Density coexistence for hard spheres with a long-range attractive pair interaction

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Abstract

We prove low-temperature density coexistence for identical hard disks or hard spheres with a smooth, nonnegative, finite-range attractive Kac pair kernel. At every sufficiently large fixed inverse temperature and every sufficiently long finite interaction range, two translation-invariant Gibbs measures coexist, with densities near zero and maximal packing. The pressure is therefore nondifferentiable in chemical potential. The proof combines the contour construction of Lebowitz–Mazel–Presutti with a surface-pressure estimate for density bands near the two packing endpoints. A packing bound controls the fine density profile; centering the interaction makes its residual boundary contribution quadratic in the band width. Compatible near-packed configurations and conditional entropy then control the hard-core boundary contribution. These estimates allow the dense restricted ensemble to retain positional order and long-range correlations.

1 Proof strategy and main result

The central task is to control the free energy of a density interface while allowing arbitrary positional organization within the dense phase. We use two restricted ensembles: one with density close to zero in every mesoscopic cell, and one with density close to maximal packing. At the dense endpoint, a local upper bound on particle counts turns the density constraint into an L^1 bound on the fine density profile. This produces a small surface correction to the restricted pressure. The correction is the input needed to close the usual contour argument.

1.1 Model and theorem

Fix $d \in \{2, 3\}$ and an exclusion distance $a > 0$. An admissible configuration is a locally finite set $q \subset \mathbb{R}^d$ with $|x - y| \geq a$ for distinct $x, y \in q$. Let r be the supremum of the upper densities of admissible configurations, measured in expanding cubes. Choose a radial function

$$J \in C_c^\infty(\mathbb{R}^d), \qquad J \geq 0, \qquad \int_{\mathbb{R}^d} J = 1, \qquad \operatorname{supp} J \subset B(0, 1), \tag{1.1}$$

positive near the origin, and put $J_R(x) = R^{-d} J(x/R)$. The pair potential is

$$v_R(x) = \begin{cases} +\infty, & |x| < a, \\ -J_R(x), & |x| \geq a. \end{cases}$$

For a bounded region A, write $N_A = \#q_A$ and $n_A = \sum_{x \in q_A} \delta_x$. Given an exterior configuration q_{A^c}, the grand-canonical Hamiltonian is

$$H_{A,\mu}(q_A \mid q_{A^c}) = H_A^{\mathrm{hc}}(q_A \mid q_{A^c}) - \mu N_A - \frac{1}{2} \sum_{\substack{x,y \in q_A \\ x \neq y}} J_R(x - y) - \sum_{x \in q_A, y \in q_{A^c}} J_R(x - y). \tag{1.2}$$

Here H_A^{hc} is zero when all hard-core constraints involving a particle in A hold, and $+\infty$ otherwise. Integration uses

$$\mathrm{d}q_A = \sum_{n\ge 0} \frac{1}{n!}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n, \qquad (x_1,\dots,x_n) \in A^n.$$

The free partition function has no exterior particles. Its pressure, in energy units, is

$$p_R(\beta,\mu) = \lim_{A\uparrow\mathbb{R}^d} \frac{1}{\beta|A|} \log Z^{\mathrm{f}}_{A,\mu},$$

where the limit is along cubes. Packing bounds imply stability of the attractive interaction; the thermodynamic limit follows from the standard theory of stable finite-range potentials [12]. For a translation-invariant measure ν, its intensity $\rho(\nu)$ is defined by $\mathbb{E}_\nu N_A = \rho(\nu)|A|$ for bounded measurable A.

Theorem 1.1. *There exist $\delta \in (0, r/10)$ and $\beta_0 < \infty$ such that, for every fixed $\beta \ge \beta_0$, all sufficiently large finite R have the following property. At some chemical potential μ_R there are two translation-invariant Gibbs measures ν_0, ν_r with intensities*

$$\rho(\nu_0) \le 2\delta, \qquad \rho(\nu_r) \ge r - 2\delta. \tag{1.3}$$

Consequently $p_R(\beta,\cdot)$ is nondifferentiable at μ_R.

The theorem concerns coexistence of a dilute phase and a dense phase that may have crystalline order. The two measures have different expected particle counts in every bounded region of positive volume.

1.2 Relation to the earlier arguments

Lebowitz–Penrose [7] identify the convexified free energy in the Kac limit. Coexistence at a fixed finite range requires additional control of interfaces. Lebowitz–Mazel–Presutti [6] obtain that control for an attractive pair interaction stabilized by a repulsive four-body interaction. Presutti [10, §10.1.4] identifies the difficulty for hard cores: ratios of interior partition functions must be estimated to surface order with a coefficient small enough to be absorbed by the interface cost.

The contour expansion with the microscopic background fixed is already present in [10, §11.4.3]. The support enlargement needed to regard its logarithm as an interaction is likewise part of [10, §11.9.1]. We use these constructions below.

The change here is the estimate for the remaining microscopic background. Close to maximal packing, saturation of the packing bound controls the fine density in L^1. Exact centering then gives a residual boundary interaction of order δ^2. A replacement of boundary cells by compatible near-packed configurations has the same energetic order; its entropy cost is made small by low temperature. Conditional entropy converts this comparison into the required surface-pressure estimate. Presutti already observes low-temperature nonconvexity of the hard-core mean-field free energy in [10, §10.1.3]; the additional ingredient here is the quantitative boundary estimate at the packing endpoints.

This changes the regime as well as the information required about the reference system. We work near vacuum and maximal packing, using the fact that leaving either endpoint band costs energy of order δ, and retain the complete restricted pressure, including any internal phase transitions. The hard-core entropy is bounded by explicit configuration-volume estimates. The modified pair potential of He–Jauslin–Lebowitz–Peled [3] provides another route to finite-range density coexistence; the interaction in (1.2) is the original translation-invariant Kac pair interaction.

OpenAI [8, 9] give two further constructions in $\mathbb{R}^3$. Their potentials combine a repulsive core containing a narrow shell of small energy, which permits tetrahedral clusters, with a summable

sequence of attractive Kac kernels of increasing ranges. In [8], the core diverges at zero but is finite at every positive separation, and the tail is integrable with $\varphi(s) = o(s^{-3})$. In [9], the potential is bounded and continuous and satisfies $|\varphi(s)| \le Cs^{-3-1/32}$ for $s \ge 1$. Both final potentials have infinite range.

Their method combines successive Lebowitz–Penrose approximations with a convex perturbation argument. Three competing regimes—vacuum, individual particles, and tetrahedral clusters—supply separated groups of supporting slopes in inverse temperature and the field conjugate to density. The construction preserves a two-dimensional ball of such slopes for the ideal pressure transforms and passes it to the final physical pressure. Partial convex duality then gives a jump in the inverse-temperature derivative of the canonical free energy at one common inverse temperature throughout an open density interval. Here the interaction has a hard core and one compactly supported Kac attraction in $d = 2, 3$; contours and surface estimates at fixed finite range yield distinct-density Gibbs measures and nondifferentiability of the pressure in chemical potential.

1.3 Order of the proof

The sections follow the inputs needed for the contour estimate.

1. **Packing and density bands (Section 2).** We prove the fine-density bound and construct positive-volume sets of near-packed configurations compatible with arbitrary exterior particles.

2. **Boundary comparison (Section 3).** We center the Kac interaction at density 0 or r, retain its explicit linear boundary field, and bound the remaining interaction and the cost of replacing boundary cells.

3. **Surface pressure (Section 4).** Two conditional-entropy inequalities give a pressure expansion for every finite union of macrocells. This step also permits the small interaction generated by contours.

4. **Contours and their effective interaction (Sections 5–6).** We define correctness, introduce recursively truncated contour weights, and expand their partition function with the microscopic background fixed. The resulting interaction satisfies the preceding surface theorem.

5. **Pressure equality (Section 7).** Packing and trial configurations give continuous restricted pressures close to 0 and rt at $\mu = -r/2 + t$. They cross for a small t.

6. **Interface cost (Sections 8–9).** Every incorrect region contains density defects with an energy cost of order δ. At equal restricted pressures, all interior-volume factors cancel. The remaining errors have order δ^2 and $\beta^{-1}C_\delta$, so the true contour weights lie below their truncations.

7. **Gibbs measures (Section 10).** The Peierls estimate yields Gibbs measures with separated densities. Their intensities are distinct subgradients of the pressure.

The parameters are chosen in the order: geometry, density-band width, inverse temperature, interaction range. We make the final numerical choices after deriving the contour estimate.

2 Packing and restricted phases

We first record the geometric information about hard spheres. The periodic approximation used here is standard; see Cohn–Elkies [1, Appendix A].

Lemma 2.1 (Particle counts in a cube)**.** *An admissible configuration with n centers in a cube of side u satisfies*

$$n \le r(u+a)^d. \tag{2.1}$$

Proof. Repeat the finite configuration on $(u+a)\mathbb{Z}^d$. Different copies are separated by at least a, and the resulting periodic packing has density $n/(u+a)^d \le r$. □

Lemma 2.2 (Periodic approximation)**.** *For every $\varepsilon > 0$ there is a periodic admissible configuration of density at least $r - \varepsilon$.*

Proof. The definition of r supplies arbitrarily large cubes containing configurations with density arbitrarily close to r. Repeat one such configuration with gaps of width a, as in Lemma 2.1. The density loss tends to zero as the cube grows. □

2.1 The three grids

Fix $B > 20$ and an integer m; set $b = B/m$, $L = BR$, and $\ell = bR$. We choose m sufficiently large in Section 8; until then $0 < b < 1$. Macrocells are

$$Q_i = L(i + [0,1)^d), \qquad i \in \mathbb{Z}^d.$$

Each macrocell is divided into m^d phase cells of side ℓ. Divide each phase cell further into fine cubes of side

$$h = \ell/n_R, \qquad n_R = \max\{1, \lfloor \ell/R^{1/2} \rfloor\}.$$

Thus $h \asymp R^{1/2}$ and $a \ll h \ll \ell < R < L$ for large R.

Two macro indices are adjacent when their ℓ^∞ distance is one. All connected components, boundaries, and graph distances below use this adjacency. For $\Lambda \subset \mathbb{Z}^d$, let $A_\Lambda = \bigcup_{i\in\Lambda} Q_i$ and

$$\partial\Lambda = \{i \in \Lambda : \text{some neighbor of } i \text{ is outside } \Lambda\}.$$

We distinguish physical volume $|A_\Lambda| = L^d|\Lambda|$ from the number $|\Lambda|$ of macrocells. For a physical macrocell domain $A = A_\Lambda$, write $|\partial_L A| = |\partial\Lambda|$.

Fix for now $0 < \delta < \min\{1, r/100\}$ and $0 < w \le \delta$. Throughout the estimates,

$$\mu = -r/2 + t, \qquad |t| \le w. \tag{2.2}$$

The constants denoted C depend only on d, a, J, b, B, r. Constants written C_δ may also depend on the near-packed configuration chosen below. All bounds are uniform in the finite domain and the boundary configuration.

2.2 Density constraints

Spread the counting measure uniformly within every fine cube, and denote the resulting density by ρ_h. Lemma 2.1 gives

$$0 \le \rho_h \le r + \epsilon_h, \qquad \epsilon_h = r[(1 + a/h)^d - 1] = O(h^{-1}). \tag{2.3}$$

Label each phase cell C by

$$\eta_C(q) = \begin{cases} 0, & N_C/|C| \le \delta, \\ r, & N_C/|C| \ge r - \delta, \\ *, & \delta < N_C/|C| < r - \delta. \end{cases} \tag{2.4}$$

For a union A of phase cells, $\mathcal{X}_A^c$ is the set of internally admissible configurations with every phase-cell label equal to $c \in \{0, r\}$.

Lemma 2.3 (Density control at an endpoint)**.** *If a phase cell C has label c, then*

$$\int_C |\rho_h - c| \le (\delta + 2\epsilon_h)|C|. \tag{2.5}$$

Proof. For $c = 0$ this is the density constraint. For $c = r$, use

$$\int_C |\rho_h - r| = r|C| - N_C + 2\int_C (\rho_h - r)_+ \le (\delta + 2\epsilon_h)|C|.$$

□

The upper packing bound makes this implication possible at the dense endpoint. At an interior density, the mean alone permits order-one positive and negative deviations of the fine profile.

2.3 Configurations compatible with every exterior

We call a configuration in a macrocell *safe* if its particles stay at least $2a$ from the cell boundary. Such a configuration is compatible with every admissible configuration outside the cell. This use of distinguished compatible states is related to the use of compatible boundary states in Dobrushin–Warstat [2, §2.1]; the following estimate quantifies its cost near maximal packing.

Lemma 2.4 (Safe dense configurations)**.** *For all sufficiently large R, each Q_i has a measurable set $\mathcal{S}_i^r \subset \mathcal{X}_{Q_i}^r$ of safe configurations such that*

$$(r - C\delta^2)L^d \le N_{Q_i} \le rL^d, \qquad \int_{\mathcal{S}_i^r} \mathrm{d}q \ge e^{-C_\delta L^d}. \tag{2.6}$$

The sets can be chosen as translates of one another. Set $\mathcal{S}_i^0 = \{\varnothing\}$.

Proof. Choose a periodic packing of density at least $r - c\delta^2$. Dilate it by $1 + c'\delta^2$, with fixed positive constants c, c'. Its density lies between $r - C\delta^2$ and $r - c''\delta^2$, and its particle separation exceeds a by a fixed multiple of δ^2. Around each center place a ball of radius a smaller fixed multiple of δ^2. Moving each center independently inside its ball preserves the hard-core constraint.

Within Q_0, retain only balls at distance at least $2a$ from its boundary. If P_δ is the fixed packing period, counting centers in a phase cell eroded by the displacement radius gives its periodic density times the cell volume, with an error $O_\delta(\ell^{d-1})$. The discarded boundary layer has the same order. For sufficiently large R, every phase cell therefore has density at least $r - \delta$, and the total number satisfies (2.6), including the upper bound because the periodic density is strictly below r.

Let $v_\delta > 0$ be the volume of one displacement ball and N the number retained. The $N!$ assignments of labeled particles to these disjoint balls cancel the factor $1/N!$ in $\mathrm{d}q$, giving configuration volume $v_\delta^N \ge e^{-C_\delta L^d}$. Translate this construction to every macrocell. One may take $C_\delta \le C(1 + |\log \delta|)$; the period affects the required lower bound on R. □

Using these sets independently in a union of macrocells gives globally admissible configurations compatible with every unchanged exterior. The replacement is grand-canonical: the new particle count may differ from the old one within its density band.

3 Centering and comparison of boundary conditions

For measures m, m' write

$$\langle m, J_R m' \rangle = \iint J_R(x - y) m(\mathrm{d}x) m'(\mathrm{d}y).$$

The same notation applies to densities. All integrals below have at least one argument supported in a bounded region. Set

$$e_R = h/R + h^{-1} + R^{-d}, \tag{3.1}$$

which tends to zero with R.

Lemma 3.1 (Replacing particles by fine densities)**.** *Let A, B be unions of fine cells, with A bounded. Replacing either counting measure in a Kac integral over $A \times B$ by its fine density changes the integral by at most $C(h/R)|A|$. For an integral crossing a macrocell domain A, the bound is*

$$C(h/R)L^d|\partial_L A|. \tag{3.2}$$

The same bounds hold when one measure already has density bounded by $2r+1$.

Proof. Moving one argument within a fine cell changes J_R by at most ChR^{-d-1}. Only points within distance $R + 2\sqrt{d}\,h$ can contribute. The packing bound gives $O(R^d)$ particles in such a neighborhood. Summing first over that neighborhood and then over particles in A gives $C(h/R)|A|$. For a crossing integral, the relevant particles in A belong to boundary macrocells, since $L > R + 2\sqrt{d}\,h$. Integration gives the same calculation for bounded densities. This is the usual Kac coarse-graining estimate; compare [10, Lemma 11.1.3.4]. □

For $c \in \{0, r\}$ define

$$\xi_A = n_A - c\mathbf{1}_A\,\mathrm{d}x, \qquad h_c = \mu + c - \tfrac{1}{2}J_R(0), \qquad B_A = \int_{A\times A^c} J_R(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

For large R, (2.2) gives $h_0 < 0 < h_r$. Define the centered Hamiltonians

$$\widetilde{H}^{c,\mathrm{f}}_A(q_A) = H^{\mathrm{hc}}_A(q_A) - h_c N_A - \tfrac{1}{2}\langle \xi_A, J_R\xi_A\rangle,$$
$$\widetilde{H}^{c}_A(q_A \mid q_{A^c}) = \widetilde{H}^{c,\mathrm{f}}_A(q_A) - \langle \xi_A, J_R\xi_{A^c}\rangle + H^{\mathrm{hc}}_{\mathrm{cross}}(q_A, q_{A^c}),$$

where the last term imposes only hard-core constraints across ∂A.

Lemma 3.2 (Exact boundary field)**.** *For every admissible interior and exterior,*

$$H_{A,\mu}(q_A \mid q_{A^c}) = \widetilde{H}^{c}_A(q_A \mid q_{A^c}) + \frac{c^2}{2}(|A| + B_A) - c\langle \mathbf{1}_A, J_R n_{A^c}\rangle. \tag{3.3}$$

Proof. Expand both centered pair integrals. Normalization of J_R gives

$$\langle n_A, J_R\mathbf{1}_A\rangle + \langle n_A, J_R\mathbf{1}_{A^c}\rangle = N_A, \qquad \langle \mathbf{1}_A, J_R\mathbf{1}_A\rangle = |A| - B_A.$$

The diagonal term in $\langle n_A, J_R n_A\rangle$ is $J_R(0)N_A$, which is canceled by the correction in h_c. The remaining terms give (3.3). □

The last two terms in (3.3) are explicit boundary and volume terms. The estimate below concerns the centered interaction that remains.

Lemma 3.3 (Small centered interaction)**.** *On the product of individually admissible phase-c macrostates, the absolute sum of centered pair terms incident to one macrocell, including its internal term, is at most*

$$C(\delta^2 + e_R)L^d. \tag{3.4}$$

Consequently a centered interaction across a macrocell domain A has absolute value at most $C(\delta^2 + e_R)L^d|\partial_L A|$.

Proof. After coarse graining, Lemma 2.3 bounds the total variation of the centered density in each phase cell by $(\delta + 2\epsilon_h)$ times its volume. The fine-cell packing bound holds separately in every macrostate, including when different macrostates clash across their faces. A ball of radius $2R$ meets phase cells of total volume at most CR^d, so the field generated by this centered density is bounded by $\|J_R\|_\infty C(\delta + 2\epsilon_h)R^d \le C(\delta + \epsilon_h)$. Integrating against the centered density in one macrocell gives $C(\delta + \epsilon_h)^2 L^d$. Lemma 3.1 restores the particle measures with error $C(h/R)L^d$. Only boundary macrocells contribute to a crossing term. □

3.1 A summable additional interaction

The surface argument will also apply after the contour expansion. To prepare for that application, allow a dimensionless interaction $U = (U_X)$ on macrocell states. Thus U_X appears in the Gibbs factor as e^{-U_X}. Assume that U is translation covariant, that X is a finite connected subset of the macro lattice whenever $U_X \neq 0$, and that

$$\sum_{X \ni 0} \|U_X\|_\infty \le \varepsilon_U. \tag{3.5}$$

The functions are defined on the product of individually admissible phase-c macrocell states. The hard-core constraints between different macrocells are imposed separately.

Lemma 3.4 (Boundary terms of a connected interaction)**.** *For every finite* Λ*,*

$$\sum_{\substack{X \cap \Lambda \neq \varnothing \\ X \cap \Lambda^c \neq \varnothing}} \|U_X\|_\infty \le \varepsilon_U |\partial\Lambda|.$$

Changing the states in T *changes the interaction by at most* $2\varepsilon_U|T|$*.*

Proof. A connected set crossing Λ contains a site of $\partial\Lambda$. Sum (3.5) over those sites. For the second claim, sum it over sites in T and bound the change of each term by twice its supremum norm. □

Use the one-cell probability measure

$$\alpha_c(\,\mathrm{d}q_i) = z_c^{-1} \mathbf{1}_{\mathcal{X}^c_{Q_i}}(q_i) e^{\beta h_c N_{Q_i}}\,\mathrm{d}q_i. \tag{3.6}$$

Identify translated macrocell state spaces so that z_c and α_c are independent of i. The dimensionless soft interaction Φ consists of U and the terms

$$V_{\{i\}} = -\frac{\beta}{2}\langle \xi_{Q_i}, J_R \xi_{Q_i}\rangle, \qquad V_{\{i,j\}} = -\beta\langle \xi_{Q_i}, J_R \xi_{Q_j}\rangle.$$

The pair term vanishes unless i, j are adjacent. Lemma 3.3 gives

$$\sum_{X \ni 0} \|\Phi_X\|_\infty \le M_R := C\beta(\delta^2 + e_R)L^d + \varepsilon_U. \tag{3.7}$$

Let I^{f}_Λ impose all hard-core constraints between different macrocells of Λ, and let I^ω_Λ also impose those with an exterior macrostate configuration ω. Write

$$\begin{aligned} \widetilde{Z}^{\mathrm{f}}_\Lambda &= z_c^{|\Lambda|} \int I^{\mathrm{f}}_\Lambda(q) e^{-\sum_{X \subset \Lambda} \Phi_X(q_X)} \alpha_c^{\otimes\Lambda}(\,\mathrm{d}q), \\ \widetilde{Z}^\omega_\Lambda &= z_c^{|\Lambda|} \int I^\omega_\Lambda(q) e^{-\sum_{X : X \cap \Lambda \neq \varnothing} \Phi_X((q\omega)_X)} \alpha_c^{\otimes\Lambda}(\,\mathrm{d}q). \end{aligned} \tag{3.8}$$

Here $q\omega$ equals q on Λ and ω outside. Denote the corresponding normalized laws by $\gamma^{\mathrm{f}}_\Lambda$ and γ^ω_Λ.

Lemma 3.5 (Mass of the safe set)**.** *For either* c*, and all sufficiently large* R*,*

$$\alpha_c(\mathcal{S}_i^c) \ge \exp\{-C[\beta(\delta^2 + e_R) + 1 + C_\delta]L^d\}. \tag{3.9}$$

Proof. For $c = 0$, $h_0 < 0$ gives $z_0 \le e^{L^d}$, while the empty configuration has mass one before normalization. For $c = r$, packing and Lemma 2.4 give

$$z_r \le e^{\beta h_r (r + O(L^{-1}))L^d + L^d}, \qquad \int_{\mathcal{S}_i^r} e^{\beta h_r N}\,\mathrm{d}q \ge e^{\beta h_r (r - C\delta^2)L^d - C_\delta L^d}.$$

The field h_r is uniformly bounded, and $L^{-1} \le C e_R$. □

Define

$$K_R = [\beta(\delta^2 + e_R) + 1 + C_\delta]L^d + \varepsilon_U. \tag{3.10}$$

Lemma 3.6 (Comparison with a free boundary)**.** *Uniformly in exterior macrostates $\omega = (\omega_i)_{i\notin\Lambda}$ with $\omega_i \in \mathcal{X}_{Q_i}^c$ for every $i \notin \Lambda$,*

$$|\log \widetilde{Z}_\Lambda^\omega - \log \widetilde{Z}_\Lambda^{\mathrm{f}}| \le CK_R|\partial\Lambda|. \tag{3.11}$$

Moreover γ_Λ^ω has density at most $\exp(CK_R|\partial\Lambda|)$ relative to $\gamma_\Lambda^{\mathrm{f}}$.

Proof. The bounded interaction crossing the boundary has absolute value at most $M_R|\partial\Lambda|$, by Lemma 3.4 applied to Φ. Adding hard-core constraints reduces an integral. These facts give the upper partition-function bound.

For the lower bound, set $T = \partial\Lambda$ and condition the free law on the states in $\Lambda \setminus T$. Its conditional integral over T is at most $e^{M_R|T|}$ after dropping hard-core indicators. Restricting every state in T to its safe set gives a lower integral $e^{-M_R|T|}\prod_{i\in T}\alpha_c(\mathcal{S}_i^c)$, because all constraints involving these cells hold. Hence

$$\gamma_\Lambda^{\mathrm{f}}(q_i \in \mathcal{S}_i^c \text{ for all } i \in T \mid q_{\Lambda\setminus T}) \ge e^{-CK_R|T|}.$$

On this event the exterior hard-core constraints also hold. The crossing soft factor is bounded below by $e^{-M_R|T|}$, proving the lower comparison. Finally, the conditioned density relative to the free law is the crossing factor times a hard-core indicator, divided by their partition-function ratio. The two estimates bound this density as stated. □

4 Surface pressure from conditional entropy

We now derive the pressure estimate from the boundary comparison. The argument uses the entropy method of Robinson–Ruelle [11] and the finite-volume variational identities of Lanford–Ruelle [5, Appendix A].

Lemma 4.1 (A stationary restricted Gibbs measure)**.** *The specification in (3.8) has a macro-lattice translation-invariant Gibbs measure ν. Its finite marginals have bounded densities relative to $\alpha_c^{\otimes\Lambda}$, and its one-site conditional densities are bounded uniformly in the exterior.*

Proof. The safe-set comparison can be applied to every cell of any finite set F. From (3.7) and (3.9), every conditional law on F has density at most $e^{CK_R|F|}$ relative to $\alpha_c^{\otimes F}$. The same bound holds for finite marginals of Gibbs laws in larger boxes with a fixed safe exterior.

For each F, these densities have weak-* convergent subsequences in $L^\infty(\alpha_c^{\otimes F})$. The state spaces are standard Borel and their L^1 spaces are separable. A diagonal subsequence over finite sets therefore gives consistent marginals and a probability measure ν. The convergence is against all bounded measurable local functions. It preserves the hard-core constraints.

To pass the Gibbs identities to the limit, first truncate the interactions meeting F to a finite neighborhood of F. The omitted norm is at most

$$|F| \sum_{X\ni 0:\,\operatorname{diam} X > k} \|U_X\|_\infty \longrightarrow 0.$$

Changing the Hamiltonian by at most ϵ changes a normalized conditional law by total variation at most $e^{2\epsilon} - 1$. The truncated kernel depends measurably on finitely many states, so its identity passes to the local limit. Letting $k \to \infty$ proves the full identity. A monotone-class argument extends it from local exterior events to all exterior events. Finally, average macro-lattice translates over growing cubes and take another local limit. The bounds and Gibbs identities persist, and the resulting measure is stationary. □

For a probability measure with density f relative to a reference probability α, use the entropy convention $H = -D = -\int f \log f \, d\alpha$. Conditional entropy is the average of this expression for the conditional laws, with the same reference. All entropies below use the product of the one-cell measures (3.6).

Lemma 4.2 (Conditional-entropy bounds)**.** *Let ν be stationary and have uniformly bounded one-site conditional densities given all other sites. Let $P = \{i \in \mathbb{Z}^d : i \prec 0\}$ for the lexicographic order, and set $s = H_\nu(q_0 \mid q_P)$. For every finite Λ,*

$$H_\nu(q_\Lambda \mid q_{\Lambda^c}) \le s|\Lambda| \le H_\nu(q_\Lambda). \tag{4.1}$$

Proof. A conditional density bound remains valid under less conditioning, by averaging the full conditional law. If that bound is M, all one-site conditional entropies lie in $[-\log M, 0]$. The chain rule is therefore applicable. In the chain rule for $H_\nu(q_\Lambda)$, a site's preceding sites in Λ form a subset of its full lexicographic past. Conditioning on the full past can only decrease entropy; stationarity makes that entropy s. In the chain rule conditional on Λ^c, each conditioning set contains the full past, giving the reverse inequality. This is the lexicographic entropy argument used in [11, Proposition 2]. □

Proposition 4.3 (Surface estimate for the centered pressure)**.** *There is a number $\widetilde{p}_c$ such that, for every finite Λ,*

$$\left|\log \widetilde{Z}^{\mathrm{f}}_\Lambda - \beta \widetilde{p}_c L^d |\Lambda|\right| \le C K_R |\partial\Lambda|. \tag{4.2}$$

The same estimate holds for $\widetilde{Z}^{\omega}_\Lambda$.

Proof. Use the stationary measure from Lemma 4.1. Define its expected soft energy per macrocell by

$$e = \mathbb{E}_\nu \sum_{X \ni 0} \frac{\Phi_X(q_X)}{|X|}.$$

The sum is absolutely convergent. Assigning each interaction equally to its sites and using Lemma 3.4 gives

$$\left|\mathbb{E}_\nu \sum_{X \subset \Lambda} \Phi_X - e|\Lambda|\right| \le M_R |\partial\Lambda|. \tag{4.3}$$

Hard-core energies are zero ν-almost surely; their indicators remain in the partition functions.

Normalize the partition functions by $z_c^{|\Lambda|}$. The finite-volume variational inequality, obtained from nonnegativity of relative entropy, gives

$$\log \frac{\widetilde{Z}^{\mathrm{f}}_\Lambda}{z_c^{|\Lambda|}} \ge H_\nu(q_\Lambda) - \mathbb{E}_\nu \sum_{X \subset \Lambda} \Phi_X \ge (s-e)|\Lambda| - C M_R |\partial\Lambda|.$$

The conditional Gibbs identity gives the other direction:

$$\begin{aligned} \mathbb{E}_\nu \log \frac{\widetilde{Z}^{q_{\Lambda^c}}_\Lambda}{z_c^{|\Lambda|}} &= H_\nu(q_\Lambda \mid q_{\Lambda^c}) - \mathbb{E}_\nu \sum_{X : X \cap \Lambda \neq \varnothing} \Phi_X \\ &\le (s-e)|\Lambda| + C M_R |\partial\Lambda|. \end{aligned}$$

Apply Lemma 3.6 to compare this with the free partition function. The two bounds prove (4.2) with

$$\widetilde{p}_c = \frac{\log z_c + s - e}{\beta L^d}.$$

Taking growing boxes identifies this number as the pressure. The conditioned estimate follows from the same boundary comparison. □

For a physical domain $A = A_\Lambda$, let $Z_A^{c,U}(q)$ be the phase-restricted physical partition function, with the conditioned terms of U added to its dimensionless Hamiltonian. Define $p_c^U = \widetilde{p}_c - c^2/2$.

Corollary 4.4 (Physical boundary formula)**.** *For every admissible exterior configuration q whose restriction to every macrocell $Q_i \subset A^c$ belongs to $\mathcal{X}_{Q_i}^c$,*

$$\log Z_A^{c,U}(q) = \beta p_c^U |A| + \beta c\langle \mathbf{1}_A, J_R n_q\rangle - \frac{\beta c^2}{2} B_A + E_c(A,q), \qquad |E_c(A,q)| \le C K_R |\partial_L A|. \quad (4.4)$$

Proof. Apply Lemma 3.2 inside the partition function and then Proposition 4.3. The scalar terms in (3.3) give exactly the displayed boundary terms. For the free physical Hamiltonian, direct expansion gives

$$H_A^{\mathrm{f}} = \widetilde{H}_A^{c,\mathrm{f}} + \frac{c^2}{2}|A| + c\langle n_A, J_R \mathbf{1}_{A^c}\rangle - \frac{c^2}{2} B_A.$$

The last two terms have surface order at fixed R, so p_c^U is also the free physical pressure. □

Write p_c^{bare} when $U = 0$. The coefficient in (4.4), divided by βL^d, is bounded by

$$C\left[\delta^2 + e_R + \frac{1 + C_\delta}{\beta} + \frac{\varepsilon_U}{\beta L^d}\right]. \quad (4.5)$$

The linear exterior field in (4.4) remains explicit. In the contour comparison it will combine with the interaction of particles on the contour support.

5 Contours and recursive weights

We use the thick contours of Presutti [10, §§10.3–10.5]. The enlargement in the definition of correctness supplies a full macrocell of constant phase next to each contour interior.

5.1 Correctness and geometric properties

For a macrocell Q_i, set $\theta_i = c$ if all its phase cells have label c, and set $\theta_i = *$ otherwise. For a set of macro indices F, write $N_1(F) = \{i : \mathrm{dist}_\infty(i, F) \le 1\}$. Define

$$\Theta_i = c \iff \theta_j = c \text{ for every } j \in N_1(\{i\}); \qquad \Theta_i = * \text{ otherwise.} \quad (5.1)$$

A cell is *correct* when $\Theta_i \in \{0, r\}$ and *incorrect* when $\Theta_i = *$. A contour Γ is a realizable pair consisting of a finite connected component S_Γ of the incorrect macrocells and the phase-cell labels on that component. We use S_Γ also for its physical union, with $|S_\Gamma|$ denoting Lebesgue volume, and write

$$n(\Gamma) = |S_\Gamma|/L^d.$$

The unique unbounded component of its macrocell complement is $E = \mathrm{Ext}(S_\Gamma)$; the bounded components are $D_1, \dots, D_k$. Its *filled support* is $C(\Gamma) = S_\Gamma \cup \bigcup_j D_j$.

Lemma 5.1 (Constant phase along a complementary boundary)**.** *For each complementary component D, its macrocells adjacent to S_Γ are correct with one common phase. Every macrocell in S_Γ adjacent to D has that same θ label. Distinct complementary components have Euclidean separation at least $L > R$.*

Proof. The set $D \cap N_1(S_\Gamma)$ is connected in the macro adjacency graph. To verify this fact, apply Timár's boundary-connectivity lemma [13, Lemma 2] with both graphs equal to the graph of ℓ^∞ neighbors on $\mathbb{Z}^d$. Its cycle-space hypothesis holds: replace diagonal edges by nearest-neighbor

paths within their unit cubes, and then generate nearest-neighbor cycles by unit squares. Each resulting generating cycle lies in a unit-cube clique.

Every cell of that boundary is correct, since an incorrect cell adjacent to S_Γ would belong to the same component. Neighboring correct cells have the same phase by (5.1); connectedness gives a common phase on the boundary. The same definition forces the neighboring support cells to have that θ label. Distinct complementary components cannot contain adjacent macro indices, so their Euclidean separation is at least L. □

Call the phase on the exterior boundary the *type* c_e of Γ, and denote the phases of the bounded components by c_j. These phases are determined by the contour code. The preceding lemma gives a collar of support cells with the appropriate density constraints at every complementary boundary.

Lemma 5.2 (Containment of filled supports)**.** *Every bounded complementary component D_j has connected complement. If A is a finite macrocell domain with connected complement and $S_\Gamma \subset A$, then $C(\Gamma) \subset A$.*

Proof. Every complementary component of the finite connected set S_Γ attaches to S_Γ. Thus S_Γ together with all components other than D_j is connected and unbounded; this is D_j^c. For the second assertion, the connected unbounded set A^c is disjoint from S_Γ and therefore lies in its exterior component. □

A *density defect* is either a phase cell with label $*$ or a pair of face-neighbor phase cells with opposite labels $0, r$. This localizes the two kinds of incorrect region used in [10, §11.3].

Lemma 5.3 (Disjoint density defects)**.** *The support of a contour of size n contains at least $n/5^d$ pairwise disjoint density defects.*

Proof. An incorrect macrocell i sees either a cell with $\theta = *$ in $N_1(\{i\})$, or two different pure θ labels there. In the first case, within that macrocell there is either a phase cell labeled $*$ or a nearest-neighbor path of phase cells containing an opposite pair. In the second case, follow a nearest-neighbor macro path inside $N_1(\{i\})$. Either it encounters $\theta = *$ or it crosses a face between opposite pure cells. The latter face contains an opposite pair of phase cells. All macrocells used for the resulting defect are incorrect and adjacent to i, so belong to S_Γ.

Greedily select incorrect macrocells at mutual ℓ^∞ distance at least three. Each selection removes at most 5^d candidates. Their radius-one neighborhoods are disjoint, and so are the associated defects. □

Lemma 5.4 (Counting contour codes)**.** *For a specified macrocell, the number of contour codes of size n whose filled support contains that cell is at most $e^{k_0 n}$, for a constant k_0 depending only on m, d.*

Proof. Connected sets of n cells containing a fixed anchor have exponentially many possibilities: a spanning-tree traversal gives a walk of length at most $2n$ in a graph of fixed degree. The labels contribute at most $3^{m^d n}$ possibilities. A connected support of size n and its filled support lie in a box of side at most $2n + 1$ in macro units. Thus at most Cn^d possible anchors are needed when the specified cell lies in the filled support. Absorb this factor into the exponential. □

Lemma 5.5 (Sum of interior boundaries)**.** *The bounded complementary components of a contour satisfy*

$$\sum_j |\partial_L D_j| \le Cn(\Gamma). \tag{5.2}$$

Proof. Every boundary cell of a complementary component is adjacent to S_Γ. The macro graph has degree $3^d - 1$. □

5.2 Diluted ensembles and the exact replacement identity

The recursion domains are finite macrocell domains A with connected complement; they include boxes and the components D_j. The *diluted phase-c partition function* $Z_A^c(q_{A^c})$ is the physical partition function with the constraint $\Theta_i = c$ on its inner boundary. The prescribed exterior satisfies $\theta_i = c$ on $N_1(A) \setminus A$. Consequently all contour supports avoid the inner boundary.

For a contour Γ of type c, put $S = S_\Gamma$, denote its phase-cell code by η_Γ, and let q denote exterior particles in $E = \mathrm{Ext}(S)$. Define

$$N_\Gamma^c(q) = \int_{\eta(q_S)=\eta_\Gamma} \mathrm{d}q_S \, e^{-\beta H_S(q_S|q)} \prod_j Z_{D_j}^{c_j}(q_S), \tag{5.3}$$

$$D_\Gamma^c(q) = \int_{q_S \in \mathcal{X}_S^c} \mathrm{d}q_S \, e^{-\beta H_S(q_S|q)} \prod_j Z_{D_j}^{c}(q_S), \qquad W_\Gamma^c(q) = \frac{N_\Gamma^c(q)}{D_\Gamma^c(q)}.$$

Here $H_S(q_S \mid q)$ includes only support interactions and interactions with E. The interior factors contain the interactions with D_j. The collar property makes each diluted boundary condition well defined. Safe fillings show that every denominator is positive.

Only exterior particles within distance R of S enter these weights. Extend the definition to products of individually admissible phase-c exterior macrostates by the same conditional integrals: interactions wholly in the exterior are absent. Safe fillings still give positive denominators. This supplies bounded measurable functions on the product state space used in Section 4.

Assign to a contour the macrocell set

$$E_\Gamma = N_1(S_\Gamma), \qquad \mathcal{C}_c(A) = \{\Gamma : \Gamma \text{ has type } c,\ E_\Gamma \subset A\}.$$

In this notation N_1 acts on macro indices and the resulting set is identified with its union of cells. Two contours are *compatible* when their supports are disjoint and no support cell of one is adjacent to a support cell of the other. Filled supports may be nested. A physical contour is *external* when its support is not contained in the filled support of another physical contour.

Proposition 5.6 (Exact contour-gas identity)**.** *For every recursion domain,*

$$Z_A^c(q_{A^c}) = \int_{\mathcal{X}_A^c} \mathrm{d}q_A \, e^{-\beta H_A(q_A|q_{A^c})} \sum_{\substack{\mathcal{F} \subset \mathcal{C}_c(A) \\ \text{pairwise compatible}}} \prod_{\Gamma \in \mathcal{F}} W_\Gamma^c(q). \tag{5.4}$$

Here $q = q_A \cup q_{A^c}$ is the combined background used to evaluate the weights.

Proof. Decompose a configuration by its external contours. After conditioning on their support particles, the complementary interiors have no interactions with one another and contribute the factors in N_Γ^c. The replacement $N_\Gamma^c = W_\Gamma^c D_\Gamma^c$ fills the same support and interiors with a phase-c background. Repeat the replacement in these phase-c interiors. It produces the compatible contour families in (5.4); every contour in this auxiliary family has type c, because each replacement restores the same background phase.

This is the Minlos–Sinai replacement identity in [10, Theorem 10.5.1.1]. Its domain conditions follow here from Lemmas 5.1 and 5.2. The diluted condition is equivalent to excluding supports touching the inner boundary, hence to $E_\Gamma \subset A$. □

5.3 Recursive truncation

Fix $\tau > 0$. Replace the interior partition functions in (5.3) by truncated ones, denoted $\widehat{Z}$, and define

$$\widehat{W}_\Gamma^c(q) = \min\left\{ \frac{\widehat{N}_\Gamma^c(q)}{\widehat{D}_\Gamma^c(q)}, e^{-\tau n(\Gamma)} \right\}. \tag{5.5}$$

Define $\widehat{Z}_A^c$ by (5.4) with these weights. This is the recursive truncation of Zahradník [14], in the form used by Presutti [10, Theorem 10.5.1.2].

Lemma 5.7 (Well-defined truncation)**.** *These rules determine unique nonnegative weights and partition functions.*

Proof. Induct on the number of cells in the filled support. Every contour integrated in an interior D_j has its filled support inside D_j, by Lemma 5.2, and therefore has smaller filled volume. Start with domains containing no contour. Each subsequent denominator is positive because the empty contour family and safe background configurations give a positive contribution. Thus the ratio, the minimum, and the finite partition function are determined at each step. □

6 The interaction generated by contours

For fixed microscopic background q, denote the sum in the truncated version of (5.4) by $\Xi_A^c(q)$. It is a numerical hard-core polymer partition function with activities bounded by $e^{-\tau n(\Gamma)}$. The following construction is the expansion with fixed background in [10, §11.4.3].

A cluster term for $(\Gamma_1, \dots, \Gamma_k)$ is

$$\frac{1}{k!}\varphi^T(\Gamma_1, \dots, \Gamma_k) \prod_{j=1}^{k} \widehat{W}_{\Gamma_j}^c(q),$$

where φ^T is the sum over connected graphs on $\{1, \dots, k\}$ of the product of $-\mathbf{1}_{\{\Gamma_i \not\sim \Gamma_j\}}$ over their edges. For $k = 1$ it equals one. Repeated contours are incompatible with themselves.

A cluster is anchored at the origin when $0 \in \bigcup_j E_{\Gamma_j}$. Its total contour size is $\sum_j n(\Gamma_j)$.

Lemma 6.1 (Uniform numerical cluster expansion)**.** *For τ larger than a fixed constant, for each finite A the cluster expansion of $\log \Xi_A^c(q)$ converges absolutely, uniformly in q, t. The sum of absolute cluster terms anchored at the origin, each multiplied by $\exp\{(\tau/4)\sum_j n(\Gamma_j)\}$, is at most $Ce^{-c_0\tau}$, uniformly also in A.*

Proof. Every contour incompatible with a given Γ meets $N_1(S_\Gamma)$. Lemma 5.4 gives, for fixed $a_0 > 0$,

$$\sum_{\Delta \not\sim \Gamma} \sup_q \widehat{W}_\Delta^c(q) e^{a_0 n(\Delta)} \le C n(\Gamma) \sum_{k \ge 1} e^{-(\tau - a_0 - k_0)k}.$$

For large τ this is at most $a_0 n(\Gamma)$, the Kotecký–Preiss criterion [4]. The criterion also holds with activities multiplied by $e^{\tau n/4}$. Its rooted-cluster bound, summed over contours whose support meets a fixed cell neighborhood, gives the stated anchored estimate and the additional exponential weight. □

Lemma 6.2 (A small interaction on connected supports)**.** *There is a translation-covariant dimensionless interaction U^c on connected macrocell sets such that*

$$\sum_{X \ni 0} \|U_X^c\|_\infty \le \varepsilon(\tau) := Ce^{-c_0\tau}. \tag{6.1}$$

Proof. The weight of Γ depends only on its exterior interaction collar, which lies in E_Γ. For each cluster assign the set $X = \bigcup_j E_{\Gamma_j}$, and define U_X^c as the negative sum of the cluster terms with that assigned set. This is the support enlargement in [10, (11.9.0.13)–(11.9.0.15)].

Each E_Γ is connected and has at most $3^d n(\Gamma)$ cells. Connectedness of the incompatibility graph makes X connected. If $0 \in X$, some constituent support meets the fixed radius-one neighborhood of 0. Anchor that constituent and apply Lemma 6.1. Taking absolute values before grouping gives (6.1), uniformly on the product macrostate space. The definitions commute with translations, giving covariance. □

Lemma 6.3 (Exact finite-domain identification)**.** *For every recursion domain A,*

$$\log \Xi_A^c(q) = -\sum_{X \subset A} U_X^c(q_X). \tag{6.2}$$

Proof. A contour belongs to the finite catalog precisely when $E_\Gamma \subset A$. Therefore a cluster belongs to that catalog precisely when its assigned union X lies in A. Its activities and connected-graph coefficient agree with those used to define U_X^c. Absolute convergence permits grouping by X. □

Let $\widehat{p}_c = p_c^{U^c}$, with the pressure defined in Section 4. Introduce the common error expression

$$\Delta_R = \delta^2 + e_R + \frac{1 + C_\delta}{\beta}. \tag{6.3}$$

For τ above the threshold in Lemma 6.1, the norm $\varepsilon(\tau)$ is bounded by a fixed constant, and $K_R \le C\beta\Delta_R L^d$ for $L \ge 1$.

Corollary 6.4 (Surface formula for the truncated ensemble)**.** *Let A be a recursion domain and q an admissible exterior configuration satisfying $\theta_i(q) = c$ for every $i \in N_1(A) \setminus A$. Then*

$$\log \widehat{Z}_A^c(q) = \beta\widehat{p}_c|A| + \beta c\langle \mathbf{1}_A, J_R n_q\rangle - \frac{\beta c^2}{2} B_A + \widehat{E}_c(A,q), \qquad |\widehat{E}_c(A,q)| \le C\beta\Delta_R L^d |\partial_L A|. \tag{6.4}$$

Proof. Keep q unchanged on $N_1(A) \setminus A$, and fill every remaining exterior macrocell with a configuration from $\mathcal{S}_i^c$ of Lemma 2.4. The resulting exterior $\widetilde{q}$ is admissible and has phase c on all of A^c. Since $L > \max\{R, a\}$, the replacement preserves the physical Hamiltonian in A and its diluted boundary condition. Lemma 6.3 shows that $\widehat{Z}_A^c$ contains only effective terms with $X \subset A$, so

$$\widehat{Z}_A^c(q) = \widehat{Z}_A^c(\widetilde{q}).$$

Add the conditioned crossing terms of U^c evaluated at $\widetilde{q}$. Lemmas 3.4 and 6.2 give

$$\left|\log \widehat{Z}_A^c(\widetilde{q}) - \log Z_A^{c,U^c}(\widetilde{q})\right| \le \varepsilon(\tau)|\partial_L A|.$$

Apply Corollary 4.4 to $\widetilde{q}$. The finite range also gives $\langle \mathbf{1}_A, J_R n_{\widetilde{q}}\rangle = \langle \mathbf{1}_A, J_R n_q\rangle$. Substitute these identities and absorb the crossing cost into the stated error. □

Corollary 6.5 (Size of the pressure correction)**.** *The bare and truncated pressures satisfy*

$$0 \le \widehat{p}_c - p_c^{\mathrm{bare}} \le \frac{C\varepsilon(\tau)}{\beta L^d}. \tag{6.5}$$

Proof. The empty family gives $\Xi_A^c \ge 1$. Equations (6.1)–(6.2) give $|\log \Xi_A^c| \le \varepsilon(\tau)|A|/L^d$. Compare the integrals and take the pressure limit along boxes. □

7 Equality of the two pressures

We keep τ above the cluster-expansion threshold and $|t| \le w \le \delta$. Pressure equality is obtained by the continuity argument used in [10, §11.1]. Here the estimates locating the crossing use only packing and trial configurations.

Lemma 7.1 (Continuity of the truncated pressures)**.** *The functions $\widehat{p}_0(t)$ and $\widehat{p}_r(t)$ are continuous on $[-w, w]$.*

Proof. Induction on filled volume shows that each truncated contour weight is continuous in t, uniformly in its exterior macrostates. Indeed, particle counts are bounded on its finite filled support, the dependence of each integrand on t is uniform, and safe fillings give a positive lower bound on its denominator. Taking the minimum with the fixed cutoff preserves continuity. This is the recursive continuity argument of [10, §11.7.3].

The uniform cluster tails in Lemma 6.1 then imply continuity of $U^c(t)$ in the norm $\sum_{X\ni 0}\|U_X\|_\infty$: first retain clusters of bounded total contour size, then let the size bound increase. For the finite physical integrals, packing and the interaction norm give

$$|\widehat{p}_c(t)-\widehat{p}_c(s)|\le r(1+a/L)^d|t-s|+\frac{1}{\beta L^d}\sum_{X\ni 0}\|U_X^c(t)-U_X^c(s)\|_\infty.$$

The inequality follows first in boxes, then in the pressure limit. Its right side tends to zero as $s\to t$. □

Lemma 7.2 (A lower energy bound)**.** *For a configuration in a macrocell domain A with free boundary,*

$$H_{A,\mu}(q_A)\ge -tN_A-Ce_R|A|. \tag{7.1}$$

Proof. Extend ρ_h by zero outside A. Since $0\le\rho_h\le r+\epsilon_h$ and $\int J_R=1$,

$$\frac{r}{2}\int\rho_h-\frac{1}{2}\langle\rho_h,J_R\rho_h\rangle\ge-\frac{\epsilon_h}{2}\int\rho_h.$$

Apply Lemma 3.1 and drop the nonnegative self-pair correction $J_R(0)N_A/2$. □

Lemma 7.3 (Upper bounds on the bare pressures)**.** *Uniformly for $|t|\le w$,*

$$p_0^{\mathrm{bare}}(t)\le\max\{t,0\}\delta+Ce_R+\beta^{-1},$$
$$p_r^{\mathrm{bare}}(t)\le tr+w\delta+Ce_R+\beta^{-1}.$$

Proof. The unrestricted configuration measure of a box A is $e^{|A|}$. In the dilute ensemble $N_A\le\delta|A|$. In the dense ensemble $N_A\ge(r-\delta)|A|$, while Lemma 2.1 bounds its upper density by $r+o(1)$ as the box grows. Apply (7.1) in the finite integral and then take the pressure limit. For negative t, the lower count bound gives $t(r-\delta)\le tr+w\delta$; for positive t, the upper count bound gives tr in the limit. □

Lemma 7.4 (Trial lower bounds)**.** *The bare pressures satisfy*

$$p_0^{\mathrm{bare}}(t)\ge 0,\qquad p_r^{\mathrm{bare}}(t)\ge tr-C(\delta^2+e_R)-C_\delta/\beta.$$

Proof. The empty configuration gives the first inequality. For the second, fill each macrocell of a box A independently from its safe set. Their configuration volume is at least $e^{-C_\delta|A|}$. Lemmas 2.4 and 2.1 give

$$\int_A|\rho_h-r|\le C(\delta^2+\epsilon_h)|A|.$$

Normalization and positivity of J_R bound the change of the pair integral from the constant profile $r\mathbf{1}_A$ by the same quantity times a fixed constant. The chemical-potential term has the same error. The constant profile has energy $-tr|A|+r^2B_A/2$, so quadrature gives

$$\frac{1}{\beta}\log Z_A^{r,\mathrm{bare,f}}\ge\left[tr-C(\delta^2+e_R)-\frac{C_\delta}{\beta}\right]|A|-CB_A.$$

For a cube of side u, $B_A\le CRu^{d-1}$. Divide by $|A|$ and let $u\to\infty$ at fixed R. □

Corollary 7.5 (An equal-pressure point)**.** *There is a fixed constant C_p such that*

$$0 \le \widehat{p}_0(t) \le C_p\Delta_R, \qquad |\widehat{p}_r(t) - rt| \le C_p\Delta_R. \tag{7.2}$$

If $rw > 3C_p\Delta_R$, there is $t_R \in (-w, w)$ with

$$\widehat{p}_0(t_R) = \widehat{p}_r(t_R) =: \widehat{p}. \tag{7.3}$$

Proof. Combine the preceding pressure bounds with Corollary 6.5 and $w\delta \le \delta^2$. At $t = -w$ the dense pressure is below the dilute pressure; at $t = w$ it is above it. Lemma 7.1 gives the crossing. □

8 Energy of density defects

Fix a contour support S, with exterior component E, interior components D_j, and phase labels c_e, c_j. Put

$$\chi = c_e\mathbf{1}_E + \sum_j c_j\mathbf{1}_{D_j}.$$

For a density ρ on S, define

$$\begin{aligned} G_S(\rho \mid \chi) = \frac{1}{2}\int_S \rho(r-\rho) + \frac{1}{4}\int_{S\times S} J_R(x-y)(\rho(x)-\rho(y))^2\,\mathrm{d}x\,\mathrm{d}y \\ + \frac{1}{2}\int_{S\times S^c} J_R(x-y)(\rho(x)-\chi(y))^2\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{8.1}$$

For $0 \le \rho \le r$, every term is nonnegative. Let u denote the extension of ρ by χ to S^c.

Lemma 8.1 (The boundary energy identity)**.** *Let $B_E = \int_{E\times S} J_R$. Then*

$$\begin{aligned} -\mu\int_S \rho - \frac{1}{2}\langle \rho\mathbf{1}_S, J_R(\rho\mathbf{1}_S)\rangle - \langle \rho\mathbf{1}_S, J_R\chi\rangle + \frac{1}{2}\sum_j c_j^2 B_{D_j} \\ = G_S(\rho \mid \chi) - t\int_S \rho - \frac{1}{2}c_e^2 B_E. \end{aligned} \tag{8.2}$$

Proof. Expand the squares in (8.1). Their ρ^2 terms cancel the local term $-\rho^2/2$, leaving $r\int_S \rho/2$ and the two negative pair integrals. The constant term is

$$\frac{1}{2}\int_{S\times S^c} J_R(x-y)\chi(y)^2\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}c_e^2 B_E + \frac{1}{2}\sum_j c_j^2 B_{D_j}.$$

Here B_{D_j} involves only S, because distinct complementary components are farther apart than R. Substitute $\mu = -r/2 + t$ and rearrange. This retains the boundary terms in the interaction decomposition used in [10, §11.3.1]. □

Lemma 8.2 (A common lower kernel)**.** *For a density defect A, set $K_A(y) = \inf_{x\in A} J_R(x-y)$ and $k_A = \int K_A(y)\,\mathrm{d}y$. There is $C_J < \infty$, depending only on J, d, such that*

$$1 - C_J b \le k_A \le 1. \tag{8.3}$$

Proof. The diameter of A is at most $2\sqrt{d}\,bR$. Compare $K_A(y)$ with $J_R(x_0 - y)$ for any $x_0 \in A$. Their difference is at most CbR^{-d} on a ball of volume CR^d and vanishes outside that ball. Integrating gives the lower bound. The upper bound follows from $K_A \le J_R(x_0 - \cdot)$. □

From now on fix the integer m so large that $b = B/m < 1$ and $C_J b \le 1/2$. All geometric constants are henceforth fixed before choosing δ.

For $A \subset S$, define its allocated energy by

$$F_A(\rho, u) = \frac{1}{2}\int_A \rho(r-\rho) + \frac{1}{4}\int_{A\times\mathbb{R}^d} J_R(x-y)(\rho(x)-u(y))^2\,\mathrm{d}x\,\mathrm{d}y.$$

For disjoint A these contributions can be summed below G_S, since the boundary-square coefficient in G_S is $1/2$.

Lemma 8.3 (Energy from the mean density)**.** *Suppose $0 \le \rho \le r$ on a defect A, and let m_A be its mean density. Then*

$$F_A(\rho, u) \ge \frac{k_A}{4}|A|m_A(r-m_A). \tag{8.4}$$

Proof. Let v_A be the variance of ρ under normalized Lebesgue measure on A. For each y,

$$\int_A (\rho(x)-u(y))^2\,\mathrm{d}x = |A|[v_A + (m_A-u(y))^2] \ge |A|v_A.$$

Replacing $J_R(x-y)$ by $K_A(y)$ and integrating gives

$$\frac{F_A}{|A|} \ge \frac{1}{2}m_A(r-m_A) - \frac{1}{2}v_A + \frac{k_A}{4}v_A.$$

Since $v_A \le m_A(r-m_A)$ and $k_A \le 1$, the right side is at least $k_A m_A(r-m_A)/4$. □

Proposition 8.4 (Energy cost of a contour)**.** *For all sufficiently large R, a fine density realizing a contour code satisfies*

$$G_S(\rho_h \mid \chi) \ge c_*\delta\ell^d n(\Gamma) - Ce_R|S|, \qquad c_* = \frac{r}{32\,5^d}. \tag{8.5}$$

Proof. Clip ρ_h at r. By (2.3), its pointwise change is at most ϵ_h, so the change in G_S is at most $C\epsilon_h|S|$. If a phase cell has label $*$, its clipped mean lies in $[\delta/2, r-\delta/2]$ for large R. For a pair of opposite phase cells, the clipped mean lies in $[r/3, 2r/3]$, hence in the same larger interval. Thus $m_A(r-m_A) \ge r\delta/4$ on every defect. Lemmas 8.2 and 8.3 give energy at least $r\delta\ell^d/32$ per defect. Sum over the disjoint defects from Lemma 5.3 and restore the clipping error. □

The defect localization follows the Peierls argument in [10, Theorem 11.3.2.1]. The estimate here uses the endpoint polynomial $\rho(r-\rho)/2$, whose cost at distance δ from an endpoint is linear in δ.

9 The contour bound and removal of the truncation

Work at a value t satisfying (7.3). The parameter choice at the end of this section will ensure that such a value exists. Fix a contour of type c_e and an exterior phase-c_e configuration q_E.

Define two integrals over the support:

$$I_\Gamma(t) = \int_{\substack{q_S \text{ internally admissible}\\ \eta(q_S)=\eta_\Gamma}} e^{-\beta[G_S(\rho_h|\chi)-tN_S]}\,\mathrm{d}q_S,$$

$$I^{c_e}_{\mathrm{safe}}(t) = \int_{\prod_{Q_i\subset S}\mathcal{S}_i^{c_e}} e^{-\beta[G_S(\rho_h|c_e\mathbf{1}_{S^c})-tN_S]}\,\mathrm{d}q_S.$$

Lemma 9.1 (Cancellation of the interior factors)**.** *The untruncated ratio formed from truncated interior partition functions satisfies*

$$\frac{\widehat{N}^{c_e}_\Gamma(q_E)}{\widehat{D}^{c_e}_\Gamma(q_E)} \le e^{C\beta\Delta_R|S|}\frac{I_\Gamma(t)}{I^{c_e}_{\mathrm{safe}}(t)}. \tag{9.1}$$

Proof. For each interior D_j, Lemma 5.1 gives $\theta_i = c_j$ on $N_1(D_j) \setminus D_j$. Thus Corollary 6.4 applies to the actual support configuration in each numerator factor. In each denominator factor the same collar has phase c_e, so the corollary applies there as well.

At equal pressures, both products contain the identical bulk factor $\exp\{\beta\widehat{p}\sum_j |D_j|\}$. The sum of their error bounds is $C\beta\Delta_R L^d n$, by Lemma 5.5. The numerator's remaining interior factors are

$$\exp\left\{\beta\sum_j c_j\langle \mathbf{1}_{D_j}, J_R n_S\rangle - \frac{\beta}{2}\sum_j c_j^2 B_{D_j}\right\}.$$

The denominator has the same expression with every c_j replaced by c_e.

Next replace the outer particles by the constant density $c_e\mathbf{1}_E$. The factor

$$\exp\{\beta c_e\langle \mathbf{1}_S, J_R(n_{q_E} - c_e\mathbf{1}_E\,\mathrm{d}x)\rangle\}$$

is common to numerator and denominator. Its residual centered interaction has absolute value at most $C\beta(\delta^2 + e_R)|S|$, since the support cells interacting with E have phase c_e. For the numerator, drop the outer hard-core constraints to obtain an upper bound. For the denominator, restrict to safe support configurations, which satisfy those constraints.

Coarse grain the support particles and use Lemma 8.1. The resulting common scalar factor is $\exp(\beta c_e^2 B_E/2)$. The quadrature and self-pair errors are at most $C\beta e_R|S|$. After canceling the common factors, the two remaining integrals are exactly those in (9.1). All errors are bounded by the stated exponential. □

Lemma 9.2 (Numerator support integral)**.** *For every contour,*

$$\log I_\Gamma(t) \le -\beta c_*\delta\ell^d n + C\beta(w + e_R)|S| + |S|.$$

Proof. Apply Proposition 8.4, use $N_S \le C|S|$, and bound the total configuration measure on S by $e^{|S|}$. □

Lemma 9.3 (Denominator support integral)**.** *For either phase,*

$$\log I^c_{\mathrm{safe}}(t) \ge -C\beta(\delta^2 + w + e_R)|S| - C_\delta|S|.$$

Proof. For $c = 0$, the empty support has $G_S = 0$ and mass one. For $c = r$, safe fillings give $\int_S |\rho_h - r| \le C(\delta^2 + e_R)|S|$. On $0 \le \rho \le r$, each local or square term of $G_S(\rho \mid r\mathbf{1}_{S^c})$ is bounded by a fixed constant times the density deficit after integrating the normalized kernel. Clipping at r changes the energy by at most $C\epsilon_h|S|$. Hence $G_S \le C(\delta^2 + e_R)|S|$ on the safe set. Its volume is at least $e^{-C_\delta|S|}$, and $|t|N_S \le Cw|S|$. □

Proposition 9.4 (The contour ratio estimate)**.** *There is a fixed constant C_q such that, at equal truncated pressures,*

$$\log\frac{\widehat{N}^c_\Gamma}{\widehat{D}^c_\Gamma} \le -\beta g\delta|S| + C_q\beta(\Delta_R + w)|S|, \qquad g = c_*(b/B)^d > 0. \tag{9.2}$$

The bound is uniform in the contour and its exterior.

Proof. Combine the three preceding lemmas, use $\ell^d n = (b/B)^d|S|$, and absorb the configuration-volume costs into Δ_R. □

9.1 Choice of parameters

The constants g, C_p, C_q are now fixed. Set

$$A_0 = 1 + 8C_p/r, \qquad \kappa = g/8.$$

Choose $\delta > 0$ so small that

$$\delta < \min\{1, r/100, A_0^{-1}\}, \qquad C_q(2 + A_0)\delta \le g/4, \qquad w = A_0\delta^2. \tag{9.3}$$

Choose the safe packing of Lemma 2.4, and then choose β_0 so large that

$$(1 + C_\delta)/\beta_0 \le \delta^2/2.$$

For each fixed $\beta \ge \beta_0$, take R sufficiently large for all preceding geometric and packing estimates, for $e_R \le \delta^2/2$, and for the cluster-expansion threshold with

$$\tau = \beta\kappa\delta L^d. \tag{9.4}$$

Then $\Delta_R \le 2\delta^2$, $w \le \delta$, and $rw > 3C_p\Delta_R$. Corollary 7.5 supplies t_R with equal pressures. At that point (9.2) and (9.3) give

$$\frac{\widehat{N}^c_\Gamma}{\widehat{D}^c_\Gamma} \le e^{-6\tau n(\Gamma)} \le e^{-2\tau n(\Gamma)}. \tag{9.5}$$

Set $\mu_R = -r/2 + t_R$.

Corollary 9.5 (Removal of all truncations)**.** *At μ_R, the truncated weights and partition functions coincide with the physical ones, and*

$$W^c_\Gamma(q) \le e^{-2\tau n(\Gamma)}. \tag{9.6}$$

Proof. The bound (9.5) is strictly smaller than the cutoff in (5.5). Induct on filled volume. For the smallest contours, every interior partition function is already physical, so the minimum selects the physical ratio. Once this holds in smaller filled volumes, the same is true for the present contour and its partition functions. This is the cutoff-removal induction of [10, Theorem 10.5.2.1 and Corollary 10.5.2.2]. □

10 Gibbs measures and the density jump

Fix $c \in \{0, r\}$. In growing macrocell boxes impose the diluted phase-c boundary condition, with the empty exterior for $c = 0$ and a periodic safe exterior for $c = r$. By Corollary 9.5, these are the physical finite-volume ensembles. We now use the Peierls consequence of the contour-ratio estimate, as in [10, Theorem 10.3.4.2].

Lemma 10.1 (Probability of leaving the imposed density band)**.** *Uniformly in the finite boxes A and phase cells $C \subset A$,*

$$\Pr(\eta_C \ne c) \le Ce^{-c_1\tau}. \tag{10.1}$$

Proof. Use the decomposition into external physical contours. For the event that a prescribed Γ is external, replace its numerator by $N^c_\Gamma \le (\sup_q W^c_\Gamma)D^c_\Gamma$. The denominator fills the same support and interiors with a physical diluted phase-c configuration. The other external contours have disjoint filled supports and keep their factors; the collars separate their interactions. The resulting integral is over a subset of the full physical ensemble. Therefore

$$\Pr(\Gamma \text{ is an external contour}) \le \sup_q W^c_\Gamma(q) \le e^{-2\tau n(\Gamma)}.$$

If $\eta_C \ne c$, its macrocell lies in the filled support of some external contour. Lemma 5.4 bounds the number of such size-n contours by $e^{k_0 n}$. Summing gives $\sum_{n\ge1} e^{k_0 n - 2\tau n} \le Ce^{-c_1\tau}$. □

Lemma 10.2 (Infinite-volume limits)**.** *Each sequence of diluted phase-c ensembles has an infinite-volume Gibbs subsequential limit for the physical interaction. The limit retains* (10.1) *for every fixed phase cell.*

Proof. In every fixed bounded region, packing bounds the number of particles and the attractive insertion field. The empty configuration gives a positive lower bound on the conditional partition function. Local conditional densities are therefore bounded uniformly in the outer volume, relative to the normalized configuration measure in that region. Once the region is far from the box boundary, the diluted constraint has no effect on its conditional kernel.

Apply the bounded-density compactness argument in Lemma 4.1, now to the unrestricted physical macrostates. The physical interaction has fixed finite range, so local setwise convergence passes its conditional Gibbs identities directly. The event in (10.1) is local and measurable, so its bound passes to the limit as well. □

Lemma 10.3 (Translation-invariant states of different densities)**.** *For large R, the limits give two translation-invariant Gibbs measures satisfying* (1.3)*.*

Proof. Let ν_c be a limit from Lemma 10.2, and put $\varepsilon_R = Ce^{-c_1\tau}$ and $V = \ell^d$. For any fixed phase cell, the packing bound and Lemma 10.1 give

$$V^{-1}\mathbb{E}_{\nu_0}N_C \le \delta + r(1 + a/\ell)^d\varepsilon_R,$$
$$V^{-1}\mathbb{E}_{\nu_r}N_C \ge (r - \delta)(1 - \varepsilon_R).$$

Average macro-lattice translates over growing cubes and take a local limit. The Gibbs identities and these bounds persist, giving macro-stationary measures. Average each measure over translations by $[0, L)^d$ to obtain full translation invariance. Its intensity is the expected count per macrocell divided by L^d. Summing the displayed bounds over the phase cells of a macrocell proves (1.3) for large R. □

Lemma 10.4 (Density is a pressure subgradient)**.** *If ν is a translation-invariant Gibbs measure of the physical system at μ, then*

$$p_R(\beta, \mu + u) \ge p_R(\beta, \mu) + u\rho(\nu), \qquad u \in \mathbb{R}. \tag{10.2}$$

Proof. Apply the entropy calculation of Proposition 4.3 to the unrestricted macrocell state space, taking the empty configuration as a safe state. At fixed R, β, μ, packing makes all soft interaction norms and replacement costs finite. The same two variational bounds therefore identify the pressure with the entropy minus energy of ν, with boundary errors vanishing on boxes. The finite-volume variational inequality at chemical potential $\mu + u$, tested against the marginal of ν, adds $u\mathbb{E}_\nu N_A/|A|$. Passing to the limit gives (10.2). This is the Gibbs variational argument of [5, Appendices A–B]. □

Proof of Theorem 1.1. The parameter choice in Section 9 gives β_0, δ, and, for each $\beta \ge \beta_0$, a sufficiently large range threshold. Lemmas 10.2 and 10.3 construct the two claimed measures at μ_R. Their intensities are distinct subgradients by Lemma 10.4. Convexity of the pressure yields

$$\partial_- p_R(\beta, \mu_R) \le \rho(\nu_0) < \rho(\nu_r) \le \partial_+ p_R(\beta, \mu_R),$$

which proves nondifferentiability. □

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