ON A FAMILY OF COUNTEREXAMPLES TO RICHARD GUY'S MEX SEQUENCE PROBLEM

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Abstract

We give a family of counterexamples to Richard Guy's problem E27 with initial terms of length five adding onto the recent example of length eight. Additionally, we prove that for length one, one always has a ultimately periodic sequence. Lastly we pose a problem regarding the notion of a z-th mex sequence.

1. Introduction and main results

Take some integer $l \geq 1$, then the nonnegative integers $a_0, ..., a_{l-1}$ will be called the initial terms of length l. In the book Unsolved Problems in Number Theory, Richard Guy defines the mex sequence by extending the initial terms by

$$a_{n+1} = \min\{a : a \in \mathbb{Z}_{\geq 0} \setminus S_n\}, \quad S_n = \{a_i + a_{n-i} : 0 \leq i \leq n\}. \tag{1.1}$$

That is, a_{n+1} is the smallest nonnegative integer for which the value is not in the set of the summed terms [2]. An example with 1,4,3,2 is shown in OEIS entry number A067017 [1]. The sequence is related to octal games among other works, a detailed description for which can be found in problem E27 [2].

Guy asks if every sequence following (1.1) is ultimately periodic [2]. Interestingly, Guy never specifies the length of the initial terms [2]. Recently, a Lean verified pre-print provided an unbounded counterexample generated by the eight initial terms given by 1,1,1,0,1,0,1,1 [3].

In this paper, we provide a family of ultimately non-periodic sequences with initial terms having length five in Theorem 1.1. As a corollary, we prove the existence of infinitely many ultimately non-periodic mex sequences with initial terms having length five or more. We also show any mex sequence generated by a single initial term is ultimately periodic. As a result, the minimal length of initial terms containing a ultimately non-periodic mex sequence must have length two, three, four, or five. Lastly, we define the notion of a z-th mex sequence, and state a problem similar to that of Guy's original problem.

Theorem 1.1. *The family of mex sequence generated by 2,0,c,0,2 with $c \geq 2$ is not ultimately periodic and is unbounded. Additionally, the sequence is given by*

$$a_n = \begin{cases} 0 & \text{if } n \equiv 0 \text{ mod } 3 \\ \frac{1}{3}(n+2) & \text{if } n \equiv 1 \text{ mod } 3 \\ 1 & \text{if } n \equiv 2 \text{ mod } 3 \end{cases}$$

for $n \geq 5$

Corollary 1.2. *For all mex sequence generated by initial terms with length $l \geq 5$, there are infinitely many sequences for which the mex sequence is ultimately non-periodic.*

Lemma 1.3. *Sequence of length one is ultimately periodic with periods two or five.*

Corollary 1.4. *For a mex sequence to be ultimately non-periodic, the minimal length of the starting terms is two, three, four, or five.*

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Conjecture 1.5. *The smallest length of the initial terms for a ultimately non-periodic mex sequence to exist is five. That is, any mex sequence generated with initial terms of length up to four are ultimately periodic.*

Theorem 1.6. *The sequence 2,0,2,0,2 is ultimately periodic for the generalized mex sequence with $z = 2$.*

2. Proof of Theorem 1.1 and Corollary 1.2

Before we prove Theorem 1.1, the first few terms of the mex sequence generated by initial terms $a_0 = 2, a_1 = 0, a_2 = c, a_3 = 0, a_4 = 2$ with $c \geq 2$ are shown below

$$2, 0, c, 0, 2, 1, 0, 3, 1, 0, 4, 1, 0, 5... \tag{2.1}$$

The pattern reflects a unit growth every three terms past the first five terms.

Proof. Suppose $n \leq 11$ then one observes from 2.1 that the terms grow every three terms. Consider the case when $n \geq 12$. Let us denote $a_0 = 2, a_1 = 0, a_2 = c, a_3 = 0, a_4 = 2$ and suppose for all m with $5 \leq m \leq n$, we have

$$a_m = \begin{cases} 0 & \text{if } m \equiv 0 \text{ mod } 3 \\ \frac{1}{3}(m+2) & \text{if } m \equiv 1 \text{ mod } 3 \\ 1 & \text{if } m \equiv 2 \text{ mod } 3 \end{cases} \tag{2.2}$$

Let us introduce the sets S_1^k, S_2^k, and S^k defined by

$$S_1^k = \{a_i + a_{n-i} : 0 \leq i < 5\} \quad S_2^k = \{a_i + a_{n-i} : 5 \leq i \leq n-5\}, \quad S^k = S_1^k \cup S_2^k$$

whenever $n \equiv k$ mod 3. We consider the three cases for n and for each case, we use strong induction to prove that 2.2 holds.

Case 1:
Suppose $n \equiv 0$ mod 3 then a direct calculation yields

$$S_1^0 = \left\{0, 1, 2, 3, \frac{1}{3}n + c\right\} \tag{2.3}$$

In order to calculate the elements in S_2^0 we consider $a_i + a_{n-i}$ for each congruence class of i. Computing each case on i gives us

$$a_i + a_{n-i} = \begin{cases} 0 & \text{if } i \equiv 0 \text{ mod } 3, \quad 5 \leq i \leq n-6 \\ a_i + 1 & \text{if } i \equiv 1 \text{ mod } 3, \quad 5 \leq i \leq n-5 \\ a_{n-i} + 1 & \text{if } i \equiv 2 \text{ mod } 3, \quad 5 \leq i \leq n-7 \end{cases}$$

and applying the induction hypothesis from 2.2 gives us

$$a_i + a_{n-i} = \begin{cases} 0 & \text{if } i \equiv 0 \text{ mod } 3, \quad 5 \leq i \leq n-6 \\ \frac{1}{3}(i+2) + 1 & \text{if } i \equiv 1 \text{ mod } 3, \quad 5 \leq i \leq n-5 \\ \frac{1}{3}(n-i+2) + 1 & \text{if } i \equiv 2 \text{ mod } 3, \quad 5 \leq i \leq n-7 \end{cases}$$

and hence S_2^0 is the set

$$S_2^0 = \left\{0, 4, ..., \frac{1}{3}n\right\}$$

Taking the union of the two sets S_1^0, S_2^0 leaves us with

$$S^0 = \left\{0, 1, 2, 3, ..., \frac{1}{3}n, \frac{1}{3}n + c\right\}$$

and observe that the smallest integer not inside S^0 is $\frac{1}{3}n + 1$ since $c \geq 2$ and hence we have $a_{n+1} = \frac{1}{3}n + 1 = \frac{1}{3}(n + 1 + 2)$.

Case 2:
Suppose $n \equiv 1$ mod 3. Calculating S_1^1 by using 2.2 gives us

$$S_1^1 = \left\{0, 2, 1 + c, \frac{1}{3}(n - 1), \frac{1}{3}(n + 2) + 2\right\}$$

Considering each case on i, we have

$$a_i + a_{n-i} = \begin{cases} \frac{1}{3}(n - i + 2) & \text{if } i \equiv 0 \text{ mod } 3, \quad 5 \leq i \leq n - 7 \\ \frac{1}{3}(i + 2) & \text{if } i \equiv 1 \text{ mod } 3, \quad 5 \leq i \leq n - 6 \\ 2 & \text{if } i \equiv 2 \text{ mod } 3, \quad 5 \leq i \leq n - 5 \end{cases}$$

and S_2^1 is given by

$$S_2^1 = \left\{2, 3, ..., \frac{1}{3}(n - 4)\right\}$$

and S^1 is the set

$$S^1 = \left\{0, 2, .., \frac{1}{3}(n - 4), \frac{1}{3}(n - 1), \frac{1}{3}(n + 2) + 2, 1 + c\right\}$$

The smallest nonnegative integer not in S^1 is 1. Therefore, $a_{n+1} = 1$.

Case 3:
Suppose $n \equiv 2$ mod 3 then by (2.2), we have S_1^2 to be

$$S_1^2 = \{1, 3, c, \frac{1}{3}(n + 1), \frac{1}{3}(n - 2) + 2\}$$

and the sums are

$$a_i + a_{n-i} = \begin{cases} 1 & \text{if } i \equiv 0 \text{ mod } 3, \quad 5 \leq i \leq n - 5 \\ \frac{1}{3}(n + 4) & \text{if } i \equiv 1 \text{ mod } 3, \quad 5 \leq i \leq n - 7 \\ 1 & \text{if } i \equiv 2 \text{ mod } 3, \quad 5 \leq i \leq n - 6 \end{cases}$$

Therefore, S_2^2 is given by

$$S_2^2 = \left\{1, \frac{1}{3}(n + 4)\right\}$$

and S^2 is the set

$$S^2 = \left\{1, 3, \frac{1}{3}(n - 2) + 2, \frac{1}{3}(n + 1), \frac{1}{3}(n + 4), c\right\}$$

We see that the minimum nonnegative integer not inside S^2 is 0 and hence we have $a_{n+1} = 0$. □

Corollary 1.2 follows immediately.

Proof. Consider one of the infinitely many initial terms given in Theorem 1.1. Since these terms form mex sequences which are unbounded, one can take the first l terms with $l \geq 5$ of the sequence and take the l-th terms as the initial terms. The mex sequence generated by these l-th terms are then also unbounded. □

3. Proof of Lemma 1.3

In this section, we prove Lemma 1.3.

Proof. Take some nonnegative integer as the first term of the sequence, a_0. We prove by induction that the mex sequence formed by a_0 is ultimately periodic by considering cases for when $a_0 = 0, a_0 = 1$, and $a_0 > 1$.

Case 1:
Suppose $a_0 = 0$ with $n \geq 6$. The first few terms are given by

$$0, 1, 0, 1, 0, 1...$$

Therefore, suppose for all integers m with $0 \leq m \leq n$, we have

$$a_m = \begin{cases} 0 & \text{if } m \equiv 0 \bmod 2 \\ 1 & \text{if } m \equiv 1 \bmod 2 \end{cases}$$

Suppose $n \equiv 0 \bmod 2$ then we have

$$a_i + a_{n-i} = \begin{cases} 0 & \text{if } i \equiv 0 \bmod 2 \\ 2 & \text{if } i \equiv 1 \bmod 2 \end{cases} \tag{3.1}$$

and hence we have the set given by $S = \{a_i + a_{n-i} : 0 \leq i \leq n\}$ is $\{0, 2\}$. Therefore, we have $a_{n+1} = 1$.

Now suppose $n \equiv 1 \bmod 2$ then the values of (3.1) are always one, so $S = \{1\}$. Therefore $a_{n+1} = 0$.

The case with $n \leq 5$ falls immediately. As a result, the sequence has period two.

Case 2:
Suppose $a_0 = 1$ then the first few terms are as follows

$$1, 0, 0, 2, 1, 1, 0, 0, 2, 1, 1, 0, 0, ...$$

Suppose for integers m with $1 \leq m \leq n$ with $n \geq 6$, we have

$$a_m = \begin{cases} 1 & \text{if } m \equiv 0 \bmod 5 \\ 0 & \text{if } m \equiv 1 \bmod 5 \\ 0 & \text{if } m \equiv 2 \bmod 5 \\ 2 & \text{if } m \equiv 3 \bmod 5 \\ 1 & \text{if } m \equiv 4 \bmod 5 \end{cases}$$

then we consider all five congruence classes n can have. Let us denote

$$S_1^k = \{a_0 + a_n\} \tag{3.2}$$

$$S_2^k = \{a_i + a_{n-i} : 1 \leq i \leq n-1\} \tag{3.3}$$

$$S^k = S_1^k \cup S_2^k \tag{3.4}$$

with $n \equiv k$ mod 5 for $k \in \{0, 1, 2, 3, 4\}$ then we have the following

$$\begin{aligned}
S_1^0 &= \{2\}, \quad S_2^0 = \{1, 2\}, \quad S^0 = \{1, 2\} \\
S_1^1 &= \{1\}, \quad S_2^1 = \{1, 4\}, \quad S^1 = \{1, 4\} \\
S_1^2 &= \{1\}, \quad S_2^2 = \{0, 1, 3\}, S^2 = \{0, 1, 3\} \\
S_1^3 &= \{3\}, \quad S_2^3 = \{0, 2, 3\}, S^3 = \{0, 2, 3\} \\
S_1^4 &= \{2\}, \quad S_2^4 = \{0, 2\}, \quad S^4 = \{0, 2\}
\end{aligned}$$

We see that taking the minimum nonnegative integer not in S^k for each k gives us a_{n+1}, which correctly matches all the cases and hence the sequence is periodic. The case for $n \leq 5$ falls immediately by computation.

Case 3:
Now consider $a_0 > 1$ then the sequences are as follows

$$a_0, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, ...$$

Suppose we have for m with $1 \leq m \leq n$ with $n \geq 6$,

$$a_m = \begin{cases} 2 & \text{if } m \equiv 0 \text{ mod } 5 \\ 0 & \text{if } m \equiv 1 \text{ mod } 5 \\ 0 & \text{if } m \equiv 2 \text{ mod } 5 \\ 1 & \text{if } m \equiv 3 \text{ mod } 5 \\ 1 & \text{if } m \equiv 4 \text{ mod } 5 \end{cases}$$

Let us borrow the same notation of S_1^k, S_2^k, and S^k then we have

$$\begin{aligned}
S_1^0 &= \{a_0 + 2\}, \quad S_2^0 = \{1, 4\}, \quad S^0 = \{1, 4, a_0 + 2\} \\
S_1^1 &= \{a_0\}, \quad S_2^1 = \{1, 2\}, \quad S^1 = \{1, 2, a_0\} \\
S_1^2 &= \{a_0\}, \quad S_2^2 = \{0, 2\}, \quad S^2 = \{0, 2, a_0\} \\
S_1^3 &= \{a_0 + 1\}, \quad S_2^3 = \{0, 2, 3\}, \quad S^3 = \{0, 2, 3, a_0 + 1\} \\
S_1^4 &= \{a_0 + 1\}, \quad S_2^4 = \{0, 1, 3\}, \quad S^4 = \{0, 1, 3, a_0 + 1\}
\end{aligned}$$

Considering each S^k, we see that the minimum integer value not in each S^k provides the correct a_{n+1} term for each case, and hence we have the sequence to be ultimately periodic. The case when $n \leq 5$ falls by direct calculation.

Combining all the cases gives us the result. □

The first and second cases in the above proof also proves that sequences with initial terms set to $1, 0$ or $0, 1$ are ultimately periodic. However, for a more general set of values given by a_0, a_1, computational results shows that a more careful case analysis is needed. We refer the reader to Conjecture 1.5.

Now we prove Corollary 1.4.

Proof. By Lemma 1.3 we have that any mex sequence generated from a single initial term is ultimately periodic and from Corollary 1.2, any mex sequence generated by five or more initial terms contains a ultimately non-periodic sequence. Therefore, the minimal length of the initial terms to generate a ultimately non-periodic mex sequences is those with length two, three, four, or five. □

After extensive computation brute forcing all cases involving 0 and 1 as well as small values, we conjecture in Conjecture 1.5 the minimal length to be five.

4. Proof of Theorem 1.6

Before we prove Theorem 1.6, we generalize the mex sequence defined by Guy by introducing the zth mex sequence.

For a given sequence, let us extend the sequence by $a_{n+1} = \min_z(\mathbb{Z}_{\geq 0} \setminus S_n)$ with $S_n = \{a_i + a_{n-i} : 0 \leq i \leq n\}$ where $\min_z$ denotes taking the z-th minimum value.

For example, consider the z-th mex sequence of $2, 0, 2, 0, 2$ when $z = 2$ given by

$$2, 0, 2, 0, 2, 2, 1, 5, 3, 1, 2, 2, 1, 6, 7, 1, 5, 2, 1, 4, 2, 1, 5, 2, 1, 5, 2, ... \tag{4.1}$$

We prove that under this new definition, the sequence $2, 0, 2, 0, 2$ is ultimately periodic when $z = 2$, proving Theorem 1.6.

Proof. A direct computation gives us (4.1). Let us denote $a_0 = 2, a_1 = 0, ..., a_{20} = 2$ then suppose for all integer m with $21 \leq m \leq n$ we have

$$a_m = \begin{cases} 1 & \text{if } m \equiv 0 \bmod 3 \\ 5 & \text{if } m \equiv 1 \bmod 3 \\ 2 & \text{if } m \equiv 2 \bmod 3 \end{cases} \tag{4.2}$$

and we prove periodicity holds for all $n \geq 44$ by induction. We again let $S_1^k = \{a_i + a_{n-i} : 0 \leq i \leq 20\}, S_2^k = \{a_i + a_{n-i} : 21 \leq i \leq n - 21\}$ and $S^k = S_1^k \cup S_2^k$ where $n \equiv k \bmod 3$, $k \in \{0, 1, 2\}$. Consider when we have $n \equiv 0 \bmod 3$ then a direct computation gives us

$$S_1^0 = \{1, 2, 3, 4, 6, 7, 8, 12\}$$

and for S_2^0 observe that applying 4.2, we have

$$a_i + a_{n-i} = \begin{cases} 2 & \text{if } i \equiv 0 \bmod 3 \\ 7 & \text{if } i \equiv 1, 2 \bmod 3 \end{cases}$$

and hence we have

$$S_2^0 = \{2, 7\}$$

and hence the second smallest nonnegative integer not in the set S^0 is 5.

The same argument holds for the other two cases providing us with

$$S_1^1 = \{1, 3, 4, 5, 6, 7, 9\}, \quad S_2^1 = \{4, 6\}$$

and hence we have 2 as the second smallest nonnegative integer not in the set S^1. Lastly, we have

$$S_1^2 = \{2, 3, 4, 5, 7, 8, 9, 10, 11\}, \quad S_2^2 = \{3, 10\}$$

and hence we have 1 as the second smallest nonnegative integer not in S^2.

Each case gives exactly a_{n+1} and for the case when $n \leq 43$ direct computation proves ultimate periodicity. Therefore, the sequence is ultimately periodic. □

It would be intriguing to see if there exists some z for which every mex sequence with initial terms of any length is ultimately periodic. Additionally, one might also study the bounds on the length of the periods.

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