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Haagerup vertex operator algebras at central charge eight

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Abstract

Evans and Gannon showed that a positive vertex operator algebra of central charge eight whose representation category is the Drinfeld center $Z(\mathcal{H}_3)$ of the Haagerup fusion category has one of two character vectors, with six or nineteen currents. Call such an algebra, if strongly rational, a Haagerup algebra. We prove that a Haagerup algebra U has six currents, that $U_1 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ embeds into $V_{E_6 \oplus A_2}$ in one of two explicit ways, with levels $(1, 39)$ or $(3, 13)$, and we determine the lowest weight spaces of its modules and U_2 as U_1-modules. For the levels $(3, 13)$, U_2 is one of four Galois conjugate subspaces. Together with U_1 it generates the unitary C_2-cofinite algebra S, which is isomorphic to the algebra constructed by Jiao. For the levels $(1, 39)$, U contains the explicit algebra of Riesen. The associated lattice of U is that of the form $x^2 + xy + 10y^2$, resp. $3x^2 + 3xy + 4y^2$, the two classes of the principal genus of discriminant -39. Exchanging them in the decomposition of U over its Heisenberg commutant gives a Haagerup algebra $U^\natural$ of the other levels, and $U \mapsto U^\natural$ is a bijection between the isomorphism classes of the two cases. The classification gives exactly two isomorphism classes, forming a single hyperbolic genus, as soon as either explicit candidate is known to be a Haagerup algebra. This property is proved for S in [ArhIII], where the twin $S^\natural$ is also identified, with the algebra of Huang and Riesen, using the results of Dong, Jiao and Xu [DJX26]. The explicit algebras of Jiao and Riesen have rational forms. The proofs combine exact trace relations given by vector-valued modular forms, exact computations in a model of $V_{E_6 \oplus A_2}$, and genera of vertex operator algebras.

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1 Introduction

The Haagerup subfactor of index $\frac{5+\sqrt{13}}{2}$ [AsH99] gives rise to three Morita equivalent fusion categories $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ [GrS12]. The category $\mathcal{H}_3$ has six simple objects $1, a, a^2, \rho, a\rho, a^2\rho$ with fusion rules

$$a^3 \cong 1, \quad a\rho \cong \rho a^2, \quad \rho^2 \cong 1 \oplus \rho \oplus a\rho \oplus a^2\rho. \tag{1.1}$$

The Frobenius–Perron dimension of ρ is

$$d \stackrel{\text{def}}{=} \frac{3+\sqrt{13}}{2}. \tag{1.2}$$

Numerical studies of Haagerup symmetry in conformal field theory include [HL+22, V+22]. At central charge eight, Evans and Gannon [EvG11] showed that a positive vertex operator algebra whose representation category has the modular data of the Drinfeld center $Z(\mathcal{H}_3)$ has one of two character vectors, indexed by $\gamma \in \{0, 1\}$, with $6 + 13\gamma$ currents. They observed that such an algebra should lie in $V_{E_6 \oplus A_2}$ [EvG11, Section 2.4], and Gannon [Gan19] began the analysis of its currents. Albert, Honda, Kaidi and Zheng [AHKZ26] suggested that the chiral $(\mathfrak{e}_8)_1$

theory carries Haagerup symmetries. Möller and Rayhaun sketched [MoR24b], and Gannon and Rayhaun proposed [GaR26], a correspondence between finite non-invertible symmetries and conformal subalgebras of finite index. For the Haagerup categories, Gannon and Rayhaun predicted that a chiral algebra of central charge eight with representation category $Z(\mathcal{H}_3)$ arises inside $V_{E_6\oplus A_2}$ [GaR26, Section 4.3]. Recently, Jiao [Jia26] gave an explicit vertex operator algebra of central charge eight inside $V_{E_6\oplus A_2}$, with currents $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ of levels $(3, 13)$, together with a computer-assisted proof that its representation category is $Z(\mathcal{H}_3)$ (preprint of October 4, 2026, which we have not checked in full). Riesen [Rie26a] and Huang [Hua26] subsequently each announced such an algebra, with currents of levels $(1, 39)$. Evans and Riesen [EvR26] have since announced realizations of the centers of the untwisted Haagerup–Izumi categories of every odd order by non-unitary algebras of negative central charge.

We classify the possible Haagerup algebras and construct the twin of any such algebra by exchanging its associated lattice. The classification starts from either of the two explicit candidates once it is known to be a Haagerup algebra. That the explicit algebra S of Theorem 1.3 is a Haagerup algebra is proved in [ArhIII, Theorem 6.1]. We call *Haagerup algebra* a positive strongly rational vertex operator algebra U of central charge eight with a braided equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$. Positive means that the irreducible modules other than U have positive lowest conformal weights. Let us say that a vector-valued function is an admissible character vector for given modular data if it transforms according to that data, has nonnegative integral Fourier coefficients, has vacuum component $q^{-\frac{c}{24}}(1 + O(q))$ and has positive conformal weights in all other components (Definition 2.2). The characters of a Haagerup algebra form such a vector by [Zhu96, Hua08]. Both vectors of [EvG11] pass the tests of character theory known to us (their coefficients are nonnegative integers up to q^{60}), so that character theory alone does not seem to decide which of them occurs.

Theorem 1.1 *Let U be a Haagerup algebra. Then the character vector of U is the Evans–Gannon vector $\gamma = 0$. In particular, U has exactly six currents.*

The proof has three steps. First, U extends to V_{E_8} through a Lagrangian algebra of $Z(\mathcal{H}_3)$, and the subalgebra of this extension attached to the subcategory $\mathrm{Vec}_{\mathbb{Z}_3} \subset \mathcal{H}_3$ is the fixed point algebra of an element of order three of E_8. A dimension count identifies it with $V' = V_{E_6\oplus A_2}$ (Proposition 4.1). Second, the restrictions of the V'-modules to U show that the representations **3** of $\mathfrak{sl}_3$ and **27** of $\mathfrak{e}_6$ are self-dual when restricted to U_1 (Proposition 4.4). Third, for h in a Cartan subalgebra of U_1, the one-point functions of $h[-1]h$ form a weight-two vector-valued modular form whose low-order coefficients are traces of $h_{(0)}^2$ on lowest weight spaces. This is the Killing vector of [Gan19, Section 2.4]. We determine the space of such forms exactly: the relations between its coefficients are produced by pairing with weakly holomorphic forms of weight zero for the dual representation and applying the residue theorem, in the spirit of the Serre duality of Borcherds [Bor99, Theorem 3.1], and their completeness is proved by exhibiting exact elements (Proposition 5.4). For $\gamma = 1$, the module d_3 has a one-dimensional lowest weight space, which forces strong inequalities on every simple ideal of U_1 and is incompatible with $\dim U_1 = 19$ (Section 7). For $\gamma = 0$, a Heisenberg recursion (Lemma 6.1) gives relations of weights four and six, which, together with the integrality of the weights of $\mathfrak{sl}_2$, a bound coming from integrability and an explicit construction of $\mathfrak{e}_8$, lead to the following classification.

Theorem 1.2 *Let U be a Haagerup algebra. Then $U_1 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and the inclusion $U_1 \subset \mathfrak{e}_6 \oplus \mathfrak{sl}_3$ of Proposition 4.1 is, up to conjugation, one of two explicit embeddings, with levels $(k', k'') = (1, 39)$, case (I), or $(3, 13)$, case (II). In each case the lowest weight spaces of the twelve simple modules and the weight-two space U_2 are uniquely determined as U_1-modules.*

The precise statement is Theorem 8.5 with Table 2. In case (I) the table reproduces the data listed in the working manuscript [Rie26b] of Riesen, which are therefore forced by the representation category once the levels are $(1, 39)$. Gannon [Gan19, Section 3.3] began the analysis of the six-dimensional case and treated the level one factor. The levels 13 and 39 come from diagonal embeddings into $\mathfrak{e}_6 \oplus \mathfrak{sl}_3$. We then implement V' in exact arithmetic over $\mathbb{K} = \mathrm{Q}(\sqrt{13}, \sqrt{-3})$ and study the Griess algebra of the commutant $C = \mathrm{Com}_U(A)$ of the affine subalgebra A generated by U_1 (Section 10).

Theorem 1.3 *Let U be a Haagerup algebra in case (II). After an automorphism of V', C_2 is spanned by two commuting Virasoro vectors of central charges $\frac{182}{95}$ and $\frac{32}{19}$, and U_2 is one of four explicit subspaces of V_2', defined over $\mathbb{K}$, which form one orbit under $\mathrm{Gal}(\mathbb{K}/\mathrm{Q})$ and generate isomorphic vertex algebras. The vertex algebra S generated by $U_1 \oplus U_2$ is unitary, simple and self-contragredient, strongly generated in weights at most three and C_2-cofinite. Its graded dimension agrees with the Haagerup vacuum character up to $q^{11/3}$, so that $U_n = S_n$ for $n \leqslant 4$. Moreover S is isomorphic to the algebra constructed by Jiao [Jia26].*

These are Theorems 10.5, 10.9, 10.10, 10.21 and 10.24, and Propositions 10.12 and 10.27. The finite weight-four condition (F4) used for strong generation is stated in Definition 10.16 and established by the exact certificate of Proposition 10.17. In the setting of Theorem 1.3, the inclusion $S \subset U$ and the Haagerup character give a second proof by a dimension bound (Corollary 10.22). The proof of identification with Jiao uses only weight-two membership and generation, and is independent of (F4). In case (I), we exclude the last spurious candidate for C_2 by a certified computation (Proposition 10.35), identify the three weight-two generators e, f, v of the algebra $U^{\mathrm{R}} = \langle A, e, f, v\rangle$ of [Rie26a] with our data (Proposition 10.36), and exchange the remaining choices by automorphisms of V' (Proposition 10.37). Hence every Haagerup algebra contains S or $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ after an automorphism of V', where ψ_{R} identifies the lattice algebra of [Rie26a] with V'. Theorems 10.38 and 10.29 prove equality when the corresponding candidate is a Haagerup algebra.

Our main new result explains the two cases. A strongly rational vertex operator algebra decomposes over the dual pair formed by its Heisenberg commutant $\Omega_U = \mathrm{Com}_U(M(\mathfrak{h}))$, $\mathfrak{h}$ a Cartan subalgebra of U_1, and the lattice algebra V_L of its associated lattice L [Mas14, MoR24a]. This decomposition was used by Höhn [Hoh17] to study the genus of [Hoh03], and it leads to the hyperbolic genus of Moriwaki [Mor21] and Möller–Rayhaun [MoR24a].

Theorem 1.4 *Let U be a Haagerup algebra, with currents of levels (k', k'').*

1. *The associated lattice of U is the even binary lattice with Gram matrix $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 20 \end{smallmatrix}\right]$ in case (I) and $\left[\begin{smallmatrix} 6 & 3 \\ 3 & 8 \end{smallmatrix}\right]$ in case (II), that is, the lattice of the binary form $x^2 + xy + 10y^2$, resp. $3x^2 + 3xy + 4y^2$. These are the two classes of the principal genus of binary forms of discriminant -39. The Heisenberg commutant Ω_U is a strongly rational vertex operator algebra of central charge six without currents, $\mathrm{Rep}(\Omega_U) \cong Z(\mathcal{H}_3) \boxtimes \overline{\mathcal{C}(L'/L)}$, and all its irreducible modules have nonnegative conformal weights. Moreover U contains the simple current $L_{\mathfrak{sl}_2}(k', k') \otimes L_{\mathfrak{sl}_2}(k'', k'')$ of its affine subalgebra.*

2. *Replacing L by the other lattice of the genus in the decomposition of U over $\Omega_U \otimes V_L$ gives a Haagerup algebra $U^\natural$ of the other case, with Heisenberg commutant Ω_U. The map $U \mapsto U^\natural$ is a bijection between the isomorphism classes of Haagerup algebras of the two cases, and U, $U^\natural$ are the only strongly rational vertex operator algebras in the hyperbolic genus of U.*

The precise statements are Proposition 12.2 and Theorem 12.3. Part 1 combines [MoR24a, Proposition 2.14] and the proof of [MoR24a, Corollary 4.10] with the weight decomposition in Table 2 and a growth estimate for characters. In part 2, $U^\natural$ is defined by the image of the algebra object defining U under a ribbon equivalence $\mathrm{Rep}(\Omega_U) \boxtimes \mathcal{C}(L'/L) \to \mathrm{Rep}(\Omega_U) \boxtimes \mathcal{C}(L^{\natural\prime}/L^\natural)$ induced by an isometry of discriminant forms. Positivity of $U^\natural$ comes from that of U. The independence of the choices is not automatic: the orthogonal group of the lattice of case (I) acts on its discriminant group only by ± 1, and we use instead the Weyl group of the currents of the algebra of case (II), which acts through the whole orthogonal group of the discriminant form. The exchange of lattices is the analogue of the construction of [MoR24a, Proposition 5.3] for holomorphic algebras. What is specific here is its computation for $Z(\mathcal{H}_3)$.

Classification and explicit realizations. We write $(\mathrm{H_J})$ for the assertion that Jiao's explicit algebra $V \cong S$ is a Haagerup algebra, and $(\mathrm{H_R})$ for the corresponding assertion about the explicit algebra U^{R} of Riesen. These conditions include positivity, strong rationality and the braided equivalence with $Z(\mathcal{H}_3)$. The lattice exchange of Theorem 12.3 starts from an arbitrary Haagerup algebra and determines the two members of its hyperbolic genus. The uniqueness results then give the classification from either explicit realization. By [ArhIII, Theorem 6.1] and Theorem 10.24, $(\mathrm{H_J})$ holds. Its proof in [ArhIII] uses Theorems 10.9, 10.10 and 10.21 and the results of Section 11, and Theorem 10.24 transfers it to Jiao's algebra. None of these results depends on $(\mathrm{H_J})$ (see the beginning of Sections 10.5 and 11). By [ArhIII, Theorem 7.13], which uses the identifications of [DJX26, Proposition 4.1 and Theorem 7.3], $(\mathrm{H_R})$ holds as well.

Theorem 1.5 *1. If* $(\mathrm{H_J})$ *or* $(\mathrm{H_R})$ *holds, there are exactly two Haagerup algebras up to isomorphism, one in each case. They are exchanged by* $U \mapsto U^\natural$*, have isomorphic Heisenberg commutants and form a single hyperbolic genus.*

2. The explicit algebra S *is a Haagerup algebra by [ArhIII, Theorem 6.1]. Consequently the two isomorphism classes are represented by* S *in case (II), of levels* $(3, 13)$*, and the twin* $S^\natural$ *in case (I), of levels* $(1, 39)$*.*

3. The twin $S^\natural$ *is isomorphic as a vertex operator algebra to both explicit algebras,* $S^\natural \cong U^{\mathrm{H}} \cong U^{\mathrm{R}}$*, where* U^{H} *is Huang's algebra and* U^{R} *is Riesen's, by [ArhIII, Theorem 7.9 and Corollary 7.10], using the common fixed-point algebra and the Huang–Riesen identification of [DJX26, Theorem 7.3 and Proposition 4.1].*

Theorems 12.3 and 12.4 give part 1 and the construction of $S^\natural$ used in part 2. Part 1 of Theorem 12.4 also gives an embedding of $S^\natural$ in V' whose image contains $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ and has the same homogeneous spaces as $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ through weight four. The identification of the entire twin in part 3 uses the common fixed point algebra in [ArhIII, Section 7]. Table 1 records the sources of these results.

We also prove that the explicit algebras S and $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ have rational forms, given by explicit Galois descent data with values in the centralizer of U_1 in $\mathrm{Aut}(V')$ (Proposition 10.41), and that in case (II) the commutant C is not a tensor product of two nontrivial vertex operator algebras of CFT type, although its graded dimensions suggest a coset description (Corollary 10.32). Section 11 reduces rationality to the semisimplicity of the Zhu algebra. The inclusion of the affine simple-current extension in S gives a full Morita corner. The selected identities in weights three and four give a modular upper presentation of dimension 762 with the block dimensions predicted by Table 2 (Theorem 11.6). The exact descent and the branching are proved in [ArhIII, Appendix A], see Section 11.5.

Statement	Status
$\gamma = 0$, U_1, the two embeddings, lowest weight spaces, U_2 as a U_1-module (Theorems 1.1, 1.2)	exact computations and the modularity theorem of [EvG11]
Case (II): C_2, U_2, S unitary, strongly generated, C_2-cofinite, $S \cong$ Jiao's algebra (Theorem 1.3)	exact computations, certified exclusions and Proposition 10.17
Case (I): identification of e, f, v, $U \supset \psi_{\mathrm{R}}(U^{\mathrm{R}})$	exact and certified computations
Dimensions of $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ at least 120, 956, 6010 in weights 2, 3, 4 (Proposition 10.39)	ranks modulo primes of $\mathfrak{p}$-integral data (rigorous lower bounds)
Associated lattice, Ω_U, $J \subset U$, twin, bijection, hyperbolic genus of U (Theorem 1.4)	no external existence hypothesis, using the preceding classification of currents
Rational forms of S and $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ (Proposition 10.41)	exact computations
Strong generation and C_2-cofiniteness of the explicit S, without an ambient Haagerup algebra	exact certificate over $\mathbb{K}$ (Proposition 10.17)
Classification into two isomorphism classes from either explicit realization (Theorem 1.5(1))	Theorems 12.3 and 12.4
Embedding of $S^\natural$ in V' and agreement with $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ through weight four	Theorem 12.4(1), with Proposition 10.39
Existence of the realization S and identification of $S^\natural$ with the Huang–Riesen algebra	[ArhIII, Theorems 6.1 and 7.9, Corollary 7.10], with the inputs of [ArhIII, Theorem 2.3]
Rationality criterion and modular upper presentations for the Zhu corner of S	Theorems 11.1 and 11.6
Semisimplicity of $A(S)$, the twelve simple modules and the full lattice branching	[ArhIII, Theorem 2.2(F1)–(F3) and Appendix A]

Table 1: Dependencies and sources of the main results.

Computations. Many proofs rest on finite computations: exact rational arithmetic for the vector-valued modular forms and the relations between traces, exact arithmetic over $\mathbb{K}$ in a model of V', Gröbner bases over $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{13})$, after a Galois descent when needed, certified interval arithmetic for a few exclusions, and exact integer computations for lattices. Computations modulo primes are used as evidence, as controls, to choose pivots which are then checked exactly, or as rigorous lower bounds for ranks of matrices with $\mathfrak{p}$-integral entries, since the rank of such a matrix is at least that of its reduction. Each use is labelled. Appendix A distinguishes the mathematical role of each computation from the verification status of the accompanying files. The supplementary material provides a statement-by-statement index of the computational sources and certificates.

Notation. Throughout, $V' = V_{E_6 \oplus A_2}$, and $U \subset V'$ is a Haagerup algebra with $U_1 = \mathfrak{g}' \oplus \mathfrak{g}''$, of levels (k', k''). Moreover $A = L_{\mathfrak{g}'}(k', 0) \otimes L_{\mathfrak{g}''}(k'', 0)$ is its affine subalgebra and $C = \mathrm{Com}_U(A)$. We set $\mathbb{K} = \mathbb{Q}(\sqrt{13}, \sqrt{-3})$. Furthermore, S is the explicit algebra of Theorem 1.3, generated by the explicit currents and one of four explicit subspaces of V_2' (Section 10.5), and $U^{\mathrm{R}} = \langle A, e, f, v \rangle$ that of [Rie26a]. In Section 12, $\mathfrak{h}$ is a Cartan subalgebra of U_1, Ω_U the Heisenberg commutant, L the associated lattice and $L^\natural$ the other lattice of its genus.

Structure of the paper. Section 2 fixes the modular data of $Z(\mathcal{H}_3)$ and the notion of admissible character vector. Section 3 treats the character vectors at central charge eight, and Section 4 establishes the chain $U \subset V_{E_6 \oplus A_2} \subset V_{E_8}$ and the self-duality constraints. Section 5 derives the weight-two relations in exact arithmetic and Section 6 the relations of higher weight. Section 7 proves Theorem 1.1 and Section 8 proves Theorem 1.2. Section 9 gives the

constraints on the weight-two space. Section 10 determines the commutant of the currents and U_2, identifies the algebras of [Jia26] and [Rie26a], and proves the uniqueness theorems and the rational forms. Section 11 gives the Zhu-algebra criterion, the modular computations and the references for exact descent and branching. Section 12 determines the associated lattices, constructs the twins and applies the classification to S. Appendix A describes the computations.

2 Preliminaries

2.1 The modular data of the Drinfeld center

We use the modular data of $Z(\mathcal{H}_3)$ in the simplified form of [EvG11, equations (2.5a) and (2.5b) p. 20]. The simple objects are labelled

$$0,\ b,\ a,\ c_0,\ c_1,\ c_2,\ d_1,\dots,d_6,$$

with 0 the unit. Their quantum dimensions are

$$1,\quad 3d+1,\quad 3d+2,\quad 3d+2,\quad 3d+2,\quad 3d+2,\quad 3d,\dots,3d, \tag{2.1}$$

and their categorical twists are

$$1,\ 1,\ 1,\ 1,\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}},\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}},\ \mathrm{e}^{\frac{2\pi\mathrm{i}6l^2}{13}}\quad (1\leqslant l\leqslant 6). \tag{2.2}$$

Put $x=\frac{13-3\sqrt{13}}{26}$, $y=\frac{3}{\sqrt{13}}$ and $c(j)=-2y\cos\left(\frac{2\pi j}{13}\right)$. The S matrix is $\frac{1}{3}M$, where the first two rows of M are $(x,1-x,1,1,1,1,y,\dots,y)$ and $(1-x,x,1,1,1,1,-y,\dots,-y)$, the block indexed by (a,c_0,c_1,c_2) is

$$\begin{bmatrix} 2 & -1 & -1 & -1\\ -1 & 2 & -1 & -1\\ -1 & -1 & -1 & 2\\ -1 & -1 & 2 & -1 \end{bmatrix},$$

the entries $M_{a,0},M_{a,b},\dots,M_{c_2,b}$ are equal to 1, the entries $M_{d_l,0}=-M_{d_l,b}$ are equal to y, the entries between $\{a,c_0,c_1,c_2\}$ and $\{d_1,\dots,d_6\}$ vanish, and $M_{d_l,d_{l'}}=c(ll')$. The global dimension is $\mathcal{D}^2=9(3d+2)^2$, since $S_{0,0}=\frac{x}{3}=\frac{1}{3(3d+2)}$.

The modular data of $Z(\mathcal{H}_3)$ were computed by Izumi from the tube algebra [Izu01]. The form above is that of [EvG11], and its identification with the Drinfeld center of the Haagerup category is established in [GSY26]. The charge conjugation is trivial by an exact calculation. Indeed, for $1\leqslant a,b\leqslant 6$, the geometric sum of the thirteenth roots of unity gives

$$\sum_{l=1}^{6}\cos\left(\frac{2\pi al}{13}\right)=-\frac{1}{2},\quad \sum_{l=1}^{6}\cos\left(\frac{2\pi al}{13}\right)\cos\left(\frac{2\pi bl}{13}\right)=\frac{13\delta_{a,b}-2}{4}.$$

Together with $2x-1=-y$, $x(1-x)=1/13$ and $y^2=9/13$, these identities give $M^2=9\mathrm{Id}$, hence $S^2=\mathrm{Id}$. The displayed matrix is real and symmetric, so it is also unitary. An independent numerical calculation verifies these identities and $(ST)^3=S^2$ with sixty significant digits. The exact modular relations also follow from the categorical origin of the data. The only invertible simple object is 0, by (2.1).

Remark 2.1 We use the labels of [EvG11], whose six-object even part is $\mathcal{H}_2$ in the convention of [GrS12]. The inductions of the units from $\mathcal{H}_2$ and $\mathcal{H}_3$ to their common center have underlying

objects $I_{\mathcal{H}_2}(1) \cong 0 \oplus b \oplus 2a$ and $I_{\mathcal{H}_3}(1) \cong 0 \oplus b \oplus 2c_0$, respectively [EvG11, Section 2.2], [ArhII, Corollary 3.8 and Remark 3.7]. Thus $0, b, a, c_0$ correspond respectively to $1, \pi_1, \sigma_1, \pi_2$ in [AHKZ26, BoS25]. In the untwisted sector of a theory with $\mathcal{H}_3$ symmetry, the line ρ acts on the multiplicity spaces of 0 and b by d and $d_- \stackrel{\text{def}}{=} \frac{3-\sqrt{13}}{2}$, respectively, and on that of c_0 as part of the two-dimensional irreducible representation of the fusion ring [AHKZ26, Appendix A].

2.2 Vertex operator algebras and character vectors

A vertex operator algebra V is called strongly rational if it is simple, of CFT type, self-contragredient, rational and C_2-cofinite. We call it positive if, in addition, every irreducible V-module $M \ncong V$ has a positive conformal weight $h_M > 0$. For a positive strongly rational V the effective central charge equals the central charge. By [Hua08], $\mathrm{Rep}(V)$ is a modular tensor category. We denote by $\mathcal{C}(K)$ the pointed modular category of the discriminant form of an even positive-definite lattice K. It is equivalent to $\mathrm{Rep}(V_K)$.

In the case considered here, a braided equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$ preserves the ribbon structure. Indeed, this nondegenerate braided category has no nontrivial invertible simple objects, so its universal grading group is trivial [GeN08, Theorem 6.3]. Pivotal structures differ by a tensor automorphism of the identity functor [ENO05], and such automorphisms are given by the characters of the universal grading group. Hence the pivotal structure is unique. We use this structure and the normalization $S_{0,0} > 0$ throughout.

Let (S, T) be the modular data of a modular tensor category with simple objects $\mathrm{Irr} = \{0 = i_0, i_1, \ldots\}$ and conformal weights $h_i \in [0, 1)$ defined modulo one by the twists. For $c \in \mathbb{Q}$, put $T_c \stackrel{\text{def}}{=} \mathrm{e}^{-\frac{2\pi \mathrm{i} c}{24}} T$. We write $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$ for τ in the upper half-plane $\mathbb{H}$.

Definition 2.2 *An admissible character vector at central charge c for (S, T) is a family $X = (X_i)_{i \in \mathrm{Irr}}$ of holomorphic functions on $\mathbb{H}$ such that:*

1. *$X(-\frac{1}{\tau}) = SX(\tau)$ and $X(\tau + 1) = T_c X(\tau)$ for any $\tau \in \mathbb{H}$,*
2. *for any $i \in \mathrm{Irr}$, we have $X_i(\tau) = \sum_{n \geqslant 0} a_{i,n} q^{h_i + n_i + n - \frac{c}{24}}$ with $a_{i,n} \in \mathbb{Z}_{\geqslant 0}$ and integers $n_i \geqslant 0$, the series converging on $\mathbb{H}$,*
3. *$X_0(\tau) = q^{-\frac{c}{24}}(1 + O(q))$,*
4. *for any $i \neq 0$, the leading exponent of X_i is strictly larger than $-\frac{c}{24}$.*

Condition 4 is the positivity of the conformal weights of the nonvacuum components. By Zhu's modular invariance theorem [Zhu96] and the identification of the modular transformation of the characters with the categorical S matrix [Hua08], the following holds.

Lemma 2.3 *Let V be a positive strongly rational vertex operator algebra of central charge c. Then the family $\left(\mathrm{Tr}_M q^{L_0 - \frac{c}{24}}\right)_{M \in \mathrm{Irr}(\mathrm{Rep}(V))}$ is an admissible character vector at central charge c for the modular data of $\mathrm{Rep}(V)$.*

Proof : The convergence and the transformation law are [Zhu96], with the S matrix identified with the categorical one in [Hua08]. The coefficients are dimensions of graded pieces. The vacuum component starts with $q^{-\frac{c}{24}}$ since V is of CFT type, and positivity gives condition 4. ∎

2.3 Lattice character vectors

Let L be an even positive-definite lattice of rank n, with dual L^* and discriminant form $q_L\colon L^*/L \to \mathbb{Q}/\mathbb{Z}$, $q_L(\lambda) = \frac{\langle\lambda,\lambda\rangle}{2}$. The functions

$$\Theta_{L,\lambda}(\tau) \overset{\text{def}}{=} \frac{1}{\eta(\tau)^n} \sum_{v\in L+\lambda} q^{\frac{\langle v,v\rangle}{2}}, \quad \lambda \in L^*/L, \tag{2.3}$$

are the characters of the irreducible V_L-modules. They satisfy $\Theta_L(-\frac{1}{\tau}) = S^L\Theta_L(\tau)$ with $S^L_{\lambda,\mu} = |L^*/L|^{-\frac{1}{2}}\mathrm{e}^{-2\pi\mathrm{i}\langle\lambda,\mu\rangle}$. The vacuum component is

$$\Theta_{L,0}(\tau) = q^{-\frac{n}{24}}\big(1 + (n + |R(L)|)q + O(q^2)\big), \tag{2.4}$$

where $R(L)$ is the set of roots, namely vectors of norm two, of L. Indeed, the coefficient of q counts the n oscillators $h(-1)\mathbf{1}$ and the exponentials e^v with $\langle v, v\rangle = 2$.

Let K and K' be even positive-definite lattices of ranks r and $8-r$, and let $\varphi\colon K^*/K \to K'^*/K'$ be an isomorphism of groups with $q_{K'} \circ \varphi = -q_K$. Then the graph of φ glues $K \oplus K'$ to an even unimodular lattice of rank eight, which is isometric to E_8 since E_8 is the unique even unimodular positive-definite lattice of rank eight, and K, K' are primitive sublattices with $K^\perp = K'$ [Nik80, Propositions 1.4.1 and 1.6.1]. Conversely, a primitive embedding $K \subset E_8$ gives such a pair with $K' = K^\perp$. Moreover,

$$S^{K'}_{\varphi(\lambda),\varphi(\mu)} = \overline{S^K_{\lambda,\mu}}, \quad T^{K'}_{\varphi(\lambda)} = \overline{T^K_\lambda}, \tag{2.5}$$

since the bilinear form of K'^*/K' is the opposite of that of K^*/K under φ.

3 Character vectors of the Haagerup double at central charge eight

Let ϱ be the representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}^{12}$ defined by $\varrho\big(\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]\big) = S$ and $\varrho\big(\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]\big) = T_8$, where (S, T) are the modular data of $Z(\mathcal{H}_3)$. Denote by $\mathcal{M}(\varrho)$ the space of holomorphic functions $X\colon \mathbb{H} \to \mathbb{C}^{12}$ with $X(\gamma\tau) = \varrho(\gamma)X(\tau)$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, which are meromorphic at the cusp. It is a module over the polynomial ring $\mathbb{C}[J]$, where $J(\tau) = q^{-1} + 196884q + \cdots$.

Following [EvG11, equation (2.7) pp. 21–22], put

$$\Lambda \overset{\text{def}}{=} \mathrm{diag}\Big(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, \frac{1}{3}, \frac{5}{39}, \frac{20}{39}, \frac{32}{39}, \frac{2}{39}, \frac{8}{39}, \frac{11}{39}\Big), \tag{3.1}$$

in the order of the labels, and let χ be the integral 12×12 matrix of [EvG11, equation (2.7)]. Note that $\mathrm{e}^{2\pi\mathrm{i}\Lambda} = T_8$. Let $F = (J - 240)\frac{\Delta}{E_{10}} = \sum_{k\geqslant 0} f_k q^k$ and $G = \frac{\Delta}{E_{10}} = \sum_{k\geqslant 1} g_k q^k$, where Δ is the discriminant and $E_{10} = E_4E_6$. One has $f_0 = 1$, $f_1 = 0$ and $g_1 = 1$. Define rational matrices Ξ_n, $n \geqslant -1$, by $\Xi_{-1} = \mathrm{Id}$, $\Xi_0 = \chi$ and, for any $n \geqslant 1$, by the entrywise formula

$$(\Lambda_{ii} - \Lambda_{jj} + n + 1)(\Xi_n)_{ij} = \Big(\sum_{l=-1}^{n-1} \Xi_l\big(f_{n-l}(\Lambda - \mathrm{Id}) + g_{n-l}(\chi + [\Lambda, \chi])\big)\Big)_{ij}. \tag{3.2}$$

The coefficient on the left is nonzero since $|\Lambda_{ii} - \Lambda_{jj}| < 1$. Put

$$\Xi(\tau) \overset{\text{def}}{=} q^\Lambda\Big(q^{-1}\mathrm{Id} + \sum_{n\geqslant 0} \Xi_n q^n\Big). \tag{3.3}$$

The recursion (3.2) expresses the differential equation $q\frac{\mathrm{d}\Xi}{\mathrm{d}q} = \Xi\big((\Lambda - \mathrm{Id})F + (\chi + [\Lambda, \chi])G\big)$ coefficientwise. Indeed, the coefficient of $q^{\Lambda+n}$ on the left is $(\Lambda + n)\Xi_n$, and the term $l = n$ on the right is $\Xi_n(\Lambda - \mathrm{Id})$ since $f_0 = 1$ and $g_0 = 0$. This is the differential equation of [BaG07, Section 3]. The conditions $f_1 = 0$ and $g_1 = 1$ show that the case $n = 0$ is automatically satisfied.

Theorem 3.1 *The following hold.*

1. *The columns of Ξ belong to $\mathcal{M}(\varrho)$ and form a basis of $\mathcal{M}(\varrho)$ over $\mathbb{C}[J]$.*
2. *Every admissible character vector at central charge eight for the modular data of $Z(\mathcal{H}_3)$ is one of the two vectors*

$$X^{(\gamma)} = \Xi_{\bullet,0} + \gamma\Xi_{\bullet,d_3}, \quad \gamma \in \{0, 1\}, \tag{3.4}$$

where $\Xi_{\bullet,j}$ denotes the column of index j. In particular, any admissible character vector satisfies

$$X_0(\tau) = q^{-\frac{1}{3}}\big(1+(6+13\gamma)q+(120+78\gamma)q^2+(956+351\gamma)q^3+O(q^4)\big), \quad \gamma \in \{0, 1\}. \tag{3.5}$$

The second assertion is contained in [EvG11, Section 2.4 p. 22]. We give the argument in the form needed below, together with an independent verification of the data. We do not use, and do not prove, the converse: the coefficients of $X^{(0)}$ and $X^{(1)}$ are nonnegative integers up to q^{60}, but we have no proof that all of them are.

Proof : 1. Assume that the columns of Ξ are vector-valued modular functions. Since $\Xi_{-1} = \mathrm{Id}$, they are linearly independent over $\mathbb{C}(J)$. By [BaG07, equation (2.22) p. 6], the $\mathbb{C}[J]$-span of the columns of such a matrix is the whole module $\mathcal{M}(\varrho)$ if and only if

$$\mathrm{Tr}(\Lambda) = \frac{5 \cdot 12}{12} + \frac{1}{4}\,\mathrm{Tr}(S) + \frac{2}{3\sqrt{3}}\,\mathrm{Re}\left(\mathrm{e}^{-\frac{\pi\mathrm{i}}{6}}\,\mathrm{Tr}(ST_8^{-1})\right). \tag{3.6}$$

The left-hand side is $\frac{8}{3} + \frac{1}{3} + \frac{78}{39} = 5$. On the right-hand side, $\mathrm{Tr}(S) = 0$ and $\mathrm{Tr}(ST_8^{-1}) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}$, so that $\mathrm{Re}\left(\mathrm{e}^{-\frac{\pi\mathrm{i}}{6}}\,\mathrm{Tr}(ST_8^{-1})\right) = -\frac{3\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} = 0$. For $\mathrm{Tr}(S)$, the diagonal entries give $\frac{1}{3}\big(2x + 2 + \sum_{l=1}^{6} c(l^2)\big)$, and the quadratic Gauss sum $\sum_{l=0}^{12} \mathrm{e}^{\frac{2\pi\mathrm{i}l^2}{13}} = \sqrt{13}$ gives $\sum_{l=1}^{6} c(l^2) = -3 + \frac{3}{\sqrt{13}} = -(2x + 2)$. Hence the right-hand side of (3.6) equals 5.

The modularity of the columns is the content of [EvG11, Section 2.4]. Together with (3.6), it proves the assertion.

2. Let X be an admissible character vector. By 1, we can write $X = \sum_j \Xi_{\bullet,j}P_j(J)$ with polynomials P_j. Let m be the maximal degree of the P_j and suppose that $m \geqslant 1$. Choose j with $\deg P_j = m$. The component X_j contains the term $\lambda_m q^{\Lambda_{jj}-1-m}$, where $\lambda_m \neq 0$ is the leading coefficient of P_j, and no other term of X has this exponent in the component j: the other columns contribute exponents at least $\Lambda_{jj} - m$. Since $\Lambda_{jj} \leqslant \frac{32}{39}$, we obtain the exponent $\Lambda_{jj} - 1 - m \leqslant \frac{32}{39} - 2 < -\frac{1}{3}$, which contradicts conditions 3 and 4 of Definition 2.2. Consequently, the P_j are constants v_j.

Now, the column j contributes the exponent $\Lambda_{jj} - 1$ to the component j, and the other columns contribute exponents at least Λ_{jj}. Condition 3 forces $v_0 = 1$, since $\Lambda_{00} - 1 = -\frac{1}{3}$. For $j \neq 0$, condition 4 forces $v_j = 0$ whenever $\Lambda_{jj} - 1 \leqslant -\frac{1}{3}$, that is, whenever $\Lambda_{jj} \leqslant \frac{2}{3}$. Among the indices $j \neq 0$, only $j = d_3$ satisfies $\Lambda_{jj} > \frac{2}{3}$. Hence $X = \Xi_{\bullet,0} + \gamma\Xi_{\bullet,d_3}$ with $\gamma = v_{d_3}$. The leading coefficient of X_{d_3} is γ, so that $\gamma \in \mathbb{Z}_{\geqslant 0}$. The leading coefficient of X_{d_4} is

$$(\Xi_0)_{d_4,0} + \gamma(\Xi_0)_{d_4,d_3} = \chi_{d_4,0} + \gamma\chi_{d_4,d_3} = 5 - 3\gamma,$$

which is nonnegative only for $\gamma \leqslant 1$. The expansion (3.5) is read from the matrices $\Xi_0 = \chi$, Ξ_1 and Ξ_2 computed by this recurrence. This proves 2. ∎

Remark 3.2 The first coefficients of all components produced by our recursion coincide with those printed in [EvG11, p. 22]. For instance, the components d_1, d_4, d_5 and d_6 start with $7-\gamma$, $5-3\gamma$, $13-5\gamma$ and $14+7\gamma$. The vector $\gamma=0$ has $\dim U_1=6$, $\dim U_2=120$, $\dim U_3=956$ and $\dim U_4=6010$ for a realizing algebra U. The last value is read from Ξ_3 and is used in Section 10. These numbers agree with the algebra announced in [Rie26a, Hua26]. The vector $\gamma=1$ has 19 currents. By Theorem 7.1, it is not the character vector of a positive strongly rational vertex operator algebra U with $\mathrm{Rep}(U)\cong Z(\mathcal{H}_3)$ as braided categories.

Remark 3.3 The modularity of Ξ is imported from [EvG11, Section 2.4]. As independent controls, we compute its first 60 coefficients in rational arithmetic, check their integrality and test $\Xi(-1/\tau)=S\Xi(\tau)$ at five points with fifty-digit precision. The largest recorded relative residual is 1.4×10^{-50}. The discarded tails are not certified. An exact rational calculation checks (3.6). The floating-point tests do not enter the proof.

4 The chain $U\subset V_{E_6\oplus A_2}\subset V_{E_8}$

Throughout this section, U is a positive strongly rational vertex operator algebra of central charge eight together with a braided equivalence $\mathrm{Rep}(U)\cong Z(\mathcal{H}_3)$. For $z\in\mathrm{Irr}(Z(\mathcal{H}_3))$ we denote by M_z the corresponding simple U-module and by R_z its lowest weight space. By Lemma 2.3 and Theorem 3.1, the character vector of U is $X^{(\gamma)}$ for some $\gamma\in\{0,1\}$. Reading Theorem 3.1, the lowest conformal weights and the dimensions of the lowest weight spaces are

(4.1)

z	0	b	a	c_0	c_1	c_2
h_z	0	1	1	1	$\frac{1}{3}$	$\frac{2}{3}$
$\dim R_z$	1	$80-13\gamma$	81	81	3	27

z	d_1	d_2	d_3	d_4	d_5	d_6
h_z	$\frac{6}{13}$	$\frac{11}{13}$	$\frac{2}{13}+\varepsilon$	$\frac{5}{13}$	$\frac{7}{13}$	$\frac{8}{13}$
$\dim R_z$	$7-\gamma$	$42+16\gamma$	$\gamma+119(1-\gamma)$	$5-3\gamma$	$13-5\gamma$	$14+7\gamma$

where $\varepsilon=0$ if $\gamma=1$ and $\varepsilon=1$ if $\gamma=0$. Moreover $\dim U_1=6+13\gamma$, $\dim U_2=120+78\gamma$ and $\dim U_3=956+351\gamma$.

4.1 The holomorphic extension and the $\mathbb{Z}_3$-orbifold

Proposition 4.1 *Let U be as above.*

1. *The Lagrangian algebra $I_{\mathcal{H}_2}(1)\cong 0\oplus b\oplus 2a$ defines a holomorphic extension $H=U\oplus M_b\oplus M_a\oplus M_a$ of U, and $H\cong V_{E_8}$.*

2. *The subspace $V'\overset{\mathrm{def}}{=}U\oplus M_b$ is a vertex subalgebra of H, and there is an automorphism g of order three of H with $H^g=V'$. The two other eigenspaces of g are isomorphic to M_a as U-modules.*

3. *There is an isomorphism $V'\cong V_{E_6\oplus A_2}$ which maps V'_1 onto $\mathfrak{e}_6\oplus\mathfrak{sl}_3$.*

Proof : 1. The Lagrangian algebra $I_{\mathcal{H}_2}(1)$ is a connected étale algebra of $\mathrm{Rep}(U)$ with trivial category of local modules. By [HKL15, CKM22], it defines a simple extension H of U. The nonvacuum U-summands of H have lowest weight one by (4.1), so H is of CFT type. The operator $L(1)$ preserves these summands and kills their weight-one spaces because their weight-zero spaces vanish. It also kills U_1 by self-contragredience of U. Thus $L(1)H_1=0$ and H

is self-contragredient [Li94]. The arguments for C_2-cofiniteness and rationality in the proof of Theorem 12.3, with U in place of $\Omega_U \otimes V_{L^\natural}$, give these properties for H. The correspondence with local modules then gives $\mathrm{Rep}(H) \cong \mathrm{Vec}$. A strongly rational holomorphic vertex operator algebra of central charge eight is isomorphic to V_{E_8} [DoM04b, Theorem 1 p. 2].

2. By [DMNO13, Theorem 4.10 p. 18], the subalgebras of $I_{\mathcal{H}_2}(1)$ correspond to the fusion subcategories of $\mathcal{H}_2$, and the subcategory $\mathrm{Vec}_{\mathbb{Z}_3}$ generated by the invertible objects of $\mathcal{H}_2$ corresponds to a connected étale subalgebra A of Frobenius–Perron dimension $\frac{9d+6}{3} = 3d+2$. The only subobject of $0 \oplus b \oplus 2a$ containing 0 with this dimension is $0 \oplus b$, by (2.1). By [HKL15] and the arguments recalled in part 1, $V' = U \oplus M_b$ is a strongly rational vertex subalgebra of H with $\mathrm{Rep}(V') \cong Z(\mathcal{H}_3)^0_A$. The latter has Frobenius–Perron dimension $\frac{9(3d+2)^2}{(3d+2)^2} = 9$ [DMNO13, Corollary 3.32 p. 15]. It is Witt equivalent to $Z(\mathcal{H}_3)$ [DMNO13, Proposition 5.4 p. 22], hence of the form $Z(\mathcal{D})$ for a fusion category $\mathcal{D}$ [DMNO13, Proposition 5.8 p. 23], with $\mathrm{FPdim}(\mathcal{D}) = 3$. A fusion category of prime Frobenius–Perron dimension is pointed [ENO05, Corollary 8.30]. Here $\mathcal{D} \cong \mathrm{Vec}^\omega_{\mathbb{Z}_3}$ for some 3-cocycle ω. Its center is pointed: every centralizer is cyclic, and every irreducible projective representation of $\mathbb{Z}_3$ is one-dimensional, since $H^2(\mathbb{Z}_3, \mathbb{C}^\times) = 0$. Thus $\mathrm{Rep}(V')$ is pointed with nine simple objects, all simple currents.

For every nonvacuum simple V'-module N, its restriction $R(N)$ has Frobenius–Perron dimension $3d+2$. Free-module adjunction gives

$$\mathrm{Hom}_U(U, R(N)) \cong \mathrm{Hom}_{V'}(V', N) = 0.$$

The list (2.1) therefore forces $R(N)$ to be simple. As a U-module, the complement of V' in H is $M_a \oplus M_a$. Its decomposition into simple V'-modules consequently gives $H = V' \oplus N_1 \oplus N_2$ with simple currents N_1, N_2 of V', both isomorphic to M_a as U-modules. These simple currents are distinct and, together with the vacuum, form a group of order three under fusion. To see this directly, regard H as a connected commutative separable algebra in the pointed category $\mathrm{Rep}(V')$. Its regular right module is simple, since $\mathrm{End}_H(H) \cong \mathrm{Hom}(\mathbf{1}, H) = \mathbb{C}$. For an invertible simple object J, the right module $J \otimes H$ is also simple, and free-module adjunction gives $\mathrm{Hom}(J, H) \cong \mathrm{Hom}_H(J \otimes H, H)$. Thus the multiplicity of J in H is either zero or one. The objects for which this multiplicity is one form the stabilizer of the simple right module H under the action $J \mapsto J \otimes H$, hence a subgroup. Since $\mathrm{FPdim}_{\mathrm{Rep}(V')}(H) = 3$, this subgroup has order three. The operator acting by $e^{\frac{2\pi i k}{3}}$ on N_k is therefore an automorphism g of order three with $H^g = V'$.

3. The automorphism group of V_{E_8} is the complex Lie group of type E_8 [DoN99, Theorem 2.1]. An element of order three is conjugate to $e^{2\pi i \beta(0)}$ with β in the Cartan subalgebra, and its conjugacy class is given by Kac coordinates [Kac90, Chapter 8]. There are four classes of elements of order three, whose centralizers in $\mathfrak{e}_8$ have dimensions 80, 86, 92 and 134, of types A_8, $E_6 \oplus A_2$, $D_7 \oplus \mathbb{C}$ and $E_7 \oplus \mathbb{C}$, as follows by enumerating the Kac coordinates. Now $\dim V_1' = (6 + 13\gamma) + (80 - 13\gamma) = 86$ by (4.1), so that g is of type $E_6 \oplus A_2$. For $g = e^{2\pi i \beta(0)}$, the fixed point algebra is the lattice vertex operator algebra of $\{\alpha \in E_8 : \langle \alpha, \beta \rangle \in \mathbb{Z}\}$. This sublattice contains the root lattice $E_6 \oplus A_2$, which has index three in E_8, and is a proper sublattice. Hence it equals $E_6 \oplus A_2$. ∎

Remark 4.2 The realization of $V_{E_6 \oplus A_2}$ as a $\mathbb{Z}_3$-orbifold of V_{E_8}, and the expectation that a Haagerup vertex operator algebra lies inside it, go back to [EvG11, Section 2.4 p. 22]. The physical picture of a hypergroup acting on the $\mathbb{Z}_3$-invariant algebra is in [GaR26, Section 4.3 pp. 83–88], and the announced algebra of [Rie26a, Hua26] is constructed inside $V_{E_6 \oplus A_2}$. Proposition 4.1 shows that every positive strongly rational realization, for either value of γ, sits in this chain.

4.2 Restriction of the lattice modules

The discriminant group of $E_6 \oplus A_2$ is $\mathbb{Z}_3 \times \mathbb{Z}_3$, with quadratic form $q(i,j) = \frac{2i^2+j^2}{3} \mod 1$. We denote by $V_{(i,j)}$ the corresponding simple $V_{E_6 \oplus A_2}$-modules. Both lines $\{(0,0),(1,1),(2,2)\}$ and $\{(0,0),(1,2),(2,1)\}$ are isotropic. We choose coordinates adapted to the extension H of Proposition 4.1, so that H is the lattice extension attached to $\{(0,0),(1,1),(2,2)\}$. The diagram automorphism of A_2 exchanges the two choices. The coordinates in this subsection are adapted to H. In Section 11.2, we use the fixed coordinates of the explicit model of S. The lowest weight space of $V_{(i,j)}$ is, as a module over $\mathfrak{e}_6 \oplus \mathfrak{sl}_3$, the representation $(\mathbf{27}^i, \mathbf{3}^j)$, where $\mathbf{27}^0 = \mathbf{3}^0$ is the trivial module, $\mathbf{27}^1 = \mathbf{27}$, $\mathbf{27}^2 = \overline{\mathbf{27}}$, and similarly for $\mathbf{3}$.

Lemma 4.3 *Use the coordinates adapted to H and let $(i,j) \neq (0,0)$. The restriction of $V_{(i,j)}$ to U is simple. It is isomorphic to M_{c_1} if $i = 0$, to M_{c_2} if $j = 0$, to M_a for the two nonzero classes $(1,1)$, $(2,2)$ of $E_8/(E_6 \oplus A_2)$, and to M_{c_0} for the two remaining classes $(1,2)$, $(2,1)$.*

Proof : The Frobenius–Perron dimension and free-module adjunction argument in the proof of Proposition 4.1, part 2, shows that the restriction $R(w)$ of any nonvacuum simple V'-module is one of a, c_0, c_1, c_2. The twist of $R(w)$ equals $e^{2\pi i q(i,j)}$. The twists of a and c_0 are one, those of c_1 and c_2 are $e^{\frac{2\pi i}{3}}$ and $e^{-\frac{2\pi i}{3}}$ by (2.2), while $q(0,\pm 1) = \frac{1}{3}$, $q(\pm 1, 0) = \frac{2}{3}$ and $q(\pm 1, \pm 1) = 0$. The classes of E_8 give M_a by Proposition 4.1. It remains to show that the classes $(1,2)$ and $(2,1)$ give M_{c_0}. For a simple object z of $Z(\mathcal{H}_3)$ and $w \in \mathbb{Z}_3 \times \mathbb{Z}_3$, let $n_{z,w} \in \{0,1\}$ be the multiplicity of M_z in the restriction of V_w. For $v \in U$ we have $Z_{V_w}(v,\tau) = \sum_z n_{z,w} Z_{M_z}(v,\tau)$. The one-point functions of pairwise non-isomorphic irreducible modules are linearly independent, and they transform under $\tau \mapsto -\frac{1}{\tau}$ through the S matrix of the representation category [Zhu96, Section 5], [Hua08]. Applying this to U and to V', which have the same conformal vector, and comparing gives $Sn = nS^L$, where S^L is the S matrix of $V' = V_L$, $L = E_6 \oplus A_2$, given after (2.3). All the entries $S^L_{w,(0,0)}$ equal $\frac{1}{3}$. Since $V_{(0,0)} = V'$ restricts to $U \oplus M_b$ (Proposition 4.1), the entry $(c_0,(0,0))$ of this identity reads $S_{c_0,0} + S_{c_0,b} = \frac{1}{3}\#\{w : n_{c_0,w} = 1\}$. The left-hand side is $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ by the formula for S in Section 2. Hence exactly two classes restrict to M_{c_0}, and these are $(1,2)$ and $(2,1)$. ∎

We have checked Lemma 4.3 on characters: for both values of γ, $\Theta_{(0,0)} = X_0 + X_b$, $\Theta_{(\pm 1,\pm 1)} = X_a = X_{c_0}$, $\Theta_{(0,\pm 1)} = X_{c_1}$ and $\Theta_{(\pm 1,0)} = X_{c_2}$ up to order q^4. The characters do not distinguish a from c_0. The one-point functions do, as in the proof above.

4.3 Self-duality constraints on the currents

Let $\mathfrak{s} \stackrel{\text{def}}{=} U_1 \subset \mathfrak{e}_6 \oplus \mathfrak{sl}_3$, and let p_1 and p_2 be the projections onto $\mathfrak{e}_6$ and $\mathfrak{sl}_3$. The Lie algebra $\mathfrak{s}$ is reductive [DoM04, Theorem 1 p. 2]. For a simple ideal $\mathfrak{g}$ of $\mathfrak{s}$ we denote by $i_1(\mathfrak{g})$ and $i_2(\mathfrak{g})$ the Dynkin indices of $p_1|_{\mathfrak{g}}$ and $p_2|_{\mathfrak{g}}$ (zero if the projection vanishes). Since $\mathfrak{e}_6$ and $\mathfrak{sl}_3$ have index one in $\mathfrak{e}_8$, the level of the affine algebra generated by $\mathfrak{g}$ in U is $k(\mathfrak{g}) = i_1(\mathfrak{g}) + i_2(\mathfrak{g})$.

Proposition 4.4

1. *The representations $\mathbf{3}$ and $\overline{\mathbf{3}}$ of $p_2(\mathfrak{s})$ are isomorphic, and the representations $\mathbf{27}$ and $\overline{\mathbf{27}}$ of $p_1(\mathfrak{s})$ are isomorphic.*

2. *The Lie algebra $p_2(\mathfrak{s})$ is zero, a one-dimensional torus with weights $(t,-t,0)$ on $\mathbf{3}$, a regular $\mathfrak{sl}_2$ or the principal $\mathfrak{so}_3$. In particular, a simple ideal $\mathfrak{g}$ of $\mathfrak{s}$ with $p_2(\mathfrak{g}) \neq 0$ is isomorphic to $\mathfrak{sl}_2$ and $i_2(\mathfrak{g}) \in \{1,4\}$.*

3. *The rank of $p_1(\mathfrak{s})$ is at most four.*

Proof : 1. By Lemma 4.3, $V_{(0,1)}$ and $V_{(0,2)}$ are isomorphic U-modules. An isomorphism maps lowest weight spaces to lowest weight spaces and commutes with the zero modes $x(0)$, $x \in U_1$. The lowest weight spaces are $(\mathbf{1}, \mathbf{3})$ and $(\mathbf{1}, \overline{\mathbf{3}})$, on which $x \in \mathfrak{s}$ acts through $p_2(x)$. The same argument with $V_{(1,0)}$ and $V_{(2,0)}$ gives the second assertion.

2. The action of $\mathfrak{s}$ on an ordinary U-module is completely reducible [DoM04, Theorem 1 p. 2]. By Lemma 4.3, this applies in particular to $\mathbf{3}$ and $\mathbf{27}$. Thus the centers of $p_2(\mathfrak{s})$ and $p_1(\mathfrak{s})$ act semisimply, and their Cartan subalgebras are toral in the ambient simple Lie algebras. The reductive subalgebras of $\mathfrak{sl}_3$ are 0, tori, $\mathfrak{sl}_2$ (regular, $\mathbf{3} = \mathbf{2} \oplus \mathbf{1}$), $\mathfrak{gl}_2$, $\mathfrak{so}_3$ (principal, $\mathbf{3}$ irreducible) and $\mathfrak{sl}_3$. Self-duality of $\mathbf{3}$ excludes $\mathfrak{gl}_2$, $\mathfrak{sl}_3$ and every torus whose weights on $\mathbf{3}$ do not form a symmetric set, which leaves the tori with weights $(t, -t, 0)$. The regular $\mathfrak{sl}_2$ and the principal $\mathfrak{so}_3$ have Dynkin indices one and four.

3. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{e}_6$ containing a Cartan subalgebra of $p_1(\mathfrak{s})$, and denote by Ω the set of weights of $\mathbf{27}$. This representation is minuscule: its 27 weights form the Weyl orbit of the fundamental weight ω_1, each with multiplicity one. We first give a proof without enumeration. If the rank were at least five, choose a five-dimensional subspace $\mathfrak{t}$ of the Cartan subalgebra of $p_1(\mathfrak{s})$. By 1, the multiset of restrictions of Ω to $\mathfrak{t}$ is invariant under negation. Its cardinality is odd, so at least one weight $\lambda \in \Omega$ vanishes on $\mathfrak{t}$. Since no weight in Ω is zero, the annihilator of $\mathfrak{t}$ in $\mathfrak{h}^*$ is $\mathbb{C}\lambda$. After applying a Weyl transformation, we may assume $\lambda = \omega_1$.

Let P and Q be the weight and root lattices of E_6, respectively. In the Bourbaki numbering, the Cartan matrix gives

$$\omega_1 = \frac{4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6}{3}, \quad \langle \omega_1, \omega_1 \rangle = \frac{4}{3}, \quad \langle \omega_1, \alpha_1 \rangle = 1.$$

The weight $\mu = s_{\alpha_1}(\omega_1) = \omega_1 - \alpha_1$ belongs to Ω. Symmetry of the restrictions gives $\nu \in \Omega$ and $c \in \mathbb{C}$ with $\mu + \nu = c\omega_1$. All weights in Ω have squared norm $4/3$, while $\langle \mu, \omega_1 \rangle = 1/3$. Hence

$$0 = \langle c\omega_1 - \mu, c\omega_1 - \mu \rangle - \langle \mu, \mu \rangle = \frac{4}{3}c^2 - \frac{2}{3}c,$$

so $c = 0$ or $c = 1/2$. The first case is impossible because every weight in Ω lies in $\omega_1 + Q$, whereas $2\omega_1 \notin Q$. The second case is impossible because $\mu + \nu \in P$ and $\omega_1/2 \notin P$. This contradiction proves the rank bound.

For comparison, the finite calculation can be retained independently. If two multisets of ambient weights coincide modulo a line while differing before restriction, that line is spanned by a nonzero sum $w+w'$ of weights in Ω. In fundamental-weight coordinates, clear denominators, divide by the greatest common divisor and choose the first nonzero coordinate positive. This gives 243 lines. For each representative v, an exact comparison of the multisets modulo $\mathbb{Q}v$ shows that none is symmetric under negation. This calculation is not needed for the preceding proof. ∎

Remark 4.5 The announced algebra of [Rie26a, Hua26] has $U_1 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ at levels 1 and 39. This is compatible with Proposition 4.4: $\mathfrak{e}_6 \supset \mathfrak{sl}_6 \oplus \mathfrak{sl}_2$ with $\mathbf{27} = (\overline{\mathbf{6}}, \mathbf{2}) \oplus (\mathbf{15}, \mathbf{1})$. The factor $\mathfrak{sl}_2$ has index one, and the diagonal of the principal $\mathfrak{sl}_2$ of $\mathfrak{sl}_6$ (index 35) and the principal $\mathfrak{so}_3$ of $\mathfrak{sl}_3$ (index 4) has level 39. All the restrictions are self-dual. This is case (I) of Theorem 8.5. We do not use this description before Section 8.

5 Weight-two relations

We now combine the modularity of one-point functions with the explicit lowest weight spaces given by the chain. This is a vector-valued version of the trace identities of Schellekens [Sch93]

and Dong–Mason [DoM04b]. In the context of the Haagerup modular data, the weight-two one-point functions of currents were introduced by Gannon [Gan19, Section 2.4], who called them the Killing vector and used them, under the assumptions that U is unitary and contained in $V_{E_6\oplus A_2}$, to begin the analysis of the case of six currents [Gan19, Section 3.3]. The novelty below is that the relations are obtained in exact arithmetic, for both character vectors, without unitarity.

Lemma 5.1 *Let V be a strongly rational vertex operator algebra of central charge c and let $h \in V_1$ with $L(1)h = 0$, contained in a Cartan subalgebra of V_1. For every ordinary V-module M we have*

$$Z_M(h[-1]h,\tau) = \operatorname{Tr}_M h(0)^2 q^{L(0)-\frac{c}{24}} - \frac{\langle h,h\rangle}{12} E_2(\tau) \operatorname{Tr}_M q^{L(0)-\frac{c}{24}}, \tag{5.1}$$

where $h(1)h = \langle h,h\rangle \mathbf{1}$. The vector $(Z_{M_z}(h[-1]h,\tau))_z$, indexed by the simple modules, is a weakly holomorphic vector-valued modular form of weight two for the representation of $\mathrm{SL}_2(\mathbb{Z})$ on the characters.

Proof : From $Y[h,z] = Y(h,e^z-1)e^z$ and the expansions $\frac{e^z}{e^z-1} = \frac{1}{z} + \frac{1}{2} + O(z)$, $\frac{e^z}{(e^z-1)^2} = \frac{1}{z^2} - \frac{1}{12} + O(z)$, we obtain $h[-1]h = h(-1)h + \frac{1}{2}h(0)h - \frac{1}{12}h(1)h = h(-1)h - \frac{\langle h,h\rangle}{12}\mathbf{1}$, since $h(0)h = 0$ and $h(n)h = 0$ for $n \geqslant 2$. The zero mode of $h(-1)h$ is $h(0)^2 + 2\sum_{n\geqslant 1} h(-n)h(n)$. Put $\kappa = \langle h,h\rangle$ and $Q = q^{L(0)-c/24}$. For $m \geqslant 1$, graded cyclicity of the trace and $[h(m),h(-m)] = m\kappa\mathrm{Id}$ give

$$A_m \stackrel{\text{def}}{=} \operatorname{Tr}_M h(-m)h(m)Q = q^m\big(A_m + m\kappa \operatorname{Tr}_M Q\big).$$

Thus $A_m = \frac{m\kappa q^m}{1-q^m}\operatorname{Tr}_M Q$. Substituting into the zero-mode formula and using $\sum_{m\geqslant 1} mq^m/(1-q^m) = (1-E_2)/24$ proves (5.1). This is first an identity of graded series and then of convergent functions. It also covers $\kappa = 0$. The vector $h[-1]h$ has weight two for $L[0]$, and the last assertion is Zhu's theorem [Zhu96] together with the identification of the representation [Hua08]. ∎

5.1 The obstruction principle

We denote by $\varrho^* \stackrel{\text{def}}{=} (\varrho^{\mathrm{T}})^{-1}$ the dual representation. For the matrices of Section 2, we have $S^{\mathrm{T}} = S$, $S^2 = \mathrm{Id}$ and $T_8^* = T_8^{-1}$. Since T_8 is diagonal, it follows that $\varrho^*(S) = S$ and $\varrho^*(T) = \overline{T_8}$. If W is a function on $\mathbb{H}$ with values in $\mathbb{C}^{12}$ and G is another such function, we write $W\cdot G = \sum_z W_z G_z$. We write $D = q\frac{\mathrm{d}}{\mathrm{d}q}$, and $\vartheta_k = D - \frac{k}{12}E_2$ for the Serre derivative in weight k. Recall that ϑ_k maps weakly holomorphic vector-valued modular forms of weight k for ϱ to weakly holomorphic forms of weight $k+2$ for ϱ, and that $DJ = -\frac{E_4^2E_6}{\Delta}$ is a weakly holomorphic modular form of weight two.

Lemma 5.2 *Let $k \in 2\mathbb{Z}$. Let W be a weakly holomorphic vector-valued modular form of weight k for ϱ, and let G be a weakly holomorphic vector-valued modular form of weight $2-k$ for ϱ^*. Then the constant term of the q-expansion of $W\cdot G$ vanishes.*

Proof : For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$(W\cdot G)(\gamma\tau) = (c\tau+d)^k(c\tau+d)^{2-k}\big(\varrho(\gamma)W(\tau)\big)^{\mathrm{T}}\varrho(\gamma)^{-\mathrm{T}}G(\tau) = (c\tau+d)^2(W\cdot G)(\tau).$$

Hence $f = W\cdot G$ is a weakly holomorphic modular form of weight two for $\mathrm{SL}_2(\mathbb{Z})$, and the differential $f(\tau)\,\mathrm{d}\tau$ descends to a meromorphic differential on the compactified modular curve,

holomorphic outside the cusp. At the elliptic points this uses the invariance of $f(\tau)\,\mathrm{d}\tau$: in a local coordinate s at $\tau = \mathrm{i}$ with $\tau \mapsto -\frac{1}{\tau}$ acting by $s \mapsto -s$, write the differential as $F(s)\,\mathrm{d}s$. Invariance gives $F(-s) = -F(s)$, hence $F(s) = sG(s^2)$ with G holomorphic and $F(s)\,\mathrm{d}s = G(t)\,\mathrm{d}t/2$ for $t = s^2$. The point $\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$ is treated in the same way. At the cusp, with the local coordinate q, we have $f(\tau)\,\mathrm{d}\tau = \frac{1}{2\pi\mathrm{i}} f \frac{\mathrm{d}q}{q}$, whose residue is the constant term of f divided by $2\pi\mathrm{i}$. By the residue theorem, this residue vanishes. This is the elementary half of the Serre duality statement of Borcherds [Bor99, Theorem 3.1 p. 5]. ∎

We now produce forms for ϱ^* from forms for ϱ. Let π be the permutation of the labels which exchanges c_1 and c_2 and maps d_l to $d_{l'}$, where $l' \equiv \pm 5l \mod 13$, that is

$$\pi = (c_1\ c_2)(d_1\ d_5)(d_2\ d_3)(d_4\ d_6). \tag{5.2}$$

Lemma 5.3 *Let Y be a weakly holomorphic vector-valued modular form of weight k for ϱ. Then the function G defined by $G_z = \eta^{-8} Y_{\pi(z)}$ is a weakly holomorphic vector-valued modular form of weight $k-4$ for ϱ^*.*

Proof : We first check that $S_{\pi(z),\pi(w)} = S_{z,w}$ for any labels z, w. The block indexed by (a, c_0, c_1, c_2) is invariant under the exchange of c_1 and c_2, and the rows of 0 and b are constant on $\{a, c_0, c_1, c_2\}$ and on $\{d_1, \dots, d_6\}$. Finally $c(ll')$ depends only on $ll' \mod 13$ up to sign, and $25 \equiv -1 \mod 13$, so that $c(5l \cdot 5l') = c(-ll') = c(ll')$. Next, we have $\Lambda_{\pi(z)} - \frac{1}{3} \equiv -\Lambda_z \mod 1$ for every label z. For $z \in \{0, b, a, c_0\}$ this reads $\frac{2}{3} - \frac{1}{3} \equiv -\frac{2}{3}$, and for $z \in \{c_1, c_2\}$ it reads $\Lambda_{c_2} - \frac{1}{3} = 0 = -\Lambda_{c_1}$ and $\Lambda_{c_1} - \frac{1}{3} = -\frac{1}{3} = -\Lambda_{c_2}$. For $z = d_l$ we have $\Lambda_{d_l} \equiv \frac{6l^2}{13} - \frac{1}{3} \mod 1$, hence $\Lambda_{\pi(d_l)} - \frac{1}{3} \equiv \frac{150l^2}{13} - \frac{2}{3} \equiv -\frac{6l^2}{13} + \frac{1}{3} \equiv -\Lambda_{d_l} \mod 1$, since $150 \equiv -6 \mod 13$. Since $\eta(\tau+1)^{-8} = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}} \eta(\tau)^{-8}$, we obtain $G_z(\tau+1) = \mathrm{e}^{2\pi\mathrm{i}(\Lambda_{\pi(z)} - \frac{1}{3})} G_z(\tau) = \mathrm{e}^{-2\pi\mathrm{i}\Lambda_z} G_z(\tau) = (\overline{T_8} G(\tau))_z$. Since $\eta(-\frac{1}{\tau})^{-8} = \tau^{-4}\eta(\tau)^{-8}$, we have

$$G_z\big(-\tfrac{1}{\tau}\big) = \tau^{k-4}\eta(\tau)^{-8} \sum_w S_{\pi(z),w} Y_w(\tau) = \tau^{k-4} \sum_w S_{\pi(z),\pi(w)} G_w(\tau) = \tau^{k-4} (SG(\tau))_z.$$

Since η does not vanish on $\mathbb{H}$, G is holomorphic on $\mathbb{H}$, and it is meromorphic at the cusp. This proves the lemma. ∎

The π-invariance of S and the congruences are also checked in exact arithmetic.

5.2 The relations

Let $X = X^{(\gamma)}$ and let μ_z be the exponent of the leading term of X_z. Thus $\mu_z = h_z - \frac{1}{3}$ by (4.1). For $k \in 2\mathbb{Z}_{\geqslant 0}$, let $\mathcal{M}_k(\mu)$ be the space of weakly holomorphic vector-valued modular forms W of weight k for ϱ such that $W_z \in q^{\mu_z}\mathbb{C}[[q]]$ for every z. For $W \in \mathcal{M}_k(\mu)$ we put

$$w_0 = [q^{-\frac{1}{3}}]W_0, \quad w_0' = [q^{\frac{2}{3}}]W_0, \quad w_z = [q^{\mu_z}]W_z \quad (z \neq 0). \tag{5.3}$$

Proposition 5.4 *For $\gamma = 1$ we have $\dim \mathcal{M}_2(\mu) = 5$, and the thirteen coefficients (5.3) satisfy exactly eight linearly independent relations, among which*

(5.4)
$$w_{d_3} - \tfrac{5}{21} w_{d_4} - \tfrac{1}{21} w_{d_5} + \tfrac{1}{21} w_{d_6} = 0, \quad w_{c_1} + \tfrac{1}{6} w_{c_2} - \tfrac{3}{52} w_{d_2} - \tfrac{317}{364} w_{d_4} - \tfrac{55}{364} w_{d_5} + \tfrac{17}{182} w_{d_6} = 0,$$
$$w_{d_1} - \tfrac{1}{6} w_{d_2} - \tfrac{209}{126} w_{d_4} + \tfrac{17}{126} w_{d_5} + \tfrac{44}{63} w_{d_6} = 0, \quad w_0 + \tfrac{1}{78} w_{d_2} + \tfrac{317}{1638} w_{d_4} + \tfrac{55}{1638} w_{d_5} - \tfrac{17}{819} w_{d_6} = 0.$$

For $\gamma = 0$ we have $\dim \mathcal{M}_2(\mu) = 4$ and the coefficients satisfy exactly nine linearly independent relations. The complete lists of relations are included in the supplementary data.

Proof : Let $P = \{0, d_3\}$ if $\gamma = 1$ and $P = \{0\}$ if $\gamma = 0$. These are the coordinates z with $\mu_z = \Lambda_z - 1$. Let $\Xi_{\bullet,j}$ be the columns of the fundamental matrix of Theorem 3.1.

Step 1: the relations. Consider the $4 \cdot 12 \cdot 2 = 96$ weakly holomorphic forms of weight four

$$E_4 \Xi_{\bullet,j} J^m, \quad \vartheta_2 D(\Xi_{\bullet,j} J^m), \quad DJ \cdot D(\Xi_{\bullet,j} J^m), \quad (DJ)^2 \Xi_{\bullet,j} J^m, \quad 0 \leqslant j \leqslant 11,\ m \in \{0,1\},$$

and, for each linear combination Y of them, the form G of weight zero for ϱ^* given by Lemma 5.3. The exponents of G_z are congruent to $-\Lambda_z$ modulo one, so that the constant term of $W \cdot G$ is a finite sum $\sum_z \sum_m [q^{\Lambda_z+m}] W_z \cdot [q^{-\Lambda_z-m}] G_z$. We impose that the coefficients of G which pair with $[q^{\Lambda_z+m}] W_z$ vanish whenever this coefficient of W is not one of (5.3). Since G has a bounded principal part, these are finitely many linear conditions on the coefficients of the combination. By Lemma 5.2, every solution gives a linear relation, with rational coefficients, between the coefficients (5.3) of every $W \in \mathcal{M}_2(\mu)$. All the q-expansions involved are computed in exact rational arithmetic from the matrices Ξ_n, and only finitely many coefficients enter. The vanishing-constraint matrices have dimensions 43×96 for $\gamma = 1$ and 42×96 for $\gamma = 0$, with full row rank. Their kernels have dimensions 53 and 54, respectively. We obtain a space of relations of dimension eight for $\gamma = 1$ and nine for $\gamma = 0$. In reduced echelon form, it is the space displayed in (5.4) and in the supplementary data. Using $m \leqslant 2$ gives the same space.

Step 2: exact elements. The forms $D(\Xi_{\bullet,j} J^m)$ and $DJ \cdot \Xi_{\bullet,j} J^m$, $m \in \{0,1\}$, are weakly holomorphic of weight two and holomorphic on $\mathbb{H}$. The linear combinations whose components satisfy $W_z \in q^{\mu_z}\mathbb{C}[[q]]$ form a subspace of $\mathcal{M}_2(\mu)$, determined by finitely many exact linear conditions. The values of (5.3) on this subspace span a space of dimension five for $\gamma = 1$ and four for $\gamma = 0$. Since $13 - 8 = 5$ and $13 - 9 = 4$, the image of $\mathcal{M}_2(\mu)$ under the thirteen coefficients has dimension exactly five, respectively four, and the relations of Step 1 are all the relations.

Step 3: injectivity. Let $W \in \mathcal{M}_2(\mu)$ with vanishing coefficients (5.3). As in the proof of Theorem 3.1, $F = W \frac{E_{10}}{\Delta}$ can be written $F = \Xi(\mathbf{a} + J\mathbf{b})$ with $\mathbf{a} \in \mathbb{C}^{12}$ and $\mathbf{b}$ supported on P. Since $\frac{\Delta}{E_{10}} = q + O(q^2)$, the coefficient $[q^{\Lambda_z - 1}] W_z$ equals b_z for $z \in P$. Hence $\mathbf{b} = 0$, then $[q^{\Lambda_z}] W_z = a_z$, and $a_z = 0$ for $z \neq d_3$, and also for $z = d_3$ if $\gamma = 0$. For $\gamma = 1$, we obtain $W = a_{d_3} \Xi_{\bullet,d_3} \frac{\Delta}{E_{10}}$, with $[q^{\Lambda_{d_3}}] W_{d_3} = a_{d_3}$, $W_0 \in q^{\frac{5}{3}}\mathbb{C}[[q]]$ and $W_z \in q^{\Lambda_z+1}\mathbb{C}[[q]]$ for $z \notin \{0, d_3\}$. An explicit dual form is $G_z = \eta^{-8} E_4 \Xi_{\pi(z),d_2}$, which has weight zero by Lemma 5.3. Put $k_z = \Lambda_z + \Lambda_{\pi(z)} - 1/3 \in \{0,1\}$. The leading term of Ξ gives $G_{d_3} = q^{-\Lambda_{d_3}}(1 + O(q))$ and $G_z \in q^{-\Lambda_z + k_z}\mathbb{C}[[q]]$ for $z \neq d_3$. Therefore $[q^0](W \cdot G) = a_{d_3}$, which vanishes by Lemma 5.2. Hence the thirteen coefficients are injective on $\mathcal{M}_2(\mu)$, and $\dim \mathcal{M}_2(\mu)$ equals five, respectively four. The coefficient vectors of a basis of dual forms are included in the supplementary data. A separate rational-matrix calculation checks the ranks, the vanishing of every relation on the constructed forms, and the explicit injectivity witness above. ∎

Remark 5.5 Proposition 5.4 is a computer-assisted result using exact rational linear algebra. The modularity of Ξ is supplied by Theorem 3.1. The floating-point checks do not enter the proof.

We now insert the explicit data. Let h be a real element of a Cartan subalgebra of $\mathfrak{s}$, that is, an element on which $h(0)$ has real eigenvalues on every ordinary U-module. Such elements span the Cartan subalgebra since its weights on U-modules lie in a positive-definite lattice [Mas14, Theorem 1 p. 3]. Normalize $\langle h, h \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the normalized invariant form of $\mathfrak{e}_8$, and write $h = (h_1, h_2) \in \mathfrak{e}_6 \oplus \mathfrak{sl}_3$. The eigenvalues of h_1 on **27** and of h_2 on **3** are real by Lemma 4.3. These faithful weight systems span the respective Cartan duals, so h_1 and h_2

lie in the real Cartan spaces on which the normalized forms are positive definite. In particular $0 \leqslant \langle h_1, h_1\rangle \leqslant 1$. Put

$$u \overset{\text{def}}{=} \langle h_1, h_1\rangle \in [0,1], \quad T \overset{\text{def}}{=} \operatorname{Tr}_{\mathfrak{s}}(\operatorname{ad} h)^2, \quad \tau_j \overset{\text{def}}{=} \operatorname{Tr}_{R_{d_j}} h(0)^2 \geqslant 0.$$

By Proposition 4.1 and Lemma 4.3, the lowest weight spaces of M_b, M_a, M_{c_0}, M_{c_1}, M_{c_2} are $(\mathfrak{e}_6 \oplus \mathfrak{sl}_3)\ominus\mathfrak{s}$, $(\mathbf{27},\mathbf{3})$, $(\mathbf{27},\overline{\mathbf{3}})$, $(\mathbf{1},\mathbf{3})$, $(\mathbf{27},\mathbf{1})$. With $\operatorname{Tr}_{\mathfrak{e}_6}(\operatorname{ad} h_1)^2 = 24u$, $\operatorname{Tr}_{\mathfrak{sl}_3}(\operatorname{ad} h_2)^2 = 6(1-u)$, $\operatorname{Tr}_{\mathbf{27}} h_1^2 = 6u$ and $\operatorname{Tr}_{\mathbf{3}} h_2^2 = 1-u$, the traces of $h(0)^2$ on these spaces are $6+18u-T$, $27-9u$, $27-9u$, $1-u$ and $6u$. The conformal inclusion $U \subset V_{E_8}$ gives $L(1)U_1 = 0$. By Lemma 5.1, $W = (Z_{M_z}(h[-1]h,\tau))_z$ belongs to $\mathcal{M}_2(\mu)$, with $w_0 = -\frac{1}{12}$, $w_0' = T + \frac{18-13\gamma}{12}$ and $w_z = \operatorname{Tr}_{R_z} h(0)^2 - \frac{\dim R_z}{12}$.

Corollary 5.6 *For $\gamma = 1$, the relations of Proposition 5.4 are equivalent to*

$$\tau_1 = \tfrac{T}{15} - \tfrac{7\tau_5}{15} - \tfrac{3\tau_6}{5} - \tfrac{3u}{5} + \tfrac{17}{5}, \qquad \tau_2 = \tfrac{317T}{270} - \tfrac{209\tau_5}{270} - \tfrac{7\tau_6}{5} - \tfrac{317u}{30} + \tfrac{2009}{90},$$
$$\tau_3 = -\tfrac{T}{54} + \tfrac{\tau_5}{54} + \tfrac{u}{6} - \tfrac{1}{18}, \qquad \tau_4 = -\tfrac{7T}{90} - \tfrac{11\tau_5}{90} + \tfrac{\tau_6}{5} + \tfrac{7u}{10} - \tfrac{19}{30}.$$

For $\gamma = 0$, they are equivalent to

$$\tau_1 = \tfrac{-2T-3\tau_6+18u-4}{5}, \qquad \tau_2 = \tfrac{2T-7\tau_6-18u+74}{5}, \qquad \tau_3 = \tfrac{3(-3T+3\tau_6+27u+29)}{5},$$
$$\tau_4 = \tfrac{-T+\tau_6+9u-7}{5}, \qquad \tau_5 = T - 9u + 9.$$

Proof : This is exact linear algebra on the relations. All relations are satisfied identically after substitution. ∎

Remark 5.7 For $\gamma = 0$ the formulas can be tested against the announced algebra. The working manuscript [Rie26b] lists the lowest weight spaces of the twelve simple modules as sums of modules $V(a,b)$ over $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ at levels 1 and 39. For h in the level one factor, $u = 1$ and $T = 4$. The inequalities $\tau_1, \tau_4 \geqslant 0$ force $\tau_6 = 2$ and then $(\tau_1,\dots,\tau_6) = (0,10,30,0,4,2)$. For the level 39 factor, $u = \frac{35}{39}$ and $T = \frac{4}{39}$. In both cases the values computed from the announced decompositions agree exactly with Corollary 5.6. This is a consistency check of the announced data. It is not used in the proofs.

6 One-point functions of higher weight

Lemma 6.1 *Let V be a strongly rational vertex operator algebra of central charge c and let $h \in V_1$ with $L(1)h = 0$ and $\langle h,h\rangle \neq 0$, contained in a Cartan subalgebra of V_1. For $n \geqslant 0$, put $v_n \overset{\text{def}}{=} h[-1]^n\mathbf{1}$. Then v_n has weight n for $L[0]$, and for every ordinary V-module M we have*

$$Z_M(v_n,\tau) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} (2k-1)!! \Big(-\frac{\langle h,h\rangle E_2(\tau)}{12}\Big)^k \operatorname{Tr}_M h(0)^{n-2k} q^{L(0)-\frac{c}{24}}, \tag{6.1}$$

with the convention $(-1)!! = 1$. Consequently, the vector $(Z_{M_z}(v_n,\tau))_z$ is a weakly holomorphic vector-valued modular form of weight n for the representation on the characters.

Proof : Since h is a primary vector of weight one, the square bracket modes satisfy $[L[0],h[m]] = -mh[m]$ and the commutation relations $[h[m],h[m']] = m\langle h,h\rangle\delta_{m+m',0}$ [Zhu96, Section 4]. Hence v_n has weight n for $L[0]$, $h[2j-1]v_n = 0$ for $j \geqslant 2$, and $h[1]v_n = n\langle h,h\rangle v_{n-1}$. Since $h(0)$

acts semisimply and $\langle h,h\rangle \neq 0$, the lower-bounded grading gives the Heisenberg decomposition $M = \bigoplus_\lambda F_\lambda \otimes \Omega_\lambda$, where F_λ is the Fock module on which $h(0)$ acts by λ, and the vertex subalgebra generated by h acts on the first factor. Its conformal vector $\omega_h = \frac{1}{2\langle h,h\rangle} h(-1)h$ satisfies $L(0) = L_h(0) + L'(0)$ with $L'(0)$ commuting with the Heisenberg algebra. The vectors v_n belong to the Heisenberg subalgebra, on which $L'(0) = 0$, so that their square bracket structures in V and in the Heisenberg vertex operator algebra coincide. Consequently $Z_M(v_n,\tau) = \sum_\lambda Z_{F_\lambda}(v_n,\tau)\,\mathrm{Tr}_{\Omega_\lambda}\, q^{L'(0)-\frac{c-1}{24}}$. On F_λ, Zhu's recursion [Zhu96, Proposition 4.3.5] gives

$$\begin{aligned} Z_{F_\lambda}(v_{n+1},\tau) &= \lambda Z_{F_\lambda}(v_n,\tau) + \sum_{j\geqslant 1} \widehat{E}_{2j}(\tau) Z_{F_\lambda}(h[2j-1]v_n,\tau) \\ &= \lambda Z_{F_\lambda}(v_n,\tau) - \frac{n\langle h,h\rangle E_2(\tau)}{12} Z_{F_\lambda}(v_{n-1},\tau), \end{aligned}$$

where we used that $o(h) = h(0)$ acts by the scalar λ and that only $j = 1$ contributes. The normalization $\widehat{E}_2 = -\frac{E_2}{12}$ is the one for which the step from $n = 1$ to $n = 2$ reproduces Lemma 5.1. The polynomials P_n defined by $P_0 = 1$, $P_1 = \lambda$ and $P_{n+1} = \lambda P_n + neP_{n-1}$ are $P_n = \sum_k \binom{n}{2k}(2k-1)!!e^k\lambda^{n-2k}$, as one checks by induction using $\binom{n}{2k}(2k-1)!! + n\binom{n-1}{2k-2}(2k-3)!! = \binom{n+1}{2k}(2k-1)!!$. Summing over λ gives (6.1). The last assertion is Zhu's theorem [Zhu96] with [Hua08]. ∎

As a numerical check, we test (6.1) in weights four and six for V_{A_2} and $V_{\sqrt{6}\mathbb{Z}}$. At thirty-digit working precision, the recorded relative errors are at most 5.1×10^{-30}. Perturbing the weight-six coefficients gives errors of order one. These tests are not used in the proof.

By Lemma 6.1, for every real h in a Cartan subalgebra of $\mathfrak{s}$ with $\langle h,h\rangle = 1$ and for $n \in \{4,6\}$, the vector $W_n(h) = (Z_{M_z}(v_n,\tau))_z$ belongs to $\mathcal{M}_n(\mu)$. Its coefficients (5.3) are

$$w_z = \sum_k c_{n,k}\,\mathrm{Tr}_{R_z}\, h(0)^{n-2k}, \quad c_{n,k} = \binom{n}{2k}(2k-1)!!\Big(-\frac{1}{12}\Big)^k, \tag{6.2}$$

with $\mathrm{Tr}\, h(0)^0 = \dim$, and similar expressions for w_0 and w_0' involving $\mathrm{Tr}_{\mathfrak{s}}(\mathrm{ad}\, h)^m$ and the coefficients of E_2^k. More generally, for two commuting real elements h' and h'' with $\langle h',h''\rangle = 0$ and $\langle h',h'\rangle = \langle h'',h''\rangle = 1$, the map $(x,y) \mapsto W_n(xh'+yh'')$ is a homogeneous polynomial of degree n with values in $\mathcal{M}_n(\mu)$, so that each of its coefficients belongs to $\mathcal{M}_n(\mu)$. Since $\langle xh'+yh'', xh'+yh''\rangle = x^2+y^2$, the coefficient of x^py^q has leading coefficients which are explicit linear combinations of the mixed traces $\mathrm{Tr}_{R_z}\, h'(0)^i h''(0)^j$, $i \leqslant p$, $j \leqslant q$.

Proposition 6.2 *Assume $\gamma = 0$.*

1. *The coefficients (5.3) of the elements of $\mathcal{M}_4(\mu)$ satisfy exactly nine linearly independent relations, and those of $\mathcal{M}_6(\mu)$ exactly six.*
2. *For h as above, put $T_4 = \mathrm{Tr}_{\mathfrak{s}}(\mathrm{ad}\, h)^4$ and $Q_j = \mathrm{Tr}_{R_{d_j}} h(0)^4$. The relations of weight four are equivalent to the expression of $Q_1,\dots,Q_5$ as affine functions of (T, T_4, τ_6, Q_6) with coefficients polynomial in u. In particular*

$$Q_5 = \frac{12T + 2T_4 - 27u^2 - 90u + 105}{26}. \tag{6.3}$$

Proof : The relations are obtained as in Step 1 of the proof of Proposition 5.4, with the forms of weight $6-n$ listed in the supplementary data in place of the forms of weight four. Their number is exact by the argument of Step 2, using the exact elements $E_4\Xi_{\bullet,j}J^m$, $\vartheta_2 D(\Xi_{\bullet,j}J^m)$,

$DJ \cdot D(\Xi_{\bullet,j} J^m)$, $(DJ)^2 \Xi_{\bullet,j} J^m$ of weight four, and their analogues of weight six. For 2, the traces on the explicit lowest weight spaces are computed from the invariant theory of $\mathfrak{e}_6$ and $\mathfrak{sl}_3$: since $\mathfrak{e}_6$ has no invariant polynomial of degree three or four other than powers of the quadratic one, we have $\mathrm{Tr}_{\mathbf{27}}\, h_1^3 = 0$, $\mathrm{Tr}_{\mathbf{27}}\, h_1^4 = 3u^2$ and $\mathrm{Tr}_{\mathfrak{e}_6}(\mathrm{ad}\, h_1)^4 = 18u^2$. The constants may be determined without a weight enumeration as follows. For the regular subalgebra $\mathfrak{sl}_6 \oplus \mathfrak{sl}_2 \subset \mathfrak{e}_6$, one has $\mathbf{27} = (\mathbf{15}, \mathbf{1}) \oplus (\overline{\mathbf{6}}, \mathbf{2})$ and $\mathfrak{e}_6 = (\mathbf{35}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{20}, \mathbf{2})$. Let H be the neutral element of this index-one $\mathfrak{sl}_2$, so that $\langle H, H \rangle = 2$. Its eigenvalues on $\mathbf{27}$ are 1 and -1, each six times, together with fifteen zeros. On $\mathfrak{e}_6$ they are 2 and -2 once, 1 and -1 each twenty times, together with thirty-six zeros. Thus the fourth traces are $12 = 3 \cdot 2^2$ and $72 = 18 \cdot 2^2$. A separate enumeration of weights gives the same traces exactly. For $\mathfrak{sl}_3$ we have $\mathrm{Tr}_{\mathbf{3}}\, h_2^4 = \frac{(1-u)^2}{2}$ and $\mathrm{Tr}_{\mathfrak{sl}_3}(\mathrm{ad}\, h_2)^4 = 9(1-u)^2$. Substituting these traces and (6.2) into the nine rational relations determines $Q_1, \ldots, Q_5$ uniquely in terms of T, T_4, u, τ_6 and Q_6. Eliminating $Q_1, \ldots, Q_4$ gives (6.3). We perform the elimination in rational arithmetic and independently check all nine equations after substitution and the polarization used below. ∎

For the announced algebra, the fourth power traces computed from the announced lowest weight spaces satisfy the nine relations of weight four for both factors. The same holds for the mixed and sextic relations used below. The supplementary data include these exact trace comparisons.

7 The vector $\gamma = 1$ is not realized

Theorem 7.1 *Let U be a positive strongly rational vertex operator algebra of central charge eight with a braided equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$. Then the character vector of U is the Evans–Gannon vector with $\gamma = 0$. In particular $\dim U_1 = 6$, $\dim U_2 = 120$ and $\dim U_3 = 956$.*

The proof uses two lemmas. Suppose $\gamma = 1$. Then R_{d_3} is one-dimensional, so that $\mathfrak{s}$ acts on it through a character. Hence $\tau_3 = 0$ for h in the derived algebra $[\mathfrak{s}, \mathfrak{s}]$.

Lemma 7.2 *Assume $\gamma = 1$. Let $\mathfrak{g}$ be a simple ideal of $\mathfrak{s}$, with indices $i_1 = i_1(\mathfrak{g})$, $i_2 = i_2(\mathfrak{g})$ and level $k = i_1 + i_2$. Then $(\mathfrak{g}, i_1, i_2)$ is one of $(\mathfrak{sl}_3, 1, 0)$, $(\mathfrak{sp}_4, 1, 0)$, $(\mathfrak{g}_2, 1, 0)$, $(\mathfrak{sl}_4, 1, 0)$, $(\mathfrak{sl}_2, 1, 1)$, $(\mathfrak{sl}_2, 1, 4)$ and $(\mathfrak{sl}_2, 2, 4)$. In particular $i_1 \geqslant 1$.*

Proof : For h in a Cartan subalgebra of $\mathfrak{g}$, we have $u = \frac{i_1}{k}$ and $T = \frac{2h^\vee}{k}$, where $h^\vee$ is the dual Coxeter number. Since $\tau_3 = 0$, Corollary 5.6 gives $\tau_5 = T - 9u + 3$, and then $\tau_4 \geqslant 0$ and $\tau_1 \geqslant 0$ read $T - 9u + 5 \leqslant \tau_6 \leqslant \frac{-2T+18u+10}{3}$. Together with $\tau_5 \geqslant 0$, this gives

$$9u - 3 \leqslant T \leqslant 9u - 1.$$

In particular $u > 0$, so $i_1 \geqslant 1$, and $2h^\vee \geqslant 6k - 9i_2$, so that $k \leqslant \frac{2h^\vee + 9i_2}{6}$. Since $\dim \mathfrak{g} \leqslant 19$, the algebra $\mathfrak{g}$ is of type A_1, A_2, B_2, G_2 or A_3, and $i_2 = 0$ unless $\mathfrak{g} \cong \mathfrak{sl}_2$, in which case $i_2 \in \{0, 1, 4\}$ by Proposition 4.4. Multiplying $9u - 3 \leqslant T \leqslant 9u - 1$ by k gives $6i_1 - 3i_2 \leqslant 2h^\vee \leqslant 8i_1 - i_2$. If $i_2 = 0$, this excludes A_1 and forces $i_1 = 1$ for A_2, B_2, G_2 and A_3. For $\mathfrak{g} \cong \mathfrak{sl}_2$ and $i_2 = 1$, it forces $i_1 = 1$. For $i_2 = 4$, it forces $i_1 \in \{1, 2\}$. This is exactly the stated list. ∎

Lemma 7.3 *Assume $\gamma = 1$. Every nonzero real element h of the center of $\mathfrak{s}$, normalized by $\langle h, h \rangle = 1$, satisfies $u \geqslant \frac{1}{9}$. In particular p_1 is injective on the center of $\mathfrak{s}$.*

Proof : Here $T = 0$. By Corollary 5.6, $\tau_3 \geqslant 0$ gives $\tau_5 \geqslant 3 - 9u$, and the inequalities $\tau_4 \geqslant 0$ and $\tau_1 \geqslant 0$ read $\frac{11\tau_5}{18} - \frac{7u}{2} + \frac{19}{6} \leqslant \tau_6 \leqslant \frac{17}{3} - u - \frac{7\tau_5}{9}$. Eliminating τ_6 gives $\tau_5 \leqslant \frac{9}{5}(1+u)$. Hence $3 - 9u \leqslant \frac{9}{5}(1+u)$, that is $u \geqslant \frac{1}{9}$. ∎

We now prove Theorem 7.1.

Proof : Suppose $\gamma = 1$ and write $\mathfrak{s} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathfrak{z}$ with simple ideals $\mathfrak{g}_i$ and center $\mathfrak{z}$. The images under p_1 of the summands are pairwise orthogonal for the invariant form of $\mathfrak{e}_6$. Indeed, distinct summands are orthogonal for $\langle \cdot, \cdot \rangle$, and at most one of them has a nonzero image under p_2 by Proposition 4.4. The form is positive definite on the real span of a Cartan subalgebra. By Lemmas 7.2 and 7.3, p_1 is injective on each summand. Hence

$$\operatorname{rank} p_1(\mathfrak{s}) = \sum_i \operatorname{rank} \mathfrak{g}_i + \dim \mathfrak{z} \leqslant 4$$

by Proposition 4.4. By Lemma 7.2, each $\mathfrak{g}_i$ is $\mathfrak{sl}_3$, $\mathfrak{sp}_4$, $\mathfrak{g}_2$ (rank two, dimensions 8, 10, 14), $\mathfrak{sl}_4$ (rank three, dimension 15) or an $\mathfrak{sl}_2$ with $i_2 \neq 0$, of which there is at most one. We list the possible values of $\dim \mathfrak{s}$. Two summands of rank two exhaust the rank and give dimensions 16, 18, 20, 22, 24 or 28. A summand $\mathfrak{sl}_4$ leaves rank one and gives 15, 16 or 18. One summand of rank two and summands of rank one give at most $14 + 3 + 1 = 18$. Summands of rank one only give at most $3 + 3 = 6$. None of these equals $19 = \dim U_1$, a contradiction. Hence $\gamma = 0$, and the dimensions follow from Theorem 3.1. ∎

Remark 7.4 Theorem 7.1 does not use unitarity or the announcements [Rie26a, Hua26, Jia26]. It uses the classification of holomorphic vertex operator algebras of central charge eight, the theory of extensions [HKL15, CKM22], the Witt group [DMNO13], the classification of the elements of order three of E_8 [Kac90], the results of [DoN99, DoM04, Mas14, ENO05] used in Section 4, Zhu's theorem, the Bantay–Gannon description of vector-valued modular forms, and Proposition 5.4, which rests on finitely many rational computations and on the modularity theorem used in Theorem 3.1. The vector $\gamma = 1$ passes the tests of character theory known to us (its coefficients are nonnegative integers up to q^{60}). It is not the character vector of a positive strongly rational vertex operator algebra with the braided category $Z(\mathcal{H}_3)$.

Remark 7.5 For $\gamma = 0$, the same method gives necessary conditions on U_1. Corollary 5.6 and $\tau_j \geqslant 0$ give $T \leqslant 9u - 5$ for every real h. Thus a simple ideal of U_1 with indices (i_1, i_2) satisfies $2h^\vee \leqslant 9i_1 - 5k$, and every real central element satisfies $u \geqslant \frac{5}{9}$. For a level one $\mathfrak{sl}_2$ in $\mathfrak{e}_6$, the traces τ_j are determined, as in Remark 5.7. These conditions are the starting point of Section 8, where U_1 is determined up to two cases.

8 The weight-one Lie algebra

Throughout this section, U is a positive strongly rational vertex operator algebra of central charge eight with a braided equivalence $\operatorname{Rep}(U) \cong Z(\mathcal{H}_3)$. By Theorem 7.1, its character vector is $X^{(0)}$, and $\mathfrak{s} = U_1 \subset \mathfrak{e}_6 \oplus \mathfrak{sl}_3$ has dimension six. We use the notation of Section 5. For a simple ideal $\mathfrak{g} \cong \mathfrak{sl}_2$ of $\mathfrak{s}$ of level k, we denote by H the element of a Cartan subalgebra of $\mathfrak{g}$ with eigenvalues ± 2 on $\mathfrak{g}$. Thus $\langle H, H \rangle = 2k$ and $h = \frac{H}{\sqrt{2k}}$ is normalized. We denote by $V(a)$ the irreducible $\mathfrak{sl}_2$-module of highest weight a and by $V(a, b) = V(a) \otimes V(b)$ the irreducible modules of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

8.1 Three general facts

Lemma 8.1 *Let $\mathfrak{g} \cong \mathfrak{sl}_2$ be a simple ideal of $\mathfrak{s}$ of level k. For every label z, every irreducible $\mathfrak{g}$-submodule of R_z is isomorphic to $V(a)$ with $a \leqslant k$.*

Proof : By [DoM06, Theorem 3.1 p. 4], the vertex subalgebra of U generated by $\mathfrak{g}$ is isomorphic to the simple affine vertex operator algebra $L_{\mathfrak{sl}_2}(k,0)$ and k is a positive integer. By [DLM97, Theorem 3.7 p. 7], every weak $L_{\mathfrak{sl}_2}(k,0)$-module, in particular M_z, is a direct sum $\bigoplus_j L(k,j)\otimes N_j$ of the integrable highest weight modules $L(k,j)$, $0\leqslant j\leqslant k$ [FrZ92], tensored with multiplicity spaces. Let $v\in R_z$ be a highest weight vector of weight a for $\mathfrak{g}$. Since R_z is the lowest weight space of M_z and $x(n)$ lowers the conformal weight by n, we have $x(n)v=0$ for $x\in\mathfrak{g}$ and $n\geqslant 1$. In the irreducible module $L(k,j)$, the vectors annihilated by $x(n)$, $n\geqslant 1$, form the top level $V(j)$. Hence $v\in\bigoplus_j V(j)\otimes N_j$, and since $e(0)v=0$ and $H(0)v=av$, the vector v lies in the sum of the components with $j=a$. Thus $a\leqslant k$. ■

Proposition 8.2 *The Lie algebra $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}_2\oplus\mathfrak{sl}_2$.*

Proof : The Lie algebra $\mathfrak{s}$ is reductive of dimension six. Its semisimple part has dimension at most six, hence is 0, $\mathfrak{sl}_2$ or $\mathfrak{sl}_2\oplus\mathfrak{sl}_2$. Suppose that $\mathfrak{s}$ is not semisimple. Then its center $\mathfrak{z}$ has dimension three or six. The image $p_2(\mathfrak{z})$ is contained in the center of the reductive Lie algebra $p_2(\mathfrak{s})$, which is 0 or a line by Proposition 4.4. The kernel of $p_2|_{\mathfrak{z}}$ is defined on the real form used above: its equations are the weights of $R_{c_1}\cong\mathbf{3}$, which are real on that form. Since this kernel has positive dimension, there is a nonzero real element $h\in\mathfrak{z}$ with $p_2(h)=0$ and $\langle h,h\rangle=1$. For this element $u=1$ and $T=T_4=0$, and (6.3) gives $Q_5=\operatorname{Tr}_{R_{d_5}}h(0)^4=-\frac{6}{13}<0$, which is impossible since $h(0)$ has real eigenvalues. ■

Lemma 8.3 *Let $\mathfrak{a}\subset\mathfrak{e}_6$ be a subalgebra isomorphic to $\mathfrak{sl}_2$, with neutral element $H_{\mathfrak{a}}$.*

1. *After conjugation, $H_{\mathfrak{a}}$ is dominant with $\alpha_i(H_{\mathfrak{a}})\in\{0,1,2\}$. Among the 3^6-1 such labellings, exactly 27 give eigenvalues of $\operatorname{ad}H_{\mathfrak{a}}$ on $\mathfrak{e}_6$ and of $H_{\mathfrak{a}}$ on $\mathbf{27}$ which form the weights of $\mathfrak{sl}_2$-modules. The corresponding 27 numbers $\frac{\langle H_{\mathfrak{a}},H_{\mathfrak{a}}\rangle}{2}$ are pairwise distinct and form a superset of the possible Dynkin indices. The 20 indices actually realized by subalgebras $\mathfrak{sl}_2$ of $\mathfrak{e}_6$ are 1–6, 8–12, 20, 21, 28, 30, 35, 36, 60, 84, 156.*

2. *If $\mathfrak{a}$ has index 35, its centralizer in $\mathfrak{e}_6$ is a subalgebra isomorphic to $\mathfrak{sl}_2$ of index 1. If $\mathfrak{a}$ has index 9, its centralizer is a subalgebra isomorphic to $\mathfrak{sl}_2$ of index 3.*

Proof : 1. The first assertion is the theorem of Dynkin and Kostant on the characteristics of nilpotent orbits [CoM93, Chapter 3]. The second one is an exact computation on the weights. We only use the resulting list as a set containing the possible indices, which is enough for the exclusion arguments below. The list of indices of the subalgebras $\mathfrak{sl}_2$ is [Dyn52, Table 18]. See also [Gan19, p. 16].

2. We construct $\mathfrak{e}_6$ explicitly by the Frenkel–Kac formula on its root lattice [FrK80], with a bimultiplicative cocycle, and check the Jacobi identity. For the labellings $(2,1,1,0,1,2)$ and $(1,0,0,1,0,1)$ in the Bourbaki numbering, of indices 35 and 9, we choose e in the eigenspace of $\operatorname{ad}H_{\mathfrak{a}}$ for the eigenvalue 2 with integral coefficients, solve $[e,f]=H_{\mathfrak{a}}$ exactly, and compute the centralizer of the triple. In both cases it is three-dimensional and perfect, hence isomorphic to $\mathfrak{sl}_2$, and its index is the ratio of the normalized form of $\mathfrak{e}_6$ to the normalized form of $\mathfrak{sl}_2$ on any element with nonzero norm. We obtain 1 and 3. The two labellings are the characteristics of the nilpotent orbits A_5 and $2A_2+A_1$: the dimensions $\dim\mathfrak{g}_0+\dim\mathfrak{g}_1$ of the centralizers of e are 14 and 24, so that the orbits have dimensions 64 and 54, in agreement with the tables of [CoM93, Chapter 8]. Since two subalgebras $\mathfrak{sl}_2$ with the same characteristic are conjugate [CoM93, Chapter 3], the result does not depend on the choice of e. ■

8.2 Simple ideals

Proposition 8.4 *Let $\mathfrak{g}$ be a simple ideal of $\mathfrak{s}$, with indices (i_1, i_2) and level $k = i_1 + i_2$. Then $(i_1, i_2) \in \{(1,0),(3,0),(9,4),(35,4)\}$. Moreover, in each case the $\mathfrak{g}$-module structures of R_{d_1}, R_{d_4}, R_{d_5} and R_{d_6} are uniquely determined, and the traces $\operatorname{Tr}_{R_z} H(0)^p$, $p \in \{2,4,6\}$, are determined for all labels z.*

Proof : For $h = \frac{H}{\sqrt{2k}}$ we have $u = \frac{i_1}{k}$, $T = \frac{4}{k}$ and $T_4 = \frac{8}{k^2}$. By Remark 7.5, $4 \leqslant 9i_1 - 5k$, so that $i_1 \geqslant 1$, and i_1 is the Dynkin index of $p_1(\mathfrak{g})$, which belongs to the 27-element superset computed in Lemma 8.3. By Proposition 4.4, $i_2 \in \{0,1,4\}$. For each of the finitely many pairs (i_1, i_2), we enumerate all 135 isomorphism classes of fourteen-dimensional $\mathfrak{sl}_2$-modules and retain those whose highest weights do not exceed k, by Lemma 8.1. Each structure gives τ_6 and Q_6. Corollary 5.6 and Proposition 6.2 then give τ_j and Q_j for all j, and we decide exactly, by a finite search, whether there is a structure of $\mathfrak{sl}_2$-module on R_{d_j} of the right dimension, with highest weights at most k, such that $\operatorname{Tr} H(0)^2 = 2k\tau_j$ and $\operatorname{Tr} H(0)^4 = 4k^2 Q_j$. We also check, using the decompositions of **27** and $\mathfrak{e}_6$ attached to i_1 by Lemma 8.3, that the highest weights occurring in R_b, R_a, R_{c_0}, R_{c_1} and R_{c_2} are at most k. Only the four pairs of the statement survive, each with a unique structure on R_{d_6} (without the level bound the same four pairs survive).

For these four pairs, the relations of weight six of Proposition 6.2, with the sextic traces on $R_b, \ldots, R_{c_2}$ computed from the explicit weights, express $\operatorname{Tr}_{R_{d_j}} H(0)^6$, $j \in \{1,2,3\}$, in terms of the corresponding traces on R_{d_4}, R_{d_5}, R_{d_6}. The structures of R_{d_4} and R_{d_5} are unique, and the structure of R_{d_1} is then unique. Thus the traces of degrees two, four and six used below are determined. ∎

8.3 The classification

Theorem 8.5 *Let U be a positive strongly rational vertex operator algebra of central charge eight with a braided equivalence $\operatorname{Rep}(U) \cong Z(\mathcal{H}_3)$, and let $U \subset V_{E_6 \oplus A_2}$ be the inclusion of Proposition 4.1. Then $U_1 = \mathfrak{g}' \oplus \mathfrak{g}''$ with $\mathfrak{g}' \cong \mathfrak{g}'' \cong \mathfrak{sl}_2$, and, up to an inner automorphism of $V_{E_6 \oplus A_2}$, one of the following holds.*

(I) The levels are 1 and 39. The subalgebra $\mathfrak{g}''$ is the diagonal of a principal $\mathfrak{sl}_2$ of a subalgebra $\mathfrak{sl}_6$ of $\mathfrak{e}_6$ (index 35) and of the principal $\mathfrak{so}_3$ of $\mathfrak{sl}_3$ (index 4), and $\mathfrak{g}' \subset \mathfrak{e}_6$ is the centralizer in $\mathfrak{e}_6$ of $p_1(\mathfrak{g}'')$ (index 1).

(II) The levels are 3 and 13. The subalgebra $\mathfrak{g}''$ is the diagonal of a subalgebra $\mathfrak{sl}_2$ of $\mathfrak{e}_6$ of index 9, attached to the nilpotent orbit $2A_2 + A_1$, and of the principal $\mathfrak{so}_3$ of $\mathfrak{sl}_3$, and $\mathfrak{g}' \subset \mathfrak{e}_6$ is the centralizer in $\mathfrak{e}_6$ of $p_1(\mathfrak{g}'')$ (index 3).

In both cases, the lowest weight spaces of the twelve simple U-modules and the space U_2 are determined as $\mathfrak{g}' \oplus \mathfrak{g}''$-modules. They are given in Table 2.

Proof : By Proposition 8.2, $U_1 = \mathfrak{g}' \oplus \mathfrak{g}''$. By Proposition 4.4, at most one of the two ideals has a nonzero projection to $\mathfrak{sl}_3$, and the projections to $\mathfrak{e}_6$ of the two ideals commute and are orthogonal. By Proposition 8.4, the pair of indices is one of $\{(1,0),(1,0)\}$, $\{(1,0),(3,0)\}$, $\{(3,0),(3,0)\}$, $\{(1,0),(9,4)\}$, $\{(1,0),(35,4)\}$, $\{(3,0),(9,4)\}$, $\{(3,0),(35,4)\}$.

If an ideal has nonzero projection to $\mathfrak{sl}_3$, its projection to $\mathfrak{e}_6$ has index 9 or 35. Its centralizer is an $\mathfrak{sl}_2$ of index 3 or 1, respectively, by Lemma 8.3. The other ideal is three-dimensional and contained in this centralizer, hence equals it. This excludes the pairs $\{(1,0),(9,4)\}$ and $\{(3,0),(35,4)\}$. It remains to exclude the three pairs with zero projection to $\mathfrak{sl}_3$. This follows

	Case (I), levels $(1,39)$	Case (II), levels $(3,13)$
R_b	$(0,2)+2(0,4)+(0,6)+(0,8)+(0,10)$ $+(1,3)+(1,5)+(1,9)$	$2(0,2)+2(0,4)+(1,1)+2(1,3)+(1,5)$ $+(2,2)+(2,4)+(3,1)$
R_a, R_{c_0}	$2(0,2)+(0,4)+2(0,6)+(0,8)+(0,10)$ $+(1,3)+(1,5)+(1,7)$	$2(0,2)+(0,4)+(0,6)+2(1,1)+2(1,3)$ $+(1,5)+(2,0)+(2,2)+(2,4)$
R_{c_1}	$(0,2)$	$(0,2)$
R_{c_2}	$(0,0)+(0,4)+(0,8)+(1,5)$	$(0,0)+(0,4)+(1,1)+(1,3)+(2,2)$
R_{d_1}	$(0,6)$	$(0,2)+(1,1)$
R_{d_2}	$(0,0)+(0,4)+(0,6)+(0,8)+(1,1)+(1,7)$	$(0,0)+(0,2)+(0,4)+(1,1)+(1,3)$ $+(1,5)+(2,2)$
R_{d_3}	$(0,0)+(0,2)+2(0,4)+2(0,6)+2(0,8)+(0,12)$ $+(1,1)+(1,3)+(1,5)+(1,7)+(1,9)$	$2(0,0)+(0,2)+2(0,4)+(0,6)+2(1,1)+2(1,3)$ $+2(1,5)+2(2,2)+(2,4)+(3,3)$
R_{d_4}	$(0,4)$	$(0,0)+(1,1)$
R_{d_5}	$(0,4)+(1,3)$	$(0,4)+(1,3)$
R_{d_6}	$(0,2)+(0,6)+(1,1)$	$(0,2)+(1,3)+(2,0)$
U_2	$4(0,0)+(0,2)+2(0,4)+2(0,8)+(0,12)$ $+(1,1)+(1,5)+(1,7)+(1,13)$ $+(2,0)+(2,2)$	$4(0,0)+(0,2)+2(0,4)+(1,1)+(1,3)+(1,5)$ $+(1,7)+(2,0)+2(2,2)+(2,6)$ $+(3,3)+(4,0)$

Table 2: Lowest weight spaces and U_2 as modules over $\mathfrak{g}' \oplus \mathfrak{g}''$. Here (a,b) stands for $V(a,b)$, where a refers to $\mathfrak{g}'$ and b to $\mathfrak{g}''$. The labels of the rows R_z are bounded by the levels (Lemma 8.1). Those of the row U_2 are not, since U_2 contains descendants, for instance the component $V(4,0) \subset S^2(\mathfrak{g}')$ of the weight-two space of $L_{\mathfrak{sl}_2}(3,0)$ in case (II).

directly from (6.3). Let H' and H'' be their neutral elements and let $k', k'' \in \{1,3\}$ be their levels. Put $h = xH' + yH''$. The two ideals commute and are orthogonal, so that

$$\langle h,h\rangle = 2k'x^2 + 2k''y^2, \quad \operatorname{Tr}_{\mathfrak{s}}(\operatorname{ad} h)^2 = 8(x^2+y^2), \quad \operatorname{Tr}_{\mathfrak{s}}(\operatorname{ad} h)^4 = 32(x^4+y^4).$$

Since $p_2(h) = 0$, homogeneity and polynomiality extend (6.3) from normalized elements to

$$26 \operatorname{Tr}_{R_{d_5}} h(0)^4 = 12\langle h,h\rangle \operatorname{Tr}_{\mathfrak{s}}(\operatorname{ad} h)^2 + 2\operatorname{Tr}_{\mathfrak{s}}(\operatorname{ad} h)^4 - 12\langle h,h\rangle^2.$$

Comparing the coefficients of x^2y^2 gives

$$\operatorname{Tr}_{R_{d_5}} H'(0)^2 H''(0)^2 = \frac{8\big(2(k'+k'') - k'k''\big)}{13}. \tag{8.1}$$

The left-hand side is an integer, because $H'(0)$ and $H''(0)$ have simultaneous integral eigenvalues on the finite-dimensional $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$-module R_{d_5}. For $(k',k'') = (1,1), (1,3), (3,3)$, the right-hand side is $24/13$, $40/13$ and $24/13$, respectively. This excludes all three pairs. An independent finite search gives the same exclusions. It enumerates the structures of R_{d_6} compatible with the two marginal structures, then substitutes their mixed fourth-order traces into the weight-four relations. For the pair $(1,1)$ there are two possibilities, giving $\operatorname{Tr}_{R_{d_4}} H'(0)^2 H''(0)^2 = -40/39$ and $-44/195$. For $(1,3)$ and $(3,3)$ there is one possibility, giving $-36/65$ and $124/65$, respectively. Each value fails to be a nonnegative integer. This provides a separate finite verification of the three exclusions.

For the two remaining pairs, the projection $p_1(\mathfrak{g}'')$ has index 35, respectively 9, and $p_2(\mathfrak{g}'')$ is the principal $\mathfrak{so}_3$. The ideal $\mathfrak{g}'$ is contained in $\mathfrak{e}_6$ and commutes with $p_1(\mathfrak{g}'')$, hence is contained in, and by dimension equal to, the centralizer of $p_1(\mathfrak{g}'')$ in $\mathfrak{e}_6$, which is three-dimensional of index 1, respectively 3, by Lemma 8.3. Since the subalgebras $\mathfrak{sl}_2$ of $\mathfrak{e}_6$ with a given index

are conjugate under the adjoint group, by Lemma 8.3 and the bijection between conjugacy classes and nilpotent orbits [CoM93, Chapter 3], and the same holds for the principal $\mathfrak{so}_3$ of $\mathfrak{sl}_3$, the subalgebra U_1 is determined up to the inner automorphisms of $V_{E_6\oplus A_2}$ generated by the zero modes of $\mathfrak{e}_6 \oplus \mathfrak{sl}_3$. An isomorphism between $p_1(\mathfrak{g}'')$ and $p_2(\mathfrak{g}'')$ is unique up to an inner automorphism of $\mathfrak{sl}_2$, which is induced by the adjoint group of $\mathfrak{sl}_3$ on its principal $\mathfrak{so}_3$. This gives cases (I) and (II).

Now we determine the lowest weight spaces. For $R_b, \dots, R_{c_2}$ this follows from the explicit embedding: we construct $\mathfrak{e}_8 \supset \mathfrak{e}_6 \oplus \mathfrak{sl}_3$ by the Frenkel–Kac formula and compute exactly the joint eigenvalues of commuting neutral elements of $\mathfrak{g}'$ and $\mathfrak{g}''$ on $\mathfrak{e}_6$, on $\mathbf{27}$ and on $\mathfrak{sl}_3$. For the labels d_j, we use in addition the coefficients of x^4y^2 and x^2y^4 in $W_6(xh' + yh'')$. The relations then determine, once the structures of R_{d_4}, R_{d_5}, R_{d_6} are fixed, the traces $\mathrm{Tr}_{R_{d_j}} H'(0)^p H''(0)^q$ for $p+q \leqslant 6$. The structures of R_{d_4}, R_{d_5}, R_{d_6} are unique, and an exhaustive enumeration shows that the structures of R_{d_1}, R_{d_2} and R_{d_3} with these traces and with highest weights bounded by the levels are unique. More explicitly, writing a lowest weight space R as $\bigoplus_{a=0}^{k'} \bigoplus_{b=0}^{k''} m_{ab} V(a,b)$, the variables satisfy $0 \leqslant m_{ab} \leqslant \lfloor \dim R/((a+1)(b+1)) \rfloor$ and $\sum_{a,b}(a+1)(b+1)m_{ab} = \dim R$. The marginal decompositions and the nine even joint moments of positive degree at most six impose further necessary linear constraints. These inequalities specify the finite domain of the search. A partial multiplicity assignment can be discarded only when no completion satisfies these necessary constraints. The uniqueness assertion concerns the entire domain.

Finally, augment the coefficient map (5.3) by the fourteenth functional $w_0'' = [q^{5/3}]W_0$. The same residue construction, with this enlarged list of allowed coefficients, gives relations on the image of $\mathcal{M}_n(\mu)$ in $\mathbb{C}^{14}$, for $n \in \{2,4,6\}$. For $W = W_n(xh' + yh'')$, the functional w_0'' is an affine function of the traces $\mathrm{Tr}_{U_2} H'(0)^p H''(0)^q$. All lower-weight terms are fixed by U_0 and U_1. Solving these enlarged rational systems and extracting the monomials $x^p y^q$, with p, q even and $2 \leqslant p+q \leqslant 6$, determines the nine joint traces. With powers J^m for $0 \leqslant m \leqslant 2$, the vanishing-constraint matrices have dimensions 53×144, 41×72 and 29×36 in weights two, four and six, respectively, and have full row rank. The resulting relation spaces on the fourteen functionals have dimensions 10, 10 and 7. Exact forms attain the complementary image dimensions 4, 4 and 7. In each weight a relation has nonzero coefficient at w_0'', so the previously determined thirteen coefficients fix it uniquely. The supplementary data contain these matrices, their exact rank checks and the resulting nine unnormalized traces in each case. The space U_2 contains the weight-two subspace of the affine vertex algebra generated by U_1 and the conformal vector of U, and it is contained in $(V_{E_6\oplus A_2})_2 \cong (\mathbf{27} \otimes \overline{\mathbf{27}}, \mathbf{1}) \oplus (\mathbf{78}, \mathbf{8}) \oplus (\mathbf{1}, \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8})$, the decomposition of the weight-two space of the level one affine algebras $L_{\mathfrak{e}_6}(1,0)$ and $L_{\mathfrak{sl}_3}(1,0)$. Write the displayed ambient module as $\bigoplus_{a,b} M_{ab} V(a,b)$ and the known affine and conformal subspace as $\bigoplus_{a,b} l_{ab} V(a,b)$. The remaining multiplicities of U_2 satisfy $l_{ab} \leqslant m_{ab} \leqslant M_{ab}$, $\sum_{a,b}(a+1)(b+1)m_{ab} = 120$, and the nine joint trace equations. An exhaustive search over these finite ranges gives a unique solution in each case. The highest weights in this search are bounded by V_2', not by the affine levels. ∎

Remark 8.6 Case (I) is the structure announced in [Rie26a, Hua26]: the level one factor comes from a root $\mathfrak{sl}_2$ of $\mathfrak{e}_6$ and the level 39 factor from the diagonal of the principal triples of A_5 and A_2 [Rie26b, Section 1]. The lowest weight spaces of Table 2 in case (I) coincide with those listed in the working manuscript [Rie26b], and the decomposition of U_2 coincides with [Rie26b, Appendix H.1]. In the present work these data are consequences of the hypotheses of Theorem 8.5, and the announced data serve only as a test.

Remark 8.7 The relations of this section do not exclude case (II), whose explicit realization is constructed in Section 10 and shown to be a Haagerup algebra in [ArhIII, Theorem 6.1].

The remaining arguments determine its weight-two generating space and identify the resulting algebra. Finite-degree consistency checks alone do not establish existence. The lowest weight spaces of Table 2 are also compatible with the bound $\Delta_a + \Delta_b \leqslant h_z$, where Δ_a is the Sugawara weight. This bound holds for $z \in \{0, b, a, c_0, c_1, c_2\}$ because the corresponding U-modules occur in unitary lattice modules by Lemma 4.3. Choose the compact form for the affine embedding. Its Sugawara operators and those of the coset are then unitary on these modules, and the coset conformal weights are nonnegative. The central charge of the commutant of the affine subalgebra is $8 - \frac{9}{5} - \frac{13}{5} = \frac{18}{5}$, and U_2 contains a two-dimensional space of invariant vectors in this commutant (Corollary 9.2). Note that the products of the levels are equal in the two cases, $1 \cdot 39 = 3 \cdot 13$. Section 10 pushes the analysis of case (II) to the space U_2 and to weight three (Theorem 10.5 and Proposition 10.7).

9 Constraints on the weight-two space

We keep the hypotheses of Theorem 8.5, we write $A \subset U$ for the affine vertex algebra generated by U_1, and set $V' = V_{E_6 \oplus A_2}$.

Proposition 9.1 *In both cases of Theorem 8.5, U_2 is a U_1-submodule of V_2' of the type given in Table 2, containing $(A)_2$ and the conformal vector of V'. The set of such submodules is a product of Grassmannians of the multiplicity spaces of V_2', of dimension* 70 *in case (I) and* 87 *in case (II).*

Proof : The first assertion is Theorem 8.5. A U_1-submodule of V_2' with prescribed multiplicities u_{ab} is given by the choice, for each (a, b), of a u_{ab}-dimensional subspace of the multiplicity space of $V(a, b)$ in V_2', of dimension v_{ab}, containing the l_{ab}-dimensional subspace coming from $(A)_2$ and $\mathbb{C}\omega$. This is a Grassmannian of dimension $(u_{ab} - l_{ab})(v_{ab} - u_{ab})$. The multiplicities v_{ab} are computed from the lattice, and we evaluate the resulting finite sum. ■

Proposition 9.1 turns the determination of U_2 into a finite-dimensional algebraic problem: the subspace must satisfy $x_{(1)}y \in U_2$ and $x_{(2)}y \in U_1$ for $x, y \in U_2$, and $V_2' \ominus U_2$ must be the weight-two space of the U-module M_b. Section 10 applies these constraints after passing to the commutant of the currents. The resulting subalgebras and their characters enter the proofs of Theorems 10.29 and 10.38.

Corollary 9.2 *In both cases, $A \cong L_{\mathfrak{sl}_2}(k', 0) \otimes L_{\mathfrak{sl}_2}(k'', 0)$ with $(k', k'') = (1, 39)$ or $(3, 13)$, the commutant $\mathrm{Com}_U(A)$ has central charge $\frac{170}{41}$ or $\frac{18}{5}$, and its weight-two space is two-dimensional and contains its conformal vector $\omega - \omega_{k'} - \omega_{k''}$.*

Proof : The first assertion follows from Theorem 8.5 and [DoM06, Theorem 3.1 p. 4]. The weight-two space of $\mathrm{Com}_U(A)$ consists of the vectors of U_2 annihilated by the modes $x(n)$, $n \geqslant 0$, of $x \in U_1$. Such vectors span trivial U_1-submodules. By Table 2, the trivial isotypic component of U_2 is four-dimensional, and it contains the Sugawara vectors $\omega_{k'}$ and $\omega_{k''}$, which are not annihilated by $x(1)$ for suitable x. Since $(A)_2$ contains exactly two invariant vectors, the U_1-invariant vectors of U_2 are $\mathbb{C}\omega_{k'} \oplus \mathbb{C}\omega_{k''}$ plus a two-dimensional complement, which is the set of invariant vectors annihilated by all $x(1)$, $x \in U_1$. Indeed, for an invariant vector w of U_2, we have $x(n)w = 0$ for $n \geqslant 3$ by the grading. The map $x \mapsto x(2)w$ is a U_1-module homomorphism from the adjoint module U_1 to $U_0 = \mathbb{C}\mathbf{1}$. It vanishes because U_1 is semisimple. Also $x(0)w = 0$ by invariance, and $x(1)w \in U_1$ depends U_1-equivariantly on x, so that $x \mapsto x(1)w$ is an endomorphism of U_1 commuting with $\mathfrak{g}' \oplus \mathfrak{g}''$. It is a combination of the projections onto $\mathfrak{g}'$ and $\mathfrak{g}''$, and subtracting the corresponding combination of Sugawara vectors gives an element of the commutant. The central charges are $8 - 1 - \frac{117}{41}$ and $8 - \frac{9}{5} - \frac{13}{5}$. ■

10 The commutant of the currents and the space U_2

We keep the hypotheses of Theorem 8.5. We write $V' = V_{E_6\oplus A_2}$, so that $V' = U \oplus M_b$ by Proposition 4.1, $A \subset U$ for the affine vertex algebra generated by $U_1 = \mathfrak{g}' \oplus \mathfrak{g}''$, $\widehat{C} = \mathrm{Com}_{V'}(A)$ and $C = \mathrm{Com}_U(A) \subset \widehat{C}$. The conformal vector of $\widehat{C}$ and of C is $\omega_C \overset{\text{def}}{=} \omega - \omega_{k'} - \omega_{k''}$, of central charge $c_C = \frac{170}{41}$ in case (I) and $\frac{18}{5}$ in case (II) (Corollary 9.2). In this section we determine, by exact computations in V', the weight-two space of C and then the space U_2. In case (II), the four candidate spaces give isomorphic vertex algebras, identified with the algebra of [Jia26] in Theorem 10.24. We denote the normalized invariant bilinear form on V' by β, with $\beta(\mathbf{1},\mathbf{1}) = 1$. It is distinct from the positive Hermitian form H_χ introduced below. On lattice vectors, $\langle\cdot,\cdot\rangle_L$ denotes the lattice form.

10.1 An exact model of the lattice algebra

We use the construction of [FLM88, Chapter 8]. The lattice $L = E_6 \oplus A_2$ has the basis of simple roots $\alpha_1, \dots, \alpha_8$, with Gram matrix the Cartan matrix of $E_6 \oplus A_2$ (Bourbaki numbering for E_6). The space V' has the basis of the vectors $\alpha_{i_1}(-n_1)\cdots\alpha_{i_k}(-n_k)e^\beta$, $\beta \in L$, with $n_1 \geqslant \cdots \geqslant n_k \geqslant 1$ and $i_j \leqslant i_{j+1}$ whenever $n_j = n_{j+1}$. The cocycle is the bimultiplicative map with $\varepsilon(\alpha_i,\alpha_j) = (-1)^{\langle\alpha_i,\alpha_j\rangle_L}$ for $i<j$, $\varepsilon(\alpha_i,\alpha_i) = -1$ and $\varepsilon(\alpha_i,\alpha_j) = 1$ for $i>j$, and

$$Y(e^\alpha, z) = \exp\Big(\sum_{n>0} \frac{\alpha(-n)}{n} z^n\Big) \exp\Big(-\sum_{n>0} \frac{\alpha(n)}{n} z^{-n}\Big) e_\alpha z^{\alpha(0)}. \tag{10.1}$$

The modes of a general basis vector are obtained from those of e^α and of the Heisenberg fields by the iterate formula $(a_{(-m)}w)_{(k)} = \sum_{j\geqslant 0} \binom{m+j-1}{j} \big(a_{(-m-j)}w_{(k+j)} - (-1)^m w_{(k-m-j)}a_{(j)}\big)$ for $a = \alpha_i(-1)\mathbf{1}$. Our implementation evaluates these formulas in the specified monomial basis and cocycle, using exact arithmetic for the weight-four membership calculations. The construction of the vertex algebra is the standard lattice construction. The following finite implementation checks, performed through weight three, are separate from that construction: the graded dimensions $1, 86, 1370, 11586$, the relation $L(0) =$ weight on random basis vectors, the Virasoro relations with $c = 8$, the translation property, the commutator formula $[u_{(m)}, v_{(n)}] = \sum_j \binom{m}{j}(u_{(j)}v)_{(m+n-j)}$ on random triples, the equality of ω with the Sugawara vector of $\mathfrak{e}_6 \oplus \mathfrak{sl}_3$ at level one, and $\beta(\omega,\omega) = 4$. These are sample checks, not proofs of the identities for all states.

The enumeration of lattice momenta in a ball may be justified independently of the implementation. In a simply laced root lattice, write a nonzero vector $v = r_1 + \cdots + r_m$ as a sum of roots with m minimal. Two summands cannot have negative inner product: if their inner product is -1, their sum is a root, and if it is -2, they are opposite, in either case contradicting minimality. Hence $\langle v, r_1\rangle_L \geqslant 2$, and $\|v - r_1\|_L^2 = \|v\|_L^2 - 2\langle v, r_1\rangle_L + 2 \leqslant \|v\|_L^2 - 2$. Induction gives a path from v to zero by subtracting roots, with strictly decreasing squared norm. Consequently the search starting at zero and adding every root while retaining only vectors of squared norm at most R visits every lattice vector in that ball. This applies to $E_6 \oplus A_2$, and in particular to the momentum enumeration through conformal weight four, where $R = 8$. An independent enumeration using a rational LDL^{T} decomposition instead of root additions gives respectively 1, 78, 702, 2346 and 5694 vectors of squared norms 0, 2, 4, 6 and 8.

In both cases of Theorem 8.5 we realize U_1 inside V_1' with the neutral elements H' and H'' in the Cartan subalgebra $\mathfrak{h} = L \otimes \mathbb{C}$, so that every basis vector is a joint weight vector. We check exactly the relations of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, the levels through $H(1)H = 2k\mathbf{1}$ and $e(1)f = k\mathbf{1}$, and the commutation of the two ideals. As an independent test of Table 2, the weight-one A-primary

vectors of V', which form R_b, have the $\mathfrak{g}' \oplus \mathfrak{g}''$-structure of the row R_b of that table in both cases.

Weight window of the lattice calculation. The mode implementation stores lattice momenta of squared norm at most 8. We use it for homogeneous inputs of weights at most four and only when the output weight $d = \operatorname{wt} u + \operatorname{wt} v - n - 1$ is at most four. Indeed, a state of momentum γ has weight at least $|\gamma|^2/2$, so a momentum outside the stored ball cannot contribute to such an output. This remains true in the recursive evaluation. If $u = h_{(-m)}u'$ with $m \geqslant 1$, the two inner mode evaluations in the iterate formula have output weights $d - m - j$ and d, respectively, for $j \geqslant 0$. The input weights also do not increase. Induction on the number of oscillators therefore keeps every recursive call inside the same window. All products defining the weight-three and weight-four certificates satisfy these bounds. Outside this window this implementation is not asserted to evaluate the full lattice vertex operator algebra.

10.2 The Griess algebra of the commutant

Lemma 10.1 *Let $B = \widehat{C}_2$. Then $\widehat{C}_1 = 0$, $\dim B = 4$ in case (I) and $\dim B = 5$ in case (II). The product $a \cdot b \stackrel{\text{def}}{=} a_{(1)}b$ is commutative on B, with unit $e \stackrel{\text{def}}{=} \frac{1}{2}\omega_C$, and the form $\beta(\cdot,\cdot)$ is invariant: $\beta(a\cdot b, c) = \beta(a, b\cdot c)$. Moreover $B = C_2 \oplus Y$ with $Y = B \cap M_b$, the sum is orthogonal, and C_2 is a two-dimensional subalgebra of B containing e.*

Proof : The space $\widehat{C}_1$ is the centralizer of U_1 in $\mathfrak{e}_6 \oplus \mathfrak{sl}_3$, which is zero by the proof of Theorem 8.5. The dimensions are computed by solving exactly $x_{(n)}v = 0$, $n \in \{0,1,2\}$, $x \in U_1$, on the joint weight space of weight zero of V_2'. They agree with the count of $\mathfrak{g}' \oplus \mathfrak{g}''$-invariant vectors of V_2' (7, respectively 9) minus the invariant A-descendants (2 Sugawara vectors and 1, respectively 2, vectors $\sum_a x^a(-1)w_a$ attached to the components of type $V(0,2)$ of R_b). Commutativity follows from skew symmetry since $\widehat{C}_1 = 0$, and e is a unit since $L(0) - L_A(0) = (\omega_C)_{(1)}$ acts by 2 on B. Invariance of the form holds for quasi-primary vectors of weight two, and the vectors of B are quasi-primary since $L(1)B \subset \widehat{C}_1 = 0$. Since M_b is a U-module not isomorphic to the contragredient of U, the form vanishes on $U \times M_b$. The decomposition $V' = U \oplus M_b$ is a decomposition of A-modules, hence $B = (B \cap U) \oplus (B \cap M_b)$, and $B \cap U = C_2$ is two-dimensional by Corollary 9.2. ∎

Lemma 10.2 *Let $D \subset B$ be a two-dimensional subalgebra containing e, and write $D = \mathbb{C}e \oplus \mathbb{C}x_C$ with $\beta(x_C, e) = 0$. Then $x_C \cdot x_C = \lambda x_C + \mu e$ with $\mu = \frac{\beta(x_C,x_C)}{\beta(e,e)}$. If $\Delta \stackrel{\text{def}}{=} \lambda^2 + 4\mu \neq 0$, then D is spanned by two orthogonal idempotents f and $e - f$, and $\omega_1 \stackrel{\text{def}}{=} 2f$ and $\omega_2 \stackrel{\text{def}}{=} 2(e - f)$ are commuting Virasoro vectors with $\omega_1 + \omega_2 = \omega_C$ and central charges*

$$c_{1,2} = \frac{c_C}{2}\Big(1 \mp \frac{\lambda}{\sqrt{\Delta}}\Big). \tag{10.2}$$

Conversely, $x_C \in e^{\perp}$ spans with e a subalgebra if and only if x_C is an eigenvector of the quadratic map $q(x_C) = (x_C \cdot x_C)^{\perp}$ of $e^{\perp}$.

Proof : Since e is a unit and D is a subalgebra, $x_C \cdot x_C = \lambda x_C + \mu e$, and $\beta(x_C \cdot x_C, e) = \beta(x_C, x_C \cdot e) = \beta(x_C, x_C)$ gives μ. The element $f = ae + bx_C$ is idempotent if and only if $2a + b\lambda = 1$ and $a^2 + b^2\mu = a$, that is $b^2\Delta = 1$ and $a = \frac{1 - b\lambda}{2}$. Then $(e - f)\cdot(e - f) = e - f$ and $f \cdot (e - f) = 0$. Since $\widehat{C}_1 = 0$, a vector ω_1 of weight two with $(\omega_1)_{(1)}\omega_1 = 2\omega_1$ is a Virasoro vector of central charge $2\beta(\omega_1, \omega_1)$ [Miy96]. Put $v = 2f$ and $w = \omega_C - v$. Skew symmetry

and $\widehat{C}_1 = 0$ give $v_{(0)}v = L(-1)v$ and $v_{(0)}\omega_C = L(-1)v$, so $v_{(0)}w = 0$. The products $v_{(1)}w$, $v_{(2)}w$ and $v_{(3)}w$ vanish by the idempotent relation, the grading and orthogonality, respectively. All higher products vanish by the grading. Thus the two Virasoro fields commute. Finally $c_1 = 2\beta(2f, 2f) = 8\big(a^2\beta(e,e) + b^2\mu\beta(e,e)\big) = c_C\big(a^2 + b^2\mu\big) = c_C a$, since $\beta(e,e) = \frac{c_C}{8}$. The last assertion is a reformulation. ∎

Proposition 10.3 *The two-dimensional subalgebras of B containing e are exactly 7 in case (I) and 15 in case (II). For all of them $\Delta \neq 0$. The central charges (c_1, c_2) of (10.2) are listed in Table 3. In case (I), the subalgebras 5 and 6 give the central charges $\frac{2400}{1271}$ and $\frac{70}{31}$. In case (II), the subalgebras 9 and 10 give $\frac{182}{95}$ and $\frac{32}{19}$. Within each pair, the two subalgebras are exchanged by the automorphism $\sqrt{13} \mapsto -\sqrt{13}$ of $\mathbb{Q}(\sqrt{13})$.*

Proof : We compute the structure constants of B exactly in a basis $e, x_1, \dots, x_{n-1}$ with $x_i \perp e$, and solve $q(x) = \lambda x$ on the affine charts of $\mathbb{P}(e^\perp)$ by Gröbner bases over $\mathbb{Q}$. The solution sets are finite and explicit. By [CaS13, Theorem 1.2 p. 2] a quadratic map of $\mathbb{C}^n$ with finitely many classes of eigenvectors has exactly $2^n - 1$ of them, counted with multiplicity. Our counts $7 = 2^3 - 1$ and $15 = 2^4 - 1$ show that all of them are simple and that the lists are complete. Independently, we compute the dimensions of the coordinate algebras of the disjoint projective charts: they are $(7, 0, 0)$ in case (I) and $(11, 2, 1, 1)$ in case (II). We also check that the listed points are distinct and satisfy the equations. Their number equals the sum of the chart dimensions, which proves exhaustiveness and simplicity directly from the finite polynomial systems. ∎

Case (I), $c_C = \frac{170}{41}$	Case (II), $c_C = \frac{18}{5}$
0, 3: $\frac{85}{41} \mp \frac{31\sqrt{385}}{2255}$ (excluded)	0, 14: $\frac{6}{5}, \frac{12}{5}$ (excluded)
1: $\frac{3010}{1517}, \frac{80}{37}$ (excluded)	1, 8: $\frac{10}{7}, \frac{76}{35}$ (excluded)
2: $\frac{4}{5}, \frac{686}{205}$ (excluded)	2: $\frac{62}{35}, \frac{64}{35}$, 3: $\frac{58}{35}, \frac{68}{35}$ (excluded)
4: $\frac{252}{185}, \frac{21118}{7585}$ (excluded)	4, 5: $\frac{9}{5} \mp \frac{31\sqrt{409}}{2045}$ (excluded)
5, 6: $\frac{2400}{1271}, \frac{70}{31}$ (survive)	6: $\frac{6}{7}, \frac{96}{35}$, 7: $\frac{34}{35}, \frac{92}{35}$ (excluded)
	9, 10: $\frac{182}{95}, \frac{32}{19}$ (survive)
	11: $\frac{52}{55}, \frac{146}{55}$, 12: $\frac{96}{55}, \frac{102}{55}$, 13: $\frac{4}{5}, \frac{14}{5}$ (excluded)

Table 3: The two-dimensional subalgebras of B containing e, with the central charges of the two Virasoro vectors and their fate under Theorem 10.5 and Propositions 10.34 and 10.35. The supplementary data use the same numbering.

10.3 Linear and quadratic conditions on the primary lines

For a pair (a, b), let $\Pi_{ab} \subset V_2'$ be the space of A-primary highest weight vectors of type $V(a, b)$, that is, vectors v of joint weight (a, b) with $e'(0)v = e''(0)v = 0$ and $x(n)v = 0$ for $x \in U_1$ and $n \geqslant 1$. The primary types of U_2 are the types of Table 2 not accounted for by $(A)_2 \oplus C_2$: in case (I) the types $(0,4)$, $(0,8)$ (twice), $(0,12)$, $(1,1)$, $(1,5)$, $(1,7)$, $(1,13)$, and in case (II) the types $(0,4)$, $(1,1)$, $(1,3)$, $(1,5)$, $(1,7)$, $(2,2)$, $(2,6)$, $(3,3)$. Indeed $U = \bigoplus_\lambda L_A(\lambda) \otimes C^\lambda$, the weight-two vectors coming from $\lambda \neq 0$ are lowest weight vectors of $L_A(\lambda)$ since $U_1 = A_1$, and the descendant part is $(A)_2 \oplus C_2$. We put $p_{ab} = U_2 \cap \Pi_{ab}$, whose dimension is the multiplicity of the type. On Π_{ab} we use the bilinear pairing $P_{ab}(u, v) = \beta(u, \theta_{ab}v)$, where $\theta_{ab}v = f'(0)^a f''(0)^b v$.

Lemma 10.4 *Let $C_2 = \mathbb{C}e \oplus \mathbb{C}x_C$ and $Y = B \cap C_2^\perp$. For every primary type (a, b) and $u, u' \in p_{ab}$:*

1. $e_{(2)}u = (x_C)_{(2)}u = 0$, *that is,* u *is quasi-primary for* ω_1 *and* ω_2,

2. $(x_C)_{(1)}p_{ab} \subset p_{ab}$, *and in particular, if* p_{ab} *is a line, it is spanned by an eigenvector of* $(x_C)_{(1)}$, *and the eigenvalues of* $(\omega_1)_{(1)}$, $(\omega_2)_{(1)}$ *on it are the weights* h_1, h_2 *of* u *for the two Virasoro algebras, with* $h_1 + h_2 = 2 - h_{ab}$, *where* h_{ab} *is the Sugawara weight,*

3. $P_{ab}(u, y_{(1)}u') = 0$ *for every* $y \in Y$, *and the pairing is nondegenerate on* p_{ab},

4. *for every primary type* (c, d) *and every* w *in the* $\mathfrak{g}' \oplus \mathfrak{g}''$*-module generated by* p_{cd}, $u_{(1)}w \in U_2$ *and* $u_{(2)}w \in U_1$.

Proof : 1. The vectors $e_{(2)}u$ and $(x_C)_{(2)}u$ belong to U_1, are annihilated by the modes $z(n)$, $z \in U_1$, $n \geqslant 1$, since e and x_C commute with A, and are highest weight vectors of type $(a, b) \neq (0, 0)$. Since $U_1 = A_1$ has no nonzero A-primary vector, they vanish. 2. The operator $(x_C)_{(1)}$ preserves U_2, the A-primary vectors and the joint weights. 3. Since $y \in M_b$ and M_b is a U-module, skew symmetry gives $y_{(1)}u' \in M_b$, and the form vanishes on $U \times M_b$. Moreover θ_{ab} commutes with $y_{(1)}$. The invariant form β is contravariant for the affine algebra A, so pairings between nonisomorphic irreducible integrable A-modules vanish. In weight two, the affine-primary summands are orthogonal to the vacuum descendants $(A)_2$. Every irreducible integrable module for the two affine $\mathfrak{sl}_2$ factors is self-contragredient. Nondegeneracy on U_2 therefore gives nondegeneracy on each primary multiplicity space. The map θ_{ab} identifies its highest and lowest weight spaces up to a nonzero scalar, which proves the assertion. 4. This is the closure of U. ∎

10.4 Case (II)

Theorem 10.5 *Assume the hypotheses of Theorem 8.5 in case (II). Then* C_2 *is one of the subalgebras* 9, 10 *of Table 3, so that* ω_C *is the sum of two commuting Virasoro vectors of central charges* $\frac{182}{95}$ *and* $\frac{32}{19}$. *For each of these two subalgebras,* U_2 *is one of two explicit subspaces of* V_2', *exchanged by complex conjugation, and defined over* $\mathbb{Q}(\sqrt{13}, \sqrt{-3})$. *The weights of the two Virasoro algebras on the primary lines are*

(a,b)	$(0,4)$	$(1,1)$	$(1,3)$	$(1,5)$	$(1,7)$	$(2,2)$	$(2,6)$	$(3,3)$
h_1 $(c = \frac{182}{95})$	$\frac{112}{95}$	$\frac{111}{95}$	$\frac{147}{95}$	$\frac{7}{95}$	$\frac{56}{95}$	$\frac{56}{95}$	$\frac{56}{95}$	$\frac{7}{19}$
h_2 $(c = \frac{32}{19})$	$\frac{8}{19}$	$\frac{12}{19}$	$\frac{1}{19}$	$\frac{68}{57}$	$\frac{4}{19}$	$\frac{50}{57}$	$\frac{4}{19}$	$\frac{12}{19}$

Proof : The proof consists of four finite steps. Exact linear algebra and Gröbner bases construct the admissible spaces. Certified interval evaluations are used only for exclusions.

Step 1: exclusion of thirteen subalgebras. It suffices to use the primary type $(3, 3)$, whose multiplicity in U_2 is one. The space Π_{33} has dimension three. For each of the candidates 0–8 and 11–14, the coordinates of x_C in the fixed basis of B are rational. Let X be the matrix of $(x_C)_{(1)}$ on Π_{33}, P the matrix of P_{33}, and Y_1, Y_2, Y_3 the matrices of $y_{(1)}$ for a rational basis of $B \cap C_2^{\perp}$. All these matrices are rational. A primary line with coordinate vector $u \in \mathbb{C}^3$ must satisfy

$$\operatorname{rank}[u, Xu] = 1, \quad u^{\mathrm{T}}PY_j u = 0 \ (j = 1, 2, 3), \quad u^{\mathrm{T}}Pu \neq 0.$$

These are only some of the necessary conditions in Lemma 10.4. Their inconsistency is enough. On the projective chart whose first nonzero coordinate is u_k, put $u_0 = \cdots = u_{k-1} = 0$ and $u_k = 1$. Adjoin a variable z and the equation $zu^{\mathrm{T}}Pu - 1 = 0$, and express the rank condition by the 2 by 2 minors of $[u, Xu]$. For each of the thirteen candidates and each of the three charts, the resulting ideal over Q has Gröbner basis $\{1\}$. The supplementary data contain the matrices,

chart polynomials and all 39 unit bases. Hence none of these thirteen candidates admits the required primary line. No eigenvalue approximation is used in this step.

Step 2: the admissible spaces. We call admissible space of type (a, b) an eigenspace E_ξ of $(x_C)_{(1)}$ on $\Pi_{ab} \cap \mathrm{Ker}\, e_{(2)} \cap \mathrm{Ker}(x_C)_{(2)}$ containing a vector u with $P_{ab}(u, u) \neq 0$ and $P_{ab}(u, y_{(1)}u) = 0$ for every $y \in Y$. By Lemma 10.4, the line p_{ab} lies in an admissible space. For the subalgebra 9, every primary type has exactly one admissible space E_ξ, of dimension two, and the eigenvalues ξ give the weights of the statement. We treat all characteristic-polynomial factors, including the nonlinear ones, and exclude the remaining eigenspaces by exact equations or certified enclosures. We then construct the surviving spaces and lines. A separate verification checks both the exclusions and the eight displayed weights exactly. Nonlinear factors with multiplicity greater than one, or an eigenspace for which the adjugate test is inconclusive, are retained rather than discarded. None remains among the exclusions used here.

Step 3: the lines. In each two-dimensional E_ξ, the isotropic lines are the common roots of the binary forms $P_{ab}(u, y_{(1)}u)$, $y \in Y$. Their greatest common divisor over $\mathbb{Q}(\sqrt{13})$ has degree two, and both roots satisfy $P_{ab}(u, u) \neq 0$. This leaves $2^8 = 256$ choices of U_2. The coefficients lie in $\mathbb{Q}(\sqrt{13}, \sqrt{-3})$. Note that $\sqrt{6}\sqrt{-7 + \sqrt{13}} = \sqrt{-3}(\sqrt{13} - 1)$.

Step 4: pairwise closure. For two lines of types (a, b) and (c, d), condition 4 of Lemma 10.4 implies that $u_{(2)}w \in U_1$ and that $u_{(1)}w$ lies in the span of $(A)_2$, C_2 and the modules generated by all the admissible spaces. When one of these conditions fails, we exhibit a rational linear form vanishing on U_1, or on a rational space containing the latter span, whose value on $u_{(1)}w$ or $u_{(2)}w$ is an algebraic number enclosed by rational operations and certified radical evaluations in Arb [Joh17]. For each exclusion, the certificate records the rational annihilator, the product tested and a nonzero interval, giving 62 certified rejections. The algebraic inputs are evaluated directly, without an intermediate rounded decimal approximation. The finite coverage check uses only the 62 pairs marked as certified. The ten inconclusive pairs are not used. In the order of primary types displayed in the theorem, the two remaining choices are $(0, 1, 1, 0, 0, 1, 0, 1)$ and $(1, 0, 0, 1, 1, 0, 1, 0)$, where the bits index the two lines with the ordering specified in the supplementary data. They are complex conjugate. By the Galois symmetry $\sqrt{13} \mapsto -\sqrt{13}$, which fixes all the rational data, the same holds for the subalgebra 10. ∎

Remark 10.6 Theorem 10.5 is a computer-assisted result. Step 1 uses the exact rational equations displayed above. In Step 2, characteristic-polynomial factors and their multiplicities are retained in the eigenspace computation. A multiple nonlinear factor or an inconclusive enclosure cannot be discarded. The stored calculation for subalgebra 9 leaves exactly the displayed two-dimensional space for each type. The generators x_C of candidates 4 and 5 are rational: $\sqrt{409}$ occurs only in their normalized Virasoro vectors and is irrelevant to Step 1. Step 4 uses exact algebraic inputs in the interval calculation. Its 62 certified pair exclusions cover 254 configurations. The ten inconclusive pairs are not used. These are separate checks of nonvanishing and of finite coverage.

Reduction conventions. For a prime p used below, let $\mathfrak{p}$ be the prime of $\mathcal{O}_\mathbb{K}$ specified by the chosen images of $\sqrt{13}$ and $\sqrt{-3}$ in $\mathbb{F}_p$. Reduction is the residue map $\mathrm{red}_\mathfrak{p} \colon (\mathcal{O}_\mathbb{K})_\mathfrak{p} \to \mathbb{F}_p$. Only coefficients integral at $\mathfrak{p}$ are reduced. In particular, every rational denominator in a matrix or a generating word must be invertible modulo p. For such a matrix M, a nonzero minor of its reduction lifts to a nonzero minor, so $\mathrm{rank}_{\mathbb{F}_p}(\overline{M}) \leqslant \mathrm{rank}_\mathbb{K}(M)$. A modular rank can therefore prove a lower bound in characteristic zero. A modular membership identity alone cannot prove the corresponding identity over $\mathbb{K}$.

Proposition 10.7 *Let U_2 be one of the four subspaces of Theorem 10.5, and let $S \subset V'$ be the vertex subalgebra generated by $U_1 \oplus U_2$. The following statements hold modulo the primes $p_1 = 134217613$ and $p_2 = 133217647$, for the residue maps at the primes $\mathfrak{p}_i$ above p_i specified in the supplementary data. All dimensions, decompositions and inclusions in this proposition refer to the reduced spaces.*

1. $\dim U_2 = 120$, *and U_2 is closed in weight two: $u_{(1)}w \in U_2$ and $u_{(2)}w \in U_1$ for all $u, w \in U_2$.*

2. *The vectors $x(-3)\mathbf{1}$, $x(-2)y$, $x(-1)u$, $L(-1)u$ and $u_{(0)}w$ ($x, y \in U_1$, $u, w \in U_2$, u a highest weight vector), together with their images under $\mathfrak{g}' \oplus \mathfrak{g}''$, span a subspace denoted by S_3^{gen}, of dimension $956 = \dim U_3$, isomorphic as a $\mathfrak{g}' \oplus \mathfrak{g}''$-module to*

$$\begin{aligned} &4V(0,0) + 11V(0,2) + 4V(0,4) + 6V(0,6) + V(0,10) + 6V(1,1) + 8V(1,3) + 7V(1,5) \\ &+ 5V(1,7) + V(1,9) + 8V(2,0) + 6V(2,2) + 7V(2,4) + 4V(2,6) + 2V(2,8) + 2V(3,1) \\ &+ 5V(3,3) + 2V(3,5) + V(3,7) + V(4,0) + 2V(4,2) + V(4,6) + V(6,0). \end{aligned}$$

   Its nine joint traces $\operatorname{Tr} H'(0)^p H''(0)^q$, with p, q even and $2 \leqslant p+q \leqslant 6$, are exactly those of U_3 determined by the relations of Proposition 6.2.

3. *For every $u \in U_2$ and every w in this subspace, $u_{(2)}w \in U_2$ and $u_{(3)}w \in U_1$. For every $y \in U_1$, $y_{(1)}w \in U_2$ and $y_{(2)}w \in U_1$.*

Proof : Exact computations in $\mathbb{F}_{p_i}$: ranks of matrices over $\mathbb{F}_{p_i}$, the closure under the zero modes of e', f', e'', f'' being reached after three iterations. In part 3 we test $u = e$, $u = x_C$, the highest weight vectors of the eight primary types and the elements y of a basis of U_1 (modulo p_1). The other elements of U_2 follow by $\mathfrak{g}' \oplus \mathfrak{g}''$-equivariance, the subspace being stable under the zero modes, and, for $(A)_2$, by expanding the modes of $x(-1)y$ and $x(-2)\mathbf{1}$ as normally ordered products of modes of currents. ■

Remark 10.8 The modular ranks give lower bounds for the characteristic-zero ranks under the reduction conventions above. The membership tests and the reduced module decomposition are statements over the residue fields. Theorems 10.9 and 10.19 establish the corresponding characteristic-zero closure and dimensions. Proposition 10.7 by itself does not prove that S is a Haagerup algebra.

10.5 Case (II): exact weight-two closure, unitarity and symmetries

From now on, and until the end of Section 10.7, $U_1 = \mathfrak{g}' \oplus \mathfrak{g}''$ is the explicit Lie algebra of currents of case (II) fixed in Section 10 (its raising currents are displayed in (10.5)), and U_2 denotes one of the four explicit subspaces of V_2' spanned by $(A)_2$, by the subalgebra 9 or 10 of Table 3 and by the $\mathfrak{g}' \oplus \mathfrak{g}''$-modules generated by the eight lines selected in Steps 2–4 of the proof of Theorem 10.5. This construction is a finite computation in V'. Except for Corollaries 10.22 and 10.25 and Theorem 10.29, the results of Sections 10.5–10.7 use neither the existence of a Haagerup algebra nor ($\mathrm{H_J}$). In particular, Theorems 10.9, 10.10, 10.21 and 10.24, which are used in [ArhIII], do not depend on [ArhIII].

Let $\mathbb{K} = \mathbb{Q}(\sqrt{13}, \sqrt{-3})$. Let $V_{\mathbb{K}}'$ be the $\mathbb{K}$-span of the fixed monomial basis, and write $U_{j,\mathbb{K}} = U_j \cap V_{\mathbb{K}}'$ for the forms specified by the rational currents and the displayed primary generators. Statements about ranks, bases and closure over $\mathbb{K}$ refer to these forms. Their extensions of scalars to $\mathbb{C}$ recover the corresponding complex spaces. We suppress the subscript

$\mathbb{K}$ in the finite computations. The four subspaces of Theorem 10.5 are spanned by vectors of V_2' whose coordinates in the monomial basis lie in $\mathbb{K}$. We write $c \mapsto \overline{c}$ for the complex conjugation of $\mathbb{K}$ ($\sqrt{-3} \mapsto -\sqrt{-3}$), $v \mapsto \overline{v}$ for the antilinear map of V' which conjugates the coordinates in the monomial basis, and σ for the automorphism $\sqrt{13} \mapsto -\sqrt{13}$ of $\mathbb{K}$ fixing $\sqrt{-3}$.

Theorem 10.9 *Each of the four subspaces U_2 of Theorem 10.5 has dimension* 120 *and is closed in weight two over $\mathbb{K}$: for all $u, w \in U_2$ we have $u_{(1)}w \in U_2$ and $u_{(2)}w \in U_1$.*

Proof : We work in exact arithmetic in $\mathbb{K}$, an element being a quadruple of rationals in the basis $1, \sqrt{13}, \sqrt{-3}, \sqrt{-39}$. Through the restriction of scalars $\mathbb{K} \cong \mathbb{Q}^4$, every $\mathbb{K}$-subspace becomes a $\mathbb{Q}$-subspace of four times its dimension, and membership is decided by ranks of rational matrices, computed weight space by weight space for the Cartan subalgebra of $\mathfrak{g}' \oplus \mathfrak{g}''$. The rank of U_2 is $480 = 4 \cdot 120$. We test the 681 nonzero products $u_{(n)}w$, $n \in \{1, 2\}$, where u runs over e, x_C and the highest weight vectors of the eight primary types, and w over e, x_C and the eight modules: none increases the rank. The general case follows: the zero modes of $\mathfrak{g}' \oplus \mathfrak{g}''$ satisfy $[y_{(0)}, u_{(n)}] = (y_{(0)}u)_{(n)}$ and preserve U_1 and U_2, which transports the statement from highest weight vectors to the whole modules. For $z \in U_1$, the construction gives $z_{(n)}U_2 \subset U_{2-n}$ for $n \geqslant 0$, with $U_0 = \mathbb{C}\mathbf{1}$ and $U_j = 0$ for $j < 0$. This follows separately on the affine descendants and on the affine-primary summands. The normal-order formula then gives $((A)_2)_{(1)}U_2 \subset U_2$ and $((A)_2)_{(2)}U_2 \subset U_1$: the negative-mode terms in the first inclusion belong to $U_1(-1)U_1 + U_1(-2)U_0 \subset (A)_2$. Finally $(x_{(-2)}\mathbf{1})_{(n)} = -nx_{(n-1)}$, and skew symmetry exchanges the two arguments. We have done this for the two choices attached to the subalgebra 9. The other two are their images under σ, which acts on coefficients and commutes with all the rational structure constants of V'. ∎

This proves part 1 of Proposition 10.7 over $\mathbb{K}$.

We now show that the algebra generated is unitary, in the sense of Dong and Lin [DoL14] and Carpi, Kawahigashi, Longo and Weiner [CKLW18, Section 5]. Recall that V' has a rational form, spanned by the states $h_1(-n_1)\cdots h_k(-n_k)e^{\lambda}$ with h_i in the root lattice, since our cocycle takes the values ± 1. For a character $\chi\colon L \to \mathbb{R}_{>0}$ of the root lattice $L = E_6 \oplus A_2$, let ϕ_χ be the antilinear map defined on this basis by

$$\phi_\chi\big(ch_1(-n_1)\cdots h_k(-n_k)e^{\lambda}\big) = (-1)^k\chi(\lambda)\overline{c}h_1(-n_1)\cdots h_k(-n_k)e^{-\lambda}, \quad c \in \mathbb{C}. \tag{10.3}$$

Theorem 10.10 *Let χ be the character taking the values $\frac{2}{15}, 1, 1, \frac{5}{3}, \frac{9}{25}, \frac{5}{24}$ on the simple roots $\alpha_1, \ldots, \alpha_6$ of E_6 (Bourbaki numbering) and $2, 2$ on the simple roots of A_2, in the simple-root coordinates fixed above.*

1. *ϕ_χ is an antilinear involutive automorphism of V', and $H_\chi(u, v) = \beta(u, \phi_\chi(v))$ is a positive definite invariant Hermitian form: (V', ϕ_χ) is a unitary vertex operator algebra.*
2. *$\phi_\chi(e') = -f'$, $\phi_\chi(f') = -e'$, $\phi_\chi(H') = -H'$, and similarly for $\mathfrak{g}''$. In particular $\phi_\chi(U_1) = U_1$.*
3. *$\phi_\chi(U_2) = U_2$ for each of the four subspaces of Theorem 10.5.*

Consequently, the vertex subalgebra $S \subset V'$ generated by $U_1 \oplus U_2$ is a unitary vertex operator algebra of central charge 8. *It is simple, of CFT type and self-contragredient ($S \cong S'$).*

Proof : 1. Let ϕ_1 be the map (10.3) for the trivial character. With a bimultiplicative cocycle we have $\varepsilon(-\alpha, -\beta) = \varepsilon(\alpha, \beta)$, and ϕ_1 is the antilinear involution for which lattice vertex operator algebras are unitary [DoL14, Theorem 4.12], [CKLW18, Example 5.9]. We have checked our

sign conventions directly: $\phi_1(u_{(n)}v) = \phi_1(u)_{(n)}\phi_1(v)$ for 5050 products of basis states of weights at most two, and the Gram matrix of $H_1(u,v) = \beta(u,\phi_1(v))$ is positive definite on each lattice sector of V_1', V_2' and V_3'. Write $\chi(\lambda) = e^{\langle h_\chi,\lambda\rangle_L}$ with h_χ in the real Cartan subalgebra, and $g = \exp(h_\chi(0))$, an automorphism of V' acting by $\chi(\lambda)$ on $M(1)\otimes e^\lambda$. Then $\phi_\chi = \phi_1 \circ g = g^{-1/2}\circ\phi_1\circ g^{1/2}$, so that (V',ϕ_χ) is the transport of (V',ϕ_1) by the automorphism $g^{-1/2}$, with Hermitian form $H_\chi(u,v) = H_1(g^{1/2}u, g^{1/2}v)$.

2. Direct computation. The character χ was found by solving the ten equations $\chi(\alpha) = -\kappa d_\alpha/c_\alpha$ with $\kappa > 0$ (κ' for the roots of $\mathfrak{g}'$, κ'' for those of $\mathfrak{g}''$), where c_α and d_α are the coefficients of e^α in e' (or e'') and of $e^{-\alpha}$ in f' (or f''). All the ratios d_α/c_α are negative, so that the right-hand sides are positive, and the solutions form a two-parameter family. The statement holds for the member displayed.

3. Exact computation over $\mathbb{K}$: for each of the 120 spanning vectors v of U_2 constructed in the proof of Theorem 10.9, the restriction of scalars of $\phi_\chi(v)$ lies in that of U_2. This is done for the two choices attached to the subalgebra 9. Since χ is rational, ϕ_χ commutes with σ, and the statement for the subalgebra 10 follows.

Finally, S is stable under ϕ_χ, and contains the conformal vector $\omega = \omega_A + 2e$ of V', so that $L(1)S \subset S$. By [CKLW18, Proposition 5.23], S is a unitary subalgebra of (V',ϕ_χ). Since $S_0 = \mathbb{C}\mathbf{1}$, it is simple by [CKLW18, Proposition 5.3]. The bilinear form β is nondegenerate on S, because $\beta(s,\phi_\chi(s)) > 0$ for $s \neq 0$ and $\phi_\chi(s) \in S$. Hence $S \cong S'$ [FHL93]. ■

Remark 10.11 Theorem 10.10 does not use any information on S beyond its generators: it holds whatever the dimensions of the homogeneous components of S are. It is the analogue of [Rie26b, Section 3] for case (I). The trivial character does not work: $\phi_1(e')$ is not proportional to f', and a positive rescaling of the lattice exponentials is needed. All four subspaces give unitary algebras. In particular, the sign of $\sqrt{13}$ in the choice of C_2 (subalgebras 9 and 10) has nothing to do with the sign of the quantum dimension $d = \frac{3\pm\sqrt{13}}{2}$: S and $\sigma(S)$ have the same graded dimensions, and the non-unitary Galois conjugates of $Z(\mathcal{H}_3)$ [EvG17] cannot be the representation category of a strongly rational algebra of CFT type whose nonvacuum simple modules have positive lowest weights, since its categorical dimensions are then positive [DJX13, Lemma 4.2], [Hua08].

Proposition 10.12 *1. The two subspaces U_2 attached to the subalgebra 9 are exchanged by complex conjugation, and by the linear automorphism $\theta_\chi\colon v \mapsto \phi_\chi(\overline{v})$ of V', which normalizes U_1. Hence they generate isomorphic vertex operator algebras. The same holds for the subalgebra 10.*

2. The group of automorphisms of the algebra B fixing e and preserving the form has order two. Its nontrivial element is an orthogonal reflection s. It exchanges the subalgebras 9 and 10 of Table 3, as well as 0 and 14, 1 and 8, 4 and 5, and fixes the seven others.

Consequently the four subspaces give at most two isomorphism classes of vertex operator algebras, and exactly one if s is induced by an automorphism of V' normalizing U_1. This is the case by Proposition 10.27.

Proof : 1. Exact computation: the conjugate of each spanning vector of the first subspace lies in the second. Since χ is rational, ϕ_χ commutes with coefficient conjugation. For either subspace W, Theorem 10.10 gives $\theta_\chi(W) = \phi_\chi(\overline{W}) = \overline{\phi_\chi(W)} = \overline{W}$.

2. An automorphism g of B fixing e permutes the fifteen two-dimensional subalgebras of Proposition 10.3 and preserves their central charges. The pairs of central charges of the subalgebras 2, 3, 6, 7, 11, 12, 13 occur only once, so g maps each corresponding line $\mathbb{C}x_C$ to itself, and $g(x_C) = x_C$ since $g(x_C\cdot x_C) = g(x_C)\cdot g(x_C)$ and $\lambda \neq 0$ since $c_1 \neq c_2$ by (10.2).

These seven vectors span the hyperplane $H = \mathrm{span}\{x_1, x_2, x_3\}$ of $e^\perp$ in the fixed basis. The restriction of β to H is nondegenerate: its Gram determinant is $\frac{147968}{2025}$, by direct evaluation. The form on B is nondegenerate because B is stable under ϕ_χ and H_χ is positive definite. Thus $N = H^\perp \cap e^\perp$ is one-dimensional and $e^\perp = H \oplus N$. An orthogonal map fixing H pointwise acts by 1 or -1 on N. Therefore g is the identity or the orthogonal reflection s in H. A direct computation shows that s is an automorphism, with the stated action. An automorphism of V' normalizing U_1 preserves B, e, the product and the form. ∎

Proposition 10.27 constructs an automorphism of V' fixing U_1 whose restriction to B is s.

Proposition 10.13 *Let U_2 be the first subspace attached to the subalgebra 9. The following statements hold modulo the prime $p_3 = 8388571$, for the residue map at the prime above p_3 specified in the supplementary data.*

1. *The space S_3^{gen} of Proposition 10.7 has dimension 956 and $u_{(1)}S_3^{\mathrm{gen}} \subset S_3^{\mathrm{gen}}$ for all $u \in U_2$. Together with Proposition 10.7, this gives the finite mode closure $y_{(n)}S_i \subset S_{i-n}$ $(n \geqslant 0)$ and $u_{(n)}S_i \subset S_{i+1-n}$ $(n \geqslant 1)$ for $y \in U_1$, $u \in U_2$, $i \in \{2,3\}$, where $S_2 = U_2$ and $S_3 = S_3^{\mathrm{gen}}$. This is the analogue of [Rie26b, Proposition 4.1].*

2. *Let $R_b \subset V_1'$ be the orthogonal of U_1, of dimension 80, and M the subspace generated from R_b by the modes of U_1, $L(-1)$ and the modes $u_{(0)}$, $u_{(-1)}$ $(u \in U_2)$, as in part 2 of Proposition 10.7. Then $\dim M_2 = 1250$, $\dim M_3 = 10630$, and $U_2 + M_2 = V_2'$, $S_3^{\mathrm{gen}} + M_3 = V_3'$. These are the dimensions of the module M_b of Section 4, and the sums are direct by dimension count.*

3. *The space S_4^{gen} spanned by $y(-1)S_3$, $L(-1)S_3$, $y(-2)U_2$, $y(-3)U_1$, $u_{(-1)}U_2$, $u_{(-2)}U_1$ and $u_{(0)}S_3$ $(y \in U_1,\ u = e,\ u = x_C$ or a highest weight vector of one of the eight types), closed under $\mathfrak{g}' \oplus \mathfrak{g}''$, has dimension 6010, the coefficient of $q^{11/3}$ in the Haagerup vacuum character, and decomposes as*

$$\begin{aligned}
&24V(0,0) + 27V(0,2) + 35V(0,4) + 18V(0,6) + 11V(0,8) + 3V(0,10) + V(0,12)\\
&+ 28V(1,1) + 43V(1,3) + 37V(1,5) + 24V(1,7) + 9V(1,9) + 2V(1,11) + 17V(2,0)\\
&+ 39V(2,2) + 31V(2,4) + 28V(2,6) + 10V(2,8) + 4V(2,10) + 15V(3,1) + 23V(3,3)\\
&+ 16V(3,5) + 10V(3,7) + 2V(3,9) + V(3,13) + 9V(4,0) + 7V(4,2) + 7V(4,4)\\
&+ 5V(4,6) + 2V(4,8) + V(5,1) + 2V(5,3) + V(5,5) + V(5,7) + V(6,0)\\
&+ V(6,2).
\end{aligned}$$

   In particular it contains the $\mathfrak{g}' \oplus \mathfrak{g}''$-module $V(3,13)$ once.

4. *For $y \in U_1$, $y_{(1)}S_4^{\mathrm{gen}} \subset S_3^{\mathrm{gen}}$ and $y_{(2)}S_4^{\mathrm{gen}} \subset U_2$. For u the highest weight vector of the types $(0,4)$, $(1,1)$, $(1,3)$, $(1,5)$, $(1,7)$, $(2,2)$, $(2,6)$, $(3,3)$ and for $u = e$, $u = x_C$, we have $u_{(1)}S_4^{\mathrm{gen}} \subset S_4^{\mathrm{gen}}$, $u_{(2)}S_4^{\mathrm{gen}} \subset S_3^{\mathrm{gen}}$ and $u_{(3)}S_4^{\mathrm{gen}} \subset U_2$.*

5. *The subspace N_4' spanned by $L(-1)S_3$, $y(-1)S_3$, $y(-2)U_2$, $y(-3)U_1$ and $u_{(-1)}U_2$, for u in the eight primary modules, closed under $\mathfrak{g}' \oplus \mathfrak{g}''$, has dimension 6010 and contains $e(-1)e$, $x_C(-1)x_C$ and $e(-1)x_C$. The corresponding characteristic-zero consequence for the reduced C_2 Poisson algebra is proved in Theorem 10.21.*

Remark 10.14 The ranks modulo p_3 are lower bounds for the ranks over $\mathbb{K}$. The membership statements are statements in characteristic p_3 and are not proofs over $\mathbb{K}$. Part 2 is a modular check of the decomposition $V' = S \oplus M_b$ required by Proposition 4.1: by unitarity, $V' =$

$S \oplus S^{\perp}$, and the space M is orthogonal to S as soon as $S_1 = U_1$, since $H_\chi(a_{(n)}r, s)$ is the Hermitian pairing of $r \in R_b$ with a vector of S_1. The occurrence over $\mathbb{K}$ of the simple current $L(3,3)\otimes L(13,13)$, whose affine conformal weight is $\frac{3}{4}+\frac{13}{4}=4$, is proved directly in Proposition 11.4: an explicit weight-four vector is shown to be affine-primary. Thus no inference from the modular module decomposition to characteristic-zero occurrence or affine primarity is needed.

10.6 Case (II): strong generation and C_2-cofiniteness

In this subsection U_2 is the first subspace attached to the subalgebra 9. By Proposition 10.12 and the Galois symmetry σ, the results hold for the four subspaces. We write $(A)_2 \subset U_2$ for the span of the vectors $x(-1)y$ and $x(-2)\mathbf{1}$ ($x, y \in U_1$), of dimension 27, and we put

$$N_3 = L(-1)U_2 + U_1(-1)U_2 \subset V_3', \quad S_3 = N_3 + (U_2)_{(0)}U_2,$$
$$N_4' = L(-1)S_3 + U_1(-1)S_3 + U_1(-2)U_2 + U_1(-3)U_1 + \textstyle\sum_\tau M_\tau(-1)U_2 \subset V_4',$$

where $M_\tau \subset U_2$ is the $\mathfrak{g}' \oplus \mathfrak{g}''$-module generated by the primary line of type τ, and $Z(n)W$ denotes the span of the $z_{(n)}w$. These spaces are defined over $\mathbb{K}$, stable under the zero modes of $\mathfrak{g}' \oplus \mathfrak{g}''$, and contained in S. Note that $x(-2)y$ and $x(-3)\mathbf{1}$ lie in N_3.

Lemma 10.15 *For each of the four choices of U_2, we have* $\dim N_3 = 725$.

Proof : We use the positive Hermitian form of Theorem 10.10. By part 2 of that theorem, its restriction to the vertex subalgebra generated by U_1 is a positive definite contravariant form for the affine Lie algebra, so that this vacuum module is irreducible, that is, $A \cong L_{\mathfrak{sl}_2}(3,0)\otimes L_{\mathfrak{sl}_2}(13,0)$. We use the decomposition of S into integrable A-modules [DLM97, Theorem 3.7 p. 7]. The regularity of A identifies the module generated by each nontrivial affine-primary summand with the corresponding simple integrable module. Write $L_A(-1)$ and $L_C(-1)$ for the translation operators of the affine and commutant Virasoro algebras, so that $L(-1) = L_A(-1) + L_C(-1)$. The affine vacuum module has no integrability relation before degree four. Its degree-three subspace therefore has dimension $6+6^2+\binom{8}{3} = 98$, by ordering the six currents according to their negative modes.

The two vectors of C_2 contribute $6 \cdot 2$ current descendants and two independent vectors $L_C(-1)C_2$. The eight nontrivial affine-primary summands have total dimension $5+4+8+12+16+9+21+16 = 91$. For an affine module of highest weight (a,b), its degree-one piece is the quotient of $(\mathfrak{g}' \oplus \mathfrak{g}'') \otimes V(a,b)$ by the integrability relations in that degree. Among these eight types, only $(3,3)$ has such a relation: the highest vector $e'(-1)v$ generates the submodule $V(5,3)$, of dimension 24. Indeed the first integrability relations in the two affine factors occur in degrees $4-a$ and $14-b$, respectively. Thus the current descendants of the eight primary summands have total dimension $6 \cdot 91 - 24$.

The Sugawara formula gives $L_A(-1)M_\tau \subset U_1(-1)M_\tau$. Thus the remaining derivatives are the vectors $L_C(-1)v$, with $v \in M_\tau$. The vector v has commutant weight $h_C = 2 - h_{ab} > 0$. Since the commutant Virasoro modes satisfy $(L_C(n))^* = L_C(-n)$, we have

$$\|L_C(-1)v\|^2_{H_\chi} = \|L_C(1)v\|^2_{H_\chi} + 2h_C\|v\|^2_{H_\chi}.$$

Consequently $L_C(-1)$ is injective on each such summand and on C_2, where $h_C = 2$. Its images are independent of the current descendants: within an affine isotypic component they have different $L_A(0)$-eigenvalues. Distinct affine highest weights give independent isotypic components. In the vacuum component the contributions from A_3, $U_1(-1)C_2$ and $L_C(-1)C_2$ have commutant weights 0, 2 and 3, respectively. These account for all terms of N_3. Hence $\dim N_3 = 98 + 14 + (6 \cdot 91 - 24) + 91 = 725$. ∎

Definition 10.16 (weight-four certificate condition) *Fix the specified bases of U_1, U_2 and S_3, and the ten vectors consisting of e, x_C and one highest weight vector of each primary type. Condition* (F4) *asserts that there is a family $\mathcal{B}_4$ of* 6010 *words from the five families defining N_4' with the following properties.*

1. *the vectors in $\mathcal{B}_4$ are integral at the chosen prime above p_3, and their reduced coordinate matrix has rank* 6010*,*
2. *every $u_{(0)}s$, for the ten specified u and all basis vectors s of S_3, and every $e_{(-1)}w$ and $(x_C)_{(-1)}w$, for basis vectors w of U_2, belongs to the $\mathbb{K}$-span of $\mathcal{B}_4$,*
3. *every word of the five defining families of N_4', with each argument ranging over the full specified basis, belongs to that span.*

The membership assertions mean the exact coordinate identities $Bc = v$ on all rows after restriction of scalars. They include zero products and concern the complete list of input words.

Proposition 10.17 *Condition* (F4) *holds for the first subspace attached to the subalgebra* 9 *of Table 3. The characteristic-zero identities and dimensions are transported to the other three subspaces by coefficient conjugation and the Galois automorphism σ.*

Proof : We use the exact lattice model over $\mathbb{K} = \mathbb{Q}(\sqrt{13}, \sqrt{-3})$. The weight-three basis is regenerated from U_1 and U_2 and checked to agree exactly with the basis used below. Its certification, also described in parts 1 and 2 of Theorem 10.19, does not use any weight-four assertion. The verification of weight four is independent of any identification with an externally constructed algebra.

All mode evaluations below have homogeneous inputs and outputs of weights at most four, so the weight-window argument in Section 10 applies. There are 956 basis vectors in S_3 and 120 in U_2, and the eight primary modules have total dimension 91. The five defining families of N_4' therefore have

$$956 + 6 \cdot 956 + 6 \cdot 120 + 6^2 + 91 \cdot 120 = 18368$$

labelled input words. All of them are evaluated from the lattice mode formulas, including the zero products. Among the nonzero values, the calculation selects 6010 words with independent reductions at the chosen prime above $p_3 = 8388571$, with $\sqrt{13} \mapsto 1649360$ and $\sqrt{-3} \mapsto 2036966$. Every denominator inverted in the reduction is checked to be nonzero modulo p_3. These labelled words form $\mathcal{B}_4$.

The remaining membership assertions have $10 \cdot 956 + 2 \cdot 120 = 9800$ labelled inputs. They too are all evaluated. The verifier checks the complete input domains, the equality of each pivot with its labelled family word, and every coordinate of each membership identity. In a Cartan block with r pivots, restriction of scalars gives $4r$ rational columns. A nonzero $4r \times 4r$ minor is selected modulo p_3, the corresponding square system is solved over Q, and the resulting equality is checked on every row in exact integer arithmetic. Thus every tested vector lies in the $\mathbb{K}$-span of the pivots.

The complete verification covers all 131 Cartan blocks carrying nonzero input vectors. The families contain 18194 nonzero words and 174 zero words, and the additional products contain 9088 nonzero values and 712 zero values. All 886 zero inputs are independently checked from the mode formulas or from the Cartan energy bound in the lattice. The complete verification reports no failure, and the exact coordinate data and verification results are included in the supplementary material. Consequently all three requirements in Definition 10.16 hold.

Finally, coefficient conjugation and σ commute with the rational lattice products and fix the current algebra. They transport U_2, the five defining families, the exact coordinate identities and linear independence over $\mathbb{K}$ to the other three choices. For the reduction statement one transports the chosen prime as well as the bases, rather than imposing the same residue values on the conjugate coefficients. The resulting characteristic-zero statements therefore hold for all four choices. This argument does not use Theorem 10.24. ∎

Remark 10.18 The exact membership assertions in (F4) admit a representation-theoretic reduction. The values of $H' + H''$ on the eight simple roots are $(0, -2, 2, 0, 2, 0, 2, 2)$. Thus every Cartan weight (a, b) of V' satisfies $a \equiv b \pmod 2$. Each irreducible $\mathfrak{g}' \oplus \mathfrak{g}''$-module in V_4' therefore contains either weight $(0, 0)$ or weight $(1, 1)$. For two stable subspaces $W, X \subset V_4'$, complete reducibility of $(W + X)/W$ gives

$$X \subset W \quad \Longleftrightarrow \quad X_{(0,0)} \subset W_{(0,0)} \text{ and } X_{(1,1)} \subset W_{(1,1)}.$$

This applies to $W = N_4'$ and $X = (U_2)_{(0)}S_3 + (U_2)_{(-1)}U_2$, since zero modes of currents act as derivations and all their argument spaces are stable. Here the subscripts denote Cartan weight spaces. Applying this criterion requires the full spaces W and X, or generators of them with their full current orbits. Restricting the ten highest weight inputs alone to these two blocks would not suffice. This reduction is not used to replace the membership calculations reported here, and it does not establish the dimension or integrality assertions of (F4).

Theorem 10.19 *The following statements hold over* $\mathbb{K}$.

1. $\dim S_3 = 956$, *and* S_3 *is stable under the zero modes of* $\mathfrak{g}' \oplus \mathfrak{g}''$.
2. $L(1)S_3 \subset U_2$, $L(2)S_3 \subset U_1$, *and* $y_{(1)}S_3 \subset U_2$, $y_{(2)}S_3 \subset U_1$ *for* $y \in U_1$.
3. $u_{(0)}S_3 \subset N_4'$ *for all* $u \in U_2$, *and* $e(-1)U_2 + x_C(-1)U_2 \subset N_4'$. *Consequently* $(U_2)_{(0)}S_3 + (U_2)(-1)U_2 \subset N_4'$.
4. $\dim N_4' = 6010$. *Hence* $N_4' = L(-1)S_3 + U_1(-1)S_3 + U_1(-2)U_2 + U_1(-3)U_1 + (U_2)(-1)U_2 + (U_2)_{(0)}S_3$ *has dimension* 6010.

Proof : All vectors are represented as $\mathbb{K}$-linear combinations of monomial states using the lattice formulas of Section 10. Membership of a vector v in the span of vectors $b_1, \dots, b_r$ is decided in the weight space of v for the Cartan subalgebra of $\mathfrak{g}' \oplus \mathfrak{g}''$, through the restriction of scalars $\mathbb{K} \cong \mathbb{Q}^4$: the vectors $b_i, \sqrt{13}b_i, \sqrt{-3}b_i, \sqrt{-39}b_i$ give $4r$ rational columns. A set of $4r$ rows on which they are independent modulo $p_3 = 8388571$ is found, so that the corresponding square minor is nonzero over $\mathbb{Q}$. The square system is solved exactly over $\mathbb{Q}$, and the solution is checked on *all* rows in exact integer arithmetic. A vector passes only if the identity $Bc = v$ holds exactly.

1. We choose 956 vectors among the generating words $x(-1)w$, $L(-1)w$, $x(-2)y$, $x(-3)\mathbf{1}$ (x, y in a basis of U_1, w in a basis of U_2), $u_{(0)}w$ ($u = e$, x_C or a highest weight vector of one of the eight types) and their images under the zero modes $e'_{(0)}, f'_{(0)}, e''_{(0)}, f''_{(0)}$, independent modulo p_3, hence over $\mathbb{K}$. All the 1871 nonzero generating words and the 2862 nonzero images of the chosen vectors under these four zero modes lie in their span, exactly. Hence the span Σ of the chosen vectors is stable under $\mathfrak{g}' \oplus \mathfrak{g}''$ and contains N_3 and $u_{(0)}U_2$ for the ten vectors u above. The zero mode identity $[y_{(0)}, u_{(0)}] = (y_{(0)}u)_{(0)}$ then gives $M_{\tau(0)}U_2 \subset \Sigma$ for each type. For $u \in (A)_2$, the modes $u_{(0)}$ are normally ordered products of modes of currents, and $u_{(0)}U_2 \subset U_1(-1)U_2 + U_1(-2)U_1 + L(-1)U_2 \subset N_3$. Thus $\Sigma = S_3$.

2. These inclusions also follow without a matrix computation. Put $T = L(-1)$. Theorem 10.9, applied to the conformal vector $\omega \in U_2$, gives $L(1)U_2 \subset U_1$, while $L(2)U_2 \subset \mathbb{C}\mathbf{1}$ by the grading. The Virasoro relation $[L(j), T] = (j+1)L(j-1)$, the current relations and the translation identity give the required inclusions on $N_3 = TU_2 + U_1(-1)U_2$. More explicitly, for $y, z \in U_1$ and $u \in U_2$, use $[L(j), y_{(-1)}] = y_{(j-1)}$, $[z_{(j)}, T] = j z_{(j-1)}$ and $[z_{(j)}, y_{(-1)}] = [z, y]_{(j-1)} + \delta_{j,1}\kappa(z, y)\,\mathrm{Id}$, for $j = 1, 2$, where $z_{(1)}y = \kappa(z, y)\mathbf{1}$.

For $u, v \in U_2$, the Virasoro commutator formula gives

$$L(1)(u_{(0)}v) = 2u_{(1)}v + (L(1)u)_{(0)}v + u_{(0)}L(1)v,$$
$$L(2)(u_{(0)}v) = 3u_{(2)}v + 3(L(1)u)_{(1)}v.$$

The terms involving $L(2)u$ or $L(2)v$ vanish because the zero mode of the vacuum is zero and $u_{(0)}\mathbf{1} = 0$. Similarly, for $y \in U_1$,

$$y_{(1)}(u_{(0)}v) = (y_{(0)}u)_{(1)}v + (y_{(1)}u)_{(0)}v + u_{(0)}y_{(1)}v,$$
$$y_{(2)}(u_{(0)}v) = (y_{(0)}u)_{(2)}v + 2(y_{(1)}u)_{(1)}v.$$

All the terms belong to the spaces asserted in part 2, by weight-two closure and skew symmetry. This proves the inclusions on $S_3 = N_3 + (U_2)_{(0)}U_2$. Independent matrix checks on the 956 basis vectors give the same inclusions: they record 772, 60, 2116 and 129 nonzero tests, respectively, with no failure.

3. For the ten vectors u of part 1 and the 956 basis vectors s of S_3, and for $u \in \{e, x_C\}$ and w in a basis of U_2, the vectors $u_{(0)}s$ and $u(-1)w$ lie in the span of 6010 words of N_4', chosen independent modulo p_3, by Proposition 10.17. Its complete verification covers 9088 nonzero tests and 712 zero tests, with no failure. The extension to all $u \in U_2$ follows as in part 1: N_4' is stable under $\mathfrak{g}' \oplus \mathfrak{g}''$, since it is spanned by stable families, S_3 is stable by part 1, and for $u \in (A)_2$ the modes $u_{(0)}$, $u_{(-1)}$ are normally ordered products of modes of currents, which map S_3 and U_2 into N_4' by part 2. Finally $M_\tau(-1)U_2 \subset N_4'$ by definition.

4. Proposition 10.17 gives membership for all 18368 words of the five defining families and independence of the 6010 pivots, hence $\dim N_4' = 6010$. ∎

The next lemma gives the passage from these finite identities to all conformal weights. It is a variant of [Rie26b, Lemma 4.3], where $N(G)$ is the span of all parenthesized normal products. With right-nested monomials the proof has to be organized differently, and we give it in full. We know of no published statement containing hypothesis 2. The case $A = G$ is classical.

Lemma 10.20 (triangular strong generation) *Let $W = \bigoplus_{n \geqslant 0} W_n$ be a vertex algebra with a grading such that $\mathrm{wt}(a_{(n)}b) = \mathrm{wt}\,a + \mathrm{wt}\,b - n - 1$ and $W_0 = \mathbb{C}\mathbf{1}$. Let $A \subset G$ be finite sets of homogeneous vectors of positive weight. Let $N(G)$ be the span of $\mathbf{1}$ and of the vectors $g^1_{(-n_1)} \cdots g^k_{(-n_k)}\mathbf{1}$ with $g^i \in G$ and $n_i \geqslant 1$. Assume:*

1. *$a_{(n)}g \in N(G)$ for all $a \in A$, $g \in G$ and $n \geqslant 0$,*
2. *every $g \in G \setminus A$ of weight d is the sum of an element of $N(G_{<d})$ and of vectors $a_{(0)}b$ with $a \in A$, $\mathrm{wt}\,a \geqslant 2$, and $b \in N(G_{<d})$ of weight $d - \mathrm{wt}\,a + 1$, where $G_{<d}$ is the set of elements of G of weight less than d.*

Then $N(G)$ is a vertex subalgebra of W, strongly generated by G.

Proof : All vectors are homogeneous, and a monomial is a word $g^1_{(-n_1)} \cdots g^k_{(-n_k)}\mathbf{1}$ as above, of length k. For $T \geqslant 0$ consider the statements

$P(T)$: $g_{(n)}w \in N(G)$ for all $g \in G$, $n \geqslant 0$ and $w \in N(G)$ with $\operatorname{wt} g + \operatorname{wt} w \leqslant T$,

$Q(T)$: $z_{(s)}w \in N(G)$ for all $z, w \in N(G)$ and $s \in \mathbb{Z}$ with $\operatorname{wt} z + \operatorname{wt} w \leqslant T$,

and $P_{<d}(T)$, $Q_{<d}(T)$ the same statements with $g \in G_{<d}$, respectively $z \in N(G_{<d})$.

Step 1: if $P(T)$ holds for the elements of $G' \in \{G, G_{<d}\}$, then $Q(T)$ holds for all $z \in N(G')$. By induction on the length of z, the case $z = \mathbf{1}$ being trivial. If $z = k_{(-p)}z'$, the iterate formula gives

$$z_{(s)}v = \sum_{i \geqslant 0} \binom{p+i-1}{i} \Big(k_{(-p-i)}\big(z'_{(s+i)}v\big) - (-1)^p z'_{(s-p-i)}\big(k_{(i)}v\big) \Big),$$

a finite sum. Here $z'_{(s+i)}v \in N(G)$ since z' is shorter, and $k_{(-p-i)}$ preserves $N(G)$ by definition. Moreover $k_{(i)}v \in N(G)$ by $P(T)$, and $z'_{(s-p-i)}(k_{(i)}v) \in N(G)$ since z' is shorter and $\operatorname{wt} z' + \operatorname{wt}(k_{(i)}v) = \operatorname{wt} z + \operatorname{wt} v - p - i \leqslant T$.

Step 2: $Q(T-1)$ implies $P(T)$ for $g \in A$. Write $w = h_{(-m)}w'$. By the commutator formula,

$$g_{(n)}h_{(-m)}w' = h_{(-m)}g_{(n)}w' + \sum_{j \geqslant 0} \binom{n}{j} (g_{(j)}h)_{(n-m-j)}w'.$$

The first term lies in $N(G)$ by $P(T-1)$, which follows from $Q(T-1)$. In the second, $z = g_{(j)}h \in N(G)$ by hypothesis 1, and $\operatorname{wt} z + \operatorname{wt} w' \leqslant T - m - j \leqslant T - 1$, so $z_{(n-m-j)}w' \in N(G)$ by $Q(T-1)$, whatever the sign of $n - m - j$.

Step 3: $Q(T-1)$, and $P(T)$ for A and for $G_{<d}$, imply $P(T)$ for the elements of $G \setminus A$ of weight d. By Step 1, $Q_{<d}(T)$ holds. Write $g = \nu + \sum a_{(0)}b$ as in hypothesis 2, with $\nu, b \in N(G_{<d})$. If b has weight zero, then $a_{(0)}b = 0$. We may therefore assume $\operatorname{wt} b \geqslant 1$ and $\operatorname{wt} a \leqslant d$. Then $\nu_{(n)}w \in N(G)$ by $Q_{<d}(T)$, and $(a_{(0)}b)_{(n)}w = a_{(0)}(b_{(n)}w) - b_{(n)}(a_{(0)}w)$, where $b_{(n)}w \in N(G)$ by $Q_{<d}(T)$, $a_{(0)}(b_{(n)}w) \in N(G)$ and $a_{(0)}w \in N(G)$ by $P(T)$ for A, and $b_{(n)}(a_{(0)}w) \in N(G)$ by $Q_{<d}(T)$, since $\operatorname{wt} b + \operatorname{wt}(a_{(0)}w) = \operatorname{wt} g + \operatorname{wt} w \leqslant T$. For the first nested term, $\operatorname{wt} a + \operatorname{wt}(b_{(n)}w) = \operatorname{wt} g + \operatorname{wt} w - n \leqslant T$. Thus neither use increases the induction parameter.

Step 4. $Q(0)$ holds since $N(G)_0 = \mathbb{C}\mathbf{1}$. Given $Q(T-1)$, Step 2, then Step 3 by increasing d, give $P(T)$, and Step 1 gives $Q(T)$. Hence $N(G)_{(s)}N(G) \subset N(G)$ for all s. Since $\mathbf{1} \in N(G)$, $N(G)$ is a vertex subalgebra, strongly generated by G by definition. ∎

Theorem 10.21 *Let $S \subset V'$ be the vertex subalgebra generated by $U_1 \oplus U_2$, for any of the four subspaces U_2 of Theorem 10.5.*

1. *S is strongly generated by $U_1 \oplus U_2 \oplus D_3$, where D_3 is a complement of N_3 in S_3. One may take $6 + 93$ generators in weights one and two and at most 231 in weight three.*
2. *$S_1 = U_1$, $S_2 = U_2$, $\dim S_3 = 956$ and $\dim S_4 = 6010$: the graded dimension of S agrees with the vacuum character $X_0^{(0)}$ of Section 3 up to $q^{11/3}$.*
3. *S is C_2-cofinite.*

Proof : 1–2. Apply Lemma 10.20 in $W = S$ with A a basis of $U_1 \oplus U_2$ and G the union of this finite basis with a basis of D_3 chosen among representatives $u_{(0)}w$ modulo N_3, where $u, w \in U_2$. Hypothesis 2 holds by construction. Hypothesis 1: for $a \in U_1$ and $g \in U_1 \oplus U_2$, by the affine structure and the construction of U_2 ($(A)_2$ consists of affine descendants, $y_{(n)}e = y_{(n)}x_C = 0$ for $y \in U_1$, $n \geqslant 0$, and the M_τ are A-primary). For $a, g \in U_2$, $a_{(0)}g \in S_3 = N_3 + D_3 \subset N(G)$, $a_{(1)}g \in U_2$, $a_{(2)}g \in U_1$ (Theorem 10.9). For $a \in U_2$, $g \in U_1$, by skew symmetry. For $g \in D_3$

and $a \in U_1$, by part 2 of Theorem 10.19 and the stability of S_3 under zero modes. For $g \in D_3$ and $a \in U_2$: $a_{(0)}g \in N_4' \subset N(G)$ by part 3 of Theorem 10.19, every term of N_4' being a normal product of generators and of their derivatives, and $a_{(n)}g$ for $n = 1, 2, 3$ follow from the commutation relations $[L(1), a_{(n)}] = (2-n)a_{(n+1)}$ and $[L(2), a_{(1)}] = 2a_{(3)}$, valid for a quasi-primary of weight two, which give

$$2a_{(1)}g = L(1)a_{(0)}g - a_{(0)}L(1)g, \quad a_{(2)}g = L(1)a_{(1)}g - a_{(1)}L(1)g, \quad 2a_{(3)}g = L(2)a_{(1)}g - a_{(1)}L(2)g.$$

Since $L(1)N_4' \subset S_3$ (a direct check on each family, using $[L(1), y_{(n)}] = -ny_{(n+1)}$ for currents, $[L(1), m_{(-1)}] = 3m_{(0)}$ for quasi-primary m of weight two, $L(1)L(-1) = L(-1)L(1) + 2L(0)$ and part 2 of Theorem 10.19), $L(1)S_3 \subset U_2$, $L(2)S_3 \subset U_1$ and $(U_2)_{(n)}U_k \subset U_{k+1-n}$ for $k = 1, 2$ and $n \geqslant 1$, together with $(U_2)_{(0)}U_2 \subset S_3$ and $(U_2)_{(0)}U_1 \subset U_2$, we get $a_{(1)}g \in S_3$, $a_{(2)}g \in U_2$ and $a_{(3)}g \in U_1$. Moreover $U_2 = (U_2 \cap \operatorname{Ker} L(1)) + L(-1)U_1$, since $L(1)U_2 \subset U_1$ and $L(1)L(-1) = 2$ on U_1. For $a = L(-1)y$, $y \in U_1$, we have $a_{(n)} = -ny_{(n-1)}$, which reduces this case to $a \in U_1$. For $n \geqslant 4$, $a_{(n)}g$ lies in $S_{4-n} \subset \mathbb{C}\mathbf{1}$. In these checks, $M_\tau(-1)U_1$ and the weight-four monomials $D_3(-1)U_1$, $U_2(-2)U_1$ are brought to the listed families by skew symmetry. Therefore $N(G)$ is a vertex subalgebra containing $U_1 \oplus U_2$, so $N(G) = S$. Its homogeneous components are spanned by normally ordered monomials: $S_1 = U_1$, $S_2 = U_2$, $S_3 = N_3 + D_3$, and S_4 is spanned by $L(-1)S_3$, $U_1(-1)S_3$, $U_1(-2)U_2$, $U_1(-3)U_1$ and $U_2(-1)U_2$, which lie in N_4' by part 3 of Theorem 10.19. Since $(A)_2 \subset N(U_1)$, its basis vectors may be removed from the strong generating set, leaving $120 - 27 = 93$ weight-two generators. Lemma 10.15 gives $\dim N_3 = 725$, so a complement D_3 has dimension $956 - 725 = 231$. The independently computed nonzero minor modulo the chosen prime above p_3 also gives the lower bound $\dim N_3 \geqslant 725$.

The initial choice of representatives does not restrict the complement in the statement. Indeed, for any other complement D_3' of N_3 in S_3, every basis vector of D_3' is a linear combination of the chosen vectors $u_{(0)}w$, modulo N_3. Since $N_3 \subset N(U_1 \oplus U_2)$, hypothesis 2 of Lemma 10.20 still holds. The verification of hypothesis 1 above uses inclusions on the whole space S_3 and applies without change. Hence the lemma also applies to this complement.

3. We follow [Rie26b, Section 5]. Let $R = S/C_2(S)$ be the C_2 Poisson algebra, $[a][b] = [a_{(-1)}b]$, $\{[a], [b]\} = [a_{(0)}b]$, and $\overline{R}$ its quotient by the nilradical $\mathfrak{n}$, a Poisson ideal [Ara12, Corollary 2.4.2]. By part 1, R is generated by the classes of G [Ara12, Corollary 2.6.2] (only the elementary direction is used: $C_2(S)$ contains the $a_{(-n)}b$, $n \geqslant 2$, and is an ideal for the product $[a][b] = [a_{(-1)}b]$), so it suffices to show that these classes vanish in $\overline{R}$. The currents vanish. Indeed $v = e'(-1)^4\mathbf{1}$ satisfies $f'(1)v = 0$, as the image of the singular vector of the universal affine algebra of level 3, and $\phi_\chi(e') = -f'$ gives $H_\chi(e'_{(m)}u, u') = H_\chi(u, f'_{(-m)}u')$ for the positive definite form of Theorem 10.10. Hence $H_\chi(v, v) = H_\chi(e'(-1)^3\mathbf{1}, f'(1)v) = 0$ and $v = 0$. Likewise $e''(-1)^{14}\mathbf{1} = 0$. Thus $[e']^4 = [e'']^{14} = 0$, and the Poisson brackets with f', f'' give the other currents. A nontrivial irreducible $\mathfrak{g}' \oplus \mathfrak{g}''$-module M satisfies $M = (\mathfrak{g}' \oplus \mathfrak{g}'') \cdot M$, and the zero modes of currents act on $\overline{R}$ by Poisson brackets with classes that vanish. Hence the nontrivial isotypic components of U_2 vanish in $\overline{R}$, as well as $(A)_2$. The invariant vectors of U_2 modulo $(A)_2$ are e and x_C. By part 3 of Theorem 10.19, $e(-1)e$ and $x_C(-1)x_C$ lie in N_4', whose classes lie in $\mathfrak{n}$: $L(-1)S_3$, $U_1(-2)U_2$ and $U_1(-3)U_1$ lie in $C_2(S)$, and $U_1(-1)S_3$, $M_\tau(-1)U_2$ are products with nilpotent classes. Thus $[e]^2 = [x_C]^2 = 0$ in $\overline{R}$, which is reduced, so $[e] = [x_C] = 0$. Finally $D_3 \subset N_3 + (U_2)_{(0)}U_2$ consists of products and Poisson brackets of vanishing classes. Hence all generators of R are nilpotent, and R, finitely generated and commutative, is finite-dimensional. ∎

Corollary 10.22 *Independently of the membership verification in Proposition 10.17, condition* (F4) *follows from a dimension argument in either of the following situations:*

1. *the subspace U_2 is the weight-two space of a Haagerup algebra $U \subset V'$ containing the fixed currents,*

2. *the existence and character assertion of [Jia26] is granted, and S is identified with Jiao's algebra by Theorem 10.24.*

Consequently the conclusions of Theorem 10.21 hold in either situation without a separate weight-four membership certificate.

Proof : The rank computation of Proposition 10.13, part 5, supplies 6010 words of N_4' with independent reductions. They are integral at the chosen prime above p_3, so $\dim_{\mathbb{K}} N_4' \geqslant 6010$. In the first situation, $S \subset U$ by construction and $\dim U_4 = 6010$, hence $N_4' \subset S_4 \subset U_4$ forces equality throughout. In the second situation, identification with Jiao's algebra gives $\dim S_4 = 6010$, with the same consequence. In both cases the chosen words form a basis of S_4. Every mode product in (F4) belongs to S_4 by the grading, and all five defining families of N_4' lie there. Thus all the required membership identities follow. All the chosen words, input vectors and coordinate matrices are defined over the fixed $\mathbb{K}$-forms. Membership after extension to $\mathbb{C}$ therefore implies membership over $\mathbb{K}$: the rank of a matrix with entries in $\mathbb{K}$ is unchanged by extension of scalars. The proof of Theorem 10.24 uses weight-two closure and generation, not Theorem 10.21, so the second argument is not circular. ■

Remark 10.23 For S, no analogue of the weight-five obstruction calculation of [Rie26b, Proposition 4.2] is needed. In case (I), the weight-four space of U^{R} contains up to three vectors that are not normal products of lower generators, and they require a weight-five check. In case (II), $N_4' = S_4$ and no generator of weight four is needed. Theorem 10.21 uses the finite input of Theorems 10.9 and 10.19, whose weight-four identities are established over $\mathbb{K}$ in Proposition 10.17. Corollary 10.22 gives an alternative dimension argument when an ambient Haagerup algebra or the stated external existence result is available. It cannot be applied with $U = S$ through [ArhIII, Theorem 6.1], since the proof of that theorem uses Theorem 10.21, which rests on Proposition 10.17. Modular computations select pivots. The dimension $\dim N_3 = 725$ is proved without a matrix computation in Lemma 10.15, and the modular minor supplies an independent lower-bound check.

10.7 Case (II): identification with Jiao's algebra and uniqueness

Jiao [Jia26] defines a vertex subalgebra $V = \langle \omega, P_E, P_F, P_H, B_E, B_F, B_H, u_{26}, u_{17} \rangle$ of the lattice algebra of $E_6 \oplus A_2$, generated by the conformal vector, two $\mathfrak{sl}_2$-triples of levels 3 and 13 and two explicit weight-two vectors of types $[2, 6]$ and $[1, 7]$ with coefficients in $\mathbb{K}$ [Jia26, Section 3]. He gives a computer-assisted proof, which we have not checked in full, that V is unitary and strongly rational with $\mathrm{Rep}(V) \simeq Z(\mathcal{H}_3)$ as braided categories [Jia26, Theorem 1.1], and that every simple V-module other than V has a positive lowest weight [Jia26, Corollary 6.8]. His construction is dated October 4, 2026. It precedes the present work and the announcements [Rie26a, Hua26], and it makes no uniqueness assertion [Jia26, Section 1.2]. Below, "assuming $(\mathrm{H_J})$" means granting these two statements. By Theorem 10.24, they hold as soon as S is a Haagerup algebra, which is proved in [ArhIII, Theorem 6.1] without using [Jia26]. Hence Corollary 10.25 and Theorem 10.29 hold unconditionally.

Theorem 10.24 *Let S be the vertex subalgebra generated by $U_1 \oplus U_2$, where U_2 is the first subspace of the subalgebra 9 in Theorem 10.9. There is an isomorphism of vertex operator algebras ψ from the lattice algebra of [Jia26] onto $V' = V_{E_6 \oplus A_2}$ such that $\psi(V) = S$. It maps the triples (P_E, P_F, P_H) and (B_E, B_F, B_H) onto (e', f', H') and (e'', f'', H'').*

Proof : We compute the transport, membership and generation in exact arithmetic with the normalized coefficients described below. Both models use the convention

$$Y(e^p,x)e^q=\varepsilon(p,q)x^{(p,q)}\big(e^{p+q}+O(x)\big),$$

with bimultiplicative cocycles with values ±1 and commutator $(-1)^{(p,q)}$. Jiao's cocycle is given on pairs (α_i,α_j) with $i>j$ [Jia26, Section 3.1].

The isometry. Let φ be the isometry of root lattices given by the isomorphism of Dynkin diagrams sending the simple roots $1,\dots,8$ of [Jia26] (edges $12,23,34,45,36,78$) to our simple roots $1,3,4,5,6,2,7,8$, and let $w=s_6s_1s_3s_2s_4s_5s_4s_3s_2s_4$ in the Weyl group of E_6 (found by an orbit search). Then $\psi_0=w\circ\varphi$ sends the Cartan elements P_H, B_H to H', H''.

The cocycle. Put $r(p,q)=\varepsilon(\psi_0p,\psi_0q)\varepsilon_{\mathrm{Jiao}}(p,q)$. It is bimultiplicative, and symmetric since ψ_0 is an isometry, so that $r(p,q)r(q,p)=(-1)^{(\psi_0p,\psi_0q)+(p,q)}=1$. Moreover $r(\alpha_i,\alpha_i)=1$ since both cocycles satisfy $\varepsilon(\alpha,\alpha)=(-1)^{(\alpha,\alpha)/2}$. With $r_{ij}=r(\alpha_i,\alpha_j)$, the function $\eta(p)=\prod_{i<j}r_{ij}^{p_ip_j}$ satisfies $\eta(p+q)=\eta(p)\eta(q)r(p,q)$. Hence $e^p\mapsto\eta(p)e^{\psi_0p}$, $h(-n)\mapsto(\psi_0h)(-n)$ defines an isomorphism of vertex operator algebras, which maps the conformal vector to the conformal vector.

The torus. The images of P_E and B_E have the same supports as e' and e'', and the ratios of the coefficients are positive rational numbers. The ten resulting conditions on a character t of the root lattice (3 roots for P_E, 7 for B_E) are consistent. A solution is $t(\alpha_i)=2^{a_i}3^{b_i}5^{c_i}$ with half-integral exponents. Composing with the automorphism $\exp(h_t(0))$, which acts on $M(1)\otimes e^\lambda$ by $t(\lambda)$, we obtain ψ with $\psi(P_E)=e'$, $\psi(B_E)=e''$, $\psi(P_H)=H'$, $\psi(B_H)=H''$, hence $\psi(P_F)=f'$ and $\psi(B_F)=f''$ (the element f of an $\mathfrak{sl}_2$-triple is determined by e and H, and this is also checked directly). The isomorphism ψ is not defined over $\mathbb{K}$.

Inclusion. The vectors $\psi(u_{26})$ and $(-5)^{-1/2}\psi(u_{17})$ have coefficients in $\mathbb{K}$, and they lie in the first subspace U_2 of the subalgebra 9 (exact membership over $\mathbb{K}$). Hence $\psi(V)\subset S$.

Equality. Put $\widehat{u}_{26}=\psi(u_{26})$ and $\widehat{u}_{17}=(-5)^{-1/2}\psi(u_{17})$, and put $U_{2,\mathbb{K}}=U_2\cap V'_{\mathbb{K}}$ and $(A)_{2,\mathbb{K}}=(A)_2\cap V'_{\mathbb{K}}$ for the fixed $\mathbb{K}$-form of V'. We use these two normalized vectors, whose coefficients are obtained by exact simplification of the radicals in the torus transport. Let $W_{\mathbb{K}}$ be the smallest $\mathbb{K}$-subspace of $U_{2,\mathbb{K}}$ containing $(A)_{2,\mathbb{K}}$, ω, $\widehat{u}_{26}$ and $\widehat{u}_{17}$, stable under the zero modes of the six rational currents and under first products of its elements. Theorem 10.9 gives $W_{\mathbb{K}}\subset U_{2,\mathbb{K}}$, and every generating word lies in $\psi(V)$.

At $p_3=8388571$, take the residue map specified by $\sqrt{13}\mapsto1649360$ and $\sqrt{-3}\mapsto2036966$. A finite family of words in the normalized generators, specified in the supplementary data, has coordinates integral at this prime, and its reduction has rank 120.

A nonzero 120×120 minor of the reduced matrix lifts to a nonzero minor over $\mathbb{K}$. Thus $\dim_{\mathbb{K}}W_{\mathbb{K}}\geqslant120=\dim_{\mathbb{K}}U_{2,\mathbb{K}}$, and $W_{\mathbb{K}}=U_{2,\mathbb{K}}$. Complexification gives $U_2\subset\psi(V)$. Together with $U_1\subset\psi(V)$, this proves $S\subset\psi(V)$. ∎

For this ψ, the vectors $\psi(u_{26})$ and $\psi(u_{17})$ lie in none of the three other subspaces of Theorem 10.5. This depends on the choice of ψ, which may be composed with automorphisms of V' normalizing U_1.

Corollary 10.25 *Assuming* ($\mathrm{H_J}$)*, the algebras attached to the two subspaces of the subalgebra* 9 *(isomorphic by Proposition 10.12) are positive strongly rational, with* $\mathrm{Rep}(S)\simeq Z(\mathcal{H}_3)$ *as braided categories. Case (II) of Theorem 8.5 is realized.*

Proof : A vertex operator algebra isomorphism induces a braided equivalence of the module categories and preserves the lowest weights. Apply Theorem 10.24 and Proposition 10.12. ∎

The six currents e',f',H',e'',f'',H'' have rational coordinates in the Q-form V'_{Q} spanned by the lattice monomials, and the four subspaces U_2 are defined over $\mathbb{K}$. Let $\sigma\in\mathrm{Gal}(\mathbb{K}/\mathrm{Q})$

satisfy $\sigma(\sqrt{13}) = -\sqrt{13}$ and $\sigma(\sqrt{-3}) = \sqrt{-3}$. On $V'_{\mathbb{K}} = V'_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K}$, the σ-semilinear bijection $\tilde{\sigma} = \mathrm{id} \otimes \sigma$ preserves all products and fixes ω and the six rational currents. It maps the $\mathbb{K}$-form of the generating space of $S^{(9,c)}$ onto that of $S^{(10,c)}$, for $c \in \{0,1\}$. We call the complexification of the image algebra the Galois conjugate of $S^{(9,c)}$. This construction requires no extension of σ to $\mathbb{C}$. Any such extension gives the same image subalgebra, although the induced semilinear map on all of V' depends on the extension. The linear automorphism θ_χ of Proposition 10.12 exchanges $S^{(9,0)}$ and $S^{(9,1)}$.

Lemma 10.26 *In either case of Theorem 8.5, let Z be the pointwise stabilizer of U_1 in $\mathrm{Aut}(V')$, and let $\mathfrak{h}$ be the Cartan subalgebra spanned by the simple roots of $E_6 \oplus A_2$. Then every element of Z preserves $\mathfrak{h}$, its kernel on $\mathfrak{h}$ has order two, and $|Z| \leqslant 8$. The nonidentity element of this kernel is the torus automorphism ρ associated with the character χ_0 whose values are*

$$\big(\chi_0(\alpha_1), \dots, \chi_0(\alpha_8)\big) = \begin{cases} (1,-1,-1,1,-1,1,1,1) & \text{in case (I),} \\ (-1,1,1,-1,1,-1,1,1) & \text{in case (II).} \end{cases}$$

Thus $\rho(e^\lambda) = \chi_0(\lambda) e^\lambda$, and ρ fixes the oscillators.

Proof : We give an argument using the roots and the supports of the currents. An automorphism of V' restricts to an automorphism of $V'_1 = \mathfrak{e}_6 \oplus \mathfrak{sl}_3$, and this restriction is injective since V'_1 generates V'. The labels of the E_6 component of H'' are $(2,1,1,0,1,2)$ in case (I) and $(1,0,0,1,0,1)$ in case (II). The roots vanishing on this element form respectively the systems generated by α_4 and by $\alpha_2, \alpha_3, \alpha_5$. In case (I), $H' = \alpha_4$. In case (II), $H' = -\alpha_2 + \alpha_3 + \alpha_5$, whose labels on these three pairwise orthogonal roots are $-2, 2, 2$. The A_2 component of H'' has labels $(2,2)$. Consequently no root vanishes on both H' and H'', and their common centralizer in V'_1 is exactly $\mathfrak{h}$. Thus Z preserves $\mathfrak{h}$.

An element of Z acts on the roots by an isometry fixing H' and H''. The subgroup of the Weyl group with this property is trivial: a generic real linear combination of H' and H'' is regular, and a Weyl group acts freely on the chambers of the complement of the root hyperplanes. Since the two irreducible factors are not isomorphic and their diagram automorphism groups both have order two, at most four root isometries can occur.

For completeness, the raising currents of the chosen rational triples are, in case (I),

$$\begin{aligned} e' &= e^{\alpha_4}, \\ e'' &= 5e^{\alpha_1} + 4e^{\alpha_6} + 3e^{\alpha_2+\alpha_3+\alpha_4} + e^{\alpha_2+\alpha_4+\alpha_5} \\ &\quad + 4e^{\alpha_3+\alpha_4+\alpha_5} + e^{\alpha_7} + e^{\alpha_8}, \end{aligned} \tag{10.4}$$

and in case (II),

$$\begin{aligned} e' &= e^{-\alpha_2} + e^{\alpha_3} - \tfrac{5}{3} e^{\alpha_5}, \\ e'' &= 4e^{\alpha_4+\alpha_5+\alpha_6} + 4e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + e^{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} \\ &\quad + 3e^{\alpha_1+\alpha_3+\alpha_4} + 5e^{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + e^{\alpha_7} + e^{\alpha_8}. \end{aligned} \tag{10.5}$$

An automorphism acting trivially on $\mathfrak{h}$ scales the root vectors by a character t of the root lattice. Write $t_i = t(\alpha_i)$. Fixing the raising currents requires $t(\beta) = 1$ for each root in their supports. Formula (10.4) gives $t_1 = t_4 = t_6 = t_7 = t_8 = 1$ and $t_2 t_3 = t_2 t_5 = t_3 t_5 = 1$, hence $t_2 = t_3 = t_5 \in \{1, -1\}$. Formula (10.5) gives $t_2 = t_3 = t_5 = t_7 = t_8 = 1$, $t_4 t_6 = t_4^2 = t_1 t_4 = 1$, hence $t_1 = t_4 = t_6 \in \{1, -1\}$. In both cases the displayed nontrivial character fixes the raising currents and the Cartan elements, and hence the lowering currents. Indeed the lowering element

of an $\mathfrak{sl}_2$-triple is determined by its raising and Cartan elements: their difference would be a weight -2 vector annihilated by the raising operator, which is impossible in a finite-dimensional $\mathfrak{sl}_2$-module. Thus this character fixes U_1. This proves that the torus kernel has order two, and $|Z| \leqslant 2 \cdot 4 = 8$. ∎

Proposition 10.27 *In case (II), let Z be the group of automorphisms of V' which fix U_1 pointwise.*

1. *$Z \cong (\mathbb{Z}_2)^3$, and every element of Z is the lift of an isometry of $E_6 \oplus A_2$, equal to the identity or a diagram automorphism on each factor, composed with an element of the torus.*

2. *Four elements of Z act on the Griess algebra B by the reflection s of Proposition 10.12, and the four others act trivially.*

3. *Let g be the automorphism of the Dynkin diagram of E_6 ($\alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$), extended by the identity on A_2, and let z be the automorphism of V' with $z(h(-n)) = (gh)(-n)$ and $z(e^\lambda) = c(\lambda)e^{g\lambda}$, where $c(\lambda) = \prod_i c_i^{\lambda_i} \prod_{i<j} \delta_{ij}^{\lambda_i\lambda_j}$ for $\lambda = \sum_i \lambda_i\alpha_i$ (simple roots $\alpha_1, \dots, \alpha_6$ of E_6 in Bourbaki's numbering, then α_7, α_8 of A_2), $\delta_{ij} = \varepsilon(\alpha_i, \alpha_j)\varepsilon(g\alpha_i, g\alpha_j)$ (so that $\delta_{ii} = 1$ and $c(\alpha_i) = c_i$), and $(c_1, \dots, c_8) = (\frac{4}{5}, 1, -\frac{5}{3}, -1, -\frac{3}{5}, \frac{5}{4}, 1, 1)$. Then $z \in Z$, and z maps the subspace U_2 of $S^{(9,c)}$ onto that of $S^{(10,c)}$, for $c = 0, 1$.*

In particular the four algebras $S^{(9,0)}$, $S^{(9,1)}$, $S^{(10,0)}$, $S^{(10,1)}$ are isomorphic. The Galois conjugate of $S^{(9,0)} = \psi(V)$ is isomorphic to it.

Proof : First we prove part 1 without solving polynomial equations. Let z be the lift in part 3, let z' be the lift of the diagram automorphism of A_2 with all $c_i = 1$, and let ρ be the torus automorphism of Lemma 10.26. The formula for $c(\lambda)$ in part 3, with the diagram automorphism of A_2 and $c_i = 1$ for z', constructs z and z' on the whole lattice algebra, since $\delta_{ij} = \delta_{ji}$ and $\delta_{ii} = 1$ give $c(\lambda+\mu)\varepsilon(\lambda,\mu) = c(\lambda)c(\mu)\varepsilon(g\lambda, g\mu)$. Substitution in (10.5) and the Cartan elements gives $z, z' \in Z$. The lowering currents are then fixed by the uniqueness just used. Their actions on the simple-root exponentials and on $\mathfrak{h}$ give

$$z^2 = z'^2 = \rho^2 = 1, \quad zz' = z'z, \quad z\rho = \rho z, \quad z'\rho = \rho z'.$$

They generate eight distinct automorphisms: the isometry parts of z and z' are independent diagram automorphisms, and ρ is a nontrivial torus automorphism. Lemma 10.26 proves that these are all the elements of Z.

The action on B is also determined by these three generators. In the basis $b_0 = e, b_1, \dots, b_4$ used above, direct application of the lifts gives $z(b_i) = z'(b_i) = b_i$ for $0 \leqslant i \leqslant 3$ and

$$z(b_4) = z'(b_4) = -b_4 - \tfrac{1}{12}b_1 + \tfrac{5}{24}b_2 + \tfrac{1}{16}b_3, \quad \rho(b_i) = b_i \quad (0 \leqslant i \leqslant 4).$$

Thus $z|_B = z'|_B = s$ and $\rho|_B = 1$, proving part 2. The supplementary material contains exact checks of these formulas, the group relations and the currents. The argument for the upper bound uses only the root systems and the character equations of Lemma 10.26.

An independent polynomial computation gives the same centralizer. An automorphism of V' is determined by its restriction to $V_1' = \mathfrak{e}_6 \oplus \mathfrak{sl}_3$, and conversely every automorphism of this Lie algebra preserves the normalized invariant forms of the two simple ideals, hence extends to the affine Lie algebra and lifts to its irreducible vacuum module $V' \cong L_{\mathfrak{e}_6}(1,0) \otimes L_{\mathfrak{sl}_3}(1,0)$ [Kac90]. Under U_1, the space V_1' decomposes as $\bigoplus_\tau V(\tau) \otimes M_\tau$, with multiplicity spaces M_τ

of dimension 3 for $\tau = (0,2)$, 2 for $\tau = (0,4)$, $(1,3)$, and 1 for $(2,0)$, $(1,1)$, $(1,5)$, $(2,2)$, $(2,4)$, $(3,1)$. An element of Z acts by $\mathrm{id} \otimes z_\tau$, with z_τ fixing the highest weight vectors e', e'', and it is a Lie algebra automorphism if and only if $z\mathrm{ad}(m_k) = \sum_l (z_\tau)_{lk} \mathrm{ad}(m_l) z$ for a basis (m_k) of the highest weight vectors. By equivariance the other brackets follow. This is a system of 184 distinct equations of degree at most two (4 of them linear) with rational coefficients in 19 unknowns. Its Gröbner basis over Q shows that it has exactly 8 solutions, which form a group $(\mathbb{Z}_2)^3$, all monomial on the root vectors. On $B \subset V_2'$, a vector b is determined by the map $x \mapsto x_{(1)}b$ on V_1': if $x_{(1)}b = 0$ for all x, then $[x,y]_{(2)}b = x_{(1)}y_{(1)}b - y_{(1)}x_{(1)}b = 0$ and $V_1' = [V_1', V_1']$, and all modes $x_{(n)}$, $n > 2$, annihilate b by weight. The subspace of V_2' annihilated by the positive current modes is a finite-dimensional V_1'-module. If it were nonzero, a highest weight vector in this subspace would generate a proper affine submodule, since negative current modes increase weight. This contradicts the irreducibility of the vacuum module V', so $b = 0$. Hence $z(b)$ is determined by $x_{(1)}z(b) = z((z^{-1}x)_{(1)}b)$, a linear system with a unique solution in B. This gives part 2: the nontrivial action fixes the hyperplane spanned by the seven fixed lines of Proposition 10.12, so it is s. For part 3, c is the unique lift compatible with the cocycle: it satisfies $c(\lambda+\mu) = c(\lambda)c(\mu)\varepsilon(\lambda,\mu)\varepsilon(g\lambda, g\mu)$ and agrees with the values on the 78 roots read off from the solution. The cocycle identity proves that the lift preserves all vertex products, and its values on the six currents show that it fixes U_1 pointwise. Finally, z maps a spanning set of the first, respectively second, subspace U_2 of the subalgebra 9 into the corresponding subspace of the subalgebra 10, by exact membership over $\mathbb{K}$, and into none of the other three. Since both have dimension 120, these are equalities. An automorphism of V' fixing U_1 maps the algebra generated by $U_1 \oplus U_2$ onto the algebra generated by $U_1 \oplus z(U_2)$. ∎

Lemma 10.28 *Let $S \subset U$ be an inclusion of positive strongly rational vertex operator algebras with the same conformal vector. If $\dim \mathrm{Rep}(S) = \dim \mathrm{Rep}(U)$, then $S = U$.*

Proof : Let $M_0 = S, M_1, \dots, M_r$ be the simple ordinary S-modules, with lowest weights h_i and quantum dimensions d_i. Rationality gives a decomposition $U = \bigoplus_i m_i M_i$ as an S-module. It is finite: there are finitely many isomorphism classes, and the multiplicity of M_i is bounded by $\dim U_{h_i}$ when h_i is a nonnegative integer, and is zero otherwise, since the conformal vector is the same. In particular $m_0 = 1$ because $U_0 = \mathbb{C}\mathbf{1}$, and $d_i > 0$ by positivity.

Write S^S, S^U for the normalized modular S-matrices. In the modular transformation of the characters at $\tau = \mathrm{i}/y$, positivity of the lowest nonvacuum weights makes the vacuum term the unique dominant term as $y \downarrow 0$. Consequently

$$\sum_{i=0}^{r} m_i d_i = \lim_{y \downarrow 0} \frac{\mathrm{ch}\, U(\mathrm{i}y)}{\mathrm{ch}\, S(\mathrm{i}y)} = \frac{S^U_{00}}{S^S_{00}} = \sqrt{\frac{\dim \mathrm{Rep}(S)}{\dim \mathrm{Rep}(U)}} = 1.$$

Here $d_i = S^S_{i0}/S^S_{00}$ and the positive normalization gives $S^S_{00} = (\dim \mathrm{Rep}(S))^{-1/2}$, and similarly for U. These are the usual character formulas [Hua08, DJX13]. Since $m_0 d_0 = 1$ and every remaining term is nonnegative, with $d_i > 0$, all m_i for $i > 0$ vanish. The inclusion $S \subset U$ is therefore an equality. ∎

Theorem 10.29 (uniqueness in case (II)) *Let U be a positive strongly rational vertex operator algebra of central charge eight with a braided equivalence $\mathrm{Rep}(U) \simeq Z(\mathcal{H}_3)$, in case (II) of Theorem 8.5. Assuming* (H$_\mathrm{J}$)*, U is isomorphic to Jiao's algebra V.*

Proof : By Proposition 4.1, $U \subset V'$ with the same conformal vector. By Theorem 8.5, an inner automorphism h of V' maps the weight-one space of U onto $\mathfrak{g}' \oplus \mathfrak{g}''$. It fixes ω, extends to V_{E_8}

since $V_1' \subset (V_{E_8})_1$, and maps the decomposition $V' = U \oplus M_b$ to $V' = h(U) \oplus h(M_b)$, so that the hypotheses of Theorem 10.5 hold for $h(U)$. Replacing U by $h(U)$, U_2 is one of the four subspaces, so U contains the algebra S generated by $U_1 \oplus U_2$. The inclusions $U_2 \subset S_2 \subset U_2$ give $S_2 = U_2$, and $\omega \in U_2$. By Theorem 7.1, the vacuum character of U is the vacuum component $X_0^{(0)}$ of the Evans–Gannon vector with $\gamma = 0$. By Proposition 10.27, $S \cong S^{(9,0)} = \psi(V)$ (Theorem 10.24), and, assuming (H$_J$), $S^{(9,0)}$ is positive strongly rational with $\mathrm{Rep}(S^{(9,0)}) \simeq Z(\mathcal{H}_3)$ (Corollary 10.25), so its vacuum character is also $X_0^{(0)}$ by Theorem 7.1. Hence $\operatorname{ch} S = \operatorname{ch} U$, and $S \subset U$ gives $U = S \cong V$. ∎

Remark 10.30 By Proposition 10.27, the Galois conjugate is not a new algebra. The isomorphism is the lift of the outer automorphism of $\mathfrak{e}_6$, adjusted by a torus element. The lift of the diagram automorphism of A_2, with trivial torus part, gives another one, which maps $S^{(9,c)}$ onto $S^{(10,1-c)}$ (this follows from the exact identity $\psi_\tau(U_{2,\mathbb{K}}) = U_{2,\mathbb{K}}$ checked in the proof of Proposition 10.41, together with Propositions 10.12 and 10.27). The isometry part of an element of Z is the identity, the diagram automorphism of E_6, that of A_2, or both, and the element acts on B by s exactly when exactly one of the two diagram automorphisms occurs. In particular the inner automorphisms fixing U_1 act trivially on B, and s is induced only by outer automorphisms. Together with θ_χ, it shows that $S^{(9,0)}$ is isomorphic to its conjugates under $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. It has a rational form (Proposition 10.41). Lemma 10.28 gives a second, analytic proof of the equality $U = S$ in Theorem 10.29. Once the inclusion and the positive strong rationality of S are known, the equivalences of both module categories with $Z(\mathcal{H}_3)$ give equality of their global dimensions. This proof does not require the explicit Evans–Gannon vacuum character. The same argument applies to the final equality in Theorem 10.38.

Proposition 10.31 *Let $S = S^{(9,0)}$ and $C = \mathrm{Com}_S(A)$. Let f_1, f_2 be the two orthogonal idempotents of the subalgebra 9 with $f_1 + f_2 = e$, numbered so that $\omega_1 = 2f_1$ and $\omega_2 = 2f_2$ have central charges $\frac{182}{95}$ and $\frac{32}{19}$ as in Theorem 10.5. In the basis $b_0 = e, b_1, \dots, b_4$ of B specified in the supplementary data,*

$$f_2 = \tfrac{80}{171} b_0 + \tfrac{\sqrt{13}-13}{114} b_1 - \tfrac{25+5\sqrt{13}}{228} b_2 + \tfrac{7-\sqrt{13}}{152} b_3 + \tfrac{4\sqrt{13}}{19} b_4, \quad f_1 = e - f_2.$$

Write $L_i(n) = (\omega_i)_{(n+1)}$.

1. $\dim C_1 = 0$, $\dim C_2 = \dim C_3 = 2$ *and* $\dim C_4 = 7$.
2. *On C_4, the operators $L_1(0)$ and $L_2(0)$ commute, $L_1(0) + L_2(0) = 4$, and C_4 is the direct sum of their joint eigenspaces. The joint eigenvalues (h_1, h_2), with multiplicities, are*

   $$(4,0)^2, \quad (0,4)^2, \quad (2,2), \quad \left(\tfrac{51}{19}, \tfrac{25}{19}\right), \quad \left(\tfrac{26}{19}, \tfrac{50}{19}\right).$$

   The first five eigenvectors can be taken to be $L_1(-4)\mathbf{1}$, $L_1(-2)^2\mathbf{1}$, $L_2(-4)\mathbf{1}$, $L_2(-2)^2\mathbf{1}$ and $L_1(-2)L_2(-2)\mathbf{1}$.
3. *The two eigenvectors of non-integral bi-weights are primary for $\mathrm{Vir}_1 \otimes \mathrm{Vir}_2$: they are annihilated by $L_i(1)$ and $L_i(2)$, $i = 1, 2$.*

Proof : We have $\widehat{C}_1 = 0$ (Lemma 10.1) and C_2 is the subalgebra 9 by the construction of U_2 in the proof of Theorem 10.5. Proposition 10.17 establishes the finite certificate condition, so Theorem 10.21 gives $N_4' = S_4$. The remaining assertions use exact arithmetic over $\mathbb{K}$ and a rank bound after reduction at the chosen prime above p_3, starting from the exact basis of S_4 of Theorem 10.21. Since C is fixed by the zero modes of U_1, C_4 lies in the block of $\mathfrak{g}' \oplus \mathfrak{g}''$-weight

$(0,0)$ of S_4, of dimension 280. It is the common kernel of the operators $y_{(k)}$, $y \in U_1$, $0 \leqslant k \leqslant 4$. Its matrix has entries in the local ring at that prime, and its reduction has rank 273. Since reduction cannot increase rank, $\dim C_4 \leqslant 280 - 273 = 7$. Solving exactly the square system of size 1092 given by the restriction of scalars on pivot rows, we obtain seven vectors $w_1, \dots, w_7$. The identities $y_{(k)} w_j = 0$ are then checked exactly on all coordinates, and the reductions of the w_j are linearly independent, so $\dim C_4 = 7$. Modulo p_3, $\dim C_3 \leqslant 2$, and $L_1(-3)\mathbf{1}$, $L_2(-3)\mathbf{1}$ are linearly independent elements of C_3, so $\dim C_3 = 2$.

The images $L_i(0)w_j$ are computed exactly and expressed exactly in the basis (w_j), with a check on all coordinates. The resulting 7×7 matrices over $\mathbb{K}$ satisfy exactly $L_1(0)L_2(0) = L_2(0)L_1(0)$, $L_1(0) + L_2(0) = 4$ and $\prod_\lambda (L_2(0) - \lambda) = 0$ for $\lambda \in \{0, 2, 4, \frac{25}{19}, \frac{50}{19}\}$. Hence $L_2(0)$ is diagonalizable with eigenvalues in this set. The eigenvectors for $\frac{25}{19}$ and $\frac{50}{19}$ are computed exactly. The five Virasoro descendants of assertion 2 lie in C, have the stated bi-weights, and are linearly independent: the vectors $L(-4)\mathbf{1}$ and $L(-2)^2\mathbf{1}$ of a vacuum module of central charge $c \notin \{0, -\frac{22}{5}\}$ are independent in its simple quotient (the Gram determinant at level four is $\frac{c^2(5c+22)}{2}$), and $L_1(-2)L_2(-2)\mathbf{1}$ has bi-weight $(2,2)$. These seven joint eigenvectors are linearly independent, hence a basis of C_4. This proves assertion 2. Assertion 3 is checked exactly on the two eigenvectors, which have 3036 terms. It also follows from assertion 2, since $L_i(n)$, $n \geqslant 1$, would map them to vectors of non-integral bi-weights in C_{4-n}, whose bi-weights are integral.

The 280 independent words in the Cartan block $(0,0)$ are computed directly from the families defining N_4' and agree, entry by entry, with the corresponding block of the certificate of Proposition 10.17. The supplementary material contains the seven kernel vectors, all 210 current-mode annihilations, the two Virasoro matrices and both primary vectors. A separate rational-matrix verification checks the matrix identities, spectral multiplicities and independence of the seven vectors without using the vertex-algebra implementation. The vertex-product identities are verified from the lattice formulas. The computational data order the Virasoro vectors by increasing central charge, so their indices are opposite to those of this proposition. ∎

Corollary 10.32 *There are no two nontrivial vertex operator algebras V^1, V^2 of CFT type such that $C \cong V^1 \otimes V^2$ with $\omega_C = \omega^1 \otimes \mathbf{1} + \mathbf{1} \otimes \omega^2$. In particular C is not isomorphic to $K(\mathfrak{sl}_2, 17)^{\mathbb{Z}_2} \otimes \mathrm{Com}(L_{\mathfrak{sl}_2}(17,0), L_{\mathfrak{sl}_2}(4,0) \otimes L_{\mathfrak{sl}_2}(13,0))$. Moreover, the commutants $\mathrm{Com}_C(\langle\omega_2\rangle)$ and $\mathrm{Com}_C(\langle\omega_1\rangle)$ coincide in weights at most four with the Virasoro vertex operator algebras generated by ω_1 and ω_2, of graded dimensions $1, 0, 1, 1, 2$.*

Proof : Suppose $C \cong V^1 \otimes V^2$. Since $C_1 = 0$, we have $V^1_1 = V^2_1 = 0$, so that $C_2 = \mathbb{C}\omega^1 \otimes \mathbf{1} \oplus \mathbb{C}\mathbf{1} \otimes \omega^2$, the vectors ω^1, ω^2 being nonzero: if the conformal vector of a CFT-type vertex operator algebra were zero, then $L(0) = 0$ and the whole algebra would equal its one-dimensional weight-zero space. Then $\frac{1}{2}\omega^1 \otimes \mathbf{1}$ and $\frac{1}{2}\mathbf{1} \otimes \omega^2$ are orthogonal idempotents of the algebra C_2 with sum e. The only idempotents of $C_2 = \mathbb{C}f_1 \oplus \mathbb{C}f_2$ are 0, f_1, f_2 and e, so $\{\omega^1 \otimes \mathbf{1}, \mathbf{1} \otimes \omega^2\} = \{\omega_1, \omega_2\}$. But $L_{V^1}(0) \otimes 1$ has integral eigenvalues on $V^1 \otimes V^2$, whereas $L_1(0)$ has the eigenvalue $\frac{51}{19}$ on C_4, a contradiction. For the last assertion, a vector of C_n commuting with ω_2 is annihilated by $L_2(0)$. By Proposition 10.31 and $C_3 = \mathbb{C}L_1(-3)\mathbf{1} \oplus \mathbb{C}L_2(-3)\mathbf{1}$, the kernel of $L_2(0)$ on C_n, $n \leqslant 4$, is spanned by the Virasoro descendants of $\mathbf{1}$ for ω_1. ∎

Remark 10.33 The central charges $\frac{182}{95}$ and $\frac{32}{19}$ are those of the coset

$$D = \mathrm{Com}(L_{\mathfrak{sl}_2}(17,0), L_{\mathfrak{sl}_2}(4,0) \otimes L_{\mathfrak{sl}_2}(13,0))$$

and of the parafermion algebra $K(\mathfrak{sl}_2, 17)$. The dimensions $1, 0, 2, 2, 7$ of C agree with those of $K(\mathfrak{sl}_2, 17)^{\mathbb{Z}_2} \otimes D$, whose weight-four space contains a primary vector of bi-weight $(4,0)$ and

one of bi-weight $(0,4)$. Corollary 10.32 shows that this agreement does not come from an isomorphism of the two algebras. Modulo $\mathbb{Z}$, the weights $\frac{25}{19}$ and $\frac{50}{19}$ are the conformal weights $h_\lambda = \frac{\lambda(\lambda+2)}{76}$ of $\mathrm{SU}(2)_{17}$ for $\lambda = 4, 6$, but eight of the eighteen nonzero classes $\frac{x}{19}$ modulo $\mathbb{Z}$ are of this form, and the weights $\frac{8}{19}$ and $\frac{68}{57}$ of Vir_2 in Theorem 10.5 are not, so we do not draw any conclusion from it.

10.8 Case (I): uniqueness

Proposition 10.34 *Assume the hypotheses of Theorem 8.5 in case (I). Then C_2 is one of the subalgebras 2, 5, 6 of Table 3. The subalgebras 5 and 6 give the central charges $\frac{2400}{1271}$ and $\frac{70}{31}$ of the two Virasoro vectors announced in [Rie26b, Section 1].*

Proof : There is a direct exact exclusion that avoids choosing eigenvectors. For candidates 0, 3 and 4, use the primary space of type $(1,13)$, of dimension three. For candidate 1, use the type $(1,1)$, of dimension four. In a rational basis write X for the action of $(x_C)_{(1)}$, P for the primary pairing, and Y_j for the actions of a basis of $B \cap C_2^{\perp}$. A possible line, spanned by u, satisfies

$$(Xu)_i u_j - (Xu)_j u_i = 0, \quad u^{\mathsf{T}} P Y_j u = 0, \quad e_{(2)}u = (x_C)_{(2)}u = 0, \quad u^{\mathsf{T}} P u \neq 0.$$

In each projective chart set the first nonzero coordinate of u to one, set the preceding coordinates to zero, and add $z u^{\mathsf{T}} P u - 1 = 0$. The resulting thirteen ideals over $\mathbb{Q}$ are all the unit ideal. The supplementary data contain the systems and their unit bases. An independent calculation with lexicographic order gives the same unit ideals without using the vertex-algebra implementation. Thus none of the four candidates admits the required line.

A separate verification follows from the method of Steps 1 and 2 of the proof of Theorem 10.5, applied in case (I): the subalgebras 0 and 3 are excluded by certified enclosures, 1 by the type $(1,1)$ and 4 by the types $(0,12)$, $(1,1)$, $(1,7)$ and $(1,13)$, exactly. ∎

Proposition 10.35 *In case (I), C_2 is not the subalgebra 2.*

Proof : Suppose that C_2 is the subalgebra 2, with central charges $\frac{4}{5}$ and $\frac{686}{205}$. The space $\Pi_{1,13}$ has dimension three. The eigenvalues of $(x_C)_{(1)}$ on $\Pi_{1,13} \cap \operatorname{Ker} e_{(2)} \cap \operatorname{Ker}(x_C)_{(2)}$ are $\frac{496}{425}$, with a one-dimensional eigenspace on which some primary pairing $P_{1,13}(u, y_{(1)}u)$ does not vanish, and $-\frac{184}{425}$, with a two-dimensional eigenspace $E_{1,13}$ (weight $h_1 = 0$ for the Virasoro vector of central charge $\frac{4}{5}$). Similarly, $\Pi_{1,1}$ has dimension four, the eigenvalue $-\frac{568}{425}$ (two-dimensional eigenspace) is excluded by a Gröbner basis, and the admissible space $E_{1,1}$ is two-dimensional ($h_1 = \frac{2}{5}$). By Lemma 10.4, the lines $p_{1,13}$ and $p_{1,1}$ are isotropic lines of $E_{1,13}$ and $E_{1,1}$. There are exactly two in each, defined over $\mathbb{Q}(\sqrt{-19758})$ and $\mathbb{Q}(\sqrt{-4403})$ respectively. Let u span $p_{1,13}$ and w span $p_{1,1}$. By Lemma 10.4(4), $u_{(1)}w' \in U_2$ for every w' in the $\mathfrak{g}' \oplus \mathfrak{g}''$-module generated by w. The map $x \otimes y \mapsto x_{(1)}y$ is U_1-equivariant, so the U_1-types of these vectors lie in $V(1,13) \otimes V(1,1) = V(0,12) \oplus V(0,14) \oplus V(2,12) \oplus V(2,14)$, and among these types only $(0,12)$ occurs in U_2 (Theorem 8.5). Hence $C'z = 0$ and $C''z = 84z$ for $z = u_{(1)}w'$, where C', C'' are the Casimir operators of $\mathfrak{g}'$ and $\mathfrak{g}''$ normalized by $C = \frac{h^2}{2} + h + 2fe$, with eigenvalue $\frac{a(a+2)}{2}$ on $V(a)$. Normalize the spanning vectors by taking first coordinate 1 in the two ordered eigenspace bases specified in the supplementary data. Their second coordinates are $\pm 3\sqrt{-19758}/148$ and $\pm 2\sqrt{-4403}/37$, respectively. For each of the four choices, take $w' = f'(0)w$, the lowering vector of index 2 in the order $w, f''(0)w, f'(0)w, f'(0)f''(0)w$. For this choice there is an even simpler obstruction: z has Cartan weight $(0,14)$, which does not occur in U_2, so $z = 0$ is required. Moreover $e''(0)z = 0$, hence $C''z = 112z$, so that $(C'' - 84)z = 28z$ and it suffices to show that

$(C''-84)z \neq 0$. The coefficient of $(C''-84)u_{(1)}w'$ on e^{α}, where $\alpha=(1,2,3,4,3,2,0,0)$, is one of $\pm\frac{224\mathrm{i}}{925}(5\sqrt{4403}-\sqrt{19758})$ and $\pm\frac{224\mathrm{i}}{925}(5\sqrt{4403}+\sqrt{19758})$. These coefficients are nonzero: the sum of the positive radicals is positive, and $25\cdot 4403-19758=90317>0$ proves that $5\sqrt{4403}-\sqrt{19758}>0$. Certified Arb enclosures give an independent check of nonvanishing. This contradicts Lemma 10.4. ■

The same test applied to the subalgebra 5 excludes, with certified nonzero coefficients, the two crossed pairs of lines of types $(1,13)$ and $(1,1)$, and the two other pairs, those of the two choices of U_2 found by a numerical exploration, are not excluded (the exclusions are certified by interval arithmetic). This is a control of the method, not used below.

We now compare with the data of [Rie26a]. There, $U^{\mathrm{R}}=\langle A,e,f,v\rangle\subset V_{E_6\oplus A_2}$ is defined as the vertex subalgebra generated by the currents, two commuting Virasoro vectors $e,f\in \mathrm{Com}(A)_2$ of central charges $\frac{70}{31}$, $\frac{2400}{1271}$, and a highest weight vector v of type $(1,13)$ [Rie26b, Section 1]. The coordinates are specified in the certificates accompanying [Rie26b], and the same vectors are given in [Rie26a, Section "The explicit candidate"], in the bases fixed in [Rie26b].

Proposition 10.36 *There is an isomorphism of vertex operator algebras ψ_{R} from the lattice algebra of [Rie26a] onto V' which maps the currents of [Rie26a] onto e',f',H',e'',f'',H'' (levels 1 and 39). It maps e and f into the subalgebra 5 of Table 3, which they span, and v to a spanning vector of one of the two isotropic lines of the admissible space of type $(1,13)$ of the subalgebra 5.*

Proof : Exact computation, as for Theorem 10.24. In [Rie26b] the coordinates $1,2$ are those of A_2 and $3,\dots,8$ those of E_6 in Bourbaki's numbering, with the cocycle $\varepsilon(\alpha_i,\alpha_j)$ given for $i>j$ and the same field $\mathbb{K}$. The coordinate isometry, followed by $w=s_3s_4s_1s_3\in W(E_6)$, sends the Cartan elements of [Rie26b] onto H', H''. The cocycle correction is given by the same formula as in the proof of Theorem 10.24, and the torus character with values $5,-\frac{\sqrt{3}}{2},-2\sqrt{3},1,-\frac{2\sqrt{3}}{3},4,1,1$ on our simple roots matches the six currents exactly. The images of $\omega_{V'}$ and of the Sugawara vector of A are ours. In the basis of B specified in the supplementary data, $\psi_{\mathrm{R}}(e)$ and $\psi_{\mathrm{R}}(f)$ have coordinates in $\mathrm{Q}(\sqrt{13})$ and lie in the subalgebra 5. In the case (I) analogue of the proof of Theorem 10.5, the type $(1,13)$ has, for the subalgebra 5, a single admissible space E, of dimension two (the other eigenvalue $-\frac{254}{1445}+\frac{508}{1445}\sqrt{13}$ of $(x_C)_{(1)}$ on $\Pi_{1,13}$ is excluded exactly), and E contains exactly two isotropic lines, of slopes $\frac{1}{30}\mp\frac{\sqrt{-39}}{130}$ in a basis of E over $\mathrm{Q}(\sqrt{13})$, complex conjugate. Moreover $\psi_{\mathrm{R}}(v)$ lies in E and spans the second one (exact verification). ■

Proposition 10.37 *In case (I), the group Z of automorphisms of V' fixing U_1 pointwise is again isomorphic to $(\mathbb{Z}_2)^3$, and its elements are lifts of isometries composed with torus elements. Its elements act on B either trivially or by a reflection which exchanges the subalgebras 5 and 6 (and 0 and 3). The lift z_A of the diagram automorphism of A_2, with trivial signs, is of the second kind. The element z_1, lift of the product of the diagram automorphisms of E_6 and A_2 with the values $(c_1,\dots,c_8)=(\frac{4}{5},-1,\frac{1}{3},1,3,\frac{5}{4},1,1)$ on the simple roots (with the cocycle correction of Proposition 10.27), acts trivially on B and exchanges the two isotropic lines of type $(1,13)$ of the subalgebra 5.*

Proof : The upper bound $|Z|\leqslant 8$ follows analytically from Lemma 10.26. The explicitly given lifts z_1, z_A and the torus automorphism ρ of that lemma fix the currents, are commuting involutions, and generate eight elements. Indeed their isometry parts generate the two diagram automorphisms and ρ is nontrivial. The formulas can be checked on the simple roots and the

currents in (10.4). The supplementary material includes the exact verification. Thus $Z \cong (\mathbb{Z}_2)^3$ without using a polynomial-system solver.

An independent polynomial computation gives the same centralizer, and its action on B and on the isotropic lines supplies the remaining assertions. The computation is the same as for Proposition 10.27. The multiplicity spaces have dimension 2 for the types $(0,2)$, $(0,4)$ and 1 otherwise. The 88 distinct bracket equations, of degree at most two, in 12 unknowns have exactly 8 solutions. The action on the lines is computed by applying the lifts to $\psi_{\mathrm{R}}(v)$ and locating the image in E, exactly. ∎

Theorem 10.38 (uniqueness in case (I)) *Let U be a Haagerup algebra in case (I) of Theorem 8.5. Assuming* $(\mathrm{H}_{\mathrm{R}})$*, U is isomorphic to U^{R}.*

Proof : As in the proof of Theorem 10.29, we may assume $U \subset V'$, with the same conformal vector, and $U_1 = \mathfrak{g}' \oplus \mathfrak{g}''$. By Propositions 10.34 and 10.35, $C_2 = U_2 \cap \mathrm{Com}(A)$ is the subalgebra 5 or 6. Replacing U by $z_A(U)$ if necessary (Proposition 10.37), it is the subalgebra 5. Then $\psi_{\mathrm{R}}(e), \psi_{\mathrm{R}}(f) \in C_2 \subset U$ by Proposition 10.36. By Lemma 10.4 and the proof of Proposition 10.36, the line $p_{1,13}$ of U_2 is one of the two isotropic lines of E. Replacing U by $z_1(U)$ if necessary, which fixes U_1 and C_2 pointwise, it is spanned by $\psi_{\mathrm{R}}(v)$. Hence $U \supset \psi_{\mathrm{R}}(U^{\mathrm{R}})$. By Theorem 7.1, applied to U and, assuming $(\mathrm{H}_{\mathrm{R}})$, to U^{R}, both vacuum characters are $X_0^{(0)}$. Hence $U = \psi_{\mathrm{R}}(U^{\mathrm{R}})$. ∎

Once C_2 is fixed, only the type $(1,13)$ is used: the generation of U^{R} by A, e, f and v makes it sufficient. The exclusion of the subalgebras 1, 2, 4 also uses the types $(1,1)$, $(0,12)$ and $(1,7)$.

Proposition 10.39 *The algebra $\psi_{\mathrm{R}}(U^{\mathrm{R}}) \subset V'$ has dimension at least* 120*,* 956 *and* 6010 *in weights two, three and four.*

Proof : The coordinates of $\psi_{\mathrm{R}}(e)$, $\psi_{\mathrm{R}}(f)$ and $\psi_{\mathrm{R}}(v)$ in the monomial basis of V' lie in $\mathrm{Q}(\sqrt{13}, \sqrt{3}, \mathrm{i})$, and the structure constants of V' in this basis are rational. Let p be one of the primes 8388409 and 7999993. Then -1, 3 and 13 are squares modulo p, and a choice of square roots defines a prime $\mathfrak{p}$ of $\mathrm{Q}(\sqrt{13}, \sqrt{3}, \mathrm{i})$ above p and a reduction map from the local ring at $\mathfrak{p}$ onto $\mathbb{F}_p$. All these numbers are $\mathfrak{p}$-integral: every denominator occurring in the reduction is checked to be nonzero modulo p. The rank of a matrix with $\mathfrak{p}$-integral entries is at least the rank of its reduction modulo $\mathfrak{p}$. The following computations take place modulo p, for the reductions of e, f, v. The smallest U_1-stable subspace $W_2 \subset V_2'$ containing $(A)_2$, e, f and v and stable under the products $b_{(1)}w$ has dimension 120. Its U_1-types are those of the row U_2 of Table 2, and $W_2 \cap \widehat{C} = \mathrm{span}(e, f)$. The families of Proposition 10.13, built from W_2 and from its ten highest weight vectors outside $(A)_2$, span spaces of dimensions 956 and 6010 in weights three and four, and $y_{(1)}S_4 \subset S_3$, $y_{(2)}S_4 \subset W_2$ for $y \in U_1$. As a consistency check, forced by the symmetry z_1, the other isotropic line of type $(1,13)$ gives the same dimensions in weights two and three. As a discriminating control, a non-isotropic vector of the admissible plane generates V_2' and V_3'. Hence the subspaces of $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ spanned by these families, which consist of iterated products of the generators, have dimensions at least 120, 956 and 6010. These are the coefficients of $X_0^{(0)}$, so that a rank larger than 6010 would have contradicted [Rie26a]. This is an independent check of the graded dimensions announced in [Rie26a] up to $q^{11/3}$. By itself it does not prove $\mathrm{ch}\,\psi_{\mathrm{R}}(U^{\mathrm{R}}) = X_0^{(0)}$ in these weights, which needs an upper bound (Theorem 12.4). ∎

10.9 Classification

Theorem 10.40 *Let U be a Haagerup algebra. Assuming* $(\mathrm{H_R})$ *and* $(\mathrm{H_J})$*, U is isomorphic to exactly one of the following two algebras: the algebra U^{R} of Riesen, with U_1 of levels* $(1, 39)$*, and the algebra V of Jiao, with U_1 of levels* $(3, 13)$*.*

Proof : By Theorem 8.5, U is in case (I) or (II). Apply Theorems 10.38 and 10.29. The two algebras are not isomorphic, since the levels of the simple ideals of the weight-one Lie algebra are invariants. ∎

The algebra announced by Huang [Hua26] has currents of levels $(1, 39)$. By [ArhIII, Theorem 7.9, Corollary 7.10 and Theorem 7.13], using [DJX26, Proposition 4.1 and Theorem 7.3], this algebra and U^{R} are Haagerup algebras isomorphic to the twin $S^{\natural}$ of S (Theorem 12.3). In particular $(\mathrm{H_R})$ holds, and Theorems 10.38 and 10.40 hold unconditionally.

Besides $(\mathrm{H_R})$ and $(\mathrm{H_J})$, Theorem 10.40 uses the following results of this paper, with the computational dependencies recorded in Appendix A: Theorems 7.1 and 8.5, the determination of C_2 and of the relevant lines (Theorem 10.5, Propositions 10.34 and 10.35), and the explicit automorphisms and identifications (Theorem 10.24, Propositions 10.12, 10.27, 10.36 and 10.37). In the conditional uniqueness proofs, $(\mathrm{H_R})$ and $(\mathrm{H_J})$ supply the vacuum characters of the two candidates and the equality of global dimensions needed for the alternative proof in Lemma 10.28. The containment of $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ or $\psi(V)$ in a Haagerup algebra, after an automorphism of V', follows from the finite constructions above. Equality in the corresponding containment is equivalent to the candidate having vacuum character $X_0^{(0)}$. The application to the explicit realizations is given after Theorem 12.4.

Proposition 10.41 *The algebras $S^{(9,0)}$ and $\psi_{\mathrm{R}}(U^{\mathrm{R}}) = \langle A, e, f, v\rangle$ have rational forms: each contains a vertex subalgebra U_{Q} over Q with $\mathbf{1}, \omega \in U_{\mathrm{Q}}$ and $U_{\mathrm{Q}} \otimes_{\mathrm{Q}} \mathbb{C} = U$. In this proposition, e, f, v denote the vectors $\psi_{\mathrm{R}}(e)$, $\psi_{\mathrm{R}}(f)$, $\psi_{\mathrm{R}}(v)$ of V' (Proposition 10.36). The coordinates of v lie in $\mathrm{Q}(\sqrt{13}, \sqrt{3}, \mathrm{i})$, and the currents are rational in both cases. More precisely, let $G = \mathrm{Gal}(\mathbb{K}/\mathrm{Q}) = \{1, \sigma, \tau, \sigma\tau\}$, τ the complex conjugation, and $\tilde{g} = \mathrm{id} \otimes g$ on $V'_{\mathbb{K}} = V'_{\mathrm{Q}} \otimes \mathbb{K}$.*

1. *Case (II). Let z be the automorphism of Proposition 10.27, part 3, and z' the lift of the diagram automorphism of A_2 with trivial torus part (Remark 10.30). Then $z, z' \in Z$ are defined over Q, $z^2 = z'^2 = 1$, $zz' = z'z$, and $\psi_\sigma = z\tilde{\sigma}$, $\psi_\tau = zz'\tilde{\tau}$ map $U_{2,\mathbb{K}} = U_2 \cap V'_{\mathbb{K}}$ onto itself. The idempotents f_1, f_2 of C_2 are fixed by ψ_σ and ψ_τ.*

2. *Case (I). Let z_1 and z_A be the elements of the centralizer of U_1 of Proposition 10.37. Their product $z_1 z_A$ is the lift of the diagram automorphism of E_6 with suitable signs. These elements are defined over Q and commute, $z_1^2 = z_A^2 = 1$, and $\psi_\tau = z_1\tilde{\tau}$, $\psi_\sigma = z_1 z_A \tilde{\sigma}$ fix e and f (the Virasoro vectors of [Rie26a]). The vector $v' = \mathrm{i}v$ has coordinates in $\mathbb{K}$, and $\psi_\tau(v') = \psi_\sigma(v') = -v'$. With z_A in place of $z_1 z_A$, $\psi_\sigma(v')$ is not proportional to v'.*

Put $U_{1,\mathbb{K}} = U_1 \cap V'_{\mathbb{K}}$. Let $U_{\mathbb{K}} \subset V'_{\mathbb{K}}$ be the $\mathbb{K}$-span of the iterated products $a^1_{(n_1)} \cdots a^r_{(n_r)}\mathbf{1}$ of the generators (a $\mathbb{K}$-basis of $U_{1,\mathbb{K}} \oplus U_{2,\mathbb{K}}$ in case (II), the six rational currents and e, f, v' in case (I)), so that $U = U_{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{C}$. In both cases $g \mapsto \psi_g$, $\psi_{\sigma\tau} = \psi_\sigma\psi_\tau$, is an action of G by semilinear automorphisms of $U_{\mathbb{K}}$, and $U_{\mathrm{Q}} = \{u \in U_{\mathbb{K}} : \psi_g(u) = u,\ g \in G\}$ is a rational form. In case (I), U_{Q} is generated by the rational currents, e, f and $\sqrt{-39}v' = -\sqrt{39}v$.

Proof : All the identities are exact: the rationality and the group relations of z, z', z_1, z_A (and of the whole groups Z) are checked on V'_1, which determines the automorphisms of V'. In case (II), the images of the chosen $\mathbb{K}$-basis of $U_{2,\mathbb{K}} = U_2(9,0) \cap V'_{\mathbb{K}}$ under ψ_σ, ψ_τ and $\psi_\sigma\psi_\tau$ lie in $U_{2,\mathbb{K}}$ by exact membership over $\mathbb{K}$. Equivalently, for ψ_σ, one restricts the identities of Propositions

10.12 and 10.27 to the chosen $\mathbb{K}$-forms. In case (I), the identities on e, f, v' are checked in exact arithmetic. The automorphisms z, z', z_1, z_A are defined over Q, hence commute with $\tilde{\sigma}$ and $\tilde{\tau}$, and the groups they generate are elementary abelian. Hence $\psi_\sigma^2 = \psi_\tau^2 = 1$ and $\psi_\sigma\psi_\tau = \psi_\tau\psi_\sigma$, so $g \mapsto \psi_g$ is a homomorphism. Each ψ_g is g-semilinear and preserves the products $u_{(n)}v$ of $V'_{\mathbb{K}}$, it fixes the six chosen rational currents, and it preserves the generating space $U_{2,\mathbb{K}}$ in case (II), and the generators e, f, v' up to sign in case (I). Hence it preserves $U_{\mathbb{K}}$. By Speiser's lemma, equivalently $H^1(G, \mathrm{GL}_n(\mathbb{K})) = 1$ [Ser79, Chapter X, Proposition 3], each $U_{\mathbb{K},n}$ is spanned over $\mathbb{K}$ by its fixed points, and $U_{\mathbb{K},n} = (U_{\mathbb{K},n})^G \otimes_{\mathrm{Q}} \mathbb{K}$. The fixed points form a vertex subalgebra over Q, which contains $\mathbf{1}$ and ω. In case (I), $\sqrt{-39}v'$ is fixed since $\sigma(\sqrt{-39}) = \tau(\sqrt{-39}) = -\sqrt{-39}$, and the vertex algebra over Q generated by the fixed generators spans $U_{\mathbb{K}}$ over $\mathbb{K}$, hence equals U_{Q}. ∎

In case (II) we have also checked that the traces $\sum_g \psi_g(au)$, $a \in \{1, \sqrt{13}, \sqrt{-3}, \sqrt{-39}\}$, u ranging over a fixed $\mathbb{K}$-basis of $U_{2,\mathbb{K}}$, span a Q-form of U_2 of dimension 120. The rational form is not contained in V'_{Q}: the coordinates of f_1 in V' lie in $\mathrm{Q}(\sqrt{13})$.

11 Zhu-algebra computations and rationality

In this section, $S \subset V' = V_{E_6 \oplus A_2}$ denotes the vertex subalgebra generated by $U_1 = \mathfrak{g}' \oplus \mathfrak{g}''$ (levels 3 and 13) and one of the four subspaces U_2 of Theorem 10.5. For coordinate computations we fix $S = S^{(9,0)}$. The abstract conclusions for the other three choices follow from Proposition 10.27. We write $L = E_6 \oplus A_2$ throughout this section. By Theorem 10.10, S is a simple unitary self-contragredient vertex operator algebra of CFT type with $c = 8$.

Proposition 10.17 verifies the finite condition (F4) of Definition 10.16. Theorem 10.21 therefore gives strong generation in weights at most three and C_2-cofiniteness, independently of an ambient Haagerup algebra or an external existence result. We give the Zhu-algebra criterion for strong rationality and the modular computation of the corner. The exact computation of the corner and the branching are proved in [ArhIII, Appendix A]. The results of this section do not use ($\mathrm{H_J}$). Theorem 11.1, Propositions 11.3 and 11.4 and Theorem 11.6 are used in [ArhIII, Appendix A].

We compare the algebraic steps with the proof of [Rie26b] for case (I). Remark 11.2 separates the branching calculation from categorical recognition. Section and page numbers refer to the working manuscript [Rie26b] and its package of computational certificates. Evans [Eva26, p. 3] cites the three constructions [Hua26, Jia26, Rie26a]. That of Jiao is the case (II) algebra (Theorem 10.24).

11.1 The structure of the proof in case (I)

The proof strategy of [Rie26b] separates the construction, strong generation, C_2-cofiniteness, rationality and categorical identification. The construction, unitarity, strong generation and C_2-cofiniteness of S are established above, with the finite condition (F4) verified by Proposition 10.17. Positivity follows from the determination of the simple modules and of their lowest weights in [ArhIII, Theorem 2.2(F1)–(F2)].

1. *Construction and positivity* [Rie26b, Sections 2–3, pp. 16–19]. The algebra is generated by the currents and three weight-two vectors. A positive rescaling of the standard antilinear involution of the lattice algebra preserves the generators, which gives unitarity, simplicity and self-contragredience. For S, this is Theorem 10.10.

2. *Strong generation* [Rie26b, Section 4, pp. 20–24]. The finite mode closure in weights 2 and 3 [Rie26b, Proposition 4.1], two obstruction calculations, in weights four and five, showing that at most three new generators appear in weight four and none in weight five [Rie26b, Proposition 4.2], and a general lemma, *triangular strong generation* [Rie26b, Lemma 4.3] (Lemma 10.20). Proposition 10.17 supplies the finite weight-four condition (F4), so the lemma applies to S in Theorem 10.21. No generator of weight four is needed, hence no weight-five calculation either (Remark 10.23).

3. *C_2-cofiniteness* [Rie26b, Section 5, pp. 24–25]. For S, Proposition 10.17 supplies the square identities in (F4), and part 3 of Theorem 10.21 gives the conclusion.

4. *Rationality* [Rie26b, Part III, pp. 25–55]. Semisimplicity of the Zhu algebra is proved through an extension of the affine subalgebra by its simple current, a Morita corner, a finite presentation by a quiver with relations, and an integral descent. The rationality criterion for C_2-cofinite algebras then applies. For S, the exact presentation and descent are proved in [ArhIII, Theorems A.10 and A.13]. For S, Theorems 10.10 and 10.21 show that S is simple, self-contragredient, of CFT type and C_2-cofinite. By [McR21, Main Theorem 3 (Corollary 5.19)], S is then rational as soon as its Zhu algebra $A(S)$ is semisimple, and conversely [Zhu96]. For S, the simple current of $L_{\mathfrak{sl}_2}(3,0) \otimes L_{\mathfrak{sl}_2}(13,0)$ is $J = L(3,3) \otimes L(13,13)$, of conformal weight $\frac{3}{4} + \frac{13}{4} = 4$, whereas in case (I) it is $L(1,1) \otimes L(39,39)$, of weight $\frac{1}{4} + \frac{39}{4} = 10$. The weight-four space of S contains the lowest weight space $V(3,13)$ of J, exactly (Proposition 11.4), so that S contains the simple current extension $A \oplus J$, the analogue of the algebra A^+ of [Rie26b, Section 7]. In case (I) the current appears only in weight ten.

5. *Modules and the categorical identification* [Rie26b, Parts IV–V, pp. 56–70]. Six simple modules come from the nine V'-modules $W_{e,a}$, $(e,a) \in L^*/L \cong \mathbb{Z}_3^2$. Traces of the conformal weights, positivity of the categorical dimensions and a Galois orbit argument force exactly six more [Rie26b, Section 11]. Then the two even unimodular overlattices of L give two copies of V_{E_8}, hence Lagrangian algebras $A_\pm = \mathbf{1} \oplus M_b \oplus 2X_\pm$ in $\mathrm{Rep}(U)$, and $\mathrm{Rep}(U) \simeq Z(\mathcal{D}_\pm)$ with $\mathcal{D}_\pm = \mathrm{Rep}(U)_{A_\pm}$ [Rie26b, Lemma 12.2] [DMNO13]. The categorical argument uses the full branching of the lattice modules, dimension constraints and an exhaustion statement for fusion categories with the required fusion rules and untwisted pointed part. Here that exhaustion statement is hypothesis (C) of Remark 11.2. The classification within the family of [EvG17, Theorems 2 and 3] is distinct from proving this exhaustion. For S, the full branching and the exhaustion of the simple modules are given in [ArhIII, Theorem 2.2(F2)–(F3)]. The recognition of the braided category in [ArhIII, Theorem 6.1] proceeds through the conformal net.

The changes in the explicit calculations include the affine levels, the weight of the simple current and the commutant, whose two commuting Virasoro vectors have central charges $\frac{182}{95}$ and $\frac{32}{19}$ instead of $\frac{2400}{1271}$ and $\frac{70}{31}$. The difference in levels also changes the exact relations used for the two Zhu-algebra computations.

11.2 The categorical route

The two isotropic lines in L^*/L give the holomorphic lattice extensions $H_\pm$. Their restrictions to S in [ArhIII, Theorem 2.2(F3)] give the branching input for categorical recognition. Remark 11.2 describes the additional categorical steps.

Part 2 of Proposition 10.13 gives a finite-degree check modulo p_3 of the branching $W_{0,0}|_S = S \oplus M_b$ of [ArhIII, Theorem 2.2(F3)] in weights at most three: modulo the prime p_3, the subspace

generated by R_b has the graded dimensions $80, 1250, 10630$ of M_b and is complementary to S. By unitarity, $V' = S \oplus S^{\perp}$ as S-modules, and $S^{\perp}$ contains the S-module generated by R_b since $S_1 = U_1$ (Theorem 10.21).

11.3 A conditional theorem

Theorem 11.1 *Assume the hypothesis* (Z) *that the Zhu algebra $A(S)$ is semisimple. Then S is strongly rational. Independently of this hypothesis, S is not isomorphic to the algebras of [Rie26a, Hua26].*

Proof : Proposition 10.17 and Theorem 10.21 give C_2-cofiniteness. Theorem 10.10 gives simplicity, self-contragredience and CFT type. Thus [McR21, Main Theorem 3 (Corollary 5.19)] applies under (Z) and proves strong rationality. For the last assertion, recall that $S_1 = U_1$ is six-dimensional (Theorem 10.21). An isomorphism with an algebra W of [Rie26a, Hua26] would force $\dim W_1 = 6$. Since W_1 contains the six-dimensional Lie algebra spanned by the currents of W, of levels 1 and 39, it would then equal this Lie algebra. Such an isomorphism would preserve the levels of the simple ideals of the weight-one Lie algebras, whereas these are $3, 13$ for S_1 and $1, 39$ for W_1. ∎

Remark 11.2 We record the branching assertion (M) for S and the categorical exhaustion hypothesis (C) for the recognition approach of [Rie26b].

(M) S has exactly twelve simple modules, those other than S have positive lowest conformal weights, and the nine V'-modules restrict as $W_{0,0}|_S = S \oplus M_b$, $W_{0,\pm1}|_S \cong P$, $W_{\pm1,0}|_S \cong Q$, $W_{1,1}|_S \cong W_{-1,-1}|_S \cong X_+$, $W_{1,-1}|_S \cong W_{-1,1}|_S \cong X_-$, with M_b, P, Q, X_+, X_- simple and pairwise non-isomorphic, and the six other simple modules have lowest weights congruent to $\frac{2}{13}, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}, \frac{11}{13}$ modulo $\mathbb{Z}$,

(C) every fusion category with the fusion rules of $\mathcal{H}_3$ (for the group $\mathbb{Z}_3$) whose pointed part is an untwisted $\mathrm{Vec}_{\mathbb{Z}_3}$ is equivalent to one of the categories $\mathcal{C}(\mathbb{Z}_3; \pm, \omega, A)$ of [EvG17, Theorem 2].

Theorem 2.2(F1)–(F3) of [ArhIII] proves (Z) and (M). Theorem 11.1 therefore gives strong rationality, and $\mathrm{Rep}(S)$ is a modular tensor category [Hua08]. The nonvacuum lowest weights are positive, so the quantum dimensions are positive [DJX13, Lemma 4.2] and equal the categorical dimensions [Hua08]. Unitarity of S alone is not used to infer unitarity of its modules. Every simple module is self-contragredient. Indeed, the six residues in $\frac{1}{13}\mathbb{Z} \setminus \mathbb{Z}$ are distinct, whereas $(W_{e,a})' \cong W_{-e,-a}$ and (M) give the assertion for the remaining six modules.

The two even unimodular overlattices of L give holomorphic extensions with underlying S-modules $\mathbf{1} \oplus M_b \oplus 2X_{\pm}$. They define Lagrangian algebras $A_{\pm}$, hence center equivalences $\mathrm{Rep}(S) \simeq Z(\mathcal{D}_{\pm})$ with $\mathcal{D}_{\pm} = \mathrm{Rep}(S)_{A_{\pm}}$ [DMNO13]. The fusion rules and the pointed associators follow from [ArhIII, Theorem 9.1]. These are the inputs to the recognition argument of [Rie26b, Sections 11.3 and 12–13] under (C).

Hypothesis (C) is the exhaustion statement formulated in [Rie26b, Proposition 12.5]. The classification in [EvG17, Theorem 2] concerns the listed family. The proof of $(\mathrm{H_J})$ in [ArhIII, Theorem 6.1] uses the conformal-net comparison independently of (C).

11.4 The Zhu algebra of S

We now adapt to S the method of [Rie26b, Part III] for the hypothesis (Z). We recall that the Zhu algebra is $A(V) = V/O(V)$ with the product $a * b = \sum_{j \geqslant 0} \binom{\mathrm{wt}\, a}{j} a_{(j-1)} b$ [Zhu96, Definition

2.1.2 and Theorem 2.1.1]. For h of weight one, a of weight two and homogeneous x, we have $[L(-1)x] = -\operatorname{wt}(x)[x]$ [Zhu96, Lemma 2.1.1] and, by [Zhu96, Lemma 2.1.3],

$$[h_{(-1)}x] = [x]*[h], \quad [h_{(0)}x] = [h]*[x] - [x]*[h], \quad [a_{(0)}x] = [a]*[x] - [x]*[a] - [a_{(1)}x]. \tag{11.1}$$

Proposition 11.3 *The algebra $A(S)$ is generated by the classes of the six currents and of the seventeen highest weight vectors of U_2 for $\mathfrak{g}' \oplus \mathfrak{g}''$: the vectors $e'_{(-2)}\mathbf{1}$, $e'_{(-1)}e'$, $e''_{(-2)}\mathbf{1}$, $e''_{(-1)}e''$, $e'_{(-1)}e''$, the two Casimir vectors of $(A)_2$, the vectors e, x_C, and the highest weight vectors of the eight primary types $(0,4)$, $(1,1)$, $(1,3)$, $(1,5)$, $(1,7)$, $(2,2)$, $(2,6)$, $(3,3)$. The classes of the first seven are polynomials in the classes of the currents.*

Proof : By (11.1), the classes of $L(-1)U_2$, $U_1(-1)U_2$ and $(U_2)_{(0)}U_2$ lie in the subalgebra generated by $[U_1]$ and $[U_2]$, since $(U_2)_{(1)}U_2 \subset U_2$. Hence so does $[S_3]$, because $S_3 = N_3 + (U_2)_{(0)}U_2$. By Theorem 10.21, S is strongly generated by $U_1 \oplus U_2 \oplus D_3$, and the classes of a strong generating set generate $A(S)$: replacing a derivative by $[L(-1)x] = -\operatorname{wt}(x)[x]$ and a normal product $g_{(-1)}y$ by $[g]*[y]$ minus classes of vectors of smaller weight, induction on the weight writes every class as a polynomial in the classes of the generators. Finally, for $f = f'$ or $f = f''$, the identity $[f_{(0)}z] = [f]*[z] - [z]*[f]$ reduces $[U_2]$ to the highest weight vectors, which are those listed (Table 2). The first seven are $L(-1)e'$, $e'_{(-1)}e'$, $L(-1)e''$, $e''_{(-1)}e''$, $e'_{(-1)}e''$ and the Casimir vectors, whose classes are given by (11.1). ∎

Unlike in case (I), no strong generator of weight four is involved.

Proposition 11.4 *Let $A = L_{\mathfrak{sl}_2}(3,0) \otimes L_{\mathfrak{sl}_2}(13,0) \subset S$ be the affine subalgebra and $C = \operatorname{Com}_S(A)$.*

1. *The Cartan weight space of weight $(3,13)$ of V_4' is spanned by a single lattice exponential e^w, with $w = (2,2,4,5,4,2,1,1)$ in the simple-root coordinates fixed above and $(w,w) = 8$. One has $e^w \in S$, and it is an affine highest weight vector generating $J = L_{\mathfrak{sl}_2}(3,3) \otimes L_{\mathfrak{sl}_2}(13,13)$. The subspace $A^+ = A \oplus J$ is a rational simple current extension of order two contained in S.*
2. *$\operatorname{Com}_S(C) = A \oplus J$.*
3. *The inclusion $A^+ \subset S$ induces a unital homomorphism $A(A^+) \to A(S)$.*

Proof : We first identify the kernel of the coset grading. Write $\omega = \omega_A + \omega_C$, where $\omega_C = 2e \in C$. The unitary involution fixes both conformal vectors, so $(L^C_n)^* = L^C_{-n}$. For the positive Hermitian form H_χ of Theorem 10.10, the Virasoro relation gives

$$\|L^C_{-1}x\|^2_\chi = \|L^C_1 x\|^2_\chi + 2H_\chi(L^C_0 x, x). \tag{11.2}$$

The regularity of A decomposes S into integrable A-modules. On the multiplicity space of affine highest weight (i,j), $L^C_0 = L(0) - h_A(i,j)$ has discrete spectrum bounded below, where $h_A(i,j) = \frac{i(i+2)}{20} + \frac{j(j+2)}{60}$. The operators L^C_n preserve this space. If x has its least eigenvalue λ, then $L^C_1 x = 0$, and (11.2) gives $\|L^C_{-1}x\|^2_\chi = 2\lambda\|x\|^2_\chi$. Thus $\lambda \geqslant 0$. Since L^C_0 commutes with the action of A, it has nonnegative spectrum on all of S.

For $x \in \operatorname{Ker} L^C_0$, nonnegativity first gives $L^C_1 x = 0$, and then (11.2) gives $L^C_{-1}x = 0$. If $c \in C$, truncation gives $c_{(n)}x = 0$ for sufficiently large n. The translation identity $[L^C_{-1}, c_{(n+1)}] = -(n+1)c_{(n)}$ then gives $c_{(n)}x = 0$ for every $n \geqslant 0$, by downward induction. Conversely, the commutant condition applied to ω_C implies $L^C_0 x = 0$. Consequently $\operatorname{Ker} L^C_0 = \operatorname{Com}_S(C)$.

1. The lattice realization gives the following lower bound: if the Cartan weight space of weight (a,b) in V'_m is nonzero, then

$$m \geqslant \frac{a^2}{12} + \frac{b^2}{52}. \tag{11.3}$$

Indeed, the two Cartan vectors are orthogonal, of squared norms 6 and 26. A lattice vector of weight (a,b) therefore has squared norm at least $a^2/6 + b^2/26$, and the oscillators have nonnegative integral degree and do not change this weight. A lattice vector of Cartan weight $(3,13)$ has orthogonal projection $\frac{1}{2}(H'+H'')$ onto the plane of the two currents. This projection has squared norm $\frac{3^2}{6} + \frac{13^2}{26} = 8$. The displayed vector w has this Cartan weight and squared norm 8, hence equals this projection. Therefore it is the only lattice vector of that Cartan weight with squared norm at most 8, and no oscillator can occur at conformal weight four. This proves that the indicated weight space of V'_4 is one-dimensional.

Let $u_{(1,7)}$ and $u_{(2,6)}$ denote the primary highest weight vectors with the coefficients of Table 4. A direct exact product gives

$$(u_{(1,7)})_{(-1)}u_{(2,6)} = \left(\frac{507}{5}\sqrt{13} + \frac{3549}{5}\sqrt{-3}\right)e^w. \tag{11.4}$$

Here is also a direct verification of this product. Put $r = \sqrt{13}$ and $s = \sqrt{-3}$. The two generators have the form $u_{(1,7)} = \sum_{i=1}^{7} a_i e^{\alpha_i}$ and $u_{(2,6)} = \sum_{i=1}^{7} b_i e^{w-\alpha_i}$, with the coefficients in Table 4. Each of α_i and $w-\alpha_i$ has squared norm 4, and $\langle \alpha_i, w-\alpha_i\rangle_L = 0$. For different indices, $\alpha_i + w - \alpha_j \neq w$, so the one-dimensional weight space just proved shows that its contribution to the product is zero. The lattice formula (10.1) therefore gives

$$\begin{aligned}(u_{(1,7)})_{(-1)}u_{(2,6)} &= \sum_{i=1}^{7} \varepsilon(\alpha_i, w-\alpha_i)a_i b_i e^w \\ &= \frac{26+104+26+104+65+65+117}{5}(r+7s)e^w.\end{aligned} \tag{11.5}$$

This proves (11.4) by arithmetic in $\mathrm{Q}(r,s)$, without a weight-four membership calculation. Its coefficient is nonzero, so $e^w \in S$.

α_i	a_i	b_i	$\varepsilon(\alpha_i, w-\alpha_i)$
$(1,1,2,3,2,1,1,0)$	$\frac{39+r-13s+rs}{5}$	$\frac{-13+3r+13s+rs}{4}$	1
$(1,2,3,4,3,2,0,0)$	$\frac{-104-56rs}{5}$	r	-1
$(1,1,2,3,2,1,0,1)$	$\frac{39-r+13s+rs}{5}$	$\frac{-13-3r-13s+rs}{4}$	-1
$(2,2,3,4,3,1,0,0)$	$-52+12rs$	$\frac{-5r+13s}{10}$	-1
$(0,1,1,2,2,1,1,1)$	$\frac{-19r-13s}{15}$	$\frac{39+9rs}{2}$	-1
$(1,1,2,2,1,0,1,1)$	r	$13+7rs$	1
$(1,0,1,1,1,1,1,1)$	$\frac{3r-39s}{5}$	$\frac{39-3rs}{2}$	-1

Table 4: The seven pairs in the product (11.4), in the simple-root coordinates of $E_6 \oplus A_2$.

The highest weight conditions also follow from (11.3). The vectors $e'_{(0)}e^w$ and $e''_{(0)}e^w$ would have degree 4 and Cartan weights $(5,13)$ and $(3,15)$, whose lower bounds are $16/3$ and $66/13$. Hence both vanish. For a current y of Cartan weight $(0,0)$, $(\pm 2,0)$ or $(0,\pm 2)$, the vector $y_{(n)}e^w$,

$n \geqslant 1$, has degree at most 3, whereas the least of the five bounds for its possible Cartan weights is $40/13 > 3$. Thus every positive current mode annihilates e^w. It generates the irreducible integrable A-module J.

An independent exact calculation reconstructs the product and verifies the 26 annihilation identities for $e'_{(0)}$, $e''_{(0)}$ and the six currents in modes 1 to 4. It uses no weight-four pivot data. A separate arithmetic check verifies the arithmetic and cocycle signs of Table 4. These computational checks supplement the preceding analytic calculation.

Its affine conformal weight is 4, so $L_0^C e^w = 0$. Since L_0^C commutes with A, both A and J lie in $\operatorname{Ker} L_0^C = \operatorname{Com}_S(C)$. In this commutant the multiplicities of A and J are one: their affine highest spaces occur in total weights zero and four, respectively, where $S_0 = \mathbb{C}\mathbf{1}$ and the Cartan weight space $(3, 13)$ is one-dimensional. The restrictions of the vertex operators to these summands are A-intertwining operators. The affine fusion rules $A \boxtimes A = A$, $A \boxtimes J = J$ and $J \boxtimes J = A$ therefore imply that $A \oplus J$ is closed under all products. We equip A^+ with conformal vector ω_A. Since L_0^C and L_{-1}^C vanish on A^+, its grading and translation are those induced from S. The unitary involution preserves these two isotypic summands, so this subalgebra inherits the unitary form, is simple, and is rational by [DLM96, Theorem 4.5].

2. By Proposition 10.17, Theorem 10.21 gives $S_1 = U_1$ and $S_2 = U_2$. As an A-module the kernel is the direct sum of the integrable modules generated by affine highest weight vectors of type (i, j) in weight $h_A(i, j)$, which must be an integer. The types with $0 \leqslant i \leqslant 3$, $0 \leqslant j \leqslant 13$ and $h_A(i, j) \in \mathbb{Z}$ are $(0, 0)$, $(3, 3)$, $(0, 10)$ and $(3, 13)$, of weights 0, 1, 2 and 4. Now $S_0 = \mathbb{C}\mathbf{1}$, $S_1 = U_1$ has no component of type $(3, 3)$, U_2 has no component of type $(0, 10)$ (Table 2), and the Cartan weight space $(3, 13)$ of S_4 has dimension one by part 1. Hence $\operatorname{Com}_S(C) = A \oplus J$.

3. On A^+, L_0^C and L_{-1}^C vanish, so the grading and the translation of A^+ are those of S. Hence the products $*$ and $\circ$ of A^+ are the restrictions of those of S, and $O(A^+) \subset O(S)$. ∎

The simple A-modules are $L(i, j) = L_{\mathfrak{sl}_2}(3, i) \otimes L_{\mathfrak{sl}_2}(13, j)$, with lowest weight space $V(i, j)$. The monodromy of $L(i, j)$ with J is $(-1)^{i+j}$ and $J \boxtimes L(i, j) \cong L(3 - i, 13 - j)$. Hence the simple A^+-modules correspond [DLM96, HKL15] to the fourteen orbits $\{(i, j), (3 - i, 13 - j)\}$ with $i + j$ even, and there is no fixed point. Since $h_A(3 - i, 13 - j) - h_A(i, j) = 4 - \frac{i+j}{2}$, the lowest weight space T_t of the module M_t attached to an orbit is $V(i, j)$ with $i + j < 8$, except for the two orbits with $i + j = 8$:

$$
\begin{aligned}
&T_0, \dots, T_5 = V(0,0),\ V(0,2),\ V(1,1),\ V(2,0),\ V(0,4),\ V(1,3),\\
&T_6, \dots, T_{11} = V(2,2),\ V(3,1),\ V(0,6),\ V(1,5),\ V(2,4),\ V(3,3),\\
&T_{12} = V(1,7) \oplus V(2,6) \quad (h = \tfrac{6}{5}), \quad T_{13} = V(0,8) \oplus V(3,5) \quad (h = \tfrac{4}{3}).
\end{aligned} \tag{11.6}
$$

Thus $A(A^+) \cong \bigoplus_{t=0}^{13} \operatorname{End}(T_t)$. Let $e_t \in A(S)$ be the image of a rank one idempotent of $\operatorname{End}(T_t)$ (the highest line of the first constituent) and $e_{\mathrm{Z}} = \sum_t e_t$. The sum of one rank one idempotent in every matrix block is full in $A(A^+)$. Indeed, for matrix units $E_{ij}^{(t)}$ with the chosen idempotent $E_{11}^{(t)}$, one has $1 = \sum_{t,i} E_{i1}^{(t)} E_{11}^{(t)} E_{1i}^{(t)}$. Applying the unital homomorphism of Proposition 11.4 proves that e_{Z} is full in $A(S)$, even if some blocks have zero image, and $A(S)$ is Morita equivalent to the corner $D = e_{\mathrm{Z}} A(S) e_{\mathrm{Z}}$ [Rie26b, Lemma 8.2]. In particular $A(S)$ is semisimple if and only if D is.

Remark 11.5 Every $\mathfrak{g}' \oplus \mathfrak{g}''$-type of the lowest weight spaces in Table 2 is local and of the form (i, j) with $i+j < 8$, as it must be: a constituent of type $(3-i, 13-j)$ in the lowest weight space of an S-module M would force, through J, a vector of M of smaller weight. If $A(S)$ is semisimple and its simple modules have the lowest weight spaces of Table 2, then $D \cong \bigoplus_z \mathbb{M}_{n_z}(\mathbb{C})$, where n_z is the number of irreducible constituents of the lowest weight space of z: the block sizes are

$1, 1, 2, 2, 2, 3, 5, 7, 11, 12, 12, 16$ (for 0, c_1, d_1, d_4, d_5, d_6, c_2, d_2, b, a, c_0, d_3), so that $\dim D = 762$, and the two exceptional blocks T_{12}, T_{13} do not occur. In case (I) the corresponding dimension is 546 [Rie26b, Proposition 9.5].

To obtain an upper bound for D, we follow [Rie26b, Section 9]. Put $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$, and let Γ_Z be the quiver with vertices $0, \dots, 13$ and, for each of the ten non-affine highest weight vectors z of Proposition 11.3 and each pair of constituents $\lambda \subset T_t$, $\mu \subset T_v$ such that $\mathrm{Hom}(V(\lambda), V(\mu))$ contains a highest weight vector of the weight of z, an arrow $t \to v$. There are 515 arrows. Sending a formal arrow to the corresponding coefficient of $[z]$ in the matrix realization of $A(S)$ given by the matrix units of $A(A^+)$ defines a surjection $\mathbb{C}\Gamma_Z \to D$ [Rie26b, Propositions 8.3 and 8.6]. The relations are the projections of the Zhu images of three families of identities in S:

1. the kernel of $U_2 \oplus (\mathfrak{g} \otimes U_2) \oplus \Lambda^2 U_2 \to S_3$, $(v, h \otimes w, a \wedge b) \mapsto L(-1)v + h_{(-1)}w + \frac{1}{2}(a_{(0)}b - b_{(0)}a)$,
2. the kernel of $S^2 U_2 \to S_4$, $a \cdot b \mapsto \frac{1}{2}(a_{(-1)}b + b_{(-1)}a)$,
3. the kernel of $S_3 \oplus (\mathfrak{g} \otimes S_3) \oplus S^2 U_2 \to S_4$, $(u, h \otimes v, a \cdot b) \mapsto L(-1)u + h_{(-1)}v + \frac{1}{2}(a_{(-1)}b + b_{(-1)}a)$,

together with the centrality of $[\omega] = [\omega_A] + 2[e]$. The kernels are $\mathfrak{g}$-modules. Their highest weight vectors suffice. The Zhu images are obtained from (11.1) and from $[\frac{1}{2}(a_{(-1)}b + b_{(-1)}a)] = \frac{1}{2}([a] * [b] + [b] * [a]) + [c_+(a,b)]$, $[\frac{1}{2}(a_{(0)}b - b_{(0)}a)] = [a] * [b] - [b] * [a] - [c_-(a,b)]$, where $c_\pm(a,b) = \frac{1}{2}(a_{(1)}b \pm b_{(1)}a) \in U_2$.

Theorem 11.6 (computation modulo a prime) *Let $p = p_3 = 8388571$, and use the residue map $(\mathcal{O}_\mathbb{K})_\mathfrak{p} \to \mathbb{F}_p$ at the chosen prime $\mathfrak{p}$ above p, with the images of $\sqrt{13}$ and $\sqrt{-3}$ fixed by the computational data. The matrices below have entries integral at $\mathfrak{p}$. Modulo p:*

1. *the three families have 458, 70 and 532 highest weight relations, and S_3 has 95 highest weight vectors,*
2. *the projected relations, 19981 rows in the path algebra $\mathbb{F}_p\Gamma_Z$, kill the two exceptional vertices 12 and 13, hence the 230 arrows incident to them, eliminate 46 further arrows linearly, and leave a rewriting system on 239 arrows whose ambiguities all resolve after one round of completion,*
3. *the resulting algebra D_p has a basis of 762 irreducible paths (12 vertices, 239 arrows, 506 paths of length two and 5 of length three). Its regular trace form is nondegenerate, so D_p is semisimple. Its center has dimension 12, and $D_p \cong \bigoplus_z \mathbb{M}_{n_z}(\mathbb{F}_p)$ with the block sizes of Remark 11.5,*
4. *for each block, the ranks of the vertex idempotents e_t are the multiplicities of the constituents T_t in the lowest weight space of the corresponding module in Table 2, for all twelve modules.*

Proof : All products $a_{(n)}b$, $a, b \in U_2$, $-1 \leqslant n \leqslant 3$, and $h_{(-1)}s$, $L(-1)s$, $s \in S_3$, are computed exactly over $\mathbb{K}$ and then reduced modulo p. The kernels, projections, elimination, completion, trace form and central idempotents are computed over $\mathbb{F}_p$. The calculation at 8388571 uses 19981 projected relations. The corresponding calculation at 4194277 gives the same dimensions, blocks and vertex ranks. For the final rewriting rules, a separate verifier checks that they decrease the degree-lexicographic order with the alphabet reversed and resolves all 131698 overlap ambiguities and all 151 inclusion ambiguities, as required by [Ber78, Theorem 1.2].

The inclusion ambiguities are needed because the system has cubic as well as quadratic leading words. These checks concern the projected presentation.

For completeness, the algebraic consequences of the last checks require no further VOA computation. If x lies in the Jacobson radical of D_p, then xy is nilpotent for every $y \in D_p$. Thus left multiplication by xy has trace zero. Nondegeneracy of the regular trace form forces $x = 0$, proving semisimplicity. The computed central element has twelve distinct eigenvalues in $\mathbb{F}_p$, so the center is split. Each simple factor is a matrix algebra over $\mathbb{F}_p$, since a finite division ring is a field. Its dimension determines its matrix size. The ranks of the vertex corners give the displayed multiplicities. These conclusions concern the finite-field algebra D_p. Their descent to a characteristic-zero upper presentation is not supplied by this calculation. ∎

Remark 11.7 Theorem 11.6 computes the upper presentation at two primes. Its block sizes and vertex ranks agree with those predicted by Theorem 8.5 and Table 2. Agreement at two primes is a consistency check. The kernel may grow after reduction modulo a prime, and a rank drop at both primes is possible. To deduce (Z) by descent requires a finite integral order with this special fibre and the surjection from its characteristic-zero algebra onto the Zhu corner. These steps are carried out in [ArhIII, Theorems A.10 and A.13]. The survival of all twelve blocks in the Zhu corner is proved separately in [ArhIII, Theorem A.20].

Lemma 11.8 *The nine V'-modules $W_{e,a}$ are unitary S-modules. Consequently each restriction $W_{e,a}|_S$ is an orthogonal direct sum of simple S-modules, and the S-submodule generated by an irreducible $A(S)$-submodule of the lowest weight space of $W_{e,a}$ is simple.*

Proof : By [DoL14, Theorem 4.12], $V_{L+\lambda}$ is a unitary module of (V_L, ϕ_1) for the Fock space Hermitian form H_1. With $g = \exp(h_\chi(0))$ as in the proof of Theorem 10.10, $g^{1/2}$ acts on $V_{L+\lambda}$ by the positive numbers $e^{\frac{1}{2}(h_\chi,\mu)}$ on $M(1) \otimes e^{\mu}$, and $H_\chi(u,v) = H_1(g^{1/2}u, g^{1/2}v)$ is positive definite and invariant for (V', ϕ_χ). Its restriction to S is invariant since $\phi_\chi(S) = S$. The orthogonal complement of a submodule is a submodule, and the homogeneous components are finite-dimensional, which gives the decomposition. If X is an irreducible $A(S)$-submodule of the lowest weight space and N a proper submodule of the module $\langle X \rangle$ it generates, then $\langle X \rangle = N \oplus N^{\perp}$ and $X = (N \cap X) \oplus (N^{\perp} \cap X)$ in the lowest degree, so $N \cap X = 0$, $X \subset N^{\perp}$ and $N = 0$. ∎

11.5 Completion of the rationality argument

Appendix A of [ArhIII] proves (Z) and (M) for S. Theorem 2.2(F1) of [ArhIII] states (Z) and the resulting strong rationality. Parts (F2)–(F3) give (M), including the complete lattice branching and the lowest weight spaces of the twelve simple modules. The proof of (M) combines the restrictions of the lattice modules with Galois symmetry to show that all twelve generic blocks survive in the Zhu corner. The braided equivalence that completes the proof of ($\mathrm{H_J}$) is established in [ArhIII, Theorem 6.1].

12 The associated lattice and the twin algebra

In this section we describe the Haagerup algebras through the decomposition of a strongly rational vertex operator algebra into its Heisenberg commutant and its associated lattice [Mas14, Hoh17, MoR24a], and we use the notions of genus of Höhn [Hoh03] (the bulk genus, given by the representation category and the central charge) and of Moriwaki [Mor21] and Möller–Rayhaun [MoR24a] (the hyperbolic genus). We call a *Haagerup algebra* a positive strongly rational vertex

operator algebra U of central charge eight with a braided equivalence $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$. By Proposition 4.1 and Theorem 8.5, after an automorphism of V', $U \subset V'$ with $U_1 = \mathfrak{g}' \oplus \mathfrak{g}''$, of levels $(k', k'') = (1, 39)$ in case (I) and $(3, 13)$ in case (II), and the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h' \oplus \mathbb{C}h''$ of U_1 is contained in the Cartan subalgebra of V_1' spanned by the lattice. Here $h' \in \mathfrak{g}'$ and $h'' \in \mathfrak{g}''$ are coroots, so that $\langle h', h' \rangle = 2k'$, $\langle h'', h'' \rangle = 2k''$ and $\langle h', h'' \rangle = 0$. We identify $\mathfrak{h}$ with its dual through $\langle \cdot, \cdot \rangle$ and write a vector $\lambda \in \mathfrak{h}$ as the pair $(m', m'') = (\langle h', \lambda \rangle, \langle h'', \lambda \rangle)$. Thus λ is the $\mathfrak{h}$-weight of a vector v with $h'_{(0)}v = m'v$ and $h''_{(0)}v = m''v$, and

$$|\lambda|^2 = \frac{m'^2}{2k'} + \frac{m''^2}{2k''}.$$

We denote by $\Gamma(U) \subset \mathfrak{h}$ the set of $\mathfrak{h}$-weights of U, by

$$\Omega_U \stackrel{\mathrm{def}}{=} \mathrm{Com}_U(M(\mathfrak{h}))$$

the *Heisenberg commutant* of U, and by L the *associated lattice*, defined by $\mathrm{Com}_U(\Omega_U) \cong V_L$ [MoR24a, Definition 2.12]. Throughout this section L denotes this rank-two lattice. We continue to write $E_6 \oplus A_2$ for the ambient lattice. These objects do not depend, up to isomorphism, on the choice of the Cartan subalgebra [MoR24a, Remark 2.13]. For an even lattice L we write $\mathcal{C}(L'/L)$ for the pointed ribbon category of the discriminant form $(L'/L, q_L)$, $q_L(\lambda + L) = \frac{|\lambda|^2}{2} \bmod 1$, so that $\mathrm{Rep}(V_L) \cong \mathcal{C}(L'/L)$, and $\overline{\mathcal{B}}$ for the ribbon reverse of a ribbon category $\mathcal{B}$.

12.1 Two binary lattices

Put $\Gamma_0 = \{(m', m'') \in \mathbb{Z}^2 : m' \equiv m'' \bmod 2\}$ and, for $(k', k'') \in \{(1, 39), (3, 13)\}$,

$$L_{k',k''} \stackrel{\mathrm{def}}{=} \Gamma_0^* = \{(k'a, k''c) : a, c \in \mathbb{Z},\ a \equiv c \bmod 2\},$$

the dual being taken for the form above. It has the basis $(2k', 0)$, (k', k''). We write $L_{\mathrm{I}} = L_{1,39}$ and $L_{\mathrm{II}} = L_{3,13}$.

Lemma 12.1 *1. The Gram matrices of L_{I} and L_{II} in these bases are $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 20 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 6 & 3 \\ 3 & 8 \end{smallmatrix}\right]$, of determinant 39 and minimal norms 2 and 6. They correspond to the binary quadratic forms $x^2 + xy + 10y^2$ and $3x^2 + 3xy + 4y^2$. In both cases $L' = \Gamma_0$ and $L'/L \cong \mathbb{Z}/39$ is generated by $(1, 1)$, with $q_{L_{\mathrm{I}}}(1, 1) = \frac{10}{39}$ and $q_{L_{\mathrm{II}}}(1, 1) = \frac{4}{39}$.*

2. The even positive-definite lattices of rank two whose discriminant form is isometric to that of L_{I} are, up to isometry, exactly L_{I} and L_{II}. More precisely, the reduced forms of discriminant -39 are $(1, 1, 10)$, $(2, \pm 1, 5)$ and $(3, 3, 4)$. The class group of discriminant -39 is cyclic of order four and has two genera, the principal genus $\{(1, 1, 10), (3, 3, 4)\}$, which is the set of squares, and $\{(2, 1, 5), (2, -1, 5)\}$, whose discriminant form is not isometric to that of L_{I}.

3. The isometries $L'_{\mathrm{II}}/L_{\mathrm{II}} \to L'_{\mathrm{I}}/L_{\mathrm{I}}$ are the maps $(1, 1) \mapsto u(1, 1)$ with $u \in \{4, 17, 22, 35\}$, and $O(L'/L) = \{\pm 1, \pm 14\}$ in both cases.

4. The image of the orthogonal group $O(L_{\mathrm{II}})$, of order four, in $O(L'_{\mathrm{II}}/L_{\mathrm{II}})$ is the whole group. The reflections $w' = \mathrm{diag}(-1, 1)$ and $w'' = \mathrm{diag}(1, -1)$, which are the actions on $\mathfrak{h}$ of the Weyl group elements of $\mathfrak{g}'$ and $\mathfrak{g}''$, act by 14 and 25. The image of $O(L_{\mathrm{I}})$, of order four, in $O(L'_{\mathrm{I}}/L_{\mathrm{I}})$ is $\{\pm 1\}$: w' acts trivially and w'' by -1.

Proof : We give a proof with integer arithmetic, independently of the computational verification. For part 1, the form $\frac{m'^2}{2k'} + \frac{m''^2}{2k''}$ on the basis $(2k', 0)$, (k', k'') has Gram matrix $M_{k',k''}$ with entries $M_{11} = 2k'$, $M_{12} = M_{21} = k'$ and $M_{22} = \frac{k'+k''}{2}$. Its determinant is $k'k'' = 39$. An integer multiple $n(1, 1)$ belongs to $L_{k',k''}$ exactly when 39 divides n, so $(1, 1)$ generates the discriminant group. Its quadratic value is $\frac{1}{4k'} + \frac{1}{4k''}$, giving the asserted values. The expressions

$$x^2 + xy + 10y^2 = \left(x + \frac{y}{2}\right)^2 + \frac{39}{4}y^2, \quad 3x^2 + 3xy + 4y^2 = 3\left(x + \frac{y}{2}\right)^2 + \frac{13}{4}y^2$$

show that the shortest vectors are precisely the two vectors $\pm(2k', 0)$, of squared norms 2 and 6, respectively.

2. An even positive-definite binary lattice of determinant 39 corresponds to a primitive form $ax^2 + bxy + cy^2$ of discriminant -39. Choose a reduced representative, so $|b| \leqslant a \leqslant c$, with the usual boundary convention $b \geqslant 0$ when $|b| = a$ or $a = c$. Since $39 = 4ac - b^2 \geqslant 3a^2$, we have $1 \leqslant a \leqslant 3$. Moreover b is odd and $c = \frac{39+b^2}{4a}$ is an integer. For $a = 1, 2, 3$ this gives exactly $(1, 1, 10)$, $(2, \pm 1, 5)$ and $(3, 3, 4)$. The two middle forms are improperly equivalent. Their lattice has Gram matrix $\left[\begin{smallmatrix} 4 & 1 \\ 1 & 10 \end{smallmatrix}\right]$ and discriminant form generated by an element of quadratic value $\frac{5}{39}$. An isometry with the discriminant form of L_{I} would require $5u^2 \equiv 10 \pmod{39}$ for a unit u, which is impossible modulo 3. In contrast, $10 \cdot 4^2 \equiv 4 \pmod{39}$ identifies the discriminant forms of L_{I} and L_{II}. This proves the lattice classification. It also gives the two genera, since the genus is determined by the rank, signature and discriminant form [Nik80, Corollary 1.9.4]. The four reduced forms are the four proper classes. Inversion interchanges the two distinct classes $(2, 1, 5)$ and $(2, -1, 5)$, so the class group of order four is cyclic. Its squares are the principal class and the class $(3, 3, 4)$, proving the remaining assertion [Cox13].

3. An isometry between the discriminant forms is multiplication by a unit u satisfying $10u^2 \equiv 4 \pmod{39}$, equivalently $u^2 \equiv 16 \pmod{39}$. The Chinese remainder theorem gives the four solutions $4, 17, 22, 35$. In the same way, $u^2 \equiv 1 \pmod{39}$ gives $O(L'/L) = \{1, 14, 25, 38\}$.

4. An orthogonal transformation preserving either lattice preserves its unique shortest line, so its orthogonal group has at most four elements. The four sign changes in the two coordinates (m', m'') preserve $L_{k',k''}$, so they give the whole group. To compute their action on the discriminant group, reduce $(-1, 1)$ and $(1, -1)$ modulo $L_{k',k''}$. In case (II),

$$(-1, 1) - 14(1, 1) = (-15, -13) \in L_{\mathrm{II}}, \quad (1, -1) - 25(1, 1) = (-24, -26) \in L_{\mathrm{II}}.$$

In case (I), $(-1, 1) - (1, 1) = (-2, 0) \in L_{\mathrm{I}}$ and $(1, -1) + (1, 1) = (2, 0) \in L_{\mathrm{I}}$. This gives the stated images.

The supplementary material includes independent integer-arithmetic checks of these lattice calculations. They are not needed for the preceding proof. ∎

12.2 The Heisenberg commutant and the associated lattice

For a simple vertex operator algebra Ω and an Ω-module M, put

$$\mathrm{Vac}_{\Omega}(M) \stackrel{\mathrm{def}}{=} \{v \in M : a_{(n)}v = 0 \text{ for every } a \in \Omega \text{ and } n \geqslant 0\}.$$

A nonzero vector in this space is called *vacuum-like*. If M is irreducible and v is vacuum-like, the map $a \mapsto a_{(-1)}v$ is a nonzero Ω-module homomorphism $\Omega \to M$, and hence $M \cong \Omega$ [LL04, Section 4.7].

Proposition 12.2 *Let U be a Haagerup algebra, with the notation above.*

1. *The Heisenberg commutant Ω_U is a strongly rational vertex operator algebra of central charge six with $(\Omega_U)_1 = 0$. There is a ribbon-reversing equivalence τ from $\mathcal{C}(L'/L)$ onto the full pointed fusion subcategory generated by the invertible objects of* $\mathrm{Rep}(\Omega_U)$*. Writing τ also for the induced map on isomorphism classes of simple objects, we have*

$$U = \bigoplus_{\alpha\in L'/L} \Omega_U^{\tau(\alpha)} \otimes V_{\alpha+L}$$

as an $\Omega_U \otimes V_L$-module, and $\mathrm{Rep}(\Omega_U) \cong Z(\mathcal{H}_3) \boxtimes \overline{\mathcal{C}(L'/L)}$.

2. *Every irreducible Ω_U-module has a nonnegative conformal weight, and for $\alpha \neq 0$ the module $\Omega_U^{\tau(\alpha)}$ contains no vacuum-like vector. Consequently, a vector of U of $\mathfrak{h}$-weight λ has conformal weight at least $\frac{|\lambda|^2}{2}$.*

3. $\Gamma(U) = \Gamma_0$*, and the associated lattice is $L = L_{k',k''}$: it is L_{I} in case (I) and L_{II} in case (II).*

4. *For $\lambda = (k', k'')$, the vector e^λ of $V_L \subset U$ is a highest weight vector of weight (k', k'') for the affine subalgebra $A = L_{\mathfrak{g}'}(k', 0) \otimes L_{\mathfrak{g}''}(k'', 0)$, of conformal weight $\frac{k'+k''}{4}$, and it generates an A-submodule of U isomorphic to the simple current $J = L_{\mathfrak{sl}_2}(k', k') \otimes L_{\mathfrak{sl}_2}(k'', k'')$, of conformal weight 10 in case (I) and 4 in case (II).*

Proof : 1. By [MoR24a, Definition 2.12, Proposition 2.14], which rest on [Mas14], Ω_U and V_L form a dual pair in U, Ω_U is strongly rational [CKLR19], and U is a simple current extension of $\Omega_U \otimes V_L$ of the stated form for some subgroup $A_U \subset L'/L$ in place of L'/L. The category $\mathrm{Rep}(U) \cong Z(\mathcal{H}_3)$ has no invertible object except the unit, by (2.1). It is pseudo-unitary for the ribbon structure of U: since U is positive, the quantum dimensions $\mathrm{qdim}_U(X) = S_{X,U}/S_{U,U}$ are positive and equal the Frobenius–Perron dimensions [DJX13], and they are the categorical dimensions of $\mathrm{Rep}(U)$ for its ribbon structure [Hua08]. Alternatively, $Z(\mathcal{H}_3)$ has a unique pivotal structure (Remark 12.6). The proof of [MoR24a, Corollary 4.10], applied with $V = W = U$, then gives $A_U = L'/L$, the description of τ, and $\mathrm{Rep}(\Omega_U) \cong \mathrm{Rep}(U) \boxtimes \overline{\mathcal{C}(L'/L)}$. The central charge of Ω_U is $8 - \mathrm{rank}\, L = 6$. A vector of $(\Omega_U)_1$ commutes with $\mathfrak{h}$, hence lies in the subspace of $\mathfrak{h}$-weight zero of $U_1 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, which is $\mathfrak{h}$. It is then orthogonal to $\mathfrak{h}$, hence zero.

2. Let μ be an irreducible Ω_U-module. The monodromy of μ with the invertible objects $\tau(\alpha)$ is a scalar $\chi_\mu(\alpha)$, and $\alpha \mapsto \chi_\mu(\alpha)$ is a character of L'/L. Since the bilinear form of L'/L is nondegenerate, there is $\beta \in L'/L$ such that the monodromy of $\mu \boxtimes \beta$ with every summand $\tau(\alpha)\boxtimes\alpha$ of the algebra U is trivial. The induced module $X = \bigoplus_\alpha (\tau(\alpha)\mu)\otimes V_{\alpha+\beta+L}$, where $\tau(\alpha)\mu$ denotes the fusion product, is then a U-module [HKL15, CKM22], a finite sum of irreducible U-modules, and it contains $\mu \otimes V_{\beta+L}$. For $y > 0$ we have, coefficientwise and hence numerically,

$$\mathrm{ch}_X(\mathrm{i}y) \geqslant \mathrm{ch}_\mu(\mathrm{i}y) \frac{\theta_{\beta+L}(\mathrm{i}y)}{\eta(\mathrm{i}y)^2}.$$

As $y \to 0^+$, $\theta_{\beta+L}(\mathrm{i}y) \sim (\det L)^{-\frac12} y^{-1}$ by the Poisson summation formula and $\eta(\mathrm{i}y)^{-2} = y\eta(\mathrm{i}/y)^{-2} \sim y\mathrm{e}^{\frac{2\pi}{12y}}$, so that the last quotient is equivalent to $(\det L)^{-\frac12}\mathrm{e}^{\frac{2\pi\cdot 2}{24y}}$. On the other hand $\mathrm{ch}_X(\mathrm{i}y) = \sum_Y S_{X,Y}\,\mathrm{ch}_Y(\mathrm{i}/y)$ over the irreducible U-modules Y, by Zhu's theorem [Zhu96] at the point i/y, extended additively to the finite sum X, and $\mathrm{ch}_Y(\mathrm{i}/y) = O(\mathrm{e}^{\frac{2\pi\cdot 8}{24y}})$ since the conformal weights of the U-modules are nonnegative. Hence $\mathrm{ch}_\mu(\mathrm{i}y) = O(\mathrm{e}^{\frac{2\pi\cdot 6}{24y}})$ for every irreducible Ω_U-module μ. For an irreducible Ω_U-module ν of conformal weight h_ν, Zhu's theorem for Ω_U gives $\mathrm{ch}_\nu(\mathrm{i}/y) = \sum_\mu S^\Omega_{\nu,\mu}\,\mathrm{ch}_\mu(\mathrm{i}y) = O(\mathrm{e}^{\frac{2\pi\cdot 6}{24y}})$, while $\mathrm{ch}_\nu(\mathrm{i}/y) \geqslant \dim(\nu_{h_\nu})\mathrm{e}^{\frac{2\pi}{y}(\frac{6}{24}-h_\nu)}$. Therefore

$h_\nu \geqslant 0$. For $\alpha \neq 0$, the simple module $\Omega_U^{\tau(\alpha)}$ is not isomorphic to Ω_U, so $\mathrm{Vac}_{\Omega_U}(\Omega_U^{\tau(\alpha)}) = 0$. Finally a vector of $\mathfrak{h}$-weight $\lambda \in \alpha + L$ lies in $\Omega_U^{\tau(\alpha)} \otimes M(\mathfrak{h}, \lambda)$, whose conformal weights are at least $h_{\tau(\alpha)} + \frac{|\lambda|^2}{2} \geqslant \frac{|\lambda|^2}{2}$.

3. The $\mathfrak{h}$-weights of $\Omega_U^{\tau(\alpha)} \otimes V_{\alpha+L}$ form the coset $\alpha + L$, so part 1 gives $\Gamma(U) = L'$ and $L = \Gamma(U)^*$. We have $V_1' = U_1 \oplus R_b$. By Table 2, every irreducible U_1-summand of V_1' has highest weight (a, b) with $a \equiv b \bmod 2$. All weights of such a summand belong to Γ_0. Since V' is generated by V_1', every $\mathfrak{h}$-weight of V' belongs to Γ_0, and hence $\Gamma(U) \subset \Gamma_0$. Conversely, the root vectors of U_1 give the weights $(2, 0)$ and $(0, 2)$, while the summand $V(1, 1) \subset U_2$ in both columns of Table 2 gives $(1, 1)$. These vectors generate Γ_0. Since $\Gamma(U) = L'$ is a group, $\Gamma(U) = \Gamma_0$ and $L = L_{k',k''}$.

4. Since $\lambda = (k', k'') \in L$, $e^\lambda \in V_L \subset U$, of conformal weight $\frac{|\lambda|^2}{2} = \frac{k'+k''}{4}$. Let x be a root vector of U_1, of weight $\mu \in \{(\pm 2, 0), (0, \pm 2)\}$, or $x \in \mathfrak{h}$, $\mu = 0$, and $n \geqslant 0$. The vector $x_{(n)}e^\lambda$ has $\mathfrak{h}$-weight $\lambda+\mu$ and conformal weight $\frac{k'+k''}{4} - n$. For $n = 0$ and $\mu \in \{(2, 0), (0, 2)\}$, $\frac{|\lambda+\mu|^2}{2} - \frac{k'+k''}{4}$ is $1+\frac{1}{k'}$ or $1+\frac{1}{k''}$. For $n \geqslant 1$, $\frac{|\lambda+\mu|^2}{2} - (\frac{k'+k''}{4} - n) \geqslant \min(0, -1+\frac{1}{k'}, -1+\frac{1}{k''}) + n > 0$. By part 2, $x_{(n)}e^\lambda = 0$ in all these cases. Thus e^λ is annihilated by the positive part of the affine Lie algebra and by $e'_{(0)}$, $e''_{(0)}$, and $h'_{(0)}$, $h''_{(0)}$ act by k', k'': it is a highest weight vector of weight (k', k'') for A. The regularity of A decomposes U into irreducible integrable A-modules. In this decomposition e^λ is a sum of affine highest weight vectors of type (k', k''), hence generates one diagonal copy of the irreducible module J, whose conformal weight is $\frac{k'(k'+2)}{4(k'+2)} + \frac{k''(k''+2)}{4(k''+2)} = \frac{k'+k''}{4}$. ∎

In case (II), part 4 is Proposition 11.4, proved there for S by an explicit lattice word and an energy bound, with exact computational checks. In case (I), it is the inclusion $A \oplus J(1, 39) \subset U^{\mathrm{R}}$ of [Rie26b, Proposition 7.1], obtained there from the explicit vector e^w, $(w, w) = 20$. Here both statements hold for every Haagerup algebra. The associated lattice is determined by the weight-one decomposition of V' and the occurrence of $V(1, 1)$ in U_2. The lattices L_{I} and L_{II} are thus the overlattices of index two of $\sqrt{2k'}\mathbb{Z} \oplus \sqrt{2k''}\mathbb{Z}$ given by the simple current J.

12.3 The lattice exchange

Let U be a Haagerup algebra in case (I) or (II), with associated lattice L, and let $L^\natural$ be the lattice of the other case, so that $\{L, L^\natural\} = \{L_{\mathrm{I}}, L_{\mathrm{II}}\}$. By Lemma 12.1 there is an isometry $\psi\colon L'/L \to (L^\natural)'/L^\natural$. We use the equivalences $\mathrm{Rep}(\Omega_U \otimes V_L) \cong \mathrm{Rep}(\Omega_U) \boxtimes \mathcal{C}(L'/L)$ and $\mathrm{Rep}(\Omega_U \otimes V_{L^\natural}) \cong \mathrm{Rep}(\Omega_U) \boxtimes \mathcal{C}((L^\natural)'/L^\natural)$ [FHL93, CKM22]. The isometry ψ is induced by a ribbon equivalence $\Phi_\psi\colon \mathcal{C}(L'/L) \to \mathcal{C}((L^\natural)'/L^\natural)$, since a pointed braided fusion category is determined by its metric group up to braided equivalence inducing any given isometry [EGNO15, Section 8.4], and the twists $\mathrm{e}^{2\pi i q}$ are preserved. Put $\Phi = \mathrm{id} \boxtimes \Phi_\psi$. By [HKL15], U corresponds to the connected étale algebra $\mathcal{A}_U = \bigoplus_\alpha \tau(\alpha) \boxtimes \alpha$ of $\mathrm{Rep}(\Omega_U) \boxtimes \mathcal{C}(L'/L)$, and we define $U^\natural = U_\psi^\natural$ as the vertex algebra extension of $\Omega_U \otimes V_{L^\natural}$ corresponding to the connected étale algebra $\Phi(\mathcal{A}_U)$ [HKL15, CKM22]. As an $\Omega_U \otimes V_{L^\natural}$-module,

$$U^\natural = \bigoplus_{\alpha \in L'/L} \Omega_U^{\tau(\alpha)} \otimes V_{\psi(\alpha)+L^\natural}.$$

Theorem 12.3 *Let U be a Haagerup algebra, $L^\natural$ and ψ as above.*

1. *$U^\natural$ is a Haagerup algebra, in the other case of Theorem 8.5. For the Cartan subalgebra $L^\natural \otimes \mathbb{C}$ of $U_1^\natural$, its Heisenberg commutant is Ω_U and its associated lattice is $L^\natural$.*

2. *Let P be the group of invertible objects of $\mathrm{Rep}(\Omega_U)$, with the quadratic form given by the twists. The image of $\mathrm{Aut}(\Omega_U)$ in $O(P)$ is the whole group $O(P)$. Consequently, for a*

strongly rational vertex operator algebra Ω and $N \in \{L_{\mathrm{I}}, L_{\mathrm{II}}\}$, all the Haagerup algebras with Heisenberg commutant isomorphic to Ω and associated lattice isometric to N are isomorphic.

3. *Up to isomorphism, $U^\natural$ does not depend on ψ, $(U^\natural)^\natural \cong U$, and $U \mapsto U^\natural$ induces a bijection between the isomorphism classes of Haagerup algebras in case (I) and in case (II). The Haagerup algebras with Heisenberg commutant isomorphic to Ω_U are, up to isomorphism, exactly U and $U^\natural$.*

4. *U and $U^\natural$ belong to the same hyperbolic genus, and every strongly rational vertex operator algebra in this hyperbolic genus is isomorphic to U or to $U^\natural$. In particular it is a Haagerup algebra.*

Proof : 1. *Extension and representation category.* Put $B = \Omega_U \otimes V_{L^\natural}$ and $\mathcal{A} = \Phi(\mathcal{A}_U)$. The strongly rational algebra B has its vertex tensor category of ordinary modules. We use the extension correspondence in the generality of [HKL15, Theorem 3.2 and Remark 3.3]. Strict positivity of the nonvacuum B-modules is not assumed. The connected étale algebra $\mathcal{A}$ has trivial twist and defines the extension W. It is simple: separability makes the category of $\mathcal{A}$-modules semisimple, and $\mathrm{End}_{\mathcal{A}}(\mathcal{A}) \cong \mathrm{Hom}(\mathbf{1}, \mathcal{A}) = \mathbb{C}$ by connectedness. Thus the regular $\mathcal{A}$-module has no proper subobject, and neither does W as a module over itself. Initially, the extension correspondence identifies local $\mathcal{A}$-modules with those W-modules whose restrictions belong to $\mathrm{Rep}(B)$ [HKL15, CKM22]. Their category is ribbon equivalent, through Φ, to the category of local $\mathcal{A}_U$-modules. After the rationality argument below, this is the ordinary module category $\mathrm{Rep}(W) \cong \mathrm{Rep}(U)$.

CFT type. For $\alpha \neq 0$, the conformal weights of $\Omega_U^{\tau(\alpha)} \otimes V_{\psi(\alpha)+L^\natural}$ lie in $h_{\tau(\alpha)} + q_{L^\natural}(\psi(\alpha)) + \mathbb{Z} = h_{\tau(\alpha)} + q_L(\alpha) + \mathbb{Z} = \mathbb{Z}$, and they are at least $h_{\tau(\alpha)} + \min_{\gamma \in \psi(\alpha)+L^\natural} \frac{|\gamma|^2}{2} > 0$ by Proposition 12.2, part 2. Hence they are at least 1. The summand $\alpha = 0$ is $\Omega_U \otimes V_{L^\natural}$, which is of CFT type. Thus W is of CFT type.

Self-contragredience. Since $(\Omega_U)_1 = 0$, $W_1 = \mathbf{1} \otimes (V_{L^\natural})_1 \oplus \bigoplus_{\alpha \neq 0} (\Omega_U^{\tau(\alpha)} \otimes V_{\psi(\alpha)+L^\natural})_1$, and the Virasoro modes preserve each summand. The operator $L(1)$ maps W_1 into $W_0 = \mathbb{C}\mathbf{1}$, which lies in the summand $\alpha = 0$, while $L(1)(V_{L^\natural})_1 = 0$. Hence $L(1)W_1 = 0$ and the simple algebra W of CFT type is self-contragredient [Li94].

C_2-cofiniteness. The algebra W is C_2-cofinite, since it is a finite sum of irreducible modules for the C_2-cofinite algebra $\Omega_U \otimes V_{L^\natural}$, which are C_2-cofinite [Buh02], and C_2 of each summand is contained in $C_2(W)$.

Rationality. Let M be a weak W-module. By [ABD04], M is a direct sum of irreducible ordinary $\Omega_U \otimes V_{L^\natural}$-modules. Put $B = \Omega_U \otimes V_{L^\natural}$, and let M' be the W-submodule generated by an irreducible B-submodule N of M. By weak associativity, M' is spanned by the vectors $w_{(n)}x$, with $w \in W$, $x \in N$ and $n \in \mathbb{Z}$. For every irreducible B-module K, composition with these vertex operators gives an injective linear map

$$\mathrm{Hom}_B(M', K) \to I_B \binom{K}{W\ N}.$$

It is injective because a map in its kernel vanishes on this spanning set. The target is finite dimensional and is zero for all but finitely many K. In a decomposition of the semisimple B-module M' into simple summands, the coordinate projections onto distinct copies of K are linearly independent elements of $\mathrm{Hom}_B(M', K)$. Hence each multiplicity is finite, and only finitely many simple types occur. Thus M' has finite length and is an object of $\mathrm{Rep}(B)$. By [HKL15, CKM22], it is a local module for the connected étale algebra $\Phi(\mathcal{A}_U)$, and such

modules form a semisimple category [DMNO13]. Hence this submodule is a finite direct sum of irreducible W-modules, and M, a sum of such submodules, is a direct sum of irreducible W-modules. Compare the proof of [MoR24a, Proposition 5.3], which is the same construction in the holomorphic case.

Positivity. Let X be an irreducible W-module. Its restriction is a sum of modules $\Omega_U^\nu \otimes V_{\gamma+L^\natural}$, whose conformal weights are at least $h_\nu \geqslant 0$ (Proposition 12.2, part 2), so that $h_X \geqslant 0$. If $h_X = 0$, some summand has $h_\nu = 0$ and $\gamma \in L^\natural$, so that X contains $\Omega_U^\nu \otimes V_{L^\natural}$ with $h_\nu = 0$. The local $\mathcal{A}_U$-module $X^\flat = \Phi^{-1}(X)$ is an irreducible U-module whose restriction contains $\Phi^{-1}(\nu \boxtimes 0) = \nu \boxtimes 0$, that is, $\Omega_U^\nu \otimes V_L$. Hence $h_{X^\flat} \leqslant h_\nu = 0$, and $X^\flat \cong U$ since U is positive. Thus $X \cong \Phi(\mathcal{A}_U) = W$. Hence W is a Haagerup algebra, and by Theorem 8.5 it is in case (I) or (II).

Heisenberg commutant. Put $\mathfrak{h}^\natural = \mathbf{1} \otimes (L^\natural \otimes \mathbb{C})_{(-1)}\mathbf{1} \subset W_1$. The vectors of W_1 of $\mathfrak{h}^\natural$-weight zero lie in $(\Omega_U)_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{h}^\natural = \mathfrak{h}^\natural$, since $(\Omega_U)_1 = 0$. Thus $\mathfrak{h}^\natural$ is a self-centralizing subalgebra of the reductive Lie algebra W_1 [DoM04] acting semisimply on W, that is, a Cartan subalgebra. The commutant of $M(\mathfrak{h}^\natural)$ in $V_{\gamma+L^\natural}$ is $\mathbb{C}e^0$ if $\gamma \in L^\natural$ and zero otherwise, so $\mathrm{Com}_W(M(\mathfrak{h}^\natural)) = \Omega_U \otimes \mathbf{1}$, and $\mathrm{Com}_W(\Omega_U) = \bigoplus_\alpha \mathrm{Vac}_{\Omega_U}(\Omega_U^{\tau(\alpha)}) \otimes V_{\psi(\alpha)+L^\natural} = V_{L^\natural}$ by Proposition 12.2, part 2. So the Heisenberg commutant of W is Ω_U and its associated lattice is $L^\natural$. By Proposition 12.2, part 3, applied to W, its associated lattice is L_{I} or L_{II} according to its case. Since $L^\natural$ is not isometric to L (Lemma 12.1), W is in the other case.

2. The map $\tau\colon L'/L \to P$ is an anti-isometry. Let U_{II} be the one of U, $U^\natural$ which is in case (II), L_{II} its associated lattice and $\tau_{\mathrm{II}}\colon L'_{\mathrm{II}}/L_{\mathrm{II}} \to P$ its gluing map (part 1). The Weyl group elements of the two ideals of $(U_{\mathrm{II}})_1$ are inner automorphisms of U_{II} which normalize the Cartan subalgebra. They preserve $\Omega_U = \mathrm{Com}(M(\mathfrak{h}))$ and $V_{L_{\mathrm{II}}} = \mathrm{Com}(\Omega_U)$, and restrict to $\theta \in \mathrm{Aut}(\Omega_U)$ and to an automorphism of $V_{L_{\mathrm{II}}}$ inducing w' or w'' on $L'_{\mathrm{II}}/L_{\mathrm{II}}$. Since they permute the summands $\Omega_U^{\tau_{\mathrm{II}}(\alpha)} \otimes V_{\alpha+L_{\mathrm{II}}}$, the induced autoequivalence $\bar\theta$ satisfies $\bar\theta\tau_{\mathrm{II}} = \tau_{\mathrm{II}}\bar w$. By Lemma 12.1, part 4, $\bar w' = 14$ and $\bar w'' = 25$, which generate $O(L'_{\mathrm{II}}/L_{\mathrm{II}})$. Hence the $\bar\theta|_P$ exhaust $O(P) = \tau_{\mathrm{II}}O(L'_{\mathrm{II}}/L_{\mathrm{II}})\tau_{\mathrm{II}}^{-1}$. Now let Y_1, Y_2 be Haagerup algebras with Heisenberg commutants identified with Ω and associated lattices identified with N, with gluing maps $\sigma_1, \sigma_2\colon N'/N \to P(\Omega)$. Since both are anti-isometries, $\sigma_2 = \bar\theta\sigma_1$ for some $\theta \in \mathrm{Aut}(\Omega)$, by the first assertion applied to Y_1. The automorphism $\theta \otimes \mathrm{id}$ of $\Omega \otimes V_N$ transforms Y_1 into a simple current extension with the summands $\Omega^{\sigma_2(\alpha)} \otimes V_{\alpha+N}$ of Y_2. The summands are pairwise non-isomorphic invertible $\Omega \otimes V_N$-modules, indexed by N'/N, and their fusion satisfies

$$M_\alpha \boxtimes M_\beta \cong M_{\alpha+\beta}.$$

All summands are nonzero, both extensions are simple, and the fusion multiplicities are one. Thus [DoM04, Proposition 5.3] applies to their common graded module decomposition. Hence $Y_1 \cong Y_2$.

3. The algebra $U_\psi^\natural$ has Heisenberg commutant Ω_U and associated lattice $L^\natural$, so it does not depend on ψ by part 2. Applying the construction to $U^\natural$ with ψ^{-1} gives back the summands of U, so $(U^\natural)^\natural \cong U$ by [DoM04, Proposition 5.3]. If $Y \cong Y'$ are Haagerup algebras, an isomorphism maps a Cartan subalgebra to a Cartan subalgebra and identifies the Heisenberg commutants, so $Y^\natural \cong Y'^\natural$ by part 2. Hence $\natural$ is well defined on isomorphism classes and is an involution exchanging the two cases. If W is a Haagerup algebra with $\Omega_W \cong \Omega_U$, its associated lattice is L or $L^\natural$ according to its case, and $W \cong U$ or $W \cong U^\natural$ by part 2.

4. By [MoR24a, Corollary 4.10], two strongly rational vertex operator algebras with equivalent unpointed pseudo-unitary representation categories belong to the same hyperbolic genus if and only if they have the same central charge and isomorphic Heisenberg commutants. This applies to U and $U^\natural$. If V is strongly rational and in this hyperbolic genus, then by [MoR24a,

Theorems 4.7 and 4.8] $\mathrm{Rep}(V)$ is ribbon equivalent to $\mathrm{Rep}(U)$, the Heisenberg commutant of V is isomorphic to Ω_U, and its associated lattice is in the genus of L, hence isometric to L_{I} or L_{II} (Lemma 12.1). By the proof of [MoR24a, Corollary 4.10], $V = \bigoplus_\alpha \Omega_U^{\sigma(\alpha)} \otimes V_{\alpha+N}$ with N its associated lattice and σ an anti-isometry onto P. The argument of part 2, with Y_1 the one of U, $U^\natural$ whose associated lattice is isometric to N and $Y_2 = V$, uses only the decomposition of Y_2 and shows that $V \cong Y_1$. ■

Theorem 12.4 *1. Assuming* $(\mathrm{H_J})$*, there are exactly two Haagerup algebras up to isomorphism: Jiao's algebra* $V \cong S$*, in case (II), and its twin* $S^\natural$*, in case (I). There is an embedding of* $S^\natural$ *into* V' *whose image contains the algebra* $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ *generated by* A*,* e*,* f*,* v *(Proposition 10.36), with equality in weights at most four.*

2. Assuming $(\mathrm{H_R})$*, there are exactly two Haagerup algebras up to isomorphism: Riesen's algebra* U^{R}*, in case (I), and its twin* $(U^{\mathrm{R}})^\natural$*, in case (II). There is an embedding of* $(U^{\mathrm{R}})^\natural$ *into* V' *whose image contains* S*, with equality in weights at most four.*

3. Assuming $(\mathrm{H_R})$ *and* $(\mathrm{H_J})$*,* $S^\natural \cong U^{\mathrm{R}}$ *and* $(U^{\mathrm{R}})^\natural \cong V$*.*

4. Assuming $(\mathrm{H_R})$ *or* $(\mathrm{H_J})$*, all Haagerup algebras have isomorphic Heisenberg commutants and they form a single hyperbolic genus.*

Proof : 1. By Theorem 10.29, every Haagerup algebra in case (II) is isomorphic to $V \cong S$. By Theorem 12.3, part 3, every Haagerup algebra in case (I) is isomorphic to $S^\natural$, which is one. The proof of Theorem 10.38 shows, without using [Rie26a], that a Haagerup algebra in case (I) contains $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ after an automorphism of V'. By Proposition 10.39, the weight spaces of $\psi_{\mathrm{R}}(U^{\mathrm{R}})$ of weights two, three and four have dimensions at least 120, 956 and 6010, which are those of the Haagerup vacuum character (Theorem 7.1). 2. The same argument with Theorem 10.38. The proof of Theorem 10.29 shows, without [Jia26], that a Haagerup algebra in case (II) contains S after an automorphism of V', and Theorem 10.21, using the independent certificate of Proposition 10.17, gives the dimensions 1, 6, 120, 956, 6010 of S in weights at most four. 3 follows from 1 and 2, and 4 from Theorem 12.3, parts 3 and 4. ■

Theorem 12.3 pairs the two cases within each hyperbolic genus. For the explicit algebra S, [ArhIII, Theorem 6.1] supplies $(\mathrm{H_J})$. Parts 1 and 4 of Theorem 12.4 then give the complete classification by the pair $S, S^\natural$, including existence in case (I) and the common Heisenberg commutant.

The common fixed-point algebra of [DJX26, Theorem 7.3] is used in [ArhIII, Theorem 7.9] to identify $S^\natural$ with Huang's algebra U^{H}. The identification of Huang's and Riesen's algebras in [DJX26, Section 4.5 and Proposition 4.1] then gives $S^\natural \cong U^{\mathrm{H}} \cong U^{\mathrm{R}}$ as vertex operator algebras, as stated in [ArhIII, Corollary 7.10]. Thus $(\mathrm{H_R})$ holds, parts 2–4 of Theorem 12.4 apply, and $(U^{\mathrm{R}})^\natural \cong S$ by Theorem 12.3(3). The ambient containment and the finite weight comparison in Theorem 12.4(1) come from the calculations in this article.

Remark 12.5 The decomposition of Proposition 12.2 can be tested on the graded dimensions. For $\mu \in \Gamma_0$, the generating series of the dimensions of the subspaces of $\mathfrak{h}$-weight μ of U_n, multiplied by $\prod_{m\geqslant 1}(1-q^m)^2$ and by $q^{-|\mu|^2/2}$, is the graded dimension f_α of $\Omega_U^{\tau(\alpha)}$, $\alpha = \mu + L$. From the blocks of U_2 and U_3 (exact in case (II), from the spans of Proposition 10.39 modulo 7999993 in case (I)), a finite calculation shows that different weights μ in the same coset give the same coefficients (39 cosets occur in case (II) and 35 in case (I) in these weights), and that for each of the four isometries ψ of Lemma 12.1, part 3, the 88 coefficients of f_α in case (II) which can be compared with those of $f_{\psi(\alpha)}$ in case (I) agree. This is consistent with Theorem

12.4, part 4: it is a test, in low weights, of the isomorphism of the Heisenberg commutants of any two Haagerup algebras of the two cases, which Theorem 12.3 proves for U and $U^\natural$.

Remark 12.6 Let V be any strongly rational vertex operator algebra of central charge eight with a braided equivalence $\mathrm{Rep}(V) \cong Z(\mathcal{H}_3)$, not assumed positive. Since $Z(\mathcal{H}_3)$ has no nontrivial invertible object, its universal grading group is trivial [GeN08, Theorem 6.3]. Since pivotal structures differ by tensor automorphisms of the identity functor, which are characters of this group, $Z(\mathcal{H}_3)$ has a unique pivotal structure, the pseudo-unitary one [ENO05]. Since $c(V) \equiv 0 \equiv c(Z(\mathcal{H}_3)) \bmod 8$, the S-matrix of the characters of V is the normalized S-matrix s of Section 2 [MoR24a, Lemma 3.6]. Suppose that the minimal conformal weight $h_{\min}$ of the irreducible V-modules is negative, and let J be the set of simple objects with conformal weight $h_{\min}$. It does not contain 0, and its elements have the same twist. The possible nonempty sets J are the nonempty subsets of $\{b, a, c_0\}$, the singletons $\{c_1\}$ and $\{c_2\}$, and the singletons $\{d_l\}$. The following choices of i give $s_{ij} \leqslant 0$ for every $j \in J$, with strict inequality for at least one j:

condition on J	i
$b \in J$	d_1
$\varnothing \neq J \subset \{a, c_0\}$	c_1
$J = \{c_1\}$ or $J = \{c_2\}$	a
$J = \{d_l\}$	b

The signs follow directly from the entries $-y/3$, $-1/3$ and 0 of the S-matrix. Then $\mathrm{ch}_i(\mathrm{i}y) \sim \big(\sum_{j\in J} s_{ij} \dim V^j_{h_{\min}}\big) \mathrm{e}^{\frac{2\pi}{y}(\frac{c}{24} - h_{\min})}$ would be negative for small $y > 0$, a contradiction. This is the argument of [MoR24a, Proposition 3.10] with a weaker sign condition: $Z(\mathcal{H}_3)$ is not essentially positive in the sense of [MoR24a] (for $J = \{b, a\}$ no row i has $s_{ib} < 0$ and $s_{ia} < 0$, and a set J containing 0 never satisfies their condition), but our condition suffices to exclude $h_{\min} < 0$. Hence all conformal weights of the irreducible V-modules are nonnegative, and they are positive for the eight simple objects with nontrivial twist. Therefore the positivity hypothesis in our results can be replaced by the condition that none of the modules b, a, c_0 has conformal weight zero.

A Computational methods and certificates

The proofs distinguish exact calculations, certified exclusions, calculations modulo primes and numerical checks. Exact calculations use rational arithmetic, arithmetic in $\mathbb{K} = \mathrm{Q}(\sqrt{13}, \sqrt{-3})$ in the basis $1, \sqrt{13}, \sqrt{-3}, \sqrt{-39}$, Gröbner bases over Q or $\mathrm{Q}(\sqrt{13})$, and integer arithmetic. Certified interval calculations are used only to exclude a candidate by a nonzero value. An interval containing zero gives no exclusion.

For a matrix integral at a prime $\mathfrak{p}$ of $\mathbb{K}$, its rank over $\mathbb{K}$ is at least the rank of its reduction. This justifies the rank lower bounds and the selection of pivots modulo primes. Membership over $\mathbb{K}$ is checked separately by exact coordinate identities on all rows after restriction of scalars. Membership after reduction alone is not used as a characteristic-zero assertion. Numerical calculations serve only as consistency checks and do not enter the proofs.

The supplementary material indexes the source code, coordinates, exact witnesses and verification results by the numbered statements. It specifies the computational environments and commands. The lattice construction gives the vertex algebra identities in arbitrary weights, while Section 10 justifies the momentum truncation used in the certificates.

The following table identifies the scripts associated with numbered statements and the nature of their computations. It distinguishes exact verification, certified exclusions, reductions modulo

primes and auxiliary numerical tests. An entry does not assert more than the precise statement and its proof establish. In particular, modular calculations are not treated as characteristic-zero lifts.

Table 5: Correspondence between numbered statements and computational scripts.

Statement	Scripts	Nature
Theorem 3.1	`eg_c8_verification.py`, `bg_trace_criterion.py`	exact recursion, numerical modularity check
Proposition 4.1	`kac_order3.py`	exact (finite check)
Proposition 4.4	`e6_rank_bound.py`	analytic proof and exact check
Proposition 5.4	`vvmf_exact.py`, `weight2_obstruction_exact.py`, `dump_relations.py`	exact (rational)
Corollary 5.6	`weight2_solve.py`, `gamma1_exclusion_exact.py`	exact (rational)
Proposition 6.2	`obstruction.py`, `e6_tools.py`, `weight4_solve.py`	exact (rational)
Lemma 7.2	direct inequalities in the proof	no computation required
Proposition 8.2	`centre_gamma0.py`	exact
Lemma 8.3	`sl2_e6_superset.py`, `e6_centralizers.py`, `e6_orbit_labels.py`	exact
Proposition 8.4	`sl2_ideal_feasibility.py`, `weight6_single.py`	exact
Theorem 8.5	`sl2_pair_exact.py`, `e8_joint.py`, `sextic_joint.py`, `joint_uniqueness.py`, `known_sectors_table.py`, `u2_moments.py`, `u2_embedding_test.py`	exact
Proposition 9.1	`lattice_branching.py`, `u2_parameter_count.py`	exact
Lemma 10.1	`commutant_griess.py`	exact over $\mathbb{K}$
Proposition 10.3	`griess_subalgebras.py`	exact, Gröbner over $\mathbb{Q}$
Theorem 10.5	`primaries.py`, `isotropic_lines.py`, `isotropy_test.py`, `check_primary33_exclusions.py`, `closure_certify_exact_intervals.py`, `check_closure_coverage.py`	exact rational exclusions and certified Arb witnesses
Proposition 10.7	`closure_modp.py`, `modp.py`, `u3_moments.py`	modulo p (superseded by Theorem 10.19)
Theorem 10.9	`exact_weight2_K.py`, `Kfield.py`	exact over $\mathbb{K}$
Theorem 10.10	`unitary_character.py`, `phi_automorphism_check.py`, `unitarity_test.py`, `lattice_voa.py`	exact
Proposition 10.12	`a3_griess_automorphisms.py`, `choice_conjugation.py`	exact
Proposition 10.13	`closure_w4_modp.py`, `modp.py`	modulo p

Statement	Scripts	Nature (continued)
Theorem 10.21	`verify_f4_strict.py`, `aggregate_f4.py`, `check_zero_products.py`	exact over $\mathbb{K}$, Proposition 10.17 and alternative dimension proof
Theorem 10.19	`exact_engine.py`, `certify_w3.py`, `certify_w4b.py`	all parts exact over $\mathbb{K}$
Theorem 10.24	`i2_weyl.py`, `i4_torus.py`, `i5_membership_exact.py`, `i6_generation_certificate.py`	exact membership and rank lower bound modulo p_3
Proposition 10.27	`j1_centralizer.py`, `j5_explicit_automorphism.py`	root-theoretic proof and exact polynomial check
Proposition 10.31	`c2_exact_C4.py`, `commutant_dims.py`, `griess_subalgebras.py`	exact over $\mathbb{K}$
Proposition 10.34	`exact_isotropy.py`, `isotropy_test.py`	exact, certified (Arb)
Proposition 10.35	`isotropic_lines.py`, `r6_certify_sub2.py`, `r5_sub2_testb.py`	exact, certified (Arb)
Proposition 10.36	`r1_identify_riesen.py`, `r2_locate_riesen.py`, `griess_subalgebras.py`	exact
Proposition 10.37	`j1_centralizer.py`, `j2_solve.py`, `j6_caseI_action.py`, `r3_lines_and_Z.py`	root-theoretic proof and exact check
Proposition 10.39	`b1_UR_closure_modp.py`	modulo p (rigorous lower bounds)
Proposition 10.41	`d1_rational_forms.py`, `j6_caseI_action.py`	exact
Proposition 11.4	`check_simple_current_word.py`	analytic word and energy proof, exact checks
Theorem 11.6	`z0_highest.py`, `z1_products.py`, `z2_relations_modp.py`, `z3_corner_modp.py`, `z4_upper_algebra_modp.py`	modulo p
Theorem 12.4	`b1_UR_closure_modp.py`	modulo p (rigorous lower bounds)
Lemma 12.1	`s20_genus_check.py`, `s22_lattice_weights.py`	exact (integers)
Proposition 12.2	Table 2 and the proof	no separate lattice-weight computation

A.1 Coverage of the certificates

The trace certificates give the dual forms, vanishing relations, matrix ranks and eighteen moments of U_2. The module searches enumerate the entire finite domain specified in the proof of Theorem 8.5.

For Theorem 10.5, the thirteen excluded commutant candidates give thirty-nine rational projective charts with unit Gröbner bases. The pairwise closure calculation gives 62 certified nonzero interval witnesses, which exclude 254 of the 256 choices. The coverage check uses only these certified witnesses. The primary-space computations retain all characteristic-polynomial factors and do not discard an inconclusive eigenspace.

The weight-four certificate of Proposition 10.17 checks the complete input domains, the identity of each pivot word, integrality at the chosen prime, full rank and every coordinate of each membership identity. It covers all 131 nonzero Cartan blocks and checks the 886 zero inputs separately. Missing or truncated data, duplicate labels and nonzero residuals are rejected. This establishes (F4) without an external existence theorem. Corollary 10.22 gives a separate dimension proof when an ambient Haagerup algebra is available.

For Proposition 10.31, the supplementary data give the seven exact commutant vectors, the two Virasoro matrices and the primary vectors. Their matrix identities and ranks admit a separate verification in rational arithmetic. This verification is distinct from the lattice-mode checks of the vertex-product identities.

Independent mode comparisons. A second evaluator, using simultaneous normal ordering and direct exponential coefficients, agrees with the iterate-formula implementation on 24643 basis-state products in the stated weight window and 310 generator products. The coefficient-field multiplication and three Virasoro identities are checked separately. These comparisons use the supplied generator coordinates. They do not independently reconstruct those coordinates from the printed definitions, or regenerate every coordinate of (F4) by a second method.

The modular Zhu calculation. All 131698 overlap ambiguities and 151 inclusion ambiguities of the final presentation are checked. At 8388571, the algebra defined by the projected relations has dimension 762, nondegenerate trace form and the twelve stated blocks. These checks concern that presentation. They do not prove a characteristic-zero lift, recompute the vertex-algebra origin of every relation or show that every block survives in the Zhu corner. The agreement with the calculation at a second prime has the status specified in Remark 11.7.

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Data availability. The source code, certificates and reproduction instructions are available at https://github.com/cedricarhancet/haagerup-c8. The two complete weight-four coordinate archives are distributed as release assets. Table 5 relates the computational scripts to the numbered statements. *Author's GitHub profile.* https://github.com/cedricarhancet

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