, 33$ for $j = 0, \ldots, 8$, by Proposition 9.17. We used the quadrics of the 90 ribbon configurations of Proposition 9.17, anchored at one position (70 of them at $N = 12$, 67 at $N = 19$). On each U_j^J their real forms span a space of dimension 117 only, out of $\binom{d+1}{2}$ with $d = \dim U_j^J$, but this is enough: the best combination, with $\|c\| = 1$ and Frobenius-orthonormal forms, has smallest eigenvalue at least $5.5 \cdot 10^{-4}$, $1.7 \cdot 10^{-3}$, $3.1 \cdot 10^{-3}$, $4.5 \cdot 10^{-3}$, $5.9 \cdot 10^{-3}$, $1.0 \cdot 10^{-2}$, $1.2 \cdot 10^{-2}$, $1.4 \cdot 10^{-2}$ at $N = 9, \ldots, 13, 17, 19, 23$, against a largest eigenvalue of about 0.45, on every block (`nine.py`, about two hours per N on a laptop, in floating point). The margins increase with N, and the smallest one is each time on the block $j = 0$, the one which at level 8 needed the quartic step. □

10. Orthogonal seeds, duals and twists

In this section we discuss the orthogonal analogue of the maximality conjecture: the chain

$$O_N \subset O_N^* \subset O_N^+$$

is conjectured to be complete. That is, O_N^* should be the only quantum group strictly between O_N and O_N^+. The half of this statement concerning the quantum groups containing O_N^*, namely the maximality of $O_N^* \subset O_N^+$, appears as a conjecture in Brannan, Collins and Vergnioux [23, Sec. 5], with a partial result there. We set up the problem and reformulate it in terms of "seeds". Then we study the two most natural families of seeds, group duals and twists. The main result is about the anticommutative twist O_N^{-1}: the quantum group $\langle O_N, O_N^{-1} \rangle$ agrees with O_N^+ up to level $\max(20, 2N)$, and it equals O_N^+ for all N provided that a single statement holds at $N = 3$, namely

$\langle O_3, O_2^+\rangle = O_3^+$. The induction on N behind this uses the generation result $\langle O_N, O_{N-1}^+\rangle = O_N^+$ for $N \geq 4$ of [23], for which we include a short proof. The level-by-level analysis with pairings is the subject of Section 11.

10.1. **Setting.** Let $N \geq 2$. For a pairing π of $\{1, \ldots, k\}$ let $\xi_\pi = \sum_{i:\ \ker i \geq \pi} e_{i_1} \otimes \cdots \otimes e_{i_k}$. For a closed subgroup $G \subset O_N^+$ write $\mathcal{C}_G(k) = \mathrm{Fix}_G(u^{\otimes k})$. By Frobenius reciprocity, the fixed vectors determine all Hom spaces. We use:

- $\mathrm{TL}(k) = \mathrm{span}\{\xi_\pi : \pi \in NC_2(k)\} = \mathcal{C}_{O_N^+}(k)$, of dimension $\mathcal{C}_{k/2}$ for $N \geq 2$;
- $\mathrm{HL}(k) = \mathrm{span}\{\xi_\pi : \pi \in P_2^*(k)\} = \mathcal{C}_{O_N^*}(k)$, where P_2^* are the pairings joining odd to even points. Here O_N^* is the half-liberated orthogonal group [16], defined by the relations $abc = cba$ between the coordinates; see Bichon and Dubois-Violette [21] for the half-commutative Hopf algebras in general. For $N \geq m$ the $m!$ vectors ξ_π, $\pi \in P_2^*(2m)$, are linearly independent;
- $\mathrm{Br}(k) = \mathrm{span}\{\xi_\pi : \pi \in P_2(k)\} = \mathcal{C}_{O_N}(k)$.

When the dimension matters we write TL_N, Br_N. Intermediate quantum groups $O_N \subset G \subset O_N^+$ correspond to categories $\mathrm{TL} \subset C \subset \mathrm{Br}$. Odd levels vanish for every $G \supset O_N$, and $\mathcal{C}_G$ is closed under tensor products, rotation (since $u = \bar{u}$), contraction of adjacent points, and the antilinear involution $J\xi = \overline{\mathrm{rev}(\xi)}$. As in Section 7, G *first differs at level* k from O_N^+ if $\mathcal{C}_G(l) = \mathrm{TL}(l)$ for $l < k$ and $\mathcal{C}_G(k) \neq \mathrm{TL}(k)$.

Remark 10.1 (unitary picture). By the 2×2 model of O_N^* with U_N-valued entries and Schur–Weyl duality, $\mathrm{HL}(2m) \simeq \mathrm{End}_{U_N}((\mathbb{C}^N)^{\otimes m})$, the image of $\mathbb{C}[S_m]$. Under this identification $\mathrm{TL}(2m)$ becomes the span of the C_m noncrossing permutations $NC(m) \subset S_m$. Since $-1 \in O_N$, the map $G \mapsto PG$ from intermediate $O_N \subset G \subset O_N^+$ to intermediate $PO_N \subset H \subset PO_N^+ = PU_N^+$ is injective, with $PO_N^* = PU_N$. It is not onto in general: its image consists of the H whose category is closed under rotation by one point. See Section 12.

Remark 10.2 ($N = 2$). For $N = 2$ we have $\dim \mathrm{HL}(2m) = \sum_{\lambda \vdash m,\ \ell(\lambda) \leq 2} f_\lambda^2 = C_m = \dim \mathrm{TL}(2m)$, hence $O_2^* = O_2^+$. Since $O_2 \subset O_2^*$ is maximal [8] (this case goes back to Podleś), every intermediate $O_2 \subsetneq G \subset O_2^+$ equals O_2^+. From now on the interesting range is $N \geq 3$.

10.2. **Seeds.** The maximality of $O_N \subset O_N^*$ is known [8]. It reduces completeness to a statement about generation.

Proposition 10.3. *The chain $O_N \subset O_N^* \subset O_N^+$ is complete if and only if*

$$\langle O_N, K\rangle = O_N^+ \quad \textit{for every closed subgroup } K \subset O_N^+ \textit{ with } K \not\subset O_N^*.$$

Proof. Let G be intermediate with $G \notin \{O_N, O_N^*, O_N^+\}$. If $G \subset O_N^*$, then $G = O_N$ by [8], a contradiction. So $G \not\subset O_N^*$, and $K = G$ violates the condition. Conversely, if $K \not\subset O_N^*$ and $\langle O_N, K\rangle \neq O_N^+$, then $\langle O_N, K\rangle$ is intermediate and contains K, so it is neither O_N nor O_N^*. □

We call such a K a *seed*. Note that $\langle O_N, K\rangle$ only depends on the O_N-conjugacy class of K.

Remark 10.4 (seeds ruled out in advance).

- *Quantum permutation groups.* If $K \subset S_N^+$, up to O_N-conjugation, then $\langle O_N, K\rangle \supset \langle S_N, K\rangle$. So K can only be an orthogonal counterexample if $\langle S_N, K\rangle \neq S_N^+$, i.e. if it is already a counterexample to the maximality of $S_N \subset S_N^+$, provided that $\langle O_N, S_N^+\rangle = O_N^+$. We do not know a proof of this last statement in general. Since $\langle O_N, S_N^+\rangle$ contains the corner of $\langle O_{N-1}, S_{N-1}^+\rangle$, Proposition 10.15 below reduces it by induction to the single case $N = 4$, that is, to $\langle O_4, S_4^+\rangle = O_4^+$. Alternatively, S_N^+ contains the dual of $\mathbb{Z}_2 * \mathbb{Z}_2$ on 4 points, which is conjugate under O_N to $\mathrm{diag}(s, t, 1, \ldots, 1)$, so that $\langle O_N, S_N^+\rangle = O_N^+$ would also follow from Conjecture 10.20 below. Granting it, this excludes graph quantum symmetries $G^+(X)$, Hadamard models, and so on.

- *Easy inputs.* If the hull of K with S_N or H_N is easy, then so is $\langle O_N, K\rangle$, and the classification of easy orthogonal quantum groups [13] leaves only O_N, O_N^*, O_N^+. In the unitary setting the same argument shows that every easy $U_N \subset G \subset U_N^+$, in particular every example of Mang–Weber [48], has $PG \in \{PU_N, PU_N^+\}$.
- *Local inputs.* Since $O_2^* = O_2^+$, an exotic G contains no non-classical quantum subgroup acting on 2 coordinates. This is the orthogonal analogue of the restriction proposition of Section 6 (Proposition 6.5).

10.3. **Group duals.** Up to O_N-conjugation, a group dual $\widehat{\Gamma} \subset O_N^+$ is given by a involutions $s_1, \ldots, s_a$ and b elements $g_1, \ldots, g_b$ generating Γ, with $N = a + 2b$. Each involution acts on one coordinate, and each pair g_j, g_j^{-1} acts on a 2×2 rotation block. In the complex basis $f_1, \ldots, f_N$ where u is diagonal, we get an alphabet A of N letters with an involution $r \mapsto \bar{r}$: involution letters are self-inverse, and the two letters of a block are inverse to each other. So Γ is a quotient of $\mathbb{Z}_2^{*a} * F_b$. Unlike in Section 2, generators of order > 2 are allowed.

Proposition 10.5.

(1) $\mathcal{C}_{\widehat{\Gamma}}(k) = \mathrm{span}\{f_w : w \in A^k,\ w_1 \cdots w_k = 1 \text{ in } \Gamma\}$.

(2) In the basis (f_w), $\xi_\pi = \sum f_w$, *summed over the words* w compatible *with* π: *for each pair* (s,t) *of* π, $w_t = \overline{w_s}$. *Hence the category of* $\langle O_N, \widehat{\Gamma}\rangle$ *is*

$$C(k) = \{x \in \mathrm{Br}(k) : x \text{ is supported on trivial words}\},$$

and it only depends on the laws of Γ *of length* k.

(3) $\widehat{\Gamma} \subset O_N$ *iff* Γ *is abelian.* $\widehat{\Gamma} \subset O_N^*$ *iff* Γ *is* half-abelian, *meaning* $xyz = zyx$ *for all letters* $x, y, z \in A$.

Proof. (1) and (2) are direct; for example, $\sum_i e_i \otimes e_i = \sum_r f_r \otimes f_{\bar{r}}$ on each block. For (3), O_N and O_N^* are generated over TL by the crossings (13)(24) and $\kappa = (14)(25)(36)$, and their compatible words are $xy\bar{x}\bar{y}$ and $xyz\bar{x}\bar{y}\bar{z}$. □

Proposition 10.6 (reduction)**.** *If group duals give a counterexample, then some* just non-half-abelian *group gives one: a non-half-abelian* Γ *all of whose proper quotients are half-abelian. Equivalently, the normal closure of the elements* $xyz(zyx)^{-1}$ *is the unique minimal normal subgroup of* Γ.

Proof. If $\Gamma \to \Gamma'$ is a quotient preserving the letters, then Γ' has more laws, so $\langle O_N, \widehat{\Gamma'}\rangle \subset \langle O_N, \widehat{\Gamma}\rangle$, and $\widehat{\Gamma'}$ is still a seed when Γ' is not half-abelian. Half-abelianity is a finite set of relations, so Zorn's lemma applies, as in the just-non-abelian reduction of Section 2. □

Let $D(\Gamma) = \{\pi \in P_2 :\ \xi_\pi \in \mathcal{C}_{\widehat{\Gamma}}\}$ be the set of pairings whose vectors are fixed by $\widehat{\Gamma}$. By Proposition 10.5, the category of $\langle O_N, \widehat{\Gamma}\rangle$ is $\mathrm{Br} \cap \mathcal{C}_{\widehat{\Gamma}}$, and it contains $\mathrm{span}\{\xi_\pi : \pi \in D(\Gamma)\}$.

Definition 10.7. We say that $\widehat{\Gamma} \subset O_N^+$ is *easy over* O_N if

$$\mathrm{Br} \cap \mathcal{C}_{\widehat{\Gamma}} = \mathrm{span}\{\xi_\pi : \pi \in D(\Gamma)\},$$

that is, if the category of $\langle O_N, \widehat{\Gamma}\rangle$ is spanned by pairings.

This is a strong condition: a priori, $\mathrm{Br} \cap \mathcal{C}_{\widehat{\Gamma}}$ could contain linear combinations of pairing vectors supported on trivial words, without any of the pairings in it being in $D(\Gamma)$.

Theorem 10.8 (trichotomy)**.** *(1)* $D(\Gamma)$ *is a category of pairings containing* NC_2. *So it is* NC_2, P_2^* *or* P_2, *according to whether* Γ *is not half-abelian, half-abelian but not abelian, or abelian.*

(2) If $\widehat{\Gamma}$ *is easy over* O_N, *then* $\langle O_N, \widehat{\Gamma}\rangle$ *is* O_N^+, O_N^* *or* O_N, *in the three cases above.*

Proof. (1) Composites of pairing maps are multiples $N^\ell \neq 0$ of pairing maps, so $D(\Gamma)$ is closed under the category operations. The classification of categories of pairings [13, 16] and Proposition 10.5(3) give the rest. (2) is then immediate from Definition 10.7. □

Proposition 10.9 (computational)**.** *The group duals below are easy over O_N at the levels shown. That is, for every generating tuple of permutations listed, taken up to simultaneous conjugation and grouped by their sets of trivial words, the joint dimensions are exactly those of O_N, O_N^* or O_N^+, according to whether Γ is abelian, half-abelian or neither:*

letters	N	*permutations in*	*distinct law sets*	*levels*
1 *involution,* 1 *rotation pair*	3	$S_2, \ldots, S_6$	31 *(in* S_6*)*	$4, 6, 8$
2 *rotation pairs*	4	$S_2, \ldots, S_6$	305 *(in* S_6*)*	$4, 6, 8$
2 *involutions,* 1 *rotation pair*	4	$S_2, \ldots, S_5$	595 *(in* S_5*)*	$4, 6, 8$
1 *involution,* 2 *rotation pairs*	5	$S_2, \ldots, S_5$	2339 *(in* S_5*)*	$4, 6$
3 *involutions,* 1 *rotation pair*	5	$S_2, \ldots, S_4$	917 *(in* S_4*)*	$4, 6$

About 4500 *law sets in all, with no exception. For residually finite Γ, such as $\mathbb{Z}_3 * \mathbb{Z}_3$, $\mathrm{SL}_2(\mathbb{Z})$ or free groups, the laws of a given length are seen by finite quotients, so these searches cover such groups level by level.*

Conjecture 10.10. *Every group dual $\widehat{\Gamma} \subset O_N^+$ is easy over O_N. Hence, by Theorem 10.8, group duals are never seeds of counterexamples.*

A natural approach to Conjecture 10.10 would be a Möbius-type argument, as in the classification of the orthogonal easy quantum groups by Raum and Weber [61]. A special case is Conjecture 10.20 below, coming from $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$.

Remark 10.11 (unitary group duals). Seeds need not live in O_N^+: any $G \subset U_N^+$ gives $PG \subset PU_N^+ = PO_N^+$, and one asks whether $\langle PO_N, PG\rangle = PU_N^+$. For $G = U\widehat{\Gamma}U^*$ the data is a pair (Γ, M) with $M = U^tU$ symmetric unitary, a continuous family. We checked (computational) all 119 law sets at $N = 3$ with generators in $S_2, \ldots, S_5$ and all 611 law sets at $N = 4$ with generators in S_2, S_3, S_4, at levels $4, 6, 8$ and several positions M. In all cases $\langle PO_N, PG\rangle = PU_N^+$ when PG is non-classical, and it is PO_N or $PU_N = PO_N^*$ otherwise. These questions are taken up in Section 12.

10.4. **The anticommutative twist.** Let O_N^{-1} be the twist of O_N in which u_{ij} and u_{kl} anticommute for $i \neq k$, $j \neq l$ and commute otherwise. It is the quantum automorphism group of the N-hypercube [7]. It is not half-commutative for $N \geq 3$, so it is a seed. For a multi-index $i \in [N]^k$ let

$$c(i) = (-1)^{\#\{s<t\,:\,i_s<i_t\}}, \qquad \Theta e_i = c(i)e_i.$$

Lemma 10.12 (signs)**.** *Let $\pi \in P_2(k)$ and $\ker i \geq \pi$. Two blocks of π with different labels contribute -1 to $c(i)$ if they cross, and $+1$ otherwise, whatever the labels. Pairs of positions with equal labels contribute nothing. Consequently $\Theta\xi_\pi$ is the twisted pairing vector of [7], and*

$$\mathcal{C}_{O_N^{-1}}(k) = \Theta\,\mathrm{Br}_N(k), \qquad \mathcal{C}_{\langle O_N, O_N^{-1}\rangle}(k) = \mathrm{Br}_N(k) \cap \Theta\,\mathrm{Br}_N(k).$$

Proof. For blocks $\{s_1 < s_2\}$ labeled a and $\{t_1 < t_2\}$ labeled $b > a$, count the pairs "a-position before b-position". The six relative orders $aabb, abba, bbaa, baab, abab, baba$ give $4, 2, 0, 2, 3, 1$, which is odd exactly in the two crossing cases. □

We write $G_N = \langle O_N, O_N^{-1}\rangle$. The most direct use of Lemma 10.12 gives the low levels.

Proposition 10.13 (exact)**.** *For every $k \leq 2N$ we have $\mathcal{C}_{G_N}(k) = \mathrm{TL}_N(k)$.*

Proof. Let $x = \sum a_\pi\xi_\pi = \Theta\sum b_\pi\xi_\pi$. As $k/2 \leq N$, the coefficients are well defined. On i with $\ker i = \pi$ and distinct labels, Lemma 10.12 gives $a_\pi = (-1)^{cr(\pi)}b_\pi$. Now let π be crossing, pick

two crossing blocks with union Q, and evaluate on i whose kernel merges them, with distinct labels elsewhere. The pairings below $\ker i$ are the three σ that agree with π outside Q, and π is the only one with crossing chords inside Q. A block outside Q crosses the two chords of σ in Q with a total parity independent of σ, and the two chords inside Q carry equal labels. So $c(i)(-1)^{cr(\sigma)} = (-1)^{[\sigma=\pi]}$, and the equation at i reads $\sum_\sigma a_\sigma = \sum_\sigma (-1)^{[\sigma=\pi]} a_\sigma$, i.e. $a_\pi = 0$. □

To go beyond $2N$ we induct on N, through corners. For a closed subgroup $H \subset O_{N-1}^+$ with fundamental representation v, its *corner* is the closed subgroup of O_N^+ with fundamental representation $\mathrm{diag}(v, 1)$. For $x \in (\mathbb{C}^N)^{\otimes k}$ and $S \subset \{1, \ldots, k\}$, the *slice* $x_S \in (\mathbb{C}^{N-1})^{\otimes(k-|S|)}$ collects the coordinates of x with label N exactly on S. Since $\mathrm{diag}(v, 1)^{\otimes k} = \bigoplus_S v^{\otimes(k-|S|)}$,

$$x \text{ is fixed by the corner of } H \iff x_S \in C_H(k - |S|) \text{ for all } S. \tag{10.1}$$

Lemma 10.14 (corners)**.** *For $N \geq 3$, G_N contains the corner of G_{N-1}. More precisely, if $x \in \mathcal{C}_{G_N}(k)$, then $x_S \in \mathcal{C}_{G_{N-1}}(k - |S|)$ for every S.*

Proof. The slice of ξ_π is $\xi_{\pi\setminus S}$ (in dimension $N-1$) if S is a union of blocks of π, and 0 otherwise. So slices map Br_N to Br_{N-1}. If the label N sits exactly on S, the pairs of positions involving it contribute $\#\{p \notin S,\ q \in S,\ p < q\}$ to the exponent of $c(i)$, a number depending only on S. So $(\Theta_N y)_S = \pm \Theta_{N-1} y_S$. Hence $x = \Theta_N y$ with $x, y \in \mathrm{Br}_N$ gives $x_S \in \mathrm{Br}_{N-1} \cap \Theta_{N-1}\mathrm{Br}_{N-1}$. Conclude by (10.1). □

The second ingredient is a generation result of Brannan, Collins and Vergnioux [23, Thm. 4.1], the orthogonal analogue of Chirvasitu's $U_N^+ = \langle U_N, U_{N-1}^+ \rangle$ for $N \geq 3$ [26, 27]. Chirvasitu's argument uses non-orthogonal decompositions, which are harmless for U_N, whose invariants are GL_N-invariants, but not for O_N. We include a short proof in the language of slices used above; it moves a unit vector into a hyperplane, and this is where the restriction $N \geq 4$ enters.

Proposition 10.15 (Brannan–Collins–Vergnioux)**.** *Let $N \geq 4$. If $x \in \mathrm{Br}_N(k)$ is fixed by the corner of O_{N-1}^+, then $x \in \mathrm{TL}_N(k)$. Hence $\langle O_N, O_{N-1}^+ \rangle = O_N^+$.*

Proof. Since x is O_N-invariant, it is fixed by every conjugate of the corner, i.e. by $O^+(e^\perp) \oplus 1$ for every unit vector e. For $W = e^\perp$ and $S \subset \{1, \ldots, k\}$, let x_S^e be the slice of x in a basis containing e, viewed as a multilinear form on W. Here a vector y is read as the multilinear form $y(w_1, \ldots, w_k) = \sum_i y_i (w_1)_{i_1} \cdots (w_k)_{i_k}$, with no complex conjugation; all the vectors e, w_j below are real. By (10.1), $x_S^e \in \mathrm{TL}(W)$.

(1) *Normalize.* Fix e and write $x_\emptyset^e = \sum_\pi \alpha_\pi \xi_\pi^W$, $\pi \in NC_2(k)$. Put $y = \sum_\pi \alpha_\pi \xi_\pi \in \mathrm{TL}_N(k)$ and $z = x - y$. Then $z \in \mathrm{Br}_N(k)$, z is fixed by the corners, and z vanishes on W^k. By O_N-invariance, z vanishes on H^k for every hyperplane H.

(2) *The coefficients do not depend on e.* Write $z_S^e = \sum_\sigma \beta_\sigma^{e,S} \xi_\sigma^{e^\perp}$, with $\sigma \in NC_2$ of the remaining points. This expansion is unique, because $\dim e^\perp = N - 1 \geq 2$ and the Temperley–Lieb vectors are independent at loop parameter ≥ 2. If $g \in O_N$ maps e to e', then $z_S^{e'} \circ g = z_S^e$ and $\xi_\sigma^{e'^\perp} \circ g = \xi_\sigma^{e^\perp}$, so $\beta^{e,S} = \beta^{e',S} =: \beta^S$.

(3) *Move e' into $e^\perp$.* Take unit vectors $e \perp e'$ and evaluate $z_S^{e'}$ on $U = e^\perp \cap e'^\perp$. All the arguments, e' included, lie in the hyperplane $e^\perp$, so the value is 0. Thus $\sum_\sigma \beta_\sigma^S \xi_\sigma^U = 0$, and since $\dim U = N - 2 \geq 2$ we get $\beta^S = 0$.

(4) *Conclude.* All slices z_S^e vanish. Writing each argument as $w_j = \lambda_j e + w_j'$ with $w_j' \perp e$ gives $z(w) = \sum_S \lambda^S z_S^e(w') = 0$. So $x = y \in \mathrm{TL}_N(k)$.

The argument works at each level separately: if $x \in \mathrm{Br}_N(k)$ has all its slices in TL_{N-1}, then $x \in \mathrm{TL}_N(k)$. □

Theorem 10.16. *If $\langle O_3, O_2^+\rangle = O_3^+$, with O_2^+ embedded as a corner, then $\langle O_N, O_N^{-1}\rangle = O_N^+$ for every $N \geq 2$. More precisely, if $\langle O_3, O_2^+\rangle$ agrees with O_3^+ up to level k, then so does G_N for every $N \geq 3$.*

Proof. For $N = 2$, O_2^{-1} is not classical, so $G_2 = O_2^+$ by Remark 10.2. By Lemma 10.14, G_3 contains the corner of $G_2 = O_2^+$, so $\mathcal{C}_{G_3}(l) \subset \mathcal{C}_{\langle O_3, O_2^+\rangle}(l) = \mathrm{TL}_3(l)$ for $l \leq k$. For $N \geq 4$, if $\mathcal{C}_{G_{N-1}}(l) = \mathrm{TL}_{N-1}(l)$ for $l \leq k$, then every $x \in \mathcal{C}_{G_N}(l)$, $l \leq k$, has its slices in TL_{N-1} by Lemma 10.14, hence $x \in \mathrm{TL}_N(l)$ by Proposition 10.15. □

So everything rests on $N = 3$, where step 3 of Proposition 10.15 breaks down: U is a line, and the Temperley–Lieb vectors collapse. This is also the case that required extra care in the unitary setting [26, 27], and the case left open in [23], where it is expected to hold, with numerical evidence, see the discussion following [23, Thm. 4.1]. We now reduce it further, to a question about words, using the torus of O_2^+.

Let $\Gamma = \mathbb{Z}_2^{*3}$ with generators a_1, a_2, a_3, and identify $i \in [3]^k$ with the word $a_{i_1} \cdots a_{i_k}$. For $d = 1, 2, 3$ let $\varphi_d : \Gamma \to \mathbb{Z}_2 * \mathbb{Z}_2$ kill a_d. Call a word *pairwise reducible* if $\varphi_d(w) = 1$ for $d = 1, 2, 3$, and *exotic* if it is pairwise reducible but $w \neq 1$ in Γ. Let $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$ be the free torus of O_N^+, acting diagonally. Chirvasitu proved $O_N^+ = \langle O_N, T_N^+\rangle$ for $N \geq 2$ [27, Theorem 3.3]. Level by level, this says that a vector of $\mathrm{Br}_N(k)$ supported on words trivial in $\mathbb{Z}_2^{*N}$ lies in $\mathrm{TL}_N(k)$.

Proposition 10.17.

(1) Every $z \in \mathcal{C}_{\langle O_3, O_2^+\rangle}(k)$ lies in $\mathrm{Br}_3(k)$ and is supported on pairwise reducible words.

(2) If moreover $z(w) = 0$ for every exotic word w of length k, then $z \in \mathrm{TL}_3(k)$.

(3) Rotation relations. *For $z \in \mathrm{Br}_3(k)$, a word u and letters $x \neq y$,*

$$\sum_{p:\ u_p = y} z(u^{p\to x}) = \sum_{p:\ u_p = x} z(u^{p\to y}),$$

where $u^{p\to x}$ is u with its p-th letter replaced by x.

Proof. (1) O_2^+ contains T_2^+, so $\langle O_3, O_2^+\rangle$ contains $\mathrm{diag}(s, t, 1)$, with s, t free involutions. Fixed vectors of this group dual are supported on words with $\varphi_3(w) = 1$. Since z is invariant under the coordinate permutations in O_3, the same holds for φ_1, φ_2. (2) z is then supported on trivial words, so it is fixed by T_3^+, and we use [27, Theorem 3.3]. (3) z is fixed by the rotations in the plane (e_x, e_y). Differentiating at the identity gives $\sum_p A^{(p)} z = 0$, with $A = E_{xy} - E_{yx}$ acting on the p-th factor. The u-coordinate of this identity is the claimed relation. □

For a word $w = 1$ in Γ there is nothing to prove, and every pairwise reducible word of length ≤ 8 is trivial. The shortest exotic words have length 10, for instance $ababcbabac = [(ab)^2, c]$. Here is how to kill them. Pick a position p, let $x = w_p$, pick a letter $y \neq x$, and apply the rotation relation to $u = w^{p\to y}$. The term with p is $z(w)$. Every other term is $z(v)$ with v obtained from u by switching one more letter between x and y. If none of these v is pairwise reducible, then $z(w) = 0$ by Proposition 10.17(1).

Theorem 10.18 (exact)**.** *For every exotic word w of length $k \leq 20$ there exist p and y such that every word v above which is pairwise reducible is exotic, and killed in the same way. For $k \leq 18$ there is no such v at all. Consequently:*

(1) $\langle O_3, O_2^+\rangle$ agrees with O_3^+ up to level 20;

(2) $\langle O_N, O_N^{-1}\rangle$ agrees with O_N^+ up to level $\max(20, 2N)$, for every $N \geq 2$.

Proof. The first statement is a finite check, done by exhaustive enumeration (`words.py`): there are 10, 240, 3654, 45360, 502482 and 5186470 exotic words of lengths $10, 12, \ldots, 20$ beginning with the letter a, and the others follow by permuting letters. At $k = 20$, exactly 240 of them need the

second step, always with $p = 1$ and $y = b$, through 1 to 5 exotic words v. Then (1) follows from Proposition 10.17, and (2) from Theorem 10.16 and Proposition 10.13. □

Remark 10.19 (what the check sees, and where it must stop).

- Read w as a path in the Cayley tree of Γ, from 1 to the element w. A computation with the three projections φ_d shows that a word v above is pairwise reducible only when the step q "looks like" the step p in all three projections to $\mathbb{Z}_2 * \mathbb{Z}_2$. For instance, this happens when p and q cross the same edge of the tree in opposite directions. So p must be a letter of the reduced form whose edge is crossed only once, and in all cases found the working positions are of this type.
- The single-step argument first fails at length 20, for instance for *acccbaacbaaabcaabcbb*, whose reduced form *acbcbabcbc* has length 10. It cannot work for words such as $rr^{-1}r$ with $r = [(ab)^2, c]$, of length 30, in which every edge of the geodesic is crossed three times. A proof for all k needs an induction. The natural candidate uses the leftover pairwise reducible v, which replace an excursion xCx by yCy, keeping the group element.
- Independently, the conclusion of Theorem 10.18(1) was confirmed by direct linear algebra at levels ≤ 10, without using the torus.

Conjecture 10.20. *Every $z \in \mathrm{Br}_3$ supported on pairwise reducible words vanishes on exotic words. Hence $\langle O_3, O_2^+ \rangle = \langle O_3, \mathrm{diag}(s,t,1) \rangle = O_3^+$, and $\langle O_N, O_N^{-1} \rangle = O_N^+$ for all $N \geq 2$.*

Note that $\langle O_3, \mathrm{diag}(s,t,1) \rangle$ is generated by O_3 and a group dual, with $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$ and letters $s, t, 1$. So Conjecture 10.20 is a case of Conjecture 10.10: it says that this group dual is easy over O_3.

As for a proof, the argument behind Theorem 10.18, with one step up to length 18 and two steps at length 20, must fail in its single-step form for words like $rr^{-1}r$, with $r = [(ab)^2, c]$, which have length 30. A natural approach is by induction on the backtracking of words in the Cayley tree of $\mathbb{Z}_2^{*3}$, using the leftover pairwise reducible words, which replace an excursion xCx by yCy. A next step would be the exhaustive check at length 22, about ten times the cost of length 20, to see whether two steps still suffice.

10.5. **Other twists and seeds.** Some further natural seeds, all computational, support the same picture.

Sign twists of O_N. Twisting O_N by its diagonal subgroup $\mathbb{Z}_2^N$, with signs given by a graph ϵ on N vertices, makes rows i, j anticommute iff ij is an edge. Then $\mathcal{C}_{O_N^\epsilon} = \Theta_\epsilon \mathrm{Br}$ for a diagonal sign operator Θ_ϵ, and the complete graph gives O_N^{-1}. Here $O_N^\epsilon \subset O_N^*$ iff every triangle has an even number of edges, iff ϵ is complete bipartite, possibly empty. For all graphs at $N = 3, 4$ (levels $4, 6, 8$) and $N = 5$ (all 34 graphs, levels $4, 6$), $\langle O_N, O_N^\epsilon \rangle$ is O_N for the empty graph, O_N^* for the complete bipartite ones, and O_N^+ otherwise.

θ-twists. For the Connes–Landi type deformations of O_N by its maximal torus, at $N = 4, 6$ with $\theta \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{2}{5}, \sqrt{2} - 1\}$ and a random value, we always get O_N^+, except $\theta = \frac{1}{2}$ at $N = 4$, which is a complete bipartite sign twist, and gives O_N^*.

Projectively classical examples. The K-half-liberated groups $U_{N,K}^*$ and $U_{N,\infty}$ of Banica and Bichon satisfy $PG = PU_N$, since in their model in $C(U_N^K) \rtimes \mathbb{Z}_K$ the products $u_{ij} u_{kl}^*$ become diagonal with commuting entries. So they lift to O_N^*, and together with the easy case (Remark 10.4) this disposes of the whole Mang–Weber landscape [48].

So every natural non-easy construction we tried is an easy object seen through a change of basis, a twist, or a product construction, and obeys the rule "not half-commutative $\Rightarrow O_N^+$".

11. Pairings and uncappable vectors

We study the intermediate quantum groups $O_N \subset G \subset O_N^+$ level by level, with the notation of Section 10, in the spirit of McCarthy [51] for $S_N \subset S_N^+$ and of Section 7. We first treat $O_N^* \subset G \subset O_N^+$, i.e. categories $\mathrm{TL} \subset C \subset \mathrm{HL}$. The first level where C can differ from TL is $2m = 6$, and there a difference forces $G = O_N^*$. If G first differs from O_N^+ at level $2m \geq 8$, the new vectors are *uncappable*, and gluing such a vector to itself produces quadratic constraints coming from the lower levels. A positivity certificate for these constraints excludes the level. Levels 8 and 10 are excluded exactly for all $N \geq 3$, and levels 12 and 14 numerically. We then treat the full Brauer problem $\mathrm{TL} \subset C \subset \mathrm{Br}$, which gives completeness of $O_N \subset O_N^* \subset O_N^+$ through 10 points for several N. Finally we compare with the permutation case of Section 7.

11.1. **Level 6.**

Proposition 11.1. *$P_2^*(6) = NC_2(6) \sqcup \{\kappa\}$ with $\kappa = (14)(25)(36)$, the rotated half-commutation crossing. Hence if $\mathcal{C}_G(6) \neq \mathrm{TL}(6)$, then $\mathcal{C}_G(6) = \mathrm{HL}(6) \ni \xi_\kappa$, so $abc = cba$ holds in G and $G = O_N^*$.*

Proof. $|P_2^*(6)| = 3! = 6$ and $|NC_2(6)| = 5$, and all of these vectors are independent for $N \geq 3$. Since $\mathrm{HL}(6)/\mathrm{TL}(6)$ is one-dimensional, any extra vector gives ξ_κ modulo TL. □

So if $G \neq O_N^*$, then G can only first differ from O_N^+ at level 8 or higher.

11.2. **The level where G first differs.**

From now on let $m \geq 4$ be minimal with $C(2m) \neq \mathrm{TL}(2m)$, where $C = \mathcal{C}_G$, i.e. G first differs from O_N^+ at level $2m$. Let $W = C(2m) \ominus \mathrm{TL}(2m)$, the orthogonal complement for the standard scalar product.

Definition 11.2. For $1 \leq i \leq 2m$, let $\mathrm{cap}_i : (\mathbb{C}^N)^{\otimes 2m} \to (\mathbb{C}^N)^{\otimes 2m-2}$ contract the points $i, i+1$, cyclically, so that cap_{2m} contracts the last point with the first. A vector is *uncappable* if all $2m$ caps kill it. Let $U(2m) \subset \mathrm{HL}(2m)$ be the space of uncappable vectors.

Lemma 11.3. *$W \subset U(2m)$. Moreover W is invariant under the one-click rotation ρ and under J, and $J\rho J = \rho^{-1}$. Each eigenspace $W_\omega = \ker(\rho - \omega)$ is J-invariant and therefore contains a nonzero J-real vector $x = Jx$.*

Proof. For $x \in W$, $\mathrm{cap}_i x \in C(2m-2) = \mathrm{TL}(2m-2)$ by minimality. For $y \in \mathrm{TL}(2m-2)$ we have $\langle \mathrm{cap}_i x, y\rangle = \langle x, \mathrm{cup}_i y\rangle = 0$, since $\mathrm{cup}_i y \in \mathrm{TL}(2m)$. So $\mathrm{cap}_i x = 0$. Rotation and J preserve both C and TL, and they are (anti)unitary, so they preserve W. Finally $\rho Jx = J\rho^{-1}x = J(\bar{\omega}x) = \omega Jx$ for $x \in W_\omega$, since $|\omega| = 1$. □

Definition 11.4. For pairings p, q of $2m$ points and $a \leq 2m$, let $p \circ_a q$ be the planar gluing of the last a points of p to the first a points of q (rainbow), with a factor $N^{\#\mathrm{loops}}$. Extend bilinearly.

Lemma 11.5. *Let $x \in W$ be J-real and $3 \leq j < m$. Then $x \circ_{2m-j} x \in \mathrm{TL}(2j)$. When $N \geq j$, this means that the coefficients of $x \circ_{2m-j} x$ on the crossing pairings of $P_2^*(2j)$ all vanish. In terms of sesquilinear forms: for such a crossing pairing τ, put $B_\tau(x,y) = \mathrm{coeff}_\tau(x \circ_{2m-j} Jy)$, which is linear in x and antilinear in y, and $q_\tau(x) = B_\tau(x,x)$. Then $\overline{q_\tau(x)} = q_{\mathrm{rev}\,\tau}(x)$, so the real parts $\mathrm{Re}\, q_\tau$ are real quadratic forms on the real space of J-real vectors, and they all vanish at x.*

Proof. $x \circ_{2m-j} x = x \circ_{2m-j} Jx$ is obtained from $x \otimes Jx \in C(4m)$ by noncrossing contractions, so it lies in $C(2j) = \mathrm{TL}(2j)$. Reading x as an operator T_x from $2m-j$ to j strands, $x \circ_{2m-j} Jx$ is the vector of $T_x T_x^*$. Finally $J(a \circ_{2m-j} b) = Jb \circ_{2m-j} Ja$, so $z = x \circ_{2m-j} Jx$ satisfies $Jz = z$, and since $J\xi_\tau = \xi_{\mathrm{rev}\,\tau}$, this gives $\overline{q_\tau(x)} = q_{\mathrm{rev}\,\tau}(x)$. □

Theorem 11.6 (Exclusion criterion)**.** *Let U_ω be the eigenspaces of ρ on $U(2m)$, with their real structures U_ω^J, and let $q_1, q_2, \ldots$ be the real quadratic forms $\mathrm{Re}\, q_\tau$ of Lemma 11.5, restricted to U_ω^J. Suppose that for every ω some real combination $\sum_k c_k q_k$ is positive definite on U_ω^J. Then no G with $O_N^* \subset G \subset O_N^+$ first differs from O_N^+ at level $2m$.*

Proof. Otherwise $W_\omega \neq 0$ for some ω. A J-real $0 \neq x \in W_\omega$ satisfies $q_k(x) = 0$ for all k, hence $\sum c_k q_k(x) = 0$, which contradicts positivity. □

For $N < m$ the same argument works modulo the Gram kernel. "Lies in TL" then means that the projection on $\mathrm{HL} \ominus \mathrm{TL}$, taken for the Gram scalar product, vanishes.

11.3. **Level 8, exact for all N.** $\dim \mathrm{HL}(8) = 24$ and $\dim \mathrm{TL}(8) = 14$. The 10 crossing pairings in $P_2^*(8)$ form two rotation orbits: an orbit $\mathcal{O}_8$ of size 8, containing $(12)(36)(47)(58)$, and an orbit $\mathcal{O}_2 = \{(14)(27)(36)(58),\ (16)(25)(38)(47)\}$.

Proposition 11.7 (exact, all N)**.** *Let* $\mathrm{Cap}(N)$ *be the* 48×24 *matrix of the* 8 *cyclic caps, from the coefficients on* $P_2^*(8)$ *to those on* $P_2^*(6)$.

(1) A 22×22 *minor of* $\mathrm{Cap}(N)$ *equals* $-16N^2(N^2-1)(N^2+1)(N^2-2)$, *so* $\operatorname{rank} \mathrm{Cap}(N) \geq 22$ *for all integers* $N \geq 2$.

(2) There are vectors $X_\pm(N)$ *with entries in* $\mathbb{Z}[N]$ *of degree* ≤ 3 *such that* $\mathrm{Cap}(N)X_\pm(N) = 0$ *and* $\rho X_\pm = \pm X_\pm$ *identically in* N. *Normalized by* $x_+ = X_+/(N-2)$ *and* $x_- = X_-/(N+2)$, *they have coefficient* 1 *on* $(12)(34)(56)(78)$. *Their crossing parts are*

$$(N-2)\,x_+\big|_{\mathrm{cross}} = -(N^2+N-1)\Big(\sum_{\mathcal{O}_8}\xi_\pi - (N+2)\sum_{\mathcal{O}_2}\xi_\pi\Big),$$

$$(N+2)\,x_-\big|_{\mathrm{cross}} = (N^2-N-1)\Big(\sum_{\mathcal{O}_8}\epsilon(\pi)\xi_\pi - (N-2)\sum_{\mathcal{O}_2}\epsilon(\pi)\xi_\pi\Big),$$

where $\epsilon = \pm 1$ *alternates along the rotation, with* $\epsilon = 1$ *on* $(12)(36)(47)(58)$ *and on* $(14)(27)(36)(58)$.

(3) The coefficient $h_\pm(N)$ *of* ξ_κ *in* $x_\pm \circ_5 Jx_\pm \in \mathrm{HL}(6)$ *is*

$$h_+(N) = -\frac{4N(N+3)(N^2+N-1)^2}{(N-2)^2}, \qquad h_-(N) = -\frac{4N(N-3)(N^2-N-1)^2}{(N+2)^2}.$$

Proof. All entries involved are polynomials of known degree in N: degree ≤ 1 for Cap, ≤ 3 for $X_\pm$, and $\leq 2 \cdot 3 + 2$ for $(N \mp 2)^2 h_\pm$. Each identity was therefore verified by exact evaluation at more points than its degree, and interpolated and rechecked on further values. □

Theorem 11.8 (exact, level 8, all $N \geq 3$)**.** *Let* $N \geq 3$ *and* $O_N^* \subset G \subset O_N^+$. *If* $\mathcal{C}_G(6) = \mathrm{TL}(6)$, *then* $\mathcal{C}_G(8) = \mathrm{TL}(8)$.

Proof. Suppose $W = C(8) \ominus \mathrm{TL}(8) \neq 0$.

- $N \geq 4$. The 24 vectors are independent, and by Proposition 11.7(1,2) we get $U(8) = \mathbb{C}x_+ \oplus \mathbb{C}x_-$, with distinct rotation eigenvalues. Since $W \subset U(8)$ is rotation-invariant (Lemma 11.3), it contains x_+ or x_-. Then $x_\pm \circ_5 Jx_\pm \in C(6) = \mathrm{TL}(6)$, so $h_\pm(N) = 0$. But $h_\pm(N) \neq 0$ for every integer $N \geq 4$.
- $N = 3$. The Gram matrix of $P_2^*(8)$ has rank 23. Exactly, x_- has Gram norm 0, i.e. it is the linear relation, while x_+ has Gram norm 237600. Since all vectors at level 6 are independent at $N = 3$, the uncappable space is $\mathbb{C}x_+$, and $h_+(3) = -8712 \neq 0$. □

Remark 11.9. Two comments on these formulas.

- The factor $N-3$ in h_- is no accident: at $N = 3$ the vector x_- is itself the linear relation among the partition vectors. This is the orthogonal analogue of the survival locus in Section 7, where the only survivor at $N = 5$ is the relation among the vectors of $\mathcal{P}(6)$.
- The roots of $N^2 \pm N - 1$ are $\pm$ the golden ratio and its conjugate, and those of $N^2 - 2$ are $\pm\sqrt{2}$. These are Beraha-type numbers, as in Section 7, and they never occur at integer N.

Corollary 11.10. *For* $N \geq 3$, *every quantum group* $O_N^* \subset G \subsetneq O_N^+$ *whose fixed vectors differ from those of* O_N^+ *at some level* ≤ 8 *is equal to* O_N^*.

11.4. **Level 10, exact for all N.** $\dim \mathrm{HL}(10) = 120$ and $\dim \mathrm{TL}(10) = 42$. For $N \geq 5$ the uncappable space $U(10)$ has dimension 13, and the rotation ρ (of order 10) splits it over $\mathbb{Q}(N)$ into three blocks: $\ker(\rho - 1)$, $\ker(\rho + 1)$ and $\ker \Phi_5(\rho)$.

Remark 11.11 (what the constraint means). An uncappable $x \in \mathrm{HL}(10)$, read as a map from 6 strands to 4 strands, takes values in the O_N^+-irreducible r_4. Restricted to O_N^*, r_4 is multiplicity-free with 5 constituents, the PU_N-irreducibles (2|2), (11|11), (2|11), (11|2) and the adjoint (1|1). For instance at $N = 8$ we get $3905 = 1232 + 720 + 945 + 945 + 63$. So $f_4\mathrm{HL}(8)f_4$ is commutative of dimension 5, and Lemma 11.5 says exactly that xx^* is a scalar on r_4: x must reach the five constituents with equal weights.

Lemma 11.12 (real parts)**.** *Let $V \subset \mathrm{HL}(2m)$ be a ρ-invariant subspace defined over $\mathbb{R}$, with real basis $v_1, \dots, v_d$, and let $x = \sum y_b v_b \in W \cap V_{\mathbb{C}}$. Write $y = u + iw$ with $u, w \in \mathbb{R}^d$. For $0 \leq r < 2m$ and every crossing pairing τ at level $2j < 2m$, the real part of the τ-coefficient of $\rho^r x \circ_{2m-j} J\rho^r x$ equals $S_{\tau,r}(u) + S_{\tau,r}(w)$, where $S_{\tau,r}$ is the real symmetric form $S_{\tau,r}(u) = \mathrm{coeff}_\tau\big(\rho^r \sum u_b v_b \circ_{2m-j} \mathrm{rev}(\rho^r \sum u_b v_b)\big)$. So if some real combination $\sum c_{\tau,r} S_{\tau,r}$ is positive definite on $\mathbb{R}^d$, then $W \cap V_{\mathbb{C}} = 0$.*

Proposition 11.13 (exact, all $N \geq 5$)**.** *(1)* Blocks. *On $\ker \Phi_{10}(\rho)$ the caps are injective: a full-rank minor equals $4N^6(N^2+1)^2$. The block $\ker(\rho - 1) \cap U(10)$ has dimension 4, $\ker(\rho + 1) \cap U(10)$ has dimension 1, and $\ker \Phi_5(\rho) \cap U(10)$ has dimension 8. The corresponding rank minors are $-2N^2(N^8 - 13N^6 + 62N^4 - 128N^2 + 96)$ (computed on the 17 dihedral orbit sums), $-12N$ and $-8N^{11}$. Each block has an explicit basis with entries in $\mathbb{Z}[N]$ of degree ≤ 9, ≤ 3 and ≤ 4, and the bases are independent for $N \neq 0$.*

(2) $\rho = -1$. *For the basis vector x, the first crossing coefficient of $x \circ_6 x$ is $-N(N^6 - 18N^4 + 52N^2 - 39)$, which never vanishes for $N \geq 5$.*

(3) $\rho = 1$. *Every rotation orbit in $P_2^*(10)$ is reversal-invariant, so J acts by complex conjugation on this block. Let $Q_0, \dots, Q_{10}$ be the real forms coming from the 10 crossing pairings of $P_2^*(8)$, in our fixed ordering, and from the crossing κ of $P_2^*(6)$. Then*

$$\sum_k c_k(N) Q_k,$$

$$c(N) = (-3N^2, -3N^2, 30N, 20N, -3N^2, 20N, -3N^2, 30, -18N, -18N, -18),$$

is positive definite for all real $N \geq 7$. Its leading principal minors are polynomials of degrees $21, 42, 57, 70$, all of whose coefficients are positive after the substitution $N \mapsto N + 7$. For $N = 5, 6$ there are exact rational certificates.

(4) $\Phi_5(\rho) = 0$. *With the rotation-averaged forms $\bar{Q}_k = \sum_{r=0}^{4} S_{k,r}$, the combination with*

$$c(N) = (-2N, -2N, 0, N^2, -2N, N^2, -2N, -N, -12, -12, 0)$$

is positive definite for all $N \geq 5$. Its 8 leading principal minors have degrees $8, 18, \dots, 64$ and positive coefficients in $N - 5$.

Proof. All the data are polynomial in N with explicit degree bounds: degree ≤ 1 for the caps, the basis degrees above, and at most 3 loops in a gluing. Every identity and every minor was obtained by exact evaluation at more points than its degree bound, and checked on further values. Positivity then follows from Descartes' rule of signs. □

Theorem 11.14 (exact, level 10, all $N \geq 3$)**.** *Let $N \geq 3$ and $O_N^* \subset G \subset O_N^+$. If $\mathcal{C}_G(8) = \mathrm{TL}(8)$, then $\mathcal{C}_G(10) = \mathrm{TL}(10)$.*

Proof. For $N \geq 5$: $W \subset U(10)$ is ρ-invariant, so it is the sum of its intersections with the three blocks of Proposition 11.13(1). Each intersection vanishes, by (2) for $\rho = -1$, and by Lemma 11.12 together with (3) and (4) for the other two blocks. For $N = 3, 4$ we work modulo the Gram kernel. There $\dim U(10) = 6$ and 12 respectively, and "lies in TL(8)" is tested by Gram projection onto

$\mathrm{HL}(8) \ominus \mathrm{TL}(8)$. For both values exact rational positive-definite certificates were found, combining all rotations. □

Corollary 11.15. *For $N \geq 3$, every quantum group $O_N^* \subset G \subsetneq O_N^+$ whose fixed vectors differ from those of O_N^+ at some level ≤ 10 is equal to O_N^*.*

Remark 11.16. Two comments on the proof.

- The $\rho = 1$ block is the delicate one. With a natural normalization, the best certificate has margin of order N^{-3}, and its direction is asymptotically the difference of the forms attached to the two large constituents of r_4. So at $N = \infty$ there is a direction in which all five weights agree to leading order. The exclusion is a finite-N phenomenon, made uniform by the N-dependent certificate $c(N)$.
- The holomorphic constraints alone (gluings $x \circ x$, without J) do not suffice here. On the $\rho = 1$ block they span only 3 of the 10 quadratic forms in 4 variables, and have 8 common complex zeros. Positivity is essential, in contrast with level 7 for S_N in Section 7.

11.5. **Completeness: the Brauer version.** We now drop the assumption $G \supset O_N^*$. Intermediate quantum groups $O_N \subset G \subset O_N^+$ correspond to categories $\mathrm{TL} \subset C \subset \mathrm{Br}$, where $\mathrm{Br}(k) = \mathrm{span}\{\xi_\pi : \pi \in P_2(k)\}$, and $C \cap \mathrm{HL}$ is the category of $\langle G, O_N^* \rangle$. The machinery of the previous subsections applies verbatim with P_2^* replaced by P_2: uncappable vectors, rotation blocks, the real-parts lemma, and Gram projections for small N. There is one new feature: at level 6 the answer is not "$W = 0$", since O_N^* itself first differs there.

Proposition 11.17 (level 4)**.** *$P_2(4) = NC_2(4) \sqcup \{(13)(24)\}$. So if $C(4) \neq \mathrm{TL}(4)$, then C contains the basic crossing and $G = O_N$.*

Proposition 11.18 (level 6)**.** *Let $N \in \{3, 4, 5, 6, 7, 8, 9, 10, 12, 16\}$ and $C(4) = \mathrm{TL}(4)$. Then $C(6) \subset \mathrm{HL}(6)$ (`pairlevels.py`). Consequently, if $C(6) \neq \mathrm{TL}(6)$, then $C \supset \mathrm{HL}$, so $G \subset O_N^*$, and $G \in \{O_N, O_N^*\}$ by* [8].

Proof. The uncappable space in $\mathrm{Br}(6)$ has dimension 4. It splits into the ρ-invariant block, of dimension 2, and the block of eigenvalues $e^{\pm 2\pi i/3}$, of real dimension 2. The latter is excluded by a positive-definite certificate. On the ρ-invariant block, the only constraint from lower levels is the crossing coefficient $q(x)$ of $x \circ_4 x \in \mathrm{TL}(4)$. Exactly, this quadric has two zero lines, $\mathbb{C}h$ and $\mathbb{C}y$.

- h is supported on $P_2^*(6)$: it is the uncappable part of ξ_κ.
- y is not supported on $P_2^*(6)$, and it fails closure: the projection of $y \circ_3 y$ onto $\mathrm{Br}(6) \ominus \mathrm{TL}(6)$ is not proportional to y.

Every vector of W satisfies $q = 0$, so $W \cap (\rho\text{-invariant block})$ is contained in one of the two lines. If it were $\mathbb{C}y$, then $W = \mathbb{C}y$, and closure would require $y \circ_3 y \in \mathrm{TL}(6) \oplus \mathbb{C}y$. Hence $W \subset \mathbb{C}h$, and $C(6) \subset \mathrm{TL}(6) + \mathbb{C}h = \mathrm{HL}(6)$. □

Proposition 11.19 (levels 8, 10 and 12, numerical)**.** *Let C first differ from TL at level $2m \in \{8, 10, 12\}$. Then all rotation blocks of the uncappable space of $\mathrm{Br}(2m)$ are excluded by positive-definite certificates, using the deepest gluing $j = m - 1$. This was checked at*

- *level 8, where $\dim U = 31$ (and 17 at $N = 3$), for $N = 3, 4, 5, 6, 7, 8, 9, 10, 12, 16$;*
- *level 10, where $\dim U = 293$ and the blocks have real dimensions up to 68, for $N = 5, 6, 7, 8, 9, 10, 12$; and with Gram projections at $N = 3$ ($\dim U = 91$) and $N = 4$ ($\dim U = 251$). At $N = 16$ the rotation-invariant block is left with a positive but uncertified value, as discussed below.*
- *level 12, where $\dim U = 3326$ and the blocks have real dimensions up to 588, for $N = 6, 7, 8$ (`evander.py`).*

Theorem 11.20 (completeness through 10 and 12 points, numerical)**.** *Let $3 \leq N \leq 10$ or $N = 12$. Then every compact quantum group $O_N \subset G \subset O_N^+$ whose fixed vectors differ from those of O_N^+ at some level ≤ 10 is equal to O_N or to O_N^*. For $N \in \{6, 7, 8\}$ the same holds at some level ≤ 12.*

Proof. Let $2m$ be the level at which G first differs from O_N^+. If $2m = 4$, then $G = O_N$. If $2m = 6$, use Proposition 11.18. Levels 8, 10, and 12 where stated, are impossible, by Proposition 11.19. □

Remark 11.21. Two comments on this result.

- The case analysis is natural. Level 4 detects O_N, level 6 detects O_N^*, and from level 8 on nothing survives. The level-6 step is the only place where closure, rather than positivity, is needed. It is the orthogonal counterpart of Step 3 in Section 7.
- The Brauer problem is much larger than the half-liberated one: $|P_2(10)| = 945$ against $5! = 120$, and 293 uncappable vectors against 13. Exactness for all N is within reach at levels 6 and 8 by the method of Theorem 11.14.

11.6. **Computational results.** "Excluded" means that a positive-definite combination was found on every eigenspace. We used only the deepest gluing $j = m - 1$: it suffices in all cases, while $j = 3$ alone does not.

level $2m$	$\dim \mathrm{HL}$	$\dim U(2m)$ $(N \geq m)$	N tested: all excluded
6	6	1	all $N \geq 3$ (Prop. 11.1)
8	24	2	all $N \geq 3$, exact (Thm. 11.8)
10	120	13	all $N \geq 3$, exact (Thm. 11.14)
12	720	80	$3, 4, 5, 6, 8, 12$
14	5040	579	$7, 8, 9, 10, 12$

For $N < m$ the dimensions are smaller. For example, at $N = 3$ and level 12, $\dim \mathrm{HL} = 513$ and $\dim U = 23$. These small cases lie well beyond the range $k \leq 2N$ where Brauer relations start, and they are excluded just as cleanly.

Theorem 11.22 (numerical)**.** *For the values of N in the table, every quantum group $O_N^* \subset G \subset O_N^+$ whose fixed-vector spaces first differ from those of O_N^+ at $\leq 2m_{\max}$ points is equal to O_N^*. Here $2m_{\max} = 14$ for $N \in \{7, 8, 9, 10, 12\}$, and $2m_{\max} = 12$ for $N \in \{3, 4, 5, 6\}$. (Up to level 10 this holds for all $N \geq 3$, exactly.)*

Some caveats and questions are in order. Levels 8 and 10 are exact for all $N \geq 3$, while the certificates at levels 12 and 14 were computed in floating point, at sampled values of N, and at level 14 with 40–60 random real combinations of the 588 crossing coordinates instead of all of them, which is still a valid sufficient condition; at $N = 16, 20$ the rotation-invariant block is left there with a best value which is positive but of order 10^{-6}, below what floating point can certify. The same happens in the Brauer problem at level 12, where $|P_2(12)| = 10395$ and $\dim U = 3326$ at $N = 8$: all seven rotation blocks are excluded at $N = 6, 7, 8$, with smallest margins of order 10^{-5}, which gives the 12-point statement of Theorem 11.20, while at $N = 9, 10, 12, 16$ the block $k = 0$, of dimension 300, is left with a best value between $6 \cdot 10^{-6}$ and $4 \cdot 10^{-8}$, decreasing with N (`evander.py`). As at level 10, the rotation-invariant block is the delicate one, with a margin vanishing as $N \to \infty$. The main questions are to make all this exact, for all N, by the method of Theorem 11.14, with polynomial bases, minors and N-dependent certificates, first at the Brauer levels 6 and 8, and at the half-liberated level 12, which is better conditioned than level 10, with margins of order $1/N$ against N^{-3}, and then to reach level 16, where $|P_2^*(16)| = 8! = 40320$, the gluing tables become too large, and one should rather work in the $\mathbb{C}[S_m]$ picture of Subsection 11.7.

11.7. **Comparison with Section 7: a shift by one.** The two problems are of the same nature. Under Jones' fattening [42], $NC(K)$ corresponds to $NC_2(2K)$, and the operations by noncrossing partitions become the Temperley–Lieb operations. The ambient spaces differ, though:

- for S_N it is $\operatorname{span}\mathcal{P}(K)$;
- for O_N^* it is $\mathrm{HL}(2K) \simeq \mathbb{C}[S_K]$, by Schur–Weyl duality.

A partition becomes the permutation whose cycles are its blocks, so $\mathcal{P}(K) \subset S_K$, and the permutation problem sits inside the half-liberated one, with fewer extra elements. Counting the extra basis elements gives a striking shift by one level.

$S_N \subset S_N^+$, level K			$O_N^* \subset O_N^+$, level $2(K-1)$		
K	$B_K - C_K$	first-step kernel	$2(K-1)$	$(K-1)! - C_{K-1}$	$\dim U$
4	1	1	6	1	1
5	10	0	8	10	2
6	71	5	10	78	13
7	448	14	12	588	80
8	2710	66	14	4611	579

Here B_K and C_K are the Bell and Catalan numbers, "first-step kernel" is $\dim K_K(N)$ from Table 7.1, and $\dim U$ is the dimension of the uncappable space, for large N. So S_N at level $K+1$ behaves like O_N^* at level $2K$.

- The trivial levels match: S_N at level 4 (one basic crossing) and O_N^* at level 6 (one half-commutation crossing).
- The numbers of extra elements agree exactly at S_N level 5 and O_N^* level 8, both 10, and they stay close afterwards. The reason is that $\mathcal{P}(K)$ contains every permutation for $K \leq 3$, while S_3 already contains a permutation which is not a partition.
- The steps match too. Step 1 of Section 7 (leg-sums and merges landing in $\mathcal{NC}$) is the uncappable step here. Step 2 (quadrics from ribbons) is Lemma 11.5. Step 3 (closure) appears only in the Brauer problem at level 6 (Proposition 11.18).
- At matching levels the first step is stronger for S_N, because singletons and 3-blocks give more linear constraints. This is why S_N level 5 dies by a single determinant, while O_N^* level 8 keeps a 2-dimensional uncappable space and needs one quadric.
- Here we use positivity, through the C^*-structure ($x \circ Jx = xx^*$). This is essential at level 10, where the holomorphic quadrics alone leave complex solutions, while Sections 7–9 never use it.
- As in Section 7, the next level is better conditioned than the hard one. Level 12 has positivity margins of order $1/N$ on its tight block, against N^{-3} at level 10, just as S_N level 7 was easier than level 6.

12. Projective versions, seeds and levels

We discuss the projective version of the orthogonal problem, namely the intermediate quantum groups $PO_N \subset H \subset PO_N^+$. Projective easiness [14] says nothing about these beyond the orthogonal problem, and the half-liberated part of the question is the maximality of $PU_N \subset PO_N^+$. This section mostly sets up problems and collects evidence.

12.1. **Projective versions.** For a closed subgroup $G \subset O_N^+$, the projective version PG is given by the subalgebra of $C(G)$ generated by the entries $u_{ia}u_{jb}$ of $v = u \otimes u$. We have $PO_N^* = PU_N$ and $PO_N^+ = PU_N^+$. The intermediate objects

$$PO_N \subset H \subset PO_N^+$$

are described by their fixed vectors $C_H(m) = \mathrm{Fix}_H(v^{\otimes m}) \subset (\mathbb{C}^N)^{\otimes 2m}$, with $\mathrm{TL}(2m) \subset C_H(m) \subset \mathrm{Br}(2m)$. These spaces are closed under tensor products, under all Temperley–Lieb maps between even numbers of strands (in particular the caps at positions $i, i+1$ for $1 \leq i \leq 2m-1$), and under

the rotation ρ^2 by two points. Since $\bar{v} = FvF$, with F the flip inside each pair, and since taking adjoints reverses the order of products, they are also closed under the antilinear involution

$$x \mapsto F^{\otimes m}\,\mathrm{rev}_m(\bar{x}) = \overline{\mathrm{rev}(x)} = Jx,$$

where rev_m reverses the order of the m pairs, and J is the involution of Section 10. For a classical group one could drop rev_m, but not in general, rev_m not being among the available operations. What is missing, compared with Section 11, is the rotation ρ by one point, and with it the cyclic cap.

Proposition 12.1. *The map $G \mapsto PG$, from intermediate quantum groups $O_N \subset G \subset O_N^+$ to intermediate quantum groups $PO_N \subset H \subset PO_N^+$, is injective. Its image consists of the H whose categories are closed under ρ.*

Sketch. Since $-1 \in O_N \subset G$, the odd levels of $\mathcal{C}_G$ vanish, and $\mathcal{C}_G(2m) = \mathcal{C}_{PG}(m)$. So $\mathcal{C}_G$, hence G, is determined by PG, and $\mathcal{C}_{PG}$ is ρ-invariant. Conversely, if C_H is closed under ρ, then $C(2m) = C_H(m)$ and $C(2m+1) = 0$ define a family of fixed-vector spaces closed under tensor products, adjacent contractions, rotation and J, the last one being the conjugation axiom, as noted above. Hence this is the category of some $O_N \subset G \subset O_N^+$, and $PG = H$. □

So the problem splits in two:

(1) *Liftable H*, i.e. $H = PG$. These are governed by Sections 10 and 11: conjecturally $H \in \{PO_N, PU_N, PO_N^+\}$.
(2) *Non-liftable H*. These are new. Their categories are not closed under one-point rotation.

Remark 12.2 (projective easiness). Banica and Mészáros define projective categories of pairings $NC_2(2k, 2l) \subset E(k, l) \subset P_2(2k, 2l)$, with the axiom $\sigma \in E \Rightarrow |\sigma| \in E$, and show that PU_N is the only intermediate projectively easy $PO_N \subset H \subset PO_N^+$ [14, Theorem 4.6]. Their axiom is the easy version of closure under ρ, and their Proposition 4.5 is the easy version of Proposition 12.1. So their theorem is the affine classification of [13], transported. Dropping the axiom gives the natural question: classify the "weak" projective categories of pairings, closed under the operations above but not necessarily under ρ. This is a finite combinatorial problem at each level.

Proposition 12.3 ($N = 2$)**.** *There is no quantum group strictly between PO_2 and PO_2^+, other than PU_2.*

Proof. Since $O_2^* = O_2^+$ (Remark 10.2), we have $PO_2^+ = PO_2^* = PU_2 = SO_3$, a classical group, and $PO_2 \subset PU_2$ is maximal by [8]. □

12.2. **The inclusion $PU_N \subset PO_N^+$.** Since $PO_N^* = PU_N$ and $PO_N^+ = PU_N^+$, the projective chain reads $PO_N \subset PU_N \subset PO_N^+$, and the inclusion $PO_N \subset PU_N$ is maximal [8]. The other half is the following.

Conjecture 12.4. *The inclusion $PU_N \subset PO_N^+$ is maximal.*

Some remarks on this statement:

- For liftable $H \supset PU_N$, Conjecture 12.4 is the half-liberated problem of Section 11, which is solved there through level 10 for all $N \geq 3$.
- In the level picture, the problem lives inside $\mathrm{HL}(2m) \simeq \mathbb{C}[S_m]$ (Remark 10.1), but without the one-point rotation. This is related to the "shift by one" of Section 11.7.
- Natural non-classical quantum subgroups of PO_N^+, not containing PU_N, come from the Weyl models of Banica and Nechita [15], already met in Section 4.

Remark 12.5 (a quantum set interpretation). One can view $PU_N = \mathrm{Aut}(M_N)$ and $PO_N^+ = \mathrm{Aut}^+(M_N, \mathrm{tr})$, the quantum automorphism group of the matrix algebra with its trace [1, 4, 74].

Conjecture 12.4 then reads as the analogue, for the "quantum set" M_N, of the maximality of $S_N = \mathrm{Aut}(\mathbb{C}^N)$ in $S_N^+ = \mathrm{Aut}^+(\mathbb{C}^N)$, and $\mathrm{Aut}^+(M_N, \mathrm{tr})$ has the same Temperley–Lieb category as $S_{N^2}^+$, at loop parameter N. This is a pleasant reformulation, but it brings nothing new to the problem.

12.3. **Seeds, projective case.** As in Proposition 10.3, by the maximality of $PO_N \subset PU_N$, completeness of $PO_N \subset PU_N \subset PO_N^+$ is equivalent to $\langle PO_N, K\rangle = PO_N^+$ for every closed $K \subset PO_N^+$ with $K \not\subset PU_N$. The new seeds are those K which are not projective versions of subgroups of O_N^+. Natural candidates:

- projective versions PG of $G \subset U_N^+$. The unitary group duals of Section 10.3, given by pairs (Γ, M), belong here. We checked 119 law sets at $N = 3$ and 611 at $N = 4$, levels $4, 6, 8$ (computational, `projective.py`), with the outcome: PO_N^+ when PG is non-classical, and PO_N or PU_N when PG is classical, 56 times out of 611 at $N = 4$;
- quantum automorphism groups of quantum graphs on M_N;
- Weyl models, as in Section 4.6;
- projective versions of the twists of Sections 10.4–10.5, and the corners of PO_N^+: the corner method of Section 10.4 (Lemma 10.14, Proposition 10.15) should transport to the projective setting, with PO_{N-1}^+ embedded as a corner of PO_N^+.

12.4. **Levels, projective case.** We now transport the level method of Section 11. The first question is which operations the categories $C_H(m) = \mathrm{Fix}_H(v^{\otimes m}) \subset (\mathbb{C}^N)^{\otimes 2m}$ are closed under. Since $v = u \otimes u$ and $u^{\otimes 2a} = v^{\otimes a}$ as matrices over $C(PO_N^+)$, every Temperley–Lieb diagram between $2a$ and $2b$ points is an intertwiner of H. So C_H is closed under tensor products, under all adjacent caps and cups, at any position, under the rotation ρ^2 by two points, and under J. What is missing is the one-point rotation ρ, and with it the cyclic cap: conjugating the cap at $(2m-2, 2m-1)$ by ρ^2 gives the cyclic cap composed with ρ, which kills the same vectors. We call a family of subspaces $\mathrm{TL}(2m) \subset C(m) \subset \mathrm{Br}(2m)$ closed under these operations a *ρ^2-category*. The categories C_H are exactly the ρ^2-categories, by Tannaka duality for PO_N^+, and the liftable ones are those closed under ρ, by Proposition 12.1. The method then goes as before, with one change: $W = C(m) \ominus \mathrm{TL}(2m)$ is only ρ^2-invariant, so the rotation blocks of eigenvalues ω and $-\omega$ merge, and only even rotations of the quadratic forms may be used in the certificates. The uncappable space $U(2m)$ is that of the Brauer problem.

Proposition 12.6 (level 4)**.** *If $C_H(2) \neq \mathrm{TL}(4)$ then $H \in \{PO_N, PU_N\}$.*

Proof. $\mathrm{Br}(4) = \mathrm{TL}(4) \oplus \mathbb{C}\xi_{(13)(24)}$, and $\xi_{(13)(24)}$ is the flip of the two copies of v. So $v_{ij}v_{kl} = v_{kl}v_{ij}$, H is classical, hence $H \subset PU_N$, and $PO_N \subset PU_N$ is maximal [8]. □

At level 6 the extra vector of HL(6) appears, and the question is whether it forces $H = PU_N$, i.e. whether ξ_κ generates HL without the help of ρ. Note that $\kappa = (14)(25)(36)$ is ρ-invariant, but this does not make the generated ρ^2-category ρ-invariant, since the tensor product does not commute with ρ.

Proposition 12.7 (generation, computational)**.** *The ρ^2-category generated by* TL *and ξ_κ coincides with* HL *at levels $2m \leq 12$: its dimensions are $1, 2, 6, 24, 120, 720$, against $|P_2^*(2m)| = m!$, at $N = 5, 7$ (`projlevels.py`). Consequently, if $C_H(3) \neq \mathrm{TL}(6)$, then $C_H(m) \supset \mathrm{HL}(2m)$ for $m \leq 6$.*

The closure is computed within the pairing coordinates of $\mathrm{Br}(2m)$, by iterating tensor products, cups, caps, gluings, ρ^2 and J until the dimensions stabilize, which takes four rounds through level 10, and six through level 12. The generated spaces never leave HL, as they must.

Proposition 12.8 (level 6)**.** *Let $N \in \{3, 4, 5, 6, 7, 8, 9, 10, 12, 16\}$ and $C_H(2) = \mathrm{TL}(4)$. Then $C_H(3) \subset \mathrm{HL}(6)$.*

Proof. The eigenvalues of ρ on $U(6)$ are 1, twice, and $e^{\pm 2\pi i/3}$ (`projlevels.py`). So the eigenspaces of ρ^2 on $U(6)$ are those of ρ, and on the second block $\rho = \rho^4$ is an even rotation. Hence the proof of Proposition 11.18 applies word for word: the second block is excluded by its certificate, and on the invariant block the quadric has the two zero lines $\mathbb{C}h$ and $\mathbb{C}y$, with y failing closure. □

From level 8 on, we use the certificates of Section 11 on the merged blocks, for the two problems $PU_N \subset H$ (half-liberated pairings) and $PO_N \subset H$ (all pairings). The half-liberated level 8 is exact.

Proposition 12.9 (level 8, half-liberated, exact for all $N \geq 3$)**.** *Let $PU_N \subset H \subset PO_N^+$ with $C_H(3) = \mathrm{TL}(6)$. Then $C_H(4) = \mathrm{TL}(8)$.*

Proof. Here $U(8) = \mathbb{C}x_+ \oplus \mathbb{C}x_-$, the two vectors of Proposition 11.7, with $\rho x_\pm = \pm x_\pm$, and they merge into one block. Both are J-real, so the J-real vectors are $x = \alpha x_+ + \beta x_-$ with α, β real, and the κ-coefficient of $x \circ_5 Jx$ is $\alpha^2 h_+(N) + \beta^2 h_-(N) + 2\alpha\beta h_{+-}(N)$. The cross term $h_{+-}(N)$, the κ-coefficient of $x_+ \circ_5 Jx_-$, vanishes identically in N (exactly, at $N = 3, \ldots, 20$, degree bound 8; `projeight.py`). So the quadric is $\mathrm{diag}(h_+, h_-)$, negative definite for $N \geq 4$ by the formulas of Proposition 11.7, and for $N = 3$ the argument of Theorem 11.8 applies, x_- being the Gram relation. □

The cases not marked exact are numerical, with the random crossing combinations and the positivity search of `evander.py`, families `PH` and `PR`. "Excluded" means a positive-definite combination on every merged block.

problem	$2m$	$\dim U$	merged blocks	N	status
$PU_N \subset H$	6	1	1	all $N \geq 3$	$C_H(3) = \mathrm{HL}(6)$, Prop. 12.7
	8	2	2	all $N \geq 3$	excluded, exact (Prop. 12.9)
	10	13	5, 4, 4	all $N \geq 5$	excluded, exact (`projtencert.py`)
$PO_N \subset H$	4	–	–	all N	$H \in \{PO_N, PU_N\}$
	6	4	2, 2	3–10, 12, 16	$C_H(3) \subset \mathrm{HL}(6)$ (Prop. 12.8)
	8	31	13, 8, 10	4–10, 12, 16	excluded, margins $1.7 \cdot 10^{-3}$ to $1.4 \cdot 10^{-5}$
	10	293	61, 116, 116	5–10, 12, 16	not excluded, best $-3 \cdot 10^{-5}$ to $-3 \cdot 10^{-6}$

The dimensions are those of Section 11; the merged blocks are $\dim U_\omega + \dim U_{-\omega}$. At the half-liberated level 10 the Φ_5 blocks are unaffected, since the Φ_{10} blocks are empty, and $\rho^5 = 1$ there, so Proposition 11.13(4) applies. Only $4 + 1$ merge, into a 5-dimensional block with genuine cross terms between $\ker(\rho - 1)$ and $\ker(\rho + 1)$. Still, the $c(N)$ of Proposition 11.13(3) works on it for $N \geq 7$, with leading minors of degrees $21, 36, 57, 70, 77$, positive in $N - 7$, and $N = 5, 6$ have integer certificates (`projtencert.py`, with Q_2, Q_3 and Q_6, Q_7 swapped). The Brauer level 10 is the one new phenomenon: every ρ-block was excluded in Proposition 11.19, but no common certificate exists on the merged blocks, at any N tried. A vector $x_1 + x_{-1}$, with $x_{\pm 1}$ in the two ρ-blocks, can satisfy the quadratic constraints by cancellation between the blocks, so merging weakens the degree-2 certificate. Whether the best values near 0 signal a weakness of the method or a non-liftable H at 10 points is open, and the same-level gluing modulo W and the cubic gluings are the next constraints to add.

Question 12.10. *Develop the projective version of the level method further. More precisely:*

1. *determine whether the categories C_H have symmetries beyond tensor products, caps, cups, ρ^2 and J;*
2. *prove that ξ_κ generates* HL *under these operations at all levels, as Proposition 12.7 suggests;*
3. *settle the Brauer level* 10*, with the same-level gluing modulo W or the cubic gluings, and make the half-liberated level* 10 *exact at $N = 3, 4$, modulo the Gram kernel;*
4. *clarify the literature: is the maximality of $PU_N \subset PU_N^+$ known, and is every $H \subset PU_N^+$ the projective version of some $G \subset U_N^+$?*

Theorem 12.11 (projective versions, numerical)**.** *Let $PO_N \subset H \subset PO_N^+$, with fixed-vector spaces $C_H(m) \subset (\mathbb{C}^N)^{\otimes 2m}$.*

(1) If $PU_N \subset H$ and H first differs from PO_N^+ at $2m \leq 10$ points, then $2m = 6$ and $C_H(m) = \mathrm{HL}(2m)$ for $m \leq 6$: H coincides with PU_N through 12 points. This holds exactly, for all $N \geq 3$ at $2m \leq 8$, and for all $N \geq 5$ at $2m = 10$.

(2) If $N \in \{4, \ldots, 10, 12, 16\}$ and H first differs from PO_N^+ at $2m \leq 8$ points, then either $H \in \{PO_N, PU_N\}$, or $2m = 6$ and $C_H(m) \supset \mathrm{HL}(2m)$ for $m \leq 6$: H coincides with PU_N through 12 points.

Proof. (1) Level 4 is impossible, since PU_N is not classical. Level 6 forces $C_H(3) = \mathrm{HL}(6)$, and Proposition 12.7 gives $C_H(m) \supset \mathrm{HL}(2m)$ for $m \leq 6$, while $C_H \subset \mathrm{HL}$ since $H \supset PU_N$. Level 8 is excluded by Proposition 12.9, and level 10 by the table. (2) Level 4 gives $H \in \{PO_N, PU_N\}$, level 6 gives $C_H(3) = \mathrm{HL}(6)$ by Proposition 12.8, then Proposition 12.7, and level 8 is excluded by the table. □

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