hexagon:2610.00728v1

WEAK POSITIVITY OF SCHUR FORMS UNDER DECOMPOSABLE CURVATURE

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Abstract. Griffiths's 1969 conjecture predicts positivity of Schur forms for Griffiths-positive vector bundles. Finski reduced its weak form to a sign question for the double mixed discriminant $\Phi_n(H)$ of a positive linear map. Du's rank-nine counterexample shows that $\Phi_n(H) \geq 0$ fails in general. We prove $\Phi_n(H) \geq 0$ for every decomposable map H, a sum of a completely positive and a completely copositive map, using a quaternionic Gaussian determinant representation. Combined with Finski's incidence formula, this inequality gives weak nonnegativity of every Schur form under decomposable curvature, without splitting assumptions. Beyond the decomposable cone, nonnegativity of $\Phi_k(H_{V,U})$ on every square compression is equivalent to $\Phi_n(H+S) \geq 0$ for every entanglement-breaking map S. This class contains explicit indecomposable maps, but is nonconvex in every size $n \geq 9$. We prove the nonconvexity by expressing Du's map as an average of four members, and the obstruction persists when all compression inequalities are strict.

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1. Introduction

Griffiths asked in 1969 whether positivity of the curvature of a Hermitian holomorphic vector bundle forces positivity of its characteristic forms (Griffiths, 1969, p. 247). In the Schur basis, the weak form of the question asks whether every Schur form $s_\lambda(E,h)$ is weakly positive under Griffiths-positive curvature. The corresponding numerical statement for Schur classes of ample bundles is the theorem of Fulton and Lazarsfeld (1983). Du's counterexample shows that the statement for forms

2020 *Mathematics Subject Classification.* Primary 15B48, 15A15; Secondary 47L07, 32L05, 05E05, 81P40.

Key words and phrases. Positive linear map, quantum permanent, mixed discriminant, entanglement-breaking map, decomposable map, Breuer–Hall map, Chern form, Schur form.

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First public release: 8 October 2026, as v1 in the author's preprint repository, https://github.com/suvrit/preprints/tree/main/preprints/qperm; every version is kept there with its timestamp.

fails: for every $r \geq 9$, the bundle $\mathcal{O}(1)^{\oplus r}$ over $\mathbb{P}^r$ carries a strictly Griffiths-positive metric whose top Chern form is negative on a nonempty open set (Du, 2026). This counterexample leaves the problem of identifying curvature conditions that do ensure pointwise positivity.

Such restricted cases have a longer history. Bott and Chern proved nonnegativity of Chern forms under the condition now called Bott–Chern nonnegativity (Bott and Chern, 1965, Lemma 5.3 and (5.5)), and Li extended the conclusion to Schur forms (Li, 2021, Proposition 3.1). Finski formulated the equivalent positive-map problem, proved Schur-form positivity under Nakano and dual Nakano positivity, and introduced decomposable curvature (Finski, 2022). Wan subsequently proved weak positivity for strongly decomposable curvature of type I and positivity for type II (Wan, 2024, Theorems 1.2 and 1.4). These hypotheses impose splittings on the tangent or fiber space. We consider ordinary decomposable curvature, with no such splitting.

Let (E, h) have rank r over a complex manifold of dimension n. In Finski's operator dictionary, the curvature at x is a map $H_x : M_n(\mathbb{C}) \to M_r(\mathbb{C})$ (Section 5). We call it *decomposable* if

$$H_x(X) = \sum_a A_a X A_a^* + \sum_b B_b X^{\mathsf{T}} B_b^*, \tag{1.1}$$

a sum of a completely positive and a completely copositive map, with no orthogonality or splitting imposed between the two families.

Schur forms under decomposable curvature. Theorem 5.3 proves that every Schur form is weakly nonnegative at a point where the curvature is decomposable. Thus a negative Schur-form evaluation requires indecomposable curvature. The proof uses Finski's quotient stability and incidence formula (Finski, 2022, Proposition 2.22 and Theorem 3.2): a Schur form is the fiber integral of the top Chern form of an auxiliary quotient bundle, whose curvature is again decomposable. It therefore suffices to control the top Chern form in every dimension.

The coefficient of a top Chern form is a determinant invariant of the curvature map. For a complex-linear map $H : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, put

$$\Gamma_n(H) = \det(\partial_X) \det H(X), \qquad \Phi_n(H) = \frac{\Gamma_n(H)}{n!}, \tag{1.2}$$

where $X = (x_{ij})$ has algebraically independent entries and $\partial_X = (\partial/\partial x_{ij})$. For $n = r$, the curvature convention in Section 5 gives

$$c_n(E, h)|_x = \Gamma_n(H_x)\, dV_x,$$

with dV_x the coordinate volume form. Griffiths semipositivity is equivalent to positivity of H_x, meaning $H_x(X) \succeq 0$ for $X \succeq 0$. With the unnormalized mixed-discriminant convention, $\Gamma_n(H)$ is Gurvits's quantum permanent $\mathrm{QP}(C_H)$ of the Choi matrix of H (Gurvits, 2002, Definition 1.7). It is also Cayley's first hyperdeterminant of order four of the coefficient tensor. The normalized invariant Φ_n is the double mixed discriminant in Finski's formulation (Finski, 2022, (1.6), (3.47)–(3.48)). His Theorem 1.3 identifies the universal weak Schur-form problem with the inequality $\Phi_n(H) \geq 0$ for positive H.

Wan proved this inequality for all positive maps in size three (Wan, 2026); sizes one and two follow as well. Du's example corresponds to a positive *bistochastic* map H_9 on M_9, meaning $H_9(I) = H_9^*(I) = I$, with $\Gamma_9(H_9) = -3/64$; Proposition 4.2 gives a self-contained determinant calculation for it.

Quaternionic representation. The sign needed for Theorem 5.3 follows from a Gaussian determinant formula. For a decomposable map (1.1) with $n = r$, put $A(g) = \sum_a g_a A_a$ and $B(h) = \sum_b h_b B_b$, where all g_a, h_b are independent standard complex Gaussians. Theorem 2.3 gives

$$\Gamma_n(H) = \mathbb{E} \det \begin{pmatrix} A(g) & -B(h) \\ B(h) & A(g) \end{pmatrix} \geq 0.$$

The block determinant is nonnegative for each realization. For $B = 0$, the formula is Gurvits's identity $\Gamma_n(H) = \mathbb{E}|\det A(g)|^2$ (Gurvits, 2004, Theorem 3.8). The Gaussian matrices $A(g)$ and $B(h)$

are independent, and the entries of each may have arbitrary covariance. Equivalently, the formula evaluates $\mathrm{QP}(P + Q^{\mathsf{T}_{\mathrm{in}}})$ for arbitrary $P, Q \succeq 0$, where T_{in} transposes the input tensor factor.

Beyond the decomposable cone, the sign need not survive compression or addition. Proposition 4.7 gives positive maps with $\Gamma_n(H) > 0$ but $\Gamma_k(U^*H(V \cdot V^*)U) < 0$, and an entanglement-breaking map S for which $\Gamma_n(H + tS) < 0$ for large t. This failure motivates imposing the sign condition on every compression.

We ask which positive perturbations preserve determinant positivity, and why that stability stops short of arbitrary convex mixing. For isometries $V, U : \mathbb{C}^k \to \mathbb{C}^n$, chosen independently, write $H_{V,U}(X) = U^*H(VXV^*)U$ for the *square compression* of H, and define

$$\mathcal{K}_n = \{H : M_n \to M_n \text{ positive} : \ \Gamma_k(H_{V,U}) \geq 0 \text{ for every } k, V, U\}. \tag{1.3}$$

We call this property *hereditary determinant positivity*. Since compressions of decomposable maps are decomposable, Theorem 2.3 places the decomposable cone in $\mathcal{K}_n$. An *entanglement-breaking* map is one of the form $S(X) = \sum_{\ell=1}^{N} (b_\ell^* X b_\ell) a_\ell a_\ell^*$.

Entanglement-breaking stability. Theorem 3.2 proves

$$H \in \mathcal{K}_n \quad \Longleftrightarrow \quad \Gamma_n(H + S) \geq 0 \quad \text{for every entanglement-breaking } S. \tag{1.4}$$

For the forward implication, Proposition 3.1 expands $\Gamma_n(H + S)$ over subsets of the rank-one summands of S. Each term is a complementary compression invariant of H multiplied by two nonnegative Gram determinants. The converse exposes a negative compression as the leading coefficient of the polynomial $t \mapsto \Gamma_n(H + tS)$ for a suitable S. The class is closed under square compression, adjoints, independent congruences, and further entanglement-breaking addition (Proposition 3.3), and a coordinate version of the theorem characterizes stability under all nonnegative classical additions (Proposition 3.8). Two consequences are given by Corollary 3.4. For positive H, Wan's theorem gives

$$\min\{\operatorname{rank} H(I), \operatorname{rank} H^*(I)\} \leq 3 \quad \Longrightarrow \quad H \in \mathcal{K}_n.$$

A positive map K on M_4 with $\Gamma_4(K) \geq 0$, embedded in M_n by congruence, also belongs to $\mathcal{K}_n$, by congruence covariance for its size-four compressions. Thus the single inequality $\Gamma_4(K) \geq 0$ certifies the embedded map in every dimension $n \geq 4$.

Hereditary determinant positivity is not convex. Theorem 4.4 constructs

$$A_0, A_1, A_2, A_3 \in \mathcal{K}_9, \qquad H_9 = \frac{1}{4}\sum_{j=0}^{3} A_j, \qquad \Gamma_9(H_9) = -\frac{3}{64}.$$

Three summands are embedded four-dimensional Breuer–Hall maps; the fourth is decomposable. Padding gives nonconvexity for every $n \geq 9$. The summands can even lie in $\operatorname{int} \mathcal{K}_n$ while their average has negative invariant (Corollary 4.5). Consequently, no convex cone invariant under simultaneous unitary conjugation can contain all decomposable maps and a padded four-dimensional Breuer–Hall map while keeping $\Gamma_n \geq 0$ (Corollary 4.6). Convex subclasses with a fixed input or output support remain available (Proposition 4.1).

On the bundle side, every Chern form is weakly nonnegative wherever all square compressions of the curvature map have nonnegative invariant (Proposition 5.1). The regularized maps of Corollary 4.5 give strictly Griffiths-positive curvature tensors, each with every Chern form strictly positive in the weak sense, whose average has a negative top Chern form (Corollary 5.4). Even simultaneous strict pointwise positivity of all Chern forms is thus not a convex condition on Griffiths-positive curvature.

Two certificates for single maps are proved in the appendices. A Hilbert–Schmidt sector condition, two linear matrix inequalities on H, gives $\operatorname{Re} \Gamma_n(H) \geq 0$ and, for positive bistochastic H, the bound $\Phi_n(H) \geq n^{-n}$ (Theorem A.1). An exact moment formula gives $\Gamma_{2m}(B_{2m}) > 0$ for the skew-unitary Breuer–Hall maps in every even size $2m \geq 4$ (Theorem B.1); its size-four value enters the proof of Theorem 4.4. These two results are certificates for single maps, not membership

tests for $\mathcal{K}_n$: Proposition 4.7 exhibits a positive map whose invariant becomes negative after an entanglement-breaking addition, although the map is positive semidefinite as a Hilbert–Schmidt operator.

1.1. **Related work.** ***Quantum permanents and Gaussian representations.*** Gurvits introduced the quantum permanent and its matroidal specialization (Gurvits, 2002, 2004); his Gaussian polynomial norm, Kraus-pencil determinant formula, mixed determinant correlation, and Hilbert–Schmidt square representation are the calculus of Section 2. Barvinok's quaternionic Gaussian estimators give nonnegative determinant representations of permanents and of real positive-semidefinite mixed discriminants (Barvinok, 1999), and the sign of the complex-adjoint quaternionic determinant is classical (Cohen and De Leo, 2000). Theorem 2.3 establishes the determinant representation for an arbitrary CP-plus-coCP map, by identifying its Wick expansion with Γ_n.

Minor sums and stability under additions. The addition expansions of Section 3 specialize the hyperdeterminantal minor-summation formula of Luque–Thibon (Luque and Thibon, 2003), recorded with proof in (Amanov and Yeliussizov, 2023, Proposition 3.9); for zero background, the entanglement-breaking expansion is Gurvits's matroidal permanent. We put these sums in positive-map form, with complementary subspaces and nonnegative Gram weights, and prove the converse: hereditary determinant positivity is equivalent to stability under every entanglement-breaking addition. The nonconvexity theorem combines the Breuer–Hall and Robertson maps (Breuer, 2006; Hall, 2006; Chruściński and Kossakowski, 2008) with Du's rank-nine construction (Du, 2026); Du's negative map is the input, and its assembly from four members of $\mathcal{K}_9$ and the resulting convex obstruction are the conclusions. Mapping cones in the sense of Girard et al. (2021) are also closed under completely positive compositions; the obstruction of Corollary 4.6 uses only simultaneous unitary conjugation.

Additional certificates. Positive maps fixing the diagonal algebra were studied by Kye and Kim–Kye (Kye, 1995; Kim and Kye, 1994); Proposition 2.5 is the phase-permanent formula for their invariant, and Marcus's inequality (Marcus, 1963) gives its lower bound. The sector certificate of Appendix A applies the tensor-phase calculus of Wang et al. (2020) to the determinant vector in Gaussian Fock space; the Hilbert–Schmidt positive-semidefinite square case is in (Gurvits, 2004, Appendix C). The Breuer–Hall theorem of Appendix B uses the characteristic-polynomial duality of Forrester (2024), attributed there to Kulkarni et al. (2025); the three-exponential moment formula and the sign of its odd moments are the further steps.

Existence of Griffiths-positive metrics. Whether every ample bundle admits a Griffiths-positive metric is a separate question. Du and Xie construct a rank-two ample bundle on an abelian surface admitting no smooth Griffiths-semipositive Hermitian metric (Du and Xie, 2026, Theorem 1.1). Their obstruction concerns existence, whereas the rank-nine construction starts with a strictly Griffiths-positive metric and yields a negative top Chern form.

1.2. **What is not proved.** The results leave open the unrestricted size-four inequality, convexity of the bistochastic slice of $\mathcal{K}_n$, and the sign on the cone of sums of a 2-positive and a 2-copositive map. Section 6 discusses these questions. For Schur forms, the theorem applies to decomposable curvature; the other subclasses of $\mathcal{K}_n$ give Chern-form positivity, and extending them to all Schur forms requires further closure properties of the auxiliary quotient curvature (Section 5).

1.3. **Organization.** Section 2 fixes conventions, records Gurvits's Gaussian identities, and proves Theorem 2.3. Section 3 develops the perturbation calculus and proves Theorem 3.2 with its consequences. Section 4 proves the nonconvexity theorem and the convex obstruction. Section 5 contains the curvature applications, and Section 6 the boundary cases and open problems. Appendices A–C prove the sector and Breuer–Hall certificates and separate the sufficient classes. Appendix D records explicit CP curvature tensors that extract individual Schur coefficients.

2. Determinant calculus and the decomposable cone

To prove Schur-form nonnegativity, we first establish the sign of Γ_n on decomposable maps. We fix its normalization through Gurvits's Gaussian identities, prove the quaternionic representation, and then collect the compression and scaling facts needed for entanglement-breaking additions.

2.1. **Positive maps and conventions.** We use the Hilbert–Schmidt inner product $\langle X, Y\rangle_{\mathrm{HS}} = \mathrm{tr}(X^*Y)$, conjugate-linear in its first variable. A positive complex-linear map is Hermiticity preserving. Its Hilbert–Schmidt adjoint is positive as well, by self-duality of the positive-semidefinite matrix cone.

For $H : M_p \to M_q$, the Choi matrix is

$$C_H = \sum_{i,j=1}^{p} E_{ij} \otimes H(E_{ij}).$$

A completely positive map is a map of the form $H(X) = \sum_\ell A_\ell X A_\ell^*$, with A_ℓ of size $q \times p$. Equivalently, $C_H \succeq 0$ (Choi, 1975a). A completely copositive map has this form after input transposition. A decomposable map is a sum of a completely positive and a completely copositive map. These definitions apply in rectangular dimensions.

An entanglement-breaking (EB) map is a map with the representation

$$S(X) = \sum_{\ell=1}^{N} (b_\ell^* X b_\ell) a_\ell a_\ell^*.$$

Equivalently, it is completely positive with a rank-one Kraus representation; this equivalence is the finite-dimensional measure-and-prepare characterization of Horodecki et al. (2003, Theorem 4). We use its conic version, without requiring trace preservation. Every such map is decomposable.

For isometries $V : \mathbb{C}^k \to \mathbb{C}^p$ and $U : \mathbb{C}^k \to \mathbb{C}^q$, define the *square compression*

$$H_{V,U}(X) = U^* H(VXV^*)U. \tag{2.1}$$

Here "square" means only that the two compressed spaces have the same dimension. No relation between the input and output subspaces is imposed. Coordinate sets $I \subseteq [p]$ and $J \subseteq [q]$ of equal size give $H_{I,J}$ by the corresponding coordinate embeddings. Set $\Gamma_0 = 1$, and give the determinant and permanent of the empty matrix the value one.

2.2. **Apolar and Gaussian identities.** Gurvits's Gaussian polynomial isometry (Gurvits, 2004, Theorem 3.8, Appendix C) turns determinant differentiation into a polynomial inner product. This isometry gives both the correlation formula for Γ_n and a square formula for Hilbert–Schmidt positive-semidefinite maps.

Let G be a matrix with independent circular complex Gaussian entries, normalized by $\mathbb{E}|G_{ij}|^2 = 1$. If $p(z) = \sum_\alpha p_\alpha z^\alpha$ and $q(z) = \sum_\alpha q_\alpha z^\alpha$, their Fischer Hermitian product is

$$\langle p, q\rangle_F = \sum_\alpha \alpha!\, \overline{p_\alpha} q_\alpha = \mathbb{E}[\overline{p(G)} q(G)]. \tag{2.2}$$

The final equality follows from $\mathbb{E}[\bar{z}^a z^b] = \delta_{ab} a!$ in one complex variable and independence. Consequently

$$\Gamma_n(H) = \langle \det, \det \circ H\rangle_F = \mathbb{E}[\overline{\det G} \det H(G)]. \tag{2.3}$$

The last expression is a correlation of two polynomials evaluated on the same random matrix. It is not in general a squared norm.

Lemma 2.1 (Fock-space calculus)**.** *On a fixed homogeneous polynomial space, let $\mathcal{U}_L p = p \circ L$. Then*

$$\mathcal{U}_L^* = \mathcal{U}_{L^*}, \qquad \mathcal{U}_B \mathcal{U}_C = \mathcal{U}_{C \circ B}.$$

In particular, for any complex-linear $L : M_n \to M_n$,

$$\Gamma_n(L \circ L^*) = \mathbb{E}|\det L(G)|^2. \tag{2.4}$$

For a Hermiticity-preserving H, the number $\Gamma_n(H)$ is real, and $\Gamma_n(H^*) = \Gamma_n(H)$.

Proof. Monomial orthogonality identifies the homogeneous polynomial space with a symmetric tensor power; the induced adjoint is the symmetric tensor power of L^*. Alternatively the adjoint identity follows directly on products of linear forms from (2.2). The order of composition reverses because the action is by pullback. Thus

$$\Gamma_n(LL^*) = \langle \det, \mathcal{U}_{L^*}\mathcal{U}_L \det\rangle_F = \|\mathcal{U}_L \det\|_F^2.$$

In an orthonormal basis of Hermitian matrices, the matrix of a Hermiticity-preserving map is real. The determinant polynomial is also real in these coordinates: it is real on all real coordinate vectors, so its coefficients are real. These two facts prove reality of the correlation. The adjoint identity gives $\Gamma_n(H^*) = \overline{\Gamma_n(H)}$. □

For later normalization checks, let $\Delta_n(X) = \operatorname{tr}(X)I_n/n$. Direct differentiation gives

$$\Gamma_n(\mathrm{Id}) = \Gamma_n(\mathsf{T}) = n!, \qquad \Gamma_n(\Delta_n) = \frac{n!}{n^n}. \tag{2.5}$$

Input and output congruences change Γ_n by nonnegative determinant factors. We use this relative invariance of the quantum permanent (Gurvits, 2004, (18)) when changing compression frames and normalizing maps.

Lemma 2.2 (Covariance and pinched sums)**.** *For square matrices A, B, with $\mathrm{Ad}_A(X) = AXA^*$,*

$$\Gamma_n(\mathrm{Ad}_A \circ H \circ \mathrm{Ad}_B) = |\det A|^2 |\det B|^2 \Gamma_n(H), \tag{2.6}$$

$$\Gamma_n(\mathsf{T} \circ H) = \Gamma_n(H \circ \mathsf{T}) = \Gamma_n(H). \tag{2.7}$$

For the pinched sum

$$(H \boxplus K)\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} H(X) & 0 \\ 0 & K(W) \end{pmatrix},$$

one has

$$\Gamma_{a+b}(H \boxplus K) = \Gamma_a(H)\Gamma_b(K). \tag{2.8}$$

Proof. The output congruence in (2.6) contributes $|\det A|^2$. For the input congruence, the chain rule replaces the derivative matrix by the corresponding left and right matrix factors, contributing $|\det B|^2$. First take invertible factors and then use polynomial continuity for singular factors. Transposing the entries of X preserves $\det(\partial_X)$; transposing the output preserves its determinant. For (2.8), the output determinant factors. Every surviving input differential permutation preserves the two blocks, and hence the differential factors as well, without a multinomial coefficient. □

2.3. **The decomposable cone.** For a completely positive map, Gurvits's identity expresses Γ_n as a squared Gaussian determinant. To include a completely copositive summand, we combine two independent Kraus pencils in a quaternionic block matrix. Its determinant is nonnegative: it is the square of the Study determinant in the convention of Cohen and De Leo (2000, Sections 3 and 7). Barvinok used related quaternionic estimators for permanents and real positive-semidefinite mixed discriminants (Barvinok, 1999, Sections 2.3 and 3.3). Here the matrix entries of each Gaussian pencil may have arbitrary covariance.

Theorem 2.3 (Quaternionic determinant representation)**.** *Suppose*

$$H(X) = \sum_a A_a X A_a^* + \sum_b B_b X^{\mathsf{T}} B_b^*.$$

For independent standard circular complex Gaussians g_a, h_b, put $A(g) = \sum_a g_a A_a$ and $B(h) = \sum_b h_b B_b$. Then

$$\Gamma_n(H) = \mathbb{E} \det \begin{pmatrix} A(g) & -B(h) \\ B(h) & \overline{A(g)} \end{pmatrix} \geq 0. \tag{2.9}$$

Proof. Write the displayed block matrix as M. Its top n rows are holomorphic in g, h, and its bottom n rows are antiholomorphic. Wick contraction pairs each top row α with a bottom row $\tau(\alpha)$, for some $\tau \in S_n$. Nonzero covariances require the two paired columns to lie in opposite column blocks. A compatible determinant permutation is specified by $i, j \in S_n$ and a subset $S \subseteq [n]$: if $\alpha \notin S$, the top row uses column $i(\alpha)$ and the paired bottom row uses column $n + j(\alpha)$; if $\alpha \in S$, those column blocks are exchanged. The determinant sign is

$$(-1)^{|S|} \operatorname{sgn}(i) \operatorname{sgn}(j) \operatorname{sgn}(\tau).$$

The top-right block supplies another $(-1)^{|S|}$. Summing over the choices of S therefore gives

$$\mathbb{E} \det M = \sum_{\tau, i, j \in S_n} \operatorname{sgn}(\tau) \operatorname{sgn}(i) \operatorname{sgn}(j) \prod_{\alpha=1}^{n} H(E_{i(\alpha), j(\alpha)})_{\alpha, \tau(\alpha)}. \tag{2.10}$$

Here a term with $\alpha \notin S$ uses $\sum_a (A_a)_{\alpha, i(\alpha)} \overline{(A_a)_{\tau(\alpha), j(\alpha)}}$, while a term with $\alpha \in S$ uses $\sum_b (B_b)_{\alpha, j(\alpha)} \overline{(B_b)_{\tau(\alpha), i(\alpha)}}$, as required. Setting the input permutation equal to $j \circ i^{-1}$ identifies (2.10) with the full Leibniz expansion of $\det(\partial_X) \det H(X)$. This identification proves the identity with its normalization.

For the sign, put

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

One has $M J_0 = J_0 \bar{M}$, so M commutes with the antiunitary $v \mapsto J_0 \bar{v}$, whose square is $-I$. Nonreal eigenvalues occur with their conjugates. The generalized eigenspace for a real eigenvalue is invariant under this antiunitary and has even complex dimension. Indeed, on a d-dimensional such space its matrix C satisfies $C\bar{C} = -I_d$; taking determinants forces d even. Each real eigenvalue therefore has even algebraic multiplicity. It follows that $\det M \geq 0$, including singular and nondiagonalizable cases. □

For $B_b = 0$ the theorem reduces to the completely positive identity of Gurvits (2004, Theorem 3.8),

$$\Gamma_n \left(\sum_a A_a(\,\cdot\,) A_a^* \right) = \mathbb{E} \left| \det \left(\sum_a g_a A_a \right) \right|^2. \tag{2.11}$$

Although the block determinant is nonnegative, the formula does not imply that all mixed coefficients of a sum of a completely positive and a completely copositive map are nonnegative.

2.4. **Rank-one resolutions and permanents.**

Lemma 2.4 (Hermitian Kadison inequality)**.** *If T is positive and unital and $X = X^*$, then $T(X^2) \succeq T(X)^2$. A positive bistochastic map is a contraction for the Hilbert–Schmidt norm.*

Proof. Write $X = \sum_j \lambda_j P_j$, put $A_j = T(P_j)$, and define $V\xi = (A_j^{1/2} \xi)_j$. Since $\sum_j A_j = I$, V is an isometry. For $D = \operatorname{diag}(\lambda_j I)$,

$$T(X^2) - T(X)^2 = V^* D (I - V V^*) D V \succeq 0.$$

If T preserves trace, then for Hermitian X,

$$\|T(X)\|_{\mathrm{HS}}^2 = \operatorname{tr} T(X)^2 \leq \operatorname{tr} T(X^2) = \operatorname{tr} X^2 = \|X\|_{\mathrm{HS}}^2.$$

For $X = A + iB$ with A, B Hermitian, apply this bound to both terms in $\|T(X)\|_{\mathrm{HS}}^2 = \|T(A)\|_{\mathrm{HS}}^2 + \|T(B)\|_{\mathrm{HS}}^2$. □

Positive maps that fix the diagonal algebra, and their Hadamard representations, were studied by Kye (1995); indecomposable examples in dimension at least four appear already in Kim and Kye (1994). For this class the invariant is an average of permanents of positive semidefinite matrices with unit diagonal, which gives the lower bound one.

Proposition 2.5 (Phase-permanent representation)**.** *If $T : M_n \to M_n$ is positive and $T(E_{ii}) = E_{ii}$, then*

$$T(E_{ij}) \in \operatorname{span}\{E_{ij}, E_{ji}\}, \qquad \Gamma_n(T) = \mathbb{E}_\theta \operatorname{per} T(v_\theta v_\theta^*) \geq 1,$$

where the θ_i are independent uniform phases and $v_\theta = (e^{i\theta_i})_i$.

Proof. Apply Lemma 2.4 to $E_{ii} + tX$, for Hermitian X. The constant term in its positive difference is zero. Both signs of arbitrarily small real t force

$$T(E_{ii}X + XE_{ii}) = E_{ii}T(X) + T(X)E_{ii}.$$

Complex linearity extends this identity to all X. Applying it to matrix units yields the asserted support. We can therefore write

$$(T(Z))_{ij} = a_{ij}z_{ij} + b_{ij}z_{ji} \quad (i \neq j), \qquad (T(Z))_{ii} = z_{ii}.$$

Expand the permanent by an output permutation σ. On a nontrivial cycle, encode a forward or reversed choice of the input variable by $\varepsilon_i \in \{1, -1\}$. The exponent of θ_i is $\varepsilon_i - \varepsilon_{\sigma^{-1}(i)}$. Phase averaging retains exactly the choices of a constant orientation on each cycle. In (2.3), a selected input monomial pairs with $\det G$ exactly when every input row and every input column is used once. This requirement is the same cycle condition. The input permutation reverses the cycles on which the reversed orientation was chosen. Reversing a cycle preserves its sign, so the two determinant signs multiply to $+1$. Every retained Gaussian monomial is squarefree and has moment one. The cycle analysis thus proves the exact permanent identity, including two-cycles and fixed points.

The next step is Marcus's permanent analogue of Hadamard's inequality (Marcus, 1963); the short Gaussian norm proof is included. For a positive-semidefinite matrix C, $\operatorname{per} C \geq \prod_i C_{ii}$. To verify this inequality directly, realize C as the covariance matrix of complex Gaussian linear forms L_i. Wick contraction gives $\operatorname{per} C = \| \prod_i L_i \|_F^2$. If $L(z) = \sum_j a_j z_j$ and $D = \sum_j \bar{a}_j \partial_{z_j}$, then the adjoint relation and $[D, L] = \sum_j |a_j|^2$ give

$$\|Lf\|_F^2 = \|Df\|_F^2 + \|a\|_2^2 \|f\|_F^2 \geq \|a\|_2^2 \|f\|_F^2.$$

Iterating proves the permanent inequality. For $C_\theta = T(v_\theta v_\theta^*)$, positivity and the fixed diagonal give $C_\theta \succeq 0$ and $(C_\theta)_{ii} = 1$. Hence $\Gamma_n(T) = \mathbb{E}_\theta \operatorname{per} C_\theta \geq 1$. □

2.5. **Strict normalization.** Sign statements for Γ_n can often be reduced to bistochastic maps. The next lemma is the strictly positive case of operator scaling, whose capacity and attainment formulation is due to Gurvits (Gurvits, 2004, Theorem 4.7). We include the proof because a boundary map need not admit an exact invertible scaling. Gurvits's use of "indecomposable" in that scaling theorem denotes a rank-increasing condition, not indecomposability in the CP-plus-coCP sense used here.

Lemma 2.6 (Bistochastic normalization)**.** *Every strictly positive map $H : M_n \to M_n$ admits a bistochastic normalization by invertible input and output congruences. Here strict positivity means $H(X) \succ 0$ for every nonzero $X \succeq 0$. These congruences preserve the sign of Γ_n.*

Proof. Compactness of the unit spheres gives $\delta > 0$ such that $H(X) \succeq \delta \operatorname{tr}(X) I$ for $X \succeq 0$. Minimize $\log\det H(X)$ over $X \succ 0$, $\det X = 1$. The lower bound

$$\log\det H(X) \geq n \log(\delta \operatorname{tr} X)$$

is coercive on this constraint. A bounded trace and determinant one bound every eigenvalue above and away from zero, so a minimizer X_0 exists. The Lagrange equation is

$$H^*(H(X_0)^{-1}) = \lambda X_0^{-1}.$$

Pairing with X_0 gives $\lambda = 1$. Thus

$$\widehat{H}(Z) = H(X_0)^{-1/2} H(X_0^{1/2} Z X_0^{1/2}) H(X_0)^{-1/2}$$

satisfies $\widehat{H}(I) = \widehat{H}^*(I) = I$. Since $\det X_0 = 1$, the covariance identity (2.6) gives

$$\Gamma_n(\widehat{H}) = \frac{\Gamma_n(H)}{\det H(X_0)}, \qquad \det H(X_0) > 0.$$

□

A positive map on the boundary need not admit an invertible bistochastic normalization. Universal sign statements can nonetheless be reduced to the strictly positive bistochastic case by first adding $\epsilon \operatorname{tr}(\,\cdot\,)I$, normalizing, and finally using polynomial continuity as $\epsilon \downarrow 0$.

We use Wan's theorem (Wan, 2026, Theorem 0.3) in the following form:

$$H : M_k \to M_k \text{ positive}, \quad k \le 3 \quad \Longrightarrow \quad \Gamma_k(H) \ge 0. \tag{2.12}$$

The non-strict case follows by trace augmentation and continuity. Sizes one and two also follow by pinched identity padding into size three.

3. Perturbation calculus and hereditary determinant positivity

To determine which additions preserve the sign of Γ_n, we expand it in rank-one perturbations. Each coefficient is a determinant invariant of a complementary compression, multiplied by nonnegative Gram factors. These coefficients prove both directions of the characterization of $\mathcal{K}_n$. The coordinate version gives block constructions with classical additions. Both expansions specialize the hyperdeterminantal minor-summation formula (Luque and Thibon, 2003; Amanov and Yeliussizov, 2023).

3.1. Complementary compressions and entanglement-breaking additions.

When the background map is zero, the following formula reduces to Gurvits's matroidal permanent (Gurvits, 2002, Definition 1.1). For a nonzero background, each remaining factor is the invariant of a complementary input/output compression. We include a direct proof so that the arbitrary nonorthogonal rank-one terms and the factorial normalization are explicit.

Proposition 3.1 (Entanglement-breaking addition formula)**.** *Let*

$$S(X) = \sum_{a=1}^{N} (b_a^* X b_a) a_a a_a^*$$

be entanglement breaking. For $L \subseteq [N]$ of cardinality $m \le n$, put $A_L = [a_a]_{a\in L}$ and $B_L = [b_a]_{a\in L}$. If both have rank m, choose isometries U_L, V_L onto the orthogonal complements of their respective column spaces. Then

$$\Gamma_n(H + S) = \sum_{\substack{L\subseteq[N]\\ |L|\le n}} \det(A_L^* A_L) \det(B_L^* B_L) \Gamma_{n-|L|}(H_{V_L,U_L}). \tag{3.1}$$

Rank-deficient subsets contribute zero. The empty subset contributes $\Gamma_n(H)$, and subsets of cardinality n use $\Gamma_0 = 1$. The formula holds for every complex-linear H, with no positivity assumption on it.

Proof. Introduce distinct parameters t_a in the sum defining S. The output determinant is affine in each t_a, since its associated matrix has rank one. For a fixed subset L its coefficient is

$$\det(A_L^* A_L)\det(U_L^* H(X) U_L)\prod_{a\in L}(b_a^* X b_a). \tag{3.2}$$

To verify this coefficient, choose a unitary basis whose first m coordinates span the columns of A_L. The coefficient using all m rank-one terms exhausts these rows and columns. Its factor is the squared volume $\det(A_L^* A_L)$ times the complementary determinant. If the columns are dependent, the coefficient is zero.

For every homogeneous polynomial f of degree $n-m$, the matching input identity is

$$\det(\partial_X)\left[\prod_{a\in L}(b_a^* X b_a) f(X)\right] = \det(B_L^* B_L)\det(\partial_Y) f(V_L Y V_L^*). \tag{3.3}$$

For independent columns, choose a unitary input basis whose first m coordinates span them. All m linear factors depend on the leading $m\times m$ block. In a surviving determinant differential, every one of its m input-row derivatives must differentiate these factors; their columns consequently exhaust the same block. The first differential factor is

$$\det(\partial_Z)\prod_{a=1}^{m}(r_a^* Z r_a) = |\det[r_1\ \cdots\ r_m]|^2 = \det(B_L^* B_L),$$

by direct alternating expansion. The remaining differential acts on the complementary compression of f. If the columns are dependent, their support has dimension less than m, so this allocation of the m row derivatives is impossible and both sides vanish. This case analysis proves (3.3). Apply it to (3.2) and sum over L. This summation also proves that no factorial is missing. □

Theorem 3.2 (Characterization by entanglement-breaking additions)**.** *Let $H : M_n \to M_n$ be complex linear. The following are equivalent:*
(1) Every square compression $H_{V,U}$, in every size $1 \le k \le n$, has $\Gamma_k(H_{V,U}) \ge 0$.
(2) $\Gamma_n(H+S) \ge 0$ for every entanglement-breaking map S.
Either condition implies that H is positive.

Proof. The first condition implies the second by Proposition 3.1. Conversely, fix a size-k compression, with $k < n$, and choose orthonormal bases $a_1,\dots,a_m$ and $b_1,\dots,b_m$ of its output and input complements, where $m = n-k$. Set

$$S(X) = \sum_{j=1}^{m}(b_j^* X b_j) a_j a_j^*.$$

Both Gram matrices are identities, so (3.1) gives

$$p(t) := \Gamma_n(H+tS), \qquad \deg p \le m, \qquad [t^m]p(t) = \Gamma_k(H_{V,U}).$$

The second condition gives $p(t) \ge 0$ for $t \ge 0$, hence $p \in \mathbb{R}[t]$ and $[t^m]p \ge 0$. For $k = n$, take $S = 0$ and use unitary covariance. Finally, the size-one inequalities are $u^* H(vv^*)u \ge 0$ for all unit vectors u, v. They imply $H(vv^*) \succeq 0$, and spectral decomposition then proves positivity of H. □

The characterization reduces stability under all EB additions to compression inequalities. The next results give closure properties and ways to verify those inequalities from low-dimensional maps.

Proposition 3.3 (Closure properties of $\mathcal{K}_n$)**.** *The class $\mathcal{K}_n$ of (1.3) is closed. For $H \in \mathcal{K}_n$, $c \ge 0$, $A, B \in M_n$, and an entanglement-breaking map S,*

$$cH,\ H^*,\ \mathsf{T}\circ H,\ H\circ\mathsf{T},\ \mathrm{Ad}_A \circ H \circ \mathrm{Ad}_B,\ H+S\ \in \mathcal{K}_n.$$

For isometries $U, V : \mathbb{C}^k \to \mathbb{C}^n$, one also has $H_{V,U} \in \mathcal{K}_k$. Every coefficient of $\Gamma_n(H + \sum_a t_a S_a)$ is nonnegative when the S_a are rank-one entanglement-breaking summands.

Proof. Closedness and nonnegative scaling follow from the defining polynomial inequalities. Iterated compression satisfies $(H_{V,U})_{V',U'} = H_{VV',UU'}$. Moreover, $(H+S)_{V,U} = H_{V,U} + S_{V,U}$ and $S_{V,U}$ is entanglement breaking. Theorem 3.2 therefore proves addition stability. The coefficient assertion follows from (3.1) with a separate parameter on each summand.

Adjoints interchange the input and output isometries, and Lemma 2.1 preserves the real value of the invariant. For invertible A, B, factor $BV = V_0 R$ and $A^*U = U_0 L$, where V_0, U_0 are isometries and $R, L \in M_k$ are invertible. Then

$$(\mathrm{Ad}_A \circ H \circ \mathrm{Ad}_B)_{V,U} = \mathrm{Ad}_{L^*} \circ H_{V_0,U_0} \circ \mathrm{Ad}_R,$$

so (2.6) multiplies its invariant by $|\det L|^2 |\det R|^2$. Singular congruences follow by continuity. Whole transposition changes one compression plane to its complex conjugate and preserves its invariant by (2.7). □

Corollary 3.4 (Low-dimensional certificates)**.** *Each of the following positive maps $H : M_n \to M_n$ belongs to $\mathcal{K}_n$. Consequently, $H + S \in \mathcal{K}_n$ for every entanglement-breaking map S:*
(1) $\min\{\operatorname{rank} H(I), \operatorname{rank} H^*(I)\} \le 3$.
(2) *$H(X) = AK(B^*XB)A^*$, where A, B are $n \times d$ matrices, $d \le 4$, and $K : M_d \to M_d$ is positive. If $d = 4$, additionally assume that $\Gamma_4(K) \ge 0$ has been established.*

Proof. In the first case, if $H(I)$ has rank at most three, order bounds show that all outputs are supported on its range. Every size-k compression with $k > 3$ then has identically zero determinant. If instead $H^*(I)$ has rank at most three, let P be its support. For $x \in \ker P$, positivity and

$$\operatorname{tr} H(xx^*) = x^* H^*(I) x = 0$$

give $H(xx^*) = 0$. For any y and complex z, positivity of $H((x+zy)(x+zy)^*)$ forces the term linear in z to vanish. Thus $H(xy^*) = H(yx^*) = 0$, and $H(X) = H(PXP)$. A size-k compression with $k > 3$ therefore ignores a nonzero input direction after a unitary change of basis. Its determinant differential is zero. All size-k compressions with $k \le 3$ are ordinary positive maps covered by Wan.

For the second case, compressions of size $k \le 3$ again follow from Wan, and those with $k > d$ have deficient output rank. At $k = d = 4$, put $A_k = U^*A$ and $B_k = V^*B$. They are square, and filtering gives

$$\Gamma_4(H_{V,U}) = |\det A_k|^2 |\det B_k|^2 \Gamma_4(K) \ge 0.$$

Singular factors give zero. In each case the signs of all square compressions have been established, so Theorem 3.2 applies. □

Remark 3.5. *We call the map K in the second case a* core *of H. The four-dimensional clause applies to a core with $\Gamma_4(K) \ge 0$; it does not assume the unrestricted size-four inequality. It applies, for example, to decomposable cores, to cores satisfying the sector condition of Appendix A, and to the four-dimensional Breuer–Hall map. A Hilbert–Schmidt PSD map in higher dimension need not have nonnegative square compressions (Proposition 4.7), so the corollary does not license EB addition to that entire class.*

Corollary 3.4 already places indecomposable maps in $\mathcal{K}_n$. The next family, built from Choi's map, gives strictly positive bistochastic examples with a lower bound on Φ_n.

Proposition 3.6 (A strictly positive indecomposable family)**.** *Define the normalized Choi map (Choi, 1975b) $C : M_3 \to M_3$ by*

$$2C(X) = \begin{pmatrix} x_{11}+x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{22} \end{pmatrix}.$$

For $n \geq 3$, let K_0 be its pinched direct sum with diagonal pinching on the remaining $n-3$ coordinates, and put

$$H_\epsilon = (1-\epsilon)K_0 + \epsilon\Delta_n, \qquad \Delta_n(X) = \frac{\operatorname{tr} X}{n} I.$$

For $0 < \epsilon < n/(n+14)$, the map H_ϵ is strictly positive, bistochastic, and indecomposable, and

$$\Phi_n(H_\epsilon) \geq (\epsilon/n)^n > 0.$$

Proof. The map C is unital and trace preserving. For $X = xx^*$, put $a = |x_1|^2$, $b = |x_2|^2$, $c = |x_3|^2$. The two-by-two principal minors of $2C(xx^*)$ are

$$a^2 + ac + bc, \qquad ab + bc + c^2, \qquad ab + ac + b^2.$$

Its determinant is $a^2b + b^2c + c^2a - 3abc \geq 0$ by AM–GM. All principal minors are nonnegative, proving ordinary positivity.

Let $\Omega = \sum_{i=1}^3 |ii\rangle$, with indices read cyclically, and set

$$W = |\Omega\rangle\langle\Omega| + \frac{1}{2}\sum_i |i, i+1\rangle\langle i, i+1| + 2\sum_i |i+1, i\rangle\langle i+1, i|.$$

The matrix W is PSD. Its partial transpose has diagonal entries one and, on each off-diagonal unordered pair, the PSD matrix $\left(\begin{smallmatrix} 1/2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, up to interchanging the two basis vectors. Thus W is PPT. Direct contraction with the Choi matrix gives

$$\operatorname{tr}(C_C W) = -\frac{3}{4}, \qquad \operatorname{tr} W = \frac{21}{2}.$$

Embed this witness in the first three input and output coordinates. Its pairing with C_{H_ϵ} is

$$-\frac{3}{4}(1-\epsilon) + \frac{21}{2n}\epsilon < 0 \quad \Longleftrightarrow \quad \epsilon < \frac{n}{n+14}.$$

A decomposable Choi matrix has nonnegative pairing with every PPT matrix, proving indecomposability.

Both summands in H_ϵ are bistochastic, and $H_\epsilon(X) \succeq (\epsilon/n)\operatorname{tr}(X) I$ for $X \succeq 0$. Its core $(1-\epsilon)C$, embedded on three input and output coordinates, has marginal rank three. All remaining terms are EB. Corollary 3.4 and the exact addition formula prove the sign. More explicitly the EB remainder is classical, with every entry of its matrix at least ϵ/n. Its permanent contribution alone is at least $n!(\epsilon/n)^n$, proving the quantitative bound. □

3.2. **Coordinate expansions and rectangular blocks.** The coordinate identity below is the order-four specialization of the hyperdeterminantal minor-summation formula (Amanov and Yeliussizov, 2023, Proposition 3.9) to a diagonal perturbation. For the tensor of C_D, the surviving input index sets coincide in pairs, as do the output sets; the permutation signs square to one and the complementary hyperdeterminant becomes $\operatorname{per} D[J^c, I^c]$. The maximality statement for classical additions, that is, for the maps C_D below with D entrywise nonnegative, follows immediately afterward.

Proposition 3.7 (Exact classical addition identity)**.** *For a scalar $n \times n$ matrix D, put*

$$C_D(X) = \operatorname{diag}(D \operatorname{diag} X).$$

For every complex-linear $H : M_n \to M_n$, without positivity assumptions,

$$\Gamma_n(H + C_D) = \sum_{\substack{I,J \subseteq [n] \\ |I| = |J|}} \Gamma_{|I|}(H_{I,J}) \operatorname{per} D[J^c, I^c]. \tag{3.4}$$

The permanent of an empty matrix is one.

Proof. Expanding the output determinant by the entries of $C_D(X)$ gives

$$\det(H(X)+C_D(X))=\sum_{J\subseteq[n]}\det H(X)[J,J]\prod_{a\notin J}\left(\sum_b D_{ab}x_{bb}\right).$$

Every input derivative assigned to the second factor must be diagonal. If I is the set of input derivative indices assigned to the first factor, homogeneity forces $|I|=|J|$. The input permutation fixes I^c and permutes I. Assigning the complementary distinct diagonal derivatives to the factors from C_D gives $\operatorname{per} D[J^c,I^c]$. The remaining alternating differential gives $\Gamma_{|I|}(H_{I,J})$. These assignments exhaust the Leibniz expansion and prove the identity, without an additional factorial. □

Proposition 3.8 (Exact maximality for classical additions)**.** *For a complex-linear map $H:M_n\to M_n$, the following are equivalent:*
(1) $\Gamma_{|I|}(H_{I,J})\geq 0$ for every pair of equally sized coordinate sets $I,J\subseteq[n]$.
(2) $\Gamma_n(H+C_D)\geq 0$ for every entrywise nonnegative matrix D.

Proof. The forward implication follows from Proposition 3.7. Conversely, fix I,J of cardinality $k<n$, and let D_0 be a partial permutation matrix mapping the input coordinates I^c bijectively to the output coordinates J^c. Since $\operatorname{per} D_0[J^c,I^c]=1$, the expansion (3.4) gives

$$p(t):=\Gamma_n(H+tC_{D_0}),\qquad \deg p\leq n-k,\qquad [t^{n-k}]p(t)=\Gamma_k(H_{I,J}).$$

As before, $p(t)\geq 0$ for $t\geq 0$ implies $p\in\mathbb{R}[t]$ and $[t^{n-k}]p\geq 0$. For $k=n$, take $D=0$; for $k=0$, use $\Gamma_0=1$. □

Lemma 3.9 (Weighted rank-one resolutions)**.** *Suppose $K:M_m\to M_m$ is positive and $K(E_{ii})=\alpha_iE_{ii}$ with $\alpha_i\geq 0$. Then*

$$K(E_{ij})\in\operatorname{span}\{E_{ij},E_{ji}\},\qquad \Gamma_{|I|}(K_{I,I})\geq\prod_{i\in I}\alpha_i.$$

If I,J have the same cardinality and $I\neq J$, then $\Gamma_{|I|}(K_{I,J})=0$.

Proof. If $\alpha_i=0$, positivity on $(e_i+ze_j)(e_i+ze_j)^*$, for arbitrarily small complex z, shows that $u^*K(E_{ij})u=0$ for every output vector u. Polarization gives $K(E_{ij})=K(E_{ji})=0$. Moreover $K(I)=\operatorname{diag}\alpha$, so the order bounds on images of Hermitian matrices show that every output is supported on the positive-α coordinates.

On these coordinates, output congruence by $\operatorname{diag}(\alpha_i^{-1/2})$ gives a positive map T fixing each E_{ii}. Proposition 2.5 gives the support restriction and, applied to each principal compression, $\Gamma_{|I|}(T_{I,I})\geq 1$. The output congruence factor is $\prod_{i\in I}\alpha_i$, proving the bound when all these numbers are positive. If a number vanishes, the relevant determinant vanishes and the bound still holds.

Finally, $K(M_I)$ is supported in I. Compressing it to a different set J of the same cardinality leaves support in $I\cap J$, of smaller dimension. Its determinant is identically zero. □

Theorem 3.10 (Rectangular block enlargement)**.** *Let $A_s\subseteq[n]$ be pairwise disjoint nonempty input coordinate sets, and $B_s\subseteq[n]$ pairwise disjoint nonempty output coordinate sets. They need not exhaust the coordinates. Put $a_s=|A_s|$, $b_s=|B_s|$, and let $K_s:M_{a_s}\to M_{b_s}$ be positive. Define H_0 by*

$$H_0(X)[B_s,B_s]=K_s(X[A_s,A_s]),\qquad H_0(X)_{ij}=0\quad\text{if }(i,j)\notin\bigcup_s(B_s\times B_s).$$

Suppose each block satisfies at least one of:
(1) K_s is decomposable, with arbitrary rectangular dimensions;
(2) $\min(a_s,b_s)\leq 3$;
(3) $a_s=b_s$ and $K_s(E_{ii})=\alpha_iE_{ii}$ in its designated local bases, with $\alpha_i\geq 0$;
(4) $a_s=b_s=4$ and $\Gamma_4(K_s)\geq 0$ has already been certified.

Then, for every entrywise nonnegative D,

$$\Gamma_n(H_0 + C_D) \geq \operatorname{per} D \geq 0. \tag{3.5}$$

The nonnegative conclusion is preserved by independent input and output congruences, global input or output transposition, and limits.

Proof. In a coordinate compression, put $I_s = I \cap A_s$ and $J_s = J \cap B_s$. If either I or J uses an unused coordinate, its Γ vanishes. The output determinant factors over the blocks, with its s-th factor of degree $|J_s|$ in the variables of A_s. Every surviving input differential permutation preserves each I_s. Its degree in that same variable group is $|I_s|$. Consequently the invariant vanishes unless $|I_s| = |J_s| =: k_s$ for every s. In the remaining cases,

$$\Gamma_{|I|}((H_0)_{I,J}) = \prod_s \Gamma_{k_s}((K_s)_{I_s,J_s}). \tag{3.6}$$

There is no multinomial coefficient, since the differential factors act on disjoint groups of variables. Reordering blocks simultaneously in the rows and columns introduces no determinant sign.

Each factor is nonnegative. In the first case compression preserves decomposability. In the second $k_s \leq 3$, so the compressed ordinary positive map is covered by Wan. The third follows from Lemma 3.9. In the fourth case the proper square compressions have size at most three and the full compression is the certified map K_s itself. Apply Proposition 3.7; every permanent is nonnegative, and the $I = J = \varnothing$ term is $\operatorname{per} D$.

For nonsingular congruences the exact covariance is

$$\Gamma_n(\mathrm{Ad}_L \circ H \circ \mathrm{Ad}_R) = |\det L|^2 |\det R|^2 \Gamma_n(H).$$

Singular cases follow by continuity. Determinant transposition leaves Γ_n unchanged, and polynomial continuity proves the assertion about limits. □

Corollary 3.11 (Sharper bound for a weighted resolution)**.** *If $H_0(E_{ii}) = \alpha_i E_{ii}$ as in Lemma 3.9, then*

$$\Gamma_n(H_0 + C_D) = \sum_{I \subseteq [n]} \Gamma_{|I|}((H_0)_{I,I}) \operatorname{per} D[I^c, I^c] \tag{3.7}$$

$$\geq \operatorname{per}(D + \operatorname{diag} \alpha). \tag{3.8}$$

If $H_0 + C_D$ is bistochastic, then $D + \operatorname{diag} \alpha$ is doubly stochastic and

$$\Phi_n(H_0 + C_D) \geq n^{-n}.$$

Proof. The classical addition identity and the weighted-resolution lemma give

$$\Gamma_n(H_0 + C_D) \geq \sum_{I \subseteq [n]} \Big(\prod_{i \in I} \alpha_i \Big) \operatorname{per} D[I^c, I^c] = \operatorname{per}(D + \operatorname{diag} \alpha).$$

Bistochasticity gives $D\mathbf{1} + \alpha = D^{\mathsf{T}}\mathbf{1} + \alpha = \mathbf{1}$, so van der Waerden's inequality (Egorychev, 1981; Falikman, 1981) gives the final bound. □

Remark 3.12. *The bound n^{-n} is specific to the last corollary; a size-three reduction block can have zero invariant. For general blocks, (3.5) implies $\Phi_n \geq (\epsilon/n)^n$ whenever $D_{ij} \geq \epsilon/n$. The theorem requires pinched input blocks. It does not permit uncontrolled coherent couplings between them, or arbitrary mixing across incompatible block structures.*

4. Convex geometry and obstructions

Hereditary determinant positivity permits arbitrary entanglement-breaking additions. It does not permit arbitrary sums of members of $\mathcal{K}_n$. We first identify closed convex subclasses with fixed support, and then exhibit an explicit obstruction to combining all such certificates into a unitary-invariant convex enlargement of decomposability.

4.1. **Closed convex subclasses with prescribed support.** Fix a proper subspace $E \subset \mathbb{C}^n$, with orthogonal projection P_E, let EB_n be the cone of entanglement-breaking maps on M_n, and set

$$\mathfrak{F}_E^{\mathrm{out}} = \{K : M_n \to M_n \text{ positive} : K(I) = P_E K(I) P_E\}, \qquad \mathfrak{C}_E^{\mathrm{out}} = \mathfrak{F}_E^{\mathrm{out}} + \mathrm{EB}_n.$$

Define the input analogues by taking Hilbert–Schmidt adjoints.

Proposition 4.1 (Fixed-support convex subclasses)**.** *If* $\dim E \le 3$*, then* $\mathfrak{C}_E^{\mathrm{out}}$ *and* $\mathfrak{C}_E^{\mathrm{in}}$ *are closed convex cones contained in* $\mathcal{K}_n$*. For* $\dim E = 3$*, they contain indecomposable maps. They do not, in general, contain all decomposable maps.*

Proof. The set $\mathfrak{F}_E^{\mathrm{out}}$ is a closed convex face of the positive-map cone. Indeed, order bounds on Hermitian inputs give

$$K(I) = P_E K(I) P_E \quad \Longleftrightarrow \quad K(X) = P_E K(X) P_E \quad \text{for every } X.$$

For $K \in \mathfrak{F}_E^{\mathrm{out}}$ with $K = K_1 + K_2$ and K_i positive, the inequalities $0 \preceq K_i(I) \preceq K(I)$ put both K_i in $\mathfrak{F}_E^{\mathrm{out}}$. The sum with EB_n is convex.

To prove closedness of the sum, use the linear functional $\ell(T) = \operatorname{tr} T(I)$. It is strictly positive on every nonzero positive map: $\ell(T) = 0$ forces $T(I) = 0$, and order bounds then force $T = 0$. Consequently the set of positive maps with $\ell(T) \le M$ is compact in finite dimension. If $K_j + S_j$ converges with $K_j \in \mathfrak{F}_E^{\mathrm{out}}$ and $S_j \in \mathrm{EB}_n$, both $\ell(K_j)$ and $\ell(S_j)$ are nonnegative and bounded. A subsequence therefore converges in each closed cone, proving closedness of their sum. Closedness of EB_n follows equivalently from the compact convex base of separable Choi matrices.

The low-dimensional certificate in Corollary 3.4 proves membership in $\mathcal{K}_n$. Embedding the Choi map on three input coordinates and on E gives an indecomposable member, as witnessed by the embedded PPT matrix in Proposition 3.6. Taking adjoints gives the input statements.

For example, the identity map does not belong to $\mathfrak{C}_E^{\mathrm{out}}$ when E is proper and $n > 1$. If $\mathrm{Id} = K + S$ with both summands positive, then $0 \preceq K(xx^*) \preceq xx^*$ for every x. Linearity applied to $(e_i + ze_j)(e_i + ze_j)^*$ implies $K = c\,\mathrm{Id}$ for a scalar $0 \le c \le 1$. Its prescribed proper output support forces $c = 0$, leaving $S = \mathrm{Id}$, which is not entanglement breaking. Thus these cones are convex subclasses, not cones containing the whole decomposable cone. □

The support is fixed before taking sums. The union over supports, or the convex hull of maps certified on different supports, is not covered by this proposition.

4.2. **An exact rank-nine calculation.** The nonconvexity theorem splits Du's rank-nine map into members of $\mathcal{K}_9$. We first write that map as P_t, a Breuer–Hall part plus a decomposable part, and recompute its invariant.

For even n and a skew-symmetric unitary J, the Breuer–Hall map is

$$B_n(X) = \operatorname{tr}(X) I_n - X - J X^{\mathsf{T}} J^*. \tag{4.1}$$

Since $\langle v, J\bar{v} \rangle = 0$ and $\|J\bar{v}\| = \|v\|$,

$$B_n(vv^*) = \|v\|^2 I_n - vv^* - (J\bar{v})(J\bar{v})^* \succeq 0,$$

so B_n is positive (Breuer, 2006; Hall, 2006). Write a size-nine matrix in three size-three diagonal blocks X, Y, Z, and denote its leading size-six block by

$$S = \begin{pmatrix} X & W \\ V & Y \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}.$$

Define

$$\begin{aligned} \widetilde{B}_6(\mathcal{X}) &= \operatorname{diag}(B_6(S), 0_3), \\ Q(\mathcal{X}) &= \operatorname{diag}(Z, Z^{\mathsf{T}}, X + Y^{\mathsf{T}}), \\ P_t &= t\widetilde{B}_6 + Q. \end{aligned}$$

The first map is positive by the rank-one test for B_6. The map Q is decomposable: its CP part is $\mathrm{diag}(Z, 0_3, X)$, and its coCP part is $\mathrm{diag}(0_3, Z^{\mathsf{T}}, Y^{\mathsf{T}})$.

Proposition 4.2 (The rank-nine coefficient, after Du)**.** *For every $t \geq 0$, the map P_t is positive and*

$$\Gamma_9(P_t) = -1536t^3.$$

In particular,

$$H_9 = \frac{1}{2}P_{1/4} \quad \textit{is bistochastic, with} \quad \Gamma_9(H_9) = -\frac{3}{64}, \qquad \Phi_9(H_9) = -\frac{1}{7741440}. \tag{4.2}$$

Proof. We give the determinant computation for the construction of (Du, 2026). Put $A = X + Y^{\mathsf{T}}$ and $R(A) = \mathrm{tr}(A)I_3 - A$. For a column $k \in \mathbb{C}^3$, write

$$\widehat{k} = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}.$$

Define k, l by $\widehat{k} = W^{\mathsf{T}} - W$ and $\widehat{l} = V^{\mathsf{T}} - V$. With $M = Z + tR(A)$, the top output block is

$$\begin{pmatrix} M & t\widehat{k} \\ t\widehat{l} & M^{\mathsf{T}} \end{pmatrix}.$$

A three-row determinant expansion, equivalently a six-dimensional Pfaffian expansion, gives the polynomial identity

$$\det \begin{pmatrix} M & \widehat{k} \\ \widehat{l} & M^{\mathsf{T}} \end{pmatrix} = \left(\det M + k^{\mathsf{T}} M l\right)^2. \tag{4.3}$$

Consequently

$$\det P_t(\mathcal{X}) = \det A \left(\det(Z + tR(A)) + t^2 k^{\mathsf{T}}(Z + tR(A))l\right)^2.$$

The output uses none of the input cross blocks joining S to Z. The input determinant differential therefore factors as $\det(\partial_S)\det(\partial_Z)$. The latter selects degree three in Z from the degree-six squared factor, so homogeneity supplies exactly t^3. It suffices to evaluate $t = 1$.

Set $D = \det(\partial_A)$ and $L = \sum_{i,j} \partial_{k_i}\partial_{A_{ij}}\partial_{l_j}$. On functions of A, k, l, the chain rule and (4.3) give

$$\det(\partial_S) = (D + L)^2.$$

In fact $\partial_X = \partial_A$, $\partial_Y = \partial_A^{\mathsf{T}}$, $\partial_W = -\widehat{\partial_k}$, and $\partial_V = -\widehat{\partial_l}$; the two minus signs cancel. For three-by-three matrix variables the Leibniz formula gives

$$\det(\partial_M)(\det M)^2 = 24 \det M, \qquad \det(\partial_M)(\det M\, M_{ij}) = 8M_{ij}.$$

After differentiating in Z and setting $Z = 0$, the remaining polynomial is

$$24 \det A \det R(A) + 16 \det A\, k^{\mathsf{T}} R(A) l.$$

Thus its final constant after $(D + L)^2$ is

$$24D^2(\det A \det R(A)) + 32D \sum_{i,j} \partial_{A_{ij}}(\det A\, R(A)_{ij}). \tag{4.4}$$

The same three-row expansion gives

$$D(\det A \det R(A)) = 15 \det R(A), \qquad \sum_{i,j} \partial_{A_{ij}}(\det A\, R(A)_{ij}) = \mathrm{tr}(A)e_2(A) - 9 \det A.$$

In particular the first term of (4.4) vanishes. Since $\det R(A) = \mathrm{tr}(A)e_2(A) - \det A$, $D \det R(A) = 0$, and $D \det A = 6$, the second differential in (4.4) equals $6 - 54 = -48$. The final constant is therefore $24 \cdot 0 + 32(-48) = -1536$. All identities are in ordinary commuting polynomial variables.

Finally,

$$P_t(I_9) = P_t^*(I_9) = \operatorname{diag}((1+4t)I_6, 2I_3).$$

At $t = 1/4$, division by two is exactly bistochastic. Degree-nine homogeneity gives $-1536/(4^3\,2^9) = -3/64$, proving the normalization. □

Both summands $\widetilde{B}_6$ and Q have zero size-nine invariant. The first has a zero output block. For the second, $\det Q(\mathcal{X}) = (\det Z)^2 \det(X + Y^{\mathsf{T}})$, whose degree in Z exceeds the three available input derivatives. The next result strengthens this elementary nonconvexity obstruction by ensuring that *every* square compression of each summand used in the construction has a nonnegative invariant.

4.3. **Rank-four localization and nonconvexity of $\mathcal{K}_n$.** The family is due to Breuer and Hall (Breuer, 2006; Hall, 2006); its four-dimensional member is unitarily equivalent to Robertson's map (Chruściński and Kossakowski, 2008, Theorem 1). The block identity below writes B_{2m} as a sum of embedded copies of B_4. It lets us assemble Du's negative map from embedded four-dimensional cores, each in $\mathcal{K}_9$, which is what the nonconvexity theorem needs.

For the next lemma use adjacent symplectic pairs, so $J_{2m} = \operatorname{diag}(J_2, \dots, J_2)$, where $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let $V_{ij} : \mathbb{C}^4 \to \mathbb{C}^{2m}$ embed the i-th and j-th pairs, in that order.

Lemma 4.3 (Localization of the Breuer–Hall family)**.** *For every $m \geq 2$,*

$$B_{2m}(X) = \sum_{1 \leq i < j \leq m} V_{ij} B_4(V_{ij}^* X V_{ij}) V_{ij}^*. \tag{4.5}$$

Proof. Write $X = (X_{ij})$ in two-by-two blocks. Using $A + J_2 A^{\mathsf{T}} J_2^* = \operatorname{tr}(A) I_2$, we obtain

$$B_{2m}(X)_{ij} = \begin{cases} \displaystyle\sum_{\ell \neq i} \operatorname{tr}(X_{\ell\ell}) I_2, & i = j, \\ -X_{ij} - J_2 X_{ji}^{\mathsf{T}} J_2^*, & i \neq j. \end{cases}$$

Each off-diagonal block and each diagonal summand occurs in exactly one embedded B_4, proving (4.5). □

Theorem 4.4 (Nonconvexity of hereditary determinant positivity)**.** *For every $n \geq 9$, the class $\mathcal{K}_n$ is nonconvex. In size nine, four explicit members have equal-weight average H_9 from (4.2). The slice $\{H \in \mathcal{K}_9 : \operatorname{tr} H(I) = 9\}$ is also nonconvex, and a negative convex combination within it is bistochastic.*

Proof. Use the symplectic pairs $\operatorname{span}(e_i, e_{i+3})$, $i = 1, 2, 3$, in the first six coordinates of $\mathbb{C}^9$. Let $W_{ij} : \mathbb{C}^4 \to \mathbb{C}^9$ have columns $(e_i, e_{i+3}, e_j, e_{j+3})$, and set

$$C_{ij}(X) = W_{ij} B_4(W_{ij}^* X W_{ij}) W_{ij}^*.$$

The normalization in Theorem B.1, or its size-four calculation, gives $\Gamma_4(B_4) = 24$. Corollary 3.4 therefore places every C_{ij} in $\mathcal{K}_9$. The map Q is decomposable and hence belongs to $\mathcal{K}_9$ as well.

Lemma 4.3 gives $\widetilde{B}_6 = C_{12} + C_{13} + C_{23}$. Therefore the four maps

$$A_0 = 2Q, \qquad A_{12} = \tfrac{1}{2} C_{12}, \qquad A_{13} = \tfrac{1}{2} C_{13}, \qquad A_{23} = \tfrac{1}{2} C_{23} \tag{4.6}$$

belong to $\mathcal{K}_9$, but their average is

$$\frac{1}{4}(A_0 + A_{12} + A_{13} + A_{23}) = \frac{1}{2} Q + \frac{1}{8} \widetilde{B}_6 = H_9.$$

Proposition 4.2 proves its negative invariant.

For a common trace normalization, put $G_0 = \frac{3}{4} Q$ and $G_{ij} = \frac{9}{8} C_{ij}$. Indeed $\operatorname{tr} Q(I) = 12$ and $\operatorname{tr} C_{ij}(I) = 8$, so all four maps have $\operatorname{tr} G(I) = 9$. Their convex combination is

$$H_9 = \frac{2}{3} G_0 + \frac{1}{9}(G_{12} + G_{13} + G_{23}). \tag{4.7}$$

For $n > 9$, pad each map in (4.6) by zero and add the same diagonal pinching map on coordinates $10, \dots, n$. The first result is decomposable. Each of the other three is a four-dimensional core with nonnegative Γ_4 plus an entanglement-breaking map, so Corollary 3.4 places it in $\mathcal{K}_n$. Their average is the pinched sum of H_9 and diagonal pinching on the remaining coordinates. Diagonal pinching has invariant one. Equation (2.8) therefore gives $\Gamma_n = -3/64$ for the average. □

Corollary 4.5 (Strict compression positivity does not restore convexity)**.** *For each $n \geq 9$, there are four strictly positive maps in the interior of $\mathcal{K}_n$ whose average has negative Γ_n. Thus the nonconvexity is not confined to deficient-rank boundary maps.*

Proof. Let $A_0, \dots, A_3$ be the maps constructed in the preceding proof, including the common tail pinching when $n > 9$. For $\epsilon > 0$, set $A_j^\epsilon = A_j + \epsilon\Delta_n$. Every size-$k$ compression satisfies

$$\Gamma_k((A_j^\epsilon)_{V,U}) \geq k!(\epsilon/n)^k > 0.$$

Indeed, its added map is $(\epsilon/n)\operatorname{tr}(\,\cdot\,)I_k$, and the terms of the EB-addition formula that use only the added map give the displayed bound. Also $A_j^\epsilon(xx^*) \succeq (\epsilon/n)\|x\|^2 I_n$. These strict bounds, continuity, and compactness of the finitely many input/output Grassmannians imply that each A_j^ϵ lies in the interior of $\mathcal{K}_n$, relative to Hermiticity-preserving maps. The average is the preceding negative map plus $\epsilon\Delta_n$. Its invariant remains negative for sufficiently small positive ϵ, by polynomial continuity. □

Corollary 4.6 (No unitary-invariant convex gluing of all rank-four cores)**.** *Fix $n \geq 9$. There is no convex cone of positive maps on M_n which simultaneously*
(1) contains all decomposable maps;
(2) contains the standard zero-padded four-dimensional Breuer–Hall map and is invariant under simultaneous unitary conjugation $H \mapsto \mathrm{Ad}_U \circ H \circ \mathrm{Ad}_{U^*}$; and
(3) has $\Gamma_n \geq 0$ on every member.

Proof. Extend each isometry W_{ij} from the construction above to a unitary on $\mathbb{C}^n$, after zero padding when $n > 9$. Simultaneous conjugation by these unitaries sends the standard padded B_4 to C_{12}, C_{13}, C_{23}. Conicity and convexity permit their positive linear combinations with Q. In size nine one such combination is H_9. In larger size add the entanglement-breaking diagonal pinching on the remaining coordinates, which is already in the decomposable cone. The resulting map has $\Gamma_n = -3/64$ by the proof of Theorem 4.4, contradicting the third condition. □

4.4. **A single-map certificate need not be hereditary.** A certificate for $\Gamma_n(H) \geq 0$ alone need not survive entanglement-breaking addition, even when H is positive semidefinite as a Hilbert–Schmidt operator. This failure is why Theorem 3.2 asks for the sign on every compression.

Proposition 4.7. *Suppose $K : M_s \to M_s$ is positive and bistochastic, with $\Gamma_s(K) < 0$. There are a positive bistochastic map $R : M_{2s} \to M_{2s}$, positive semidefinite as a Hilbert–Schmidt operator, and an entanglement-breaking map D such that $\Gamma_{2s}(R + tD) < 0$ for all sufficiently large t.*

Proof. For s-by-s blocks define

$$R\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X + K(W) & Y \\ Z & W + K^*(X) \end{pmatrix}, \qquad D\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} \operatorname{diag}(W) & 0 \\ 0 & 0 \end{pmatrix}.$$

Writing $R = (\mathrm{Id} + \mathcal{S})/2$, the map $\mathcal{S}$ is positive and bistochastic. It is Hilbert–Schmidt selfadjoint, with diagonal-block operator matrix $\left(\begin{smallmatrix} 0 & K \\ K^* & 0 \end{smallmatrix}\right)$, and is zero on off-diagonal input blocks. Lemma 2.4 gives its Hilbert–Schmidt contraction property. Hence $\|\mathcal{S}\| \leq 1$ and $R \succeq 0$ in Hilbert–Schmidt order.

The map D has rank-one Kraus operators $|i\rangle\langle s + i|$, so is entanglement breaking. The polynomial $p(t) = \Gamma_{2s}(R + tD)$ has degree at most s, because D has only s nonzero output rows. The leading

output-determinant coefficient is

$$[t^s]\det(R+tD)(\mathcal{X}) = 2^{-s}\left(\prod_{i=1}^{s} W_{ii}\right)\det(K^*(X)+W).$$

It uses no cross-block input variables. The full input determinant differential therefore factors into those in X and W. Its s derivatives in X annihilate all terms using W in the second determinant. The complementary diagonal differentiation gives one, not $s!$. Therefore

$$[t^s]p(t) = 2^{-s}\Gamma_s(K^*) = 2^{-s}\Gamma_s(K) < 0. \tag{4.8}$$

The adjoint equality is Lemma 2.1. A polynomial with a negative leading coefficient is negative for all sufficiently large positive t. □

For $K = H_9$, (4.8) gives

$$[t^9]\Gamma_{18}(R+tD) = -\frac{3}{32768}.$$

An explicit, deliberately conservative parameter is $t = 2^{33}(18!)^3$. Indeed every matrix-unit coefficient of the contraction R has modulus at most one, and every coefficient of D is zero or one. The full triple-permutation expression has $(18!)^3$ terms. Thus, with $p(t) = \sum_{k=0}^{9} a_k t^k$,

$$\sum_{k<9}|a_k| \le B := 2^{18}(18!)^3, \qquad p(t) \le t^8\left(-\frac{3t}{32768}+B\right) = -2Bt^8 < 0.$$

This bound gives a finite parameter without locating the smallest admissible t.

Thus adding an entanglement-breaking map, and hence a completely positive or decomposable map, to a Hilbert–Schmidt positive-semidefinite map can make the invariant negative. The cross compressions of R are $K/2$ and $K^*/2$, so $R \notin \mathcal{K}_{2s}$.

5. Applications

We now express Chern forms through the determinant invariants of curvature compressions. Hereditary determinant positivity then controls every Chern form, and decomposability permits the incidence construction needed for all Schur forms. Throughout this section, nonnegativity is pointwise for the Chern–Weil forms of the specified metric; strict positivity will be stated explicitly.

5.1. **From hereditary determinant positivity to Chern forms.** Let (E,h) have rank r over a complex manifold of dimension n. At a point x, choose holomorphic coordinates and a unitary fiber frame, and write its normalized Chern curvature as

$$\Omega = \frac{i}{2\pi}R^h = i\sum_{a,b=1}^{n} H_x(E_{ab})\,dz_a\wedge d\bar z_b, \qquad H_x : M_n(\mathbb{C}) \longrightarrow M_r(\mathbb{C}). \tag{5.1}$$

Hermitian symmetry of curvature makes H_x Hermiticity preserving. Griffiths semipositivity is exactly ordinary positivity of H_x, since its evaluation on a tangent vector v and fiber vector ξ is $\xi^* H_x(vv^*)\xi \ge 0$. With

$$dV_k = \bigwedge_{a=1}^{k}(i\,dz_a\wedge d\bar z_a), \qquad \Gamma_k(T) = \det(\partial_X)\det T(X) = k!\Phi_k(T),$$

the top-form convention is

$$c_k(E,h)|_x = \Gamma_k(H_x)dV_k \quad \text{when } n=r=k. \tag{5.2}$$

In particular, there is no further factorial multiplying Γ_k. This convention agrees with Finski (2022, Theorem 3.4) after converting volume conventions, and with the explicit normalization in Du (2026, §2, (2.1)–(2.6)).

Every Hermiticity-preserving map $H : M_n \to M_r$ can be realized as the curvature map at the origin of

$$h_H(z) = I_r - 2\pi H(zz^*),$$

which is a positive definite metric on a sufficiently small ball. Since

$$h_H(0) = I_r, \qquad dh_H(0) = 0, \qquad \partial_{z_a}\partial_{\bar{z}_b} h_H(0) = -2\pi H(E_{ab}),$$

its curvature at zero satisfies (5.1). Consequently a negative Γ_n for a positive square map is an actual pointwise top-Chern obstruction, not a mismatch between representations of the curvature tensor.

Here is a direct verification of (5.2) that also treats lower Chern forms. For isometries $V : \mathbb{C}^k \to \mathbb{C}^n$ and $U : \mathbb{C}^k \to \mathbb{C}^r$, put

$$H_{V,U}(X) = U^* H_x(VXV^*)U.$$

The image of V is the tangent k-plane L.

Proposition 5.1 (Chern forms as sums of compression invariants)**.** *For $k \le \min(n, r)$ and any orthonormal fiber basis,*

$$c_k(E,h)|_L = \left(\sum_{\substack{J \subseteq \{1,\dots,r\} \\ |J|=k}} \Gamma_k(H_{V,U_J}) \right) dV_L, \tag{5.3}$$

where U_J is the coordinate inclusion. Equivalently, for invariant probability measure on the complex Grassmannian,

$$c_k(E,h)|_L = \binom{r}{k} \left(\int_{\mathrm{Gr}_{\mathbb{C}}(k,r)} \Gamma_k(H_{V,U})\, d\mu(U) \right) dV_L. \tag{5.4}$$

Consequently, if all square compressions of H_x have nonnegative Γ, every Chern form is weakly nonnegative at x.

Proof. The coefficient of t^k in $\det(I + t\Omega)$ is the sum of the k-by-k principal determinants of Ω. For a square map T, expand the determinant of $i\sum_{a,b} T(E_{ab})\, dz_a \wedge d\bar{z}_b$. A nonzero term uses every holomorphic and antiholomorphic index once. If their assignments to output rows are $\rho, \eta \in S_k$ and the output determinant permutation is $\sigma \in S_k$, its coefficient of dV_k is

$$\operatorname{sgn}(\sigma)\operatorname{sgn}(\rho)\operatorname{sgn}(\eta) \prod_{a=1}^{k} T(E_{\rho(a),\eta(a)})_{a,\sigma(a)}.$$

These products are exactly the terms of $\det(\partial_X) \det T(X)$: the input derivative permutation is $\eta\rho^{-1}$. This matching proves (5.2) and (5.3).

For an orthonormal k-frame U, write $u_J = \det U[J, [k]]$. Unitary invariance and Cauchy–Binet give

$$\int \overline{u_J} u_K\, d\mu(U) = \frac{\delta_{JK}}{\binom{r}{k}}, \qquad \sum_{|J|=k} |u_J|^2 = 1.$$

The entries of Ω have even degree and commute, so

$$\int \det(U^*\Omega U)\, d\mu(U) = \sum_{J,K} \det \Omega[J,K] \int \overline{u_J} u_K\, d\mu(U) = \binom{r}{k}^{-1} \sum_J \det \Omega[J,J].$$

The last display proves (5.4). Weak nonnegativity means precisely nonnegativity of the restriction to each tangent k-plane, so the final assertion follows. □

The preceding proposition proves only weak nonnegativity. It does not assert that the Hermitian form on the entire exterior power is positive semidefinite, or that the differential form is strongly nonnegative. In top degree the distinction disappears: a negative multiple of the complex volume form violates all these positivity notions.

Besides decomposable curvature, the proposition applies to the rectangular versions of the low-dimensional classes in Corollary 3.4: maps with input or output marginal rank at most three, and maps of the form

$$H_x(X) = AK(B^*XB)A^* + S(X), \qquad A : \mathbb{C}^d \to \mathbb{C}^r, \quad B : \mathbb{C}^d \to \mathbb{C}^n, \tag{5.5}$$

where S is entanglement breaking, $K : M_d \to M_d$ is positive, $d \le 4$, and $\Gamma_4(K) \ge 0$ when $d = 4$. The proof of that corollary applies to every square compression in these rectangular dimensions.

The sector and Hilbert–Schmidt criteria for a single square map, and the coordinate block criteria, give the Chern-form sign on the corresponding plane. For the top Chern form of a rank-r bundle, the certificate must hold on every tangent r-plane. For a lower Chern form, one needs the square input/output compressions in (5.3).

5.2. **From decomposability to all Schur forms.** We use the usual Chern-root convention s_λ for Schur polynomials. Finski uses the conjugate indexing

$$P_a(c) = \det(c_{a_i - i + j}) = s_{a'}.$$

In particular, $c_k = s_{(1^k)} = P_{(k)}$. The distinction matters when comparing Chern–Weil evaluations with Schur coefficients in representation theory.

Call the curvature *decomposable* at a point if its rectangular map in (5.1) has the form

$$H_x(X) = \sum_\alpha A_\alpha X A_\alpha^* + \sum_\beta B_\beta X^{\mathsf{T}} B_\beta^*. \tag{5.6}$$

The matrices and their number may depend on the point; no smooth choice of a decomposition is required. This definition is the operator form of Finski's decomposable nonnegativity (Finski, 2022, Proposition 2.18).

The quotient stability in the next lemma is Finski's (Finski, 2022, Proposition 2.22). We restate it in the present Kraus conventions, together with the elementary pullback and direct-sum operations, to identify precisely the input needed below.

Lemma 5.2 (Operations needed for the incidence construction)**.** *Decomposable curvature is preserved by holomorphic pullback, orthogonal direct sums, adjoining a flat bundle, and taking a holomorphic quotient with its induced metric.*

Proof. For a tangent pullback with derivative L,

$$H_x(LXL^*) = \sum_\alpha (A_\alpha L) X (A_\alpha L)^* + \sum_\beta (B_\beta \bar{L}) X^{\mathsf{T}} (B_\beta \bar{L})^*.$$

Output direct sums and zero blocks preserve the same Kraus form.

For a quotient, identify its fiber with the orthogonal complement of the kernel, and let P be the corresponding output isometry. In local unitary frames, the normalized second-fundamental-form contribution is

$$i \sum_\nu c_\nu \wedge \bar{c}_\nu^{\mathsf{T}}, \qquad c_\nu = \sum_a \ell_{\nu a}\, dz_a,$$

absorbing the positive factor $1/(2\pi)$ into the columns. With $L_\nu = [\ell_{\nu 1} \ \cdots \ \ell_{\nu n}]$, the quotient map is

$$H^Q(X) = P^* H_x(X) P + \sum_\nu L_\nu X L_\nu^*.$$

The added map is completely positive, so the quotient retains the form (5.6). This computation also proves directly the quotient assertion of Finski (2022, Proposition 2.22). Transposing all matrix conventions merely interchanges the two summand types. □

The closure lemma allows us to apply the determinant theorem to the auxiliary quotient in Finski's incidence formula. Fiber integration then transfers its top Chern-form sign to the required Schur form.

Theorem 5.3 (Schur forms for ordinary decomposable curvature)**.** *Let (E, h) be a Hermitian holomorphic vector bundle of rank r over a complex manifold of dimension n. If its curvature is decomposable at a point x, every Schur form $s_\lambda(E, h)$ is weakly nonnegative at x. In particular, curvature that is decomposable at every point implies weak nonnegativity of all Schur forms throughout the base.*

Proof. The empty partition gives the form 1, while $|\lambda| > n$ or $\mathrm{length}(\lambda) > r$ gives the zero form. Otherwise let $1 \le k = |\lambda| \le n$, and write $a = \lambda'$ in Finski's notation, padding it by zeros. Put $V_0 = \mathbb{C}^{r+k}$ and

$$F = \mathrm{Hom}(V_0, E), \qquad Y = \mathbb{P}(F \oplus \mathcal{O}), \qquad Q = \pi^*(F \oplus \mathcal{O})/\mathcal{O}_Y(-1).$$

Thus F is the orthogonal direct sum of $r + k$ copies of E, and Q has rank $N = r(r + k)$. Fix the flag of subspaces in V_0 with dimensions $r + i - a_i$, as in the Kempf–Laksov construction, and let

$$\Omega_a(E)_x = \{u \in \mathrm{Hom}(V_0, E_x) : \dim(\ker u \cap V_i) \ge i,\ 1 \le i \le k\}.$$

Its projective compactification is $Z_a = \mathbb{P}(\Omega_a(E) \oplus \mathcal{O}) \subset Y$. The fibers of Z_a have complex dimension $N - k$.

Finski's identity of differential forms, rather than merely classes, is

$$s_\lambda(E, h) = P_a(c(E, h)) = \pi_*(c_N(Q, h^Q) \wedge [Z_a]). \tag{5.7}$$

Here the expression on the right means integration over the regular loci of the analytic incidence fibers; equivalently it is the pushforward current. The resulting current is the smooth form on the left (Finski, 2022, Theorem 3.2).

Lemma 5.2 shows that the curvature of Q is decomposable at every point above x. Its restriction to any tangent N-plane there is therefore a decomposable square map. The decomposable-map determinant theorem gives its $\Gamma_N \ge 0$. Equation (5.2) proves that $c_N(Q, h^Q)$ is weakly nonnegative at those points.

Finally, fix a tangent k-plane on the base. At a regular point of an incidence fiber, combine that plane with the $(N - k)$-dimensional vertical tangent space. The integrand defining the restriction of (5.7) is the evaluation of the weakly nonnegative (N, N)-form on this decomposable N-plane, with the complex orientation. It is nonnegative. The singular strata have zero measure for integration over the analytic cycle, and its multiplicities are positive. Fiber integration is therefore nonnegative, proving the theorem. Only the original curvature at x enters these evaluations, so no decomposition on a neighborhood of x is required. □

The proof does not give strictness from strict Griffiths positivity on the boundary of the decomposable cone.

The same proof applies whenever the curvature of Q in (5.7) has $\Gamma_N \ge 0$ on every tangent N-plane along the regular incidence locus. Under this condition, the corresponding Schur form is weakly nonnegative.

5.3. **Nonconvexity of simultaneous pointwise Chern positivity.** The obstruction in Section 4 has a pointwise geometric consequence stronger than the negativity of a single top Chern form under ordinary positivity.

Corollary 5.4. *In complex dimension and bundle rank nine, there exist four strictly Griffiths-positive curvature tensors, on the same tangent and fiber spaces, such that every Chern form of each tensor is strictly weakly positive, whereas the top Chern form of their average is negative. The same conclusion holds in every common dimension and rank at least nine.*

Proof. Use the four maps A_j^ϵ from Corollary 4.5, with ϵ sufficiently small to retain the negative invariant of their average. Assign to each map the curvature tensor in (5.1). Its strict product-vector bound is exactly strict Griffiths positivity. For every tangent k-plane $L = \operatorname{im} V$,

$$\sum_{|J|=k} \Gamma_k((A_j^\epsilon)_{V,U_J}) \geq \binom{n}{k} k!(\epsilon/n)^k > 0, \qquad 1 \leq k \leq n,$$

so (5.3) gives strict weak positivity. The average tensor has top coefficient $\Gamma_n(\frac{1}{4}\sum_j A_j^\epsilon) < 0$. By the local metric realization in Section 5.1, the four tensors are genuine curvature tensors at a point, not merely formal arrays. □

The conclusion concerns averaging curvature tensors in fixed frames. Averaging Hermitian metrics need not average their curvatures, and the corollary makes no assertion about all Schur forms of the four tensors.

5.4. **What the rank-nine obstruction changes.** For $H = H_9$, the metric h_H of Section 5.1 realizes the negative invariant in (4.2) as a negative top Chern form. A small positive trace addition makes the curvature strictly Griffiths positive and retains the negative sign. Du's construction realizes such negativity globally on $\mathcal{O}(1)^{\oplus r}$ over $\mathbb{P}^r$ for every $r \geq 9$ (Du, 2026, Theorem 1.1 and Propositions 3.1–3.2).

This negative top Chern form disproves the unrestricted pointwise implication from Griffiths positivity to all Schur forms, since the top Chern form is itself a Schur form. It leaves intact both the structural hypotheses in Theorem 5.3 and numerical Schur positivity for ample bundles. For example, $c_r(\mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r}) = c_1(\mathcal{O}(1))^r$ has integral one, independently of the chosen metric. A negative density on one open set does not contradict that positive integral.

6. Boundary cases and open problems

This section records the boundary of what the preceding theorems prove: an exact equality case for the normalized size-four problem, the size-four form of the maximality theorem, and the open problems that the counterexamples do not decide.

6.1. **A zero in the normalized rank-four problem.** Gurvits's skew-symmetric size-three example has $\Gamma_3 = 0$ (Gurvits, 2002, Example 2.6); padding that example by a scalar block gives an equality case, which Appendix A uses to test the sector certificates:

$$T_0(X) = \operatorname{diag}\left(\frac{\operatorname{tr}(A) I_3 - A^{\mathsf{T}}}{2}, x_{44}\right), \qquad A = X_{[3],[3]}. \tag{6.1}$$

It is positive, unital, and trace preserving. In fact the first block is completely positive: its Choi matrix is $(I_9 - F_3)/2$, where F_3 is the flip. Formula (2.11) realizes its invariant Γ_3 as the expected squared determinant of a skew-symmetric Gaussian matrix of size three. That determinant vanishes identically. Pinched factorization therefore gives

$$\Gamma_4(T_0) = \Phi_4(T_0) = 0.$$

The two Choi blocks are orthogonal for the Frobenius inner product, and

$$\|C_{T_0}\|_F^2 = \frac{1}{4}\operatorname{tr}(I_9 - F_3)^2 + 1 = \frac{18-6}{4} + 1 = 4.$$

Since T_0 is also a positive map on M_3 plus an entanglement-breaking scalar block, Corollary 3.4 puts it in $\mathcal{K}_4$, with $\Gamma_4(T_0) = 0$. On the traceless diagonal direction $\operatorname{diag}(1, -1, 0, 0)$, T_0 acts by $-1/2$. It fails the Hilbert–Schmidt sector condition, so the strict bistochastic lower bound from that theorem does not apply.

6.2. **The size-four boundary of the maximality theorem.**

Corollary 6.1. *The class $\mathcal{K}_3$ is the entire positive-map cone, and*

$$\mathcal{K}_4 = \{H : M_4 \to M_4 \text{ positive} : \Gamma_4(H) \geq 0\}.$$

For any positive $H : M_4 \to M_4$ and any entanglement-breaking S, every nonconstant coefficient of $\Gamma_4(H + tS)$ is nonnegative. In particular,

$$\Gamma_4(H + tS) \geq \Gamma_4(H) + t^4\Gamma_4(S) \qquad (t \geq 0). \tag{6.2}$$

Proof. All compressions of size at most three are covered by (2.12). Full-size isometries preserve the sign by (2.6), which proves the descriptions of $\mathcal{K}_3, \mathcal{K}_4$. In (3.1), every nonempty subset compresses H to size at most three. Its Gram factors and its invariant are nonnegative. Homogeneity identifies the coefficient of t^4 with $\Gamma_4(S)$, proving the final inequality. □

The corollary does not determine the constant coefficient $\Gamma_4(H)$. In particular, a nonnegative value after depolarizing addition does not yield a nonnegative value before that addition.

6.3. **Open problems.** The first open question is the sign of Γ_4 on all positive maps, equivalently whether $\mathcal{K}_4$ is the full positive cone. In higher dimensions, Theorem 4.4 leaves the convexity of the bistochastic slice unresolved: the constructed summands have a bistochastic average but are not individually bistochastic. Another question is which convex cones can extend decomposability while preserving the sign; Corollary 4.6 excludes those containing the padded B_4 and closed under simultaneous unitary conjugation.

For the nondecomposable subclasses of $\mathcal{K}_n$, extending Chern-form positivity to all Schur forms requires control of the quotient curvatures in (5.7). Their second-fundamental-form contribution is completely positive but need not be entanglement breaking, so the addition theorem does not settle this extension.

Finally, the examples do not decide the larger cone consisting of sums of a 2-positive and a 2-copositive map. A map is 2-positive if $\mathrm{Id}_{M_2} \otimes H$ is positive, and 2-copositive if its input-transposed version is 2-positive. Breuer–Hall maps are atomic, meaning that they lie outside this sum cone (Chruściński and Kossakowski, 2008, Theorem 3). The leading size-six compression of H_9 is $B_6/8$; since the sum cone is preserved by completely positive embeddings and compressions, H_9 is atomic as well. Thus the displayed size-nine counterexample does not settle the sign problem on that cone.

Appendix A. A Hilbert–Schmidt sector certificate

Addition of matrix phases under tensor products is part of the sectorial-matrix calculus of Wang et al. (2020, Theorem 11.1). We apply that mechanism, with a singular-support endpoint argument, to the determinant vector in Gaussian Fock space. The result is an explicit sufficient condition for $\mathrm{Re}\,\Gamma_n(H) \geq 0$ and a bistochastic lower bound; the Hilbert–Schmidt positive-semidefinite square case is in Gurvits (2004, Appendix C).

All adjoints in this appendix refer to the Hilbert–Schmidt inner product on $M_n(\mathbb{C})$. A positive-semidefinite superoperator need not have a positive-semidefinite Choi matrix. We use the normalization and Fischer product from (1.2) and (2.2).

Theorem A.1 (Hilbert–Schmidt sector certificate)**.** *Let $n \geq 2$ and let $H : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be complex linear. Put*

$$S = \frac{H + H^*}{2}, \qquad A = \frac{H - H^*}{2i}, \qquad t_n = \tan\frac{\pi}{2n}.$$

Suppose the following two Hilbert–Schmidt operator inequalities hold:

$$-t_n S \preceq A \preceq t_n S. \tag{A.1}$$

Then $\operatorname{Re}\Gamma_n(H) \geq 0$. *If* H *is Hermiticity preserving, then* $\Gamma_n(H)$ *is real and* $\Phi_n(H) \geq 0$. *If in addition* $H(I) = H^*(I) = I$, *then*

$$\operatorname{Re}\Phi_n(H) \geq n^{-n}. \tag{A.2}$$

Thus an ordinary positive bistochastic map satisfying (A.1) *satisfies* $\Phi_n(H) \geq n^{-n}$.

Proof. Adding the two positive inequalities in (A.1) gives $S \succeq 0$. They also imply $A \ker S = 0$. Indeed, for $v \in \ker S$, the positive operators $t_n S \pm A$ both have zero quadratic form at v, so both annihilate v. On the support of S define

$$K = S^{-1/2} A S^{-1/2}.$$

Then $K = K^*$, $\|K\| \leq t_n$, and, extending K by zero on the kernel,

$$H = S^{1/2}(I + iK)S^{1/2}. \tag{A.3}$$

For every integer $0 \leq k \leq n$, the normal operator $(I + iK)^{\otimes k}$ has eigenvalues

$$\prod_{j=1}^{k}(1 + i\kappa_j), \qquad |\kappa_j| \leq t_n.$$

For $k \leq n$,

$$\left|\arg \prod_{j=1}^{k}(1 + i\kappa_j)\right| \leq \sum_{j=1}^{k} |\arctan \kappa_j| \leq \frac{k\pi}{2n} \leq \frac{\pi}{2}.$$

Congruence by $(S^{1/2})^{\otimes k}$ therefore proves

$$\operatorname{Re} H^{\otimes k} \succeq 0, \qquad 0 \leq k \leq n. \tag{A.4}$$

On homogeneous Gaussian Fock space, use the pullback operators $\mathcal{U}_L f = f \circ L$ from Lemma 2.1. The symmetric-tensor model transfers (A.4) to nonnegative Hermitian part of $\mathcal{U}_H$ in every homogeneous degree $k \leq n$. Equivalently, diagonalize K and apply the preceding product-of-eigenvalues calculation to monomials, then use the palindromic factorization

$$\mathcal{U}_H = \mathcal{U}_{S^{1/2}} \mathcal{U}_{I+iK} \mathcal{U}_{S^{1/2}}.$$

Its outside factors are selfadjoint, so

$$\operatorname{Re}\Gamma_n(H) = \langle \det, (\operatorname{Re}\mathcal{U}_H) \det \rangle_F \geq 0.$$

For Hermiticity-preserving H, reality follows from Lemma 2.1.

For the quantitative assertion, decompose

$$G = zI/\sqrt{n} + G_0, \qquad \det G = \sum_{k=0}^{n} z^{n-k} q_k(G_0),$$

where G_0 is the standard Gaussian on the traceless Hilbert–Schmidt subspace, independent of z. Bistochasticity makes the scalar and traceless subspaces invariant for both H and H^*. The traceless restriction has the same sector bounds. Orthogonality of the powers of z gives

$$\Gamma_n(H) = \sum_{k=0}^{n} (n-k)! \left\langle q_k, \mathcal{U}_{H|_{\mathrm{tr}=0}} q_k \right\rangle.$$

Each summand has nonnegative real part, and $q_0 = n^{-n/2}$. Thus

$$\operatorname{Re}\Gamma_n(H) \geq n! |q_0|^2 = \frac{n!}{n^n},$$

which proves (A.2). □

For fixed n, condition (A.1) defines a convex cone by two linear matrix inequalities. It contains the Hilbert–Schmidt PSD cone and allows a nonzero skewadjoint part. Intersecting it with the ordinary positive-map cone gives a sufficient class without requiring complete positivity. It does not assert that arbitrary decomposable summands may be added to a map in this class.

Remark A.2 (Recovery of the square formula)**.** *For $H = H^* \succeq 0$, Lemma 2.1 gives*

$$\Gamma_n(H) = \mathbb{E}|\det H^{1/2}(G)|^2.$$

Here $H^{1/2}$ is the Hilbert–Schmidt operator square root and need not preserve positive matrices. If $H(I) = I$, selfadjointness gives $H^*(I) = I$, so the theorem yields $\Phi_n(H) \geq n^{-n}$.

A.1. **A sharper phase certificate for normal operators.**

Proposition A.3 (Normal spectral-support certificate)**.** *Suppose H is normal as a Hilbert–Schmidt operator. Choose an orthonormal eigenbasis E_j, with eigenvalues λ_j, and write*

$$\det\left(\sum_j z_j E_j\right) = \sum_{|\alpha|=n} c_\alpha z^\alpha.$$

Then

$$\Gamma_n(H) = \sum_{|\alpha|=n} \alpha!|c_\alpha|^2\lambda^\alpha. \tag{A.5}$$

Consequently it suffices that

$$\operatorname{Re}\lambda^\alpha \geq 0 \quad \textit{whenever } c_\alpha \neq 0. \tag{A.6}$$

For Hermiticity-preserving H this condition implies $\Phi_n(H) \geq 0$; for bistochastic Hermiticity-preserving H it implies $\Phi_n(H) \geq n^{-n}$.

Proof. Use the same standard Gaussian coefficients z_j in both determinants. The coefficient of z^α in $\det H(G)$ is $c_\alpha\lambda^\alpha$. Gaussian monomial orthogonality proves (A.5), and (A.6) makes every summand's real part nonnegative. The support condition depends only on the nonzero multihomogeneous components across distinct eigenspaces, so it is unchanged by a unitary basis change within one eigenspace. In the bistochastic case choose $E_0 = I/\sqrt{n}$ and complete it within the invariant traceless subspace. The pure scalar term has coefficient $n^{-n/2}$ and contributes $n!/n^n$. □

Condition (A.6) is a finite criterion on the support of a determinant polynomial and the phases of eigenvalue products, rather than an assumption on the sign of the sum in (A.5). It can admit negative eigenvalues. For example, transposition is $+1$ on symmetric matrices and -1 on skew-symmetric matrices. Transposition invariance of the determinant excludes odd total degree in the skew variables, so every nonzero eigenvalue product is $+1$.

For the equality map T_0, put $D = \operatorname{diag}(1, 1, -2, 0)$. Then

$$T_0(D) = -D/2, \qquad T_0(E_{44}) = E_{44}, \qquad \det(tD + zE_{44}) = -2t^3z.$$

The nonzero t^3z term has eigenvalue product $(-1/2)^3 < 0$, so T_0 fails (A.6), consistently with $\Phi_4(T_0) = 0$.

Appendix B. The entire skew-unitary Breuer–Hall family

The Breuer–Hall maps (Breuer, 2006; Hall, 2006; Chruściński and Kossakowski, 2008) are positive and indecomposable in every even size at least four. We prove that their invariant Γ_n is positive in every even size, through an exact three-exponential moment formula and a sign argument for its odd moments. The moment formula comes from a skew-Gaussian duality of Forrester and Kulkarni–Kawabata–Ryu by a polynomial specialization in the coupling parameter; no Gaussian law is continued to negative variance.

Theorem B.1. *Let $n = 2m$ and let $J \in M_n(\mathbb{C})$ be skew-symmetric and unitary. Define*

$$B_n(X) = \operatorname{tr}(X) I_n - X - JX^{\mathsf{T}} J^*.$$

If X_1, X_2, X_3 are independent exponential random variables of mean one, then

$$\Gamma_{2m}(B_{2m}) = \mathbb{E}\left(X_1^2 + X_2^2 + X_3^2 - 2X_1X_2 - 2X_1X_3 - 2X_2X_3\right)^m. \tag{B.1}$$

This value is zero for $m = 1$ and strictly positive for every $m \geq 2$. For $n \geq 4$, the normalized map $\widehat{B}_n = B_n/(n-2)$ is positive, bistochastic, and indecomposable, and

$$\Phi_n(\widehat{B}_n) = \frac{\Gamma_n(B_n)}{n!(n-2)^n} > 0, \qquad \|C_{\widehat{B}_n}\|_F^2 = \frac{3n}{n-2}. \tag{B.2}$$

The next identity is a specialization of the characteristic-polynomial duality of Forrester (2024, Proposition 1.2, equation (1.13)), attributed there to Kulkarni et al. (2025). In that formula set $N = m$, $k = 1$ and identify the self-dual matrix as JA. Then $\det(tI - JA) = \det(A + tJ)$ and the dual real matrix is 2×2, with variance-one entries. We give a finite algebraic proof of this specialization to fix every sign and the common Gaussian coefficients.

Lemma B.2 (Skew Gaussian duality)**.** *Let $n = 2m$ and let J_0 have adjacent diagonal blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let A be a complex skew-symmetric matrix with independent standard complex Gaussian entries above the diagonal, normalized by $\mathbb{E}|A_{ij}|^2 = 1$. Let G be a real 2×2 matrix with independent $N(0,1)$ entries. For every $t \in \mathbb{C}$,*

$$\mathbb{E}_A |\det(A + tJ_0)|^2 = \mathbb{E}_G \det(GG^{\mathsf{T}} + |t|^2 I_2)^m. \tag{B.3}$$

Proof. We use a finite Grassmann algebra, so all exponentials and expectations below are coefficientwise finite algebraic operations. Introduce odd variables $x_{i,a}, y_{i,b}$, where $1 \leq i \leq n$ and $a, b \in \{1, 2\}$. The extraction order is

$$x_{1,1} \cdots x_{n,1}\; x_{1,2} \cdots x_{n,2}\; y_{1,1} \cdots y_{n,1}\; y_{1,2} \cdots y_{n,2}. \tag{B.4}$$

Put

$$U_{ij} = \sum_{a=1}^{2} x_{i,a} x_{j,a}, \qquad V_{ij} = \sum_{b=1}^{2} y_{i,b} y_{j,b}.$$

Since a determinant of a skew matrix is its Pfaffian squared, the left side of (B.3) is the coefficient of (B.4) in

$$\mathbb{E}_A \exp\left(\sum_{i<j} (A_{ij} + t(J_0)_{ij}) U_{ij} + \sum_{i<j} (\overline{A_{ij}} + \bar{t}(J_0)_{ij}) V_{ij} \right).$$

The quadratic Grassmann elements are even and commute. The complex Gaussian identity, with variance one, gives

$$\mathbb{E}_A \exp\left(\sum_{i<j} (A_{ij} U_{ij} + \bar{A}_{ij} V_{ij}) \right) = \exp\left(\sum_{i<j} U_{ij} V_{ij} \right).$$

If $S_{ab} = \sum_i x_{i,a} y_{i,b}$, one interchange of odd variables and the vanishing of repeated odd variables give

$$\sum_{i<j} U_{ij} V_{ij} = -\frac{1}{2} \sum_{a,b=1}^{2} S_{ab}^2.$$

The variance-one real Gaussian identity therefore yields

$$\exp\left(-\frac{1}{2} \sum_{a,b} S_{ab}^2 \right) = \mathbb{E}_G \exp\left(i \sum_{a,b} g_{ab} S_{ab} \right).$$

The remaining exponent factors over the m pairs $(i,j) = (2p-1, 2p)$ specifying J_0. Regrouping (B.4) into these pairs exchanges chunks of two odd variables, and hence introduces no sign. For one pair the extraction order is

$$x_{i,1}, x_{j,1}, x_{i,2}, x_{j,2}, y_{i,1}, y_{j,1}, y_{i,2}, y_{j,2},$$

and the exponent is

$$t\sum_a x_{i,a}x_{j,a} + \bar{t}\sum_b y_{i,b}y_{j,b} + i\sum_{a,b} g_{ab}(x_{i,a}y_{i,b} + x_{j,a}y_{j,b}).$$

The top coefficient can be checked according to the number of factors containing an entry of G:

number of such factors	contribution
0	$\lvert t\rvert^4$
2	$\lvert t\rvert^2(g_{11}^2 + g_{12}^2 + g_{21}^2 + g_{22}^2)$
4	$(g_{11}g_{22} - g_{12}g_{21})^2$.

Their sum is exactly $\det(GG^{\mathsf{T}} + |t|^2 I_2)$. Every pair uses the *same* four coefficients g_{ab}. Consequently the product is the mth power inside one common expectation, proving (B.3). □

Proof of Theorem B.1. First take $J = J_0$ as in Lemma B.2. For a real parameter $s \geq 0$, define

$$T_s(X) = \operatorname{tr}(X) I_n - JX^{\mathsf{T}}J^* + sX.$$

Write $\Omega = \sum_i e_i \otimes e_i$, $P = |\Omega\rangle\langle\Omega|$, and let F denote swap. Set $F_J = (I \otimes J)F(I \otimes J^*)$. Then

$$C_{T_s} = I - F_J + sP \succeq 0.$$

Conjugating its output factor by J^* gives

$$I - F + s|(I \otimes J)\Omega\rangle\langle(I \otimes J)\Omega|.$$

The sign $J^* = -J$ has no effect on the last rank-one operator. This operator is the covariance matrix of $A + \sqrt{s}\, gJ$, where A is the skew Gaussian matrix in the lemma and g is an independent standard complex Gaussian. The exact CP determinant identity therefore gives

$$\Gamma_n(T_s) = \mathbb{E}_{A,g} |\det(A + \sqrt{s}\, gJ)|^2 = \mathbb{E}_{G,E} \det(GG^{\mathsf{T}} + sEI_2)^m, \tag{B.5}$$

where $E = |g|^2$ is exponential of mean one and independent of G. There is no extra factorial: the CP identity is normalized by $\Gamma_n(\mathrm{Id}) = \mathbb{E}|g|^{2n} = n!$.

Both sides of (B.5) are polynomials in s of degree at most $2m$. Equality for all $s \geq 0$ is consequently a polynomial identity, valid at $s = -1$. As $T_{-1} = B_n$, the identity at $s = -1$ proves

$$\Gamma_n(B_n) = \mathbb{E}_{G,E} \det(GG^{\mathsf{T}} - EI_2)^m. \tag{B.6}$$

The probability law in (B.6) is still the original positive product law of four real Gaussians and one exponential.

The real variables

$$p = \frac{g_{11} + g_{22}}{\sqrt{2}}, \quad q = \frac{g_{12} - g_{21}}{\sqrt{2}}, \quad r = \frac{g_{11} - g_{22}}{\sqrt{2}}, \quad u = \frac{g_{12} + g_{21}}{\sqrt{2}}$$

are independent $N(0,1)$ variables. Put $X_1 = (p^2 + q^2)/2$ and $X_2 = (r^2 + u^2)/2$. These two variables are independent mean-one exponentials, and

$$\|G\|_F^2 = 2(X_1 + X_2), \qquad \det G = X_1 - X_2.$$

Substitute these identities in (B.6), and rename E as X_3, to obtain (B.1).

It remains to prove its sign. Define

$$\lambda(x_1, x_2, x_3) = \sum_i x_i^2 - 2\sum_{i<j} x_i x_j.$$

For the three independent exponentials, $S = X_1 + X_2 + X_3$ is independent of $U = (X_1, X_2, X_3)/S$. The vector U is uniform on the equilateral simplex

$$\Delta = \{u \in \mathbb{R}^3 : u_i \geq 0,\ \textstyle\sum_i u_i = 1\},$$

and $\mathbb{E}S^{2m} = (2m+2)!/2$. It therefore suffices to determine the sign of the integral of λ^m over Δ.

Set $c = \frac{1}{3}(1,1,1)$, the simplex center. In Euclidean polar coordinates write $u = c + \rho e_\theta$ with $0 \leq \rho \leq R(\theta)$. Then

$$\lambda(u) = 2\rho^2 - \frac{1}{3}.$$

On an edge $u_k = 0$, with $\{i,j,k\} = \{1,2,3\}$,

$$b(\theta) := 2R(\theta)^2 - \frac{1}{3} = \lambda(u) = (u_i - u_j)^2 \in [0,1].$$

This boundary value is nonconstant in θ. Direct radial integration gives

$$\int_\Delta \lambda(u)^m \, dA = \frac{1}{4(m+1)} \int_0^{2\pi} \left[b(\theta)^{m+1} - (-1/3)^{m+1}\right] d\theta. \tag{B.7}$$

For $m = 1$, direct exponential moments give $\mathbb{E}\lambda(X_1, X_2, X_3) = 3 \cdot 2 - 2 \cdot 3 = 0$. Equation (B.7) therefore implies $\mathbb{E}_\theta b(\theta)^2 = 1/9$, where θ is uniform on $[0, 2\pi]$. If m is even, the integrand in (B.7) is strictly positive. If $m \geq 3$ is odd, strict Jensen applied to the nonconstant variable b^2 gives

$$\mathbb{E}_\theta b^{m+1} > \left(\mathbb{E}_\theta b^2\right)^{(m+1)/2} = (1/3)^{m+1}.$$

These two cases prove the asserted strict positivity. For $m = 1$, the map itself is zero, consistently with the moment calculation.

Positivity follows from the rank-one calculation following (4.1). Also $B_n(I_n) = B_n^*(I_n) = (n-2)I_n$, proving the stated bistochastic normalization.

The Choi matrix is $C_{B_n} = I - P - F_J$. Consider $W = I + P + F_J \succeq 0$. Its partial transpose on the output factor is

$$W^{\mathsf{T}_{\text{out}}} = I + F + |(I \otimes J)\Omega\rangle\langle(I \otimes J)\Omega| \succeq 0.$$

Here $P^{\mathsf{T}_{\text{out}}} = F$ and, for real J, $F_J^{\mathsf{T}_{\text{out}}} = (I \otimes J)P(I \otimes J^{\mathsf{T}})$. Since $F_J\Omega = -\Omega$ and $P^2 = nP$,

$$(P + F_J)^2 = I + (n-2)P, \qquad \operatorname{tr}(C_{B_n} W) = -n(n-2) < 0 \quad (n \geq 4).$$

A decomposable Choi matrix pairs nonnegatively with every PPT positive semidefinite operator. The negative pairing above therefore proves indecomposability. Expanding $\operatorname{tr}(I - P - F_J)^2$, using $\operatorname{tr} F_J = n$ and $PF_J = -P$, gives

$$\|C_{B_n}\|_F^2 = 3n(n-2),$$

which proves the Choi-norm formula in (B.2).

Finally, every complex skew-unitary matrix has the form $J = UJ_0U^{\mathsf{T}}$ for a unitary U. One obtains this form by choosing orthonormal pairs for the antiunitary J followed by complex conjugation, whose square is $-I$. Direct substitution gives

$$B_J = \mathrm{Ad}_U \circ B_{J_0} \circ \mathrm{Ad}_{U^*}.$$

These unitary changes preserve the determinant invariant, positivity, bistochasticity, indecomposability, and the Choi norm. Thus all the conclusions hold for the original complex skew-unitary J. □

Exact controls and scope. Independent sparse-Pfaffian expansions give

n	2	4	6	8	10
$\Gamma_n(B_n)$	0	24	384	31104	2488320.

In particular, the bistochastic rank-four and rank-six maps have squared Choi norms 6 and 9/2, and invariant values $\Phi_4 = 1/16$ and $\Phi_6 = 1/7680$, respectively. The theorem covers skew-unitary

Breuer–Hall maps; it asserts no closure under arbitrary addition and no result for arbitrary skew contractions. Pinched direct sums preserve the product of the Γ invariants, and therefore furnish positive indecomposable examples in odd dimensions as well.

The moment theorem goes beyond the normal phase criterion. For even $n \geq 6$, use adjacent-pair J and put $D = \operatorname{diag}(1, 1, 1, 1, -2, -2, 0, \ldots, 0)$. Then $\operatorname{tr} D = 0$, $JD^{\mathsf{T}}J^* = D$, and $\widehat{B}_n(D) = -2D/(n-2)$, whereas $\widehat{B}_n(I_n) = I_n$. But

$$\det(zI_n + tD) = z^{n-6}(z+t)^4(z-2t)^2$$

has coefficient -4 at $t^3 z^{n-3}$. After orthonormal rescaling this coefficient remains nonzero. The determinant therefore has a nonzero odd-degree component in the negative Hilbert–Schmidt eigenspace, so the sufficient condition in Proposition A.3 fails. Existence of this homogeneous component is independent of the eigenbasis inside that eigenspace. The positive moment theorem thus controls cancellations that the termwise phase condition cannot certify.

The same calculation excludes the Jordan orbits of the Hilbert–Schmidt PSD class, with the terminology of Appendix C. A Jordan extension of the polar partial isometry would fix I_n and send D to $-D$, forcing the displayed determinant to be even in t. For $n = 4$, the corresponding exclusion is the Clifford argument in Section C.1.

Appendix C. Strict separation of the sufficient classes

The comparison classes in this appendix are decomposable maps, Hilbert–Schmidt positive-semidefinite positive maps, and positive maps preserving an orthogonal rank-one resolution together with their depolarizing regularizations $(1-\beta)T + \beta\Delta_n$, $0 \leq \beta \leq 1$. A Jordan symmetry means a unitary conjugation or its composition with transposition. Every reference to a filtered class means independent invertible input and output congruences, with whole input/output transposition also allowed.

C.1. **An independent rank-four Breuer–Hall control.** Let X, Y, Z be the Pauli matrices, and set

$$s_1 = X \otimes I_2, \quad s_2 = Y \otimes I_2, \quad s_3 = Z \otimes X, \quad s_4 = Z \otimes Y, \quad s_5 = Z \otimes Z.$$

These five matrices are pairwise anticommuting traceless Hermitian involutions, with $s_1 s_2 s_3 s_4 s_5 = -I_4$. Let $J = s_2 s_4 = iX \otimes Y$, a real skew orthogonal matrix, and define the normalized Breuer–Hall map

$$B(A) = \frac{\operatorname{tr}(A) I_4 - A - JA^{\mathsf{T}}J^*}{2}.$$

The involution $T_J(A) = JA^{\mathsf{T}}J^*$ fixes $I_4, s_1, \ldots, s_5$. Its fixed space has dimension six: multiplication by J identifies it with the skew matrices. Hence those six matrices span the fixed space. The Hilbert–Schmidt eigenvalues of B are 1 on I_4, -1 on the five s_i, and zero on the complementary space.

The matrices $I_4/2, s_1/2, \ldots, s_5/2$ are orthonormal. The Clifford relations yield

$$\det \frac{z_0 I_4 + \sum_{i=1}^5 z_i s_i}{2} = \frac{1}{16}\left(z_0^2 - \sum_{i=1}^5 z_i^2\right)^2. \tag{C.1}$$

This polynomial is unchanged when all five z_i change sign. Thus every nonzero eigenvalue product in (A.5) is positive, verifying (A.6). More explicitly, (C.1) is the orthogonal Fock projection of the full determinant onto these six coordinates. Its squared norm gives

$$\Gamma_4(B) = \frac{24 + 80 + 120 + 160}{256} = \frac{3}{2}, \qquad \Phi_4(B) = \frac{1}{16}.$$

This computation is an independent control for the all-even-dimensional Breuer–Hall argument.

The map is positive and bistochastic. For $\|x\| = 1$,

$$B(xx^*) = \tfrac{1}{2} P_{\{x, J\bar{x}\}^\perp}, \qquad \operatorname{rank} B(xx^*) = 2,$$

since $x \perp J\bar{x}$. Here P_F denotes the orthogonal projection onto F.

The PPT witness in the proof of Theorem B.1 has pairing -4 with the Choi matrix of this normalized map, proving indecomposability.

The same map lies outside the Hilbert–Schmidt PSD class even after input/output Jordan symmetries. On its support, its polar partial isometry fixes I_4 and negates every s_i. A unitary conjugation with this action would negate their scalar product $-I_4$, a contradiction. A transpose-conjugation gives the same contradiction, because reversing five anticommuting generators has sign $(-1)^{10} = 1$. A putative Jordan factor in a PSD factorization would have to extend this polar partial isometry. Finally, its rank-two pure outputs and failure of strict positivity exclude both the projection-preserving class and its positive depolarizing regularizations considered here.

C.2. **A strict sector example outside three comparison classes.**

Lemma C.1 (Uniqueness of strict bistochastic scaling)**.** *Let $T : M_n \to M_n$ be strictly positive and bistochastic. If $\mathrm{Ad}_L \circ T \circ \mathrm{Ad}_R$ is also bistochastic, with L, R invertible, then $L = cU$ and $R = c^{-1}V$ for a positive scalar c and unitaries U, V.*

Proof. Strict positivity and compactness give $T(X) \succeq \eta \operatorname{tr}(X) I$ for $X \succeq 0$ and some $\eta > 0$. Duality gives the same bound for T^*, so T^* is strict as well. Put $A = L^*L$ and $B = RR^*$. The two normalizations are exactly

$$T(B) = A^{-1}, \qquad T^*(A) = B^{-1}.$$

If B is nonscalar, strict positivity applied to $B - \lambda_{\min}(B) I$ and $\lambda_{\max}(B) I - B$ gives

$$\lambda_{\min}(B) I \prec T(B) \prec \lambda_{\max}(B) I, \qquad \operatorname{cond}(A) < \operatorname{cond}(B).$$

The matrix A must then also be nonscalar: otherwise unitality of T^* in the second equation would make B scalar. Applying the same argument to T^* gives the contradictory strict inequality $\operatorname{cond}(B) < \operatorname{cond}(A)$. Thus $B = bI$ and $A = b^{-1} I$, which proves the assertion. □

Corollary C.2 (Filtered PSD maps reduce to Jordan orbits after balancing)**.** *A strictly positive bistochastic map that is an invertible input/output congruence of an ordinary positive Hilbert–Schmidt PSD map is input/output unitarily equivalent to a bistochastic Hilbert–Schmidt PSD map. Allowing input/output transposes gives the corresponding Jordan equivalence.*

Proof. Invertible congruences preserve strict positivity, so the original Hilbert–Schmidt PSD map Q is strict. By the strict balancing result in Lemma 2.6, choose positive definite A, B such that $T = \mathrm{Ad}_A \circ Q \circ \mathrm{Ad}_B$ is bistochastic. Since $Q = Q^*$,

$$T^* = \mathrm{Ad}_{BA^{-1}} \circ T \circ \mathrm{Ad}_{B^{-1}A}$$

is also bistochastic. Lemma C.1 implies $A^{-1} B^2 A^{-1} = cI$ for some $c > 0$. Uniqueness of the positive square root gives $B = \sqrt{c}\, A$, and hence

$$T = c\, \mathrm{Ad}_A \circ Q \circ \mathrm{Ad}_A \succeq 0$$

as a Hilbert–Schmidt operator. Comparing any other strictly positive bistochastic filter image with T and applying the lemma again reduces its filters to scalar unitaries, whose scalar factors cancel. Transposition commutes through congruences by conjugating their matrices, proving the last assertion. □

The following construction gives a strictly positive example separating the sector theorem itself from the comparison classes. Let $K : M_s \to M_s$ be any positive bistochastic indecomposable map;

below we take $s = 3$ and the normalized Choi map C of Proposition 3.6. Put $n = 2s + 3$. First define on M_{2s}

$$\mathcal{S}\begin{pmatrix} A & C \\ D & E \end{pmatrix} = \begin{pmatrix} K(E) & 0 \\ 0 & K^*(A) \end{pmatrix}, \qquad R = \frac{\mathrm{Id} + \mathcal{S}}{2}.$$

As in the proof of Proposition 4.7, both maps are positive and bistochastic, and $\mathcal{S}$ is a selfadjoint Hilbert–Schmidt contraction. Hence $R \succeq 0$ in Hilbert–Schmidt order. These properties use only positivity and bistochasticity of K.

The map R sends no nonzero rank-one input to rank one. Otherwise, for $P = vv^*$, the positive summands P and $\mathcal{S}(P)$ would share the line of v. The second is nonzero and block diagonal, forcing v into one block. It would then be supported in the opposite block from v, a contradiction.

Let P_3 be a three-cycle permutation matrix, and put

$$\epsilon = \frac{1}{n+2}, \qquad M = (1-\epsilon)I_3 + \epsilon P_3, \qquad T_M(A) = \mathrm{diag}(M \,\mathrm{diag}\, A).$$

The map T_M is a CP bistochastic diagonal channel. With $\boxplus$ denoting a pinched direct sum, set

$$H_0 = R \boxplus T_M, \qquad H = \frac{\mathrm{Id}_n + H_0}{2}. \tag{C.2}$$

The resulting map is positive and bistochastic. On the first M_{2s} block, it acts as the Hilbert–Schmidt PSD operator $(\mathrm{Id} + R)/2$. On off-diagonal blocks and on off-diagonal entries in the final M_3 block, it acts as $\mathrm{Id}/2$. Its sole nonselfadjoint block is the classical diagonal subspace, where its matrix is

$$B_0 = (I_3 + M)/2 = (1-a)I_3 + aP_3, \qquad a = \frac{1}{2(n+2)}.$$

The eigenvalues of that normal matrix are

$$1, \quad u \pm iv, \qquad u = \frac{4n+5}{4(n+2)}, \quad v = \frac{\sqrt{3}}{4(n+2)}.$$

Consequently $S \succeq I/2$ on the full Hilbert–Schmidt space, and

$$\frac{|v|}{u} = \frac{\sqrt{3}}{4n+5} < \frac{1}{n} < \tan\frac{\pi}{2n}.$$

This bound verifies (A.1) on every block, so Theorem A.1 gives $\Phi_n(H) \geq n^{-n}$.

Embedding an input in the lower $s \times s$ block within M_{2s} and compressing the output to its upper $s \times s$ block recovers $K/4$. Since decomposability survives CP embeddings and compressions, H is indecomposable.

The Hilbert–Schmidt operator H is invertible. Its polar unitary equals the identity outside the classical diagonal subspace and the polar unitary of B_0 on that subspace. Put

$$\rho = \sqrt{1 - 3a + 3a^2} \in (0, 1).$$

Dividing the two nonconstant Fourier eigenvalues of B_0 by ρ shows that its polar unitary is a real circulant matrix

$$U_{B_0} = \alpha I_3 + \beta P_3 + \gamma P_3^2, \qquad \gamma = \frac{1 - 1/\rho}{3} < 0.$$

It sends a coordinate PSD projection to a diagonal matrix with a negative entry, and so is not a positive Jordan automorphism. If independent Hilbert–Schmidt-unitary Jordan automorphisms J_1, J_2 transformed H to a PSD operator Q, then

$$H = J_1^* Q J_2^* = (J_1^* J_2^*)(J_2 Q J_2^*)$$

would be its polar decomposition with a positive Jordan unitary factor. Uniqueness gives a contradiction. Hence H lies outside the Hilbert–Schmidt PSD class and its stated Jordan orbits.

Finally, T_M sends every nonzero rank-one input to rank at least two: the support of $M(|v_i|^2)_i$ has at least two elements, since $0 < \epsilon < 1$ and P_3 has no fixed point. Together with the rank property of

R, this support bound implies that neither H_0 nor H sends a nonzero rank-one input to rank one. On the other hand, H is not strictly positive: a rank-one input supported in the first $2s$ coordinates has output supported there. Positive depolarizing regularizations of projection-preserving maps are strict, whereas their unregularized versions preserve designated rank-one outputs. Thus neither possibility describes H.

Taking $s = 3$ and $K = C$ gives an example on M_9, with $\epsilon = 1/11$, $a = 1/22$, and

$$\frac{|v|}{u} = \frac{\sqrt{3}}{41}, \qquad \gamma = \frac{1 - 22/\sqrt{421}}{3} < 0.$$

Now make it strictly positive by setting

$$H_\delta = (1-\delta)H + \delta\Delta_9, \qquad \delta = \frac{1}{10}. \tag{C.3}$$

The sector inequalities are convex and Δ_9 is Hilbert–Schmidt PSD, so H_δ remains in the certified sector. It is bistochastic, satisfies $H_\delta(X) \succeq \operatorname{tr}(X) I_9/90$ for $X \succeq 0$, and obeys

$$\Phi_9(H_\delta) \geq 9^{-9}.$$

The witness in Proposition 3.6 has trace $21/2$ and pairs with C_C to give $-3/4$. Embed it in the lower-input, upper-output three-dimensional corner used above. That corner of H is $C/4$, so its pairing with C_{H_δ} is exactly

$$-\frac{3(1-\delta)}{16} + \frac{7\delta}{6} = \frac{-9 + 65\delta}{48}.$$

It is negative for $\delta < 9/65$ and equals $-5/96$ at $\delta = 1/10$. Thus the strictly positive map in (C.3) is indecomposable.

On the reducing decomposition $M_9 = \mathbb{C}I \oplus \{X : \operatorname{tr} X = 0\}$, write $H = 1 \oplus H_{\mathrm{tr}}$. Then

$$H_\delta = 1 \oplus (1-\delta)H_{\mathrm{tr}}.$$

Since H is invertible and $1 - \delta > 0$, the polar unitary is unchanged; in particular, it retains the negative coefficient γ. The earlier uniqueness-of-polar-decomposition argument excludes the Hilbert–Schmidt PSD class and its stated Jordan orbits for H_δ as well. Corollary C.2 strengthens this exclusion to all invertible input/output congruences of ordinary positive Hilbert–Schmidt PSD maps, including transposes.

To exclude all depolarized projection-preserving maps, define the minimum over unit product vectors by

$$\mu(L) = \min_{\|x\|=\|y\|=1} y^* L(xx^*)y.$$

The unregularized H has $\mu(H) = 0$: use an input supported in the first six coordinates and an output test vector in the last three. Hence $\mu(H_\delta) = \delta/9$ exactly. If T is positive and sends one orthogonal rank-one resolution to another, then $\mu(T) = 0$, attained by testing a designated rank-one output in a perpendicular direction. Consequently

$$\mu\big((1-\beta)T + \beta\Delta_9\big) = \beta/9 \qquad (0 \leq \beta \leq 1).$$

A representation of H_δ in this form would force $\beta = \delta$ and then $T = H$, contradicting the fact that H sends no nonzero rank-one input to rank one. The same argument covers the input/output unitary and transposition symmetries, which preserve the depolarizing map and these product-vector minima.

This argument also excludes arbitrary invertible filters of the earlier projection-preserving families. An unregularized projection-preserving map is not strict, and invertible filters cannot change that property. A positively depolarized member is already strictly positive and bistochastic. Lemma C.1 reduces any bistochastic filter image of it to unitary input/output changes, which were just excluded. Positive scalar multiples can be absorbed into the filters, and transposes commute through them.

Thus (C.3) is a fully specified strictly positive bistochastic map on M_9 outside the union of all three comparison classes even after their invertible input/output filtering and Jordan symmetries. Only the intermediate H_0 is a pinched direct sum; H retains off-diagonal identity terms, and its sign and that of H_δ are certified here by the sector theorem.

Appendix D. Ferrers tests for Schur coefficients

We record explicit completely positive curvature tensors that extract individual Schur coefficients. This construction complements the Schur–Griffiths cone identification (Fulton and Lazarsfeld, 1983, Appendix A, Proposition A.3) and Li's sum-of-squares formula (Li, 2021, Propositions 3.1 and 8.2). It uses the Frobenius character formula and exterior Cauchy decomposition; see Tamvakis (2004, Sections 2–5).

Fix $d, r, m \geq 1$, let $U = \mathbb{C}^r$ and $W = \mathbb{C}^m$, and identify $U \otimes W$ with $M_{r\times m}$ with its standard Hermitian product. For $A_1, \dots, A_d \in M_{r\times m}$, put

$$\Omega = i \sum_{j,k=1}^{d} A_j A_k^* \, dz_j \wedge d\bar{z}_k, \qquad dV_d = \bigwedge_{j=1}^{d} (i \, dz_j \wedge d\bar{z}_j). \tag{D.1}$$

For homogeneous symmetric polynomials p of degree d, define $L_\Omega(p)$ by $p(\Omega) = L_\Omega(p) dV_d$. The map $H(E_{jk}) = A_j A_k^*$ is CP, with Kraus matrices $K_s e_j = A_j e_s$; conversely, every CP map $M_d \to M_r$ has this form for some m.

For $\mu \vdash d$, let χ_μ be the irreducible character of S_d of dimension f^μ. On $(U \otimes W)^{\otimes d}$, let T_π^U permute only the U factors and set

$$\Pi_\mu^U = \frac{f^\mu}{d!} \sum_{\pi \in S_d} \chi_\mu(\pi^{-1}) T_\pi^U,$$

$$w = A_1 \wedge \cdots \wedge A_d = \frac{1}{\sqrt{d!}} \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(d)}.$$

Thus Π_μ^U is the orthogonal projection onto the row μ-isotypic component, and the exterior embedding is isometric.

Lemma D.1 (Schur-form projector identity)**.** *For the curvature tensor in* (D.1)*,*

$$L_\Omega(s_\mu) = \frac{d!}{f^\mu} \langle w, \Pi_\mu^U w \rangle = \frac{d!}{f^\mu} \|\Pi_\mu^U w\|^2. \tag{D.2}$$

In particular, every Schur form is weakly nonnegative for CP curvature.

Proof. For $\pi \in S_d$, let p_π be the product of power sums indexed by its cycle lengths. Expanding the traces in $p_\pi(\Omega)$ and alternating the holomorphic and antiholomorphic indices gives

$$L_\Omega(p_\pi) = d! \langle w, T_{\pi^{-1}}^U w \rangle.$$

Indeed, the two alternations contribute $\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$, and the normalization of w contributes $1/d!$; the factors i^d cancel against dV_d. Since the even-degree entries of Ω commute, the Frobenius identity gives

$$s_\mu = \frac{1}{d!} \sum_{\pi \in S_d} \chi_\mu(\pi) p_\pi,$$

$$L_\Omega(s_\mu) = \sum_{\pi \in S_d} \chi_\mu(\pi) \langle w, T_{\pi^{-1}}^U w \rangle = \frac{d!}{f^\mu} \langle w, \Pi_\mu^U w \rangle.$$

The norm identity follows because Π_μ^U is an orthogonal projection. It preserves the exterior power because it commutes with simultaneous row and column permutations. □

Theorem D.2 (Ferrers coefficient tests)**.** *Let $\lambda \vdash d$ have at most r rows and at most m columns. Index the tangent coordinates by*

$$D_\lambda = \{(a,s) : 1 \le a \le r,\ 1 \le s \le \lambda_a\},$$

and choose $A_{(a,s)} = E_{as} \in M_{r\times m}$ in (D.1). The resulting CP curvature tensor Ω_λ satisfies

$$L_{\Omega_\lambda}(s_\mu) = \delta_{\lambda\mu}\frac{d!}{f^\lambda} \qquad (\mu \vdash d). \tag{D.3}$$

Consequently, for $p = \sum_{\mu\vdash d,\,\ell(\mu)\le r} c_\mu s_\mu$,

$$L_{\Omega_\lambda}(p) = \frac{d!}{f^\lambda}c_\lambda. \tag{D.4}$$

Proof. The wedge w_λ of the matrix units indexed by D_λ has norm one and row and column weights (λ,λ'). Every row or column raising operator annihilates it: moving a cell upward or leftward produces a repeated exterior factor. Hence w_λ is a highest weight vector in the λ summand of the exterior Cauchy decomposition

$$\bigwedge^d (U\otimes W) \simeq \bigoplus_{\substack{\nu\vdash d\\ \ell(\nu)\le r,\ \nu_1\le m}} S_\nu U \otimes S_{\nu'} W.$$

Thus $\Pi^U_\mu w_\lambda = \delta_{\mu\lambda} w_\lambda$; Lemma D.1 and linearity give the two identities. □

The factor $d!/f^\lambda$ is the product of the hook lengths of λ. The corresponding map is

$$H_\lambda : M_d \longrightarrow M_r, \qquad H_\lambda(X)_{ab} = \sum_{s\le \min(\lambda_a,\lambda_b)} x_{(a,s),(b,s)}.$$

Corollary D.3 (Exact separation of the Schur cone)**.** *A homogeneous symmetric polynomial with real coefficients in r variables is Schur-positive if and only if its Chern–Weil form is weakly nonnegative on every CP curvature tensor. It suffices to test the Ferrers tensors of its degree with $m = d$. The universal CP quantifier may also be restricted to maps with positive-definite Choi matrices.*

Proof. Lemma D.1 proves sufficiency of Schur positivity; (D.4) detects every negative coefficient. Adding $\varepsilon\,\mathrm{tr}(\,\cdot\,)I_r$ to a detecting map makes its Choi matrix positive definite, while polynomial continuity preserves the negative evaluation for sufficiently small $\varepsilon > 0$. □

AI USAGE

The results in this paper were developed through extensive use of an AI assistant GPT-6 Astra (Pro). The choice of the questions worth answering was fully made by the author, as was the selection of the subset of results worth retaining. This choice of material evolved over many months, including up to the recent disproof of $\Phi_9 \ge 0$. None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs. The writing, polishing, and presentation were done with GPT-6 as well as Claude's (Fable 5.1 and Opus 5.5) assistance.

ACKNOWLEDGEMENTS

The author (SS) thanks the Alexander von Humboldt (AvH) Foundation for their generous support via a Humboldt Professorship in AI.

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