THE MACDONALD (q,t)-PERMANENT II: PROBABILITY LAWS AND THEIR REGIONS OF VALIDITY

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ABSTRACT. The Macdonald (q,t)-permanent replaces the constant cycle weight of the Vere-Jones α-permanent by $(1-t^k)/(1-q^k)$ for a cycle of length k. We determine which of the probability laws built on the α-permanent survive this deformation, and where. The scalar gamma-type law survives on the whole parameter square, but the matrix law, a compound Wishart matrix, is more fragile. We prove that it exists in every dimension exactly when $t \in \{0, q, q^2, \dots\}$, and in dimension two exactly when $t \le q$. We conjecture that in each fixed dimension the matrix law exists exactly on the region that we show to be necessary, a q-analogue of Gindikin's theorem.

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1. INTRODUCTION

The determinant power $\det(I-M)^{-\alpha}$ underlies several probability laws. Its coefficients are the α-permanents of Vere-Jones, which govern permanental point processes and loop soups (Vere-Jones, 1988, 1997; Shirai and Takahashi, 2003; McCullagh and Møller, 2006; Le Jan, 2011). As a Laplace transform it gives a gamma law in one variable for every $\alpha > 0$, but in dimension d far fewer. By Gindikin's theorem, $\det(I+S)^{-\alpha}$ is the Laplace transform of a random positive semidefinite $d \times d$ matrix exactly when α lies in the Wallach set $\{0, 1, \dots, d-1\} \cup (d-1, \infty)$ (Scott and Sokal, 2014, Theorems 1.3(b) and 2.2). Only the integers survive in every dimension, and there the law is the Wishart law, a sum of α independent Gaussian rank-one matrices.

Replacing the constant cycle weight α by a weight that depends on the cycle length preserves the generating-function identities, as the companion paper shows (Sra, 2026a). It need not preserve the parameter domains on which those identities define probability laws, and determining those domains is the subject of this paper.

For $0 \le q, t < 1$ and $A = (a_{ij}) \in M_n(\mathbb{C})$, define

$$\operatorname{per}_{q,t}(A) = \sum\nolimits_{\sigma \in S_n} w_\sigma \prod\nolimits_{i=1}^{n} a_{i,\sigma(i)}, \qquad w_\sigma = \prod\nolimits_{C \in \mathrm{Cyc}(\sigma)} w_{|C|}, \qquad w_k := \frac{1-t^k}{1-q^k}. \tag{1.1}$$

Here $\mathrm{Cyc}(\sigma)$ includes the fixed points. At $q = t$ every cycle weight is one, and as $q \uparrow 1$ with $t = q^\alpha$ it tends to α, so (1.1) contains the permanent and deforms the Vere-Jones α-permanent. The determinant power becomes a product of determinant ratios,

$$\mathcal{Z}_{q,t}(M) = \exp\left(\sum\nolimits_{k \ge 1} \frac{w_k}{k} \operatorname{tr} M^k\right) = \prod\nolimits_{r \ge 0} \frac{\det(I - tq^r M)}{\det(I - q^r M)},$$

which also makes sense at $M = -S$ for every $S \succeq 0$.

In one dimension the deformation costs nothing: $\mathcal{Z}_{q,t}(-s)$ is a Laplace transform on the whole parameter square, and a Bernoulli–exponential series constructs the law (Section 3). In dimension d we ask for a compound Wishart matrix, that is, a positive semidefinite random matrix W with $\mathbb{E}e^{-\operatorname{tr} SW} = \mathcal{Z}_{q,t}(-S)$ for every $S \succeq 0$. Its diagonal moments would be the values of $\operatorname{per}_{q,t}$ on Gram

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matrices, so such a matrix would explain at once the positivity studied in Part I. Positivity alone falls short, since it tests derivatives only at the origin, whereas a matrix Laplace transform must be completely monotone on the whole cone. Nor does the classical theory carry over. There the Wallach set comes from the symmetry of the cone, while $\mathcal{Z}_{q,t}(-S)$ is invariant only under unitary conjugation. Even so, the answer has the shape of Gindikin's: the region shrinks as the dimension grows, and a discrete set survives in every dimension. We collect our answers for the matrix law in the following informal theorem.

Theorem (informal; Theorems 4.5 and 4.7, Corollary 4.6)**.** *Fix $0 < q < 1$ and $0 \le t < 1$. The function $\mathcal{Z}_{q,t}(-S)$ is the Laplace transform of a positive semidefinite random matrix in every dimension if and only if*

$$t \in \mathcal{G}(q) := \{0, q, q^2, q^3, \ldots\}.$$

In dimension two it is one if and only if $0 \le t \le q$. In dimension d it is necessary that

$$t \in \mathcal{R}_d(q) := \left([0, q^{d-1}] \cup \{q, q^2, \ldots, q^{d-2}\}\right) \cap [0, 1),$$

where the finite set is empty for $d \le 2$.

Along $t = q^\alpha$ the theorem reads as Gindikin's, since $\mathcal{G}(q)$ is the image of the integers, with $t = 0$ for $\alpha = \infty$, and $\mathcal{R}_d(q)$ is the image of the positive part of the Wallach set. On $\mathcal{G}(q)$ the Wishart construction survives. At $t = q^m$ the cycle weights are the power sums of $(1, q, \ldots, q^{m-1})$, and a sum of m independent Gaussian rank-one matrices with these weights has exactly the required moments (Section 4.1). Off $\mathcal{G}(q)$ a law might still exist without such a construction. What rules it out is the moment of $(\det W)^2$ that the law would have to possess. This moment factors exactly, and it is negative in every sufficiently large dimension (Section 4.3).

In a fixed dimension more survives. Since the transform depends only on the eigenvalues of S, the Harish-Chandra–Itzykson–Zuber integral turns existence into a positivity condition on derivatives of the scalar density. In dimension two that condition is log-concavity, which the density has throughout $0 < t < q$, so the necessary region $\mathcal{R}_2(q) = [0, q]$ is exact. Whether $\mathcal{R}_d(q)$ is also sufficient for $d \ge 3$ remains open; we state it as a q-analogue of Gindikin's theorem (Conjecture 6.1).

Discrete laws need only nonnegative coefficients, not a mixing measure, so they survive on larger regions. The closed-walk expansion of $\mathcal{Z}_{q,t}$ gives infinitely divisible occupation laws for entrywise nonnegative kernels throughout the square (Theorem 5.1). The coefficient law of a rank-d positive semidefinite contraction exists on $\mathcal{R}_d(q)$, and whether it is infinitely divisible does not depend on (q, t) (Theorem 5.4). Read as correlation functions, $\mathrm{per}_{q,t}$ gives Cox processes on $\mathcal{G}(q)$, and on the whole square for rank-one kernels (Section 5.4); read as probabilities, the Schur coefficients give a determinantal measure on partitions with at most d rows on $\mathcal{R}_d(q)$ (Theorem 5.8). Coefficient positivity alone does not give a matrix mixing law (Appendix B.2).

The Macdonald permanent is introduced, and its algebra and positivity are developed, in Part I (Sra, 2026a). The facts needed here are proved locally, so this paper can be read on its own. Supplement II (Sra, 2026b) collects further count laws, matrix moments, and partition models.

1.1. **Relation to earlier work.** The scalar law used here is the remainder in the self-decomposition of a Poissonian exponential functional studied by Bertoin et al. (2004, Proposition 3.1); what is new is the Bernoulli–exponential series, which identifies its moments and Lévy measure with the Macdonald cycle weights. Shirai's Wishart representation expresses repeated α-determinants as diagonal moments and gives their Poisson mixtures (Shirai, 2007, Theorem 3.8 and Corollary 3.11), and geometric column variances give the corresponding construction here, with entry formulas that specialize the Wishart permutation calculus of Graczyk et al. (2003) and Bryc (2008a).

The Cox construction uses the permanental processes of Shirai and Takahashi (2003) and McCullagh and Møller (2006). The loop expansion is a length-weighted version of the classical permanental construction (Le Jan, 2011; Li and Zheng, 2023). The coefficient-law criterion is related

to the multivariate negative-binomial theory of Griffiths and Milne (1987). The random-partition application uses the Schur correlation identities of Okounkov (2001), with positivity supplied only through the finite number of allowed rows.

2. The generating function

Every law in this paper is read off $\mathcal{Z}_{q,t}$, so this section collects what the later sections need about it: its Taylor coefficients at matrices with repeated indices, which give the discrete laws; its invariant polynomial moments, which pose the matrix Laplace problem; and the signs of its Schur coefficients, which decide where both are positive. The cycle expansion and the Schur specialization give all three, with proofs short enough to include here.

We write $[n]=\{1,\dots,n\}$, $\mathbb{N}=\{0,1,\dots\}$, and $M_{d,p}(\mathbb{C})=\mathbb{C}^{d\times p}$; when $d=p$ we write $M_d(\mathbb{C})$. For $A\in M_n(\mathbb{C})$ and $S\subseteq[n]$, the matrix $A[S]$ is the principal submatrix indexed by S. Empty determinants and permanents, including $\mathrm{per}_{q,t}(\varnothing)$, are 1. The notation $A\succeq 0$ means that A is Hermitian positive semidefinite. We use the operator norm $\|\cdot\|_{\mathrm{op}}$, the spectral radius $\rho(\cdot)$, and inner products that are conjugate-linear in the first argument.

Repeated indices are how the coefficients of $\mathcal{Z}_{q,t}$ become values of $\mathrm{per}_{q,t}$. For $A\in M_N(\mathbb{C})$ and $\mathbf{m}\in\mathbb{N}^N$, let $A[\mathbf{m}]$ have rows and columns indexed by pairs (i,a) with $1\le a\le m_i$, and entry a_{ij} at $((i,a),(j,b))$. Write $\mathbf{m}!=\prod_i m_i!$ and $x^{\mathbf{m}}=\prod_i x_i^{m_i}$.

Proposition 2.1 (The repeated-index master identity)**.** *For $A\in M_N(\mathbb{C})$ and $X=\mathrm{diag}(x_1,\dots,x_N)$,*

$$\begin{aligned}\mathcal{Z}_{q,t}(M)&=\exp\left(\sum\nolimits_{k\ge1}\frac{w_k}{k}\operatorname{tr}M^k\right)=\prod\nolimits_{r\ge0}\frac{\det(I-tq^rM)}{\det(I-q^rM)},\\ \mathcal{Z}_{q,t}(XA)&=\sum\nolimits_{\mathbf{m}\in\mathbb{N}^N}\frac{\mathrm{per}_{q,t}(A[\mathbf{m}])}{\mathbf{m}!}x^{\mathbf{m}}.\end{aligned}\tag{2.1}$$

The coefficient identity is formal at zero. For $0\le q,t<1$ and $\rho(M)<1$, the trace series converges absolutely and equals the determinant product.

Proof. The monomials of $\mathrm{tr}(XA)^k$ are rooted closed words of length k. The factor $1/k$ removes the choice of a cyclic starting position, and exponentiating collects disjoint cycles into permutations. With m_i distinguishable copies of label i, the resulting weight is $\mathrm{per}_{q,t}(A[\mathbf{m}])$; forgetting the labels within each type divides it by $\mathbf{m}!$, which proves the formal identity for arbitrary cycle weights.

For $\rho(M)<1$, the bound $|\operatorname{tr}M^k|\le N\rho(M)^k$ gives absolute convergence. Expand $w_k=\sum_{r\ge0}(q^{rk}-t^kq^{rk})$ and use $-\log\det(I-aM)=\sum_{k\ge1}a^k\operatorname{tr}M^k/k$. Absolute convergence permits interchanging the sums and proves the product formula. At $q=0$ only the term $r=0$ is retained. □

The loop soups and Cox processes below come in families indexed by an intensity $\beta>0$, and raising $\mathcal{Z}_{q,t}$ to the power β multiplies every cycle weight by β. We therefore write

$$\mathrm{per}^{(\beta)}_{q,t}(A):=\sum\nolimits_{\sigma\in S_n}\beta^{c(\sigma)}w_\sigma\prod\nolimits_i a_{i,\sigma(i)},$$

$$\sum\nolimits_{\mathbf{m}}\frac{\mathrm{per}^{(\beta)}_{q,t}(A[\mathbf{m}])}{\mathbf{m}!}x^{\mathbf{m}}=\mathcal{Z}_{q,t}(XA)^\beta.$$

where $c(\sigma)$ counts cycles and the power is defined near zero by the logarithm that vanishes there. At $q=t$ this is the classical α-permanent with $\alpha=\beta$, and our convention for it is $\mathrm{per}_\alpha(A)=\sum_\sigma\alpha^{c(\sigma)}\prod_i a_{i,\sigma(i)}$, so that $\mathrm{per}_1=\mathrm{per}$ and $\mathrm{per}_{-1}=(-1)^n\det$. The convention of Shirai and Takahashi is related by $\det_\eta(A)=\eta^n\,\mathrm{per}_{1/\eta}(A)$ for $\eta\neq0$.

The one-variable case of (2.1) gives the constants H_n that recur below: they are the moments of the scalar law of Section 3, the weights of its negative-binomial companion, and, through the bound (2.4), the size of the Cox correlations of Section 5.4.

Proposition 2.2 (Scalar coefficients and normalizations)**.** *Put*

$$H_n(q,t) := \frac{(t;q)_n}{(q;q)_n}, \qquad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \qquad (a;q)_\infty := \prod_{j\geq 0}(1-aq^j). \tag{2.2}$$

Then

$$H(z) := \sum_{n\geq 0} H_n z^n = \exp\left(\sum_{k\geq 1} \frac{w_k z^k}{k}\right) = \frac{(tz;q)_\infty}{(z;q)_\infty}, \qquad |z|<1. \tag{2.3}$$

Moreover,

$$\operatorname{per}_{q,t}(I_n) = w_1^n, \qquad \operatorname{per}_{q,t}(uv^{\mathsf{T}}) = n!H_n(q,t)\prod_i u_i v_i,$$
$$\operatorname{per}_{q,t}(A\oplus B) = \operatorname{per}_{q,t}(A)\operatorname{per}_{q,t}(B), \qquad \operatorname{per}_{q,t}(DAE) = \Big(\prod_i D_{ii}E_{ii}\Big)\operatorname{per}_{q,t}(A)$$

for diagonal D, E, and

$$|\operatorname{per}_{q,t}(A)| \leq n!H_n(q,t)\prod_{i=1}^n a_{ii} \qquad (A\succeq 0,\ 0\leq q,t<1). \tag{2.4}$$

Proof. Removing the first factor from each scalar product in (2.1) gives $(1-z)H(z) = (1-tz)H(qz)$. Its constant coefficient is one, and its coefficients satisfy $(1-q^n)[z^n]H = (1-tq^{n-1})[z^{n-1}]H$ for $n\geq 1$. This recurrence gives (2.2) and (2.3). The exponential formula gives $\sum_{\sigma\in S_n} w_\sigma = n!H_n$. For a rank-one matrix every permutation monomial equals $\prod_i u_i v_i$, which proves its evaluation. The identity, block-diagonal, and diagonal-scaling formulas follow directly from (1.1). $|a_{ij}|\leq\sqrt{a_{ii}a_{jj}}$ for $A\succeq 0$ bounds every monomial by $\prod_i a_{ii}$; summing the positive weights gives (2.4). □

The determinant product also shows how the parameter t composes, and the scalar law of Section 3 splits accordingly into independent pieces. For $0\leq q,u,v<1$,

$$\begin{aligned}\operatorname{per}_{q,uv}(A) &= \sum_{S\subseteq[n]} u^{|S|}\operatorname{per}_{q,v}(A[S])\operatorname{per}_{q,u}(A[S^c]),\\ \mathcal{Z}_{q,uv}(M) &= \mathcal{Z}_{q,u}(M)\mathcal{Z}_{q,v}(uM).\end{aligned} \tag{2.5}$$

Indeed, the determinant products cancel in the second identity; the square-free coefficient identity in the first then follows from (2.1).

2.1. **Signs of the Schur coefficients.** The signs of the Schur coefficients of $\operatorname{per}_{q,t}$ decide which of the laws below exist, so we derive them together with their sign pattern.

A partition $\lambda = (\lambda_1,\dots,\lambda_\ell)$ is a weakly decreasing sequence of positive integers. Its size is $|\lambda| = \sum_i \lambda_i$, its length is $\ell(\lambda)=\ell$, and $\lambda\vdash n$ means $|\lambda|=n$. We identify λ with its diagram $\{(i,j): 1\leq i\leq \ell(\lambda),\ 1\leq j\leq \lambda_i\}$ and write $\lambda'_j = \#\{i:\lambda_i\geq j\}$ for the conjugate partition. The hook length of its cell (i,j) is

$$h_{ij} = \lambda_i - j + \lambda'_j - i + 1.$$

If $\mu = 1^{m_1}2^{m_2}\cdots$ records the multiplicities of its parts, then

$$z_\mu = \prod_{k\geq 1} k^{m_k} m_k!, \qquad w_\mu = \prod_i w_{\mu_i}.$$

There are $n!/z_\mu$ permutations of cycle type $\mu\vdash n$.

For an alphabet $Z=(z_1,z_2,\dots)$, let $p_k(Z) = \sum_i z_i^k$ and $p_\mu = \prod_i p_{\mu_i}$. We work degree by degree in the symmetric-function algebra $\Lambda_{\mathbb{Q}(q,t)} = \mathbb{Q}(q,t)[p_1,p_2,\dots]$, where $\deg p_k = k$. A specialization is an algebra homomorphism from this algebra, specified by its values on the p_k; it need not be evaluation at a sequence of numbers. The complete symmetric functions h_n are defined by

$$\sum_{n\geq 0} h_n(Z)u^n = \exp\left(\sum_{k\geq 1}\frac{p_k(Z)u^k}{k}\right).$$

The Schur function $s_\lambda(Z)$ is the sum of the monomials $\prod_{(i,j)\in\lambda} z_{T(i,j)}$ over semistandard tableaux T of shape λ. Thus $s_\lambda(Z)\geq 0$ on a nonnegative alphabet, and $s_\lambda(z_1,\dots,z_d)=0$ if $\ell(\lambda)>d$. These conventions follow (Macdonald, 1995, Chapter I, Sections 2–3).

Let χ^λ denote the irreducible character of S_n indexed by $\lambda \vdash n$, and put $f^\lambda = \chi^\lambda(e)$. We use the unnormalized irreducible immanant

$$\mathrm{Imm}_\lambda(A) = \sum_{\sigma\in S_n} \chi^\lambda(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}.$$

Thus $\mathrm{Imm}_{(n)} = \mathrm{per}$ and $\mathrm{Imm}_{(1^n)} = \det$. The Frobenius identity

$$s_\lambda = \sum_{\mu\vdash n} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu$$

relates this normalization to symmetric functions (Macdonald, 1995, Chapter I, Section 7).

The specialization $B_{q,t}$ is defined by $p_k[B_{q,t}] = w_k$. It need not come from a positive alphabet, so its Schur values can be negative; the next proposition finds exactly where they are not.

Proposition 2.3 (Hook-content coefficients and their signs)**.** *For $0 < q < 1$ and real t,*

$$\begin{aligned} \mathrm{per}_{q,t}(A) &= \sum_{\lambda\vdash n} c_\lambda(q,t)\, \mathrm{Imm}_\lambda(A), \\ c_\lambda(q,t) &= \prod_{(i,j)\in\lambda} \frac{q^{i-1} - tq^{j-1}}{1-q^{h_{ij}}} = s_\lambda[B_{q,t}]. \end{aligned} \tag{2.6}$$

For $d \geq 1$, define

$$\mathcal{R}_d(q) := \left([0, q^{d-1}] \cup \{q, q^2, \ldots, q^{d-2}\}\right) \cap [0,1), \tag{2.7}$$

where the finite set is empty for $d \leq 2$. For $0 \leq t < 1$, the coefficients $c_\lambda(q,t)$ are nonnegative for every partition with at most d rows if and only if $t \in \mathcal{R}_d(q)$. They are nonnegative for every partition if and only if $t \in \mathcal{G}(q) = \{0, q, q^2, \ldots\}$.

Proof. By the Frobenius identity and character orthogonality, the coefficient of χ^λ in the class function $\sigma \mapsto w_\sigma$ is $\sum_{\mu\vdash n} \chi^\lambda(\mu) w_\mu / z_\mu = s_\lambda[B_{q,t}]$, which proves the immanant expansion. At $t = q^m$, the power sums are those of $(1, q, \ldots, q^{m-1})$, whose Schur principal specialization is

$$s_\lambda(1, q, \ldots, q^{m-1}) = \prod_{(i,j)\in\lambda} \frac{q^{i-1} - q^m q^{j-1}}{1-q^{h_{ij}}};$$

see Macdonald (1995, Chapter I, Section 3). For fixed q, both sides of the proposed hook formula are polynomials in t of degree at most n. Agreement at every $t = q^m$ proves the formula identically.

If $t \leq q^{d-1}$ and $i \leq d$, then $tq^{j-1} \leq q^{d+j-2} \leq q^{i-1}$, so every numerator is nonnegative. At $t = q^m$, positivity follows from the positive alphabet. Conversely, if $t \notin \mathcal{R}_d(q)$, then $q^r < t < q^{r-1}$ for some $1 \leq r \leq d-1$; the column (1^{r+1}) has

$$c_{(1^{r+1})}(q,t) = \frac{\prod_{i=0}^{r}(q^i - t)}{\prod_{k=1}^{r+1}(1-q^k)} < 0,$$

because only its last numerator factor is negative. Intersecting over d leaves precisely $\mathcal{G}(q)$. □

The coefficient sign test also gives the Gram-matrix positivity needed for discrete coefficient laws.

Lemma 2.4 (The central operator and Gram matrices)**.** *On $(\mathbb{C}^d)^{\otimes n}$ let $U_\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$, and put $T = \sum_{\sigma\in S_n} w_\sigma U_\sigma$. On its λ-isotypic component, T acts as the scalar*

$$\frac{1}{f^\lambda} \sum_{\sigma\in S_n} w_\sigma \chi^\lambda(\sigma) = \frac{n!}{f^\lambda} c_\lambda(q,t). \tag{2.8}$$

If $0 < q < 1$, $t \in \mathcal{R}_d(q)$, and $A \succeq 0$ has rank at most d, then $\mathrm{per}_{q,t}(A) \geq 0$, in every size.

Proof. The operator is central because its coefficients depend only on cycle type, and self-adjoint because $w_{\sigma^{-1}} = w_\sigma$. Schur's lemma gives the scalar in (2.8) by taking the trace on the irreducible S_n-module of dimension f^λ. A conjugacy class of type μ has $n!/z_\mu$ elements, so the Frobenius formula gives the stated value. Schur–Weyl decomposition contains only partitions with $\ell(\lambda) \leq d$.

Proposition 2.3 therefore makes $T \succeq 0$ in the indicated region. If $A = [\langle u_i, u_j \rangle]$ with $u_i \in \mathbb{C}^d$ and $u = u_1 \otimes \cdots \otimes u_n$, then $\langle u, Tu \rangle = \mathrm{per}_{q,t}(A)$, after replacing σ by σ^{-1} in the sum, so $\mathrm{per}_{q,t}(A) \geq 0$. □

3. Laws in one dimension

We first show that the scalar law survives the deformation on the whole parameter square. It is the paper's building block: its moments are the rank-one values of $\mathrm{per}_{q,t}$, its density decides the matrix problem in dimension two (Section 4.4), and it gives Cox intensities throughout the square (Section 5.4). The transform $\mathcal{Z}_{q,t}(-s) = \prod_{r\geq 0}(1+tq^r s)/(1+q^r s)$ deforms the gamma transform $(1+s)^{-\alpha}$, and a Bernoulli–exponential series realizes it with explicit moments and Lévy measure.

Proposition 3.1 (A scalar infinitely divisible law)**.** *Let $0 \leq q, t < 1$. Take independent random variables $B_r \sim \mathrm{Bernoulli}(1-t)$ and $E_r \sim \mathrm{Exp}(1)$, for $r \geq 0$, and set*

$$S_{q,t} := \sum\nolimits_{r\geq 0} q^r B_r E_r.$$

When $q = 0$, retain only the $r = 0$ term. The series converges almost surely and in L^1, and

$$\mathbb{E}e^{-sS_{q,t}} = \frac{(-ts;q)_\infty}{(-s;q)_\infty}, \qquad s \geq 0, \tag{3.1}$$

$$\mathbb{E}S_{q,t}^n = n! H_n(q,t), \qquad n \geq 0, \tag{3.2}$$

$$\mathrm{cum}_k(S_{q,t}) = (k-1)! w_k, \qquad k \geq 1. \tag{3.3}$$

The law is infinitely divisible. For $0 < q < 1$, it has zero drift and Lévy measure

$$\nu_{q,t}(dx) = \frac{1}{x} \sum\nolimits_{r\geq 0} \left(e^{-x/q^r} - e^{-x/(tq^r)} \right) dx, \qquad x > 0,$$

where the second exponential is zero when $t = 0$; for $q = 0$ only the $r = 0$ term occurs.

Proof. The partial sums increase, with expectations converging to $(1-t)/(1-q)$, so by monotone convergence the series converges almost surely and in L^1. For $s \geq 0$, a single summand has transform

$$\mathbb{E}e^{-sq^r B_r E_r} = t + \frac{1-t}{1+sq^r} = \frac{1+tsq^r}{1+sq^r},$$

and independence with bounded convergence proves (3.1).

For $0 \leq u < 1$, monotone convergence instead gives

$$\mathbb{E}e^{uS_{q,t}} = \prod\nolimits_{r\geq 0} \frac{1-tuq^r}{1-uq^r} = H(u) < \infty.$$

Both sides are analytic for $|u| < 1$: the exponential moment dominates every derivative on compact subsets. Differentiation at zero gives (3.2), and differentiation of $\log H(u) = \sum_{k\geq 1} w_k u^k/k$ gives (3.3).

For $a > 0$ and $0 \leq t < 1$, Frullani's integral gives

$$\int_0^\infty (1 - e^{-sx}) \frac{e^{-x/a} - e^{-x/(ta)}}{x}\, dx = \log(1+as) - \log(1+tas),$$

with the second exponential absent at $t = 0$. By Tonelli, $\int_0^\infty x\, \nu_{q,t}(dx) = \sum_{r\geq 0}(q^r - tq^r) = (1-t)/(1-q)$, so $\int (1 \wedge x)\nu_{q,t}(dx) < \infty$, and summing the identities over $a = q^r$ yields

$$-\log \mathbb{E}e^{-sS_{q,t}} = \int_0^\infty (1 - e^{-sx})\, \nu_{q,t}(dx).$$

Infinite divisibility follows: a Poisson random measure on $(0,\infty)$ with intensity $\nu_{q,t}$ has total mass with this Laplace transform and finite mean, and intensity $\nu_{q,t}/h$ gives an hth convolution root. □

For $0<q<1$, Bertoin et al. (2004, Proposition 3.1(i)) obtains the law of $S_{q,t}$ as the remainder $I_t^{(q)}$ in the self-decomposition of a Poisson exponential functional: for a rate-one Poisson process N_u,

$$I^{(q)}=\int_0^\infty q^{N_u}\,du \overset{d}{=} \sum\nolimits_{r\ge 0} q^r E_r, \qquad I^{(q)} \overset{d}{=} t\, I^{(q)\prime} + I_t^{(q)}$$

with independent terms on the right, and the ratio of the two transforms is (3.1).

The parameter multiplication identity (2.5) becomes an additive decomposition of these scalar variables, and at $t=q^m$ cancellation leaves finitely many exponentials. The decomposition is what reduces the dimension-two theorem to $q^2\le t<q$ (Theorem 4.7).

Corollary 3.2 (Convolution, boundaries, and the gamma limit)**.** *For $0\le q,u,v<1$,*

$$S_{q,uv} \overset{d}{=} S_{q,u} + uS'_{q,v}, \tag{3.4}$$

where the variables on the right are independent and the prime denotes an independent copy. For every positive integer m,

$$S_{q,q^m} \overset{d}{=} \sum\nolimits_{r=0}^{m-1} q^r E_r.$$

At $q=0$, the distribution is $t\delta_0+(1-t)\,\mathrm{Exp}(1)$. For every fixed $\alpha>0$, as $q\uparrow 1$ with $t=q^\alpha$, $S_{q,t}$ converges in distribution and in moments to the gamma law of shape α and rate 1.

Proof. The product of the Laplace transforms on the right side of (3.4) is

$$\prod\nolimits_{r\ge 0}\frac{1+usq^r}{1+sq^r}\frac{1+uvsq^r}{1+usq^r}=\prod\nolimits_{r\ge 0}\frac{1+uvsq^r}{1+sq^r}.$$

Laplace-transform uniqueness proves the decomposition. For $t=q^m$, the same product cancels to $\prod_{r=0}^{m-1}(1+sq^r)^{-1}$. The $q=0$ statement follows directly from the defining series.

Since $0<(1-x^\alpha)/(1-x)\le\max(1,\alpha)$ for $0<x<1$ (by $x^\alpha\ge x$ if $\alpha\le 1$, and the mean value theorem otherwise), the weights $w_k(q,q^\alpha)$ tend to α and are bounded uniformly in q and k. Hence

$$\log \mathbb{E}e^{uS_{q,q^\alpha}}=\sum\nolimits_{k\ge 1}\frac{w_k(q,q^\alpha)u^k}{k}\longrightarrow -\alpha\log(1-u)$$

uniformly on compact subsets of $|u|<1$; the limit is the gamma moment generating function, which gives weak convergence and the moments $\alpha(\alpha+1)\cdots(\alpha+n-1)$. □

The intensity β of Section 2 has a scalar meaning too, which the rank-one Cox processes of Section 5.4 use. For $\beta>0$, the same Poisson construction with intensity $\beta\nu_{q,t}$ defines a nonnegative variable $S_{q,t}^{(\beta)}$ with $\mathbb{E}e^{uS_{q,t}^{(\beta)}}=H(u)^\beta$ for $u<1$. The exponential formula gives

$$\mathbb{E}(S_{q,t}^{(\beta)})^n=\mathrm{per}_{q,t}^{(\beta)}(J_n),$$

where J_n is the all-ones matrix. Independent addition of the Poisson random measures adds β, so the cycle fugacity is the time parameter of a scalar subordinator.

3.1. **The (q,t)-negative-binomial law.** The count laws of Section 5.1 have a one-dimensional prototype, and it already shows their two constructions. The coefficients of H, suitably tilted, are probabilities on $\mathbb{N}$ for every (q,t): for $0\le z<1$, define $N_{q,t}(z)$ by

$$\mathbb{P}(N_{q,t}(z)=n)=\frac{H_n z^n}{H(z)}, \qquad n\ge 0. \tag{3.5}$$

At $z=0$ this means the point mass at zero. We call (3.5) the (q,t)-*negative-binomial law*. At $t=0$ it is the Euler distribution, with mass $(z;q)_\infty z^n/(q;q)_n$. At $t=q^m$ it is the negative q-binomial distribution of the second kind, with mass $(z;q)_m(q^m;q)_n z^n/(q;q)_n$; these conventions agree with Charalambides (2018, Section 1 and Example 2.5).

Its generating function is $H(zs)/H(z)$, with

$$\log \frac{H(zs)}{H(z)} = \sum\nolimits_{k\geq 1} \frac{w_k z^k}{k}(s^k - 1).$$

The logarithm gives the first construction: $N_{q,t}(z)$ is the compound Poisson sum $\sum_{k\geq 1} kC_k$ with independent $C_k \sim \text{Poisson}(w_k z^k/k)$, the one-point loop soup of Theorem 5.1. The scalar law gives the second: if the law of $S_{q,t}$ is tilted by $e^{zs}/H(z)$, a conditionally Poisson variable of mean zs has the same generating function, the one-point Gaussian tilt of Theorem B.1. Further representations, factorial cumulants, and the conditioned permutation law are in Supplement II (Sra, 2026b).

4. The matrix law

This section proves the theorem of the introduction on the matrix law, the compound Wishart matrix that is the matrix analogue of the scalar law of Section 3. Gaussian columns build this law wherever they can produce the cycle weights, which happens exactly on $\mathcal{G}(q)$. Elsewhere no such construction exists, and to rule a law out we use a moment that every representing law must have.

4.1. **The Wishart construction.** We first build the law wherever Gaussian columns can produce the cycle weights. For a Wishart matrix with m independent Gaussian columns, the Wick formula assigns one column to each cycle of a pairing, so each cycle contributes the number of columns and the diagonal moments are per_m of the covariance submatrix (Shirai, 2007, Corollary 3.11). Weighting the columns geometrically replaces that count by the power sums of $(1, q, \ldots, q^{m-1})$, which are the cycle weights at $t = q^m$, and at $t = 0$ in the limit $m = \infty$. We call these weighted columns the *colors* of the matrix. Proposition 4.2 shows that colors exist only on $\mathcal{G}(q)$, which is why Section 4.3 has to look for laws without them.

We use a standard circular complex Gaussian scalar with density $\pi^{-1}e^{-|z|^2}$ on $\mathbb{C}$, so its squared modulus has law Exp(1). A centered circular complex Gaussian vector with covariance $\Sigma \succeq 0$ means a vector $\Sigma^{1/2}g$, where g has independent such coordinates. In particular, $\mathbb{E}zz^* = \Sigma$ and $\mathbb{E}zz^{\mathsf{T}} = 0$.

Let $z^{(r)}$, $r \geq 0$, be independent vectors with this law in $\mathbb{C}^d$. For a positive integer m, set

$$W_{q,m}(\Sigma) := \sum\nolimits_{r=0}^{m-1} q^r z^{(r)} z^{(r)*}. \tag{4.1}$$

For $m = \infty$ and $0 \leq q < 1$ the partial sums converge almost surely in trace norm, their traces having expectation $(1-q)^{-1}\operatorname{tr}\Sigma$. For finite m, $W_{q,m}(\Sigma) = \Sigma^{1/2} G D_m G^* \Sigma^{1/2}$ with $D_m = \text{diag}(1, q, \ldots, q^{m-1})$, a classical compound-Wishart matrix; here q sets a column covariance and does not weight pairing crossings as in the q-Wishart model of Bryc (2008b, Definitions 1.2–1.3). The Wick formula computes every entry moment, and on the diagonal they are values of $\text{per}_{q,t}$.

Theorem 4.1 (Entry moments and Laplace transform)**.** *Let $0 \leq q < 1$, let $m \in \{1, 2, \ldots\} \cup \{\infty\}$, and let Σ be a positive semidefinite $d \times d$ complex matrix. Write $W = W_{q,m}(\Sigma)$, with $t = q^m$ for finite m and $t = 0$ for $m = \infty$. For every $n \geq 1$ and all index lists $i_1, \ldots, i_n, j_1, \ldots, j_n$ in $[d]$,*

$$\mathbb{E}\prod\nolimits_{a=1}^{n} W_{i_a j_a} = \sum\nolimits_{\sigma \in S_n} w_\sigma \prod\nolimits_{a=1}^{n} \Sigma_{i_a, j_{\sigma(a)}}. \tag{4.2}$$

For every Hermitian $S \succeq 0$,

$$\mathbb{E}e^{-\operatorname{tr} SW} = \prod\nolimits_{r=0}^{m-1} \det(I + q^r \Sigma^{1/2} S \Sigma^{1/2})^{-1}, \tag{4.3}$$

with an infinite product for $m = \infty$. In particular,

$$\mathbb{E}\prod\nolimits_{a=1}^{n} W_{i_a i_a} = \text{per}_{q,t}([\Sigma_{i_a i_b}]_{a,b=1}^{n}). \tag{4.4}$$

Proof. For finite m, expand the entries in (4.1):

$$\mathbb{E}\prod\nolimits_{a=1}^{n} W_{i_a j_a} = \sum\nolimits_{r_1, \ldots, r_n = 0}^{m-1} q^{r_1 + \cdots + r_n} \mathbb{E}\prod\nolimits_{a=1}^{n} z_{i_a}^{(r_a)} \overline{z_{j_a}^{(r_a)}}.$$

The complex Wick formula gives

$$\mathbb{E}\prod_{a=1}^{n} z_{i_a}^{(r_a)}\overline{z_{j_a}^{(r_a)}} = \sum_{\sigma\in S_n}\prod_{a=1}^{n}\Sigma_{i_a,j_{\sigma(a)}}\mathbf{1}_{\{r_a=r_{\sigma(a)}\}}.$$

A surviving assignment of column indices is constant on each cycle of σ. Summing that common index over a cycle of length k gives $\sum_{r=0}^{m-1} q^{rk} = w_k(q,q^m)$. Different cycles have independent summation indices, which proves (4.2).

For a single column, diagonalize the positive semidefinite matrix $A = \Sigma^{1/2}S\Sigma^{1/2}$. Unitary invariance of the standard complex Gaussian vector gives

$$\mathbb{E}e^{-z^*Sz} = \prod_{\ell=1}^{d}\mathbb{E}e^{-\lambda_\ell(A)|g_\ell|^2} = \prod_{\ell=1}^{d}(1+\lambda_\ell(A))^{-1} = \det(I+A)^{-1}.$$

Independence proves (4.3) for finite m. For infinite m, bounded convergence passes to the limit on the left. The product on the right converges because $0 \le \log\det(I+q^rA) \le q^r \operatorname{tr} A$.

To pass entry moments to the infinite sum, choose $a>0$ with $a\|\Sigma\| < 1$ if $\Sigma \neq 0$. For every finite partial sum W_M, the same Gaussian integration with the opposite sign gives

$$\mathbb{E}e^{a\operatorname{tr} W_M} = \prod_{r=0}^{M-1}\det(I-aq^r\Sigma)^{-1}.$$

The logarithms are bounded uniformly in M by $a\operatorname{tr}\Sigma/((1-a\|\Sigma\|)(1-q))$. Monotone convergence proves $\mathbb{E}e^{a\operatorname{tr} W} < \infty$. For $A \succeq 0$, $|A_{ij}| \le \sqrt{A_{ii}A_{jj}} \le \operatorname{tr} A$. Thus products of entries of W_M are dominated by $(\operatorname{tr} W)^n$, which is integrable. Dominated convergence proves (4.2) for $m=\infty$; its cycle factors tend to $\sum_{r\ge0} q^{rk} = (1-q^k)^{-1}$. Taking $j_a = i_a$ gives (4.4). □

The permutation expansion is the classical Wishart calculus (compare Bryc (2008a, Theorem 2.1) and Graczyk et al. (2003)), with the geometric column covariance supplying the power sums w_k; cumulants and invariant moments are in Supplement II (Sra, 2026b). With $\Sigma = I_d$, (4.3) is $\mathcal{Z}_{q,t}(-S)$, so the theorem already answers the matrix question on $\mathcal{G}(q)$.

4.2. **Where Gaussian colors exist.** Can the Gaussian construction be pushed off $\mathcal{G}(q)$? It requires a single operator for all cycle lengths: a fixed positive color covariance D must realize every cycle weight at once, $w_k = \operatorname{tr} D^k$ for all k.

Proposition 4.2. *Fix $0<q<1$ and $0\le t<1$. A positive trace-class operator D satisfying $\operatorname{tr} D^k = w_k(q,t)$ for every $k\ge1$ exists if and only if $t=0$ or $t=q^m$ for a positive integer m.*

Proof. For $t=q^m$, take $D = \operatorname{diag}(1,q,\dots,q^{m-1})$. For $t=0$, take the trace-class operator $D=\operatorname{diag}(1,q,q^2,\dots)$ on $\ell^2(\mathbb{N})$. Their power sums have the required values.

Conversely, let $d_1,d_2,\dots\ge0$ be the eigenvalues of a positive trace-class operator with the asserted traces. The Schur polynomial $s_\lambda(d_1,\dots,d_M)$ is nonnegative for every M, by its semistandard-tableau expansion. As $M\to\infty$, each power sum converges to w_k. Since a fixed Schur function is a polynomial in finitely many power sums, these nonnegative values converge to $c_\lambda(q,t)$. Thus $c_\lambda(q,t)\ge0$ for every partition. Proposition 2.3 then requires $t\in\mathcal{G}(q)$. □

4.3. **Complete monotonicity in every dimension.** Off $\mathcal{G}(q)$ there are no Gaussian colors, and existence means that $\mathcal{Z}_{q,t}(-S)$ is completely monotone on the positive cone; by the Bernstein–Hausdorff–Widder–Choquet theorem on cones (Scott and Sokal, 2014, Theorem 2.2), this is equivalent to a positive semidefinite matrix Laplace representation. To exclude a law we need a property that every representing law would have and that can be computed without it. The moment of $(\det W)^2$ is such a property: its sign classifies existence in every dimension at once, and in dimension two the eigenvalue density decides the rest.

For $0<q<1$, $0\le t<1$, and S in the cone $\mathbb{P}^r$ of positive semidefinite $r\times r$ matrices, put

$$L_{q,t}^{(r)}(S) := \mathcal{Z}_{q,t}(-S) = \prod_{j\ge0}\frac{\det(I+tq^jS)}{\det(I+q^jS)}, \tag{4.5}$$

a positive function, analytic on a neighborhood of $\mathbb{P}^r$. For a Hermitian B write $D_B L(S) = \frac{d}{d\varepsilon} L(S+\varepsilon B)|_{\varepsilon=0}$. The function L is *completely monotone* on $\mathbb{P}^r$ if

$$(-1)^k D_{B_1} \cdots D_{B_k} L(S) \geq 0 \qquad (S \succ 0,\ B_i \succeq 0,\ k \geq 0),$$

the definition of Scott and Sokal (2014). The condition tests every positive basepoint, not only the origin. At the origin, with $B_i = u_i u_i^*$, the master theorem (2.1) identifies the alternating derivative with $\mathrm{per}_{q,t}([u_i^* u_j])$, so complete monotonicity of $L^{(r)}_{q,t}$ implies the nonnegativity of $\mathrm{per}_{q,t}$ on every Gram matrix of vectors in $\mathbb{C}^r$.

The moments that a representing measure would have can be defined without the measure. Let $\ell^{(r)}_{q,t}$ be the linear functional on real polynomials on the Hermitian $r \times r$ matrices determined by

$$\ell^{(r)}_{q,t}(e^{\mathrm{tr}(XW)}) = \mathcal{Z}_{q,t}(X), \tag{4.6}$$

an identity of Taylor coefficients at $X = 0$; the trace pairing is nondegenerate, so (4.6) fixes every polynomial moment, and no representing measure is assumed. By (2.1), $\ell^{(r)}_{q,t}(\prod_i u_i^* W u_i) = \mathrm{per}_{q,t}([u_i^* u_j])$. Determinant powers have rectangular Schur coefficients as moments, and the square $(\det W)^2$ is the moment that decides existence.

Proposition 4.3 (Rectangular moments)**.** *For positive integers r, k,*

$$\ell^{(r)}_{q,t}((\det W)^k) = \left[\prod_{a=0}^{k-1} \frac{(r+a)!}{a!}\right] c_{(k^r)}(q,t). \tag{4.7}$$

In particular, with $e_r := c_{(1^r)}$ and $e_0 := 1$,

$$e_r(q,t) = \prod_{j=0}^{r-1} \frac{q^j - t}{1 - q^{j+1}}, \tag{4.8}$$

$$\ell^{(r)}_{q,t}((\det W)^2) = r!\,(r+1)!\,(e_r^2 - e_{r-1}e_{r+1}). \tag{4.9}$$

The identities (4.7) and (4.9) hold for arbitrary cycle weights w, with $c_\lambda = s_\lambda[\rho_w]$ for the specialization $\rho_w \colon p_k \mapsto w_k$ and $\ell^{(r)}_w$ defined by (4.6) with $\mathcal{Z}_w(X) = \exp(\sum_{j\geq 1} w_j \,\mathrm{tr}\, X^j/j)$.

Proof. Apply the Cayley operator $\det(\partial_X)^k$ to the Cauchy expansion $\mathcal{Z}_{q,t}(X) = \sum_\lambda c_\lambda(q,t) s_\lambda(X)$, the Schur Cauchy identity for the specialization $B_{q,t}$ of (2.6), and evaluate at zero. Its action on Schur polynomials is

$$\det(\partial_X) s_\lambda(X) = \left[\prod_{i=1}^{r} (\lambda_i + r - i)\right] s_{\lambda - (1^r)}(X),$$

with value zero when $\lambda_r = 0$. In degree kr, only $\lambda = (k^r)$ survives k applications. Since $s_{(k^r)}(X) = \det(X)^k$, the Cayley identity gives the factor in (4.7); see Caracciolo et al. (2013) for that identity. The column formula (4.8) follows from (2.6), or by comparing coefficients in $(1+tz)E(z) = (1+z)E(qz)$ for $E(z) = \sum_{r\geq 0} e_r z^r = \prod_{j\geq 0}(1+q^j z)/(1+tq^j z)$. Finally, the dual Jacobi–Trudi identity gives $c_{(2^r)} = e_r^2 - e_{r-1}e_{r+1}$. Nothing in the argument uses the Macdonald form of the coefficients. □

For the Macdonald weights, the Turán expression (4.9) factors, which makes its sign explicit.

Theorem 4.4 (Exact determinant-square sign)**.** *If $0 < t < 1$ and $t \neq q^M$ for every $M \geq 1$, then*

$$c_{(2^r)}(q,t) = e_r(q,t)^2 \,\frac{(1-q)q^{r-1}(1-tq)}{(1-q^{r+1})(q^{r-1}-t)}. \tag{4.10}$$

Consequently, if $q^{m+1} < t < q^m$ with $m \geq 0$, then

$$\ell^{(r)}_{q,t}((\det W)^2) \begin{cases} > 0, & 1 \leq r \leq m+1, \\ < 0, & r \geq m+2. \end{cases} \tag{4.11}$$

Proof. No e_r vanishes at these parameters, and (4.8) gives

$$\frac{e_{r-1}e_{r+1}}{e_r^2} = \frac{(1-q^r)(q^r-t)}{(q^{r-1}-t)(1-q^{r+1})}.$$

Subtracting from one yields (4.10), because the numerator simplifies to $(1-q)q^{r-1}(1-tq)$. Every factor in (4.10) except $q^{r-1}-t$ is positive, which proves (4.11). □

At $t=q^M$ the square moments are positive for $r \le M$ and zero for $r > M$, and at $t=0$ they are positive; these values are read from (4.7), not by dividing in (4.10). Together with (4.11) they say that $\ell^{(r)}_{q,t}((\det W)^2) \ge 0$ exactly when $t \in \mathcal{R}_r(q)$, the coefficient region of Proposition 2.3.

Positivity of the square moment in every dimension excludes every parameter off the locus, while on the locus the Gaussian representation gives complete monotonicity.

Theorem 4.5 (Complete monotonicity in every dimension)**.** *Let $0<q<1$ and $0 \le t < 1$, and put $\mathcal{G}(q) = \{0, q, q^2, q^3, \dots\}$. The following are equivalent.*

(i) $t \in \mathcal{G}(q)$.

(ii) $c_\lambda(q,t) \ge 0$ *for every partition* λ.

(iii) $L^{(r)}_{q,t}$ *is completely monotone on* $\mathbb{P}^r$ *for every* $r \ge 1$.

(iv) $\ell^{(r)}_{q,t}((\det W)^2) \ge 0$ *for every* $r \ge 1$.

(v) $\ell^{(r)}_{q,t}((\det W)^2) \ge 0$ *for an unbounded set of integers* r.

Each of these conditions implies that $\mathrm{per}_{q,t}(A) \ge 0$ *for every complex Hermitian positive semidefinite* A*, of every size and rank.*

Proof. The equivalence of (i) and (ii) is Proposition 2.3. If $t=q^M$ or $t=0$, Theorem 4.1 with $\Sigma = I_r$ gives

$$L^{(r)}_{q,t}(S) = \mathbb{E}e^{-\operatorname{tr}(SW)}, \qquad W = \sum\nolimits_{j=0}^{M-1} q^j g_j g_j^*, \tag{4.12}$$

where g_j are independent standard complex Gaussian vectors in $\mathbb{C}^r$ and the sum is infinite at $t=0$, converging almost surely in trace norm. A Laplace transform of a probability measure on $\mathbb{P}^r$ is completely monotone, and its polynomial moments are those of (4.6), so (i) implies (iii) and (iv).

If (iii) holds, the Bernstein–Hausdorff–Widder–Choquet theorem on cones (Scott and Sokal, 2014, Theorem 2.2) gives a probability measure on $\mathbb{P}^r$ with Laplace transform $L^{(r)}_{q,t}$. Analyticity of $L^{(r)}_{q,t}(sI)$ at $s=0$ and monotone convergence give finite trace moments of every order, and since $(\det W)^2 \le (\operatorname{tr} W)^{2r}$ on $\mathbb{P}^r$, the square moment is finite and agrees with (4.6). Thus (iii) implies (iv), and (iv) implies (v). Theorem 4.4 excludes every $t \in (0,1) \setminus \mathcal{G}(q)$ from (v), which proves (v) implies (i). The final assertion follows from the diagonal Gaussian moment identity (4.4). □

The same moment obstruction restricts existence in a fixed dimension. A representing law has the formal polynomial moments of (4.6); the boundary-moment justification is given in Appendix A.1.1.

Corollary 4.6 (A necessary cone-valued Laplace condition)**.** *Fix $0<q<1$, $0 \le t < 1$, and a positive integer d. If a positive semidefinite random matrix $W \in M_d(\mathbb{C})$ satisfies*

$$\mathbb{E}e^{-\operatorname{tr} SW} = \mathcal{Z}_{q,t}(-S), \qquad S \succeq 0,$$

then $t \in \mathcal{R}_d(q)$.

Proof. For a representing law, $(\det W)^2$ has finite expectation equal to (4.9). Its nonnegativity and Theorem 4.4, including the geometric boundary values, give $t \in \mathcal{R}_d(q)$. □

4.4. **The eigenvalue density and dimension two.** Corollary 4.6 allows a matrix law only for $t \in \mathcal{R}_d(q)$; does one exist there? This subsection reduces the question to the scalar density and answers it in dimension two. The transform depends only on the eigenvalues of S, so any such W is unitarily invariant, and the Harish-Chandra–Itzykson–Zuber integral turns its eigenvalue law into a signed combination of derivatives of the scalar density. Existence then becomes a positivity question. Let $\psi_{q,t}$ be the density of $S_{q,t}$. Appendix A.2 writes it as an exponential series $\sum_{r\ge 0} c_r e^{-q^{-r}x}$ (Lemma A.1), whose first $d-2$ derivatives vanish at zero when $t < q^{d-1}$, and proves (Proposition A.2) that for $0 \le t < q^{d-1}$ a positive semidefinite random $W \in M_d(\mathbb{C})$ with $\mathbb{E}e^{-\operatorname{tr} SW} = \mathcal{Z}_{q,t}(-S)$ exists if and only if

$$\det\bigl[(-D)^{j-1}\psi_{q,t}(x_i)\bigr]_{i,j=1}^{d} \ge 0 \qquad (0 < x_1 < \cdots < x_d), \qquad D = \frac{d}{dx},$$

and then its ordered eigenvalues have density $\Delta(x)\det[(-D)^{j-1}\psi_{q,t}(x_i)]/\prod_{j=0}^{d-1} j!$ with $\Delta(x) = \prod_{i<j}(x_j - x_i)$; uniqueness of Laplace transforms also rules out mass on the boundary of the cone.

In dimension two the determinant is $\psi_{q,t}(x)\psi_{q,t}(y)$ times the decrease of $(\log\psi_{q,t})'$ from x to y, so the criterion is log-concavity, which holds throughout the necessary range.

Theorem 4.7 (The rank-two Laplace region)**.** *Fix $0 < q < 1$ and $0 \le t < 1$. A positive semidefinite random $W \in M_2(\mathbb{C})$ with $\mathbb{E}e^{-\operatorname{tr} SW} = \mathcal{Z}_{q,t}(-S)$ for $S \succeq 0$ exists if and only if $0 \le t \le q$. For $0 < t < q$, the density $\psi_{q,t}$ is log-concave, and the matrix law has ordered-eigenvalue density*

$$(y-x)(\psi'_{q,t}(x)\psi_{q,t}(y) - \psi_{q,t}(x)\psi'_{q,t}(y)), \qquad 0 < x < y.$$

Proof. First suppose $q^2 \le t < q$. The coefficients of (A.1) satisfy

$$c_0 > 0, \qquad c_r = \frac{t - q^r}{q(1-q^r)}c_{r-1}.$$

It gives $c_r \le 0$ for every $r \ge 1$: at $t = q^2$ the series terminates after $r = 1$, and for $q^2 < t < q$ all these coefficients are negative. Hence

$$B(x) := e^x\psi_{q,t}(x) = c_0 - \sum\nolimits_{r\ge 1}(-c_r)e^{-(q^{-r}-1)x}$$

is positive, increasing, and concave. Thus $\log B$ is concave, and so is $\log\psi_{q,t}(x) = -x + \log B(x)$.

Convolution of log-concave densities is log-concave, by the Prékopa–Leindler inequality, and so is convolution with an exponential density of any rate.

For $0 < t < q^2$, choose an integer $k \ge 1$ such that $q^2 \le t/q^k < q$. Repeatedly applying (3.4) with first parameter q gives

$$S_{q,t} \overset{d}{=} \sum\nolimits_{r=0}^{k-1} q^r E_r + q^k S'_{q,t/q^k},$$

with independent summands. The first part of the proof handles the last density, and successive exponential convolutions prove log-concavity of $\psi_{q,t}$. For $0 < x < y$, we now have

$$\det\begin{bmatrix}\psi_{q,t}(x) & -\psi'_{q,t}(x)\\ \psi_{q,t}(y) & -\psi'_{q,t}(y)\end{bmatrix} = \psi_{q,t}(x)\psi_{q,t}(y)\bigl((\log\psi_{q,t})'(x) - (\log\psi_{q,t})'(y)\bigr) \ge 0.$$

Proposition A.2 proves existence and the stated eigenvalue density.

The endpoints $t = q$ and $t = 0$ are the Gaussian constructions zz^* and $\sum_{r\ge 0} q^r z^{(r)}z^{(r)*}$ of Theorem 4.1, and Corollary 4.6 requires $t \in \mathcal{R}_2(q) = [0, q]$. □

5. Discrete laws

This section turns from the matrix law to the discrete laws, which read $\mathcal{Z}_{q,t}$ and $\mathrm{per}_{q,t}$ as probabilities or as correlation functions rather than as moments. They need only nonnegative weights, not a matrix mixing law, so they survive on larger regions, which this section determines.

5.1. **Closed walks and occupation laws.** The count laws are given by the coefficients of $\mathcal{Z}_{q,t}$, and two mechanisms make these coefficients nonnegative. If the kernel is entrywise nonnegative, the trace expansion is a sum of positive closed-walk weights, realized by a Poisson collection of weighted walks, a length-weighted loop soup (Li and Zheng, 2023), in which w_k enters through the intensity of walks of length k. If the kernel is positive semidefinite of rank d, the Schur signs of Section 2 give nonnegativity on $\mathcal{R}_d(q)$, whatever the signs of its entries. We begin with the first.

For the loop construction, let $K \in M_N(\mathbb{R})$ be entrywise nonnegative with $\rho(K) < 1$. Symmetry and positive semidefiniteness are not required. A closed word of length $k \geq 1$ is an ordered tuple $\ell = (i_1, \dots, i_k) \in [N]^k$, with $i_{k+1} = i_1$, and its weight is

$$K(\ell) := \prod_{a=1}^{k} K_{i_a i_{a+1}}.$$

Cyclic rotations are separate words, and the factor $1/k$ goes into the intensity, so that $\sum_{\ell \in [N]^k} K(\ell)/k = \operatorname{tr} K^k/k$ with no separate convention for periodic words. With these conventions, a Poisson collection of words realizes the coefficients of $\mathcal{Z}_{q,t}^{\beta}$ as probabilities.

Theorem 5.1 (A positive length-weighted loop law)**.** *Let $0 \leq q, t < 1$ and $\beta > 0$. For each closed word ℓ of length k, take an independent Poisson variable M_ℓ of mean*

$$\frac{\beta w_k}{k} K(\ell).$$

Let N_i count all occurrences of vertex i in these words, with each word counted M_ℓ times. Almost surely only finitely many words are sampled, and, for $0 \leq s_i \leq 1$,

$$\mathbb{E} \prod_i s_i^{N_i} = \frac{\mathcal{Z}_{q,t}(D_s K)^{\beta}}{\mathcal{Z}_{q,t}(K)^{\beta}}, \qquad D_s := \operatorname{diag}(s_1, \dots, s_N). \tag{5.1}$$

In particular, the vector $\mathbf{N} = (N_i)$ has the law

$$\mathbb{P}(\mathbf{N} = \mathbf{m}) = \frac{\operatorname{per}_{q,t}^{(\beta)}(K[\mathbf{m}])}{\mathbf{m}!\, \mathcal{Z}_{q,t}(K)^{\beta}}, \qquad \mathbf{m} \in \mathbb{N}^N. \tag{5.2}$$

These occupation laws are infinitely divisible in β.

Proof. Since $0 < w_k \leq (1-q)^{-1}$ and $0 \leq \operatorname{tr} K^k \leq N\rho(K)^k$, the total intensity

$$\sum_\ell \mathbb{E} M_\ell = \beta \sum_{k \geq 1} \frac{w_k}{k} \operatorname{tr} K^k = \beta \log \mathcal{Z}_{q,t}(K)$$

is finite, so almost surely finitely many words are sampled and every N_i is finite.

For a word ℓ, write $s(\ell) = \prod_{a=1}^{|\ell|} s_{i_a}$. The Poisson generating function and independence give

$$\log \mathbb{E} \prod_i s_i^{N_i} = \sum_\ell \mathbb{E} M_\ell (s(\ell) - 1) = \beta \sum_{k \geq 1} \frac{w_k}{k} \left(\operatorname{tr}(D_s K)^k - \operatorname{tr} K^k \right).$$

The series is absolutely convergent because $|s(\ell) - 1| \leq 1$, and exponentiating proves (5.1). Extracting coefficients in the master theorem with cycle weights βw_k proves (5.2). Finally, superposing independent word families adds intensities, so the law with parameter β is the r-fold convolution of the law with parameter β/r. □

Theorem 5.1 needs nonnegative entries, but a positive semidefinite 2×2 kernel becomes entrywise nonnegative after a diagonal change of phase, which leaves every closed-walk product unchanged. Two distinct Gram vectors, repeated arbitrarily often, therefore give a count law throughout the square, the two-type counterpart of Shirai (2007, Proposition 3.10).

Corollary 5.2 (A two-type infinitely divisible count law)**.** *Let* $K = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) \succeq 0$ *with* $\|K\| < 1$*, and let* $0 \le q, t < 1$ *and* $\beta > 0$*. Then*

$$\mathbb{P}(N_1 = p, N_2 = m) = \frac{\mathrm{per}_{q,t}^{(\beta)}(K[p,m])}{p!\, m!\, \mathcal{Z}_{q,t}(K)^{\beta}}, \qquad p, m \in \mathbb{N}, \tag{5.3}$$

defines a compound-Poisson distribution, where $K[p,m]$ *repeats the first index* p *times and the second* m *times. Its generating function is*

$$\mathbb{E}[s_1^{N_1} s_2^{N_2}] = \frac{\mathcal{Z}_{q,t}(\mathrm{diag}(s_1,s_2)K)^{\beta}}{\mathcal{Z}_{q,t}(K)^{\beta}}.$$

In particular, this distribution is infinitely divisible.

Proof. Choose a diagonal unitary matrix U such that UKU^* has off-diagonal entries $|b|$; if $b = 0$, take $U = I$. Its diagonal entries are $a, d \ge 0$, so UKU^* is entrywise nonnegative. Diagonal unitary conjugation preserves every closed-walk product and the spectral radius. Since $\rho(K) = \|K\| < 1$, Theorem 5.1 applies to UKU^* and gives the law and its compound-Poisson construction. □

With three or more types, the phases of a positive semidefinite kernel cannot in general be removed, so the corollary says nothing about general Gram matrices of rank two, and their coefficients need another source of positivity.

5.2. **Positive semidefinite kernels.** For $\beta = 1$, the Schur coefficient signs give a second route to nonnegative coefficients, one that allows negative or complex entries.

Corollary 5.3. *Let* $d \ge 1$ *be an integer, and suppose* $K \in M_N(\mathbb{C})$ *satisfies* $K \succeq 0$*,* $\|K\| < 1$*, and* $\operatorname{rank} K \le d$*. If* $0 < q < 1$ *and* $t \in \mathcal{R}_d(q)$*, then*

$$\frac{\mathrm{per}_{q,t}(K[\mathbf{m}])}{\mathbf{m}!\, \mathcal{Z}_{q,t}(K)}, \qquad \mathbf{m} \in \mathbb{N}^N,$$

is a probability distribution with generating function $\mathcal{Z}_{q,t}(D_s K)/\mathcal{Z}_{q,t}(K)$*.*

Proof. Write $K_{ij} = \langle v_i, v_j \rangle$ with $v_i \in \mathbb{C}^d$. Repeating the vectors according to $\mathbf{m}$ represents $K[\mathbf{m}]$ as a Gram matrix in the same space. Lemma 2.4 therefore makes every coefficient nonnegative. In degree $n = |\mathbf{m}|$, only partitions with $\ell(\lambda) \le \min(n, d)$ occur, and $\mathcal{R}_d(q) \subseteq \mathcal{R}_{\min(n,d)}(q)$. The one-variable series $\mathcal{Z}_{q,t}(uK)$ has radius of convergence greater than 1, since $\|K\| < 1$. Applying the master theorem degree by degree and setting $u = 1$ gives

$$\sum\nolimits_{\mathbf{m}} \frac{\mathrm{per}_{q,t}(K[\mathbf{m}])}{\mathbf{m}!} = \mathcal{Z}_{q,t}(K).$$

For $0 \le s_i \le 1$, nonnegativity lets us evaluate the series at s, proving the last claim. □

Corollary 5.3 produces the distribution but no mechanism behind it, such as a mixing law, that is, a random positive semidefinite matrix given which the counts are Poisson. On $\mathcal{G}(q)$ there is one. For $t = q^m$, or $t = 0$ with $m = \infty$, let $W = W_{q,m}(K)$; tilting this Gaussian matrix by $e^{\mathrm{tr}\, W}/\mathcal{Z}_{q,t}(K)$ and sampling independent Poisson counts with means W_{ii} gives the coefficient law, and the weighted colors then have covariances

$$C_r = q^r K (I - q^r K)^{-1}.$$

Theorem B.1 proves this and gives the factorial moments. Infinite divisibility of the count law is a separate question from that of this mixing law.

5.3. **Infinite divisibility of the coefficient law.** Which coefficient laws are infinitely divisible? The answer is decided by the logarithm of the generating function, and there the cycle weights enter only as positive factors $w_{|\mathbf{m}|}$ in front of closed-walk sums of K. The matrix condition therefore separates from (q,t). In the classical case this is the multivariate negative-binomial problem of Griffiths and Milne (1987), and general matrix criteria require care with zero entries (Eisenbaum and Maunoury, 2017, Section 3); for Hermitian matrices the trace expansion gives a direct criterion. For $K \in M_N(\mathbb{C})$ with $K \succeq 0$ and $\|K\| < 1$, and for $\mathbf{m} \in \mathbb{N}^N \setminus \{\mathbf{0}\}$, define

$$\Gamma_{\mathbf{m}}(K) = \frac{1}{|\mathbf{m}|}[s^{\mathbf{m}}] \operatorname{tr}((D_s K)^{|\mathbf{m}|}), \qquad D_s = \operatorname{diag}(s_1, \dots, s_N).$$

Reversing an ordered closed walk conjugates its weight, so $\Gamma_{\mathbf{m}}(K)$ is real. It sums the weights of all closed walks with occupation vector $\mathbf{m}$, including the factor $1/|\mathbf{m}|$ from the trace logarithm.

Theorem 5.4 (Infinite divisibility of coefficient laws)**.** *Let $K \in M_N(\mathbb{C})$ satisfy $K \succeq 0$ and $\|K\| < 1$, and let $0 \le q, t < 1$. Set*

$$F_{q,t;K}(s) = \frac{\mathcal{Z}_{q,t}(D_s K)}{\mathcal{Z}_{q,t}(K)}.$$

The following conditions are equivalent:

(i) *$F_{q,t;K}$ is the probability generating function of an infinitely divisible distribution on $\mathbb{N}^N$.*
(ii) *$F_{q,t;K}^{\beta}$ is a probability generating function for every $\beta > 0$.*
(iii) *$\Gamma_{\mathbf{m}}(K) \ge 0$ for every $\mathbf{m} \in \mathbb{N}^N \setminus \{\mathbf{0}\}$.*

Here infinite divisibility is with respect to addition on $\mathbb{N}^N$. When these conditions hold, the distribution in (ii) *is compound Poisson, with jump $\mathbf{m}$ having rate*

$$\nu_\beta(\mathbf{m}) = \beta \, \frac{1 - t^{|\mathbf{m}|}}{1 - q^{|\mathbf{m}|}} \, \Gamma_{\mathbf{m}}(K). \tag{5.4}$$

In particular, the class of matrices satisfying the conditions is independent of $(q,t) \in [0,1)^2$.

Proof. Choose $R > 1$ with $R\|K\| < 1$. On the polydisk $|s_i| < R$ the trace logarithm is analytic and

$$-\log\det(I - D_s K) = \sum\nolimits_{k \ge 1} \frac{1}{k} \operatorname{tr}((D_s K)^k) = \sum\nolimits_{\mathbf{m} \ne \mathbf{0}} \Gamma_{\mathbf{m}}(K) s^{\mathbf{m}}.$$

The last series is absolutely summable on $|s_i| \le 1$: Cauchy estimates on any smaller polydisk of radius greater than one bound the coefficients by a summable geometric sequence. Since $0 < w_k \le (1-q)^{-1}$, the weighted series is absolutely summable there as well. Consequently

$$\log F_{q,t;K}(s) = \sum\nolimits_{\mathbf{m} \ne \mathbf{0}} w_{|\mathbf{m}|} \Gamma_{\mathbf{m}}(K)(s^{\mathbf{m}} - 1).$$

If (iii) holds, this is the exponent of a compound Poisson generating function with finite total jump rate. Multiplication by $\beta > 0$ gives (5.4) and proves (ii). Condition (ii) implies (i) by taking $\beta = 1/r$ for every positive integer r. Conversely, under (i), let G_r generate an rth convolution root. Since $G_r(0)^r = F_{q,t;K}(0) > 0$, the root with positive constant term is unique near zero:

$$G_r(s) = \exp\left(\frac{1}{r} \log F_{q,t;K}(s)\right).$$

For each fixed $\mathbf{m} \ne \mathbf{0}$, coefficient extraction at zero therefore gives

$$[s^{\mathbf{m}}] G_r(s) = \frac{1}{r} w_{|\mathbf{m}|} \Gamma_{\mathbf{m}}(K) + O(r^{-2}).$$

The left side is nonnegative. Multiplying by r and letting $r \to \infty$ proves $w_{|\mathbf{m}|}\Gamma_{\mathbf{m}}(K) \ge 0$. Every w_k is strictly positive on $[0,1)^2$, so (iii) follows, and with it parameter independence. □

For real symmetric matrices, the coefficient criterion has a finite graph-theoretic interpretation. Its classical form is contained in the Griffiths–Milne theory and in Eisenbaum and Maunoury (2017, Theorem 3.2). The following proof also covers zero entries and reducible matrices.

Corollary 5.5 (Real symmetric matrices)**.** *Let $K \in M_N(\mathbb{R})$ be symmetric, $K \succeq 0$, and $\|K\| < 1$. The equivalent conditions of Theorem 5.4 hold if and only if there is a diagonal matrix $S = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_N)$, with $\varepsilon_i \in \{-1, 1\}$, such that SKS is entrywise nonnegative.*

Proof. Signature conjugation preserves every closed-walk product, because each vertex sign occurs an even number of times. Thus an entrywise nonnegative SKS makes all $\Gamma_{\mathbf{m}}(K)$ nonnegative. For the converse, form the undirected graph with an edge $\{i, j\}$ when $i \neq j$ and $K_{ij} \neq 0$. Suppose that some simple cycle has a negative product of edge entries, and choose one of minimum length $\ell \geq 3$. This cycle has no chord. Indeed, a chord splits it into two shorter simple cycles, and the product of their signs equals the sign of the original cycle, because the chord appears twice. One of the shorter cycles would therefore have negative sign.

Let V be the vertex set of this chordless cycle and let $\mathbf{1}_V$ be its indicator vector. A term contributing to $[s^{\mathbf{1}_V}] \operatorname{tr}((D_s K)^\ell)$ visits every vertex in V exactly once. The induced graph on V is the cycle itself, so these terms consist of its two orientations and ℓ choices of a starting vertex. No diagonal step or backtracking walk can contribute to this squarefree monomial. Hence

$$\Gamma_{\mathbf{1}_V}(K) = 2 \prod_{\{i,j\} \text{ in the cycle}} K_{ij} < 0,$$

contrary to Theorem 5.4. All simple cycles therefore have positive sign. In each connected component, choose a spanning tree and assign vertex signs successively from a root so that every tree edge becomes positive under signature conjugation. Each remaining edge closes a simple cycle with its tree path. Positivity of that cycle makes the remaining edge positive as well. The diagonal entries are nonnegative because $K \succeq 0$. Combining the signatures of the components gives S. □

The criterion concerns the count kernel K, not the covariance of a mixing law, and it differs from the inverse-M-matrix criterion for continuous multivariate gamma laws: Appendix B.2 gives a kernel whose counts are infinitely divisible while their Gaussian mixing law is not.

5.4. **Correlation functions and Cox processes.** We next ask where $\operatorname{per}_{q,t}$ is the correlation function of a point process, its third probabilistic reading after moments and probabilities. A Cox process is a Poisson point process whose intensity is itself random, and when the intensity is the squared modulus of a Gaussian field one obtains the permanental (boson) processes of Shirai and Takahashi (2003) and McCullagh and Møller (2006), whose correlation functions are α-permanents of the covariance kernel. The Gaussian colors of Section 4.1 give such processes with $\operatorname{per}_{q,t}$ in place of $\operatorname{per}_\alpha$ on $\mathcal{G}(q)$, and the scalar law gives them for rank-one kernels throughout the square.

The construction runs on a general state space. Let E be a locally compact, second-countable Hausdorff space with a diffuse Radon measure μ. Let g be a jointly measurable centered circular complex Gaussian field, with covariance $K(x, y) = \mathbb{E}[g(x)\overline{g(y)}]$. Assume that the integral operator of K is positive and locally trace class and that $\int_B K(x, x)\, d\mu(x) < \infty$ for every relatively compact Borel set B. Its restriction to B is an $L^2(B, \mu)$-valued Gaussian random variable, and its covariance operator $K_B = 1_B K 1_B$ satisfies

$$\operatorname{tr} K_B = \mathbb{E} \int_B |g(x)|^2\, d\mu(x) = \int_B K(x, x)\, d\mu(x).$$

We use this covariance version of the kernel throughout. A correlation function ρ_n is the density of the nth factorial moment measure:

$$\mathbb{E} \sum\nolimits_{(x_1,\dots,x_n) \in \Xi^n_{\neq}} F(x_1, \dots, x_n) = \int_{E^n} F(x_1, \dots, x_n) \rho_n(x_1, \dots, x_n)\, d\mu^n,$$

for nonnegative measurable F, where the sum runs over ordered tuples of distinct points. Density formulas hold almost everywhere. The Gaussian colors then give the following process.

Theorem 5.6 (Cox realization on the geometric curves)**.** *Take independent copies g_r of g and define*

$$\Lambda(x) = \sum_{r=0}^{m-1} q^r |g_r(x)|^2,$$

where $0 \le q < 1$, m is a positive integer or ∞, and $t = q^m$, or $t = 0$ if $m = \infty$. Given Λ, let Ξ be Poisson with intensity $\Lambda(x)\,d\mu(x)$. Then Ξ is locally finite and simple, with correlations

$$\rho_n(x_1, \dots, x_n) = \mathrm{per}_{q,t}([K(x_i, x_j)]_{i,j=1}^n). \tag{5.5}$$

For a nonnegative measurable compactly supported function f, put $h = 1 - e^{-f}$ and $K_h = M_{\sqrt{h}} K M_{\sqrt{h}}$, where M_v denotes multiplication by v. The Laplace functional is

$$\mathbb{E} e^{-\langle f, \Xi\rangle} = \prod_{r=0}^{m-1} \det(I + q^r K_h)^{-1} = \prod_{r \ge 0} \frac{\det(I + tq^r K_h)}{\det(I + q^r K_h)}. \tag{5.6}$$

For $m = \infty$, the middle product is infinite.

Proof. The expected integrated intensity on B is

$$\mathbb{E} \int_B \Lambda\, d\mu = \Big(\sum_{r=0}^{m-1} q^r\Big) \operatorname{tr} K_B = w_1 \operatorname{tr} K_B < \infty.$$

Thus $\Lambda\, d\mu$ is almost surely locally finite, simultaneously on a countable exhaustion by relatively compact sets, and diffuse, so the conditional Poisson process is simple.

Conditional Poisson factorial moments give $\rho_n(x_1, \dots, x_n) = \mathbb{E} \prod_i \Lambda(x_i)$. For a finite collection of locations, the vectors $(g_r(x_1), \dots, g_r(x_n))$ have covariance $[K(x_i, x_j)]$. Theorem 4.1 therefore proves (5.5). In the infinite-color case this application is justified by monotone convergence of products of the nonnegative partial intensities. The bound (2.4) gives

$$\int_{B^n} \rho_n\, d\mu^n \le n! H_n(q,t) (\operatorname{tr} K_B)^n < \infty,$$

so all local factorial moment measures are finite.

For the Laplace functional, conditional Poisson expectation gives $\mathbb{E} \exp(-\int h\Lambda\, d\mu)$. The random function $\sqrt{h}\, g_r$ is an L^2 Gaussian variable with trace-class covariance K_h. If λ_j are its nonzero covariance eigenvalues, its squared norm has the law $\sum_j \lambda_j |\xi_j|^2$, where ξ_j are independent standard circular complex Gaussians. Finite spectral truncation and bounded convergence give

$$\mathbb{E} \exp\left(-q^r \int h |g_r|^2\, d\mu\right) = \prod_j (1 + q^r \lambda_j)^{-1} = \det(I + q^r K_h)^{-1}.$$

Independence proves the color product. Since $0 \le \log\det(I + q^r K_h) \le q^r \operatorname{tr} K_h$, its logarithm converges absolutely when there are infinitely many colors. For finite m, cancellation of the numerator factors at indices $r + m$ gives the last expression in (5.6); the remaining tail tends to 1 by the same trace bound. □

Each color contributes an independent permanental process with kernel $q^r K$, so Ξ is their superposition. For fixed m, letting $q \uparrow 1$ in the finite-degree correlations recovers $\mathrm{per}_m([K(x_i, x_j)])$, which in the Shirai–Takahashi convention is $\det_{1/m}([mK(x_i, x_j)])$.

The scalar law $S_{q,t}$ of Section 3 exists for every (q,t) in the square, not only on the curves, and for a rank-one covariance kernel it supplies the random intensity directly.

Proposition 5.7 (Rank-one Cox representation)**.** *Suppose $K(x,y) = a(x)\overline{a(y)}$, with a measurable and $|a|^2$ locally integrable. For $0 \le q, t < 1$, the Cox process with random intensity $S_{q,t}|a(x)|^2\, d\mu(x)$ has correlations $\mathrm{per}_{q,t}([K(x_i, x_j)])$. Replacing $S_{q,t}$ by $S_{q,t}^{(\beta)}$ gives $\mathrm{per}_{q,t}^{(\beta)}([K(x_i, x_j)])$ for $\beta > 0$.*

Proof. The scalar variable has finite mean w_1, or βw_1 for its time-β extension. Its intensity is therefore almost surely locally finite. Conditional Poisson moments give the nth correlation as $\mathbb{E}S_{q,t}^n \prod_i |a(x_i)|^2$. By Proposition 3.1 this is $n!H_n(q,t)\prod_i |a(x_i)|^2$, which equals the matrix function by Proposition 2.2. The moments $\mathbb{E}(S_{q,t}^{(\beta)})^n = \mathrm{per}_{q,t}^{(\beta)}(J_n)$ give the final assertion. □

Further process formulas and boson–fermion superpositions, with the required determinantal contraction hypotheses, are given in the supplement (Sra, 2026b).

5.5. **Schur coefficients as probabilities.** A last probability model takes the Schur coefficients themselves as the probabilities and survives on all of $\mathcal{R}_d(q)$, where the matrix law is only conjectured to exist. On all partitions the coefficients are nonnegative only on $\mathcal{G}(q)$, but a Schur measure supported on partitions with at most d rows needs only the coefficients visible there, nonnegative on $\mathcal{R}_d(q)$. Its determinantal structure survives the restriction, since the correlation identity of Okounkov (2001) is an identity of symmetric functions, and positivity enters only when it is evaluated.

Let $a = (a_1, \dots, a_d)$ with $0 \le a_i < 1$, $0 < q < 1$, and $t \in \mathcal{R}_d(q)$, and define

$$\mathbb{P}_{\mathrm{Schur}}(\lambda) = \frac{s_\lambda(a) c_\lambda(q,t)}{\mathcal{Z}_{q,t}(\mathrm{diag}\, a)}. \tag{5.7}$$

Since $s_\lambda(a) = 0$ for $\ell(\lambda) > d$, only nonnegative coefficients from Proposition 2.3 contribute. The Schur Cauchy identity, applied degree by degree to the specialization $p_k[B_{q,t}] = w_k$, gives

$$\sum\nolimits_\lambda s_\lambda(a) c_\lambda(q,t) = \exp\left(\sum\nolimits_{k\ge1} \frac{w_k}{k} p_k(a)\right) = \mathcal{Z}_{q,t}(\mathrm{diag}\, a) < \infty.$$

To justify evaluation rather than only formal equality, first replace a by ua with $0 \le u < 1$; the trace series is analytic beyond $u = 1$, and the nonnegative degree coefficients permit passage to $u = 1$. Thus (5.7) is a probability law, and it is determinantal, by the following specialization of Okounkov (2001, Theorems 1 and 2).

Theorem 5.8 (Determinantal correlations in the finite-rank region)**.** *Let $d \ge 1$, $a = (a_1, \dots, a_d)$ with $0 \le a_i < 1$, $0 < q < 1$, and $t \in \mathcal{R}_d(q)$. Under (5.7), the random subset*

$$X(\lambda) = \{\lambda_i - i + \tfrac{1}{2} : i \ge 1\} \subset \mathbb{Z} + \tfrac{1}{2}$$

has determinantal correlations, with the following kernel. Put $a_* = \max_i a_i$ and

$$J(z) = \prod\nolimits_{i=1}^d (1 - a_i z)^{-1} \frac{(z^{-1};q)_\infty}{(tz^{-1};q)_\infty}.$$

Choose counterclockwise circles of radii $1 < r_w < r_z < a_*^{-1}$, with $a_*^{-1} = \infty$ if $a_* = 0$. Define

$$\mathsf{K}(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r_z} \oint_{|w|=r_w} \frac{J(z)}{J(w)} \frac{z^{-x-1/2} w^{y-1/2}}{z-w}\, dw\, dz. \tag{5.8}$$

For every finite set of distinct half-integers $x_1, \dots, x_k$,

$$\mathbb{P}\{x_1, \dots, x_k \in X(\lambda)\} = \det[\mathsf{K}(x_i, x_j)]_{i,j=1}^k.$$

Equivalently, the kernel is the Laurent coefficient

$$\mathsf{K}(x,y) = [z^{x-1/2} w^{-y-1/2}] \frac{J(z)}{J(w)(z-w)}, \qquad \frac{1}{z-w} = \sum\nolimits_{r\ge0} \frac{w^r}{z^{r+1}}. \tag{5.9}$$

The Laurent expansions in this formula are taken in the annulus $1 < |w| < |z| < a_*^{-1}$.

Appendix C proves the theorem, including the analytic continuation behind the contour formula.

For $0 < t < q^{d-1}$ away from the points q^m, the specialization $B_{q,t}$ is not Schur-positive in all lengths, yet Theorem 5.8 still defines a determinantal process; its points are partition coordinates, not the spatial points of the Cox process.

The size of the random partition connects back to Section 3. Its generating function is

$$\mathbb{E}_{\mathrm{Schur}} u^{|\lambda|} = \prod_{i=1}^{d} \frac{H(ua_i)}{H(a_i)},$$

so $|\lambda|$ is a sum of d independent (q,t)-negative-binomial variables $N_{q,t}(a_i)$. The Macdonald measure with alphabets a and (1) has the same size distribution but is supported on one-row partitions; the comparison is in Supplement II (Sra, 2026b).

6. Open questions

The results leave one question open for the matrix laws and one for the count laws. The matrix Laplace problem that remains is the q-analogue of Gindikin's theorem from the introduction, with $\mathcal{R}_d(q)$ in place of the Wallach set.

Conjecture 6.1 (q-Gindikin)**.** *For $d \geq 1$, $0 < q < 1$, and $0 \leq t < 1$, a random matrix $W \succeq 0$ in $M_d(\mathbb{C})$ with $\mathbb{E}e^{-\operatorname{tr} SW} = \mathcal{Z}_{q,t}(-S)$ for every $S \succeq 0$ exists if and only if $t \in \mathcal{R}_d(q)$.*

Necessity is Corollary 4.6 and dimension two is Theorem 4.7, so only sufficiency in dimensions $d \geq 3$, away from $\mathcal{G}(q)$, remains. Two reformulations are available. In the continuous range $t < q^{d-1}$, Proposition A.2 reduces it to a determinant inequality for derivatives of the scalar density, and Theorem A.4 turns it into a family of positivity conditions for tilted cycle weights.

For the count laws, the open question is a finite test for infinite divisibility.

Problem 6.2 (Infinitely divisible coefficient laws)**.** Characterize the complex Hermitian positive semidefinite matrices K whose closed-walk sums $\Gamma_{\mathbf{m}}(K)$ are all nonnegative.

For real symmetric matrices, Corollary 5.5 gives the finite criterion of signature conjugation to an entrywise nonnegative matrix. An intrinsic complex-matrix criterion would replace the infinite family of inequalities in Theorem 5.4.

AI usage

The results in this paper were developed through extensive use of an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs. The expansion of the author's notes into the present manuscript, together with its finite symbolic verification program, also used OpenAI's models, and Anthropic's Claude (Fable 5.1 and Opus 5.5) prepared this version of the manuscript. Assistant checks and reruns are not external peer review or formal verification.

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Appendix A. Analytic details for matrix Laplace transforms

The classification in Section 4.3 uses a representing law only through its moments and its eigenvalue density, and both need justification at the boundary of the positive cone. A representing law must have the formal moments (4.6), so that the determinant-square obstruction applies to it, and the scalar density must vanish to the right order at zero, so that inverting the Harish-Chandra–Itzykson–Zuber integral leaves no mass on the boundary. We also record two equivalent forms of complete monotonicity, finite witnesses near the origin (Corollary A.3) and a criterion at scalar basepoints, which reformulate the open problem of Section 6.

A.1. **Trace bounds and boundary moments.** For finite matrices, we use the formal and analytic domains of Proposition 2.1. For operators on an infinite-dimensional Hilbert space, the determinants are Fredholm determinants of trace-class perturbations of the identity.

If $T \succeq 0$ is trace class and $a \geq 0$, its eigenvalues $\lambda_j \geq 0$ satisfy

$$0 \leq \log\det(I + aT) = \sum\nolimits_j \log(1 + a\lambda_j) \leq a \sum\nolimits_j \lambda_j = a \operatorname{tr} T.$$

Thus $\sum_r q^r < \infty$ controls the products over Gaussian colors on every relatively compact set. All point-process densities are understood almost everywhere with respect to the reference measure.

A.1.1. *Polynomial moments of a representing law.* A representing law has the formal polynomial moments used in Theorem 4.5 and Corollary 4.6.

Justification of polynomial moments. Set $T_0 = \operatorname{tr} W \geq 0$ and $f(s) = \mathbb{E}e^{-sT_0} = \mathcal{Z}_{q,t}(-sI)$ for $s \geq 0$. The right side extends analytically to $|s| < 1$. For $s > 0$, differentiation under the expectation gives

$$(-1)^k f^{(k)}(s) = \mathbb{E}T_0^k e^{-sT_0}.$$

The derivative is justified because $x^k e^{-sx}$ is bounded on $[0,\infty)$, locally uniformly for $s > 0$. Monotone convergence as $s \downarrow 0$ then proves $\mathbb{E}T_0^k = (-1)^k f^{(k)}(0) < \infty$. For any fixed $0 < R < 1$, Cauchy's estimate gives $|f^{(k)}(0)| \leq M_R k! R^{-k}$, where $M_R = \max_{|s|=R} |f(s)|$. Hence, for $0 < a < R$,

$$\mathbb{E}e^{aT_0} = \sum\nolimits_{k\geq 0} \frac{a^k \mathbb{E}T_0^k}{k!} \leq \frac{M_R}{1 - a/R} < \infty.$$

For a complex matrix A with $\|A\| < a$, the estimate $|\operatorname{tr} AW| \leq \|A\| T_0$ makes $F(A) = \mathbb{E}e^{\operatorname{tr} AW}$ analytic in all matrix entries. The determinant expression $\mathcal{Z}_{q,t}(A)$ is analytic on the same sufficiently small neighborhood of zero. They agree when $A = -S$ is negative definite and small, by hypothesis, and the identity theorem, applied coordinatewise in a Hermitian basis of $M_d(\mathbb{C})$, extends the equality to a complex neighborhood of zero.

Consequently, every polynomial moment is finite and is obtained by differentiating $\mathcal{Z}_{q,t}$ at zero. These are exactly the formal moments in (4.6); in particular, this identifies the determinant-square moment used in the main proof. □

A.2. **The scalar density and the derivative criterion.** The rank-two argument requires the two analytic inputs of Theorem 4.7: the exponential series for the density of $S_{q,t}$, with the boundary values of its derivatives, and the derivative criterion for the matrix law. Write

$$\varphi_{q,t}(s) = \frac{(-ts;q)_\infty}{(-s;q)_\infty}, \qquad s \geq 0,$$

for the Laplace transform of $S_{q,t}$.

Lemma A.1 (The scalar density)**.** *Let $0 < q < 1$ and $0 \le t < 1$. The law of $S_{q,t}$ has a strictly positive smooth density on $(0,\infty)$ given by*

$$\psi_{q,t}(x) = \sum\nolimits_{r\ge 0} c_r(q,t) e^{-q^{-r}x}. \tag{A.1}$$

For $t > 0$, the coefficients are

$$c_r(q,t) = \frac{(t;q)_\infty}{(q;q)_\infty}\left(\frac{t}{q}\right)^r \frac{(q/t;q)_r}{(q;q)_r}; \tag{A.2}$$

at $t = 0$, they are

$$c_r(q,0) = \frac{(-1)^r q^{r(r-1)/2}}{(q;q)_\infty (q;q)_r}.$$

The series converges in $L^1(0,\infty)$, and it can be differentiated term by term on every compact subinterval of $(0,\infty)$. If $k \ge 0$ and $t < q^{k+1}$, its kth derivative extends continuously to zero and

$$\psi_{q,t}^{(k)}(0) = 0. \tag{A.3}$$

If $d \ge 2$ and $t < q^{d-1}$, then

$$\int_0^\infty x^\ell |\psi_{q,t}^{(j)}(x)|\, dx < \infty, \qquad \ell \ge 0, \quad 0 \le j \le d-1, \tag{A.4}$$

for every integer ℓ.

Proof. For $t > 0$, the finite products in (A.2) are bounded in absolute value, while $(q;q)_r \ge (q;q)_\infty > 0$. Consequently $|c_r(q,t)| \le C(t/q)^r$ for a constant depending on q, t. At $t = 0$, the displayed formula decays faster than any geometric sequence. These bounds give

$$\sum\nolimits_{r\ge 0} |c_r(q,t)| q^r < \infty$$

and prove L^1 convergence of (A.1). The additional factor $e^{-q^{-r}x}$ proves local uniform convergence of every differentiated series for $x > 0$.

To identify this initially signed density, let $n \ge 0$. The q-binomial theorem gives, for $t > 0$,

$$\begin{aligned}\int_0^\infty x^n \psi_{q,t}(x)\, dx &= n! \frac{(t;q)_\infty}{(q;q)_\infty} \sum\nolimits_{r\ge 0} \frac{(q/t;q)_r}{(q;q)_r} (tq^n)^r \\ &= n! \frac{(t;q)_\infty}{(q;q)_\infty} \frac{(q^{n+1};q)_\infty}{(tq^n;q)_\infty} = n! \frac{(t;q)_n}{(q;q)_n}.\end{aligned}$$

For $t = 0$, Euler's identity $\sum_{r\ge 0} (-1)^r q^{r(r-1)/2} z^r/(q;q)_r = (z;q)_\infty$ gives the same result. All termwise integrations are absolute. Moreover, the absolute value of the signed density has a finite exponential moment of every order $a < 1$, since $\sum_r |c_r|/(q^{-r} - a) < \infty$. Its moment generating function therefore agrees near zero with that of $S_{q,t}$. Laplace-transform uniqueness identifies the two measures.

The defining independent series for $S_{q,t}$ shows that $\mathbb{P}(S_{q,t} < \varepsilon) > 0$ for every $\varepsilon > 0$: choose a sufficiently long tail with small expectation and require each of the finitely many remaining terms to be small. Conditioning on $B_0 = 1$ then gives, for $x > 0$,

$$\psi_{q,t}(x) \ge (1-t) e^{-x} \mathbb{P}(qS_{q,t} < x) > 0$$

(almost everywhere by convolution, then everywhere by continuity).

If $t < q^{k+1}$, the series for $\psi_{q,t}^{(k)}(0)$ converges absolutely; for $t > 0$ the q-binomial theorem gives

$$\psi_{q,t}^{(k)}(0) = (-1)^k \frac{(t;q)_\infty}{(q;q)_\infty} \frac{(q^{-k};q)_\infty}{(t/q^{k+1};q)_\infty} = 0.$$

Euler's identity proves the same assertion at $t = 0$. Finally, termwise integration bounds the integral in (A.4) by

$$\ell! \sum\nolimits_{r\ge 0} |c_r(q,t)| q^{r(\ell+1-j)}.$$

For $t>0$, this sum is bounded by a geometric series with ratio $tq^{\ell-j}<1$; for $t=0$, it converges for every ℓ, j. The differentiated series also decay exponentially at infinity. These facts justify integration by parts with the boundary values in (A.3). □

The derivative principle for invariant Hermitian ensembles relates eigenvalues to diagonal entries (Kieburg and Zhang, 2023, Proposition 4). For the present transform, it is useful to construct a finite signed measure first. Uniqueness then determines whether the candidate is positive, including whether an additional boundary component could occur.

Proposition A.2 (Derivative criterion)**.** *Fix $d\ge 2$, $0<q<1$, and $0\le t<q^{d-1}$. Set*

$$\Delta(x)=\prod\nolimits_{1\le i<j\le d}(x_j-x_i),\qquad J_d=\prod\nolimits_{j=0}^{d-1} j!,\qquad D=\frac{d}{dx}.$$

There exists a positive semidefinite random matrix $W\in M_d(\mathbb{C})$ with Laplace transform

$$\mathbb{E}e^{-\operatorname{tr} SW}=\mathcal{Z}_{q,t}(-S),\qquad S\succeq 0,$$

if and only if

$$\det\big[(-D)^{j-1}\psi_{q,t}(x_i)\big]_{i,j=1}^{d}\ge 0,\qquad 0<x_1<\cdots<x_d. \tag{A.5}$$

When it exists, the law is unique and unitarily invariant. Its ordered eigenvalues have density

$$\rho_{q,t}^{(d)}(x)=\frac{\Delta(x)}{J_d}\det\big[(-D)^{j-1}\psi_{q,t}(x_i)\big]_{i,j=1}^{d},\qquad 0<x_1<\cdots<x_d. \tag{A.6}$$

In particular, it assigns no mass to matrices of deficient rank or with repeated eigenvalues.

Proof. The integrability bounds in (A.4) make the right side of (A.6) absolutely integrable: expand both the determinant and the Vandermonde into finitely many products of one-variable functions. It therefore defines a finite signed eigenvalue measure on the ordered chamber. Average this measure over normalized Haar measure on $\mathrm{U}(d)$ through the map $(x,U)\mapsto U\operatorname{diag}(x)U^*$, and call the resulting finite signed matrix measure μ.

For pairwise distinct $s_i>0$, the normalized Harish-Chandra–Itzykson–Zuber formula is

$$\int_{\mathrm{U}(d)} e^{-\operatorname{tr}(\operatorname{diag}(s)U\operatorname{diag}(x)U^*)}\,dU=J_d\,\frac{\det[e^{-s_ix_j}]_{i,j=1}^{d}}{\Delta(-s)\Delta(x)}. \tag{A.7}$$

The formula is the analytic continuation of Kieburg and Zhang (2023, equation (2.29)). Multiplying (A.7) by (A.6) cancels J_d and $\Delta(x)$. The product of the two remaining determinants is symmetric in the x_i. Expanding both determinants and applying Fubini's theorem (Andréief's identity) gives

$$\begin{aligned}\int e^{-\operatorname{tr}(\operatorname{diag}(s)W)}\,d\mu(W)&=\frac{\det\big[\int_0^\infty e^{-s_ix}(-D)^{j-1}\psi_{q,t}(x)\,dx\big]_{i,j=1}^{d}}{\Delta(-s)}\\&=\frac{\det[(-s_i)^{j-1}\varphi_{q,t}(s_i)]_{i,j=1}^{d}}{\Delta(-s)}=\prod\nolimits_{i=1}^{d}\varphi_{q,t}(s_i).\end{aligned}$$

The factor $d!$ in Andréief's identity is canceled by restricting the symmetric integral to the ordered chamber. Repeated integration by parts uses $\psi_{q,t}^{(k)}(0)=0$ for $0\le k\le d-2$. Continuity extends the formula to all $s_i\ge 0$. At $s=0$, it shows that μ has total mass one; conjugation invariance proves the required transform for every $S\succeq 0$.

Laplace transforms on the positive cone determine finite signed measures there, since the algebra spanned by $W\mapsto e^{-\operatorname{tr} SW}$, $S\succ 0$, is dense in C_0 of the cone by Stone–Weierstrass. Any probability law with the proposed transform must consequently equal μ. Since the ordered-eigenvalue map pushes μ back to the signed density (A.6), positivity is equivalent to nonnegativity of that density almost everywhere. Continuity on the open chamber makes this equivalent to (A.5). This argument also excludes any additional mass on the boundary. □

A.3. **Finite witnesses for the additive kernel.** The negative square moment is visible on finitely many points near the origin: it is the leading term of a quadratic form in the additive kernel $(S, S') \mapsto L(S + S')$, whose positive definiteness in every dimension is another equivalent condition.

Corollary A.3 (Finite local kernel witnesses)**.** *If $q^{m+1} < t < q^m$ and $r = m + 2$, then arbitrarily close to zero there exist positive definite $S_1, \dots, S_N \in \mathbb{P}^r$ with*

$$[L^{(r)}_{q,t}(S_i + S_j)]^N_{i,j=1} \not\succeq 0, \qquad N \le \binom{r^2 + r}{r}.$$

Positive definiteness of this kernel for every r is equivalent to the conditions of Theorem 4.5.

Proof. Choose a real basis $B_1, \dots, B_d$ of the Hermitian $r \times r$ matrices consisting of rank-one positive semidefinite matrices, so $d = r^2$. There is a real homogeneous polynomial p of degree r with $p(\operatorname{tr}(B_1 W), \dots, \operatorname{tr}(B_d W)) = \det W$. Write $p(z - \mathbf{1}) = \sum_\beta a_\beta z^\beta$ and set $S_\beta = \varepsilon(I + \sum_j \beta_j B_j)$. The formal exponential sum $\sum_\beta a_\beta e^{-\operatorname{tr}(S_\beta W)}$ equals $e^{-\varepsilon \operatorname{tr} W} p(e^{-\varepsilon \operatorname{tr}(B_1 W)} - 1, \dots, e^{-\varepsilon \operatorname{tr}(B_d W)} - 1)$, whose first nonzero homogeneous term is $(-\varepsilon)^r \det W$; squaring and applying (4.6) gives

$$\sum\nolimits_{\beta,\gamma} a_\beta a_\gamma L^{(r)}_{q,t}(S_\beta + S_\gamma) = \varepsilon^{2r} \ell^{(r)}_{q,t}((\det W)^2) + O(\varepsilon^{2r+1}).$$

The leading coefficient is negative by Theorem 4.4, and there are at most $\binom{d+r}{r}$ multi-indices β of degree at most r. On $\mathcal{G}(q)$ the representation (4.12) makes the additive kernel positive definite. □

A.4. **A scalar-basepoint criterion.** Complete monotonicity requires positivity of the derivatives at every positive basepoint. At a scalar basepoint, these derivatives are permanents with a new sequence of cycle weights. For $s \ge 0$, the tilted weights are

$$v_k(s) = \sum\nolimits_{j \ge 0} \left[\left(\frac{q^j}{1 + sq^j} \right)^k - \left(\frac{tq^j}{1 + stq^j} \right)^k \right], \qquad k \ge 1. \tag{A.8}$$

Since $x \mapsto x/(1 + sx)$ is increasing, every $v_k(s)$ is positive, and $v_k(0) = w_k(q, t)$. Normalizing (4.5) at a scalar basepoint gives

$$\frac{L^{(r)}_{q,t}(sI - X)}{L^{(r)}_{q,t}(sI)} = \exp\left(\sum\nolimits_{k \ge 1} \frac{v_k(s)}{k} \operatorname{tr} X^k \right), \qquad \|X\|_{\mathrm{op}} < s + 1,$$

The proof of the master theorem (2.1) uses only the exponential form of $\mathcal{Z}_{q,t}$, so for arbitrary cycle weights $v = (v_k)_{k \ge 1}$, with $\operatorname{per}_v(A) = \sum_\sigma \prod_{C \in \mathrm{Cyc}(\sigma)} v_{|C|} \prod_i a_{i,\sigma(i)}$ and $A \in M_n(\mathbb{C})$,

$$[x_1 \cdots x_n] \exp\left(\sum\nolimits_{k \ge 1} \frac{v_k}{k} \operatorname{tr}((XA)^k) \right) = \operatorname{per}_v(A), \qquad X = \operatorname{diag}(x_1, \dots, x_n). \tag{A.9}$$

Applied to the weights $v(s)$ and $A = [u_i^* u_j]$ with $u_i \in \mathbb{C}^r$, it gives

$$(-1)^n D_{u_1 u_1^*} \cdots D_{u_n u_n^*} L^{(r)}_{q,t}(sI) = L^{(r)}_{q,t}(sI) \operatorname{per}_{v(s)}(A). \tag{A.10}$$

Since every positive definite S lies below a scalar matrix, scalar basepoints suffice.

Theorem A.4. *For $r \ge 1$, $0 < q < 1$, and $0 \le t < 1$, the following are equivalent.*

(i) $L^{(r)}_{q,t}$ is completely monotone on $\mathbb{P}^r$.

(ii) $\operatorname{per}_{v(s)}(A) \ge 0$ for every $s \ge 0$ and every $A \succeq 0$ of rank at most r, in every size.

(iii) The positivity in (ii) holds for s in some unbounded subset of $[0, \infty)$.

More precisely, positivity at $s = s_0$ implies the alternating derivative inequalities at every S with $0 \prec S \preceq s_0 I$.

Proof. By (A.10), spectral decomposition of the directions, and multilinearity, positivity of $\mathrm{per}_{v(s)}$ on rank at most r is the set of alternating derivative inequalities at sI. Given $0 \prec S \preceq sI$, set $H = sI - S \succeq 0$. Since $\|H\|_{\mathrm{op}} < s+1$, Taylor expansion at sI gives

$$(-1)^n D_{B_1}\cdots D_{B_n} L(S) = \sum\nolimits_{a\ge 0} \frac{(-1)^{n+a}}{a!} D_{B_1}\cdots D_{B_n} D_H^a L(sI) \ge 0,$$

each summand being nonnegative. An unbounded set of values s covers every positive definite S, so (iii) implies (i). Clearly (i) implies (ii) and (ii) implies (iii), the case $s=0$ by continuity. □

Appendix B. Gaussian tilting and continuous mixing

Section 5.1 uses two facts about the Poisson mixture behind the count laws on $\mathcal{G}(q)$, proved here: the tilted Gaussian matrix is the mixing law, and infinite divisibility of the counts is not inherited by it. The second fact is why Theorem 5.4 is a statement about counts only.

B.1. **Exponential tilting and coefficient laws.** On $\mathcal{G}(q)$, tilting the Gaussian law identifies its Poisson mixture and the resolvent covariance of each color.

Theorem B.1 (Exponential tilting and the resolvent change of kernel)**.** *Suppose $K \succeq 0$ and $\|K\| < 1$. Let $0 \le q < 1$ and let $W = W_{q,m}(K)$, where either m is a positive integer and $t = q^m$, or $m = \infty$ and $t = 0$. Tilt the distribution of W by*

$$\frac{e^{\mathrm{tr}\, W}}{\mathcal{Z}_{q,t}(K)}. \tag{B.1}$$

Conditional on the tilted matrix, take independent Poisson variables of means W_{ii}. Their joint law is (5.2) at $\beta = 1$. Under this tilt the weighted Gaussian colors have covariance matrices

$$C_r = q^r K(I - q^r K)^{-1}, \qquad 0 \le r < m. \tag{B.2}$$

More precisely, if $i_1, \dots, i_n$ is a list in which index i occurs m_i times, then

$$\mathbb{E}\prod\nolimits_i (N_i)_{m_i} = \sum\nolimits_{\phi:[n]\to\{0,\dots,m-1\}} \prod\nolimits_{r\in \mathrm{im}\,\phi} \mathrm{per}\big([(C_r)_{i_a i_b}]_{a,b\in\phi^{-1}(r)}\big).$$

Here $(x)_j = x(x-1)\cdots(x-j+1)$, and for $m = \infty$ the sum runs over maps into $\mathbb{N}$.

Proof. By the Gaussian transform in Theorem 4.1, $\mathbb{E}e^{\mathrm{tr}\, W} = \mathcal{Z}_{q,t}(K) < \infty$, so (B.1) is normalized. Conditional Poisson expectation gives

$$\mathbb{E}\prod\nolimits_i s_i^{N_i} = \frac{\mathbb{E}e^{\mathrm{tr}\, W}\exp(\sum_i (s_i-1)W_{ii})}{\mathcal{Z}_{q,t}(K)} = \frac{\mathcal{Z}_{q,t}(D_s K)}{\mathcal{Z}_{q,t}(K)}.$$

The master theorem identifies the law from this generating function. To identify the colors, diagonalize K on its range. A Gaussian coordinate of variance $\lambda > 0$ has density $(\pi\lambda)^{-1}\exp(-|z|^2/\lambda)$ on $\mathbb{C}$. Multiplication by $\exp(q^r|z|^2)$ changes its variance to $\lambda/(1-q^r\lambda)$; the inequality $\|K\| < 1$ ensures that this variance is finite. Coordinates of variance zero remain zero. Thus color r has tilted covariance $K(I-q^rK)^{-1}$, and multiplying the color itself by $q^{r/2}$ gives (B.2). The tilt factors over colors, so their independence is preserved.

For infinitely many colors, let $W^{(R)}$ be the sum through color R. The densities $e^{\mathrm{tr}\, W^{(R)}}/\mathbb{E}e^{\mathrm{tr}\, W^{(R)}}$ converge in L^1 to (B.1) by monotone convergence, so every finite color marginal is the product of the tilts just computed. Moreover,

$$\sum\nolimits_{r\ge 0} \mathrm{tr}\, C_r \le \frac{\mathrm{tr}\, K}{1-\|K\|}\sum\nolimits_{r\ge 0} q^r < \infty,$$

so the tilted intensity remains finite. This bound and $\sup_r \|C\|_r < \infty$ also give a finite exponential moment of its trace near zero, by the Gaussian determinant formula. Thus all the factorial moments below are finite. Conditional Poisson factorial moments equal products of the means. Expanding those products over color assignments and applying the complex Gaussian moment formula proves

the stated permanent sum. For infinitely many colors, the products of diagonal intensities increase to their limits, so monotone convergence justifies the expansion. □

For one color, the probability coefficients involve $\mathrm{per}(K[\mathbf{m}])$, whereas factorial moments involve $\mathrm{per}(C[\mathbf{m}])$ with $C = K(I-K)^{-1}$. For several colors the C_r are not geometric multiples of one matrix, so the resolvent change applies to each color separately.

B.2. **Counts and their continuous mixing laws.** The criterion concerns the count kernel K. It is different from the inverse-M-matrix criterion for a continuous multivariate gamma law. For a real positive definite covariance C, Bapat (1989) characterizes infinite divisibility of $\det(I+CD_u)^{-1/2}$ by the existence of a signature S such that $SC^{-1}S$ is an M-matrix. In the symmetric positive definite case this means that its off-diagonal entries are nonpositive. The same criterion characterizes existence of $\det(I+CD_u)^{-\beta}$ for every $\beta > 0$; see also Eisenbaum and Maunoury (2017, Theorem 5.1). For example, take

$$K = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Its eigenvalues are $1/2$ and $(2\pm\sqrt{2})/4$, so it is a positive definite contraction. Its coefficient laws are infinitely divisible by Corollary 5.5. At the classical point $(q,t)=(0,0)$, the corresponding Gaussian mixing covariance is $C = K(I-K)^{-1}$, and

$$C^{-1} = K^{-1} - I = \begin{pmatrix} 3 & -2 & -2 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}.$$

The product of its three off-diagonal entries is positive. Signature conjugation preserves this product, whereas three nonzero nonpositive off-diagonal entries have negative product. Thus no signature makes C^{-1} an M-matrix. The counts are infinitely divisible, but their mixing law is not.

Appendix C. The finite-rank determinantal formula

Theorem 5.8 applies Okounkov's correlation formula at a specialization that is not Schur-positive in all lengths. The formula is an identity of symmetric functions, so it extends degree by degree to formal specializations; what needs proof is that it may be evaluated at the given parameters, and this is where the restriction to d rows enters.

Proof of Theorem 5.8. For a specialization B of the symmetric functions, write

$$H_B(v) = \sum\nolimits_{j\ge 0} h_j[B]v^j = \exp\left(\sum\nolimits_{j\ge 1} \frac{p_j[B]}{j} v^j\right).$$

For ordinary finite alphabets a, b, Okounkov (2001, Theorems 1 and 2) give the correlation formula with $J(z) = H_a(z)/H_b(z^{-1})$. His generating function is $\sqrt{zw}\,J(z)/(J(w)(z-w))$, with coefficient $z^x w^{-y}$; extracting this coefficient gives (5.8). In particular, the powers in the integrand are integers, so the contour formula requires no choice of a square root. The identity extends to a formal specialization B degree by degree. To see this, replace a by ua and expand at $u=0$. For a fixed set of coordinates, the coefficient of u^n on the probability side is a symmetric function of B of degree n: only partitions of size at most n enter after division by the normalizing series. The same statement holds on the determinant side. Indeed, expand

$$H_{ua}(z) = \sum\nolimits_{p\ge 0} u^p h_p(a) z^p, \qquad H_{ua}(w)^{-1} = \sum\nolimits_{s=0}^{d} (-u)^s e_s(a) w^s,$$

and expand $H_B(w^{-1})$, $H_B(z^{-1})^{-1}$, and $(z-w)^{-1}$ in their indicated powers. At a fixed power u^n, coefficient extraction for $\mathsf{K}(x,y)$ leaves finitely many terms, each of degree $n+y-x$ in B. In every determinant term the coordinate differences sum to zero. Thus its coefficient of u^n has degree n in

B. Evaluation on finite alphabets with at least n variables is injective on symmetric functions of degree n. Since the identity holds on those alphabets, it holds before evaluation and hence after the substitution $p_j[B] = w_j$. It remains to justify evaluating the resulting formal identity at $u = 1$. For $B = B_{q,t}$,

$$H_B(v) = H(v) = \frac{(tv;q)_\infty}{(v;q)_\infty}.$$

Both J and J^{-1} are analytic in $1 < |z| < a_*^{-1}$. Keep the two contours fixed and replace J by $J_u(z) = H_{ua}(z)/H(z^{-1})$. The contour integrals are analytic for $|u| < (a_* r_z)^{-1}$, a disk containing $u = 1$. On the probability side, the unnormalized correlation series converges absolutely for $|u|a_* < 1$, because its absolute value is bounded by

$$\sum\nolimits_\lambda |u|^{|\lambda|} s_\lambda(a) c_\lambda(q,t) = \prod\nolimits_{i=1}^{d} H(|u|a_i).$$

Here the nonnegativity of $c_\lambda(q,t)$ is needed only for $\ell(\lambda) \le d$. The normalizing function $\prod_i H(ua_i)$ is analytic and nonzero on this disk. The formal identity therefore gives equality of analytic functions near $u = 0$, and analytic continuation gives the claimed equality at $u = 1$. Formula (5.9) follows from Cauchy's coefficient formula and the nesting $r_w < r_z$. □