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A KOTELJANSKII INEQUALITY FOR PERMANENTS

SUVRIT SRA*

Abstract. We prove a permanental analogue of Koteljanskii's inequality. If A is an inverse M-matrix that becomes symmetric after a positive diagonal similarity, then $\operatorname{per}(A_{S\cup T})\operatorname{per}(A_{S\cap T}) \geq \operatorname{per}(A_S)\operatorname{per}(A_T)$ for all $S,T \subseteq [n]$, where A_S is the principal submatrix indexed by S. The proof expresses permanents as moments of a complex Gaussian vector and uses Ginibre's correlation inequality. Finally, an explicit counterexample shows that symmetry cannot be dropped.

1. Introduction

For an $n \times n$ matrix A and $S \subseteq [n]$, write $A_S = A[S,S]$ for the principal submatrix with row and column set S, and set $\det(A_\varnothing) = \operatorname{per}(A_\varnothing) = 1$. Koteljanskii's inequality (Koteljanskii, 1950, 1953)

$$\det(A_{S\cup T})\det(A_{S\cap T}) \leq \det(A_S)\det(A_T), \qquad S,T \subseteq [n],$$

holds for positive semidefinite matrices, and also for M-matrices and inverse M-matrices (Fallat et al., 1998). Its case $S \cap T = \varnothing$ is Fischer's inequality, and iterating Fischer's inequality gives Hadamard's inequality $\det(A) \leq \prod_i a_{ii}$. For positive semidefinite matrices the permanent reverses Hadamard's inequality, $\operatorname{per}(A) \geq \prod_i a_{ii}$ (Marcus, 1963), and Lieb's permanental inequality (Lieb, 1966) reverses Fischer's: $\operatorname{per}(A_{S\cup T}) \geq \operatorname{per}(A_S)\operatorname{per}(A_T)$ for disjoint S,T. These two reversals suggest the permanental analogue of Koteljanskii's inequality,

$$\operatorname{per}(A_{S\cup T})\operatorname{per}(A_{S\cap T}) \geq \operatorname{per}(A_S)\operatorname{per}(A_T), \qquad S,T \subseteq [n], \tag{1.1}$$

where as usual the permanent of an $n \times n$ matrix is defined as

$$\operatorname{per}(A) = \sum_{\pi \in \mathrm{Sym}_n} \prod\nolimits_{i\in[n]} A_{i,\pi(i)}. \tag{1.2}$$

Which matrices satisfy (1.1)? Contrary to the determinant, here positive semidefiniteness does not suffice, even with nonnegative entries: the positive semidefinite matrix A with rows $(1,1,0)$, $(1,2,1)$, $(0,1,1)$ has $\operatorname{per}(A)\operatorname{per}(A_{\{2\}}) = 8 < 9 = \operatorname{per}(A_{\{1,2\}})\operatorname{per}(A_{\{2,3\}})$. This obstruction led us to consider inverse M-matrices instead, as they also satisfy Koteljanskii's inequality. Numerous initial experiments in early 2015 convinced us that for inverse M-matrices the inequality (1.1) could hold.

A matrix A is an inverse M-matrix if $A \geq 0$ entrywise, A is invertible, and A^{-1} has nonpositive off-diagonal entries. We prove (1.1) for inverse M-matrices that are symmetric after a positive diagonal similarity, which we call *symmetrizable*, and show that (1.1) can fail without this hypothesis.

Theorem 1.1 (Koteljanskii for per)**.** *Let A be a symmetrizable inverse M-matrix. Then the set function $S \mapsto \operatorname{per}(A_S)$ is log-supermodular, i.e., inequality* (1.1) *holds for all $S,T \subseteq [n]$.*

Theorem 1.2 (Counterexample)**.** *The matrix*

$$A = \begin{pmatrix} 13 & 1 & 7 & 7 & 7 \\ 1 & 13 & 7 & 7 & 7 \\ 1 & 1 & 7 & 7 & 7 \\ 1 & 1 & 1 & 7 & 7 \\ 1 & 1 & 1 & 1 & 7 \end{pmatrix}$$

* TU Munich, Dept. of Mathematics, School of CIT; Garching, Germany. Email: s.sra@tum.de.
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is a non-symmetrizable inverse M-matrix, and $\operatorname{per}(A)\operatorname{per}(A_{\{3,4,5\}}) < \operatorname{per}(A_{\{1,3,4,5\}})\operatorname{per}(A_{\{2,3,4,5\}})$.

For the proof of Theorem 1.1 it suffices to consider symmetric (and hence, also positive definite) inverse M-matrices since diagonal similarity does not change principal permanents. Our starting point is the classical representation of $\operatorname{per}(A_S)$ as the moment $\mathbb{E}\prod_{i\in S}|\zeta_i|^2$ of a complex Gaussian vector. Then, Ginibre's inequality (Ginibre, 1970) helps us show that the density of $X = (|\zeta_1|^2, \ldots, |\zeta_n|^2)$ is MTP_2 (multivariate totally positive of order two)—see Proposition 2.2. This property provides a key building block for a tilting based argument that closes the proof. Our results leave open which non-symmetrizable inverse M-matrices satisfy (1.1); Section 4 lists this and other questions.

1.1. **Related work.** Johnson (1982) surveys inverse M-matrices, with a sequel by Johnson and Smith (2011); for permanental inequalities on totally positive matrices see Skandera and Soskin (2025). The MTP property and the FKG inequality for densities are due to Karlin and Rinott (1980). Eisenbaum (2014) characterized positively correlated squared Gaussian vectors and extended the characterization to the permanental vectors of Section 3.2; see the erratum (Eisenbaum, 2022) and Finner and Roters (2024), who show that positively associated squared Gaussian vectors need not be MTP_2 and extend this to multivariate χ-square distributions. Positive association alone does not give (1.1), which needs association after the tilt by $\prod_{i\in S\cap T} X_i$ (Remark 2.5); the example we provide therein also contradicts Corollary 11 of the technical report of Lovász and Saks (2003).

1.2. **Notation and definitions.** Throughout, we use the shorthand $[n] = \{1, 2, \ldots, n\}$. The central class of matrices for this paper is that of inverse M matrices; we define these below.

Definition 1.3 (M-matrices)**.** A square matrix $M = (m_{ij})$ is a nonsingular *M-matrix* if: (i) $m_{ij} \le 0$ for all $i \neq j$; and (ii) M is invertible with $M^{-1} \ge 0$ entrywise. A matrix A is an *inverse M-matrix* if $A = M^{-1}$ for some nonsingular M-matrix M.

Let us also formally define log-supermodularity.

Definition 1.4 (Log-supermodularity)**.** A set function $f : 2^{[n]} \to \mathbb{R}_{\ge 0}$ is *log-supermodular* if

$$f(S\cup T)f(S\cap T) \ge f(S)f(T), \qquad S, T \subseteq [n].$$

We are now ready to prove our main result, Theorem 1.1.

2. Proof of Theorem 1.1

The proof has four steps: Step 1 reduces to symmetric A; Step 2 writes $\operatorname{per}(A_S)$ as a squarefree moment of $X = (|\zeta_1|^2, \ldots, |\zeta_n|^2)$, where ζ is a complex Gaussian vector with covariance A; Step 3 shows that X has an MTP_2 density, and once the phases are integrated out, this reduces to a correlation inequality of Ginibre; Step 4 finishes by observing that squarefree moments of an MTP_2 density are log-supermodular.

2.1. **Step 1, symmetry.** Let Δ be a positive diagonal matrix such that $A' = \Delta A \Delta^{-1}$ is symmetric. This is again an inverse M-matrix: $A' \ge 0$, and $(A')^{-1} = \Delta M \Delta^{-1}$ has the sign pattern of $M = A^{-1}$. Moreover, for every $S \subseteq [n]$ and every permutation π of S, $\prod_{i\in S} A'_{i,\pi(i)} = \prod_{i\in S} \frac{\Delta_{ii}}{\Delta_{\pi(i),\pi(i)}} A_{i,\pi(i)} = \prod_{i\in S} A_{i,\pi(i)}$. Thus, $\operatorname{per}(A'_S) = \operatorname{per}(A_S)$; so henceforth we may assume that A is symmetric.

2.2. **Step 2, Gaussian moments.** Now that A is symmetric, we also need it to be positive definite so that we can use it as the covariance matrix of a complex Gaussian vector. Fortunately, a symmetric inverse M-matrix is automatically positive definite. Indeed, put $x = A\mathbf{1}$; then $x > 0$, since no row of the invertible nonnegative matrix A vanishes. Since $Mx = \mathbf{1}$, the symmetric matrix $\operatorname{diag}(x) M \operatorname{diag}(x)$ has nonpositive off-diagonal entries and i-th row sum $x_i > 0$. It is therefore strictly diagonally dominant with positive diagonal, hence positive definite, and so are M and A. Let $\zeta \in \mathbb{C}^n$ be a complex Gaussian vector with covariance A and density $\pi^{-n}\det M\, e^{-z^* M z}$.

Let $X = (|\zeta_1|^2, \dots, |\zeta_n|^2)$. By the moment theorem for complex Gaussian vectors (Reed, 1962), $\mathbb{E}\prod_{i\in S} X_i = \mathbb{E}\prod_{i\in S}\zeta_i\overline{\zeta_i} = \operatorname{per}(A_S)$; in particular, all squarefree moments of X are finite.

2.3. **Step 3, MTP$_2$ density.** Next, we wish to show that the density of X is MTP$_2$. Recall that for a positive density p on $(0,\infty)^n$, the MTP$_2$ property is

$$p(x\vee y)p(x\wedge y) \geq p(x)p(y). \tag{2.1}$$

For C^2 densities (2.1) is equivalent to non-negative Hessian off-diagonals, i.e., $\partial_{ij}\log p \geq 0$ for $i\neq j$.

We prove this non-negativity for the density of X. To that end, first write $M = D - C$, where $D = \operatorname{diag}(d_1,\dots,d_n)$ with $d_i = M_{ii} > 0$, and $C \geq 0$ is symmetric with zero diagonal (Berman and Plemmons, 1994, Chapter 6). Let $\mathcal{E}$ be the pairs $\{i,j\}$ with $C_{ij} > 0$; using polar coordinates $z_i = \sqrt{x_i}e^{\mathbb{i}\theta_i}$, we observe that

$$z^*Mz = \sum\nolimits_i d_i x_i - \sum\nolimits_{\{i,j\}\in\mathcal{E}} 2C_{ij}\sqrt{x_i x_j}\cos(\theta_i-\theta_j).$$

Now integrate out θ to obtain the density

$$p(x) = \det M \exp\Big(-\sum\nolimits_i d_i x_i\Big)\,\mathcal{Z}(J(x)), \qquad J_{\{i,j\}}(x) = 2C_{ij}\sqrt{x_i x_j}, \tag{2.2}$$

where $\mathcal{Z}$ is the partition function on $\mathcal{E}$ defined as follows: For $e=\{i,j\}\in\mathcal{E}$ set $\omega_e(\theta) = \cos(\theta_i-\theta_j)$, where $\theta\in\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. For nonnegative couplings J_e, define

$$\mathcal{Z}(J) := \int_{\mathbb{T}^n} \exp\Big(\sum\nolimits_{e\in\mathcal{E}} J_e\omega_e(\theta)\Big)\,d\theta, \qquad d\mu_J(\theta) = \mathcal{Z}(J)^{-1}\exp\Big(\sum\nolimits_{e\in\mathcal{E}} J_e\omega_e(\theta)\Big)\,d\theta,$$

with normalized Haar measure $d\theta$. Write $\mathbb{E}_J$ and Cov_J for expectation and covariance under μ_J; differentiation under the integral then gives

$$\frac{\partial\log\mathcal{Z}}{\partial J_e} = \mathbb{E}_J[\omega_e], \qquad \frac{\partial^2\log\mathcal{Z}}{\partial J_e\partial J_f} = \operatorname{Cov}_J(\omega_e,\omega_f).$$

The sign of $\partial_{ij}\log p$ is therefore controlled by the angular means $\mathbb{E}_J[\omega_e]$ and covariances $\operatorname{Cov}_J(\omega_e,\omega_f)$.

Inequality (2.3) below is a case of Ginibre's inequality (Ginibre, 1970) for edge observables. We include its proof to make explicit which correlation property is used.

Lemma 2.1. *For every $e,f\in\mathcal{E}$, $\mathbb{E}_J[\omega_e]\geq 0$, and $\operatorname{Cov}_J(\omega_e,\omega_f)\geq 0$.*

Proof. The absolutely convergent Fourier expansion $e^{J\cos u} = \sum_{p,q\geq 0}\frac{(J/2)^{p+q}}{p!q!}e^{\mathbb{i}(p-q)u}$ has nonnegative coefficients. The product over edges has the same property, so its integral against $\cos(\theta_i-\theta_j)$ is nonnegative. Division by $\mathcal{Z}(J)>0$ proves the first inequality.

For the covariance, introduce two independent angular variables θ,ϕ. With $H(\theta) = \sum_g J_g\omega_g(\theta)$,

$$2\mathcal{Z}(J)^2\operatorname{Cov}_J(\omega_e,\omega_f) = \int(\omega_e(\theta)-\omega_e(\phi))(\omega_f(\theta)-\omega_f(\phi))e^{H(\theta)+H(\phi)}\,d\theta\,d\phi.$$

The map $(u,v)\mapsto(u+v,u-v)$ is a surjective homomorphism of $\mathbb{T}^{2n}$, and therefore pushes normalized Haar measure to normalized Haar measure. Orient each edge arbitrarily and write $u_e = u_i - u_j$ and $v_e = v_i - v_j$. Under the substitution $\theta = u+v$, $\phi = u-v$,

$$\omega_e(\theta)-\omega_e(\phi) = -2\sin u_e\sin v_e, \qquad H(\theta)+H(\phi) = 2\sum_g J_g\cos u_g\cos v_g.$$

Expanding the exponential gives

$$\operatorname{Cov}_J(\omega_e,\omega_f) = \frac{2}{\mathcal{Z}(J)^2}\sum_{m\in\mathbb{N}_0^{\mathcal{E}}}\left(\prod_g\frac{(2J_g)^{m_g}}{m_g!}\right)\left[\int_{\mathbb{T}^n}\sin u_e\sin u_f\prod_g(\cos u_g)^{m_g}\,du\right]^2 \geq 0, \tag{2.3}$$

which completes the proof. □

We are now ready to formally state the desired MTP$_2$ property.

Proposition 2.2. *Let A be a symmetric inverse M-matrix, and let $\zeta \in \mathbb{C}^n$ be a complex Gaussian vector with covariance A. Then $X = (|\zeta_1|^2, \ldots, |\zeta_n|^2)$ has an MTP$_2$ density on $(0,\infty)^n$.*

Proof. By (2.2) and the derivatives of $\log \mathcal{Z}$ above, for $i \neq j$,

$$\partial_i \partial_j \log p(x) = \sum\nolimits_e \mathbb{E}_J[\omega_e]\, \partial_i \partial_j J_e(x) + \sum\nolimits_{e,f} \mathrm{Cov}_J(\omega_e, \omega_f)\, \partial_i J_e(x)\, \partial_j J_f(x) \geq 0.$$

Indeed, all first derivatives of the couplings are nonnegative, and their mixed derivatives at distinct coordinates vanish except for $e = \{i, j\}$, where

$$\partial_{ij} J_e(x) = \frac{J_e(x)}{4 x_i x_j} \geq 0.$$

Lemma 2.1 supplies the remaining nonnegative factors. □

2.4. **Step 4, squarefree moments.** The MTP$_2$ property proves the Koteljanskii inequality (1.1) after a change of measure, as shown below.

Lemma 2.3. *Suppose that a random vector $X \in (0,\infty)^n$ has an MTP$_2$ density and finite squarefree moments. Then, the set function $f(S) = \mathbb{E} \prod_{i \in S} X_i$ is log-supermodular.*

Proof. Fix S, T and put $R = S \cap T$. The density $p_R(x) = f(R)^{-1} \prod_{i \in R} x_i p(x)$ is MTP$_2$, since the multiplying factor is a product of functions of individual coordinates. By the FKG inequality for MTP$_2$ densities (Karlin and Rinott, 1980), increasing functions are positively correlated under this density. Apply this rule to the increasing functions $u(x) = \prod_{i \in S \setminus R} x_i$ and $v(x) = \prod_{i \in T \setminus R} x_i$, first truncated at a level N and then with $N \to \infty$ by monotone convergence; the assumed moments make all terms finite. It yields

$$\frac{f(S \cup T)}{f(R)} \geq \frac{f(S)}{f(R)} \frac{f(T)}{f(R)},$$

which proves the assertion. □

Now we have all the steps, and can now state the final proof.

Proof of Theorem 1.1. By Step 1 we may assume that A is symmetric. By Step 3's Proposition 2.2, the vector X has an MTP$_2$ density, while by Step 2 its squarefree moments are $\mathrm{per}(A_S)$. Thus, using Step 4's Lemma 2.3 we may immediately conclude (1.1). □

Remark 2.4 (Repeated indices)**.** *The same proof gives log-supermodularity of all mixed moments of X on $\mathbb{N}_0^n$: tilt by $x^{a \wedge b}$ and apply the FKG inequality to $x^{a - a \wedge b}$ and $x^{b - a \wedge b}$. By the same moment theorem (Reed, 1962), these moments are the permanents of the matrices obtained from A by repeating row and column i the same number of times.*

Remark 2.5 (Positive association is not enough)**.** *The tilt in the proof of Lemma 2.3 cannot be skipped: positive association of a random vector does not make its squarefree moments log-supermodular. For a random vector $W \in \mathbb{R}^n_{\geq 0}$ write $\Phi_W(S) = \mathbb{E} \prod_{i \in S} W_i$. For independent $N \sim \mathrm{Poisson}(\lambda)$ and $N' \sim \mathrm{Poisson}(\mu)$, a constant $a \geq 0$, and $W = (a + N, a + N + N', a + N')$, a direct moment calculation gives*

$$\Phi_W(\{1,2,3\})\Phi_W(\{2\}) - \Phi_W(\{1,2\})\Phi_W(\{2,3\}) = -\lambda\mu.$$

The vector W is positively associated, since its coordinates are nondecreasing functions of the independent variables N and N' (Esary et al., 1967).

The same example, with $a = 0$ and $\lambda = \mu = 1$, shows that disjoint convolution does not preserve log-supermodularity. Here $\Phi_W(\{1,2,3\})\Phi_W(\{2\}) = 4 \cdot 2 < 3 \cdot 3 = \Phi_W(\{1,2\})\Phi_W(\{2,3\})$. Since W is the sum of the independent vectors $(N, N, 0)$ and $(0, N', N')$, expanding $\prod_{i \in S} W_i$ over the two summands shows that Φ_W is the disjoint convolution

$$(f \dot{*} g)(S) := \sum_{R \subseteq S} f(R) g(S \setminus R)$$

of their moment functions f and g. Both are log-supermodular: f vanishes outside $2^{\{1,2\}}$, where the only nontrivial instance is $f(\{1,2\})f(\varnothing) = 2 \geq 1 = f(\{1\})f(\{2\})$, and similarly for g on $2^{\{2,3\}}$. The example therefore contradicts Corollary 11 of the technical report of Lovász and Saks (2003), which asserts that the disjoint convolution of two log-supermodular functions is log-supermodular.

3. Counterexamples

This section proves Theorem 1.2; it also briefly considers the more general setup of α-permanents, where a similar negative result holds (Theorem A.1). Both results come from inverse M-matrices whose inverses have the directed graph of two cycles that share a path; the matrix of Theorem 1.2 belongs, after a column scaling, to the family used for Theorem A.1 (Remark A.2).

3.1. Proof of Theorem 1.2.

Proof of Theorem 1.2. Direct multiplication gives

$$A^{-1} = \frac{1}{84}\begin{pmatrix} 7 & 0 & -7 & 0 & 0 \\ 0 & 7 & -7 & 0 & 0 \\ 0 & 0 & 14 & -14 & 0 \\ 0 & 0 & 0 & 14 & -14 \\ -1 & -1 & 0 & 0 & 14 \end{pmatrix}.$$

This matrix has nonpositive off-diagonal entries and inverse $A > 0$. The matrix A is not symmetrizable: if $\Delta A \Delta^{-1}$ were symmetric, so would be its inverse $\Delta A^{-1}\Delta^{-1}$, but a diagonal similarity preserves the zero pattern of A^{-1}, and $(A^{-1})_{13} \neq 0 = (A^{-1})_{31}$. Take $S = \{1,3,4,5\}$, $T = \{2,3,4,5\}$, and $R = \{3,4,5\}$. The four permanents are

$$\operatorname{per}(A) = 182784, \qquad \operatorname{per}(A_R) = 546, \qquad \operatorname{per}(A_S) = \operatorname{per}(A_T) = 9996.$$

Consequently,

$$\operatorname{per}(A)\operatorname{per}(A_R) = 99800064 < 99920016 = \operatorname{per}(A_S)\operatorname{per}(A_T),$$

and their ratio is $832/833$. □

3.2. α-permanents.

While developing Theorems 1.1 and 1.2 we wondered whether similar results might hold for the α-permanents of Vere-Jones (1997). The situation here turns out to be more nuanced, as noted in Remark 3.1 below.

For $\alpha > 0$, the α-permanent of a matrix H indexed by S is

$$\operatorname{per}_\alpha(H) = \sum_{\pi \in \mathfrak{S}(S)} \alpha^{c(\pi)} \prod_{i \in S} H_{i,\pi(i)}, \tag{3.1}$$

where $\mathfrak{S}(S)$ is the group of permutations of S and $c(\pi)$ is the number of cycles of π; the ordinary permanent is $\alpha = 1$. A random vector $Y \in [0,\infty)^n$ with Laplace transform $\det(I + A\Lambda)^{-\alpha}$, where $\Lambda = \operatorname{diag}(s_1, \dots, s_n)$, is an α-permanental vector with kernel A in the sense of Vere-Jones (1997), and its squarefree moments are the principal α-permanents,

$$\mathbb{E}\prod_{i \in S} Y_i = \operatorname{per}_\alpha(A_S), \qquad S \subseteq [n], \tag{3.2}$$

by Vere-Jones (1997, Proposition 4.2). Eisenbaum and Kaspi (2009) proved that an infinitely divisible permanental vector with kernel A exists if and only if A is, up to diagonal similarity, the Green function of a transient Markov chain, that is, an inverse M-matrix.

Remark 3.1 (Positive cases)**.** *For symmetric positive definite A there is a real Gaussian analogue. If $\eta \sim N(0, A)$, then $(\eta_i^2)_i$ is MTP_2 exactly when A^{-1} is, up to a signature, an M-matrix (Karlin and Rinott, 1981). The vector $(\eta_i^2/2)_i$ has Laplace transform $\det(I + A\Lambda)^{-1/2}$, so by (3.2) its squarefree moments are $\operatorname{per}_{1/2}(A_S)$, and $\mathbb{E}\prod_{i\in S}\eta_i^2 = 2^{|S|}\operatorname{per}_{1/2}(A_S)$. Lemma 2.3 and the diagonal similarity argument above therefore give Theorem 1.1 for $\operatorname{per}_{1/2}$, but not for the permanent itself. If A is a*

symmetric inverse M-matrix and the graph of A^{-1} is a tree, Royen (2025, Section 4) proves that the α-permanental vector with kernel A has an MTP_2 density for every $\alpha > 0$; formula (3.2) and Lemma 2.3 then give log-supermodularity of $S \mapsto \mathrm{per}_\alpha(A_S)$ for every $\alpha > 0$.

The Gaussian argument uses $\alpha = 1$ (complex) and $\alpha = \frac{1}{2}$ (real) in an essential way, and for other α the symmetrizable case is open when the graph of A^{-1} has cycles (Section 4). For general inverse M-matrices the answer is negative. Every inverse M-matrix is the kernel of an α-permanental vector for every $\alpha > 0$ (Eisenbaum and Kaspi, 2009), so one may ask whether some other order α gives the analogue of (1.1) for all inverse M-matrices. None does—see Appendix A for details.

4. Open problems

(a) Characterize the inverse M-matrices whose principal permanents are log-supermodular. By Theorem 1.1 this class contains the symmetrizable ones, and by Theorem 1.2 it is smaller than the class of all inverse M-matrices.

(b) For symmetrizable inverse M-matrices and $\alpha \notin \{\frac{1}{2}, 1\}$, decide whether $S \mapsto \mathrm{per}_\alpha(A_S)$ is log-supermodular. Theorem 1.1 and Remark 3.1 settle $\alpha = 1$, $\alpha = \frac{1}{2}$, and every $\alpha > 0$ when the graph of A^{-1} is a tree.

(c) The proof of Theorem A.1 needs the dimension to grow with α. For fixed n, is there an α_n such that $S \mapsto \mathrm{per}_\alpha(A_S)$ is log-supermodular for every $n \times n$ inverse M-matrix and every $\alpha \geq \alpha_n$?

(d) For an inverse M-matrix A, consider

$$f_A(x_0, x) = \sum\nolimits_{S \subseteq [n]} \mathrm{per}(A_S)^{-1} x_0^{n-|S|} \prod\nolimits_{i \in S} x_i.$$

For positive semidefinite A, the polynomial with coefficients $\det(A_S)$ in place of $1/\mathrm{per}(A_S)$ is $\det(x_0 I + \mathrm{diag}(x)A)$, which is real stable (Borcea and Brändén, 2008) and hence Lorentzian (Brändén and Huh, 2020). Is f_A Lorentzian for every inverse M-matrix A? Numerical tests with $n \leq 5$ found no counterexample, including the matrix of Theorem 1.2; for random nonnegative A, by contrast, f_A is usually not Lorentzian, and for symmetric inverse M-matrices the polynomial with coefficients $\mathrm{per}(A_S)$ was never Lorentzian.

AI Usage

The conjecture was formulated by the author in April 2015, with partial proofs. After a sequence of back and forth discussions with GPT-5.5 (Pro) the first proof was attained. Subsequently, an audit with GPT-6 (Astra) revealed a few small errors, that were fixed. AI assisted with the drafting of the paper (GPT-6, and Claude Opus 5.5 and Fable 5.1), while the author strove to rewrite the entire paper by hand to simplify, polish, and compactify.

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Appendix A. More on α-permanents

Theorem A.1. *For every $\alpha > 0$ there are an integer n, an entrywise positive $n \times n$ inverse M-matrix A, and sets $S, T \subseteq [n]$ such that*

$$\operatorname{per}_\alpha(A_{S\cup T}) \operatorname{per}_\alpha(A_{S\cap T}) < \operatorname{per}_\alpha(A_S) \operatorname{per}_\alpha(A_T).$$

Proof of Theorem A.1. Fix $k \geq 1$ and take the $n = k+2$ vertices $1, 2, r_1, \dots, r_k$. Let B have weight one on the edges $1 \to r_1$, $2 \to r_1$, and $r_j \to r_{j+1}$, and weight t on $r_k \to 1$ and $r_k \to 2$, with $0 < t < 1/2$. All other entries vanish. The only directed cycles are $1 \to r_1 \to \cdots \to r_k \to 1$ and $2 \to r_1 \to \cdots \to r_k \to 2$, and they share $r_1, \dots, r_k$, so the permutation expansion of the determinant gives, for $U = \operatorname{diag}(u)$,

$$\det(I - BU) = 1 - t(u_1 + u_2) \textstyle\prod_{j=1}^{k} u_{r_j}.$$

Taking $U = sI$ shows that the nonzero eigenvalues of B are the $(k+1)$st roots of $2t$, so $\rho(B) = (2t)^{1/(k+1)} < 1$. The graph of B is strongly connected, so $A = (I - B)^{-1} = \sum_{m \geq 0} B^m$ is entrywise positive, and it is an inverse M-matrix.

Let (N_1, N_2) have the negative multinomial law with probability generating function

$$\mathbb{E}[v^{N_1} w^{N_2}] = \left(\frac{1 - 2t}{1 - tv - tw} \right)^{\alpha},$$

put $L = N_1 + N_2$, and let $W = (N_1, N_2, L, \dots, L)$, with coordinates indexed by $1, 2, r_1, \dots, r_k$. Conditional on W, let $Y_1, \dots, Y_n$ be independent gamma variables with shape $\alpha + W_i$ and rate one, so that $\mathbb{E}[e^{-s_i Y_i} \mid W] = q_i^{\alpha + W_i}$ with $q_i = 1/(1 + s_i)$. For $s \in [0, \infty)^n$ put $Q = \operatorname{diag}(q_1, \dots, q_n)$ and $P = \prod_{j=1}^{k} q_{r_j}$. Since $q^W := \prod_i q_i^{W_i} = (q_1 P)^{N_1} (q_2 P)^{N_2}$,

$$\mathbb{E} e^{-\sum_i s_i Y_i} = \left(\textstyle\prod_i q_i \right)^{\alpha} \left(\frac{1 - 2t}{1 - t(q_1 + q_2)P} \right)^{\alpha} = \left(\frac{\det(I - B)}{\det(I + \Lambda) \det(I - BQ)} \right)^{\alpha},$$

by the formula for $\det(I - BU)$ with $U = Q$ and with $U = I$. Here $\Lambda = \operatorname{diag}(s)$, and $(I - BQ)(I + \Lambda) = I - B + \Lambda$, so the last denominator is $\det(I - B + \Lambda) = \det(I - B) \det(I + A\Lambda)$. Hence Y has Laplace transform $\det(I + A\Lambda)^{-\alpha}$. The variable L is negative binomial, so W and Y have finite moments of all orders, and (3.2), together with the conditional independence of the Y_i and $\mathbb{E}[Y_i \mid W] = \alpha + W_i$, gives

$$\operatorname{per}_\alpha(A_S) = \mathbb{E} \textstyle\prod_{i \in S} (\alpha + W_i), \qquad S \subseteq [n].$$

Set $R = \{r_1, \dots, r_k\}$, $S = R \cup \{1\}$, and $T = R \cup \{2\}$, so that $S \cup T = [n]$ and $S \cap T = R$. Tilt the law of (N_1, N_2) by $(\alpha + L)^k$, and denote the tilted expectation by $\mathbb{E}_*$. Then

$$\operatorname{per}_\alpha(A_{S\cup T}) \operatorname{per}_\alpha(A_R) - \operatorname{per}_\alpha(A_S) \operatorname{per}_\alpha(A_T) = \left(\mathbb{E}(\alpha + L)^k \right)^2 \operatorname{Cov}_*(N_1, N_2).$$

The generating function of (N_1, N_2) depends on v, w only through $v + w$, so given L the variable N_1 is binomial with parameters $(L, 1/2)$, and $N_2 = L - N_1$. This remains so after the tilt, which depends only on L. Hence

$$\operatorname{Cov}_*(N_1, N_2) = \tfrac{1}{4}(\operatorname{Var}_*(L) - \mathbb{E}_* L).$$

Write $z = 2t$, and let $(\alpha)_m$ denote the rising factorial. The variable L has $\mathbb{P}(L = m) = (1 - z)^\alpha (\alpha)_m z^m / m!$, so the tilted law is $\mathbb{P}_*(L = m) = a_m z^m / G(z)$, where

$$G(z) = \sum_{m \geq 0} a_m z^m, \qquad a_m = \frac{(\alpha)_m}{m!} (\alpha + m)^k.$$

The series converges for $|z| < 1$. Since $\mathbb{E}_* L = z G'(z)/G(z)$ and $\mathbb{E}_*[L(L-1)] = z^2 G''(z)/G(z)$,

$$\operatorname{Var}_*(L) - \mathbb{E}_* L = z^2 (\log G)''(z).$$

At zero the numerator of $(\log G)''$ is

$$G(0)G''(0) - G'(0)^2 = \alpha^{k+1}(\alpha+1)(\alpha+2)^k - \alpha^2(\alpha+1)^{2k}.$$

This quantity is negative exactly when

$$\left(\frac{\alpha(\alpha+2)}{(\alpha+1)^2}\right)^k < \frac{\alpha}{\alpha+1}.$$

The base on the left lies strictly between zero and one, so a sufficiently large integer k satisfies this inequality. Continuity then makes $(\log G)''(2t) < 0$ for sufficiently small positive t, and the covariance above is negative. □

Remark A.2. *For $\alpha = 1$ one may take $k = 3$, because $(3/4)^3 < 1/2$. With the vertices ordered $1, 2, r_1, r_2, r_3$ and $t = 1/14$, the matrix of Theorem 1.2 is $84\,(I-B)^{-1}\operatorname{diag}(\frac{1}{7}, \frac{1}{7}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14})$; a positive column scaling multiplies $\operatorname{per}(A_S)$ by a modular factor and so does not affect (1.1). For $k = 2$ the sign at zero is that of $\alpha^2+\alpha-1$, so the four-vertex members of the family violate log-supermodularity for small t exactly when $\alpha < (\sqrt{5}-1)/2$.*