WIGNER–YANASE–DYSON METRICS HAVE MONOTONE CURVATURE

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Abstract. Gibilisco and Isola conjectured that the scalar curvature of the Wigner–Yanase–Dyson metric with parameter p on the invertible density matrices increases under mixing for all p in some interval $(1, 1+\varepsilon)$; the case $p \downarrow 1$ is Petz's conjecture for the Kubo–Mori metric (resolved in a companion paper). We prove that for $p \in (1-\varepsilon, 1+\varepsilon)$, with $\varepsilon > 0$ independent of the dimension, the scalar curvature is a concave function of the eigenvalues and hence increases under mixing. By Dittmann's formula it suffices to prove that one symmetric function H_p of three positive variables is concave, which we do by showing that $-1/H_p$ is a Stieltjes transform along every line in the positive cone, uniformly in p near 1. The value of ε is not explicit.

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1. Introduction

The Kubo–Mori metric is one of the monotone Riemannian metrics on the manifold $\mathcal{D}_n$ of invertible $n \times n$ density matrices (Petz, 1994, 1996). Call a density matrix ω_1 *more mixed* than ω_2 if, for each k, the sum of the k largest eigenvalues of ω_1 is at most the corresponding sum for ω_2, and say that a function on $\mathcal{D}_n$ *increases under mixing* if it is at least as large at the more mixed of any two such states. Petz (1994) conjectured that the scalar curvature of the Kubo–Mori metric increases under mixing. Dittmann (2000) expressed the scalar curvature of every monotone metric as a sum, over triples of eigenvalues, of one symmetric function of three positive variables, and reduced Petz's conjecture to the concavity of that function for the Kubo–Mori metric. The companion paper (Sra, 2026) proves the concavity, and with it Petz's conjecture.

The Kubo–Mori metric sits inside a one-parameter family. A monotone metric on $\mathcal{D}_n$, in the sense of Petz (1996), is a Riemannian metric of the form

$$g_\rho(A, B) = \operatorname{Tr} A\, c(L_\rho, R_\rho)(B), \qquad L_\rho B = \rho B, \quad R_\rho B = B\rho,$$

where $c(x, y) = 1/(y f(x/y))$ for an operator monotone function $f : (0,\infty) \to (0,\infty)$ with $f(1) = 1$ and $f(x) = x f(1/x)$. For $p \notin \{0, 1\}$ let $\tilde{p}$ be defined by $1/p + 1/\tilde{p} = 1$, and let

$$f_p(x) = \frac{1}{p\tilde{p}} \, \frac{(x-1)^2}{(x^{1/p} - 1)(x^{1/\tilde{p}} - 1)}, \qquad f_1(x) = \frac{x-1}{\log x}. \tag{1.1}$$

Then $f_p = f_{\tilde{p}}$, and f_p is operator monotone exactly when $p \in (-\infty, -1] \cup [1/2, \infty]$ (Gibilisco and Isola, 2005, Theorem 6.1). The monotone metric of f_p is the Wigner–Yanase–Dyson metric $\mathrm{WYD}(p)$. Since $p \mapsto \tilde{p}$ exchanges $(1, 2]$ with $[2, \infty)$ and $[1/2, 1)$ with $(-\infty, -1]$, each of these metrics is $\mathrm{WYD}(p)$ for exactly one $p \in [1/2, 2]$. The endpoint $p = 2$ is the Wigner–Yanase metric and $p = 1/2$ the metric of the right logarithmic derivative; the Kubo–Mori metric is $p = 1$, an interior point of the family, because f_p is real-analytic in $a = 1/p$ across $a = 1$.

Gibilisco and Isola (2005, Conjecture 8.1) asked whether the monotonicity of the curvature persists along this family: they conjectured that the scalar curvature of $\mathrm{WYD}(p)$ increases under mixing for every p in some interval $(1, 1+\varepsilon)$, and observed that, because the curvature at a fixed state

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depends continuously on p, this contains Petz's conjecture as the limiting case $p \downarrow 1$ (Gibilisco and Isola, 2005, Theorem 8.1). We have not found the corresponding statement for $p < 1$, on the side of the family between the Kubo–Mori metric and the right logarithmic derivative, in the literature. At $p = 2$ the curvature is constant (Gibilisco and Isola, 2005, Theorem 7.2); we are not aware of a result for any other $p \neq 1$. The proof at $p = 1$ does not transfer by itself, for reasons explained after Theorem 1.1. We prove both conjectures, with ε independent of n.

Let H_p be Dittmann's symmetrized function for $\mathrm{WYD}(p)$, recalled in Section 2.1.

Theorem 1.1. *There is $\varepsilon > 0$ such that for every $p \in (1-\varepsilon, 1+\varepsilon)$:*

(i) *H_p is jointly concave on $(0,\infty)^3$;*

(ii) *for every $n \geq 2$, the scalar curvature of $\mathrm{WYD}(p)$ on $\mathcal{D}_n$ is a symmetric concave function of the eigenvalues of the density matrix, and in particular it increases under mixing.*

The number ε does not depend on n.

Theorem 1.1 is not a corollary of the companion paper, and the two papers answer different questions. The companion proves one inequality at $p = 1$: the boundary value $\operatorname{Im} H_1(-1 + \mathrm{i}0, s, t)$ is positive for all $s, t > 0$, by explicit estimates of the logarithm on the negative axis. The question here is whether the monotonicity at $p = 1$ is an isolated fact or persists along the family, and the answer needs uniformity in two directions: in p, with a neighborhood that does not shrink with n, and in (s, t), because continuity in p keeps the boundary value positive only on compact subsets of $(0,\infty)^2$, while near its boundary the strict inequality at $p = 1$ leaves no margin and the arguments interact with p through the powers $s^{1-1/p}$, $t^{1-1/p}$. The tools are correspondingly new: an exact decomposition of H_p valid for every p (Proposition 2.2), which at $p = 1$ collapses to the companion's normal form; a simplex formula that gives the sign of its main boundary term for all $s, t > 0$ at once (Lemma 3.3); and corner estimates in which no product of $1/p - 1$ with $\log(t/s)$ is ever expanded (Section 3.4).

Three limitations should be stated at once. The proof gives no value of ε: it establishes estimates near the boundary of the eigenvalue domain, each valid for $|1/p - 1|$ below an unspecified threshold, and then argues by compactness; the auxiliary bound $|1/p - 1| \leq 1/100$ that appears in the estimates is not the neighborhood. Explicit constants in the estimates of Section 3 would give an explicit ε; we have not carried this out. The base case is not proved here: it is the boundary inequality of the companion paper, a statement about the Kubo–Mori function that is stronger than Petz's conjecture, and we use it as proved there; the argument adds nothing at $p = 1$. Finally, nothing is claimed for the rest of the family. The method rests on the inequality $\operatorname{Im} H_p(-1 + \mathrm{i}0, s, t) > 0$ for all $s, t > 0$, which fails at $p = 2/3$ (Remark 3.13); whether the scalar curvature of $\mathrm{WYD}(p)$ increases under mixing for every $p \in [1/2, 2)$ remains open.

A by-product is Proposition 2.6: the Stieltjes measures of the line restrictions, which for $p < 1$ carry atoms at the endpoints of the cut with the explicit masses (2.26).

1.1. **Idea of the proof.** Write $a = 1/p$ and $\delta = a - 1$, and let $\mathscr{H}_\delta = H_p$, so that $\mathscr{H}_0$ is the function of the companion paper. The route there was this: for all $X, Y \in (0,\infty)^3$, the function $t \mapsto -c/\mathscr{H}_0(X + tY)$ is, for a normalizing constant c, the Stieltjes transform of a positive measure; the reciprocal of a Stieltjes transform is concave; and homogeneity of degree one turns concavity along positive directions into concavity on the whole cone. Whether a line restriction is a Stieltjes transform is decided by the sign of its imaginary part on the cut, where one or two of the three arguments are negative, and by homogeneity and symmetry this comes down to the positivity of the boundary value

$$B_\delta(s, t) = \operatorname{Im} \mathscr{H}_\delta(-1 + \mathrm{i}0, s, t), \qquad s, t > 0.$$

At $\delta = 0$ this positivity is the theorem of the companion paper.

For $\delta \neq 0$ the boundary value is continuous in (δ, s, t), so $B_\delta > 0$ persists on every compact set of pairs (s, t) once $|\delta|$ is small. The work is at the boundary of $(0,\infty)^2$. We meet this with an exact

identity. In the coordinates $u = x^a$ and $w = (x^{-\delta} - 1)/\delta$, the function $\mathscr{H}_\delta$ is the sum of three terms (Proposition 2.2): the square of a three-node determinant divided by two Vandermonde products, and two second divided differences, of the functions $u \mapsto x^b$ and $w \mapsto x^{-b}$, where $b = 1 + 2\delta$. The boundary value of the last term is an integral over the two-dimensional simplex whose integrand has a definite phase, so its imaginary part $J_\delta(s,t) > 0$ for all $s,t > 0$ and $|\delta| \le 1/100$ (Lemma 3.3). The rest of the proof shows that $B_\delta/J_\delta \to 1$ in every joint limit in which $\delta \to 0$ and (s,t) escapes to the boundary of $(0,\infty)^2$, with no condition on $\delta \log s$ or $\delta \log t$ (Theorem 3.11); a sequential compactness argument then gives the neighborhood. The estimates are elementary, but they must be organized so that no product $\delta \log(t/s)$ is ever expanded. The passage from the boundary sign to Stieltjes measures follows the companion paper, with one new feature: for $p < 1$ the measures acquire atoms at the endpoints of the cut, with explicit positive masses.

Section 2 recalls Dittmann's formula, proves the decomposition, and reduces Theorem 1.1 to $B_\delta > 0$ for all $s,t > 0$ and all small $|\delta|$. Section 3 proves that positivity; the corner estimates it relies on are proved in Appendix A.

2. Dittmann's function and the reduction to a boundary sign

This section sets up Dittmann's function for WYD(p), writes it in the coordinates x^a and $(x^{-\delta} - 1)/\delta$ as a sum of three divided differences, and reduces Theorem 1.1 to one statement: that $B_\delta(s,t) = \operatorname{Im} \mathscr{H}_\delta(-1 + \mathrm{i}0, s, t) > 0$ for all $s,t > 0$ and all small $|\delta|$. The reduction follows the companion paper. Once the boundary value is positive, the line restrictions of $-1/\mathscr{H}_\delta$ are Stieltjes transforms of positive measures, and the reciprocal of a Stieltjes transform is concave.

2.1. Dittmann's formula and normalization.

Throughout, p is close to 1, and we write

$$\delta = \frac{1}{p} - 1, \qquad a = 1 + \delta = \frac{1}{p}, \qquad b = 1 + 2\delta = 2a - 1, \qquad |\delta| \le \eta_0 = \frac{1}{100}. \tag{2.1}$$

The bound $|\delta| \le \eta_0$ is the working range of the estimates, not the neighborhood of Theorem 1.1.

Let $\mathcal{P}_n$ be the cone of positive definite complex Hermitian matrices and

$$\mathcal{D}_n = \{\rho \in \mathcal{P}_n : \operatorname{Tr} \rho = 1\}.$$

In terms of eigenvalue vectors, a density matrix with eigenvalues λ is more mixed than one with eigenvalues μ exactly when $\lambda \prec \mu$ in the sense of majorization: the sums of the k largest coordinates of λ do not exceed those of μ, and the total sums agree.

For $a = 1/p$ near 1, the Morozova–Chentsov function of WYD(p), that is, $c_p(x,y) = 1/(y f_p(x/y))$ with f_p from (1.1), is

$$c_p(x,y) = \frac{(x^a - y^a)(x^{1-a} - y^{1-a})}{a(1-a)(x-y)^2}, \qquad c_1(x,y) = \frac{\log x - \log y}{x - y}. \tag{2.2}$$

All diagonal values are interpreted continuously, so $c_p(x,x) = 1/x$. The corresponding metric is

$$g_{p,\rho}(A,B) = \operatorname{Tr} A\, c_p(L_\rho, R_\rho)(B), \qquad L_\rho B = \rho B, \quad R_\rho B = B\rho.$$

That this is a monotone metric for p in a neighborhood of 1 was recalled in Section 1.

For $c = c_p$, put $\ell(z,x) = \partial_z \log c(z,x)$ and define, at distinct positive arguments,

$$h_1(x,y,z) = \frac{c(x,y) - z c(x,z) c(y,z)}{(x-z)(y-z) c(x,z) c(y,z)}, \tag{2.3}$$

$$h_2(x,y,z) = \frac{(c(x,z) - c(y,z))^2}{(x-y)^2 c(x,y) c(x,z) c(y,z)}, \tag{2.4}$$

$$h_3(x,y,z) = z \frac{\ell(z,x) - \ell(z,y)}{x-y}, \qquad h_4(x,y,z) = z \ell(z,x) \ell(z,y), \tag{2.5}$$

$$h = h_1 - \tfrac{1}{2}h_2 + 2h_3 - h_4, \qquad H_p(x,y,z) = \frac{h(x,y,z)+h(y,z,x)+h(z,x,y)}{3}. \tag{2.6}$$

Each h_j is symmetric in its first two arguments, so the cyclic average is the full symmetrization. The apparent singularities at repeated positive arguments are removable.

Dittmann's formula for the scalar curvature at a state with eigenvalues $\lambda_1, \dots, \lambda_n$ (Dittmann, 2000, Theorem 1 and Corollary 1) is

$$\mathrm{Scal}_p(\lambda) = C_n + \sum_{i,j,k=1}^{n} H_p(\lambda_i,\lambda_j,\lambda_k) - \sum_{i=1}^{n} H_p(\lambda_i,\lambda_i,\lambda_i), \qquad C_n = \frac{(n^2-1)(n^2-2)}{4}. \tag{2.7}$$

The factor $1/2$ in (2.6) multiplies h_2. When the dependence on the parameter is the point, we write

$$\mathscr{H}_\delta = H_p, \qquad \delta = \frac{1}{p} - 1. \tag{2.8}$$

Proposition 2.1 (Normalization)**.** *The function $\mathscr{H}_\delta$ is symmetric and homogeneous of degree -1, and*

$$\mathscr{H}_\delta(x,x,x) = -\frac{b^2}{8x}, \qquad \mathrm{Scal}_p(I/n) = C_n - \frac{b^2}{8}(n^4 - n^2). \tag{2.9}$$

Proof. Symmetry follows from the definition. Homogeneity follows from $c_p(tx,ty) = t^{-1}c_p(x,y)$ and (2.3)–(2.6). To check the diagonal, expand

$$c_p(x(1+q),x) = \frac{1}{x}\left(1 - \frac{q}{2} + \frac{a^2-a+4}{12}q^2 + O(q^3)\right).$$

The diagonal limit of Dittmann's formula is $h(x,x,x) = 15/(8x) - 3x^2\partial_1^2 c_p(x,x)$ (Dittmann, 2000, eq. (44)). Thus

$$h(x,x,x) = \frac{1}{x}\left(\frac{15}{8} - \frac{a^2-a+4}{2}\right) = -\frac{(2a-1)^2}{8x}.$$

There are $n^3 - n$ remaining index triples at the maximally mixed state; substituting $x = 1/n$ into (2.7) gives the second identity. □

2.2. **Power coordinates and the decomposition.** The kernel (2.2) is a product of two divided differences, of x^a and of $x^{1-a} = x^{-\delta}$. In the coordinates $u = x^a$ and $w = (x^{-\delta} - 1)/\delta$ the factors become $u_i - u_j$ and $\delta(w_i - w_j)$, and Dittmann's function becomes a sum of three terms, each a divided difference in one coordinate system. This identity is what the rest of the paper works with: the term in the w-coordinates is the one whose boundary value has a sign (Lemma 3.3), and the other two are the remainder that the estimates of Section 3 control.

For x_i in the principal slit plane, set

$$u_i = x_i^a, \qquad w_i = \begin{cases} (x_i^{-\delta} - 1)/\delta, & \delta \neq 0, \\ -\operatorname{Log} x_i, & \delta = 0. \end{cases} \tag{2.10}$$

For three distinct nodes define

$$\begin{aligned} D_u &= (u_0 - u_1)(u_1 - u_2)(u_2 - u_0), \\ D_w &= (w_0 - w_1)(w_1 - w_2)(w_2 - w_0), \\ \Delta &= \sum_{\mathrm{cyc}} w_i(u_j - u_k). \end{aligned} \tag{2.11}$$

The identities below are initially asserted where the displayed denominators are nonzero; their removable limits are always understood.

Proposition 2.2 (Decomposition)**.** *One has*

$$\mathscr{H}_\delta = \mathcal{A}_\delta + \mathcal{U}_\delta + \mathcal{V}_\delta, \tag{2.12}$$

where

$$\mathcal{A}_\delta = -\frac{a\Delta^2}{6D_uD_w}, \tag{2.13}$$

$$\mathcal{U}_\delta = -\frac{a^2}{3}\sum_i \frac{x_i^b}{(u_i-u_j)(u_i-u_k)}, \tag{2.14}$$

$$\mathcal{V}_\delta = -\frac{1}{3}\sum_i \frac{x_i^{-b}}{(w_i-w_j)(w_i-w_k)}. \tag{2.15}$$

Here and below $\{j,k\} = \{0,1,2\}\setminus\{i\}$ *in a vertex sum.*

Proof. We verify the identity algebraically from (2.6).

Temporarily put $\alpha = a$, $\beta = 1-a$, $v_i = x_i^{1-a}$, so that $x_i = u_iv_i$. Write $r_{ij} = 1/(x_i - x_j)$ and

$$C_{ij} = \frac{(u_i-u_j)(v_i-v_j)}{(x_i-x_j)^2}, \qquad Q_{ij} = \frac{\alpha u_i}{x_i(u_i-u_j)} + \frac{\beta v_i}{x_i(v_i-v_j)}.$$

Then $c_p(x_i,x_j) = C_{ij}/(\alpha\beta)$ and $\ell(x_i,x_j) = Q_{ij} - 2r_{ij}$.

Expand (2.6) with x_i as its distinguished third argument and sum over i. The terms containing only r_{ij} sum to $-9\sum_i x_i/((x_i-x_j)(x_i-x_k)) = 0$. The terms linear in Q also cancel. For example, after grouping the coefficient of $1/(x_j-x_k)$ for the u-part, its numerator is a multiple of

$$\frac{u_i}{u_i-u_j} + \frac{u_j}{u_j-u_i} - \frac{u_i}{u_i-u_k} - \frac{u_k}{u_k-u_i} = 1 - 1 = 0.$$

The v-part is identical. What remains is

$$\mathscr{H}_\delta = \alpha\beta\, G - \frac{1}{3}\sum_i x_iQ_{ij}Q_{ik}, \tag{2.16}$$

where

$$G = \frac{1}{6C_{01}C_{12}C_{20}}\left[2\sum_i \frac{C_{jk}^2}{(x_i-x_j)(x_i-x_k)} - \sum_i \frac{(C_{ij}-C_{ik})^2}{(x_j-x_k)^2}\right].$$

Let D_v be the cyclic product of the v-differences and $\Delta_v = \sum_{\mathrm{cyc}} v_i(u_j-u_k)$. Two elementary three-node identities are

$$G = -\frac{\Delta_v^2}{6D_uD_v}, \tag{2.17}$$

$$\sum_i \left[\frac{1}{(u_i-u_j)(v_i-v_k)} + \frac{1}{(v_i-v_j)(u_i-u_k)}\right] = -\frac{\Delta_v^2}{D_uD_v}. \tag{2.18}$$

For a direct verification of the first, put $L_{ij} = (u_i-u_j)(v_i-v_j)$ and $D_x = (x_0-x_1)(x_1-x_2)(x_2-x_0)$. The bracket in the formula for G, after multiplication by D_x^2, is

$$-(L_{01}^2 + L_{12}^2 + L_{20}^2 - 2L_{01}L_{12} - 2L_{12}L_{20} - 2L_{20}L_{01}) = -\Delta_v^2.$$

Also $C_{01}C_{12}C_{20} = D_uD_v/D_x^2$. For (2.18), multiplication by D_uD_v gives the same negative quadratic expression in the L_{ij}. These are polynomial identities; they can also be checked after translating $u_0 = v_0 = 0$, when the quadratic expression is $(u_1v_2 - u_2v_1)^2$.

Expanding the last term of (2.16) and using $x_i = u_iv_i$ gives

$$\mathscr{H}_\delta = \frac{\alpha\beta\Delta_v^2}{6D_uD_v} - \frac{\alpha^2}{3}\sum_i \frac{u_i/v_i}{(u_i-u_j)(u_i-u_k)} - \frac{\beta^2}{3}\sum_i \frac{v_i/u_i}{(v_i-v_j)(v_i-v_k)}.$$

Now $v_i = 1 + \delta w_i$, $\beta = -\delta$, $u_i/v_i = x_i^b$, and $v_i/u_i = x_i^{-b}$. Thus $D_v = \delta^3 D_w$ and $\Delta_v = \delta\Delta$, which gives (2.13)–(2.15) for $\delta \neq 0$. The normalized expressions (2.10) are analytic in δ at zero, so the identity extends to $\delta = 0$. □

2.3. **Analytic continuation and the Stieltjes representation.** Throughout this section δ is fixed, and

$$F_\delta = -\frac{8}{b^2}\mathscr{H}_\delta, \qquad M_\delta = \frac{1}{F_\delta}. \tag{2.19}$$

The normalization gives $M_\delta(x, x, x) = x$ by (2.9), and at $\delta = 0$ it makes F_0 the function F of the companion paper (Section 3.1). Powers and logarithms of negative numbers always mean upper principal boundary values; in particular $\operatorname{Log}(-1 + \mathrm{i}0) = \mathrm{i}\pi$. Everything reduces to $B_\delta > 0$ on $(0, \infty)^2$, where

$$B_\delta(s, t) = \operatorname{Im} \mathscr{H}_\delta(-1 + \mathrm{i}0, s, t), \qquad s, t > 0. \tag{2.20}$$

We follow the companion argument: analytic continuation of the line restrictions of F_δ, behavior at the endpoints of the cut, signs on the whole cut, and the Cauchy representation. Two points are new. The endpoint behavior depends on the sign of δ, and for $\delta > 0$ the representing measures have atoms.

Lemma 2.3 (Analytic continuation)**.** *For $X, Y \in (0, \infty)^3$, put $x_i(z) = X_i + zY_i$ and $\tau_i = X_i/Y_i$. The function*

$$f(z) = F_\delta(X + zY) \tag{2.21}$$

is holomorphic in the upper and lower half-planes and across the real interval $(-\min_i \tau_i, \infty)$. Apparent singularities at coincident nonzero arguments are removable. As $z \to \infty$,

$$f(z) = \frac{F_\delta(Y)}{z} + O(|z|^{-2}), \tag{2.22}$$

with corresponding upper and lower boundary interpretations near the negative real axis.

Proof. For $\operatorname{Im} z > 0$, all $x_i(z)$ lie in the upper half-plane. If q is real, $0 < |q| < 2$, and $x^q = y^q$ for two upper-half-plane points, then $q(\operatorname{Log} x - \operatorname{Log} y) = 2\pi\mathrm{i}k$ for an integer k. Its imaginary part has absolute value strictly below 2π, so $k = 0$ and $x = y$. Thus both powers appearing in (2.2), with exponents a and $-\delta$, are injective when the exponents are nonzero. At $\delta = 0$ use injectivity of the logarithm. Consequently $c_p(x_i, x_j)$ has no zero at distinct upper-half-plane arguments. At equality its limiting value is $1/x_i \neq 0$.

The expressions (2.3)–(2.6) therefore extend holomorphically, with their repeated-node singularities removed by analytic divided differences. Lower-half-plane analyticity follows by conjugation. Positive real arguments have ordinary analytic neighborhoods.

For large z, write $X_i + zY_i = z(Y_i + X_i/z)$. Principal logarithms add in this factorization in each open half-plane: the second factor is close to a positive real number and the product remains in the same half-plane. The degree -1 homogeneity calculation in (2.6) is therefore valid with this complex factor. Since F_δ is analytic near the positive vector Y,

$$f(z) = z^{-1}F_\delta(Y + X/z) = z^{-1}F_\delta(Y) + O(|z|^{-2}).$$

The same estimate holds for the boundary values by continuity. □

We call a pair (X, Y) *generic* when the three τ_i are distinct and the two nonzero arguments at each $z = -\tau_i$ are distinct. These conditions exclude only finitely many algebraic equalities in the positive variables. They form a dense open set; for any fixed $X > 0$, generic positive directions Y are dense. Proving the contour formula first for generic pairs avoids simultaneous zero arguments. Continuity will remove this restriction from the concavity conclusion.

Lemma 2.4 (Endpoint expansions)**.** *Let y, z be distinct nonzero arguments on the relevant principal sheets, with their power differences nonzero. As $x \to 0$ within either adjacent half-plane, the following estimates hold; they are locally uniform in y, z.*
(1) If $\delta > 0$,

$$\mathscr{H}_\delta(x,y,z) = -\frac{\delta^2}{3x} + O(|x|^{-1+\delta}) + O(1). \tag{2.23}$$

(2) If $\delta = -e < 0$,

$$\mathscr{H}_{-e}(x,y,z) = -\frac{e^2}{3y^e z^e} x^{-1+2e}(1 + O(|x|^e)) + O(1). \tag{2.24}$$

(3) If $\delta = 0$,

$$\mathscr{H}_0(x,y,z) = -\frac{1}{3x(\operatorname{Log} x - \operatorname{Log} y)(\operatorname{Log} x - \operatorname{Log} z)} + O(1). \tag{2.25}$$

For a generic affine line, the only simple pole terms of f occur when $\delta > 0$. At $z = -\tau_i$ their residues are

$$r_i = \frac{8\delta^2}{3b^2 Y_i} > 0. \tag{2.26}$$

After subtracting these pole terms, the remaining boundary singularities are absolutely integrable.

Proof. Use Proposition 2.2. The two fixed u-coordinates are nonzero and distinct, while $u_x = x^a \to 0$.

If $\delta > 0$, $w_x \sim x^{-\delta}/\delta$ diverges. The term $\mathcal{A}_\delta$ is bounded: its numerator is quadratic in w_x to leading order, as is its D_w denominator, while D_u has a nonzero limit. The term $\mathcal{U}_\delta$ is bounded. The x-vertex of $\mathcal{V}_\delta$ is exactly

$$-\frac{x^{-b}}{3(w_x - w_y)(w_x - w_z)} = -\frac{\delta^2}{3x}\frac{1}{(1 - x^\delta y^{-\delta})(1 - x^\delta z^{-\delta})}.$$

Its expansion gives (2.23); the other two vertices of $\mathcal{V}_\delta$ are bounded and in fact $O(|x|^\delta)$.

If $\delta = -e < 0$, then $w_x = (1 - x^e)/e \to 1/e$. The differences with the other two w-coordinates have nonzero limits. Again $\mathcal{A}_\delta$, $\mathcal{U}_\delta$ and the two vertices of $\mathcal{V}_\delta$ at the fixed nodes are bounded. The remaining vertex is

$$-\frac{e^2 x^{-1+2e}}{3(y^e - x^e)(z^e - x^e)},$$

which proves (2.24).

At zero, $w_x = -\operatorname{Log} x$ diverges, $\mathcal{A}_\delta$ stays bounded, and $\mathcal{U}_\delta$ vanishes, being the second divided difference of a linear function. The x-vertex of $\mathcal{V}_\delta$ is the explicit term in (2.25); the fixed vertices are $O(1/|\operatorname{Log} x|)$.

At a simple root on an affine line, $x_i = Y_i(z + \tau_i)$, so (2.23) gives (2.26). The other coordinates vary analytically and remain in compact neighborhoods of the fixed nonzero values, so the local estimates still apply. The remainders are bounded respectively by

$$C|z+\tau_i|^{-1+\delta}, \qquad C|z+\tau_i|^{-1+2e}, \qquad \frac{C}{|z+\tau_i|\,|\log|z+\tau_i||^2},$$

up to bounded terms. Each is integrable in the real boundary variable. Its integral over a small semicircle tends to zero after multiplying by arc length, as needed in the contour argument. □

The boundary value B_δ covers the part of the cut where exactly one argument is negative. The next lemma reduces the rest of the cut to it.

Lemma 2.5 (Signs on the cut)**.** *For nonzero positive r, s, t,*

$$\mathscr{H}_\delta(-r+\mathrm{i}0, -s+\mathrm{i}0, t+\mathrm{i}0) = -\overline{\mathscr{H}_\delta(r+\mathrm{i}0, s+\mathrm{i}0, -t+\mathrm{i}0)}. \tag{2.27}$$

If all three arguments are negative, the upper boundary value is real. Thus $B_\delta > 0$ implies $\operatorname{Im} f(x+\mathrm{i}0) \le 0$ for f in (2.21) and every x on its cut, apart from the isolated endpoints.

Proof. Conjugate the arguments $(r+\mathrm{i}0, s+\mathrm{i}0, -t+\mathrm{i}0)$ and rotate all three by $e^{\mathrm{i}\pi}$. During this rotation their arguments stay within a common principal-sheet continuation. Homogeneity of degree -1 gives the multiplier $e^{-\mathrm{i}\pi} = -1$, proving (2.27). Also $\operatorname{Im}(-\overline{z}) = \operatorname{Im} z$. If all three arguments are negative, the same common rotation from a positive triple gives a real value, namely the negative of the positive-triple value. For one negative argument, positive homogeneity reduces directly to $(-1, s, t)$. Finally $F_\delta = -8\mathscr{H}_\delta/b^2$, so its imaginary part has the opposite sign. □

The Cauchy integral formula now gives the representation, conditionally on the boundary sign.

Proposition 2.6 (Stieltjes representation)**.** *Fix $|\delta| \le \eta_0$ and suppose that $B_\delta(s,t) > 0$ for every $s, t > 0$. For every generic positive pair (X, Y), there is a finite nonzero positive measure $\mu_{X,Y}$ supported on $[\tau_{\min}, \tau_{\max}] \subset (0,\infty)$ such that*

$$F_\delta(X+zY) = \int \frac{\mathrm{d}\mu_{X,Y}(t)}{z+t}. \tag{2.28}$$

Its density away from the three endpoints is

$$\rho(t) = -\frac{1}{\pi} \operatorname{Im} F_\delta(X - tY + \mathrm{i}0Y) \ge 0. \tag{2.29}$$

If $\delta > 0$, it additionally has the three atoms (2.26); if $\delta \le 0$, there are no endpoint atoms.

Proof. Lemmas 2.3–2.4 justify Cauchy's integral formula on a large slit contour, with small semicircles around each $-\tau_i$. The large-circle contribution vanishes by (2.22). When $\delta > 0$, subtract the simple pole terms $r_i/(z+\tau_i)$ first; the remaining small-circle contributions vanish by Lemma 2.4. In all cases the upper and lower values are conjugate, so the two sides of the slit give

$$f(z) = \int_{\tau_{\min}}^{\tau_{\max}} \frac{\rho(t)}{z+t}\,\mathrm{d}t + \sum_{i:\,\delta>0} \frac{r_i}{z+\tau_i}.$$

On $(0, \tau_{\min})$ all arguments are positive, and on $(\tau_{\max}, \infty)$ all are negative with a real boundary value by Lemma 2.5, so $\rho = 0$ there. Integrability at the endpoints was proved above. On $(\tau_{\min}, \tau_{\max})$, away from the endpoints, one or two arguments are negative, so $\rho > 0$ by the assumed $B_\delta > 0$ and Lemma 2.5. The atoms are positive by (2.26). Thus the measure is finite, positive, and nonzero. □

2.4. **Concavity from the boundary sign.** The passage from the Stieltjes representation to concavity is the companion's, and we only cite it: the reciprocal of a Stieltjes transform is concave, and a function homogeneous of degree one that is concave along lines with positive direction is concave on the whole cone.

Lemma 2.7 (Reciprocal Stieltjes concavity)**.** *Let F be a smooth homogeneous function of degree -1 on $(0,\infty)^d$. Suppose that, for each $X > 0$ and a dense set of positive directions Y, the restriction $F(X+zY)$ is the Stieltjes transform of a finite nonzero positive measure whose support is a compact subset of $(0,\infty)$. Then $F > 0$ and $M = 1/F$ is jointly concave. For every $c_0 > 0$, the function $-c_0/M$ is jointly concave as well.*

Proof. Evaluation at $z = 0$ gives $F(X) > 0$. Along a direction with the stated representation, $z \mapsto M(X+zY)$ is concave near $z = 0$ by Sra (2026, Lemma 2.8), a Cauchy–Schwarz inequality for the representing measure. Smoothness extends this from the dense set of directions to every positive

direction, and Sra (2026, Lemma 2.9) turns concavity along positive directions into concavity of the degree-one homogeneous function M on the cone. Finally,

$$D^2(-c_0/M) = \frac{c_0}{M^2} D^2M - \frac{2c_0}{M^3}\, DM \otimes DM \preceq 0,$$

which proves the last assertion. □

Theorem 1.1 is now reduced to the boundary sign. If $B_\delta > 0$ on $(0,\infty)^2$ for all $|\delta| < \eta$, then for these δ Proposition 2.6 represents F_δ along every generic positive line, and Lemma 2.7 makes $\mathscr{H}_\delta = -(b^2/8)/M_\delta$ concave. Section 3 proves the boundary sign, Theorem 3.12, and completes the proof there.

3. The boundary sign, uniformly in the parameter

This section proves Theorem 3.12: $B_\delta > 0$ on all of $(0,\infty)^2$ for every sufficiently small $|\delta|$. At $\delta = 0$ this positivity is the boundary inequality of the companion paper (Section 3.1), and continuity extends it to every compact set of pairs (s,t) for small $|\delta|$. The work is at the boundary of $(0,\infty)^2$. Section 3.2 isolates the positive part J_δ of B_δ, an integral over a simplex, and writes B_δ in exact real coordinates as J_δ plus a remainder. Sections 3.3 and 3.4 show that $B_\delta/J_\delta \to 1$ as $\delta \to 0$ and (s,t) leaves every compact subset of $(0,\infty)^2$, and a compactness argument closes the proof.

Conventions. In the estimates, $A \lesssim B$ means $|A| \le CB$ with an absolute constant, independent of δ, s, t in the displayed ranges. We write $A \asymp B$ for two positive quantities bounded above and below by such constants. A subscript $\mathcal{K}$ permits dependence on a fixed compact interval. Constants may change between formulas. All denominators on the right of an $\asymp$ statement are positive.

3.1. The base case.

At $\delta = 0$ the decomposition reduces to the normal form of the companion paper. For positive x, y, z, set

$$A = c_1(x,y), \qquad B = c_1(x,z), \qquad C = c_1(y,z),$$

$$T = \int_0^\infty \frac{\mathrm{d}q}{(q+x)(q+y)(q+z)}, \qquad U = \int_0^\infty \frac{q\,\mathrm{d}q}{(q+x)(q+y)(q+z)}. \tag{3.1}$$

Partial fractions give $c_1(x,y) = \int_0^\infty ((q+x)(q+y))^{-1}\,\mathrm{d}q$ and therefore

$$A = U + zT, \qquad B = U + yT, \qquad C = U + xT. \tag{3.2}$$

Put

$$Q = \frac{2U}{xyz} - T^2, \qquad F_{\mathrm{KM}} = \frac{4Q}{3ABC}. \tag{3.3}$$

Proposition 3.1 (The function of the companion paper)**.** *On $(0,\infty)^3$,*

$$H_1 = \mathscr{H}_0 = -\frac{Q}{6ABC} = -\frac{1}{8} F_{\mathrm{KM}}. \tag{3.4}$$

The right side is the normal form of Dittmann's function h for the Kubo–Mori metric in Sra (2026, Proposition 2.2 and Definition 2.5); thus $H_1 = h$ and F_{KM} is the function F of that paper.

Proof. At $\delta = 0$, $u_i = x_i$ and $w_i = -\operatorname{Log} x_i$ in Proposition 2.2. Hence $D_w = -ABC\, D_u$ and $\Delta/D_u = -T$, so $\mathcal{A}_0 = T^2/(6ABC)$. The term $\mathcal{U}_0$ vanishes, being minus one third of the second divided difference of the linear function $x \mapsto x$. For $\mathcal{V}_0$, since $\operatorname{Log} x - \operatorname{Log} y = (x-y)A$ and cyclically,

$$\sum_{\mathrm{cyc}} \frac{x^{-1}}{(\operatorname{Log} x - \operatorname{Log} y)(\operatorname{Log} x - \operatorname{Log} z)} = \frac{xy(x-y)A - xz(x-z)B + yz(y-z)C}{xyz(x-y)(x-z)(y-z)ABC} = \frac{U}{xyzABC},$$

where the last equality follows by substituting (3.2): the numerator is $(x-y)(x-z)(y-z)U$. Thus $\mathcal{V}_0 = -U/(3xyzABC)$, and the three terms add up to (3.4). The identity holds for positive

arguments and hence, both sides being analytic, for the upper boundary values and their confluent limits. □

Proposition 3.2 (Base case)**.** *One has $B_0(s,t) > 0$ for every $s,t > 0$, including $s = t$.*

Proof. By Proposition 3.1, $B_0(s,t) = -\frac{1}{8}\operatorname{Im} F_{\mathrm{KM}}(-1+\mathrm{i}0, s, t)$, and Sra (2026, Theorem 3.5) proves $\operatorname{Im} F_{\mathrm{KM}}(-1+\mathrm{i}0, s, t) < 0$ for all $s,t > 0$, with the same principal-branch convention as here; the case $s = t$ is proved there separately, not inferred from a non-strict limit. □

The boundary inequality, rather than the real concavity of $\mathscr{H}_0$ or the monotonicity of the Kubo–Mori curvature, is the base case used below.

3.2. **The boundary value.** Of the three terms in (2.12), $\mathcal{V}_\delta$ has a boundary value of definite sign; it is the positive part against which everything else is measured. Set

$$J_\delta(s,t) = \operatorname{Im} \mathcal{V}_\delta(-1+\mathrm{i}0, s, t). \tag{3.5}$$

Every estimate below is relative to J_δ.

Lemma 3.3 (Simplex formula)**.** *For $|\delta| \le 1/100$ and $s,t > 0$, $J_\delta(s,t) > 0$. More precisely, for $\delta \neq 0$,*

$$\mathcal{V}_\delta(-1+\mathrm{i}0, s, t) = -\frac{ab}{3}\int_{\Delta_2} \left(\lambda_0 e^{-\mathrm{i}\pi\delta} + \lambda_1 s^{-\delta} + \lambda_2 t^{-\delta}\right)^{1/\delta} \mathrm{d}\lambda, \tag{3.6}$$

where $\Delta_2 = \{\lambda_i \ge 0 : \sum_i \lambda_i = 1\}$ has ordinary simplex measure of total mass $1/2$. At $\delta = 0$,

$$J_0(s,t) = \frac{1}{3}\int_{\Delta_2} s^{-\lambda_1} t^{-\lambda_2} \sin(\pi\lambda_0)\, \mathrm{d}\lambda > 0. \tag{3.7}$$

Proof. As a function of $w = (x^{-\delta} - 1)/\delta$, the quantity x^{-b} is $f_\delta(w) = (1+\delta w)^{2+1/\delta}$, and $f_\delta''(w) = ab(1+\delta w)^{1/\delta}$. The elementary second-divided-difference formula is

$$f[z_0, z_1, z_2] = \int_{\Delta_2} f''(\lambda_0 z_0 + \lambda_1 z_1 + \lambda_2 z_2)\, \mathrm{d}\lambda.$$

For completeness, integrate first in λ_1 with λ_2 fixed and then in λ_2. The fundamental theorem of calculus gives

$$\frac{1}{z_1 - z_0}\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1} - \frac{f(z_2) - f(z_0)}{z_2 - z_0}\right),$$

which is the same symmetric second divided difference. Confluent cases follow by continuity. The argument holds for analytic f on a neighborhood of the convex hull of the nodes.

Apply this identity to (2.15). The convex combination inside (3.6) stays in the right half-plane. At an interior point of the simplex its argument lies in $(-\pi\delta, 0)$ if $\delta > 0$ and in $(0, -\pi\delta)$ if $\delta < 0$; after division by δ the phase lies in $(-\pi, 0)$ in both cases, so the integrand has negative imaginary part. Since $ab > 0$, $J_\delta > 0$. Taking the continuous limit at zero gives (3.7). □

Real boundary coordinates. The simplex formula gives the sign of J_δ but not its size, and the comparison of B_δ with J_δ needs both in real coordinates. Put

$$\theta = \pi\delta, \quad c = \cos\theta, \quad \kappa = \frac{\sin\theta}{\delta}, \quad \kappa|_{\delta=0} = \pi. \tag{3.8}$$

For $x > 0$ define

$$z_x = \frac{x^{-\delta} - c}{\delta}, \quad D_x = z_x^2 + \kappa^2, \quad S_x = x^a, \quad E_x = S_x^2 + 2cS_x + 1, \tag{3.9}$$

with $z_x = -\log x$ at $\delta = 0$. For $s < t$, abbreviate

$$S = s^a, \quad T = t^a, \quad v = s^{-\delta}, \quad w = t^{-\delta}, \qquad K = \frac{z_s - z_t}{T - S}, \qquad g(x) = \frac{x^b}{E_x}.$$

Introduce

$$N = \frac{s^{-b}D_t - t^{-b}D_s}{z_s - z_t}, \tag{3.10}$$

$$L = -\cos(2\theta)(z_s + z_t) - 2\delta c(z_s z_t - \kappa^2), \tag{3.11}$$

$$A_* = A_1 + A_2 + A_3 + A_4, \tag{3.12}$$

$$A_1 = \frac{\delta}{K}\frac{(S+T+2c)D_sD_t}{E_sE_t}, \qquad A_2 = \frac{(S+w)D_s}{E_s},$$

$$A_3 = \frac{(T+v)D_t}{E_t}, \qquad A_4 = K(z_s + z_t),$$

$$U_* = U_1 + U_2, \quad U_1 = \frac{(S+T+2cST)D_sD_t}{E_sE_t}, \quad U_2 = D_sD_t\frac{g(s)-g(t)}{S-T}. \tag{3.13}$$

The symbols N, L, A_*, U_* depend on (δ, s, t).

Proposition 3.4 (Exact boundary formula)**.** *At distinct positive arguments,*

$$B_\delta(s,t) = \frac{\kappa}{3D_sD_t}\left(N + L - \frac{a}{2}A_* + a^2\delta U_*\right), \qquad J_\delta(s,t) = \frac{\kappa}{3D_sD_t}(N+L). \tag{3.14}$$

Both formulas extend continuously to $s = t$. In particular, at $s = t = x$,

$$K = \frac{x^{-b}}{a}, \qquad N = x^{-a}(z_x^2 - 2cz_x + b\kappa^2), \tag{3.15}$$

$$\frac{g(s)-g(t)}{S-T} \longrightarrow \frac{x^\delta(bE_x - 2aS_x(S_x+c))}{aE_x^2}. \tag{3.16}$$

Proof. At the negative node, $u_0 = -e^{\mathrm{i}\theta}$ and $w_x - w_0 = z_x + \mathrm{i}\kappa$. Put $\mathsf{U}_x = S_x + e^{\mathrm{i}\theta}$ and $\mathsf{Z}_x = z_x + \mathrm{i}\kappa$. Since $\mathsf{Z}_s + K\mathsf{U}_s = \mathsf{Z}_t + K\mathsf{U}_t$, the term $\mathcal{A}_\delta$ is

$$\mathcal{A}_\delta = \frac{a}{6}\left[\frac{1}{K\mathsf{U}_s\mathsf{U}_t} + \frac{1}{\mathsf{U}_s\mathsf{Z}_t} + \frac{1}{\mathsf{U}_t\mathsf{Z}_s} + \frac{K}{\mathsf{Z}_s\mathsf{Z}_t}\right].$$

Taking imaginary parts and using $\sin\theta = \delta\kappa$ yields $-a\kappa A_*/(6D_sD_t)$. For instance,

$$\operatorname{Im}\frac{1}{\mathsf{U}_s\mathsf{Z}_t} = -\kappa\frac{S+w}{E_sD_t},$$

because $c + \delta z_t = w$. In $\mathcal{U}_\delta$, the numerator at the negative node is $(-1+\mathrm{i}0)^b = -e^{2\mathrm{i}\theta}$, while the two positive-node terms have real numerators. Directly taking their imaginary parts gives $a^2\delta\kappa U_*/(3D_sD_t)$. In $\mathcal{V}_\delta$ the numerator at the negative node is $-e^{-2\mathrm{i}\theta}$; its imaginary part gives L, and those of the positive-node terms give N. This proves (3.14). Finally, $xz_x' = -x^{-\delta}$, $S_x' = ax^{a-1}$, and differentiating the real quotients gives (3.15)–(3.16). □

The two terms N and L carry the positive part of the boundary value; the next lemma records their signs. The remaining terms are the ones the estimates of Sections 3.3 and 3.4 must control.

Lemma 3.5 (Signs of N and L)**.** *For $|\delta| \le \eta_0$ and all $s, t > 0$, one has $N > 0$. For $x > 0$ put*

$$P_x = \frac{1 - cx^{-\delta}}{\delta}, \qquad P_x|_{\delta=0} = \log x. \tag{3.17}$$

If $s, t \ge e$, then

$$L = s^{-\delta}P_t + t^{-\delta}P_s > 0. \tag{3.18}$$

Proof. Let $f(x) = x^{-b}/D_x$. From $xz'_x = -x^{-\delta}$ and $x^{-\delta} = c + \delta z_x$,

$$-\frac{xf'(x)}{f(x)} = \frac{z_x^2 - 2cz_x + b\kappa^2}{z_x^2 + \kappa^2}. \tag{3.19}$$

The numerator is a quadratic in z_x with discriminant $4(c^2 - b\kappa^2) < 0$, because $\kappa > 3$ and $b \geq 0.98$ on the parameter interval. Hence f is strictly decreasing, as is $x \mapsto z_x$, and

$$N = D_s D_t \frac{f(s) - f(t)}{z_s - z_t} > 0 \qquad (s \neq t);$$

at $s = t = x$, (3.15) gives $N = x^{-a}(z_x^2 - 2cz_x + b\kappa^2) > 0$.

For (3.18), substitute $z_s = (s^{-\delta} - c)/\delta$, $z_t = (t^{-\delta} - c)/\delta$ and $\delta^2\kappa^2 = 1 - c^2$ in (3.11). The result is $(s^{-\delta} + t^{-\delta} - 2cs^{-\delta}t^{-\delta})/\delta = s^{-\delta}P_t + t^{-\delta}P_s$ for $\delta \neq 0$, and $L = \log s + \log t$ at $\delta = 0$. Finally $P_x > 0$ for $x \geq e$:

$$P_x = c\int_0^{\log x} e^{-\delta r}\,\mathrm{d}r + \frac{1-c}{\delta},$$

where the integral is at least $e^{-\eta_0}$ and $|1 - c|/|\delta| \leq \pi^2|\delta|/2 \leq 1/20$. □

3.3. **One escaping argument.** All limits in this section are joint limits in δ and the arguments: no bound on $|\delta| \log t$ is assumed.

For $L > 0$, define

$$A_\delta(L) = \frac{\delta}{1 - e^{-\delta L}}, \qquad C_\delta(L) = \frac{\delta}{e^{\delta L} - 1}, \qquad A_0(L) = C_0(L) = \frac{1}{L}. \tag{3.20}$$

For either sign of δ, $A_\delta(L) > 0$ and $C_\delta(L) > 0$. Their integral representations give

$$\frac{e^{-\eta_0 L}}{L} \leq A_\delta(L), C_\delta(L) \leq |\delta| + \frac{1}{L}. \tag{3.21}$$

For the upper bounds, use $q/(e^q - 1) \leq 1$ and $q/(1 - e^{-q}) \leq 1 + q$ for $q \geq 0$; the latter follows from $e^q \geq 1 + q$. The lower bounds follow by writing the reciprocals as $\int_0^L e^{\mp\delta r}\,\mathrm{d}r$.

Proposition 3.6 (One escaping argument)**.** *Let $\mathcal{K} = [m, M] \subset (0, \infty)$ be compact. Uniformly for $s \in \mathcal{K}$, as $\delta \to 0$ and $L \to \infty$,*

$$B_\delta(s, e^L) = A_\delta(L)\frac{\pi(1 + s^{-1})}{3((\log s)^2 + \pi^2)}[1 + o_{\mathcal{K}}(1)], \tag{3.22}$$

$$B_\delta(s, e^{-L}) = \frac{\pi}{3}e^{bL}C_\delta(L)^3[1 + o_{\mathcal{K}}(1)]. \tag{3.23}$$

The same leading terms hold for J_δ. In particular, $B_\delta/J_\delta \to 1$ in each regime.

Proof. Use the decomposition of Proposition 2.2, with fixed nodes $x_0 = -1 + \mathrm{i}0$ and $x_1 = s$, and escaping node $x_2 = t$. Denote their normalized coordinates by u_0, u_s, U and w_0, w_s, W. The differences $u_0 - u_s$ and $w_0 - w_s$ stay bounded away from zero, uniformly for $s \in \mathcal{K}$ and δ near zero. All coordinates of the first two nodes remain uniformly bounded.

***Large** t.* Put $t = e^L$ and $A = A_\delta(L)$. Then $U = t^a$ and $W = -1/A$. Along the joint limit, $U \to \infty$ and $|W| \to \infty$. The determinant formulas give

$$|D_u| \asymp_{\mathcal{K}} U^2, \quad |D_w| \asymp_{\mathcal{K}} |W|^2, \quad |\Delta| \lesssim_{\mathcal{K}} U + |W|,$$

so

$$|\mathcal{A}_\delta| \lesssim_{\mathcal{K}} A^2 + U^{-2}, \qquad |\mathcal{U}_\delta| \lesssim_{\mathcal{K}} t^{-a} + t^{-1}. \tag{3.24}$$

The second estimate follows by separating the two fixed vertices from the t-vertex; the latter has size $t^b/U^2 = t^{-1}$. The term $\mathcal{V}_\delta$ is exactly

$$\begin{aligned}\mathcal{V}_\delta = &-\frac{A}{3(w_0-w_s)}\left(\frac{x_0^{-b}}{1+Aw_0}-\frac{s^{-b}}{1+Aw_s}\right)\\ &-\frac{t^{-b}A^2}{3(1+Aw_0)(1+Aw_s)}.\end{aligned}\tag{3.25}$$

Now $x_0^{-b}\to -1$, $w_0\to -\mathrm{i}\pi$, and $w_s\to -\log s$. Since $A\to 0$, the first line equals

$$A\frac{1+s^{-1}}{3(\log s-\mathrm{i}\pi)}+O_{\mathcal{K}}(A(|\delta|+A)).$$

The errors in (3.24), divided by A, tend uniformly to zero: by (3.21), for example,

$$\frac{t^{-a}+t^{-1}}{A}\le 2Lt^{-1+2\eta_0}\longrightarrow 0,\qquad \frac{U^{-2}}{A}\le Lt^{-2+3\eta_0}\longrightarrow 0.$$

The last line of (3.25) is also $o(A)$. Taking imaginary parts proves (3.22), whose leading coefficient is bounded below by a positive constant on $\mathcal{K}$.

Small t. Put $t=e^{-L}$ and $C=C_\delta(L)$. Then $U=t^a\to 0$ and $W=1/C\to\infty$. The fixed u-differences are bounded away from zero, so $\mathcal{A}_\delta$ is bounded: D_u is bounded away from zero, $|D_w|\asymp_{\mathcal{K}} W^2$, and $|\Delta|\lesssim_{\mathcal{K}} W$. The term $\mathcal{U}_\delta$ is bounded as well, and the two vertices of $\mathcal{V}_\delta$ at the fixed nodes are $O_{\mathcal{K}}(C)$. Hence

$$\mathscr{H}_\delta(x_0,s,t)=-\frac{t^{-b}C^2}{3(1-Cw_0)(1-Cw_s)}+O_{\mathcal{K}}(1).\tag{3.26}$$

These bounds are uniform once $|\delta|$ is sufficiently small and L sufficiently large in terms of $\mathcal{K}$; this uniformity is enough for the asserted joint limit.

Since w_s is real and $\operatorname{Im} w_0=-\kappa$, the imaginary part of the explicit term is

$$\frac{t^{-b}C^3\kappa}{3|1-Cw_0|^2(1-Cw_s)}.\tag{3.27}$$

Its coefficient tends uniformly to $\pi/3$. Moreover,

$$t^{-b}C^3\ge\frac{t^{-1+5\eta_0}}{L^3}\longrightarrow\infty,$$

so the bounded error in (3.26) is negligible relative to (3.27). This proves (3.23) and the assertion for J_δ. □

Corollary 3.7 (Positivity with one escaping argument)**.** *For every compact $\mathcal{K}\subset(0,\infty)$, there exists $\eta_{\mathcal{K}}>0$ such that*

$$B_\delta(s,t)>0\qquad (s\in\mathcal{K},\ t>0,\ |\delta|<\eta_{\mathcal{K}}).$$

Proof. The positive leading terms of Proposition 3.6 control the two tails uniformly. Between the tails, s,t lie in a compact set; Proposition 3.2 and continuity in (δ,s,t) give a positive minimum for sufficiently small $|\delta|$. Take the smaller of the resulting neighborhoods. □

3.4. **The three corners and the global theorem.** It remains to control the joint limits in which both arguments escape. Define the error envelope

$$\Psi(r)=(1+|\log r|)^3 r^{1-6\eta_0},\qquad 0<r\le e^{-1}.\tag{3.28}$$

Then $\Psi(r)\to 0$ as $r\downarrow 0$. The three propositions below bound the terms of (3.14) other than N and L relative to the positive part. Their proofs occupy Appendix A.

Proposition 3.8 (Small–small corner)**.** *For $0 < s \le t \le e^{-1}$ and $|\delta| \le \eta_0$,*

$$\frac{|L| + |\delta U_*|}{N} \lesssim \Psi(t), \tag{3.29}$$

$$\frac{|A_*|}{N} \lesssim \frac{t^{-\delta}}{z_t} + \Psi(t). \tag{3.30}$$

The inequalities include the case $s = t$.

Proposition 3.9 (Mixed corner)**.** *For $s \le e^{-1}$, $t \ge e$, and $|\delta| \le \eta_0$,*

$$\frac{|L| + |A_*| + |\delta U_*|}{N} \lesssim \Psi(s). \tag{3.31}$$

Proposition 3.10 (Large–large corner)**.** *For $e \le s \le t$ and $|\delta| \le \eta_0$,*

$$\frac{|A_*| + |\delta U_*|}{L} \lesssim \Psi(1/s). \tag{3.32}$$

The estimate includes $s = t$.

Why the estimates close. In each corner the positive part has an explicit scale, and the terms $A_1, \dots, A_4$, U_1, U_2 are bounded one at a time against it. The parameter enters only through $x^{-\delta}$ and through integrals $\int_0^R e^{\pm\delta r}\,\mathrm{d}r$ over the logarithmic separation $R = \log(t/s)$; these quantities are kept exact until the single bound $\sup_{R\ge 0}(1+R)^k e^{-\gamma R} < \infty$ absorbs them, so no expansion in $\delta \log s$ is ever needed. A monotone form of l'Hôpital's rule controls the divided differences K and $(g(s) - g(t))/(S - T)$ up to $s = t$, which is why every estimate includes the confluent case.

Theorem 3.11 (The three corners)**.** *Along every joint limit with $\delta \to 0$ and, up to exchanging s, t,*

$$(s,t) \to (0,0), \qquad (s,t) \to (0,\infty), \qquad (s,t) \to (\infty,\infty),$$

one has

$$\frac{B_\delta(s,t)}{J_\delta(s,t)} \longrightarrow 1. \tag{3.33}$$

No condition is imposed on $\delta \log s$, $\delta \log t$, or s/t.

Proof. Lemma 3.3 gives $J_\delta > 0$. From (3.14),

$$\frac{B_\delta}{J_\delta} - 1 = \frac{-aA_*/2 + a^2 \delta U_*}{N + L}.$$

In the small–small corner assume $s \le t$. Proposition 3.8 gives $|L|/N \to 0$ and an error bounded by

$$C\left(\Psi(t) + \frac{t^{-\delta}}{z_t}\right) \le C'\left(\Psi(t) + |\delta| + \frac{1}{\log(1/t)}\right) \longrightarrow 0.$$

In the mixed corner, Proposition 3.9 gives $|L|/N \to 0$ and the required relative error directly. In the large–large corner, $N + L \ge L > 0$ by Lemma 3.5, and Proposition 3.10 gives the result. The case $s = t$ is included in each proposition. □

Theorem 3.12 (Global boundary positivity)**.** *There exists $\eta > 0$ such that*

$$B_\delta(s,t) > 0 \qquad (s,t > 0,\ |\delta| < \eta). \tag{3.34}$$

Proof. If no such neighborhood existed, one could choose $\delta_j \to 0$ and positive s_j, t_j with $B_{\delta_j}(s_j, t_j) \le 0$. After passing to a subsequence, both s_j, t_j have limits in $[0, \infty]$.

If both limits belong to $(0, \infty)$, continuity and Proposition 3.2 give a contradiction. If exactly one limit belongs to that interval, Corollary 3.7 gives a contradiction. In all remaining cases, symmetry reduces the limit to one of the three corners in Theorem 3.11. There $B_{\delta_j} = J_{\delta_j}(1 + o(1)) > 0$, again a contradiction. □

Proof of Theorem 1.1. Theorem 3.12 gives $\eta > 0$, which we may take at most η_0, such that $B_\delta > 0$ on $(0,\infty)^2$ whenever $|\delta| < \eta$. For such δ, Proposition 2.6 represents F_δ along every generic positive line, and Lemma 2.7 with $c_0 = b^2/8$ shows that $F_\delta > 0$ and that M_δ and $\mathscr{H}_\delta = -c_0/M_\delta$ are jointly concave on $(0,\infty)^3$. This is part (i).

For part (ii), cancel the fully diagonal index triples in (2.7) before using concavity:

$$\mathrm{Scal}_p(\lambda) = C_n + \sum_{\substack{i,j,k\in\{1,\dots,n\}\\ \text{not all equal}}} H_p(\lambda_i,\lambda_j,\lambda_k). \tag{3.35}$$

Every summand is $\mathscr{H}_\delta$ composed with a linear map selecting coordinates of λ; this includes the terms with two equal indices, such as $H_p(\lambda_i,\lambda_i,\lambda_j)$. Each summand is therefore concave, and the sum is invariant under permutations of λ. Equal eigenvalues change nothing, since the index sum is the same and $\mathscr{H}_\delta$ is continuous. A symmetric concave function on the simplex is Schur-concave (Marshall et al., 2011, Chapter 3, Section C), which is the increase under mixing.

Finally, $|1/p - 1| < \eta$ holds whenever $|p-1| < \eta/(1+\eta)$, so $\varepsilon = \eta/(1+\eta)$ serves in Theorem 1.1; decreasing η if necessary keeps the interval inside $[1/2, 2]$, where $\mathrm{WYD}(p)$ is a monotone metric. No step introduced a constant depending on n. □

Remark 3.13 (Failure of the boundary sign at $p = 2/3$)**.** *The boundary sign does not hold throughout the family. At $p = 2/3$, so $\delta = 1/2$, $a = 3/2$ and $b = 2$, the nodes $(-1+\mathrm{i}0, 1, 4)$ have the normalized coordinates $(u_0,u_1,u_2) = (-\mathrm{i}, 1, 8)$ and $(w_0,w_1,w_2) = (-2-2\mathrm{i}, 0, -1)$, and Proposition 2.2, an identity valid at these distinct nodes although δ lies outside the range of the estimates, gives*

$$\mathcal{A}_{1/2} = \frac{27}{364} - \frac{153}{728}\mathrm{i}, \quad \mathcal{U}_{1/2} = -\frac{18}{91} + \frac{9}{364}\mathrm{i}, \quad \mathcal{V}_{1/2} = -\frac{1}{16} + \frac{1}{8}\mathrm{i},$$

so that

$$H_{2/3}(-1+\mathrm{i}0, 1, 4) = -\frac{271}{1456} - \frac{11}{182}\mathrm{i}. \tag{3.36}$$

Hence $\operatorname{Im} H_p(-1+\mathrm{i}0, s, t) \geq 0$ fails for some $s,t > 0$ at $p = 2/3$, and a proof for every $p \in [1/2, 2)$, if one exists, cannot proceed line by line through Stieltjes representations of F_δ. The computation says nothing about the concavity of $H_{2/3}$ on $(0,\infty)^3$ or about the monotonicity of the scalar curvature of $\mathrm{WYD}(2/3)$.

AI USAGE

The results in this paper were developed with an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs.

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Appendix A. Proof of the corner estimates

This appendix proves Propositions 3.8, 3.9 and 3.10. It first collects the elementary tools: a monotone form of l'Hôpital's rule, the scale bounds for $f = x^{-b}/D_x$ and $g = x^b/E_x$, and the sizes of the logarithmic coordinates z_x and P_x in the small and large ranges.

The first tool is an elementary form of the monotone l'Hôpital rule; broader versions are discussed by Pinelis (2007, 2020).

Lemma A.1 (Monotone l'Hôpital rule)**.** *Suppose $F(0) = G(0) = 0$, $G' > 0$ on $(0, R_0)$, and F'/G' is nonincreasing. Then F/G is nonincreasing and does not exceed the right limit of F'/G' at zero, when that limit exists.*

Proof. Write

$$\frac{F(R)}{G(R)} = \frac{\int_0^R (F'/G')(r) G'(r)\,\mathrm{d}r}{\int_0^R G'(r)\,\mathrm{d}r}.$$

This is a weighted average of a nonincreasing function and is therefore at least its value at the right endpoint. Differentiating F/G now gives a nonpositive derivative. The bound at zero follows from the same weighted-average formula. □

Lemma A.2 (Scale bounds)**.** *Let $f(x) = x^{-b}/D_x$ as in Lemma 3.5. Uniformly for $|\delta| \le \eta_0$,*

$$\frac{1}{2} \le -\frac{x f'(x)}{f(x)} \le \frac{3}{2}. \tag{A.1}$$

Consequently, for $s < t$,

$$(s/t)^{3/2} \le f(t)/f(s) \le (s/t)^{1/2}, \qquad \frac{K(s,t)}{s^{-\delta}} \le \frac{s^{-a}}{a}. \tag{A.2}$$

Also

$$|g'(x)| \le C \begin{cases} x^{b-1}, & 0 < x \le e^{-1}, \\ x^{-2}, & x \ge e, \end{cases} \tag{A.3}$$

with an absolute constant C.

Proof. By (3.19), $-xf'/f = (z_x^2 - 2cz_x + b\kappa^2)/(z_x^2 + \kappa^2)$. On the parameter interval, $\kappa > 3$ and $0.98 \le b \le 1.02$. Since $2|cz_x| \le \kappa^{-1}(z_x^2 + \kappa^2)$, the bounds (A.1) follow. Integrating the logarithmic derivative gives the first inequalities in (A.2). For the remaining inequality set $R = \log(t/s)$. Then

$$\frac{K}{s^{-\delta}} = s^{-a} \frac{\int_0^R e^{-\delta r}\,\mathrm{d}r}{e^{aR} - 1}.$$

The derivative ratio of the last numerator and denominator is $a^{-1}e^{-br}$, which is decreasing. Lemma A.1 gives the bound. Finally,

$$\frac{xg'(x)}{g(x)} = b - \frac{2aS_x(S_x+c)}{E_x}$$

is uniformly bounded. Combine this with $g(x) \le Cx^b$ for $x \le e^{-1}$ and $g(x) \le Cx^{-1}$ for $x \ge e$ to prove (A.3). □

Lemma A.3 (Logarithmic coordinates)**.** *For $0 < x \le e^{-1}$,*

$$z_x \asymp \int_0^{|\log x|} e^{\delta r}\,\mathrm{d}r > 0, \quad D_x \asymp z_x^2, \quad z_x \lesssim (1+|\log x|)x^{-\eta_0}, \quad \frac{x^{-\delta}}{z_x} \lesssim |\delta| + \frac{1}{|\log x|}. \tag{A.4}$$

For $x \ge e$, with P_x as in (3.17),

$$P_x > 0, \quad P_x \asymp -z_x, \quad D_x \asymp P_x^2, \quad P_x \lesssim (1+\log x)x^{\eta_0}, \quad P_x' = cx^{-a} > 0. \tag{A.5}$$

The lower bounds for z_x and P_x on their respective ranges are absolute positive constants.

Proof. In the small range,

$$z_x = \int_0^{|\log x|} e^{\delta r}\,\mathrm{d}r + \frac{1-c}{\delta}.$$

The last term has absolute value at most $\pi^2|\delta|/2$. The integral is at least $\int_0^1 e^{-\eta_0 r}\,\mathrm{d}r$, so this error is a fixed small fraction of the integral. This proves the first comparison and the positive lower bound. The integral is at most $|\log x|e^{\eta_0|\log x|}$ and at least a constant times $\min(|\log x|, 1/|\delta|)$, with the usual infinite interpretation at zero. Since $x^{-\delta} = c + \delta z_x$, these estimates give (A.4).

For the large range, the quantities $-z_x$ and P_x are respectively

$$\int_0^{\log x} e^{-\delta r}\,\mathrm{d}r - \frac{1-c}{\delta}, \qquad c\int_0^{\log x} e^{-\delta r}\,\mathrm{d}r + \frac{1-c}{\delta}.$$

The same comparison, now with $e^{-\delta r}$, proves their positivity, comparability, and the upper bound. Differentiation gives $P_x' = cx^{-a}$. □

A.1. **The small–small corner.** A polynomial in a logarithmic separation will always be absorbed using

$$\sup_{R\ge 0}(1+R)^k e^{-\gamma R} < \infty \quad (k \text{ fixed},\ \gamma \ge \gamma_0 > 0). \tag{A.6}$$

This elementary bound follows either by differentiating, or by comparing the exponential series with a higher-degree monomial. Importantly, factors such as $e^{\delta R}$ are retained exactly until they are bounded by (A.6).

Proof of Proposition 3.8. We first handle nearby nodes and then separated nodes. In both cases $E_s, E_t \asymp 1$ and $D_x \asymp z_x^2$ by Lemma A.3.

Nearby nodes: $s \le t \le 2s$**.** The integral formulas

$$z_s - z_t = \int_s^t x^{-a}\,\mathrm{d}x, \qquad T - S = \int_s^t ax^{a-1}\,\mathrm{d}x$$

and (A.1) give, uniformly up to confluence,

$$K \asymp s^{-b}, \qquad z_s \asymp z_t, \qquad N \asymp s^{-a}z_s^2. \tag{A.7}$$

For N, apply the integral formula for $f(s) - f(t)$ and use the positive two-sided bound on $-xf'/f$ in (A.1). Set $z = z_s$. Direct substitution into (3.12)–(3.13) yields

$$\frac{|A_1|}{N} \lesssim |\delta| s^{a+b} z^2, \qquad \frac{|A_2| + |A_3|}{N} \lesssim s^{2a} + s,$$

$$\frac{|A_4|}{N} \lesssim \frac{s^{-\delta}}{z}, \qquad \frac{|L|}{N} \lesssim s^a(z^{-1}+|\delta|),$$
$$\frac{|U_1|}{N} \lesssim s^{2a}z^2, \qquad \frac{|U_2|}{N} \lesssim s^b z^2.$$

For the last bound, (A.3) implies $|(g(s)-g(t))/(S-T)| \lesssim s^{b-a} = s^{\delta}$. Using $z \lesssim (1+|\log s|)s^{-\eta_0}$, the terms other than A_4 are respectively bounded by constant multiples of

$$(1+|\log s|)^2 s^{2-5\eta_0}, \quad s, \quad s^{1-\eta_0}, \quad (1+|\log s|)^2 s^{2-4\eta_0}, \quad (1+|\log s|)^2 s^{1-4\eta_0}.$$

These terms are bounded by $C\Psi(t)$ because $t/s \le 2$. The remaining term has the form in (3.30), again by comparability of nearby nodes.

Separated nodes: $t \ge 2s$**.** Put

$$R = \log(t/s), \qquad E = \int_0^R e^{\delta r}\,\mathrm{d}r, \qquad q = \frac{z_s}{z_t}, \qquad w = t^{-\delta}.$$

The exact identities

$$z_s - z_t = wE, \qquad z_s = e^{\delta R}z_t + cE \tag{A.8}$$

give

$$E \le Re^{\eta_0 R}, \qquad q \lesssim (1+R)e^{\eta_0 R}. \tag{A.9}$$

Since $f(t)/f(s) \le 2^{-1/2}$ and $T - S \asymp T$,

$$N \asymp \frac{s^{-b}z_t^2}{wE}, \qquad K \asymp \frac{wE}{t^a}. \tag{A.10}$$

The following bounds are obtained by inserting (A.10) into the explicit terms; they display all dependence on the separation R:

$$\frac{|A_1|}{N} \lesssim |\delta| t^{a+b}e^{-bR}z_t^2 q^2, \tag{A.11}$$

$$\frac{|A_2|}{N} \lesssim \left[t^{2+2\delta}e^{-(a+b)R} + te^{-bR}\right] Eq^2, \tag{A.12}$$

$$\frac{|A_3|}{N} \lesssim \left[t^{2+2\delta}e^{-bR} + te^{-aR}\right] E, \tag{A.13}$$

$$\frac{|A_4|}{N} \lesssim \frac{w}{z_t}E^2 e^{-bR}(1+q), \tag{A.14}$$

$$\frac{|U_1|}{N} \lesssim t^{2+2\delta}e^{-bR}Ez_t^2 q^2, \tag{A.15}$$

$$\frac{|U_2|}{N} \lesssim t^b e^{-bR}Ez_t^2 q^2, \tag{A.16}$$

$$\frac{|L|}{N} \lesssim t^a e^{-bR}E\left(\frac{1+q}{z_t} + |\delta|q\right). \tag{A.17}$$

For example, A_2 divided by the scale for N is bounded by $(s^a + w)wEs^b q^2$, which is exactly the right side of (A.12). For U_2, use $|g(s)-g(t)| \lesssim t^b$ and $T - S \asymp t^a$, hence the divided difference is $O(t^{\delta})$. The other displayed inequalities follow by canceling the indicated powers of z_t in the same way; no limiting expansion is involved.

The combinations of E, q, and exponential factors in (A.11)–(A.17) are uniformly bounded by (A.9) and (A.6). The smallest decay rate required is $b - 3\eta_0 \ge 1 - 5\eta_0 > 0$. The bound on z_t in (A.4) now gives

$$|A_1|/N \lesssim (1+|\log t|)^2 t^{2-5\eta_0}, \qquad (|A_2|+|A_3|)/N \lesssim t,$$
$$|A_4|/N \lesssim w/z_t, \qquad |U_1|/N \lesssim (1+|\log t|)^2 t^{2-4\eta_0},$$

$$|U_2|/N \lesssim (1+|\log t|)^2 t^{1-4\eta_0}, \qquad |L|/N \lesssim t^{1-\eta_0}.$$

These imply (3.29)–(3.30). The nearby-node proof already included the confluent limit, so the proposition holds for $s = t$ as well. □

A.2. **The mixed corner.**

Proof of Proposition 3.9. Put $z = z_s > 0$ and $P = -z_t > 0$. By Lemma A.3, both are bounded below by absolute positive constants. Moreover,

$$D_s \asymp z^2, \quad D_t \asymp P^2, \quad E_s \asymp 1, \quad E_t \asymp T^2,$$

and $s/t \le e^{-2}$, so (A.2) gives

$$K \asymp \frac{z+P}{T}, \qquad N \asymp \frac{s^{-b}P^2}{z+P}. \tag{A.18}$$

We also have

$$v = c + \delta z \lesssim 1+z, \quad w = c - \delta P \lesssim 1+P, \quad P/T \lesssim 1, \tag{A.19}$$

where v, w remain positive. The last bound follows from $P \lesssim (1+\log t)t^{\eta_0}$ and $a \ge 1-\eta_0$.

Here are the termwise estimates after division by the scale in (A.18):

$$\begin{aligned}
|A_1|/N &\lesssim |\delta| s^b z^2,\\
|A_2|/N &\lesssim s^b z^2 \frac{(1+P)(z+P)}{P^2},\\
|A_3|/N &\lesssim s^b (z+P)\left(T^{-1} + vT^{-2}\right),\\
|A_4|/N &\lesssim s^b \frac{(z+P)^3}{TP^2},\\
|L|/N &\lesssim s^b \frac{(z+P)(z+P+|\delta| zP+1)}{P^2},\\
|U_1|/N &\lesssim s^b z^2 \frac{z+P}{T}.
\end{aligned}$$

Each right side is at most $Cs^b(1+z)^3$, by (A.19) and the positive lower bound on P. For U_2, the elementary bound

$$|g(s) - g(t)| \lesssim s^b + t^{-1}, \qquad T - S \asymp T$$

gives

$$|U_2|/N \lesssim s^b z^2 \frac{z+P}{T}(s^b + t^{-1}) \lesssim s^b (1+z)^3.$$

Finally, (A.4) implies

$$s^b(1+z)^3 \lesssim (1+|\log s|)^3 s^{1-5\eta_0} \lesssim \Psi(s),$$

which proves the claim. □

A.3. **The large–large corner.**

Proof of Proposition 3.10. Write $P = P_s$, $Q = P_t$, $v = s^{-\delta}$, and $w = t^{-\delta}$. Then $0 < P \le Q$,

$$D_s \asymp P^2, \quad D_t \asymp Q^2, \quad E_s \asymp S^2, \quad E_t \asymp T^2, \quad L = vQ + wP.$$

We use repeatedly the consequence of (A.6)

$$\sup_{t \ge s} (1+\log t)^k t^{-\gamma} \le C(1+\log s)^k s^{-\gamma} \tag{A.20}$$

for the finitely many exponents γ appearing below, all bounded away from zero.

First, (A.2) gives

$$\frac{|A_4|}{L} \lesssim \frac{KQ}{vQ} \le \frac{s^{-a}}{a}. \tag{A.21}$$

Split the two summands in each of A_2, A_3, using the corresponding positive summand of L:

$$\frac{|A_2|}{L} \lesssim \frac{P}{s} + \frac{P}{S^2}, \tag{A.22}$$

$$\frac{|A_3|}{L} \lesssim \frac{Q}{Tv} + \frac{Q}{T^2}. \tag{A.23}$$

For example, $SP^2/(S^2L) \le P/(Sv) = P/s$, while $wP^2/(S^2L) \le P/S^2$. Applying (A.5) and (A.20) gives

$$\frac{|A_2| + |A_3|}{L} \lesssim (1 + \log s)s^{-1+3\eta_0}. \tag{A.24}$$

For A_1, if $t \le 2s$, then $K \asymp s^{-b}$, $Q \asymp P$, $T \asymp S$, and $L \asymp vP$, so

$$|A_1|/L \lesssim |\delta|P^3 s^{-2}. \tag{A.25}$$

If $t \ge 2s$, then

$$K = \frac{Q - P}{c(T - S)}, \qquad Q - P = c\int_s^t x^{-a}\,\mathrm{d}x \gtrsim v,$$

and $T - S \asymp T$. Consequently

$$\begin{aligned} \frac{|A_1|}{L} &\lesssim \frac{|\delta|P^2 Q}{S^2 v(Q - P)} \\ &\lesssim |\delta|\left(\frac{P^2}{S^2 v} + \frac{P^3}{S^2 v^2}\right) = |\delta|\left(P^2 s^{-2-\delta} + P^3 s^{-2}\right). \end{aligned} \tag{A.26}$$

Thus both cases are bounded by $C(1 + \log s)^3 s^{-2+3\eta_0}$.

The first primal numerator satisfies

$$\frac{|U_1|}{L} \lesssim \frac{P^2 Q}{STv} = \frac{P^2 Q}{sT} \lesssim (1 + \log s)^3 s^{-2+4\eta_0}, \tag{A.27}$$

because $S + T + 2cST \lesssim ST$ in this range. For U_2, if $t \le 2s$, (A.3) yields

$$\left|\frac{g(s) - g(t)}{S - T}\right| \lesssim s^{-a-1}, \qquad |U_2|/L \lesssim P^3 s^{-2}.$$

If $t \ge 2s$, use $|g(s) - g(t)| \lesssim s^{-1}$ to obtain

$$|U_2|/L \lesssim \frac{P^2 Q}{sTv} \lesssim (1 + \log s)^3 s^{-2+3\eta_0}.$$

Together with (A.21) and (A.24), these estimates imply (3.32); each right side is bounded by a constant times $(1 + \log s)^3 s^{-1+6\eta_0} = \Psi(1/s)$. The near-node bounds and the exact limits in Proposition 3.4 include $s = t$. □