hexagon:2610.00729v1

# THE MACDONALD $(q,t)$-PERMANENT I: ALGEBRA AND POSITIVITY

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Abstract. We introduce the Macdonald $(q,t)$-permanent, a two-parameter cycle-weighted permanent that contains the usual permanent and determinant, as well as the $\alpha$-permanent of Vere-Jones, as degenerate cases. We prove a MacMahon-type master theorem for its generating function, and determine when it induces a valid kernel on tuples of vectors. Nonnegativity of the $(q,t)$-permanent itself on semidefinite matrices turns out to be subtler: for $t=q^\alpha$ nonnegativity holds exactly at the positive Wallach exponents; for a fixed $q$ we determine the region in rank two and for higher ranks under an additional condition, while for small $q$ positivity can even reach beyond the kernel region. Whether positivity forces $t\in\{0,q,q^2,\dots\}$ remains an intriguing open problem.

## 1. Introduction

The $\alpha$-permanent of Vere-Jones (1988) weights each cycle of a permutation by the same scalar $\alpha$. It interpolates between the permanent and the determinant, and through it gamma laws, Wishart moments, loop occupation fields, and permanental point processes are expressed (Vere-Jones, 1997; Shirai and Takahashi, 2003; McCullagh and Møller, 2006; Li and Zheng, 2023). In each of these uses it is evaluated on a positive semidefinite matrix, and its value is a moment, so it must be nonnegative. It is known that, for $\alpha>0$, $\mathrm{per}_\alpha\ge 0$ on positive semidefinite matrices of every size exactly when $\alpha$ is a positive integer (Brändén, 2012, Theorem 2.3).

We introduce a deformation in which the weight also depends on the cycle length, suggested by Macdonald's power-sum scalar product. For $0\le q,t<1$ and $A=(a_{ij})\in M_n(\mathbb{C})$ we define

$$\mathrm{per}_{q,t}(A):=\sum\nolimits_{\sigma\in S_n} w_\sigma \prod\nolimits_{i=1}^{n} a_{i,\sigma(i)},\qquad w_\sigma:=\prod\nolimits_{k\in \mathrm{Cyc}(\sigma)} w_k(q,t),\qquad w_k(q,t):=\frac{1-t^k}{1-q^k}. \tag{1.1}$$

Here $\mathrm{Cyc}(\sigma)$ includes fixed points, and $w_k=w_k(q,t)$. We call this function the *Macdonald $(q,t)$-permanent*; its dependence on cycle length distinguishes it from the inversion-weighted $q$-permanent of Bapat and Lal (1994) and from the quantum immanants of Okounkov (1996). On $q=t$ all cycle weights are one, so $\mathrm{per}_{q,q}=\mathrm{per}$, and for $t=q^\alpha$ they tend to $\alpha$ as $q\uparrow 1$, recovering $\mathrm{per}_\alpha$.

The key property of $\mathrm{per}_{q,t}$ that we study is this paper is: *for which $(q,t)$ is it nonnegative on positive semidefinite matrices?* Two identities make this question accessible. The first is a MacMahon-type master theorem, where we introduce the generating function

$$\mathcal{Z}_{q,t}(M):=\exp\left(\sum\nolimits_{k\ge1}\frac{w_k}{k}\operatorname{tr}M^k\right)=\prod\nolimits_{j\ge0}\frac{\det(I-tq^jM)}{\det(I-q^jM)}, \tag{1.2}$$

which generates $\mathrm{per}_{q,t}$ on matrices with repeated indices. Here $\mathrm{per}_{q,t}$ is the square-free polarization of its degree-$n$ component, the one-row Macdonald function $Q_{(n)}$. The second identity comes from expanding $Q_{(n)}$ in Schur functions

$$\mathrm{per}_{q,t}(A)=\sum\nolimits_{\lambda\vdash n} c_\lambda(q,t)\,\mathrm{Imm}_\lambda(A),\qquad c_\lambda(q,t)=\prod\nolimits_{(i,j)\in\lambda}\frac{q^{i-1}-tq^{j-1}}{1-q^{h_{ij}}},$$

2020 *Mathematics Subject Classification.* Primary 15A15, 05E05; Secondary 15B48, 20C30, 32M15.

*Key words and phrases.* Macdonald permanent, alpha-permanent, immanant, kernel, Wallach set.

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First public release: 29 October 2026, as v1 in the author's preprint repository, https://github.com/suvrit/preprints/tree/main/preprints/macperm1; every version is kept there with its timestamp.

where $\mathrm{Imm}_\lambda$ is the irreducible immanant and $h_{ij}$ is a hook length (Section 2).

For tuples $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ of vectors in $\mathbb{C}^d$ we define the "kernel"

$$\mathscr{K}_{q,t}^{n,d}(\mathbf{u}, \mathbf{v}) := \mathrm{per}_{q,t}([\langle u_i, v_j \rangle]_{i,j=1}^n). \tag{1.3}$$

The first problem asks whether (1.3) is a true kernel, i.e., whether it is positive semidefinite; the second one asks if $\mathrm{per}_{q,t}(A) \geq 0$ for every positive semidefinite $A$. For $0 < q < 1$ and $r \geq 1$, put

$$\mathcal{R}_r(q) := ([0, q^{r-1}] \cup \{q, q^2, \ldots, q^{r-2}\}) \cap [0, 1).$$

Then, the first main result of our paper is the following.

**Theorem 1.1** (Kernel region)**.** *Let $n, d \geq 1$, $0 < q < 1$, $0 \leq t < 1$, and $r = \min(n, d)$. The kernel (1.3) is positive semidefinite if and only if $t \in \mathcal{R}_r(q)$. In particular, $\mathcal{R}_1(q) = [0, 1)$. For every $t \leq 0$, the kernel is positive semidefinite for all $n, d \geq 1$.*

The proof is Schur–Weyl duality: since $w_\sigma$ is a class function, the operator $\sum_\sigma w_\sigma U_\sigma$ that permutes the factors of $(\mathbb{C}^d)^{\otimes n}$ acts on each isotypic component as a positive multiple of $c_\lambda(q, t)$, so positivity amounts to $c_\lambda(q, t) \geq 0$ for partitions with at most $d$ rows (Section 3). The same sign test governs a matrix-ball kernel and the total positivity of a scalar sequence (Theorem 3.5). As $r$ grows, the regions shrink to $\{0, q, q^2, \ldots\}$ (Corollary 3.2).

On $\mathcal{R}_r(q)$ the Schur expansion also gives scalar positivity on matrices of rank at most $r$, in every size (Theorem 4.1). Is this region also necessary? Along $t = q^\alpha$ it becomes the positive part of the Wallach set for $r \times r$ matrices (Faraut and Korányi, 1990), $\mathcal{W}_r := \{\alpha > 0 : \alpha \in \{1, 2, \ldots\} \text{ or } \alpha \geq r-1\}$, and across the whole deformation the answer is yes.

**Theorem 1.2** (Positivity throughout the deformation)**.** *Fix $r \geq 1$ and $\alpha > 0$, and let $A$ range over the complex positive semidefinite matrices of every size and of rank at most $r$. The following are equivalent: (i) $\alpha \in \mathcal{W}_r$; (ii) $\mathrm{per}_{q,q^\alpha}(A) \geq 0$ for every $q \in (0, 1)$ and every such $A$; (iii) $\mathrm{per}_\alpha(A) \geq 0$ for every such $A$. If $\alpha \notin \mathcal{W}_r$, one such $A$ has $\mathrm{per}_{q,q^\alpha}(A) < 0$ for all $q$ in an interval $(q_0, 1)$.*

The proof passes to the limit $q \uparrow 1$, where the negative matrices of Brändén (2012, Theorem 2.3) have rank at most $\lfloor \alpha \rfloor + 2$ (Section 4.1). Intersecting over $r$ extends the positive-exponent half of Brändén's theorem across the deformation (Corollary 4.2).

At a fixed $q$ the picture is subtler. The region $\mathcal{R}_r(q)$ is still exact in rank two, and it remains exact in every rank $r \geq 3$ once $q$ exceeds an explicit threshold (Theorem 4.3, Corollary 4.8). The negative matrices come from random binary polynomials in rank two and from random projections in higher rank. For small $q$, however, scalar positivity reaches beyond $\mathcal{R}_r(q)$ at every finite rank (Theorem 4.9). Whether scalar positivity in all ranks forces $t \in \{0, q, q^2, \ldots\}$ remains open (Conjecture 6.1). The exact regions in sizes three and four are in Appendix C.

Our tools are classical: Macdonald's symmetric-function theory (Macdonald, 1995, Chapters I and VI), the Schur-function and immanant correspondence of Goulden and Jackson (1992), the Wallach set (Faraut and Korányi, 1990); and also the notable construction of Brändén (2012). The Jack limit of our master theorem is the $\alpha$-extension of MacMahon's (Foata and Zeilberger, 1988; Vere-Jones, 1988), and the block inequalities of Theorem 5.1 transfer Lieb's permanent inequality to $\mathrm{per}_{q,t}$ along Frenkel's partition argument (Frenkel, 2010).

## 2. Cycle weights and the Schur expansion

Both positivity problems of the introduction turn on the signs of the hook-content coefficients $c_\lambda(q, t)$, so we derive these first. They appear once $\mathrm{per}_{q,t}$ is read as a symmetric function: the cycle weights $w_k$ are the power sums of a Macdonald specialization, square-free polarization turns power sums into permutation sums, and the coefficients are Schur functions evaluated at that specialization.

2.1. **Notation.** We write $[n]=\{1,\dots,n\}$, $\mathbb{N}=\{0,1,\dots\}$, and $M_{d,p}(\mathbb{C})=\mathbb{C}^{d\times p}$; when $d=p$ we write $M_d(\mathbb{C})$. For $A\in M_n(\mathbb{C})$ and $S\subseteq[n]$, the matrix $A[S]$ is the principal submatrix indexed by $S$. Empty determinants and permanents, including $\mathrm{per}_{q,t}(\varnothing)$, are 1. The notation $A\succeq 0$ means that $A$ is Hermitian positive semidefinite. We use the operator norm $\|\cdot\|_{\mathrm{op}}$, the spectral radius $\rho(\cdot)$, and inner products that are conjugate-linear in the first argument.

A partition $\lambda=(\lambda_1,\dots,\lambda_\ell)$ is a weakly decreasing sequence of positive integers. Its size is $|\lambda|=\sum_i\lambda_i$, its length is $\ell(\lambda)=\ell$, and $\lambda\vdash n$ means $|\lambda|=n$. We identify $\lambda$ with its diagram $\{(i,j):1\le i\le\ell(\lambda),\,1\le j\le\lambda_i\}$ and write $\lambda'_j=\#\{i:\lambda_i\ge j\}$ for the conjugate partition. The hook length of its cell $(i,j)$ is

$$h_{ij}=\lambda_i-j+\lambda'_j-i+1.$$

If $\mu=1^{m_1}2^{m_2}\cdots$ records the multiplicities of its parts, then

$$z_\mu=\prod\nolimits_{k\ge1}k^{m_k}m_k!,\qquad w_\mu=\prod\nolimits_i w_{\mu_i}.$$

There are $n!/z_\mu$ permutations of cycle type $\mu\vdash n$.

For an alphabet $Z=(z_1,z_2,\dots)$, let $p_k(Z)=\sum_i z_i^k$ and $p_\mu=\prod_i p_{\mu_i}$. We work degree by degree in the symmetric-function algebra $\Lambda_{\mathbb{Q}(q,t)}=\mathbb{Q}(q,t)[p_1,p_2,\dots]$, where $\deg p_k=k$. A specialization is an algebra homomorphism from this algebra, specified by its values on the $p_k$; it need not be evaluation at a sequence of numbers. The complete symmetric functions $h_n$ are defined by

$$\sum\nolimits_{n\ge0}h_n(Z)u^n=\exp\left(\sum\nolimits_{k\ge1}\frac{p_k(Z)u^k}{k}\right).$$

The Schur function $s_\lambda(Z)$ is the sum of the monomials $\prod_{(i,j)\in\lambda}z_{T(i,j)}$ over semistandard tableaux $T$ of shape $\lambda$. Thus $s_\lambda(Z)\ge0$ on a nonnegative alphabet, and $s_\lambda(z_1,\dots,z_d)=0$ if $\ell(\lambda)>d$. These conventions follow (Macdonald, 1995, Chapter I, Sections 2–3).

Let $\chi^\lambda$ denote the irreducible character of $S_n$ indexed by $\lambda\vdash n$, and put $f^\lambda=\chi^\lambda(e)$. We use the unnormalized irreducible immanant

$$\mathrm{Imm}_\lambda(A)=\sum\nolimits_{\sigma\in S_n}\chi^\lambda(\sigma)\prod\nolimits_{i=1}^n a_{i,\sigma(i)}.$$

Thus $\mathrm{Imm}_{(n)}=\mathrm{per}$ and $\mathrm{Imm}_{(1^n)}=\det$. The Frobenius identity

$$s_\lambda=\sum\nolimits_{\mu\vdash n}\frac{\chi^\lambda(\mu)}{z_\mu}p_\mu$$

relates this normalization to symmetric functions (Macdonald, 1995, Chapter I, Section 7).

Our convention for the Vere-Jones parameter is

$$\mathrm{per}_\alpha(A)=\sum\nolimits_{\sigma\in S_n}\alpha^{c(\sigma)}\prod\nolimits_i a_{i,\sigma(i)},$$

where $c(\sigma)$ is the number of cycles. In particular, $\mathrm{per}_1=\mathrm{per}$ and $\mathrm{per}_{-1}=(-1)^n\det$. The alternative convention $\det_\eta(A)=\sum_\sigma\eta^{n-c(\sigma)}\prod_i a_{i,\sigma(i)}$ satisfies $\det_\eta(A)=\eta^n\,\mathrm{per}_{1/\eta}(A)$ for $\eta\neq0$. The standard Jack parameter is the reciprocal of $\alpha$ in the degeneration considered below.

2.2. **Polarization and the master theorem.** The definition (1.1) sums over permutations, whereas Macdonald's theory works with power sums, that is, with traces of matrix powers. The bridge is polarization: evaluate a symmetric function at $XA$, with $X=\mathrm{diag}(x_1,\dots,x_n)$ a diagonal matrix of variables, and keep the coefficient of $x_1\cdots x_n$. The trace of a matrix power is a sum over closed walks on the index set, whereas a permutation is a collection of disjoint cycles, that is, of closed walks with distinct vertices. Extracting the coefficient of $x_1\cdots x_n$ keeps exactly the walks whose vertices are distinct. This observation transfers symmetric-function identities to matrix identities without assuming that the matrix is diagonalizable.

For a square matrix $M$, we evaluate a symmetric function $f$ by the algebra homomorphism $p_k\mapsto\mathrm{tr}\,M^k$. If the eigenvalues of $M$, counted with algebraic multiplicity, are $\xi_1,\dots,\xi_N$, this

evaluation agrees with $f(\xi_1,\dots,\xi_N)$. For a homogeneous symmetric function of degree $n$ and a matrix $A \in M_n(\mathbb{C})$, define its square-free matrix polarization by

$$\mathfrak{M}_n(f)(A) := [x_1 \cdots x_n] f(XA), \qquad X = \operatorname{diag}(x_1,\dots,x_n).$$

**Lemma 2.1** (Polarization dictionary)**.** *If $f = \sum_{\mu\vdash n} a_\mu p_\mu$, then*

$$\mathfrak{M}_n(f)(A) = \sum\nolimits_{\sigma\in S_n} z_{\rho(\sigma)} a_{\rho(\sigma)} \prod\nolimits_{i=1}^{n} a_{i,\sigma(i)}, \tag{2.1}$$

*where $\rho(\sigma)$ is the cycle type of $\sigma$. In particular, $\mathfrak{M}_n(s_\lambda) = \operatorname{Imm}_\lambda$.*

*Proof.* Expanding a trace gives

$$\operatorname{tr}((XA)^k) = \sum\nolimits_{i_1,\dots,i_k=1}^{n} x_{i_1}\cdots x_{i_k} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

A term contributing to $[x_1 \cdots x_n] p_\mu(XA)$ uses every label exactly once across all trace factors. Each factor therefore specifies a cycle on distinct labels, and different factors use disjoint labels. For a fixed permutation of type $\mu = 1^{m_1} 2^{m_2} \cdots$, each $k$-cycle has $k$ possible starting points in its trace. Its $m_k$ cycles of length $k$ can be assigned to the corresponding trace factors in $m_k!$ orders. The resulting multiplicity is $\prod_k k^{m_k} m_k! = z_\mu$, which proves (2.1).

The Frobenius identity gives $\mathfrak{M}_n(s_\lambda) = \operatorname{Imm}_\lambda$, as in Goulden and Jackson (1992). □

The one-row Macdonald functions are characterized by the generating function (Macdonald, 1995, Chapter VI, Sections 2 and 4)

$$\sum\nolimits_{n\ge 0} Q_{(n)}(Z;q,t) u^n = \exp\left( \sum\nolimits_{k\ge 1} \frac{w_k}{k} p_k(Z) u^k \right). \tag{2.2}$$

The identity holds over $\mathbb{C}(q,t)$ and specializes wherever its coefficients are defined; the full Macdonald basis and its polarizations are in Supplement I (Sra, 2026b).

The positivity arguments evaluate $\operatorname{per}_{q,t}$ on Gram matrices whose vectors repeat and on matrices of every size at once, and the generating function $\mathcal{Z}_{q,t}$ does both: allowing repeated labels in the polarization turns its Taylor coefficients into $\operatorname{per}_{q,t}$ of the matrices with repeated rows and columns.

**Proposition 2.2** (Macdonald polarization and repeated indices)**.** *For every $A \in M_n(\mathbb{C})$,*

$$\operatorname{per}_{q,t}(A) = [x_1 \cdots x_n] Q_{(n)}(XA;q,t). \tag{2.3}$$

*For $A \in M_N(\mathbb{C})$ and $\mathbf{m} = (m_1,\dots,m_N) \in \mathbb{N}^N$, let $A[\mathbf{m}]$ be the matrix whose rows and columns are indexed by the pairs $(i,a)$ with $1 \le i \le N$ and $1 \le a \le m_i$, and whose $((i,a),(j,b))$ entry is $a_{ij}$. Then, as a formal power series,*

$$\mathcal{Z}_{q,t}(XA) = \sum\nolimits_{\mathbf{m}\in\mathbb{N}^N} \frac{\operatorname{per}_{q,t}(A[\mathbf{m}])}{\mathbf{m}!} x^{\mathbf{m}}, \qquad \mathbf{m}! := \prod\nolimits_{i=1}^{N} m_i!, \quad x^{\mathbf{m}} := \prod\nolimits_{i=1}^{N} x_i^{m_i}.$$

*For $|q| < 1$ and fixed $t \in \mathbb{C}$, the determinant product in (1.2) agrees with the trace exponential near $M = 0$. For $0 \le q, t < 1$, both define the same analytic function on $\{M : \rho(M) < 1\}$.*

*Proof.* Expanding the exponential in (2.2) by cycle lengths gives

$$Q_{(n)}(Z;q,t) = \sum\nolimits_{\mu\vdash n} \frac{w_\mu}{z_\mu} p_\mu(Z).$$

Lemma 2.1 proves (2.3). Since $Q_{(j)}(XA)$ is homogeneous of degree $j$ in $x$, the same coefficient can be extracted from $\mathcal{Z}_{q,t}(XA) = \sum_{j\ge 0} Q_{(j)}(XA;q,t)$.

To account for repeated labels, put $B = A[\mathbf{m}]$ and introduce one variable $y_{ia}$ for each of its indices. Write $Y = \operatorname{diag}(y_{ia})$ and $z_i = \sum_{a=1}^{m_i} y_{ia}$, with an empty sum equal to zero. Expansion of the trace shows that

$$\operatorname{tr}((YB)^k) = \sum\nolimits_{i_1,\dots,i_k} z_{i_1}\cdots z_{i_k} a_{i_1 i_2}\cdots a_{i_k i_1} = \operatorname{tr}((\operatorname{diag}(z_1,\dots,z_N)A)^k).$$

Consequently $\mathcal{Z}_{q,t}(YB) = \mathcal{Z}_{q,t}(\operatorname{diag}(z_1,\dots,z_N)A)$. In a monomial $z^{\mathbf{k}}$, the coefficient of $\prod_{i,a} y_{ia}$ vanishes unless $\mathbf{k} = \mathbf{m}$; in that case it equals $\mathbf{m}!$. The square-free identity for $B$ thus gives

$$\operatorname{per}_{q,t}(A[\mathbf{m}]) = \mathbf{m}!\,[x^{\mathbf{m}}]\mathcal{Z}_{q,t}(XA),$$

including $\mathbf{m} = \mathbf{0}$ by the empty-matrix convention.

For the determinant identity, take $M$ sufficiently small that $\|M\| < 1$ and $|t|\|M\| < 1$. Absolute convergence permits the expansion

$$\begin{aligned}\log \prod\nolimits_{r\ge 0} \frac{\det(I - tq^r M)}{\det(I - q^r M)} &= \sum\nolimits_{r\ge 0}\sum\nolimits_{k\ge 1} \frac{q^{rk} - (tq^r)^k}{k} \operatorname{tr} M^k \\ &= \sum\nolimits_{k\ge 1} \frac{1 - t^k}{1 - q^k} \frac{\operatorname{tr} M^k}{k}.\end{aligned}$$

with the logarithms that vanish at $M = 0$. Exponentiation proves the product formula near zero.

For $0 \le q, t < 1$ and $\rho(M) \le a < 1$, $0 < w_k \le (1-q)^{-1}$ and $|\operatorname{tr} M^k| \le N a^k$, so the trace series converges locally uniformly on $\rho(M) < 1$ and its exponential is analytic there; the determinant product converges locally uniformly as well, its factors differing from 1 by $O(q^r)$, and the two agree by analytic continuation from zero. At $q = 0$ only the factor $r = 0$ remains. □

The rank-one case of the master theorem computes the constant $H_n(q,t)$ below, which is the one-row Schur coefficient and hence the constant in the scalar lower bound of Theorem 4.1; it also gives the elementary symmetries used throughout. We use $(a;q)_\infty = \prod_{j\ge 0}(1 - aq^j)$ when $|q| < 1$.

**Corollary 2.3** (Elementary normalizations)**.** *Put*

$$H_n(q,t) := \frac{(t;q)_n}{(q;q)_n}, \qquad (a;q)_n := \prod\nolimits_{j=0}^{n-1}(1 - aq^j). \tag{2.4}$$

*Then*

$$\operatorname{per}_{q,t}(I_n) = w_1^n, \qquad \operatorname{per}_{q,t}(uv^{\mathsf{T}}) = n!H_n(q,t)\prod\nolimits_{i=1}^n u_i v_i.$$

*For square matrices $A, B$, one has $\operatorname{per}_{q,t}(A \oplus B) = \operatorname{per}_{q,t}(A)\operatorname{per}_{q,t}(B)$. For $n \times n$ diagonal matrices $D, E$, one has*

$$\operatorname{per}_{q,t}(DAE) = \Big(\prod\nolimits_{i=1}^n D_{ii}E_{ii}\Big)\operatorname{per}_{q,t}(A).$$

*If $A \succeq 0$ and $0 \le q, t < 1$, then*

$$|\operatorname{per}_{q,t}(A)| \le n!H_n(q,t)\prod\nolimits_{i=1}^n a_{ii}. \tag{2.5}$$

*Proof.* Only the identity permutation contributes on $I_n$. To evaluate the rank-one case, define

$$F(z) := \frac{(tz;q)_\infty}{(z;q)_\infty} = \sum\nolimits_{j\ge 0} h_j z^j.$$

Removing the first factors gives $(1 - z)F(z) = (1 - tz)F(qz)$. Comparing coefficients yields

$$h_0 = 1, \qquad (1 - q^j)h_j = (1 - tq^{j-1})h_{j-1} \quad (j \ge 1),$$

and hence $h_j = H_j(q,t)$. For $M = Xuv^{\mathsf{T}}$, $\operatorname{tr} M^k = (\sum_i x_i u_i v_i)^k$. The master theorem reduces to $F(\sum_i x_i u_i v_i)$, whose square-free coefficient is $n!H_n \prod_i u_i v_i$.

The direct-sum and diagonal-scaling identities hold term by term in (1.1). For (2.5), $|a_{ij}| \le \sqrt{a_{ii}a_{jj}}$ gives $|\prod_i a_{i,\sigma(i)}| \le \prod_i a_{ii}$, and the weights $w_\sigma \ge 0$ sum to $\operatorname{per}_{q,t}(\mathbf{1}\mathbf{1}^{\mathsf{T}}) = n!H_n$. □

2.3. **The hook-content coefficients.** Since $w_\sigma$ is a class function on $S_n$, $\mathrm{per}_{q,t}$ is a linear combination of the irreducible immanants $\mathrm{Imm}_\lambda$. Its coefficients are explicit products over the cells of $\lambda$. Their signs determine the kernel region and give a sufficient condition for scalar positivity. The coefficients are evaluations at the specialization

$$p_k[B_{q,t}] = w_k, \qquad B_{q,t} = \frac{1-t}{1-q},$$

This is plethystic notation for the specialization $p_k \mapsto w_k$, rather than a quotient of alphabets. In particular, $p_k[ZB_{q,t}] = p_k(Z)w_k$.

**Proposition 2.4** (Hook-content immanant expansion)**.** *For $0 < q < 1$, arbitrary real $t$, and $A \in M_n(\mathbb{C})$,*

$$\mathrm{per}_{q,t}(A) = \sum\nolimits_{\lambda\vdash n} c_\lambda(q,t)\,\mathrm{Imm}_\lambda(A), \qquad c_\lambda(q,t) = \prod\nolimits_{(i,j)\in\lambda} \frac{q^{i-1} - tq^{j-1}}{1-q^{h_{ij}}}. \tag{2.6}$$

*The same identity holds at $q = 0$ by taking limits.*

*Proof.* By the generating function of the $h_n$ and (2.2), $Q_{(n)}(Z;q,t) = h_n[ZB_{q,t}]$, and the Schur Cauchy identity (Macdonald, 1995, Chapter I, Section 4) gives

$$h_n[ZB_{q,t}] = \sum\nolimits_{\lambda\vdash n} s_\lambda[B_{q,t}]s_\lambda(Z).$$

Lemma 2.1 therefore identifies $c_\lambda(q,t)$ with $s_\lambda[B_{q,t}]$.

To evaluate this specialization, first take $t = q^m$ for an integer $m \geq 1$. Its power sums are those of the finite alphabet $(1, q, \ldots, q^{m-1})$. The principal specialization of a Schur function (Macdonald, 1995, Chapter I, Section 3) is

$$s_\lambda(1,q,\ldots,q^{m-1}) = q^{\sum_i (i-1)\lambda_i} \prod\nolimits_{(i,j)\in\lambda} \frac{1-q^{m+j-i}}{1-q^{h_{ij}}} = \prod\nolimits_{(i,j)\in\lambda} \frac{q^{i-1} - q^m q^{j-1}}{1-q^{h_{ij}}}.$$

For fixed $q \in (0,1)$, both $s_\lambda[B_{q,t}]$ and the product in (2.6) are polynomials in $t$ of degree at most $n$, and they agree at the infinitely many points $t = q^m$, hence identically. As $q \downarrow 0$ every denominator has a nonzero limit, which gives the boundary identity. □

The diagonal specialization $q = t$ gives the permanent. For $t = q^\alpha$, $\alpha > 0$, each cycle weight tends to $\alpha$ as $q \uparrow 1$, so the finite permutation sum tends to $\mathrm{per}_\alpha$. At $t = q^m$, $m \geq 1$ integral, the weights are the power sums of the alphabet $(1, q, \ldots, q^{m-1})$:

$$w_k(q, q^m) = \sum\nolimits_{a=0}^{m-1} q^{ak}.$$

The boundary formulas and their proofs are collected in Proposition E.1.

## 3. Kernel positivity and total positivity

This section proves Theorem 1.1. Because $w_\sigma$ is a class function, the kernel (1.3) is a matrix coefficient of a central element of the group algebra of $S_n$ acting on $(\mathbb{C}^d)^{\otimes n}$, and Schur–Weyl duality diagonalizes such elements with the coefficients $c_\lambda(q,t)$ as eigenvalues; kernel positivity is therefore exactly their sign test. The same test reappears before polarization, for the determinant product on matrix balls and for total positivity of $H_n$, which shows that $\mathcal{R}_r(q)$ belongs to the specialization itself rather than to the permanent. The necessity half uses linear combinations of simple tensors, which a single Gram matrix need not detect, so it says nothing yet about scalar positivity.

In terms of quadratic forms, a kernel $\mathscr{K}$ on tuples is positive semidefinite if, for every finite family $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(L)}$ and every $z_1, \ldots, z_L \in \mathbb{C}$,

$$\sum\nolimits_{a,b=1}^{L} \overline{z_a} z_b \mathscr{K}(\mathbf{u}^{(a)}, \mathbf{u}^{(b)}) \geq 0.$$

Each immanant term of (2.6) is such a kernel, which is the sufficiency half of the sign test.

**Lemma 3.1** (Immanants on Gram matrices)**.** *For $\lambda \vdash n$, the kernel $\mathrm{Imm}_\lambda([\langle u_i, v_j\rangle])$ on $(\mathbb{C}^d)^n$ is positive semidefinite. It vanishes identically when $\ell(\lambda) > d$.*

*Proof.* Let $P_\lambda$ be the central orthogonal projection onto the $\lambda$-isotypic component of $(\mathbb{C}^d)^{\otimes n}$. The central idempotent formula gives

$$\mathrm{Imm}_\lambda([\langle u_i, v_j\rangle]) = \frac{n!}{f^\lambda}\langle u_1 \otimes \cdots \otimes u_n, P_\lambda(v_1 \otimes \cdots \otimes v_n)\rangle.$$

Applying $P_\lambda$ to both tensors makes this a Gram kernel. Schur–Weyl decomposition has no such component when $\ell(\lambda) > d$. □

***Proof of Theorem 1.1.*** Let $\mathcal{T} = (\mathbb{C}^d)^{\otimes n}$ with its tensor-product inner product, and let

$$U_\sigma(v_1 \otimes \cdots \otimes v_n) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

The $U_\sigma$ form a unitary representation of $S_n$, and $T = \sum_{\sigma \in S_n} w_\sigma U_\sigma$ is self-adjoint, since inverse permutations have the same weight, with

$$\langle u_1 \otimes \cdots \otimes u_n,\, T(v_1 \otimes \cdots \otimes v_n)\rangle = \mathscr{K}_{q,t}^{n,d}(\mathbf{u}, \mathbf{v}).$$

Simple tensors span $\mathcal{T}$, so kernel positivity is equivalent to $T \succeq 0$.

Schur–Weyl decomposition separates the irreducible $S_n$ actions:

$$\mathcal{T} \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \le d}} S_\lambda(\mathbb{C}^d) \otimes V_\lambda.$$

Here $V_\lambda$ is the irreducible $S_n$-module with character $\chi^\lambda$ and $S_\lambda(\mathbb{C}^d)$ its multiplicity space, the Schur module, which is nonzero exactly when $\ell(\lambda) \le d$.

Since $w_\sigma$ depends only on cycle type, Schur's lemma makes $T$ a scalar on each $V_\lambda$ factor. Taking its trace on that factor gives the scalar

$$\frac{1}{f^\lambda}\sum\nolimits_{\sigma \in S_n} w_\sigma \chi^\lambda(\sigma) = \frac{n!}{f^\lambda} c_\lambda(q,t). \tag{3.1}$$

Indeed, the class of cycle type $\mu$ has size $n!/z_\mu$, and the Frobenius formula evaluates the resulting class sum as $n! s_\lambda[B_{q,t}]$. Since every summand with $\ell(\lambda) \le d$ occurs, $T \succeq 0$ if and only if $c_\lambda(q,t) \ge 0$ for every $\lambda \vdash n$ with $\ell(\lambda) \le d$.

If $0 \le t \le q^{r-1}$, every relevant numerator $q^{i-1} - tq^{j-1}$ is nonnegative. At $t = q^m$, with $1 \le m \le r-2$, a partition with more than $m$ rows has a zero factor at $(m+1, 1)$, while all numerators of the remaining partitions are positive. For $t \le 0$ all numerators are positive. The denominators are positive in each case. This proves sufficiency and the final assertion.

Suppose instead that $t \notin \mathcal{R}_r(q)$. Then $r \ge 2$ and there is an integer $m \in \{1, \dots, r-1\}$ such that $q^m < t < q^{m-1}$. The hook $\lambda = (n-m, 1^m)$ has $m+1 \le d$ rows and is therefore present. Its first-row factors $1 - tq^{j-1}$ and the factors $q^{i-1} - t$ for $2 \le i \le m$ are positive, while the bottom factor $q^m - t$ is negative. Thus this coefficient is negative, and (3.1) excludes kernel positivity. □

As the rank grows the regions shrink, and their intersection is the locus on which we conjecture scalar positivity in all ranks (Conjecture 6.1).

**Corollary 3.2.** *For $0 < q < 1$ and $0 \le t < 1$, (1.3) is positive for all $n, d \ge 1$ exactly when*

$$t = 0 \quad \text{or} \quad t = q^m \quad \text{for an integer } m \ge 1.$$

*In the positive-kernel region,*

$$|\mathscr{K}(\mathbf{u}, \mathbf{v})|^2 \le \mathscr{K}(\mathbf{u}, \mathbf{u})\mathscr{K}(\mathbf{v}, \mathbf{v}).$$

*Proof.* Each $q^m$ and 0 belong to every region, while any other $t > 0$ lies between two successive powers of $q$ and is excluded for large $r$. The inequality is the $2 \times 2$ case of positivity. □

**Remark 3.3** (A necessary distinction)**.** *The negative eigenspace in the proof of Theorem 1.1 contains a linear combination of simple tensors, but need not contain a simple tensor on which the form is negative. The theorem therefore classifies kernel positivity, not nonnegativity of* $\mathrm{per}_{q,t}(A)$ *on single Gram matrices. The distinction also arises for classical* $\alpha$*-permanents (Brändén, 2012).*

*For example, every* $2 \times 2$ *positive semidefinite matrix satisfies*

$$\mathrm{per}_{q,t}(A) = w_1^2 a_{11}a_{22} + w_2|a_{12}|^2 \geq 0 \qquad (0 \leq q, t < 1).$$

*For* $n = d = 2$ *and* $0 < q < 1$*, however, Theorem 1.1 requires* $t \leq q$*. The scalar and kernel conditions already differ in this dimension.*

3.1. **Finite-order total positivity and matrix-ball kernels.** The determinant product itself defines a kernel before polarization. Its arguments lie in the operator-norm ball

$$\mathbb{B}_{d,p} := \{U \in \mathbb{C}^{d\times p} : \|U\|_{\mathrm{op}} < 1\}.$$

For $U, V \in \mathbb{B}_{d,p}$, set

$$\mathscr{H}_{q,t}(U,V) := \prod\nolimits_{j\geq 0} \frac{\det(I - tq^j UV^*)}{\det(I - q^j UV^*)}. \tag{3.2}$$

By Sylvester's identity $\det(I_d - UV^*) = \det(I_p - V^*U)$ the side is immaterial; no denominator vanishes since $\|UV^*\|_{\mathrm{op}} < 1$, and for fixed real $t$ the product converges locally uniformly.

The same coefficients also govern a purely scalar notion of positivity, the total positivity of the sequence $H_n = \mathrm{per}_{q,t}(J_n)/n!$ of Corollary 2.3. For a real sequence $h = (h_j)_{j\geq 0}$ with $h_0 = 1$, set $h_j = 0$ for $j < 0$. The sequence is a *Pólya frequency sequence of order* $r$, or $\mathrm{PF}_r$, if every minor of size at most $r$ of the matrix $(h_{j-i})_{i,j\geq 0}$ is nonnegative, and $\mathrm{PF}_\infty$ if this holds for every $r$. Total positivity is a sign condition on Toeplitz minors, and the next lemma turns it into the Schur sign test of Theorem 1.1.

**Lemma 3.4** (Toeplitz minors and Schur positivity)**.** *Let* $\rho : \Lambda_{\mathbb{R}} \to \mathbb{R}$ *be a specialization, and set* $h_j = h_j[\rho]$*. Then* $(h_j)_{j\geq 0}$ *is* $\mathrm{PF}_r$ *if and only if*

$$s_\lambda[\rho] \geq 0 \qquad \text{for every partition } \lambda \text{ with } \ell(\lambda) \leq r.$$

The proof of Lemma 3.4 is in Appendix A. Combined with the sign test in the proof of Theorem 1.1, the lemma shows that one region governs three positivity notions at once.

**Theorem 3.5** (Finite-order total positivity and matrix kernels)**.** *Let* $d, p \geq 1$*,* $0 < q < 1$*,* $0 \leq t < 1$*, and* $r = \min(d, p)$*. The following conditions are equivalent:*
*(1) The sequence* $H_n = (t;q)_n/(q;q)_n$*,* $n \geq 0$*, is* $\mathrm{PF}_r$*.*
*(2) For every* $n \geq 1$*, the kernel (1.3) is positive semidefinite on* $(\mathbb{C}^r)^n$*.*
*(3) The kernel (3.2) is positive semidefinite on* $\mathbb{B}_{d,p}$*.*
*(4)* $t \in \mathcal{R}_r(q)$*.*
*For* $t \leq 0$*, the first three conditions hold for every* $d, p \geq 1$*.*

The coefficient identity behind the theorem is

$$\mathscr{H}_{q,t}(U,V) = \sum\nolimits_\lambda c_\lambda(q,t) s_\lambda(UV^*).$$

For the map induced by $U$ on Schur modules,

$$s_\lambda(UV^*) = \mathrm{tr}(S_\lambda(U) S_\lambda(V)^*).$$

This is a positive semidefinite kernel, nonzero precisely when $\ell(\lambda) \leq \min(d,p)$. Distinct partitions give independent polynomial coefficient spaces, so positivity of the kernel also forces every contributing $c_\lambda$ to be nonnegative. Lemma 3.4 identifies the same condition with total positivity. The coefficient separation and convergence arguments are proved in Appendix A.

In every degree and rank, Corollary 3.2 reduces the positive-square locus to $t = 0, q, q^2, \ldots$. At $t = q^m$ the specialization is the finite alphabet $(1, q, \ldots, q^{m-1})$; at $t = 0$ it is the summable alphabet $(q^j)_{j\geq 0}$. These are the Edrei–Thoma specializations of the present family in the positive square. Their character interpretation and the extension to $t < 0$ are given in Supplement I (Sra, 2026b).

## 4. Scalar positivity

Sufficiency of $\mathcal{R}_r(q)$ for scalar positivity comes free from Section 3, since every immanant is nonnegative on a Gram matrix; the difficulty is necessity. A negative coefficient gives a negative direction of the kernel, but that direction is a linear combination of tensors, whereas a scalar counterexample must be a single Gram matrix. The necessity results below therefore each rest on a construction of negative matrices, and which one works depends on what is held fixed: the deformation, the rank, or the size. For small $q$ none works at a fixed rank, and we prove positivity beyond the kernel region instead. Throughout, $\mathcal{W}_r$ is the Wallach set of the introduction.

### 4.1. Scalar positivity and the Wallach set.

Where all contributing coefficients are nonnegative, the one-row term of the Schur expansion alone is a lower bound.

**Theorem 4.1** (Rank-sensitive scalar positivity)**.** *Let $A \in M_n(\mathbb{C})$ be Hermitian positive semidefinite of rank at most $r$, where $1 \leq r \leq n$. If $0 < q < 1$ and $t \in \mathcal{R}_r(q)$, then*

$$\mathrm{per}_{q,t}(A) \geq H_n(q,t)\,\mathrm{per}(A) \geq H_n(q,t)\prod_{i=1}^{n} a_{ii} \geq 0.$$

*The same inequalities hold for every real $t \leq 0$. In either range, $\mathrm{per}_{q,t}(A) = 0$ if and only if some diagonal entry of $A$ is zero.*

*Proof.* Write $A$ as a Gram matrix of vectors in $\mathbb{C}^r$. Lemma 3.1 gives $\mathrm{Imm}_\lambda(A) \geq 0$. It also gives $\mathrm{Imm}_\lambda(A) = 0$ when $\ell(\lambda) > r$, since the corresponding summand is absent from $(\mathbb{C}^r)^{\otimes n}$.

The sign analysis in the proof of Theorem 1.1 makes every contributing coefficient nonnegative in the stated ranges. The one-row term of the Schur expansion is $c_{(n)}(q,t)\,\mathrm{Imm}_{(n)}(A) = H_n(q,t)\,\mathrm{per}(A)$. Discarding the other terms proves the first inequality. The permanent lower bound is Lieb's inequality iterated over singletons (Frenkel, 2010, equation (10)).

The factor $H_n(q,t)$ is strictly positive in the stated ranges. Hence a matrix with all diagonal entries positive has $\mathrm{per}_{q,t}(A) > 0$. If $a_{ii} = 0$, positive semidefiniteness forces the entire $i$th row and column to vanish, so every permutation monomial is zero. □

The sufficiency in Theorem 4.1 holds at every $q$, and in the Jack limit $q \uparrow 1$ it becomes the classical nonnegativity of $\mathrm{per}_\alpha$ on Gram matrices of rank at most $r$. Theorem 1.2 says that across the whole deformation nothing more is true. Its negative matrices come from the classical case: the limit $q \uparrow 1$ carries a negative value of $\mathrm{per}_\alpha$ back to negative values of $\mathrm{per}_{q,q^\alpha}$, and the rank-tracked witness is Lemma B.1 in Appendix B.

***Proof of Theorem 1.2.*** If $\alpha \in \mathcal{W}_r$, then $q^\alpha \in \mathcal{R}_r(q)$ for every $q \in (0,1)$, and Theorem 4.1, applied with $\min(n,r)$ in place of $r$, gives (ii), since $\mathcal{R}_r(q) \subseteq \mathcal{R}_h(q)$ for $h \leq r$. Since $(1-q^{\alpha k})/(1-q^k) \to \alpha$ as $q \uparrow 1$ for each positive integer $k$, and the defining permutation sum for a fixed $A$ is finite, $\mathrm{per}_{q,q^\alpha}(A) \to \mathrm{per}_\alpha(A)$, so (ii) implies (iii).

If $\alpha \notin \mathcal{W}_r$, then $\alpha$ is nonintegral and $s = \lfloor \alpha \rfloor + 2 \leq r$. Lemma B.1 supplies a complex positive semidefinite matrix $A$ of rank at most $s$ with $\mathrm{per}_\alpha(A) < 0$. For this $A$ the limit above gives $\mathrm{per}_{q,q^\alpha}(A) \to \mathrm{per}_\alpha(A) < 0$, so $\mathrm{per}_{q,q^\alpha}(A) < 0$ for all $q$ close to one. This is the last assertion, and it shows that (ii) fails, so (ii) implies (i). □

Intersecting the sets $\mathcal{W}_r$ over $r$ leaves the positive integers, which gives the counterpart across the deformation of the positive-exponent half of Brändén (2012, Theorem 2.3).

**Corollary 4.2** (Unrestricted rank)**.** *Let $\alpha > 0$. Then $\mathrm{per}_{q,q^\alpha}(A) \geq 0$ for every $q \in (0,1)$, every $n \geq 1$, and every complex positive semidefinite $A \in M_n(\mathbb{C})$ if and only if $\alpha$ is a positive integer.*

*Proof.* The condition is statement (ii) of Theorem 1.2 for every $r$, hence equivalent to $\alpha \in \bigcap_r \mathcal{W}_r = \{1, 2, \ldots\}$. □

4.2. **Scalar positivity in rank two, uniformly in the size.** In rank two, an unbounded immanant ratio settles necessity at every $q$.

**Theorem 4.3** (Scalar positivity in rank two, uniformly in the size)**.** *Fix $0 \leq q, t < 1$. The inequality $\mathrm{per}_{q,t}(A) \geq 0$ holds for every complex Hermitian positive semidefinite matrix $A$ of rank at most two, in every size, if and only if $t \leq q$.*

When the rank is fixed and the size grows, more Gram configurations are tested, and the two-row immanants, whose coefficients are negative above $t = q$, get room to outweigh the permanent. The following lemma shows that in rank two the standard immanant can be arbitrarily large relative to the permanent, which fixed small sizes do not allow.

**Lemma 4.4** (A rank-two immanant ratio)**.** *Let $\mathsf{H}_n = \sum_{j=1}^n j^{-1}$. For every $n \geq 2$, there is an $n \times n$ complex positive semidefinite correlation matrix $A$ of rank two such that*

$$\frac{\mathrm{Imm}_{(n-1,1)}(A)}{\mathrm{per}\, A} \geq \frac{n\mathsf{H}_n - 1}{n+1}. \tag{4.1}$$

*In particular, these ratios are unbounded as the size varies.*

*Proof.* For a polynomial $f$ on $\mathbb{C}^2$, use the Gaussian norm

$$\|f\|_G^2 = \mathbb{E}|f(g_1, g_2)|^2,$$

where $g_1, g_2$ are independent standard circular complex Gaussians, normalized by $\mathbb{E}|g_i|^2 = 1$. Consider the random homogeneous polynomial

$$p(z_1, z_2) = \sum_{j=0}^n c_j \binom{n}{j}^{1/2} z_1^{n-j} z_2^j, \qquad c_0, \ldots, c_n \text{ independent standard complex Gaussians.}$$

Its covariance is $\mathbb{E}p(z)\overline{p(w)} = \langle w, z\rangle^n$, so its distribution is invariant under unitary changes of the two variables. Almost surely it has $n$ distinct projective zeros and factors as $p = C\prod_{i=1}^n \ell_i$, where each linear form $\ell_i$ has a coefficient vector of norm one. The Gram matrix $A$ of these coefficient vectors is a correlation matrix of rank two. The complex Gaussian moment formula gives

$$\|p\|_G^2 = |C|^2\, \mathrm{per}\, A, \qquad \|p/\ell_i\|_G^2 = |C|^2\, \mathrm{per}\, A[[n] \setminus \{i\}].$$

With these norms, define

$$Q(p) = \sum_{i=1}^n \frac{\|p/\ell_i\|_G^2}{\|p\|_G^2}.$$

The standard character satisfies $\chi^{(n-1,1)}(\sigma) = \#\{i : \sigma(i) = i\} - 1$. Since $a_{ii} = 1$, it follows that

$$\frac{\mathrm{Imm}_{(n-1,1)}(A)}{\mathrm{per}\, A} = Q(p) - 1. \tag{4.2}$$

The root-average calculation in Appendix B gives

$$\mathbb{E}Q(p) = \frac{n(\mathsf{H}_n + 1)}{n+1}.$$

Together with (4.2), this implies that some realization satisfies (4.1). □

Above $t = q$ every two-row coefficient is negative, so the standard-immanant term alone can outweigh the permanent, while all other two-row terms have the same sign.

**Lemma 4.5** (Two-row coefficients)**.** *Let $n \geq 2$ and $0 < q < t < 1$. Every coefficient $c_\lambda(q,t)$ with $\ell(\lambda) = 2$ is negative, and*

$$\frac{c_{(n-1,1)}(q,t)}{H_n(q,t)} = -\frac{(t-q)(1-q^{n-1})}{(1-q)(1-tq^{n-1})}.$$

*Hence every positive semidefinite matrix $A$ of rank at most two with positive diagonal satisfies*

$$\frac{\operatorname{per}_{q,t}(A)}{H_n(q,t)\operatorname{per} A} \leq 1 - \frac{(t-q)(1-q^{n-1})}{(1-q)(1-tq^{n-1})}\frac{\operatorname{Imm}_{(n-1,1)}(A)}{\operatorname{per} A}. \tag{4.3}$$

*Proof.* For a two-row partition, the numerator at $(2,1)$ is $q - t < 0$. Every first-row numerator is $1 - tq^{j-1} > 0$, and every other second-row numerator is $q - tq^{j-1} = q(1 - tq^{j-2}) > 0$. All hook denominators are positive, so the coefficient has exactly one negative factor.

The Jacobi–Trudi identity gives $s_{(n-1,1)} = h_{n-1}h_1 - h_n$. Since $h_j[B_{q,t}] = H_j(q,t)$ and $H_n/H_{n-1} = (1-tq^{n-1})/(1-q^n)$, it follows that

$$\begin{aligned}\frac{c_{(n-1,1)}}{H_n} &= \frac{(1-t)(1-q^n)}{(1-q)(1-tq^{n-1})} - 1 \\ &= \frac{(q-t)(1-q^{n-1})}{(1-q)(1-tq^{n-1})}.\end{aligned}$$

On a Gram matrix of rank at most two the immanants with more than two rows vanish and the others are nonnegative, as in the proof of Theorem 4.1; the one-row term of the Schur expansion is $H_n(q,t)\operatorname{per} A$, and dropping every two-row term except that of the standard immanant, all of them nonpositive, gives (4.3), whose denominator is positive because $H_n(q,t) > 0$ and $\operatorname{per} A \geq \prod_i a_{ii} > 0$. □

*Proof of Theorem 4.3.* For $0 < q < 1$ and $t \leq q$, Theorem 4.1 with $r = \min(n,2)$ gives nonnegativity in rank at most two, since $\mathcal{R}_2(q) = [0,q]$; at $q = t = 0$ the function is the ordinary permanent, which is nonnegative on positive semidefinite matrices.

Suppose $0 < q < t < 1$. The coefficient of the immanant ratio in (4.3) tends to $(t-q)/(1-q) > 0$ as $n \to \infty$, while $\mathsf{H}_n \to \infty$, so there is an $n \geq 2$ with

$$\frac{(t-q)(1-q^{n-1})}{(1-q)(1-tq^{n-1})}\frac{n\mathsf{H}_n - 1}{n+1} > 1.$$

Lemma 4.4 supplies a rank-two correlation matrix for which the last ratio in (4.3) is at least $(n\mathsf{H}_n - 1)/(n+1)$. The right side is therefore strictly negative, and $\operatorname{per}_{q,t}(A) < 0$.

At $q = 0 < t < 1$, the hook formula (E.1) and rank-two vanishing give

$$\operatorname{per}_{0,t}(A) = (1-t)\bigl(\operatorname{per} A - t\operatorname{Imm}_{(n-1,1)}(A)\bigr),$$

so an $n$ with $t(n\mathsf{H}_n - 1)/(n+1) > 1$ and the matrix of Lemma 4.4 again give $\operatorname{per}_{0,t}(A) < 0$. □

The negative matrices of Theorem 4.3 are complex, and their number of distinct Gram vectors grows with the size; in size three the region is larger (Theorem C.1), and with only two distinct vectors, repeated, positivity holds on the whole square (Corollary E.2).

4.3. **Scalar counterexamples from random projections.** In rank three and higher, generic homogeneous polynomials in more than two variables do not factor into linear forms. We obtain negative values by averaging $\operatorname{per}_{q,t}$ over the unitary orbit of an orthogonal projection of rank $r$. The average turns the immanant expansion (2.6) into explicit terms, one per partition. Each term has an exponential rate in the size, with factors that grow only polynomially; the largest rates determine the sign of the sum.

**Theorem 4.6** (Scalar counterexamples at fixed $q$)**.** *Let $r \geq 2$, $0 < q < 1$, and*

$$(r+1)q^{r-1} > \sum_{j=0}^{r-1} q^j. \tag{4.4}$$

*For every $t$ with $q^{r-1} < t < q^{r-2}$, there is an integer $n_0 = n_0(q,t,r) \geq r$ such that the random rank-$r$ projection $P_n$ in Lemma 4.7 satisfies*

$$\mathbb{E}\,\mathrm{per}_{q,t}(P_n) < 0 \qquad \text{for every } n \geq n_0.$$

*Consequently, for every $n \geq n_0$ there is a complex Hermitian positive semidefinite matrix $A \in M_n(\mathbb{C})$ of rank $r$ with $\mathrm{per}_{q,t}(A) < 0$. The integer $n_0$ can be chosen uniformly for $t$ in any compact subinterval of $(q^{r-1}, q^{r-2})$.*

Through the Schur expansion, the average of $\mathrm{per}_{q,t}$ over the orbit reduces to averages of single immanants, and these are explicit.

**Lemma 4.7** (Haar-averaged immanants)**.** *For $1 \leq r \leq n$ and Haar distributed $U \in \mathrm{U}(n)$, set*

$$Q_r = \mathrm{diag}(I_r, 0_{n-r}), \qquad P_n = UQ_rU^*.$$

*For every $\lambda \vdash n$,*

$$\mathbb{E}\,\mathrm{Imm}_\lambda(P_n) = f^\lambda \frac{s_\lambda(1^r)}{s_\lambda(1^n)} = \frac{n!\, s_\lambda(1^r)}{\prod_{(i,j)\in\lambda}(n+j-i)}. \tag{4.5}$$

*In particular, the expectation vanishes when $\ell(\lambda) > r$.*

The formula follows by averaging $P_n^{\otimes n}$ on each Schur–Weyl component (Appendix B).

For $q^{r-1} < t < q^{r-2}$, the partitions with $r$ rows have negative coefficients, and those with fewer rows have positive coefficients. Write $p_i = \lambda_i/n$, appending zero parts. Up to factors bounded above and below by powers of $n$, the absolute value of a term in the expected Schur expansion has exponential rate $\Phi(p) - 1$, where

$$\Phi(p) = (\log q)\sum_{i=1}^{r}(i-1)p_i - \sum_{i=1}^{r}\big((1+p_i)\log(1+p_i) - p_i\big).$$

On the ordered simplex this function is strictly concave, with unique maximizer under (4.4)

$$p_i^* = \frac{(r+1)q^{i-1}}{\sum_{j=0}^{r-1} q^j} - 1 > 0.$$

Thus the largest rate occurs among negative terms and is strictly larger than every rate on the boundary $p_r = 0$, where the positive terms lie. This strict gap forces the expectation to be negative for large $n$. Uniform estimates, partition rounding, and the compact-$t$ assertion are proved in Appendix B. For $r = 3$ the condition reads $q > (1+\sqrt{13})/6$, and the theorem then gives rank-three negative matrices throughout $q^2 < t < q$.

With Theorem 4.1, this theorem classifies scalar positivity at bounded rank in every size.

**Corollary 4.8** (Scalar positivity at fixed rank)**.** *Let $r \geq 3$, let $0 < q < 1$ satisfy (4.4), and let $0 \leq t < 1$. Then $\mathrm{per}_{q,t}(A) \geq 0$ for every complex Hermitian positive semidefinite matrix $A$, of every order and of rank at most $r$, if and only if*

$$t \in \mathcal{R}_r(q) = [0, q^{r-1}] \cup \{q, q^2, \ldots, q^{r-2}\}.$$

*Proof.* Sufficiency is Theorem 4.1, applied with $\min(n,r)$ in place of $r$, since $\mathcal{R}_r(q) \subseteq \mathcal{R}_h(q)$ for $h \leq r$. For necessity, the threshold can be rewritten as

$$\sum_{j=1}^{r-1}(q^{-j} - 1) < 1.$$

Every summand is positive, so the same condition holds with $r$ replaced by any $h$ with $2 \leq h \leq r$. If $t \notin \mathcal{R}_r(q)$, then $q^{h-1} < t < q^{h-2}$ for one such $h$. Theorem 4.6, applied in rank $h$, supplies a positive semidefinite matrix of rank at most $r$ with negative scalar value. □

4.4. **Scalar positivity beyond the kernel region.** For small $q$, the scalar region can extend beyond $\mathcal{R}_R(q)$ in every size at a fixed rank bound $R$. The comparison is with $t = q^2$, where two Gaussian colors realize the weights. A bound on the cost of deleting a Gaussian factor controls the omitted colors uniformly in the size.

**Theorem 4.9** (Positivity beyond any prescribed finite rank)**.** *For every integer $R \geq 3$ there are $q, t$ with $q^2 < t < q$, so that $t \notin \mathcal{R}_R(q)$, such that*

$$\operatorname{per}_{q,t}(A) \geq \tfrac{1}{2} \operatorname{per}_{q,q^2}(A) \geq 0 \tag{4.6}$$

*for every complex Hermitian positive semidefinite $A$ of rank at most $R$, in every size. One choice is $q = 1/(4R)$ and $t = q^2 + \delta_R$ with $\delta_R$ as in (4.10).*

The comparison writes both $\operatorname{per}_{q,q^2}$ and $\operatorname{per}_{q,0}$ as moments of Gaussian quadratic forms, the first with two colors and the second with infinitely many. For arbitrary cycle weights $w$, write $\operatorname{per}_w$ for the sum in (1.1) with those weights.

**Lemma 4.10** (Gaussian cycle moments)**.** *Let $A = [u_i^* u_j]_{i,j=1}^n$ with $u_i \in \mathbb{C}^R$, and let $g_a$ be independent standard circular complex Gaussian vectors in $\mathbb{C}^R$. If $(c_a)$ is a finite or summable sequence of nonnegative numbers, put $w_k = \sum_a c_a^k$. Then*

$$\mathbb{E} \prod_{i=1}^n \left( \sum_a c_a |u_i^* g_a|^2 \right) = \operatorname{per}_w(A).$$

*In particular, $c_a = q^a$ for $0 \leq a < m$ gives $\operatorname{per}_{q,q^m}(A)$, and $c_a = q^a$ for all $a \geq 0$ gives $\operatorname{per}_{q,0}(A)$.*

*Proof.* For finitely many colors, expand the product and apply the complex Wick formula. A pairing permutation $\sigma$ requires the color index to be constant on each cycle $C$, whose color sum is $\sum_a c_a^{|C|}$. The covariance factors give the permutation monomial of $A$ (replacing $\sigma$ by its inverse does not change the sum), proving the identity.

For a summable sequence, each nonnegative sum in the product is finite almost surely, since its expectation is $\|u_i\|^2 \sum_a c_a < \infty$. Products of the partial sums increase, so monotone convergence passes the expectation to the limit. Each $w_k$ is finite and is the limit of its finite-color sum; the permutation sum has only finitely many terms and therefore converges to $\operatorname{per}_w(A)$. The geometric choices give the stated Macdonald weights. □

The colors beyond the first two add a nonnegative term $u_i^* Y u_i$ to each factor, and the following bound controls the cost of that addition with a constant independent of the size.

**Lemma 4.11** (Uniform Gaussian deletion)**.** *Let $k \geq 2$, let $C = \operatorname{diag}(c_1, \dots, c_k)$ with every $c_a > 0$, and let $g_1, \dots, g_k$ be independent standard complex Gaussian vectors in $\mathbb{C}^R$. For $u_1, \dots, u_n \in \mathbb{C}^R$ with Gram matrix $A = [u_i^* u_j]$, put*

$$L_i = \sum_{a=1}^k c_a |u_i^* g_a|^2, \quad F_C(A) = \mathbb{E} \prod_{i=1}^n L_i, \quad d = Rk, \quad \beta = \operatorname{tr} C - \max_a c_a, \quad K = \frac{d-1}{\beta}.$$

*The following bounds then hold:*

$$\sum_{i=1}^n a_{ii} F_C(A[[n] \setminus \{i\}]) \leq \frac{n(d-1)}{\beta(n+d-1)} F_C(A) \leq K F_C(A), \tag{4.7}$$

$$\mathbb{E} \prod_{i=1}^n (L_i + x a_{ii}) \leq e^{Kx} F_C(A) \qquad (x \geq 0). \tag{4.8}$$

*For deterministic $Y \succeq 0$, moreover, $\mathbb{E} \prod_i (L_i + u_i^* Y u_i) \leq e^{K \operatorname{tr} Y} F_C(A)$.*

The proof of Lemma 4.11 is in Appendix B.

*Proof of Theorem 4.9.* Take $q = 1/(4R)$, write $F(A) = \operatorname{per}_{q,q^2}(A)$, and let $A = [u_i^* u_j]$ with $u_i \in \mathbb{C}^R$. With $g_0, g_1, \dots$ independent standard complex Gaussian vectors in $\mathbb{C}^R$, Lemma 4.10 gives

$$F(A) = \mathbb{E} \prod_i (|u_i^* g_0|^2 + q |u_i^* g_1|^2), \qquad \operatorname{per}_{q,0}(A) = \mathbb{E} \prod_i (|u_i^* g_0|^2 + q |u_i^* g_1|^2 + u_i^* Y u_i),$$

where $Y = \sum_{j\ge 2} q^j g_j g_j^*$ is independent of $g_0, g_1$. The last assertion of Lemma 4.11 with $C = \mathrm{diag}(1,q)$, so that $K = (2R-1)/q$, averaged over $Y$, gives $\mathrm{per}_{q,0}(A) \le F(A)\,\mathbb{E}e^{K\,\mathrm{tr}\,Y}$. Here $\mathbb{E}e^{K\,\mathrm{tr}\,Y} = \prod_{j\ge 2}(1-Kq^j)^{-R}$ and $\sum_{j\ge 2} Kq^j = (2R-1)q/(1-q) < 1$, so the inequality $\prod_j(1-b_j) \ge 1 - \sum_j b_j$ yields

$$\mathrm{per}_{q,0}(A) \le M_R F(A), \qquad M_R \le \overline{M}_R := \left(\frac{1-q}{1-2Rq}\right)^R. \tag{4.9}$$

For $\ell(\lambda) \le R$, the hook formula (2.6) gives

$$D_\lambda(t) := \frac{c_\lambda(q,t)}{c_\lambda(q,0)} = \prod\nolimits_{(i,j)\in\lambda}(1-tq^{j-i}), \qquad B_R := \frac{\sum_{i=1}^R q^{1-i}}{1-q},$$

and for $0 \le t \le 2q^2$ the product rule gives $|D'_\lambda(t)| \le C_R := B_R e^{2q^2 B_R}$, uniformly in $\lambda$ and its size. On $A$ the immanants in (2.6) are nonnegative and only partitions with at most $R$ rows contribute, with positive coefficients $c_\lambda(q,0)$, so

$$|\,\mathrm{per}_{q,t}(A) - \mathrm{per}_{q,s}(A)| \le C_R|t-s|\,\mathrm{per}_{q,0}(A) \qquad (s,t\in[0,2q^2]).$$

With these constants, choose

$$\delta_R = \min\{q^2, (2C_R\overline{M}_R)^{-1}\}, \qquad t = q^2 + \delta_R. \tag{4.10}$$

Then $q^2 < t \le 2q^2 < q$, and (4.9) gives $\mathrm{per}_{q,t}(A) \ge F(A) - C_R\delta_R\overline{M}_R F(A) \ge \frac{1}{2}F(A)$, which is (4.6). □

Remark B.2 gives the explicit rank-three example $q = 1/10$, $t = 21/2000$.

## 5. Block and diagonal inequalities

After nonnegativity, the classical question for the permanent on positive semidefinite matrices is how it compares with the product of its values on complementary principal blocks, and Lieb's inequality says that it dominates that product. The Schur expansion cannot carry this over to $\mathrm{per}_{q,t}$, since it compares $\mathrm{per}_{q,t}$ with the permanent only through its one-row term. A second expansion, in products of ordinary permanents of principal blocks, does: its coefficients are nonnegative, and Lieb's inequality applies block by block. Because the coefficients are indexed by set partitions of $[n]$, the hypothesis here depends on the matrix size, not on the rank.

**Theorem 5.1** (Block and diagonal inequalities)**.** *Let $n \ge 1$, $0 < q < 1$, $t \in \mathcal{R}_n(q)$, and let $A \in M_n(\mathbb{C})$ be Hermitian positive semidefinite. For $I \sqcup J = [n]$, set $D = A[I] \oplus A[J]$, after a simultaneous reordering of rows and columns. Then*

$$\mathrm{per}_{q,t}(A) \ge \mathrm{per}_{q,t}(D) = \mathrm{per}_{q,t}(A[I])\,\mathrm{per}_{q,t}(A[J]), \qquad \mathrm{per}_{q,t}(A) \ge w_1^n \prod\nolimits_{i=1}^n a_{ii}.$$

*Proof.* Let $\Pi_n$ denote the set partitions of $[n]$. Proposition D.2 gives

$$\mathrm{per}_{q,t}(A) = \sum\nolimits_{\pi\in\Pi_n} a_\pi(q,t) \prod\nolimits_{B\in\pi} \mathrm{per}\,A[B], \qquad a_\pi(q,t) \ge 0.$$

Lieb's inequality on each block gives $\mathrm{per}\,A[B] \ge \mathrm{per}\,A[B\cap I]\,\mathrm{per}\,A[B\cap J] = \mathrm{per}\,D[B]$ (Frenkel, 2010, equation (10)). Multiplying these nonnegative factors and summing proves the block inequality; block multiplicativity gives the equality. Iterating over singleton blocks gives the diagonal bound because $\mathrm{per}_{q,t}([a_{ii}]) = w_1 a_{ii}$. □

The coefficient signs follow from a divided-difference identity in Appendix D. Signed Fischer inequalities and a real-Gaussian comparison are proved in Supplement I (Sra, 2026b). The sufficient region $\mathcal{R}_n(q)$ is not asserted to be necessary for each inequality: the block inequality holds throughout the square in size two.

## 6. Open questions

The scalar problem that remains concerns a fixed $q$ and matrices of unbounded size and rank, where neither the limit $q \uparrow 1$ nor a rank bound is available. The kernel locus suggests its answer.

**Conjecture 6.1** (Scalar Macdonald positivity)**.** *For every $0 < q < 1$ and $0 \le t < 1$, the inequality $\mathrm{per}_{q,t}(A) \ge 0$ holds for every complex Hermitian positive semidefinite matrix $A$, of every size, if and only if $t \in \{0, q, q^2, q^3, \dots\}$.*

Sufficiency is Theorem 4.1. Necessity holds for $t > q$ by the rank-two theorem, and in the further strips reached by Theorem 4.6. The small-$q$ comparison shows that no fixed rank bound suffices to supply witnesses uniformly over all excluded parameters. In particular, the pair $q = 1/10$, $t = 21/2000$ requires a negative witness of rank at least four, if one exists. No such witness is supplied here. The endpoint $t = 1$, outside the stated range, would add the identically zero function in every positive size. Even at a fixed rank, the question is open.

**Problem 6.2** (The scalar region at a prescribed $q$ and rank)**.** For $0 < q < 1$ and $r \ge 3$, determine the set of $t$ for which $\mathrm{per}_{q,t}(A) \ge 0$ for every size and every complex $A \succeq 0$ of rank at most $r$.

The set contains $\mathcal{R}_r(q)$, equals it under the Haar threshold, and can be strictly larger for small $q$. Determining its boundary requires understanding scalar evaluations when positive and negative immanant terms coexist. Even the shape of the region above $q^2$ at small $q$ is open: the present comparison gives a right neighborhood of $q^2$ but does not determine how far it extends.

The exact fixed-size regions are likewise unknown for $n \ge 5$. The thresholds for $n = 3, 4$ and their explicit witnesses are in Appendix C. The broader immanant and signed-inequality questions accompany their corresponding extensions in Supplement I.

## AI usage

The results in this paper were developed through extensive use of an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs. The expansion of the author's notes into the present manuscript, together with its finite symbolic verification program, also used OpenAI's models, and Anthropic's Claude (Fable 5.1 and Opus 5.5) prepared this version of the manuscript. Assistant checks and reruns are not external peer review or formal verification.

## Acknowledgements

The author (SS) thanks the Alexander von Humboldt (AvH) Foundation for their generous support via a Humboldt Professorship in AI.

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## Appendix A. Total positivity and matrix-ball proofs

Theorem 3.5 identifies three positivity notions with one sign test, and the main text gives only the mechanism. Two facts make the identification rigorous. Toeplitz minors of $(h_{j-i})$ are skew Schur functions, so total positivity of order $r$ is Schur positivity in at most $r$ rows; and the Schur components of the matrix-ball kernel live in independent polynomial spaces, so positivity of the kernel forces the sign of each coefficient separately.

### A.1. **Toeplitz minors.**

*Proof of Lemma 3.4.* Consider a $k \times k$ Toeplitz minor, with increasing row indices $a_1 < \cdots < a_k$ and column indices $b_1 < \cdots < b_k$. If $a_i > b_i$ for some $i$, its first $i$ columns are supported in at most the first $i-1$ rows, so the minor vanishes. Otherwise set

$$\lambda_i = b_{k+1-i} + i - k, \qquad \mu_i = a_{k+1-i} + i - k, \qquad 1 \le i \le k.$$

Both $\lambda$ and $\mu$ are partitions, padded with zeros, and $\mu \subseteq \lambda$. Reversing both orders and transposing gives the skew Jacobi–Trudi identity (Macdonald, 1995, Chapter I, Section 5)

$$\det[h_{b_j-a_i}]_{i,j=1}^{k} = \det[h_{\lambda_i-\mu_j-i+j}]_{i,j=1}^{k} = s_{\lambda/\mu}[\rho].$$

The two reversals have cancelling signs.

The Littlewood–Richardson rule (Macdonald, 1995, Chapter I, Section 9) expands this skew function as

$$s_{\lambda/\mu} = \sum\nolimits_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}, \qquad c_{\mu\nu}^{\lambda} \in \mathbb{N}.$$

If $c_{\mu\nu}^{\lambda} > 0$, the symmetry $c_{\mu\nu}^{\lambda} = c_{\nu\mu}^{\lambda}$ implies $\nu \subseteq \lambda$. Thus every contributing $\nu$ has at most $k$ rows. Schur positivity through $r$ rows therefore makes every minor of order $k \le r$ nonnegative. Conversely, for a partition $\lambda$ with $k$ rows, choose $a_i = i-1$ and $b_j = \lambda_{k+1-j} + j - 1$. These strictly increasing indices give the minor $s_{\lambda}[\rho]$, proving necessity. □

A.2. **Matrix-ball coefficient spaces.**

*Proof of Theorem 3.5.* The generating function of the specialization $B_{q,t}$ gives $H_n = h_n[B_{q,t}]$. Lemma 3.4 therefore identifies the first condition with $c_{\lambda}(q,t) \ge 0$ for every $\ell(\lambda) \le r$. The central-operator proof of Theorem 1.1, applied in every degree, identifies the second condition with the same inequalities. Its hook sign test identifies these inequalities with the fourth condition, and proves them also for $t \le 0$. It remains to establish the equivalent matrix-ball condition.

Near the origin, the Schur Cauchy identity gives the homogeneous expansion

$$\mathscr{H}_{q,t}(U,V) = \sum\nolimits_{n\ge 0} \sum\nolimits_{\lambda\vdash n} c_{\lambda}(q,t) s_{\lambda}(UV^{*}). \tag{A.1}$$

For a linear map $U : \mathbb{C}^p \to \mathbb{C}^d$, its tensor power induces a linear map on Schur modules, $S_{\lambda}(U) : S_{\lambda}(\mathbb{C}^p) \to S_{\lambda}(\mathbb{C}^d)$. The tensor inner products induce inner products for which this construction commutes with composition and adjoints. The character of $S_{\lambda}$ is $s_{\lambda}$, so

$$s_{\lambda}(UV^{*}) = \operatorname{tr}(S_{\lambda}(U) S_{\lambda}(V)^{*}).$$

In orthonormal bases, write the polynomial entries of $S_{\lambda}(U)$ as $f_{\lambda,ab}(U)$. Then

$$s_{\lambda}(UV^{*}) = \sum\nolimits_{a,b} f_{\lambda,ab}(U) \overline{f_{\lambda,ab}(V)}.$$

The right side is a positive semidefinite kernel. It is nonzero exactly when $\ell(\lambda) \le r$, because both Schur modules must be nonzero.

To prove necessity, one must separate the coefficient spaces of different partitions. The degree-$n$ polynomials on $\mathbb{C}^{d\times p}$ have the Cauchy decomposition

$$\operatorname{Sym}^{n}((\mathbb{C}^{d})^{*} \otimes \mathbb{C}^{p}) \cong \bigoplus_{\substack{\lambda\vdash n \\ \ell(\lambda)\le r}} S_{\lambda}((\mathbb{C}^{d})^{*}) \otimes S_{\lambda}(\mathbb{C}^{p}).$$

One way to obtain it is to realize the symmetric power as the invariants under the diagonal $S_n$ action on $((\mathbb{C}^d)^*)^{\otimes n} \otimes (\mathbb{C}^p)^{\otimes n}$. Schur–Weyl decomposition in the two factors leaves $(V_{\lambda} \otimes V_{\mu})^{S_n}$, whose dimension is one for $\lambda = \mu$ and zero otherwise. The matrix entries $f_{\lambda,ab}$ span the indicated $\lambda$ summand: their span is a nonzero equivariant image of that irreducible module. In particular, these polynomial spaces form a direct sum.

Every linear functional on degree-$n$ polynomials can be written as $L = \sum_{|\gamma|=n} b_{\gamma} \partial^{\gamma}|_{0}$. If an analytic kernel is positive semidefinite near zero, then

$$L_U \overline{L_V}\, \mathscr{H}(U,V)\big|_{U=V=0} \ge 0,$$

since derivatives at zero are limits of finite-difference functionals, and applying the same one in both arguments gives nonnegative quadratic forms. Fix a nonzero $\lambda$ summand and choose $L$ to vanish on

every other degree-$n$ summand while remaining nonzero on this one. Derivatives of order $n$ at zero annihilate homogeneous terms of other degrees. Applying $L$ to (A.1) therefore gives

$$0 \le c_\lambda(q,t) \sum\nolimits_{a,b} |L f_{\lambda,ab}|^2.$$

The sum is strictly positive, so $c_\lambda(q,t) \ge 0$ is necessary for every partition with $\ell(\lambda) \le r$.

For sufficiency, it remains to justify the infinite positive sum. Suppose first that all the coefficients under consideration are nonnegative. On $\|U\|_{\mathrm{op}} \le a < 1$, the eigenvalues of $UU^*$ lie in $[0, a^2]$, and at most $r$ of them are nonzero. The tableau expansion of $s_\lambda$ has nonnegative coefficients, so

$$s_\lambda(UU^*) \le a^{2|\lambda|} s_\lambda(1^r).$$

The diagonal majorant is finite:

$$\sum\nolimits_\lambda c_\lambda(q,t) a^{2|\lambda|} s_\lambda(1^r) = \prod\nolimits_{j\ge 0} \left( \frac{1 - tq^j a^2}{1 - q^j a^2} \right)^r < \infty.$$

To justify this equality without assuming convergence of (A.1), apply its formal identity to the scalar variable $z$ in $zI_r$. The product is analytic for $|z| < 1$, and its Taylor series converges at $z = a^2$. For two arguments, the Hilbert–Schmidt Cauchy–Schwarz inequality bounds each $|s_\lambda(UV^*)|$ by $\sqrt{s_\lambda(UU^*) s_\lambda(VV^*)}$. The same diagonal majorant therefore gives normal convergence of the positive kernel sum on smaller balls.

The expansion also converges normally at the other real parameters. The hook formula gives $|c_\lambda(q,t)| \le c_\lambda(q,-|t|)$, and the right side has nonnegative coefficients. Applying the preceding bound at $-|t|$ proves absolute convergence of (A.1). Its sum agrees near zero with the determinant product and hence agrees throughout the ball. In the claimed parameter range, finite partial sums are positive semidefinite, and so are their limits, which proves sufficiency. □

For $\alpha > 0$, the specialization $t = q^\alpha$ has the locally uniform limit

$$\mathscr{H}_{q,q^\alpha}(U,V) \longrightarrow \det(I - UV^*)^{-\alpha} \qquad (q \uparrow 1). \tag{A.2}$$

with the determinant power defined by the logarithm that vanishes at $UV^* = 0$: since $0 < (1-q^{\alpha k})/(1-q^k) \le \max(1,\alpha)$ with limit $\alpha$, the logarithmic series is dominated on a ball of radius $a < 1$ by $d \max(1,\alpha) \sum_{k\ge 1} a^{2k}/k$, and dominated convergence proves (A.2).

The substitution $t = q^\alpha$ translates $t \in \mathcal{R}_r(q)$ into $\alpha \in \mathcal{W}_r$, and the same range is sharp for the limiting determinant kernel: applied to (E.3), the cellwise sign test gives it directly, since positive integers kill partitions with too many rows, $\alpha \ge r - 1$ makes every relevant factor nonnegative, and a hook gives a negative coefficient between consecutive integers. The determinant kernel and its parameter range belong to the classical theory of matrix balls and bounded symmetric domains (Faraut and Korányi, 1990); see also Sra (2026a) for the contraction-matrix formulation.

## Appendix B. Scalar witness proofs and Gaussian estimates

The scalar theorems of Section 4 are proved there from a few lemmas, each stated next to the mechanism it serves. Their proofs are calculations rather than ideas, and we collect them here so that the main text can follow the construction of negative matrices without interruption. The one estimate with content of its own is the Gaussian deletion bound of Lemma 4.11: its constant is independent of the size, and this uniformity is what lets Theorem 4.9 perturb $t$ in every size.

B.1. **The rank of a classical scalar witness.** The classical complete-monotonicity obstruction supplies the rank information needed by the across-deformation theorem.

**Lemma B.1.** *If $\alpha > 0$ is not an integer and $s = \lfloor \alpha \rfloor + 2$, there is a complex Hermitian positive semidefinite matrix $A$, of some finite size and rank at most $s$, such that* $\mathrm{per}_\alpha(A) < 0$.

*Proof.* Choose rank-one positive semidefinite matrices $B_1,\dots,B_d$ spanning the real vector space of Hermitian $s\times s$ matrices, and put

$$p(x)=\det\Bigl(\sum\nolimits_{j=1}^{d} x_jB_j\Bigr).$$

Since $\alpha$ is not an integer and $\alpha<s-1$, the complex Scott–Sokal criterion, in the form stated in Brändén (2012, Theorem 2.6(2)), says that $p^{-\alpha}$ is not completely monotone on $(0,\infty)^d$. Consequently, for some $y\in(0,\infty)^d$ and some $\nu\in\mathbb{N}^d$, the coefficient of $x^\nu$ in $p(y-x)^{-\alpha}$ is negative.

Set $C=\sum_j y_jB_j$. Since the $B_j$ are positive semidefinite and span the Hermitian matrices, $C$ is positive definite. Write $C^{-1/2}B_jC^{-1/2}=u_ju_j^*$ with $u_j\in\mathbb{C}^s$, and put $U=[u_1\ \cdots\ u_d]$ and $X=\operatorname{diag}(x)$. Sylvester's identity and the classical master theorem give

$$p(y-x)^{-\alpha}=\det(C)^{-\alpha}\det(I-XU^*U)^{-\alpha},\qquad [x^\nu]p(y-x)^{-\alpha}=\frac{\det(C)^{-\alpha}}{\nu!}\operatorname{per}_\alpha\bigl((U^*U)[\nu]\bigr).$$

Thus $A=(U^*U)[\nu]$ is a Gram matrix of repeated vectors in $\mathbb{C}^s$ and satisfies $\operatorname{per}_\alpha(A)<0$. This construction is that of Brändén (2012, Theorem 2.3), with its Gram dimension tracked. □

The last assertion of Theorem 1.2 is uniform on compact sets of noninteger exponents: for a compact $J\subset(m,m+1)$, each matrix remains negative on a neighborhood of $(q,\alpha)=(1,\alpha_0)$, and a finite cover of $J$ supplies finitely many matrices of rank at most $m+2$ and a common bound $q_J<1$. It gives nothing at a prescribed $q$, and no uniformity as $\alpha$ approaches an integer.

B.2. **The root-deletion average.** We complete the calculation in the proof of Lemma 4.4, using its polynomial $p$, Gaussian norm, and quantity $Q(p)$.

*Root-average calculation.* To compute $\mathbb{E}Q(p)$ we change variables on the zero set. For a projective zero $z$ of $p$, let $F(p,z)=\|p/\ell_z\|_G^2/\|p\|_G^2$, where $\ell_z$ is any unit linear form vanishing at $z$; its phase does not affect $F$. The quantity $F$ is invariant under simultaneous unitary changes of the polynomial and the zero, and $0\le F\le 1$, because after a unitary change of variables $\ell_z=z_2$, and multiplying each Gaussian-orthogonal monomial by $z_2$ multiplies its squared norm by a positive integer. Consequently the finite measure

$$B\longmapsto\mathbb{E}\sum\nolimits_{z:\,p(z)=0}F(p,z)\,\mathbf{1}_B(z)$$

on $\mathbf{CP}^1$ is unitarily invariant, hence a constant multiple of the probability measure

$$d\nu(z)=\frac{d^2z}{\pi(1+|z|^2)^2}$$

in the affine chart $(z_1,z_2)=(1,z)$, and the constant is $\mathbb{E}Q(p)$.

We compute the density of this measure at $z=0$. Write $P_c(z)=p(1,z)$. The incidence map

$$(z,c_1,\dots,c_n)\longmapsto\left(-\sum\nolimits_{j=1}^{n}c_j\binom{n}{j}^{1/2}z^j,c_1,\dots,c_n\right)$$

parametrizes coefficient vectors together with a distinguished zero; off the null set of multiple zeros it has one preimage for each zero, and its real Jacobian is $|P_c'(z)|^2$. The area formula, applied to the Gaussian coefficient density and the bounded weight $F$, therefore gives the density of the measure. At $z=0$ the constraint reads $c_0=0$ and $P_c'(0)=\sqrt{n}\,c_1$, so the density there is

$$\frac{n}{\pi}\mathbb{E}\bigl[|c_1|^2F(p,0)\mid c_0=0\bigr],$$

where conditioning on $c_0=0$ means that $c_0$ is set to zero while the other coefficients retain their independent Gaussian laws. The density is continuous near zero by dominated convergence, since

on compact sets of $z$ the squared Jacobian is bounded by a constant times $\sum_{j=1}^n |c_j|^2$ and $F \le 1$. Comparing with $d\nu/d^2z = 1/\pi$ at zero yields

$$\mathbb{E}Q(p) = n\mathbb{E}\big[|c_1|^2 F(p,0) \mid c_0 = 0\big]. \tag{B.1}$$

Gaussian monomial orthogonality gives, when $c_0 = 0$,

$$\|p\|_G^2 = n! \sum_{j=1}^n |c_j|^2, \qquad \|p/z_2\|_G^2 = n! \sum_{j=1}^n \frac{|c_j|^2}{j}.$$

The variables $X_j = |c_j|^2$ are independent standard exponentials, so with $S = \sum_{j=1}^n X_j$, which has mean $n$, the vector $(V_j) = (X_j/S)$ is independent of $S$ and uniform on the simplex. Thus

$$\mathbb{E}V_1^2 = \frac{2}{n(n+1)}, \qquad \mathbb{E}V_1V_j = \frac{1}{n(n+1)} \quad (j \ne 1).$$

Substituting into (B.1) gives

$$\mathbb{E}Q(p) = n\mathbb{E}\left[\frac{X_1 \sum_{j=1}^n X_j/j}{S}\right] = \frac{n(\mathsf{H}_n + 1)}{n+1},$$

and by (4.2) some realization satisfies (4.1). □

B.3. **Haar averages and exponential rates.**

*Proof of Lemma 4.7.* Let $\Pi_\lambda$ be the central orthogonal projection onto the $\lambda$-isotypic component of $(\mathbb{C}^n)^{\otimes n}$, and put $e = e_1 \otimes \cdots \otimes e_n$. The central idempotent formula gives

$$\mathrm{Imm}_\lambda(P_n) = \frac{n!}{f^\lambda}\langle e, \Pi_\lambda P_n^{\otimes n} e\rangle, \qquad \langle e, \Pi_\lambda e\rangle = \frac{(f^\lambda)^2}{n!}.$$

For the second, only the identity permutation has a nonzero matrix coefficient at $e$.

Under Schur–Weyl decomposition, $Q_r^{\otimes n}$ acts on the $\lambda$-isotypic component as $S_\lambda(Q_r) \otimes I$, where $S_\lambda(\mathbb{C}^n)$ is the irreducible polynomial representation of $\mathrm{U}(n)$ with character $s_\lambda$. Averaging its unitary conjugates and applying Schur's lemma gives

$$\mathbb{E}P_n^{\otimes n} = \sum_{\lambda \vdash n} \frac{\operatorname{tr} S_\lambda(Q_r)}{\dim S_\lambda(\mathbb{C}^n)} \Pi_\lambda = \sum_{\lambda \vdash n} \frac{s_\lambda(1^r)}{s_\lambda(1^n)} \Pi_\lambda.$$

Hence the first equality in (4.5) holds. The hook-length and hook-content formulas give

$$\frac{f^\lambda}{s_\lambda(1^n)} = \frac{n!}{\prod_{(i,j)\in\lambda}(n+j-i)},$$

which proves the second equality. Finally, $s_\lambda(1^r) = 0$ when $\ell(\lambda) > r$. □

*Proof of Theorem 4.6.* For a partition $\lambda$ with at most $r$ rows, append zero parts to write $\lambda = (\lambda_1, \dots, \lambda_r)$. The hook-content product (2.6) can be written as

$$c_\lambda(q,t) = q^{\sum_{i=1}^r (i-1)\lambda_i} \frac{\prod_{i=1}^r \prod_{j=1}^{\lambda_i} (1 - tq^{j-i})}{\prod_{(i,j)\in\lambda} (1 - q^{h_{ij}})}.$$

Under the hypotheses on $t$, the only possible negative numerator factor is the factor for the cell $(r,1)$. Thus $c_\lambda(q,t) > 0$ when $\lambda_r = 0$ and $c_\lambda(q,t) < 0$ when $\lambda_r > 0$.

The same product gives bounds uniform in the partition: for each fixed $i \le r$, the partial products $\prod_{j=1}^l (1 - tq^{j-i})$ have no zero factor and converge as $l \to \infty$ to a nonzero limit, so their absolute values lie between positive constants, including at $l = 0$. Since $h_{ij} \ge \lambda_i - j + 1$, the denominator satisfies

$$(q;q)_\infty^r \le \prod_{i=1}^r (q;q)_{\lambda_i} \le \prod_{(i,j)\in\lambda} (1 - q^{h_{ij}}) \le 1.$$

It follows that constants $a, b > 0$, depending only on $q, t, r$, satisfy

$$a\, q^{\sum_{i=1}^{r}(i-1)\lambda_i} \le |c_\lambda(q,t)| \le b\, q^{\sum_{i=1}^{r}(i-1)\lambda_i}. \tag{B.2}$$

To compare the terms in the Haar average, define

$$\mathcal{D}_r = \left\{ p \in [0,1]^r : p_1 \ge \cdots \ge p_r,\ \sum_{i=1}^{r} p_i = 1 \right\}, \qquad \varphi(x) = (1+x)\log(1+x) - x,$$

and

$$\Phi(p) = (\log q) \sum_{i=1}^{r} (i-1)p_i - \sum_{i=1}^{r} \varphi(p_i).$$

For $p_i = \lambda_i/n$ and $n \ge 2r$, comparison with a Riemann sum gives, uniformly in $\lambda$,

$$\log \prod_{(i,j)\in\lambda} (n+j-i) = n \log n + n \sum_{i=1}^{r} \varphi(p_i) + O_r(1).$$

Indeed, replacing $j - i$ by $j$ inside each logarithm changes the sum by $O_r(1)$, and the function $\log(1+x)$ has bounded derivative on $[0,1]$. Stirling's formula and the dimension formula

$$1 \le s_\lambda(1^r) = \prod_{1\le i<j\le r} \frac{\lambda_i - \lambda_j + j - i}{j-i} \le (n+r)^{\binom{r}{2}}$$

then show that constants $A, B > 0$, depending only on $q, t, r$, satisfy, for every $\lambda \vdash n$ with $\ell(\lambda) \le r$ and every $n \ge 2r$,

$$A\sqrt{n}\, e^{n(\Phi(p)-1)} \le |c_\lambda(q,t)|\, \mathbb{E}\, \mathrm{Imm}_\lambda(P_n) \le B n^{\binom{r}{2}+1/2} e^{n(\Phi(p)-1)}. \tag{B.3}$$

The function $\Phi$ is strictly concave, with Hessian $-\operatorname{diag}((1+p_i)^{-1})$. Set

$$K = \frac{r+1}{\sum_{j=0}^{r-1} q^j}, \qquad p_i^* = K q^{i-1} - 1 \quad (1 \le i \le r).$$

Condition (4.4) gives $p_1^* > \cdots > p_r^* > 0$, and $\sum_i p_i^* = 1$. At $p^*$ all coordinates of $\nabla\Phi$ equal $-\log K$, so $p^*$ is the unique maximizer of $\Phi$ on $\mathcal{D}_r$. Compactness and uniqueness imply the strict gap

$$\Phi_\partial := \max_{\substack{p\in\mathcal{D}_r \\ p_r = 0}} \Phi(p) < \Phi(p^*) =: \Phi_*. \tag{B.4}$$

For every sufficiently large $n$, choose $\lambda_i^{(n)} = \lfloor n p_i^* \rfloor$ for $2 \le i \le r$ and $\lambda_1^{(n)} = n - \sum_{i=2}^{r} \lambda_i^{(n)}$. These parts form a partition with $\lambda_r^{(n)} > 0$ and $\lambda^{(n)}/n = p^* + O_r(n^{-1})$. The gradient of $\Phi$ is bounded on $\mathcal{D}_r$, so (B.3) gives a constant $A' > 0$ with

$$|c_{\lambda^{(n)}}(q,t)|\, \mathbb{E}\, \mathrm{Imm}_{\lambda^{(n)}}(P_n) \ge A'\sqrt{n}\, e^{n(\Phi_* - 1)}.$$

Every positive term has $\lambda_r = 0$, and there are at most $(n+1)^{r-1}$ such partitions, so their sum is at most

$$B' n^{\binom{r}{2}+r-1/2} e^{n(\Phi_\partial - 1)}$$

for a constant $B' > 0$. Keeping only the negative term of (2.6) indexed by $\lambda^{(n)}$ therefore gives

$$\mathbb{E}\, \mathrm{per}_{q,t}(P_n) \le B' n^{\binom{r}{2}+r-1/2} e^{n(\Phi_\partial - 1)} - A'\sqrt{n}\, e^{n(\Phi_* - 1)} < 0$$

for every sufficiently large $n$, by the strict gap (B.4), and a negative expectation forces a negative value at some projection in the orbit.

If $t$ ranges over a compact subinterval of $(q^{r-1}, q^{r-2})$, the row products used in (B.2) converge uniformly and stay uniformly away from zero. All constants in (B.3) can therefore be chosen uniformly on that subinterval. The rate function and the gap (B.4) depend only on $q, r$, which proves the asserted uniform choice of $n_0$. □

B.4. **Gaussian deletion and an explicit rank-three instance.**

*Proof of Lemma 4.11.* If some $u_i = 0$, both sides vanish. Otherwise let $z \in \mathbb{C}^d$ collect the coordinates of $g_1, \dots, g_k$, and put $p = \prod_i L_i$, $\rho = \|z\|^2$, and $\Delta = \sum_\nu \partial_{z_\nu}\partial_{\bar{z}_\nu}$. Direct differentiation and $c_a^2 \le (\max_a c_a)c_a$ give

$$\Delta \log L_i \ge \frac{\beta a_{ii}}{L_i}.$$

Let $\mathbb{E}_p$ be the Gaussian measure weighted by $p/\mathbb{E}p$. Bihomogeneity and integration by parts give $\mathbb{E}_p(\Delta p/p) = n$, while $\rho \sim \Gamma(n+d,1)$ under $\mathbb{E}_p$ and $\|\partial \log p\|^2 \ge n^2/\rho$ by Euler's identity, so that

$$\beta \frac{\sum_i a_{ii} F_C(A[[n]\setminus\{i\}])}{F_C(A)} \le \mathbb{E}_p \Delta \log p \le n - \frac{n^2}{n+d-1} = \frac{n(d-1)}{n+d-1}.$$

Integration by parts is applied only to the polynomial $p$; the nonnegative logarithmic terms, after multiplication by $p$, are bounded in sum by $\Delta p$, hence integrable. If $T_a = \sum_{|S|=a}(\prod_{i\in S} a_{ii}) F_C(A[S^c])$, then (4.7) gives $(a+1)T_{a+1} \le KT_a$, hence $T_a \le K^a F_C(A)/a!$, and expanding the product gives (4.8). Finally, $u_i^* Y u_i \le (\operatorname{tr} Y) a_{ii}$ proves the last assertion. □

In rank three, the constants in the proof of Theorem 4.9 give an explicit pair $(q,t)$.

**Remark B.2** (A rank-three instance)**.** *At $q = 1/10$ and $t = 21/2000$ every factor of $D_\lambda(t)$ with $\ell(\lambda) \le 3$ has absolute value at most one on $[0, 1/50]$, so $|D'_\lambda| \le B_3 = 370/3$, and $M_3 \le 729/64$; with $\delta = 1/2000$ the constant is $1 - (370/3)(729/64)/2000 = 3809/12800$, and*

$$\operatorname{per}_{1/10,21/2000}(A) \ge \frac{3809}{12800} \operatorname{per}_{1/10,1/100}(A) \ge 0 \qquad (\operatorname{rank} A \le 3)$$

*in every size. As $21/2000 > q^2$, the region $\mathcal{R}_3(1/10) = [0,1/100] \cup \{1/10\}$ is strictly smaller than the scalar region in rank three, whereas the two coincide for $q > (1+\sqrt{13})/6$ (Corollary 4.8).*

## Appendix C. The exact scalar regions in sizes three and four

Holding the size fixed, instead of the rank, gives a third scalar regime, and in small sizes it can be settled completely. The answers show how far the fixed-size regions are from the all-size ones of Section 4: the thresholds in sizes three and four both lie strictly above $t = q$, and their witnesses give necessary conditions in every larger size.

In size three the scalar region can be found exactly, because once the diagonal is normalized the permutation sum depends on the matrix through two real numbers, which positive semidefiniteness confines to an explicit region. The argument works for arbitrary nonnegative cycle weights $w = (w_k)_{k\ge1}$, for which we write

$$\operatorname{per}_w(A) = \sum_{\sigma\in S_n} \prod_{C\in \mathrm{Cyc}(\sigma)} w_{|C|} \prod_{i=1}^n a_{i,\sigma(i)}.$$

For constant weights and real symmetric matrices, the minimum below was found by Osogami et al. (2013, Theorem 6.1); the argument extends to independent cycle weights and complex matrices.

**Theorem C.1** (Sharp positivity in size three)**.** *Let $w_1, w_2, w_3 \ge 0$. For every Hermitian positive semidefinite matrix $A = (a_{ij})_{i,j=1}^3$,*

$$\operatorname{per}_w(A) \ge \left(w_1^3 + \min\left\{0, \frac{3w_1w_2 - w_3}{4}\right\}\right) a_{11}a_{22}a_{33}. \tag{C.1}$$

*The constant is sharp among matrices with prescribed positive diagonal entries. If a diagonal entry vanishes, both sides of (C.1) are zero. Consequently,*

$$\operatorname{per}_w(A) \ge 0 \text{ for every Hermitian } A \succeq 0 \iff 4w_1^3 + 3w_1w_2 - w_3 \ge 0. \tag{C.2}$$

*For Macdonald weights with $0 \le q,t < 1$, this condition is equivalent to*

$$t \le \tau_3(q) := \frac{2+3q+3q^2}{3+3q+2q^2}. \tag{C.3}$$

*Proof.* If a diagonal entry of $A$ vanishes, the corresponding row and column vanish by positive semidefiniteness, every permutation monomial is zero, and (C.1) holds with both sides zero. Otherwise a diagonal congruence, which multiplies the weighted permanent by $a_{11}a_{22}a_{33}$, reduces the assertion to a correlation matrix $R = (r_{ij})$. Put

$$s = |r_{12}|^2 + |r_{23}|^2 + |r_{31}|^2, \qquad z = \mathrm{Re}(r_{12}r_{23}r_{31}).$$

The permutation expansion and determinant inequality give

$$\mathrm{per}_w(R) = w_1^3 + w_1w_2s + 2w_3z, \qquad \det R = 1 - s + 2z \ge 0.$$

If $z \ge 0$, the weighted permanent is at least $w_1^3$. Suppose $z < 0$ and write $u = -z$. The arithmetic–geometric mean inequality and the determinant inequality imply

$$0 < s < 1, \qquad u \le \min\left\{ \left(\frac{s}{3}\right)^{3/2}, \frac{1-s}{2} \right\}. \tag{C.4}$$

The two bounds agree at $s = 3/4$.

On $0 \le s \le 3/4$ the resulting lower bound

$$w_1^3 + w_1w_2s - 2w_3(s/3)^{3/2}$$

is concave in $s$, so its minimum is attained at an endpoint, where it equals $w_1^3$ and $w_1^3+3w_1w_2/4-w_3/4$. For $3/4 \le s \le 1$, the other bound in (C.4) gives

$$\mathrm{per}_w(R) \ge w_1^3 - w_3 + (w_1w_2 + w_3)s \ge w_1^3 + \frac{3}{4}w_1w_2 - \frac{1}{4}w_3.$$

Together these cases prove (C.1). The value $w_1^3$ is attained by $I_3$, and the other endpoint value by

$$R_* = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \succeq 0,$$

with eigenvalues $3/2, 3/2, 0$; diagonal congruence transfers both equality cases to any positive diagonal, which proves sharpness, and the lower bound with these examples proves (C.2).

For Macdonald weights, $w_1 > 0$ and

$$\frac{4w_1^3 + 3w_1w_2 - w_3}{w_1} = \frac{4(1-t)^2}{(1-q)^2} + \frac{3(1-t^2)}{1-q^2} - \frac{1+t+t^2}{1+q+q^2}.$$

The right side factors as

$$\frac{3(1-qt)(2+3q+3q^2-(3+3q+2q^2)t)}{(1-q)^2(1+q)(1+q+q^2)},$$

and all factors other than the last linear one are positive on $0 \le q,t < 1$, which proves (C.3). □

**Example C.2** (Scalar failure above the exact size-three threshold)**.** *The rank-two matrix $R_*$ in Theorem C.1 satisfies $\mathrm{per}_{q,t}(R_*) < 0$ exactly when $t > \tau_3(q)$. For example,*

$$\mathrm{per}_{0,3/4}(R_*) = -\frac{3}{64}, \qquad \mathrm{per}_{1/10,3/4}(R_*) = -\frac{250}{8019}.$$

The threshold exceeds $q$, since

$$\tau_3(q) - q = \frac{2(1-q^3)}{3+3q+2q^2} > 0 \qquad (0 < q < 1),$$

so the size-three scalar region is an interval that contains $\mathcal{R}_3(q) = [0, q^2] \cup \{q\}$ and extends beyond $t = q$; at $q = 0$ the threshold is $t = 2/3$. In the Jack limit the criterion becomes $\beta(4\beta-1)(\beta+1) \geq 0$, so for the $\beta$-permanent with $\beta > 0$ the size-three threshold is $\beta \geq 1/4$.

Padding $R_*$ with an identity block carries the size-three obstruction to every larger size.

**Corollary C.3** (A necessary condition in every larger size)**.** *Fix $n \geq 3$ and $0 \leq q, t < 1$. If $\mathrm{per}_{q,t}(A) \geq 0$ for every $n \times n$ Hermitian positive semidefinite matrix, then $t \leq \tau_3(q)$. More precisely, if $t > \tau_3(q)$, the real positive semidefinite matrix $R_* \oplus I_{n-3}$ satisfies*

$$\mathrm{per}_{q,t}(R_* \oplus I_{n-3}) = w_1^{n-3}\left(w_1^3 + \frac{3}{4}w_1 w_2 - \frac{1}{4}w_3\right) < 0.$$

*Proof.* Every nonzero permutation monomial respects the two diagonal blocks. The singleton blocks contribute $w_1^{n-3} > 0$, and Theorem C.1 gives the sign of the remaining factor. □

In size four the same reduction to correlation matrices leaves four cycle sums, whose values have a convex hull with only seven vertices, because seven test matrices decide every class-function inequality on $4 \times 4$ positive semidefinite matrices (Barrett et al., 1999, Theorem 2(a)). The sign of $\mathrm{per}_w$ on the cone is therefore decided by seven matrices, three of which give the exact region.

**Theorem C.4** (Sharp positivity in size four)**.** *Put $S(q) = 1 + q + q^2$, $B(q) = 2 + q + q^2 + 2q^3$, and*

$$\tau_4(q) = \frac{2S(q)}{B(q) + \sqrt{B(q)^2 - 4qS(q)^2}}.$$

*For $0 \leq q, t < 1$, the following are equivalent:*

*(i) $\mathrm{per}_{q,t}(A) \geq 0$ for every Hermitian positive semidefinite $A \in M_4(\mathbb{C})$;*

*(ii) $Q_q(t) := S(q) - B(q)t + qS(q)t^2 \geq 0$;*

*(iii) $t \leq \tau_4(q)$.*

*In particular, $\tau_4(0) = 1/2$. If $t > \tau_4(q)$, a negative witness is the rank-two correlation matrix*

$$H = \begin{pmatrix} 1 & s & s & s \\ s & 1 & \mathrm{i}s & -\mathrm{i}s \\ s & -\mathrm{i}s & 1 & \mathrm{i}s \\ s & \mathrm{i}s & -\mathrm{i}s & 1 \end{pmatrix}, \qquad s = \frac{1}{\sqrt{3}}. \tag{C.5}$$

We identify the seven test matrices directly, using the cited theorem only for their number.

**Lemma C.5** (Seven cycle-sum vectors)**.** *For a correlation matrix $A$ of order four, put*

$$x = \sum\nolimits_{i<j} |a_{ij}|^2, \qquad y = |a_{12}a_{34}|^2 + |a_{13}a_{24}|^2 + |a_{14}a_{23}|^2,$$

*and let $z$ and $v$ be the sums of the permutation monomials of cycle types $(3,1)$ and $(4)$. The convex hull of $(x, y, z, v)$ over all complex correlation matrices of order four has exactly the following seven vertices, where $J_m$ is the all-ones matrix, $R_m = \frac{m}{m-1}I_m - \frac{1}{m-1}J_m$, and $H$ is the matrix (C.5):*

| $A$ | $x$ | $y$ | $z$ | $v$ |
|---|---|---|---|---|
| $I_4$ | 0 | 0 | 0 | 0 |
| $J_4$ | 6 | 3 | 8 | 6 |
| $J_3 \oplus 1$ | 3 | 0 | 2 | 0 |
| $J_2 \oplus J_2$ | 2 | 1 | 0 | 0 |
| $R_3 \oplus 1$ | 3/4 | 0 | −1/4 | 0 |
| $R_4$ | 2/3 | 1/27 | −8/27 | 2/27 |
| $H$ | 2 | 1/3 | 0 | −2/3 |

*Proof.* Let $P$ be the convex hull, a compact set. A real class-function evaluation on a correlation matrix is an affine function of $(x,y,z,v)$; invertible diagonal congruence multiplies every permutation monomial by the same positive factor, and a positive semidefinite matrix with a zero diagonal entry has a zero row. The seven-test theorem (Barrett et al., 1999, Theorem 2(a)) with convex separation therefore makes $P$ the convex hull of at most seven points, and it remains to exhibit the seven listed points as distinct vertices. In the face arguments below, each inequality is applied first to generating correlation matrices and then to their convex combinations, and equality forces equality in every component with positive coefficient.

Every correlation matrix has $0 \le x \le 6$. At $x=0$ the matrix is $I_4$; at $x=6$, $\operatorname{tr}(A^2) = 4+2x = 16 = (\operatorname{tr} A)^2$ forces rank one, and a rank-one correlation matrix is diagonally unitarily equivalent to $J_4$. These two values give unique cycle-sum vectors, hence vertices.

The face $y=0$ consists of matrices whose graph of nonzero off-diagonal entries has no two disjoint edges, so the graph lies in a star or in a triangle with an isolated vertex. A star has $z=v=0$; a triangle has $v=0$ and $z = 2\operatorname{Re} p$ with $p = a_{12}a_{23}a_{31}$ after relabelling. The bound $z \le 2$ has equality only at the cycle vector of $J_3 \oplus 1$. If $\operatorname{Re} p = -a < 0$, the $3\times 3$ determinant and the AM–GM inequality give $3a^{2/3} \le 3|p|^{2/3} \le x \le 1-2a$, so $a \le 1/8$ and $z \ge -1/4$, with equality forcing $x = 3/4$ and the cycle vector of $R_3 \oplus 1$. The two extrema on this face are two more vertices.

The affine functional $\det A = 1 - x + y + z - v$ is nonnegative, and its zero face is generated by matrices of rank at most three, on which $4 + 2x = \operatorname{tr}(A^2) \ge (\operatorname{tr} A)^2/3 = 16/3$. Equality forces the spectrum $(4/3, 4/3, 4/3, 0)$, so $A = \frac{4}{3}(I - uu^*)$ for a unit vector $u$ with $|u_i| = 1/2$, and diagonal unitary conjugation gives $R_4$, the fifth vertex.

Finally, $e_3(A) = \sum_{|T|=3} \det A[T] = 4 - 2x + z \ge 0$ vanishes exactly at rank at most two, where $4 + 2x \ge 16/2$ with equality exactly for the spectrum $(2,2,0,0)$. Then $A^2 = 2A$, $z=0$, and $v = y - 1$, and the diagonal equations of $A^2 = 2A$ give $|a_{12}|^2 = |a_{34}|^2 = a$, $|a_{13}|^2 = |a_{24}|^2 = b$, $|a_{14}|^2 = |a_{23}|^2 = c$ with $a+b+c = 1$, so $y = a^2 + b^2 + c^2 \in [1/3, 1]$. The endpoints of this segment are attained by $H$, which satisfies $H^2 = 2H$ and so is positive semidefinite of rank two with the listed cycle sums, and by $J_2 \oplus J_2$. These are the last two vertices. □

Evaluating $\operatorname{per}_w$ at the seven vertices turns Lemma C.5 into inequalities on the weights; four vertices give positive values, and the other three give the conditions below.

**Corollary C.6** (Positive cycle weights in size four)**.** *For $w_1, w_2, w_3, w_4 > 0$, $\operatorname{per}_w$ is nonnegative on every Hermitian positive semidefinite matrix of order four if and only if*

$$4w_1^3 + 3w_1w_2 - w_3 \ge 0, \tag{C.6}$$

$$27w_1^4 + 18w_1^2w_2 + w_2^2 - 8w_1w_3 + 2w_4 \ge 0, \tag{C.7}$$

$$3w_1^4 + 6w_1^2w_2 + w_2^2 - 2w_4 \ge 0. \tag{C.8}$$

*Proof.* For a correlation matrix, $\operatorname{per}_w(A) = w_1^4 + w_1^2w_2x + w_2^2y + w_1w_3z + w_4v$, which is positive on the first four matrices of Lemma C.5; its values on $R_3 \oplus 1$, $R_4$, and $H$, multiplied by positive factors, are the left sides of (C.6)–(C.8). □

***Proof of Theorem C.4.*** A brief calculation gives

$$\operatorname{per}_{q,t}(H) = w_1^4 + 2w_1^2w_2 + \frac{w_2^2 - 2w_4}{3} = \frac{8(1-t)(1-qt)Q_q(t)}{3(1-q)^4(1+q)^2(1+q^2)}, \tag{C.9}$$

with every factor other than $Q_q(t)$ positive on the square, so $Q_q(t) \ge 0$ is necessary. Since $Q_q(0) = S(q) > 0$, $Q_q(1) = -(1-q)^2(1+q) < 0$, and $B(q)^2 - 4qS(q)^2 = (1-q)^2(4q^4+8q^3+9q^2+8q+4) > 0$, the quadratic $Q_q$ has exactly one root in $(0,1)$, namely $\tau_4(q)$, and is nonnegative on $[0,1)$ exactly to its left; the formula also holds at $q=0$.

For sufficiency, apply Corollary C.6. Inequality (C.6) is $t \le \tau_3(q)$ by Theorem C.1, and with $D = 3 + 3q + 2q^2$,

$$D^2 Q_q(\tau_3(q)) = -(1-q)^2(1+q)(3q^4 + 8q^3 + 12q^2 + 8q + 3) < 0,$$

so $\tau_4(q) < \tau_3(q)$ and (C.6) follows from $t \le \tau_4(q)$. Inequality (C.7) holds everywhere: $\mathrm{per}_{q,t}(R_4)$ equals $8(1-t)(1-qt)(a(q) - b(q)t + c(q)t^2)$ over $27(1-q)^4(1+q)^2(1+q^2)(1+q+q^2)$, where

$$a(q) = 5 + 12q + 20q^2 + 20q^3 + 16q^4 + 8q^5,$$
$$b(q) = 12 + 28q + 41q^2 + 41q^3 + 28q^4 + 12q^5,$$
$$c(q) = 8 + 16q + 20q^2 + 20q^3 + 12q^4 + 5q^5,$$

and the quadratic in $t$ is positive for every real $t$ because $c(q) > 0$ and

$$b(q)^2 - 4a(q)c(q) = -(1-q)^2(16 + 64q + 152q^2 + 240q^3 + 271q^4 + 240q^5 + 152q^6 + 64q^7 + 16q^8) < 0.$$

Inequality (C.8) is $Q_q(t) \ge 0$ by (C.9). □

Since $Q_q(q) = (1-q)(1-q^4) > 0$, the thresholds satisfy $q < \tau_4(q) < \tau_3(q)$ for $0 \le q < 1$: the size-four region is strictly larger than the all-size rank-two region $t \le q$ and strictly smaller than the size-three region; at $q = 0$ they are $t \le 1/2$ and $t \le 2/3$. Padding the witness by singleton blocks, $\mathrm{per}_{q,t}(H \oplus I_{n-4}) = w_1^{n-4}\,\mathrm{per}_{q,t}(H)$, so $t \le \tau_4(q)$ is necessary for scalar positivity in every fixed size $n \ge 4$, which sharpens Corollary C.3.

## Appendix D. Partition coefficients for the block inequalities

The proof of Theorem 5.1 expands $\mathrm{per}_{q,t}$ in products of permanents of principal blocks and needs the signs of the coefficients. At $t = q^M$ these coefficients are sums over assignments of distinct colors $q^r$ to the blocks, which makes them divided differences of $x \mapsto x^\alpha$. Their signs are then those of the falling factorial $\alpha(\alpha-1)\cdots(\alpha-k+1)$, and this is where the size hypothesis $t \in \mathcal{R}_n(q)$ comes from. Let $\Pi_n$ denote the set partitions of $[n]$, and put $z^{\underline{k}} = z(z-1)\cdots(z-k+1)$, with $z^{\underline{0}} = 1$.

For positive integers $s_1, \dots, s_k$ and a symmetric-function specialization $\rho$, define

$$\widetilde{m}_s[\rho] := \sum_{\theta \in \Pi_k} \prod_{C \in \theta} (-1)^{|C|-1}(|C|-1)!\, p_{\sum_{i \in C} s_i}[\rho]. \tag{D.1}$$

On a finite alphabet $z = (z_1, \dots, z_M)$, this is

$$\widetilde{m}_s(z) = \sum_{\substack{j_1,\dots,j_k \in [M] \\ j_1,\dots,j_k \text{ distinct}}} z_{j_1}^{s_1} \cdots z_{j_k}^{s_k}. \tag{D.2}$$

Indeed, expanding the power sums in (D.1) and fixing the indices $j_1, \dots, j_k$ chosen for the $k$ factors, the coefficient of $z_{j_1}^{s_1} \cdots z_{j_k}^{s_k}$ factors over the groups of equal indices, and for a group of size $r$ the factor is $\sum_{\sigma \in S_r} (-1)^{r - c(\sigma)}$, which equals one for $r = 1$ and zero for $r > 1$. In the usual monomial notation, $\widetilde{m}_s = (\prod_j m_j(s)!)\, m_{\mathrm{sort}(s)}$, where $m_j(s)$ is the number of entries of $s$ equal to $j$.

For pairwise distinct real nodes $x_0, \dots, x_k$, write

$$[x_0, \dots, x_k]f := \sum_{j=0}^{k} \frac{f(x_j)}{\prod_{\ell \ne j}(x_j - x_\ell)}$$

for the divided difference of order $k$. Its sign determines the sign of every coefficient in the expansion.

**Lemma D.1** (Signs of the partition coefficients)**.** *Let $0 < q < 1$, $\alpha > 0$, $k \ge 1$, and $s = (s_1, \dots, s_k) \in \mathbb{N}^k$ with $s_i \ge 1$. For $\tau \in S_k$, set $a_i = s_{\tau(i)}$, $S_j = \sum_{i=j+1}^{k} a_i$ for $0 \le j \le k$, and $x_j = q^{S_j}$. Then*

$$\widetilde{m}_s[B_{q,q^\alpha}] = \sum_{\tau \in S_k} q^{\sum_{i=1}^{k}(i-1)a_i}\, [x_0, \dots, x_k](x^\alpha), \tag{D.3}$$

*and*

$$\operatorname{sgn} \widetilde{m}_s[B_{q,q^\alpha}] = \operatorname{sgn} \alpha^{\underline{k}}. \tag{D.4}$$

*If* $t \in \mathcal{R}_n(q)$ *and* $\sum_i s_i = n$, *then* $\widetilde{m}_s[B_{q,t}] \geq 0$.

*Proof.* First take $\alpha = M$ to be an integer with $M \geq k$. The specialization $B_{q,q^M}$ is the finite alphabet $(1, q, \ldots, q^{M-1})$. In (D.2), sort the selected indices as $0 \leq r_1 < \cdots < r_k \leq M-1$ and let $\tau \in S_k$ be the permutation for which $r_i$ carries the exponent $a_i = s_{\tau(i)}$. Put

$$g_0 = r_1, \qquad g_j = r_{j+1} - r_j - 1 \quad (1 \leq j < k), \qquad g_k = M - 1 - r_k.$$

These gaps are nonnegative and sum to $M-k$, and the term is

$$q^{\sum_i a_i r_i} = q^{\sum_i (i-1)a_i} \prod_{j=0}^{k} x_j^{g_j}.$$

Summing over the gaps therefore gives $q^{\sum_i (i-1)a_i} h_{M-k}(x_0, \ldots, x_k)$. Partial fractions in the generating function $\prod_{j=0}^{k} (1 - x_j z)^{-1}$ give

$$h_{M-k}(x_0, \ldots, x_k) = \sum_{j=0}^{k} \frac{x_j^M}{\prod_{\ell \neq j} (x_j - x_\ell)} = [x_0, \ldots, x_k](x^M).$$

Summing over $\tau$ proves (D.3) for these integer values of $\alpha$.

For fixed $q$, both sides are polynomials in $t = q^\alpha$ of degree at most $\sum_i s_i$, the right side because $x_j^\alpha = t^{S_j}$, and they agree at the infinitely many points $t = q^M$, hence identically.

The nodes satisfy $0 < x_0 < \cdots < x_k = 1$, and for $f \in C^k$ on their interval the mean value theorem for divided differences gives

$$[x_0, \ldots, x_k] f = \frac{f^{(k)}(\xi)}{k!} \qquad \text{for some } \xi \in (x_0, x_k).$$

For $f(x) = x^\alpha$, $f^{(k)}(x) = \alpha^{\underline{k}} x^{\alpha - k}$, so every summand in (D.3) has the sign of $\alpha^{\underline{k}}$; if the falling factorial vanishes, $x^\alpha$ is a polynomial of degree less than $k$ and every divided difference is zero. Either way (D.4) follows.

Finally, for $0 < t < 1$, membership of $t = q^\alpha$ in $\mathcal{R}_n(q)$ means $\alpha \in \{1, 2, \ldots\}$ or $\alpha \geq n - 1$, and then $\alpha^{\underline{k}} \geq 0$ because $k \leq n$; at $t = 0$ the conclusion follows by continuity. □

The positive coefficients let Lieb's inequality act on each principal block separately.

**Proposition D.2** (Expansion into products of permanents)**.** *Let* $n \geq 1$, $0 < q < 1$, $t \in \mathbb{C}$, *and* $A \in M_n(\mathbb{C})$. *For* $\pi \in \Pi_n$, *put* $a_\pi(q,t) = \widetilde{m}_{(|B|:B\in\pi)}[B_{q,t}]$. *Then*

$$\operatorname{per}_{q,t}(A) = \sum_{\pi \in \Pi_n} a_\pi(q,t) \prod_{B \in \pi} \operatorname{per}(A[B]). \tag{D.5}$$

*If* $t \in \mathcal{R}_n(q)$, *every coefficient* $a_\pi(q,t)$ *is nonnegative.*

*Proof.* Take $t = q^M$ with an integer $M \geq n$, so that $w_{|C|} = \sum_{r=0}^{M-1} q^{r|C|}$ and each cycle $C$ of $\sigma$ chooses a color $r \in \{0, \ldots, M-1\}$ with weight $q^{r|C|}$. The indices whose cycles share a color form a block $B$ of a set partition $\pi$. For prescribed blocks and distinct colors, summing the permutations within $B$ gives $\operatorname{per}(A[B])$, and summing over the distinct colors gives $\widetilde{m}_{(|B|:B\in\pi)}(1, q, \ldots, q^{M-1})$. Thus (D.5) holds at $t = q^M$, and since both sides are polynomials in $t$ of degree at most $n$, it holds for every $t$. Lemma D.1 proves the final assertion. □

## Appendix E. Boundary formulas used in the positivity proofs

Several proofs use $\operatorname{per}_{q,t}$ at the edges of the parameter square, where its formulas simplify: the case $q = 0$ of the rank-two theorem uses the Hall–Littlewood formula, and the matrix-ball comparison the Jack limit. We also prove the two-type positivity mentioned after Theorem 4.3.

**Proposition E.1** (Limits and boundary formulas)**.** *For $A \in M_n(\mathbb{C})$, the following hold.*
*(1) If $q = e^{-\varepsilon}$ and $t = e^{-\alpha\varepsilon}$ with $\alpha > 0$, then $\mathrm{per}_{q,t} \to \mathrm{per}_\alpha$ as $\varepsilon \downarrow 0$, with Jack parameter $1/\alpha$.*
*(2) On $q = t$, one has $\mathrm{per}_{q,t}(A) = \mathrm{per}(A)$.*
*(3) At $q = 0$,*

$$\begin{aligned}\mathrm{per}_{0,t}(A) &= \sum\nolimits_{S\subseteq[n]} (-t)^{|S|} \det A[S] \,\mathrm{per}\, A[S^c] \\ &= (1-t)\sum\nolimits_{j=0}^{n-1} (-t)^j \,\mathrm{Imm}_{(n-j,1^j)}(A), \qquad n \geq 1. \end{aligned} \tag{E.1}$$

*The subset identity also holds for $n = 0$, with value 1.*
*(4) At $t = 0$, the Schur coefficients are*

$$c_\lambda(q,0) = \frac{q^{\sum_i (i-1)\lambda_i}}{\prod_{(i,j)\in\lambda}(1-q^{h_{ij}})}.$$

*(5) For an integer $m \geq 1$,*

$$\mathrm{per}_{q,q^m}(A) = \sum\nolimits_{S_0 \sqcup \cdots \sqcup S_{m-1}=[n]} \prod\nolimits_{r=0}^{m-1} q^{r|S_r|} \,\mathrm{per}\, A[S_r]. \tag{E.2}$$

*The blocks are ordered and may be empty.*

*Proof.* For fixed $k$,

$$\frac{1-e^{-\alpha k\varepsilon}}{1-e^{-k\varepsilon}} \longrightarrow \alpha.$$

The permutation sum has finitely many terms, so its limit is $\mathrm{per}_\alpha(A)$. The Macdonald power-sum norm tends to $z_\mu \alpha^{-\ell(\mu)}$; the Jack scalar product with parameter $\theta$ has norm $z_\mu \theta^{\ell(\mu)}$. Thus $\theta = 1/\alpha$. On $q = t$ every $w_k = 1$.

At $q = 0$, the determinant product reduces to $\det(I - tXA)/\det(I - XA)$. The principal-minor expansion of the numerator and the ordinary-permanent instance of the master theorem give the subset identity in (E.1). For its immanant form, take $q \downarrow 0$ in (2.6). A partition that is not a hook contains the cell $(2,2)$, whose numerator is $q(1-t)$; every other numerator stays bounded and every denominator tends to one. Its coefficient therefore vanishes. In the hook $(n-j, 1^j)$, the cell $(1,1)$ contributes $1-t$, the other first-row cells contribute one, and the remaining $j$ first-column cells contribute $-t$. The product of these factors is the stated hook coefficient. Setting $t = 0$ in (2.6) gives the fourth assertion directly.

Finally, $w_k(q, q^m) = \sum_{r=0}^{m-1} q^{rk}$. Choosing a term in this sum colors a $k$-cycle with a color $r$ and assigns it weight $q^{rk}$. Let $S_r$ contain the labels in cycles of color $r$. For a fixed ordered partition $(S_0, \ldots, S_{m-1})$, summing the permutations within each block gives $\mathrm{per}\, A[S_r]$, while the color weight is $q^{r|S_r|}$. Every colored permutation occurs once, which proves (E.2). □

As $q \uparrow 1$, each cell factor $(q^{i-1} - q^\alpha q^{j-1})/(1-q^{h_{ij}})$ of the hook product tends to $(\alpha + j - i)/h_{ij}$, and since $n$ has finitely many partitions, the immanant expansion becomes

$$\mathrm{per}_\alpha(A) = \sum\nolimits_{\lambda\vdash n} \left( \prod\nolimits_{(i,j)\in\lambda} \frac{\alpha + j - i}{h_{ij}} \right) \mathrm{Imm}_\lambda(A). \tag{E.3}$$

With two distinct Gram vectors, each repeated, every permutation monomial is nonnegative.

**Corollary E.2** (Two-type positivity)**.** *Let $K = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \succeq 0$ and $0 \leq q, t < 1$. If $K[p,m]$ repeats its first index $p$ times and its second index $m$ times, then $\mathrm{per}_{q,t}(K[p,m]) \geq 0$ for every $p, m \in \mathbb{N}$.*

*Proof.* A permutation of the repeated labels crosses from the first type to the second as many times as it crosses back. If that number is $j$, its matrix monomial is $a^{p-j} d^{m-j} |b|^{2j} \geq 0$. Every cycle weight is positive, so the defining sum is nonnegative. □