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ON MAJORIZATION INEQUALITIES FOR NORMALIZED JACK AND MACDONALD POLYNOMIALS

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ABSTRACT. We study the inequality $P_\lambda(x;q,t)/P_\lambda(1^n;q,t) \geq P_\mu(x;q,t)/P_\mu(1^n;q,t)$ for all $x \in \mathbb{R}^n_{>0}$ whenever the partition λ majorizes μ; Sra suggested it in 2015, and Chen, Khare and Sahi stated it as a conjecture in 2025. We prove the Jack case for every n from Sun's positive Gelfand–Tsetlin integral and the Dixon–Anderson recursion. For Macdonald polynomials we prove the case $n = 2$, the covers $(d) \succeq (d-1,1)$ for every n when $t \leq q$, and, by exact certificates, the case $n = 3$ with $|\lambda| \leq 8$. The HCIZ mechanism behind the Jack proof has no direct finite-q analogue: at two explicit pairs (q,t), no positive representation of the normalized ratio has a kernel multiplicative in λ, and normalizing at the principal specialization t^δ instead fails the inequality.

1. INTRODUCTION

A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ has size $|\lambda| = \sum_i \lambda_i$; we pad partitions with zeros as needed. For partitions with at most n parts we write $\lambda \succeq \mu$, and say that λ *majorizes* (or *dominates*) μ, if

$$\sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i \quad (1 \leq r < n), \qquad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i. \tag{1.1}$$

Thus $(3,1) \succeq (2,2)$: the more "spread out" partition is the larger one. The same definition applies to arbitrary real vectors once they are sorted in decreasing order; it is then the classical majorization order of Hardy, Littlewood and Pólya (Marshall et al., 2011).

For a partition λ with at most n parts, let $P_\lambda(x;q,t)$ be the monic Macdonald polynomial in the variables $x = (x_1, \ldots, x_n)$, as in Macdonald (1995, Ch. VI), and let

$$F^{(n)}_\lambda(x;q,t) := \frac{P_\lambda(x_1,\ldots,x_n;q,t)}{P_\lambda(1^n;q,t)}, \qquad 0 < q,t < 1, \quad x \in \mathbb{R}^n_{>0}, \tag{1.2}$$

be its normalization at the point $1^n = (1,\ldots,1)$. Unless a degeneration is stated, $0 < q,t < 1$; then $P_\lambda(1^n;q,t) > 0$ and P_λ has nonnegative coefficients in the monomial basis (Macdonald, 1995, Ch. VI, §§6–7). We study the conjecture

$$\lambda \succeq \mu \quad \Longrightarrow \quad F^{(n)}_\lambda(x;q,t) \geq F^{(n)}_\mu(x;q,t) \qquad (x \in \mathbb{R}^n_{>0}). \tag{M}$$

Both sides of (M) are homogeneous of the same degree and equal one at $x = 1^n$, so (M) concerns the shape of F_λ away from the diagonal. We call (1.2) the *anchored* ratio, to distinguish it from the principal-specialization ratio $P_\lambda(x)/P_\lambda(t^\delta)$, with $t^\delta = (t^{n-1},\ldots,t,1)$, which fails the analogue of (M) (Appendix C).

The Schur case $q = t$ of (M) is the conjecture of Cuttler et al. (2011), proved by Sra (2016, posted February 2015) with the Harish-Chandra–Itzykson–Zuber (HCIZ) integral (Section 1.2) and, by a different method and for real exponents, by Khare and Tao (2021, Theorems 1.14 and 1.15). The

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same paper suggested that the HCIZ approach should yield the analogous inequalities for zonal, Jack and Macdonald polynomials; (M) is the Macdonald form of that suggestion. The Jack degeneration $q \to 1$, $t = q^{1/\alpha}$, was later stated as a conjecture by McSwiggen and Novak (2022, Conjecture 4.7) for Heckman–Opdam polynomials of arbitrary root systems, with a proof in semisimple rank one, and by Chen and Sahi (2024); it is Theorem 1.1 below. For Macdonald polynomials, Chen et al. (2025) stated (M) as a conjecture, together with several stronger positivity statements (their Conjecture 5), and proved the cases $n = 2$ and $\mu = (1^{|\lambda|})$, $n \geq |\lambda|$ (their Theorem 5.5). McSwiggen and Sahi (2026b,a) proved the Jack case by two further routes and established finite-q results for the principal-specialization normalization on shifted q-lattices; Appendices C and D explain why those results do not reach (M), and correct one of them.

1.1. **Results.** Our first theorem proves (M) for (monic) Jack polynomials $P_\lambda(x;\alpha)$.

Theorem 1.1 (Jack majorization in every rank)**.** *For every $\alpha > 0$, $n \geq 1$, $x \in \mathbb{R}^n_{>0}$, and equal-size partitions $\lambda \succeq \mu$ with at most n parts,*

$$\frac{P_\lambda(x;\alpha)}{P_\lambda(1^n;\alpha)} \geq \frac{P_\mu(x;\alpha)}{P_\mu(1^n;\alpha)}.$$

If $\lambda \neq \mu$ and $x_i \neq x_j$ $\forall i \neq j$, the inequality is strict; if $x = c\mathbf{1}$, equality holds for all equal-size λ, μ.

Its proof (Section 2) uses only Sun's integral and the Okounkov–Olshanski recursion, and differs from both proofs of McSwiggen–Sahi (Remark 2.4). At finite q we prove (M) for $n = 2$ and all $0 < q, t < 1$.

Theorem 1.2 (Macdonald majorization in rank two)**.** *For $0 < q, t < 1$, $x_1, x_2 > 0$, and equal-size two-part partitions $\lambda \succeq \mu$,*

$$F^{(2)}_\lambda(x;q,t) \geq F^{(2)}_\mu(x;q,t).$$

If $\lambda \neq \mu$ and $x_1 \neq x_2$, the inequality is strict.

The statement is Theorem 5.5 of Chen et al. (2025), who obtain the stronger conclusion that $F^{(2)}_\lambda - F^{(2)}_\mu$ lies in the Muirhead cone. Our proof (Section 3) uses a short induction on Rogers' recurrence for the continuous q-ultraspherical polynomials.

For larger n we prove one family of covers and a finite range of sizes. For $n \geq 2$, $d \geq 2$, and $0 < t \leq q < 1$, $F_{(d)} \geq F_{(1)}F_{(d-1)}$, hence $F_{(d)} \geq F_{(d-1,1)}$, and the same holds for rectangular translates (Theorem A.2). The proof is a likelihood-ratio argument in which the restriction $t \leq q$ is exactly the log-concavity of the coefficients of $(tz;q)_\infty/(z;q)_\infty$. For $n = 3$ we prove (M) for all $0 < q, t < 1$ and $|\lambda| \leq 8$ (Certificate 4.2), by exact tensor-Bernstein certificates over $\mathbb{Q}(q,t)$ of every dominance cover. The three-row cover $(4,2) \succeq (4,1,1)$ is certified strictly off the diagonal (Certificate 4.1). The conjecture remains open in general; Section 7 lists cases beyond these results.

1.2. **The HCIZ mechanism, and what is missing at finite q.** The proof of the Schur case in Sra (2016) and the proof of Theorem 1.1 below rest on one mechanism. Write $x = e^z$ coordinatewise. In the Schur case, Harish-Chandra's formula for the unitary integral (Harish-Chandra, 1957; Itzykson and Zuber, 1980) gives

$$\frac{s_\lambda(e^z)}{s_\lambda(1^n)} = \frac{V(z)}{V(e^z)} \int_{U(n)} \exp\big(\mathrm{tr}(\mathrm{diag}(z)\, U\, \mathrm{diag}(\lambda+\delta)U^*)\big)\, dU, \qquad \delta = (n-1, n-2, \ldots, 0), \tag{1.3}$$

with dU the Haar probability measure and V the Vandermonde product; coincident coordinates are interpreted by continuity. The integral is the Laplace transform, evaluated at $\lambda + \delta$, of the push-forward of Haar measure under $U \mapsto \mathrm{diag}(U^* \mathrm{diag}(z) U)$, a positive measure independent of λ. This Laplace transform is log-convex and symmetric in its argument, and a symmetric convex function is Schur-convex (Marshall et al., 2011, Proposition C.2). Since $\lambda \succeq \mu$ implies $\lambda + \delta \succeq \mu + \delta$,

the Schur case of (M) follows. The same argument for spherical functions of symmetric spaces is in McSwiggen and Novak (2022).

For Jack polynomials there is no unitary integral, but the mechanism survives. Sun (2016), building on Borodin and Gorin (2015), wrote the type-A Heckman–Opdam hypergeometric function $\mathcal{F}_\theta(z,s)$, up to a positive factor independent of s, as the Laplace transform in s of a positive measure on Gelfand–Tsetlin patterns, for every $\theta>0$; and $\mathcal{F}_\theta(z,\lambda+\theta\rho)$, with ρ the Weyl vector, is the normalized Jack polynomial. Section 2 proves Theorem 1.1 this way. It obtains the specialization to Jack polynomials by identifying the Gelfand–Tsetlin measure, tilted at $s=\theta\rho$, with the iterated Dixon–Anderson links of Okounkov and Olshanski (1997).

At finite q no positive integral of this kind is known. Cherednik's global spherical function is the bispectral eigenfunction of the Macdonald difference operators, and $F_\lambda^{(n)}$ is its anchored ratio (Section 5); what is missing is a positive representation of that ratio in which λ enters through a kernel one can compare along the majorization order. One might replace $e^{\langle\lambda,\,y\rangle}$ by a real kernel $K_\lambda(y)$ with $K_{\lambda+\mu}=K_\lambda K_\mu$. Proposition 5.3 and Corollary 5.4 show that this replacement fails: at two pairs (q,t), one with $t>q$ and one with $t<q$, the values $F^{(2)}_{(r)}(2,1;q,t)$ have a negative 3×3 Hankel determinant, so they are not the moments of any positive measure on the line. A kernel valid for all $0<q,t<1$ must therefore depend on λ nonmultiplicatively; finding one is the "non-exponential HCIZ problem" of the title, and Section 6 collects the constraints on it. For $n\geq 2$, no positive Doob transform makes the dual Macdonald operator a positive combination of q-shifts on the positive chamber (Proposition 5.5).

The finite-q analogue of the Dixon–Anderson recursion that does exist, Okounkov's q-integral formula (Okounkov, 1998), is normalized at t^δ rather than at 1^n, and so are the lattice results of McSwiggen and Sahi (2026b,a). This normalization fails: $\lambda\mapsto P_\lambda(x)/P_\lambda(t^\delta)$ is neither Schur-convex nor log-convex already at $x=1^2$, a point of the dominant q-lattice whenever $t=q^k$ with $k\in\mathbb{Z}_{\geq 1}$ (Proposition C.3, Remark C.4).

A positive structure without multiplicativity does exist: $\lambda\mapsto F_\lambda(X)$ is a positive harmonic function of an Up–Down random walk on signatures (Proposition B.1). For $n=3$, coupling the Up steps from λ and from $\lambda+e_1-e_2$ reduces to two explicit inequalities between evaluations $P_\nu(1^3;q,t)$ (Proposition B.3). Total positivity of the rank-three branching kernel is equivalent to an inequality between the evaluations $P_{(m)}(1,1;q,t)$ that is stronger than their log-concavity when $t<q$ (Remark B.5). These inequalities are open; they hold on a grid of (q,t) for small indices (Remark B.6).

1.3. **Conventions.** For a vector v we write $|v|=\sum_i v_i$ and $V(v)=\prod_{i<j}(v_i-v_j)$. The q-shifted factorial is $(a;q)_k=\prod_{j=0}^{k-1}(1-aq^j)$, with $(a;q)_\infty$ the infinite product. We write $e_n(x)=x_1\cdots x_n$ and $p_1(x)=x_1+\cdots+x_n$. The rectangular-translation identity $P_{\lambda+(r^n)}=e_n^r P_\lambda$ (Macdonald, 1995, Ch. VI, §4) and $e_n(1^n)=1$ give $F_{\lambda+(r^n)}=e_n^r F_\lambda$. Vectors $z\in\mathbb{R}^n$ are *regular* if their coordinates are pairwise distinct, and *central* if they are all equal. Jack polynomials $P_\lambda(x;\alpha)$ are monic, with $\theta=1/\alpha$, and $F_\lambda^{(n)}(x;\alpha)=P_\lambda(x;\alpha)/P_\lambda(1^n;\alpha)$ is the Jack analogue of (1.2); such ratios do not depend on the choice among the usual scalar normalizations.

Statements labeled *exact certificate* concern finitely many explicit rational functions of (q,t); their proofs are finite computations in exact arithmetic over $\mathbb{Q}(q,t)$, done in SageMath.

2. The Jack case

Throughout this section $\theta>0$ and $\alpha=1/\theta$. The proof of Theorem 1.1 has two steps. First, for fixed z, the function $s\mapsto\mathcal{F}_\theta(z,s)$ is symmetric and is a positive multiple of the Laplace transform of a positive measure independent of s (Sun's integral, Subsection 2.1). Second, $\mathcal{F}_\theta(z,\lambda+\theta\rho)$ is the normalized Jack polynomial (Theorem 2.3). A symmetric convex function is Schur-convex, and

$\lambda \succeq \mu$ implies $\lambda + \theta\rho \succeq \mu + \theta\rho$, so the two steps give the inequality; Subsection 2.3 also proves the strictness clause.

2.1. **Sun's Gelfand–Tsetlin integral.** Let

$$\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2}\right), \qquad |\rho| = 0.$$

For $z_1 > \cdots > z_n$, a Gelfand–Tsetlin pattern below z is an array

$$\xi^{(1)} \prec \xi^{(2)} \prec \cdots \prec \xi^{(n)} = z, \qquad \xi_i^{(\ell+1)} \geq \xi_i^{(\ell)} \geq \xi_{i+1}^{(\ell+1)},$$

where $\xi^{(\ell)} \in \mathbb{R}^\ell$; the set of all such patterns is the Gelfand–Tsetlin polytope $\mathrm{GT}(z)$, of dimension $n(n-1)/2$. Put $\xi^{(0)} = \varnothing$ and

$$R_\ell(\xi) = |\xi^{(\ell)}| - |\xi^{(\ell-1)}| \qquad (1 \leq \ell \leq n), \tag{2.1}$$

so that $R(\xi) = (R_1(\xi), \dots, R_n(\xi))$ is a linear image of the pattern with $|R(\xi)| = |z|$. Following Sun (2016, (1.6) and Theorem 1.4), define

$$\Phi_\theta(z, s) := \Gamma(\theta)^{-n(n-1)/2} \int_{\mathrm{GT}(z)} e^{\langle s,\, R(\xi)\rangle}\, W^0_{\theta,z}(\xi)\, d\xi, \tag{2.2}$$

$$W^0_{\theta,z}(\xi) := \prod_{\ell=1}^{n-1} \frac{\displaystyle\prod_{i=1}^{\ell}\prod_{j=1}^{\ell+1} \left|e^{\xi_i^{(\ell)}} - e^{\xi_j^{(\ell+1)}}\right|^{\theta-1}}{\displaystyle\prod_{1\leq i<j\leq \ell} \left|e^{\xi_i^{(\ell)}} - e^{\xi_j^{(\ell)}}\right|^{\theta-1} \prod_{1\leq i<j\leq \ell+1} \left|e^{\xi_i^{(\ell+1)}} - e^{\xi_j^{(\ell+1)}}\right|^{\theta-1}} e^{-(\theta-1)|\xi^{(\ell)}|}. \tag{2.3}$$

All factors in (2.3) are nonnegative, and the Selberg-type singularities are locally integrable for every $\theta > 0$. Sun's Theorem 1.4, which Sun attributes to the Macdonald limit of Borodin and Gorin (2015, Definition 6.1 and Proposition 6.3), states that

$$\mathcal{F}_\theta(z, s) = \frac{\prod_{j=1}^n \Gamma(j\theta)}{\Gamma(\theta)^n \prod_{i<j} \left(2\sinh\frac{z_i - z_j}{2}\right)^\theta} \Phi_\theta(z, s) \tag{2.4}$$

is the type-A Heckman–Opdam hypergeometric function (Heckman and Opdam, 1987; Heckman and Schlichtkrull, 1994), in the normalization $\mathcal{F}_\theta(0, s) = 1$. It is real-analytic, symmetric in the coordinates of each argument, and a joint eigenfunction of the Heckman–Opdam operators; the second-order one is

$$\mathcal{D}_\theta = \sum_{i=1}^n \partial_{z_i}^2 + \theta \sum_{i<j} \coth\left(\frac{z_i - z_j}{2}\right)(\partial_{z_i} - \partial_{z_j}), \qquad \mathcal{D}_\theta \mathcal{F}_\theta(z, s) = (\|s\|^2 - \theta^2\|\rho\|^2)\mathcal{F}_\theta(z, s). \tag{2.5}$$

The proof of Theorem 1.1 uses three features of (2.2)–(2.4): $W^0_{\theta,z} \geq 0$; the spectral variable s enters only through $e^{\langle s,\, R(\xi)\rangle}$; and $\mathcal{F}_\theta(z, s)$ is symmetric in s.

Remark 2.1 (The central GL_n coordinate)**.** *There is no trace-zero restriction in* (2.4)*. Translating every row of a pattern by c maps $\mathrm{GT}(z)$ onto $\mathrm{GT}(z + c\mathbf{1})$, adds c to every R_ℓ, and leaves W^0 and the prefactor in* (2.4) *unchanged (the exponents of e^c in* (2.3) *cancel level by level); and $|R(\xi)| = |z|$. Hence*

$$\mathcal{F}_\theta(z + c\mathbf{1}, s) = e^{c|s|}\mathcal{F}_\theta(z, s), \qquad \mathcal{F}_\theta(z, s + c\mathbf{1}) = e^{c|z|}\mathcal{F}_\theta(z, s),$$

and, writing $\bar{z} = z - (|z|/n)\mathbf{1}$ for the trace-zero part,

$$\mathcal{F}_\theta(z, s) = e^{|z|\,|s|/n}\, \mathcal{F}_\theta(\bar{z}, \bar{s}). \tag{2.6}$$

At $s = \lambda + \theta\rho$, identity (2.6) *is the homogeneity of Jack polynomials.*

2.2. **Dixon–Anderson links and the spectral specialization.** For $x_1 > \cdots > x_n > 0$, write $y \prec x$ if $y \in \mathbb{R}^{n-1}$ and $x_1 \geq y_1 \geq x_2 \geq \cdots \geq y_{n-1} \geq x_n$, and define the *Dixon–Anderson link*

$$\Lambda^{\theta}_{n-1,n}(x, dy) := \frac{\Gamma(n\theta)}{\Gamma(\theta)^n} \frac{V(y)}{V(x)^{2\theta-1}} \prod_{i=1}^{n-1} \prod_{j=1}^{n} |y_i - x_j|^{\theta-1}\, dy \qquad (y \prec x). \tag{2.7}$$

The Dixon–Anderson integral (Anderson, 1991) says that (2.7) has total mass one for every $\theta > 0$, so $\Lambda^{\theta}_{n-1,n}(x, \cdot)$ is a probability measure on $\{y \prec x\}$. Jack polynomials are its eigenfunctions in the following sense.

Theorem 2.2 (Okounkov and Olshanski 1997, §6)**.** *Let ν be a partition with at most $n-1$ parts. Then*

$$\int_{y \prec x} P_\nu(y; 1/\theta)\, \Lambda^{\theta}_{n-1,n}(x, dy) = \frac{C^{(n)}_\nu}{C^{(n)}_0} P_\nu(x; 1/\theta), \qquad C^{(n)}_\nu = \prod_{i=1}^{n-1} B(\nu_i + (n-i)\theta, \theta), \tag{2.8}$$

where B is the beta function and $C^{(n)}_0 = \Gamma(\theta)^n / \Gamma(n\theta)$.

Okounkov and Olshanski obtain (2.8) as the $q \to 1$ limit of Okounkov's q-integral formula (Okounkov, 1998); $C^{(n)}_0$ is the Dixon–Anderson integral itself. Letting $x \to 1^n$ in (2.8), the link concentrates at $y = 1^{n-1}$, so $C^{(n)}_\nu / C^{(n)}_0 = P_\nu(1^{n-1}) / P_\nu(1^n)$ and (2.8) becomes

$$\frac{P_\nu(x; 1/\theta)}{P_\nu(1^n; 1/\theta)} = \int_{y \prec x} \frac{P_\nu(y; 1/\theta)}{P_\nu(1^{n-1}; 1/\theta)} \Lambda^{\theta}_{n-1,n}(x, dy). \tag{2.9}$$

Identity (2.9) is the θ-analogue of the branching rule. Its finite-q analogue, with the anchored normalization, is the object that Section 5 shows to be missing (Definition 5.2).

Iterating (2.9) produces a random Gelfand–Tsetlin pattern. Let $X^{(n)} = x$, and for $k = n, n-1, \ldots, 2$ let $X^{(k-1)}$ be drawn from $\Lambda^{\theta}_{k-1,k}(X^{(k)}, \cdot)$; write $\mathbb{E}^{\theta}_x$ for the resulting expectation, and put

$$D_k = \prod_{i=1}^{k} X^{(k)}_i, \qquad D_0 = 1, \qquad R_k = \log \frac{D_k}{D_{k-1}} \quad (1 \leq k \leq n), \tag{2.10}$$

so that $R = (R_1, \ldots, R_n)$ is the vector (2.1) of the pattern $\xi^{(k)}_i = \log X^{(k)}_i$.

Theorem 2.3 (Jack ratios as Dixon–Anderson averages)**.** *Let $x_1 > \cdots > x_n > 0$ and $z = \log x$. For every partition λ with at most n parts, with $\lambda_{n+1} := 0$,*

$$\frac{P_\lambda(x; 1/\theta)}{P_\lambda(1^n; 1/\theta)} = \mathbb{E}^{\theta}_x \prod_{k=1}^{n} D_k^{\lambda_k - \lambda_{k+1}} = \mathbb{E}^{\theta}_x\, e^{\langle \lambda,\, R \rangle}. \tag{2.11}$$

Moreover, the law of the pattern $\xi = \log X$ under $\mathbb{E}^{\theta}_x$ is the Gelfand–Tsetlin measure of (2.2) tilted at $s = \theta\rho$:

$$\mathbb{E}^{\theta}_x\, g(\xi) = \frac{\prod_{j=1}^{n} \Gamma(j\theta)}{\Gamma(\theta)^{n(n+1)/2} \prod_{i<j} \left(2 \sinh \frac{z_i - z_j}{2}\right)^{\theta}} \int_{\mathrm{GT}(z)} g(\xi)\, e^{\langle \theta\rho,\, R(\xi) \rangle}\, W^0_{\theta,z}(\xi)\, d\xi \tag{2.12}$$

for every bounded measurable g. Consequently

$$\frac{P_\lambda(e^z; 1/\theta)}{P_\lambda(1^n; 1/\theta)} = \mathcal{F}_\theta(z, \lambda + \theta\rho). \tag{2.13}$$

Proof. Let $\lambda' = \lambda - (\lambda_n^n)$, a partition with at most $n-1$ parts. Since $P_\lambda = e_n^{\lambda_n} P_{\lambda'}$ and $e_n(1^n) = 1$, we have $F^{(n)}_\lambda(x; 1/\theta) = D_n^{\lambda_n} F^{(n)}_{\lambda'}(x; 1/\theta)$, and (2.9) gives $F^{(n)}_{\lambda'}(x; 1/\theta) = \mathbb{E}^{\theta}_x F^{(n-1)}_{\lambda'}(X^{(n-1)}; 1/\theta)$. Iterating down the pattern gives the first equality in (2.11); the second is the definition of R.

For (2.12), change variables $X^{(k)}_i = e^{\xi^{(k)}_i}$ in the product of the links. The constants multiply to $\prod_{k=2}^{n} \Gamma(k\theta)/\Gamma(\theta)^k = \prod_{j=1}^{n} \Gamma(j\theta)/\Gamma(\theta)^{n(n+1)/2}$, which is the constant in (2.12). The Vandermonde

factors multiply to $\prod_{k=1}^{n-1} V(X^{(k)})^{2-2\theta} \cdot V(x)^{1-2\theta}$, because $V(X^{(1)}) = 1$ and each intermediate level appears once as a "y" and once as an "x". The corresponding product in (2.3), after $X = e^{\xi}$, is $\prod_{k=1}^{n-1} V(X^{(k)})^{2-2\theta} \cdot V(x)^{1-\theta}$, and the cross factors $\prod |X_i^{(k)} - X_j^{(k+1)}|^{\theta-1}$ agree. The Jacobian is $\prod_{k<n} \prod_i dX_i^{(k)} = e^{\sum_{k<n} |\xi^{(k)}|} d\xi$, while (2.3) carries $e^{-(\theta-1)\sum_{k<n}|\xi^{(k)}|}$. Altogether,

$$\prod_{k=2}^{n} \Lambda^{\theta}_{k-1,k}(X^{(k)}, dX^{(k-1)}) = \frac{\prod_{j=1}^{n} \Gamma(j\theta)}{\Gamma(\theta)^{n(n+1)/2}} \, V(x)^{-\theta} \, e^{\theta \sum_{k=1}^{n-1} |\xi^{(k)}|} \, W^0_{\theta,z}(\xi) \, d\xi.$$

Since $\rho_k - \rho_{k+1} = 1$ for $k < n$ and $\rho_n = (1-n)/2$, Abel summation gives $\langle \theta\rho,\, R(\xi)\rangle = \theta \sum_{k<n} |\xi^{(k)}| - \theta \frac{n-1}{2} |z|$, and $\prod_{i<j} 2 \sinh \frac{z_i - z_j}{2} = V(x)\, e^{-\frac{n-1}{2}|z|}$. Substituting both, the product of the links equals the right-hand side of (2.12) with $g \equiv 1$, which proves (2.12). Finally, taking $g = e^{\langle \lambda,\, R\rangle}$ in (2.12) and comparing with (2.2)–(2.4) gives $\mathbb{E}^{\theta}_x e^{\langle \lambda,\, R\rangle} = \mathcal{F}_\theta(z, \lambda + \theta\rho)$, which with (2.11) is (2.13). □

Identity (2.13) is classical: the Heckman–Opdam function at the spectral point $\lambda + \theta\rho$ is the normalized Jacobi polynomial, which in type A is the Jack polynomial (Heckman and Schlichtkrull, 1994, (4.4.10)); see also McSwiggen and Novak (2022, (4.16)). The proof above is included because it shows how the two positive kernels fit together. The Gelfand–Tsetlin integral (2.2) is symmetric in s, though its weight need not have mass one; the Dixon–Anderson links are probability measures but not symmetric in λ; and the shift by $\theta\rho$ converts one into the other.

2.3. **The majorization theorem.**

Proof of Theorem 1.1. The inequality. First suppose that x is regular. By symmetry we may sort x; put $z = \log x$. By (2.2)–(2.4), for fixed z the function $s \mapsto \mathcal{F}_\theta(z, s)$ is $c_\theta(z) \int e^{\langle s,\, R(\xi)\rangle} d\omega(\xi)$ for a positive constant $c_\theta(z)$ and a positive measure $\omega = W^0_{\theta,z}\, d\xi$ on $\mathrm{GT}(z)$, independent of s. A Laplace transform of a positive measure is log-convex, hence convex; and $\mathcal{F}_\theta(z, \cdot)$ is symmetric. A symmetric convex function is Schur-convex (Marshall et al., 2011, Proposition C.2). The vectors $\lambda + \theta\rho$ and $\mu + \theta\rho$ are sorted, and adding $\theta\rho$ to both sides of each inequality in (1.1) preserves it, so $\lambda \succeq \mu$ implies $\lambda + \theta\rho \succeq \mu + \theta\rho$. Hence $\mathcal{F}_\theta(z, \lambda + \theta\rho) \geq \mathcal{F}_\theta(z, \mu + \theta\rho)$, which by (2.13) is the inequality. For non-regular x the inequality follows by continuity of both sides in x.

Equality. If $x = c\mathbf{1}$, then $P_\nu(x; \alpha) = c^{|\nu|} P_\nu(1^n; \alpha)$ by homogeneity, so both ratios equal $c^{|\nu|}$, and $|\lambda| = |\mu|$.

Strictness. Let x be regular and $n \geq 2$, so that z is regular and not central. Differentiating under the integral,

$$\nabla_s^2 \mathcal{F}_\theta(z, s) = c_\theta(z) \int_{\mathrm{GT}(z)} R(\xi) R(\xi)^{\mathsf{T}} e^{\langle s,\, R(\xi)\rangle} \, d\omega(\xi),$$

so for $v \in \mathbb{R}^n$, $v^{\mathsf{T}} \nabla_s^2 \mathcal{F}_\theta(z, s)\, v = c_\theta(z) \int \langle v,\, R(\xi)\rangle^2 e^{\langle s,\, R(\xi)\rangle} d\omega(\xi) \geq 0$, with equality only if $\langle v,\, R(\xi)\rangle = 0$ for ω-almost every ξ. Since ω has a density that is positive almost everywhere on the full-dimensional polytope $\mathrm{GT}(z)$, and $\xi \mapsto \langle v,\, R(\xi)\rangle$ is affine, equality forces $\langle v,\, R(\xi)\rangle = 0$ on all of $\mathrm{GT}(z)$. Now the vertices of $\mathrm{GT}(z)$ at which each row is obtained from the row above by deleting one entry have $R(\xi)$ equal to a permutation of z, and every permutation of z arises this way. For non-central z the permutations of z affinely span the hyperplane $\{|w| = |z|\}$, whose direction space is $\mathbf{1}^{\perp}$. Hence $\langle v,\, R\rangle \equiv 0$ on $\mathrm{GT}(z)$ forces $v \in \mathbb{R}\mathbf{1}$. So the Hessian of $\mathcal{F}_\theta(z, \cdot)$ is positive definite on $\mathbf{1}^{\perp}$ at every s, and $\mathcal{F}_\theta(z, \cdot)$ is strictly convex on every affine hyperplane $\{|s| = \text{const}\}$.

A strictly convex symmetric function on such a hyperplane is strictly Schur-convex there: if $a \succeq b$ with b not a permutation of a, then b is a convex combination $\sum_k p_k \sigma_k(a)$ of permutations of a (Marshall et al., 2011, Chapter 2, B.2), at least two of which are distinct and carry positive weight, so strict convexity gives $f(b) < \sum_k p_k f(\sigma_k a) = f(a)$. Apply strict Schur-convexity with $a = \lambda + \theta\rho$ and $b = \mu + \theta\rho$: both are sorted, so they are permutations of each other only if $\lambda = \mu$. □

Remark 2.4 (Three routes to the Jack inequality)**.** *The inequality of Theorem 1.1, without strictness, is Corollary 2.4 of McSwiggen and Sahi (2026b), follows from Theorem 2.8 there, and follows from Theorem 4.8 of McSwiggen and Sahi (2026a). Theorem 2.8 proves the Schur-convexity of $s \mapsto \mathcal{F}_\theta(z,s)$ by induction on n from the positive interlacing recursion of Amri and Bedhiafi for the nonsymmetric Opdam function; Theorem 4.8 deduces it from a positive Laplace representation of the Opdam–Cherednik kernel for all root systems. The proof above is a third route, and it gives the strictness clause directly. Corollary 2.4 of McSwiggen and Sahi (2026b) rests on a finite-q lattice theorem that fails at central points (Appendix D), but their Theorem 2.8 proves the Jack statement independently.*

Remark 2.5 (Why a pointwise comparison fails)**.** *Formula (2.11) invites the attempt to prove (M) in the Jack case pointwise: under a root transfer $\lambda \mapsto \lambda + e_i - e_j$, $i<j$, the integrand $\prod_k D_k^{\lambda_k-\lambda_{k+1}}$ is multiplied by $D_iD_{j-1}/(D_{i-1}D_j)$, and one might hope that $D_iD_{j-1} \ge D_{i-1}D_j$. This inequality fails already for $n=2$ and $(2,0) \succeq (1,1)$: the two integrands are Y^2 and x_1x_2, where the Dixon–Anderson (beta) variable $Y = X_1^{(1)}$ has positive mass on both sides of $\sqrt{x_1x_2}$. The proof of Theorem 1.1 uses convexity and symmetry of the full average instead; item (2) of Section 6 records this constraint.*

3. Rank two

The proof of Theorem 1.2 is an induction on Rogers' three-term recurrence for the continuous q-ultraspherical polynomials, which are the one-row Macdonald polynomials in two variables. The same recurrence drives the results of Appendix A.

Put

$$w_k = \frac{(t;q)_k}{(q;q)_k} \qquad (k \ge 0), \tag{3.1}$$

the coefficients of $(tz;q)_\infty/(z;q)_\infty = \sum_k w_k z^k$. The one-row Macdonald polynomial in two variables is $P_{(m)}(x_1,x_2;q,t) = w_m^{-1}\sum_{j=0}^m w_j w_{m-j} x_1^{m-j}x_2^j$ (Macdonald, 1995, Ch. VI, §§2 and 4). Writing $x_1 = re^u$, $x_2 = re^{-u}$, the polynomial $P_{(m)}$ is r^m times the continuous q-ultraspherical polynomial

$$C_m(\cos\vartheta; t \mid q) = \sum_{j=0}^m w_j w_{m-j}\, e^{\mathrm{i}(m-2j)\vartheta} \tag{3.2}$$

(Koekoek et al., 2010, §14.10.1), evaluated at $\cos\vartheta = \cosh u$, divided by w_m. Let $c_m = C_m(1;t \mid q) = \sum_j w_j w_{m-j} > 0$.

Proof of Theorem 1.2. For $\lambda = (a,b)$, set $d = a+b$, $m = a-b$, and write $x_1 = re^u$, $x_2 = re^{-u}$ with $r = \sqrt{x_1x_2}$, so that $z := \cosh u \ge 1$, with $z=1$ exactly when $x_1 = x_2$. Since $P_{(a,b)} = (x_1x_2)^b P_{(m)}$,

$$F^{(2)}_{(a,b)}(x;q,t) = r^d \phi_m(z), \qquad \phi_m(z) := \frac{C_m(z;t \mid q)}{c_m}. \tag{3.3}$$

(Here $C_m(z)$ with $z = \cosh u$ means the Laurent polynomial (3.2) with $e^{\mathrm{i}\vartheta}$ replaced by e^u; since d and m have the same parity, every term $r^d e^{(m-2j)u}$ is a monomial in x_1, x_2.) The cover $(a,b) \succeq (a-1,b+1)$ keeps d and lowers m by two, so it suffices to prove $\phi_{m+1} \ge \phi_m$ for $z \ge 1$, strictly for $z > 1$.

Rogers' recurrence (Koekoek et al., 2010, §14.10.1) reads

$$2(1-tq^m)\, z\, C_m = (1-q^{m+1})C_{m+1} + (1-t^2q^{m-1})C_{m-1} \qquad (m \ge 1). \tag{3.4}$$

Divide by $2(1-tq^m)c_m$ and put

$$A_m = \frac{(1-q^{m+1})c_{m+1}}{2(1-tq^m)c_m}, \qquad B_m = \frac{(1-t^2q^{m-1})c_{m-1}}{2(1-tq^m)c_m}, \qquad A_0 = 1,\ B_0 = 0. \tag{3.5}$$

For $m \geq 1$, $A_m, B_m > 0$, and evaluating (3.4) at $z = 1$ gives $A_m + B_m = 1$. Hence $z\phi_m = A_m\phi_{m+1} + B_m\phi_{m-1}$, that is,

$$A_m(\phi_{m+1} - \phi_m) = (z-1)\phi_m + B_m(\phi_m - \phi_{m-1}). \tag{3.6}$$

Now $\phi_m(z) > 0$ for $z \geq 1$, because (3.2) has positive coefficients, and $\phi_0 = 1$, $\phi_1 = z$. Induction on m in (3.6) gives $\phi_{m+1} - \phi_m \geq 0$, with strict inequality for $z > 1$ since then $(z-1)\phi_m > 0$. □

4. Exact rank-three certificates

For $n = 3$, Theorem A.2 proves only the covers $(d) \succeq (d-1, 1)$ and their rectangular translates, and only for $t \leq q$. This section proves (M) for $n = 3$, all $0 < q, t < 1$, and $|\lambda| \leq 8$, and the cover $(4, 2) \succeq (4, 1, 1)$ strictly off the diagonal, by exact computation over $\mathbb{Q}(q, t)$.

By symmetry and homogeneity, it suffices to consider

$$x = \big((1+u)(1+v),\ 1+v,\ 1\big), \qquad u, v \geq 0, \tag{4.1}$$

which parametrizes the sorted chamber $x_1 \geq x_2 \geq x_3 = 1$, with $u = v = 0$ exactly at the diagonal. For a pair $\lambda \succeq \mu$ of partitions with at most three parts, the *cleared gap*

$$\widetilde{G}_{\lambda,\mu}(u, v) := P_\lambda(x)\, P_\mu(1^3) - P_\mu(x)\, P_\lambda(1^3), \qquad \widetilde{G}_{\lambda,\mu} = P_\lambda(1^3)P_\mu(1^3)(F_\lambda - F_\mu), \tag{4.2}$$

is a polynomial in u, v whose coefficients, the *chamber coefficients*, are rational functions of (q, t). Since $P_\lambda(1^3)P_\mu(1^3) > 0$, the inequality $F_\lambda \geq F_\mu$ holds if and only if $\widetilde{G}_{\lambda,\mu} \geq 0$ for all $u, v \geq 0$, and it holds if every chamber coefficient is nonnegative on $(0, 1)^2$.

We write each chamber coefficient in lowest terms as N/D with $N, D \in \mathbb{Q}[q, t]$. Every denominator D below factors into monomials and polynomials $h(q^a t^b)$ with h having no root in $(0, 1)$ (in practice $1 - qt$, $1 - q^2t$, $1 - q^3t^2$ and $1 + qt$), so D has constant sign on the open square. Orient N so that $D > 0$ there, and expand N in the tensor-Bernstein basis of its bidegree (r, s),

$$N(q, t) = \sum_{a=0}^{r}\sum_{b=0}^{s} \beta_{ab} \binom{r}{a} q^a (1-q)^{r-a} \binom{s}{b} t^b (1-t)^{s-b}.$$

If every $\beta_{ab} \geq 0$, then $N \geq 0$ on $[0, 1]^2$; since every basis function is positive on $(0, 1)^2$, a nonzero such N is positive there. The test is sufficient but not necessary, so it cannot certify a false inequality; no degree elevation was needed below. The counts of Bernstein coefficients reported below include those of the oriented denominators, which are expanded in the same way.

4.1. The cover $(4, 2) \succeq (4, 1, 1)$.

Exact certificate 4.1 (The cover $(4, 2) \succeq (4, 1, 1)$)**.** *For every $0 < q, t < 1$ and $x \in \mathbb{R}^3_{>0}$,*

$$F^{(3)}_{(4,2)}(x; q, t) \geq F^{(3)}_{(4,1,1)}(x; q, t), \tag{4.3}$$

with equality only when $x_1 = x_2 = x_3$.

Exact certificate. The cleared gap (4.2) has 32 nonzero chamber coefficients; the coefficients of u and v vanish. Every denominator factors as described above, and the Bernstein coefficients of the numerators and oriented denominators are nonnegative: 2966 positive and 620 zero. Hence every chamber coefficient is nonnegative on $(0, 1)^2$, which proves (4.3). The coefficients of u^2 and v^2 are nonzero, hence positive on $(0, 1)^2$; since all terms are nonnegative, the gap vanishes only at $u = v = 0$. □

4.2. **All covers of size at most eight.** The method of Certificate 4.1 applies to every cover, including the three-row covers $(a,b,0) \succeq (a,b-1,1)$, which are not rectangular translates of two-row covers. Since dominance is the transitive closure of its covers, certifying every cover of a given size proves (M) for all pairs of that size.

Exact certificate 4.2 (Every rank-three cover of size at most eight)**.** *For every dominance cover $\lambda \succeq \mu$ with $|\lambda| \le 8$ and at most three parts, every chamber coefficient of the cleared gap (4.2) is nonnegative on $(0,1)^2$. Consequently (M) holds for $n = 3$, all $0 < q,t < 1$, and $|\lambda| \le 8$.*

Exact certificate. There are 36 such covers, with between 6 and 78 nonzero chamber coefficients each. Each cleared gap is certified as in Certificate 4.1; the Bernstein test decided every coefficient without degree elevation. □

5. The global spherical function and the missing kernel

The proof of Theorem 1.1 has two inputs: an eigenfunction whose anchored ratio is F_λ, and a positive integral representation of that ratio in which λ enters through an exponential. At finite q the first input exists: it is Cherednik's global spherical function (Subsection 5.1). This section shows that the second input cannot take the forms suggested by the Jack case. An anchored analogue of (2.9), and more generally a kernel multiplicative in λ, fails at two explicit pairs (q,t) (Subsections 5.2–5.3); a positive operator in the spectral variable fails for every $n \ge 2$ (Subsection 5.4).

5.1. **Cherednik's global spherical function.** In n variables, the first Macdonald–Ruijsenaars operator is

$$D_1^{(n)} = \sum_{i=1}^{n} \prod_{j \ne i} \frac{tX_i - X_j}{X_i - X_j}\, T_{q,X_i}, \qquad D_1^{(n)} P_\lambda = \Big(\sum_{i=1}^{n} q^{\lambda_i} t^{n-i}\Big) P_\lambda, \tag{5.1}$$

where T_{q,X_i} replaces X_i by qX_i. Cherednik's global symmetric spherical function $\mathcal{P}^\circ(X,\Lambda)$ is the Weyl-invariant, bispectral reproducing kernel of the full commuting family of the corresponding type-A operators (Cherednik, 2009); its Gaussian normalization removes the periodic ambiguity that the difference equations alone leave. In Cherednik's anti-dominant convention, let $c_\lambda = w_0(\lambda - (|\lambda|/n)\mathbf{1})$, where w_0 reverses coordinates, and put $\Lambda_\lambda = q^{c_\lambda - \rho_k}$, with ρ_k the deformed Weyl vector of Cherednik (2009). Equation (3.11) of Cherednik (2009), on the type-A torus $e_n(\bar{X}) = 1$, has the form

$$\mathcal{P}^\circ(\bar{X}, \Lambda_\lambda) = C(q,t) \frac{P_\lambda(\bar{X}; q, t)}{P_\lambda(q^{-\rho_k}; q, t)}, \tag{5.2}$$

where $C(q,t)$ is independent of X and λ. Dividing (5.2) by its value at $\bar{X} = 1^n$ removes every convention-dependent constant. To recover the GL_n ratio, put $r = e_n(X)^{1/n}$ and $\bar{X} = r^{-1}X$.

Proposition 5.1 (Anchored global spherical ratio)**.** *For $n \ge 2$, every partition λ with at most n parts, and $X \in \mathbb{R}^n_{>0}$,*

$$F_\lambda^{(n)}(X; q, t) = r^{|\lambda|} \frac{\mathcal{P}^\circ(\bar{X}, \Lambda_\lambda)}{\mathcal{P}^\circ(1^n, \Lambda_\lambda)}. \tag{5.3}$$

Proof. Divide (5.2) by its value at $\bar{X} = 1^n$ and use $P_\lambda(X) = r^{|\lambda|} P_\lambda(\bar{X})$. □

For equal-size partitions the factor $r^{|\lambda|}$ is common to both sides of (M), as in (2.6). What is missing is an analogue of the positive formula (2.4).

5.2. **Anchored links.** The normalization of the Dixon–Anderson recursion (2.9) suggests the following finite-q analogue.

Definition 5.2 (Anchored link)**.** A family $\mathsf{B}_{n\to n-1}(dY \mid X)$ of positive probability measures on $\mathbb{R}_{>0}^{n-1}$, indexed by $X \in \mathbb{R}_{>0}^{n}$, is an *anchored link* if for every partition ν with at most $n-1$ parts

$$F_\nu^{(n)}(X;q,t) = \int F_\nu^{(n-1)}(Y;q,t)\,\mathsf{B}_{n\to n-1}(dY \mid X). \tag{5.4}$$

Multiplying (5.4) by $P_\nu(1^n)$ shows that an anchored link is a positive kernel with the Macdonald polynomials as eigenfunctions and with prescribed eigenvalues,

$$\int P_\nu(Y;q,t)\,\mathsf{B}_{n\to n-1}(dY \mid X) = \frac{P_\nu(1^{n-1};q,t)}{P_\nu(1^n;q,t)}\,P_\nu(X;q,t), \tag{5.5}$$

and conversely. Since the P_ν span the symmetric polynomials, (5.5) determines the link on polynomials; the question is whether the functional it defines is positive. For $t = q^K$, Okounkov's q-integral formula (Okounkov, 1998, Theorem I) satisfies an identity of the form (5.5), but with the principal ratios $P_\nu(t^{\delta_{n-1}})/P_\nu(t^{\delta_n})$ as eigenvalues, where $t^{\delta_k} = (t^{k-1},\dots,t,1)$: at $x = (1,t,\dots,t^{n-1})$ only one summand of the q-integral survives (Okounkov, 1998, Example 3.11). Iterating an anchored link down to one variable would give $F_\lambda^{(n)}(X) = \mathbb{E}\prod_k D_k^{\lambda_k-\lambda_{k+1}}$ exactly as in (2.11), a positive representation of F_λ with a kernel multiplicative in λ. The next subsection rules out such a representation at two pairs (q,t).

5.3. **The Hankel obstruction.**

Proposition 5.3 (Hankel obstruction)**.** *Let $m_r = F_{(r)}^{(2)}(2,1;q,t)$, $r \ge 0$. At $(q,t) = (\frac{1}{3},\frac{7}{8})$ and at $(q,t) = (\frac{1}{3},\frac{1}{11})$ there is no positive measure on $\mathbb{R}$ whose moments are $m_0,\dots,m_4$.*

Proof. For every positive measure ν on $\mathbb{R}$ with moments m_r, the Hankel matrix $H = (m_{i+j})_{i,j=0}^2$ is positive semidefinite, because $c^{\mathsf{T}}Hc = \int (c_0 + c_1 y + c_2 y^2)^2\,d\nu(y) \ge 0$. Exact computation from the two-variable polynomials $P_{(r)}$ gives, at $q = \frac{1}{3}$, $t = \frac{7}{8}$,

$$(m_0,\dots,m_4) = \Big(1, \tfrac{3}{2}, \tfrac{93}{38}, \tfrac{17}{4}, \tfrac{4089}{532}\Big), \qquad \det H = -\tfrac{67}{192052} < 0, \tag{5.6}$$

and at $q = \frac{1}{3}$, $t = \frac{1}{11}$,

$$(m_0,\dots,m_4) = \Big(1, \tfrac{3}{2}, \tfrac{30}{13}, \tfrac{277}{76}, \tfrac{65721}{11126}\Big), \qquad \det H = -\tfrac{832225}{70593757936} < 0. \tag{5.7}$$

Explicit witnesses are $c = (13,-20,7)$, with $c^{\mathsf{T}}Hc = -\frac{1}{76}$ in the first case, and $c = (19,-25,8)$, with $c^{\mathsf{T}}Hc = -\frac{3030}{1374061}$ in the second. □

The two pairs have $t > q$ and $t < q$, so the obstruction is not confined to either side of the Schur line $t = q$. In the Jack limit, by contrast, the values $F_{(r)}^{(2)}(2,1)$ are the moments of a Dixon–Anderson beta law on $[1,2]$.

Corollary 5.4 (No anchored link in rank two; no multiplicative kernel)**.** *At the two parameter pairs of Proposition 5.3:*

(1) there is no anchored link $\mathsf{B}_{2\to1}$ in the sense of Definition 5.2;

(2) more generally, there is no representation $F_\lambda^{(2)}(X) = \int K_\lambda(y)\,\nu_X(dy)$ valid for all partitions λ, with ν_X a positive measure and a real kernel that is multiplicative in the partition, $K_{\lambda+\mu} = K_\lambda K_\mu$.

Proof. Since $F_{(r)}^{(1)}(Y) = Y^r$, an anchored link $\mathsf{B}_{2\to1}$ would make $m_r = \int Y^r\,\mathsf{B}_{2\to1}(dY \mid (2,1))$ the moments of a positive measure. For the second statement, multiplicativity gives $K_0^2 = K_0$ and $K_0K_\lambda = K_\lambda$. Restrict ν_X to $\{K_0 = 1\}$; all kernels vanish on its complement, and the restricted

measure has mass $F_0(X) = 1$. Along the ray $\lambda = re_1$, the values $F^{(2)}_{re_1}(X) = \int K_{e_1}(y)^r \nu_X(dy)$ are the moments of the push-forward of ν_X under K_{e_1}, a positive measure on $\mathbb{R}$. Both contradict Proposition 5.3 at $X = (2, 1)$. □

The path representation of Appendix B.1 is not multiplicative in λ, so Corollary 5.4 does not apply to it.

5.4. **The dual difference equation is not termwise positive.** Cherednik's function is bispectral, so one may equally try to generate $\lambda \mapsto F_\lambda(X)$ by a positive operator acting in the spectral variable. The type-A dual Macdonald operator is

$$D_\Lambda = \sum_{i=1}^{n} A_i(\Lambda) T_{q,\Lambda_i}, \qquad A_i(\Lambda) = \prod_{j \neq i} \frac{t\Lambda_i - \Lambda_j}{\Lambda_i - \Lambda_j}. \tag{5.8}$$

Proposition 5.5 (No positive Doob transform of the dual operator)**.** *Let $n \geq 2$, $0 < q, t < 1$, and let h be a positive function on $\mathbb{R}^n_{>0}$. Then the coefficients a_i of $h^{-1} D_\Lambda h = \sum_i a_i(\Lambda) T_{q,\Lambda_i}$ are not all nonnegative on the chamber $\Lambda_1 > \cdots > \Lambda_n > 0$.*

Proof. The transformed coefficient is $a_i(\Lambda) = A_i(\Lambda)\, h(T_{q,\Lambda_i}\Lambda)/h(\Lambda)$, which has the sign of $A_i(\Lambda)$. Choose $t\Lambda_1 < \Lambda_2 < \Lambda_1$, and, when $n > 2$, choose $t\Lambda_1 > \Lambda_3 > \cdots > \Lambda_n > 0$. The factor indexed by $j = 2$ in A_1 is negative and every other factor is positive, so $A_1(\Lambda) < 0$. □

Positivity in the spectral variable can therefore emerge only after restriction to a lattice that avoids the region $t\Lambda_1 < \Lambda_2 < \Lambda_1$, or after composing several steps; it cannot be read off term by term from the difference equation.

6. Constraints on a (q, t)-HCIZ kernel

A finite-q construction combining the HCIZ comparison with the rank-reducing recursion of Section 2 would need to:

(1) have the eigenvalues $P_\nu(1^{n-1})/P_\nu(1^n)$ of (5.5), not the principal ratios $P_\nu(t^{\delta_{n-1}})/P_\nu(t^{\delta_n})$ of Okounkov's q-integral formula; the ratio normalized at t^δ fails the inequality (Appendix C);
(2) compare averages rather than integrands, since $D_i D_{j-1} \geq D_{i-1} D_j$ fails already in the Jack case (Remark 2.5);
(3) depend on λ nonmultiplicatively, whatever the support of the measure (Corollary 5.4);
(4) reduce to the HCIZ integral (1.3) at $q = t$, to the Dixon–Anderson links (2.7) as $q \to 1$, and for $n = 2$ to the recurrence (3.4).

The Up–Down walk of Proposition B.1 gives a positive path representation of $F_\lambda(X)$ for every non-central $X \in \mathbb{R}^n_{>0}$, and its Martin representation (B.4) is not multiplicative in λ. Its boundary and its limits in item (4) have not been identified, and it does not supply the rank-reducing link (5.5), which Corollary 5.4 rules out at the pairs of Proposition 5.3. A proof of (M) based on the walk would therefore need a comparison argument on signatures.

7. Open problems

(1) Conjecture (M) beyond the cases $q = t$; $\mu = (1^{|\lambda|})$ with $n \geq |\lambda|$; $n = 2$; the covers $(d) \succeq (d-1, 1)$ with $t \leq q$; and $n = 3$ with $|\lambda| \leq 8$. Open cases include $(8, 1) \succeq (7, 2)$ for $n = 3$ and all $0 < q, t < 1$, and $(3, 1) \succeq (2, 2)$ for $n = 4$ and $q \neq t$.
(2) Inequality (B.10), equivalent to the MTP_2 property of the rank-three branching kernel, and the inequalities (B.6)–(B.7) for the Up coupling of λ and $\lambda + e_1 - e_2$ when $n = 3$ (Proposition B.3); also whether (B.10) implies (B.6)–(B.7). All three inequalities hold in the tests of Remark B.6.
(3) The Martin boundary of the Up–Down walk (Remark B.2), which in the formulation of this paper is the finite-q HCIZ problem.
(4) Whether the product inequality (A.3) holds for $t > q$ when $n \geq 3$; no counterexample was found.

Appendix A. Further results at finite q

This appendix proves the remaining finite-q results: log-concavity of the evaluations $P_{(m)}(1,1;q,t)$ and the inequality $F_{(d)} \geq F_{(d-1,1)}$ for $t \leq q$. Both rest on Rogers' recurrence (3.4). The main text does not use them.

A.1. **Log-concavity of the one-row evaluations.** Let

$$\Delta_m := P_{(m)}(1,1;q,t) = \frac{c_m}{w_m} \qquad (m \geq 0), \tag{A.1}$$

with $\Delta_0 = 1$ and $\Delta_1 = 2$. For $n = 2$, the Up step of the walk in Appendix B adds a box to the first row of $(m,0)$ with probability $\Delta_{m+1}/(2\Delta_m)$; the following proposition shows that these probabilities are non-increasing in m.

Proposition A.1 (Log-concavity of Δ_m)**.** *For every $0 < q,t < 1$, the sequence $(\Delta_m)_{m\geq 0}$ is log-concave: $\Delta_m^2 \geq \Delta_{m-1}\Delta_{m+1}$ for $m \geq 1$. Equivalently, $r_m := \Delta_m/\Delta_{m-1}$ is non-increasing in m.*

Proof. Rogers' recurrence (3.4) at $z = 1$, divided by w_m and using $w_{m\pm1}/w_m$ from (3.1), becomes a recurrence for Δ_m:

$$\Delta_m = 2\Delta_{m-1} - \kappa_m \Delta_{m-2}, \qquad \kappa_m = \frac{(1-t^2q^{m-2})(1-q^{m-1})}{(1-tq^{m-1})(1-tq^{m-2})} \quad (m \geq 2).$$

Put $s_m = r_m - 1$ and

$$\varepsilon_m = 1 - \kappa_m = \frac{q^{m-2}(1-t)(q-t)}{(1-tq^{m-1})(1-tq^{m-2})}.$$

Then $s_1 = 1$, and dividing the recurrence by Δ_{m-1} gives

$$s_m = \frac{s_{m-1} + \varepsilon_m}{1 + s_{m-1}}, \qquad s_{m-1} - s_m = \frac{s_{m-1}^2 - \varepsilon_m}{1 + s_{m-1}}.$$

If $t \geq q$, then $\varepsilon_m \leq 0$ and $s_{m-1} - s_m \geq 0$ follows at once. Suppose $t < q$. Then $0 < \varepsilon_m < 1$, and $\varepsilon_{m+1}/\varepsilon_m = q(1-tq^{m-2})/(1-tq^m) < 1$, so (ε_m) is decreasing. We claim $s_{m-1} \geq \sqrt{\varepsilon_m}$ for all $m \geq 2$, which gives $s_{m-1} - s_m \geq 0$. For $m = 2$ the claim reads $1 \geq \sqrt{\varepsilon_2}$. For the induction step, the recurrence gives

$$s_m - \sqrt{\varepsilon_m} = \frac{(s_{m-1} - \sqrt{\varepsilon_m})(1 - \sqrt{\varepsilon_m})}{1 + s_{m-1}} \geq 0,$$

so $s_m \geq \sqrt{\varepsilon_m} \geq \sqrt{\varepsilon_{m+1}}$. In all cases $r_m \leq r_{m-1}$. □

A.2. **The covers $(d) \succeq (d-1,1)$ for $t \leq q$.** Theorem A.2 below compares consecutive one-row polynomials. Up to scalars, the one-row polynomials are the coefficients of a product of generating functions, so total positivity of convolutions applies. The polynomials $Q_{(m)} = w_m P_{(m)}$ have the generating function

$$\sum_{m\geq 0} Q_{(m)}(x) z^m = \prod_{i=1}^{n} W(x_i z), \qquad W(z) = \frac{(tz;q)_\infty}{(z;q)_\infty} = \sum_{k\geq 0} w_k z^k, \tag{A.2}$$

(Macdonald, 1995, Ch. VI, §2), so $Q_{(m)}(x)$ is the m-th coefficient of the convolution of the n sequences $(w_k x_i^k)_{k\geq 0}$. Two classical facts about such convolutions are needed. A nonnegative sequence with no internal zeros is called log-concave (or PF_2) if $h_k^2 \geq h_{k-1}h_{k+1}$; a sequence (f'_k) dominates (f_k) in the *likelihood-ratio order*, written $f' \geq_{\mathrm{lr}} f$, if f'_k/f_k is non-decreasing in k. Convolution with a PF_2 sequence preserves both log-concavity and the likelihood-ratio order (Karlin, 1968, Chapter 3); see Shaked and Shanthikumar (2007, Theorem 1.C.9) for the second statement. Both are instances of the basic composition formula for totally positive kernels.

Theorem A.2 (The covers $(d) \succeq (d-1,1)$ for $t \le q$)**.** *Let $n \ge 2$, $d \ge 2$, $x \in \mathbb{R}^n_{>0}$, and $0 < t \le q < 1$. Then*

$$F^{(n)}_{(d)}(x;q,t) \ge F^{(n)}_{(1)}(x)\, F^{(n)}_{(d-1)}(x;q,t) \tag{A.3}$$

and consequently

$$F^{(n)}_{(d)}(x;q,t) \ge F^{(n)}_{(d-1,1)}(x;q,t). \tag{A.4}$$

Both inequalities are strict unless $x_1 = \cdots = x_n$. For every integer $r \ge 0$, rectangular translation gives

$$F^{(n)}_{(d+r,r^{n-1})} \ge F^{(n)}_{(d-1+r,1+r,r^{n-2})}, \tag{A.5}$$

again strictly away from the diagonal.

Proof. ***Step 1: two variables.*** For $a, b > 0$ put $e_1 = a + b$, $e_2 = ab$, and $f_m = f_m(e_2) := [z^m]\, W(az)W(bz)$, a polynomial in e_1, e_2. With $a = re^u$, $b = re^{-u}$ we get $f_m = r^m C_m(\cosh u)$, so Rogers' recurrence (3.4) at index $m-1$, multiplied by r^m, gives, for $m \ge 1$ and $f_{-1} = 0$,

$$(1-q^m)f_m = (1 - tq^{m-1})\, e_1 f_{m-1} - (1 - t^2 q^{m-2})\, e_2 f_{m-2}. \tag{A.6}$$

Fix e_1 and regard $e_2 \in (0, e_1^2/4]$ as the variable; a two-coordinate spread $(a,b) \mapsto (a+\epsilon, b-\epsilon)$ with $a \ge b$ keeps e_1 and lowers e_2. Let $u_m = f_m/f_{m-1}$, $\gamma_m = (1-tq^{m-1})/(1-q^m)$ and $\eta_m = (1-t^2q^{m-2})/(1-q^m)$. Then $u_1 = \gamma_1 e_1$ does not depend on e_2, and for $m \ge 2$, (A.6) gives

$$u_m = \gamma_m e_1 - \frac{\eta_m e_2}{u_{m-1}}, \qquad \frac{du_m}{de_2} = -\frac{\eta_m (u_{m-1} - e_2\, u'_{m-1})}{u^2_{m-1}}.$$

Since $\eta_m > 0$, $u_{m-1} > 0$ and, inductively, $u'_{m-1} \le 0$, we get $u'_m < 0$ for all $m \ge 2$. Hence if $e'_2 < e_2$ then $u_m(e'_2) \ge u_m(e_2)$, strictly for $m \ge 2$; so the ratio $f_m(e'_2)/f_m(e_2) = \prod_{k \le m} u_k(e'_2)/u_k(e_2)$ is non-decreasing in m, and strictly increasing for $m \ge 2$. Thus a spread of two coordinates raises the coefficient sequence of $W(az)W(bz)$ in the likelihood-ratio order.

Step 2: the other variables. The coefficient sequence $(w_k x_i^k)_k$ of each remaining factor $W(x_i z)$ is log-concave exactly when

$$\frac{w_{k+1}}{w_k} = \frac{1 - tq^k}{1 - q^{k+1}} \tag{A.7}$$

is non-increasing in k, and a brief calculation shows that $w_{k+1}/w_k \ge w_{k+2}/w_{k+1}$ is equivalent to $(q-t)(1-q)q^k \ge 0$. So for $t \le q$ every factor is PF_2, and so is the convolution h of all factors other than the two being spread. Convolving the likelihood-ratio comparison of Step 1 with h preserves it, so a two-coordinate spread raises the sequence $(Q_{(m)}(x))_{m \ge 0}$ in the likelihood-ratio order. Strictness survives the convolution: by the basic composition formula, the 2×2 minor expressing the comparison after convolution is a nonnegative combination of the minors before it, and the term indexed by the two top positions carries the weight $h_0^2 > 0$.

Step 3: chaining. Let $\bar{x} = n^{-1} \sum_i x_i$. Since $x/\bar{x} \succeq 1^n$, the vector $x/\bar{x}$ is obtained from 1^n by finitely many two-coordinate spreads, each keeping the coordinates positive (Marshall et al., 2011, Chapter 2, B.1). Along the chain the sequence $(Q_{(m)})_m$ increases in the likelihood-ratio order, hence $Q_{(d)}(x/\bar{x})/Q_{(d)}(1^n) \ge Q_{(d-1)}(x/\bar{x})/Q_{(d-1)}(1^n)$, which is $F_{(d)}(x/\bar{x}) \ge F_{(d-1)}(x/\bar{x})$. Homogeneity and $F_{(1)}(x) = \bar{x}$ turn this inequality into (A.3), with strict inequality as soon as one genuine spread occurs, that is, unless $x = \bar{x}\mathbf{1}$.

Step 4: the cover. The Pieri rule for multiplication by $e_1 = p_1$ (Macdonald, 1995, Ch. VI, (6.24)) has positive coefficients for $0 < q, t < 1$, and evaluating it at 1^n normalizes them:

$$F_{(1)}F_{(d-1)} = \sigma_{d,n} F_{(d)} + (1 - \sigma_{d,n}) F_{(d-1,1)}, \qquad 0 < \sigma_{d,n} < 1. \tag{A.8}$$

Substituting (A.8) into (A.3) and cancelling $1-\sigma_{d,n}$ gives (A.4). Finally $P_{\lambda+(r^n)} = e_n^r P_\lambda$ with $e_n(1^n)=1$ gives (A.5). □

The restriction $t \le q$ enters only in Step 2; for $n=2$ there are no other variables, and the argument gives (A.3) for all $0<q,t<1$. Whether (A.3) holds for $t>q$ when $n\ge 3$ is open; exact checks for $n=3$, $d\le 5$, a few points x, and all $(q,t)\in\{i/12 : 1\le i\le 11\}^2$ found no violation.

Appendix B. The Up–Down walk and the rank-three coupling

Corollary 5.4 rules out positive representations of F_λ that are multiplicative in λ. This appendix constructs one that is not: a random walk on signatures with $\lambda \mapsto F_\lambda(X)$ as a positive harmonic function. It then reduces one family of couplings of this walk for $n=3$ to two open inequalities between evaluations $P_\nu(1^3;q,t)$.

B.1. **A positive Up–Down walk with F_λ as harmonic function.** Fix n and let $\mathcal{S}$ be the set of dominant integer signatures $\lambda\in\mathbb{Z}^n$, $\lambda_1\ge\cdots\ge\lambda_n$. Extend Macdonald polynomials to signatures by $P_\lambda := e_n^{-r}P_{\lambda+(r^n)}$ for any r with $\lambda+(r^n)$ a partition; the rectangular-translation identity makes the definition independent of r. Put $\Delta_\lambda = P_\lambda(1^n;q,t)>0$, so that $F_\lambda = P_\lambda/\Delta_\lambda$ as before. For $X\in\mathbb{R}^n_{>0}$ set

$$\Xi(X) = \frac{p_1(X)\,p_1(X^{-1})}{n^2} = \frac{(\sum_i X_i)(\sum_i X_i^{-1})}{n^2} \;\ge\; 1, \tag{B.1}$$

with equality exactly when X is central, by the Cauchy–Schwarz inequality.

Proposition B.1 (The Up–Down walk)**.** *Let $0<q,t<1$.*

(1) (Up step.) Writing $p_1P_\lambda = \sum_{\nu=\lambda+\square} c_{\lambda\nu}P_\nu$ for the Pieri rule, the matrix $\mathsf{U}_{\lambda\nu} = c_{\lambda\nu}\Delta_\nu/(n\Delta_\lambda)$ is stochastic on $\mathcal{S}$, and $\sum_\nu \mathsf{U}_{\lambda\nu}F_\nu(X) = (p_1(X)/n)\,F_\lambda(X)$.

(2) (Down step.) Writing $p_1(X^{-1})P_\lambda(X) = \sum_{\nu=\lambda-\square} c'_{\lambda\nu}P_\nu(X)$, the matrix $\mathsf{D}_{\lambda\nu} = c'_{\lambda\nu}\Delta_\nu/(n\Delta_\lambda)$ is stochastic, and $\sum_\nu \mathsf{D}_{\lambda\nu}F_\nu(X) = (p_1(X^{-1})/n)\,F_\lambda(X)$.

(3) (Harmonicity.) Let $\mathsf{K}=\mathsf{DU}$, a stochastic kernel on each component $\mathcal{S}_c=\{\lambda:|\lambda|=c\}$. Then

$$\sum_\nu \mathsf{K}(\lambda,\nu)F_\nu(X) = \Xi(X)\,F_\lambda(X), \tag{B.2}$$

so that for non-central X, $\lambda\mapsto F_\lambda(X)$ is a positive harmonic function of the strictly substochastic kernel $\mathsf{P}_X = \mathsf{K}/\Xi(X)$.

(4) (Feynman–Kac.) Let $(\lambda(r))_{r\ge0}$ be the Markov chain with kernel K and $\lambda(0)=\lambda$, let $D\subset\mathcal{S}_c$ be finite with $\lambda\in D$, and let τ_D be the first exit time from D. If X is not central, then

$$F_\lambda(X) = \mathbb{E}_\lambda\Big[\Xi(X)^{-\tau_D}F_{\lambda(\tau_D)}(X)\,;\;\tau_D<\infty\Big]. \tag{B.3}$$

Proof. (1) The Pieri coefficients $c_{\lambda\nu}$ are nonnegative for $0<q,t<1$ (Macdonald, 1995, Ch. VI, (6.24)), and they are unchanged by rectangular translation, so the expansion holds for signatures. Evaluating it at 1^n gives $n\Delta_\lambda = \sum_\nu c_{\lambda\nu}\Delta_\nu$, which is the row-sum condition; dividing the expansion by $n\Delta_\lambda$ gives the eigenrelation.

(2) The inversion identity $P_\lambda(X^{-1};q,t) = e_n(X)^{-\lambda_1}P_{\lambda_1\mathbf{1}-w_0\lambda}(X;q,t)$, with $w_0\lambda=(\lambda_n,\dots,\lambda_1)$, holds for Macdonald polynomials (Macdonald, 1995, Ch. VI, §4); in the signature notation it reads $P_\lambda(X^{-1}) = P_{\lambda^*}(X)$ with $\lambda^* = -w_0\lambda$. Applying (1) to λ^* and substituting $X\mapsto X^{-1}$ gives the Down expansion with $c'_{\lambda\nu} = c_{\lambda^*\nu^*}\ge0$, and $\nu^*=\lambda^*+\square$ is the same as $\nu=\lambda-\square$. The rest is as in (1).

(3) Compose the two eigenrelations. Each step changes $|\lambda|$ by ±1, so K preserves $\mathcal{S}_c$.

(4) By (B.2), $M_r = \Xi(X)^{-r}F_{\lambda(r)}(X)$ is a martingale for the K-chain, so $F_\lambda(X) = \mathbb{E}_\lambda M_{\tau_D\wedge r}$ for every r. On $\{\tau_D\le r\}$, $M_{\tau_D\wedge r} = \Xi^{-\tau_D}F_{\lambda(\tau_D)}$ is bounded by the maximum of F over the finitely many states reachable in one step from D; on $\{\tau_D>r\}$, $\lambda(r)\in D$ and $M_r\le \Xi^{-r}\max_D F\to0$ because $\Xi(X)>1$. Dominated convergence as $r\to\infty$ gives (B.3). □

Remark B.2 (The Martin boundary as the finite-q HCIZ integral)**.** *Fix non-central X, so that P_X is transient, since $\sum_{r\ge 0}\mathsf{P}_X^r(\lambda,\nu)\le\sum_{r\ge 0}\Xi(X)^{-r}<\infty$. Whenever P_X is irreducible on $\mathcal{S}_c$, the minimal Martin boundary theory of transient chains (Woess, 2000, Chapter IV) represents the positive harmonic function of Proposition B.1(3) as*

$$F_\lambda^{(n)}(X;q,t)=\int_{\partial_M^{\min}}\mathsf{K}_X^{\mathrm{Martin}}(\lambda,\zeta)\,\nu_X(d\zeta) \tag{B.4}$$

for a positive measure ν_X on the minimal Martin boundary of P_X. This representation is on a fixed-size component and imposes no multiplicativity in λ, so it does not contradict Corollary 5.4, which concerns a single multiplicative representation for all partitions. Identifying the boundary, the kernel and the measure explicitly is, in this formulation, the finite-q HCIZ problem, and (B.4) *by itself says nothing about majorization. The walk does offer a coupling strategy, which we now describe.*

B.2. **An adjacent-row Up coupling reduces to two evaluation inequalities.** Call (λ,μ) a *root pair* if $\mu=\lambda+e_i-e_j$ for some $i<j$ with both signatures dominant; then $\mu\succeq\lambda$, and every pair $\mu\succeq\lambda$ is connected by a chain of root pairs. Suppose that for every root pair the Up steps from λ and from μ can be coupled so that the outcomes are equal or again a root pair, and likewise for the Down steps. Then the gaps $G(\lambda,\mu)=F_\mu-F_\lambda$ satisfy $\Xi\,G(\lambda,\mu)=\sum\tilde{\mathsf{K}}((\lambda,\mu),(\lambda',\mu'))G(\lambda',\mu')$, where $\tilde{\mathsf{K}}$ is the coupled kernel and the sum runs over the coupled outcomes; equal outcomes contribute zero. This positive recursion is the first step toward a maximum-principle argument for (M), and for each root pair the existence of the coupling is a finite question. We treat the Up coupling for $n=3$ and $\mu=\lambda+e_1-e_2$. The other root transfers, and the boundary control needed for a maximum principle, remain open.

For integers $A\ge 0$ and $B\ge 1$ put

$$\lambda=(A+B,B,0),\qquad \mu=(A+B+1,B-1,0),\qquad \Delta_{A,B}=P_{(A+B,B)}(1^3;q,t),$$

with $\Delta_{A,0}=P_{(A)}(1^3)$ (so $\Delta_{A,B}=\Delta_{(A+B,B)}$ in the notation of Appendix B.1), and let

$$\varpi(A,B)=\psi_2(A+B)\,\psi_1(B),\qquad \psi_h(g)=\frac{(1-q^{g-1}t^{h+1})(1-q^gt^{h-1})}{(1-q^gt^h)(1-q^{g-1}t^h)}, \tag{B.5}$$

which is the Pieri coefficient of $P_{(A+B,B,1)}$ in $p_1P_{(A+B,B,0)}$.

Proposition B.3 (Hall reduction for $n=3$)**.** *For the root pair (λ,μ) above, a coupling of the one-box Up steps from λ and from μ that sends every pair of outcomes to equal signatures or to a root pair exists if and only if*

$$L_{A,B}:=\Delta_{A+1,B}\Delta_{A+2,B-1}-\Delta_{A,B}\Delta_{A+3,B-1}\ge 0, \tag{B.6}$$

$$R_{A,B}:=\varpi(A,B)\,\Delta_{A,B-1}\Delta_{A+2,B-1}-\varpi(A+2,B-1)\,\Delta_{A+2,B-2}\Delta_{A,B}\ge 0, \tag{B.7}$$

where for $B=1$ the second term of $R_{A,1}$ is defined to be zero. The same conditions characterize the Down coupling for the dual root pair $(\lambda^*,\mu^*)=(-w_0\lambda,-w_0\mu)$.

Proof. Label the outcome of an Up step by the row $i\in\{1,2,3\}$ that receives the box, and let p_i and p_i' be the probabilities of outcome i from λ and from μ. The pairs of outcomes that are equal or root pairs are

$$1\longmapsto\{1,2\},\qquad 2\longmapsto\{2\},\qquad 3\longmapsto\{2,3\}$$

(rows of λ on the left, admissible rows of μ on the right; $\lambda+e_1=\mu+e_2$ is the equal pair), after deleting any outcome that has probability zero. A coupling supported on these pairs is a transport plan between p and p' with the given bipartite support, and by the supply–demand form of Hall's theorem (Strassen, 1965) it exists if and only if $p(S)\le p'(N(S))$ for every set S of rows, $N(S)$ being its neighborhood. The conditions for $S=\{2,3\}$ and $S=\{1,2\}$ read $p_1\ge p_1'$ and $p_3\ge p_3'$; given these two, the remaining conditions follow from $p_1+p_2+p_3=p_1'+p_2'+p_3'=1$.

The monic top Pieri coefficient and rectangular translation give

$$p_1 = \frac{\Delta_{A+1,B}}{3\Delta_{A,B}}, \qquad p_1' = \frac{\Delta_{A+3,B-1}}{3\Delta_{A+2,B-1}},$$

while the bottom Pieri coefficient (B.5) and translation by $-(1^3)$ give

$$p_3 = \frac{\varpi(A,B)\Delta_{A,B-1}}{3\Delta_{A,B}}, \qquad p_3' = \frac{\varpi(A+2,B-1)\Delta_{A+2,B-2}}{3\Delta_{A+2,B-1}} \quad (B \geq 2).$$

For $B = 1$ the row-3 outcome from μ is unavailable and $p_3' = 0$; for $A = 0$ the row-2 outcome from λ has probability zero and is deleted. Cross-multiplying $p_1 \geq p_1'$ and $p_3 \geq p_3'$ gives (B.6) and (B.7). The involution $\lambda \mapsto -w_0\lambda$ exchanges Up and Down steps and maps root pairs to root pairs. It therefore transports the Up coupling just proved to the Down coupling for (λ^*, μ^*). □

Remark B.4 (The case $n = 2$)**.** *For $n = 2$ the same reduction gives one condition. For $\lambda = (m, 0)$ and $\mu = (m+1, -1)$ the admissible pairs are $1 \mapsto \{1, 2\}$ and $2 \mapsto \{2\}$, and the condition $p_2 \leq p_2'$ reads $\Delta_{m+1}/(2\Delta_m) \geq \Delta_{m+3}/(2\Delta_{m+2})$ in the notation of (A.1), that is, $r_{m+1} \geq r_{m+3}$. Proposition A.1 proves this monotonicity for all $0 < q, t < 1$, so for $n = 2$ the Up coupling exists for every root pair, and inversion gives the Down coupling. For $n = 3$ the inequalities (B.6) and (B.7) are open (Remark B.6).*

Remark B.5 (Why rank-two log-concavity does not lift)**.** *The evaluations $\Delta_{A,B}$ are given by the branching rule as a double sum over rank-two evaluations. With w_m as in (3.1) and $g_m = (qt; q)_m/(t^2; q)_m$,*

$$\Delta_{A,B} = \sum_{i=0}^{A}\sum_{j=0}^{B} \frac{w_i w_{A-i}}{w_A}\,\frac{w_j w_{B-j}}{w_B}\,\frac{g_{B+i-j}\,g_{A+B}}{g_{A+B-j}\,g_{B+i}}\,\Delta_{B+i-j}. \tag{B.8}$$

Up to factors depending on i alone and on j alone, the summand is H_{B+i-j} with $H_m = g_m\Delta_m$. A family of kernels of the form $a(i)b(j)H(i-j)$, with positive a, b, is multivariate totally positive of order two (MTP_2) on all its rectangles exactly when the corresponding consecutive terms of H are log-concave (Karlin, 1968, Chapter 3), and

$$H_m^2 - H_{m-1}H_{m+1} = g_m^2\big[\Delta_m^2 - \omega_m\Delta_{m-1}\Delta_{m+1}\big], \qquad \omega_m - 1 = \frac{tq^{m-1}(1-q)(q-t)}{(1-q^m t)(1-q^m t^2)}. \tag{B.9}$$

For $t < q$ one has $\omega_m > 1$, so the log-concavity of Proposition A.1 is not enough: the MTP_2 property of the rank-three branching kernel is equivalent to

$$\Delta_m^2 \geq \omega_m\,\Delta_{m-1}\Delta_{m+1} \qquad (m \geq 1). \tag{B.10}$$

Inequality (B.10) is open; it holds in the tests of Remark B.6, including where $\omega_m > 1$. Whether it implies (B.6)–(B.7) through MTP_2 correlation inequalities is not settled either.

Remark B.6 (Numerical evidence)**.** *The inequalities $L_{A,B} \geq 0$ and $R_{A,B} \geq 0$ of Proposition B.3 hold at all 225 points $(q, t) \in \{i/16 : 1 \leq i \leq 15\}^2$ for the twelve pairs (A, B) with $0 \leq A \leq 3$ and $1 \leq B \leq 3$, and (B.10) holds at the same points for $m \leq 7$, including where $t < q$ and $\omega_m > 1$. These checks use exact rational arithmetic at each point; they do not prove the inequalities for all (q, t). The identity (B.8) was checked exactly, as an identity in (q, t), for $0 \leq A \leq 3$ and $1 \leq B \leq 3$.*

Appendix C. The principal-specialization normalization fails

Okounkov's q-integral formula (Okounkov, 1998) and the finite-q lattice results of McSwiggen and Sahi (2026b,a) are normalized at the principal specialization rather than at 1^n. This appendix shows that the principal-specialization ratio does *not* satisfy the analogue of (M), and why. Put

$$\Omega_\lambda(x; q, t) := \frac{P_\lambda(x; q, t)}{P_\lambda(t^\delta; q, t)}, \qquad t^\delta = (t^{n-1}, t^{n-2}, \ldots, t, 1),$$

and let

$$L_n^{q,t,a} := \{(aq^{-\mu_1}t^{n-1},\, aq^{-\mu_2}t^{n-2}, \dots, aq^{-\mu_n}) : \mu_1 \geq \cdots \geq \mu_n,\ \mu_i \in \mathbb{Z}\}$$

be the scaled dominant q-lattice of McSwiggen and Sahi (2026b), on which Theorem 2.1 of that paper asserts that $\lambda \mapsto \Omega_\lambda(x; q, t)$ is log-convex and Schur-convex. Neither holds at the central point 1^2, which lies on the lattice whenever $t = q^k$ with $k \in \mathbb{Z}_{\geq 1}$ (Lemma C.2).

C.1. **The determinant shift acquires a factor.** Under the 1^n normalization, $P_\lambda(1^N) = P_{\lambda - c(1^N)}(1^N)$ gives $F_\lambda^{(N)}(y) = (y_1 \cdots y_N)^c F_{\lambda - c(1^N)}^{(N)}(y)$ for $c = \lambda_N$. Under the principal normalization the same shift acquires a factor.

Lemma C.1 (The corrected shift)**.** *Let λ have N parts and put $c = \lambda_N$. Then*

$$\Omega_\lambda^{(N)}(y; q, t) = \left(\frac{y_1 \cdots y_N}{t^{N(N-1)/2}}\right)^c \Omega_{\lambda - c(1^N)}^{(N)}(y; q, t).$$

Proof. From $P_\lambda(y) = (y_1 \cdots y_N)^c P_{\lambda - c(1^N)}(y)$, evaluation at t^δ gives $P_\lambda(t^\delta) = t^{cN(N-1)/2} P_{\lambda - c(1^N)}(t^\delta)$; divide. □

For example, $P_{(1,1)}(y_1, y_2) = y_1 y_2$ and $t^\delta = (t, 1)$ give $\Omega_{(1,1)}(y_1, y_2; q, t) = y_1 y_2 / t$, so the identity $\Omega_{\lambda + (1^n)} = e_n \Omega_\lambda$ fails for $n = 2$ unless $t = 1$.

C.2. **A counterexample for $n = 2$.**

Lemma C.2 (When the central point lies on the lattice)**.** *For $n = 2$, $(1, 1) \in L_2^{q,t,a}$ for some a if and only if $t = q^k$ for an integer $k \geq 1$. More generally, if $t = q^k$ then $1^n \in L_n^{q,q^k,1}$, by $\mu_i = k(n - i)$.*

Proof. Matching coordinates, $aq^{-\mu_2} = 1$ forces $a = q^{\mu_2}$, and then $aq^{-\mu_1}t = 1$ forces $t = q^{\mu_1 - \mu_2}$ with $\mu_1 \geq \mu_2$; $0 < t < 1$ makes the exponent at least one. Conversely $q^{-k(n-i)}t^{n-i} = 1$ when $t = q^k$. □

Proposition C.3 (Ω-majorization fails on the lattice)**.** *Let $0 < q, t < 1$ and $n = 2$. Then $(2, 0) \succeq (1, 1)$, yet*

$$\Omega_{(1,1)}(1, 1; q, t) - \Omega_{(2,0)}(1, 1; q, t) = \frac{(1 - t)^2(1 - qt)}{t(1 + t)(1 - qt^2)} > 0. \tag{C.1}$$

If $t = q^k$ with $k \in \mathbb{Z}_{\geq 1}$, then $(1, 1) \in L_2^{q,t,1}$ by Lemma C.2, and (C.1) contradicts the Schur-convexity of $\lambda \mapsto \Omega_\lambda(x; q, t)$ on the lattice.

Proof. In two variables $P_{(1,1)}(x_1, x_2) = x_1 x_2$ and $P_{(2,0)}(x_1, x_2) = x_1^2 + x_2^2 + A x_1 x_2$ with $A = (1 - t)(1 + q)/(1 - qt)$. Hence $\Omega_{(1,1)}(1, 1) = 1/t$ and $\Omega_{(2,0)}(1, 1) = (2 + A)/(1 + t^2 + At)$, so the difference is

$$\frac{1 + t^2 + At - t(2 + A)}{t(1 + t^2 + At)} = \frac{(1 - t)^2}{t(1 + t^2 + At)},$$

the At terms cancelling. Since $1 + t^2 + At = (1 + t)(1 - qt^2)/(1 - qt)$, the difference is (C.1), and it is positive for $0 < q, t < 1$. □

At $q = t = r$ the polynomials are Schur, $(1, 1) \in L_2^{r,r,1}$, and (C.1) reduces to $1/r - 3/(1 + r + r^2) = (1 - r)^2/(r(1 + r + r^2))$; at $r = 1/2$ the two values are $\Omega_{(1,1)} = 2$ and $\Omega_{(2,0)} = 12/7$. Off the Schur line, at $q = 1/2$, $t = q^2 = 1/4$, one has $A = 9/7$ and $\Omega_{(2,0)}(1, 1) = 368/155 < 4 = \Omega_{(1,1)}(1, 1)$.

Remark C.4 (Log-convexity fails as well)**.** *Along the one-row ray, $\Omega_{(0)}(1, 1) = 1$, $\Omega_{(1)}(1, 1) = 2/(1 + t)$ and $\Omega_{(2)}(1, 1) = (2 + A)/(1 + t^2 + At)$, and the same algebra gives*

$$\Omega_{(1)}(1, 1)^2 - \Omega_{(0)}(1, 1)\,\Omega_{(2)}(1, 1) = \frac{(1 - q)(1 - t)^2}{(1 + t)^2(1 - qt^2)} > 0. \tag{C.2}$$

So $m \mapsto \Omega_{(m)}(1,1;q,t)$ is not log-convex at $m=1$, for any $0<q,t<1$. In particular $\lambda \mapsto \Omega_\lambda(1^2;q,t)$ cannot be a positive exponential sum $\sum_v c_v q^{\langle \lambda, v\rangle}$ in λ, since such sums are log-convex along every line. At $q=t=1/2$ the three values are 1, $4/3$, $12/7$, and $(4/3)^2 = 16/9 > 12/7$.

C.3. **Why this leaves conjecture (M) untouched.** The anchored ratio satisfies $F^{(n)}_\lambda(1^n;q,t)=1$ for every λ, and in the smallest case

$$F^{(2)}_{(2,0)}(x;q,t) - F^{(2)}_{(1,1)}(x;q,t) = \frac{x_1^2+x_2^2+Ax_1x_2-(2+A)x_1x_2}{2+A} = \frac{(x_1-x_2)^2}{2+A} \;\geq\; 0.$$

So the anchored inequality holds, for every x, in the case where the Ω-inequality fails. The difference is the factor $t^{-cN(N-1)/2}$ of Lemma C.1, which the 1^n normalization cancels. Nothing in this appendix bears on (M) itself; it explains why the finite-q results for Ω do not transfer to (M).

Appendix D. Relation to the work of McSwiggen and Sahi

Two recent preprints of McSwiggen and Sahi bear on (M); we state what they prove and how it relates to the anchored problem.

D.1. **The Opdam–Cherednik kernel and the Jack case.** McSwiggen and Sahi (2026a) prove (their Theorem 3.2) that for a finite root system with nonnegative multiplicity k there is a probability measure $\pi_{k,x}$, supported on $\mathrm{conv}(Wx)$, such that the nonsymmetric Opdam–Cherednik kernel is its Laplace transform,

$$G_{k,s-\rho}(x) = \int e^{\langle s,\, y\rangle}\, \pi_{k,x}(dy). \tag{D.1}$$

Symmetrizing over the Weyl group gives log-convexity of the Heckman–Opdam hypergeometric function in its spectral variable and hence Weyl-majorization inequalities for it (their Theorem 4.8), for all root systems and all real x. In type A, their Theorem 4.8 is the Schur-convexity of $s \mapsto \mathcal{F}_\theta(z,s)$ used in the proof of Theorem 1.1, obtained from a different positive representation; combined with (2.13) it gives the Jack inequality. The paper's finite-q statements (Proposition 4.3 and Theorem 4.4) concern the Macdonald polynomial normalized as $\mathtt{P}_\lambda = P_\lambda/P_\lambda(-\rho)$, where evaluation of e^ν at x means $q^{\langle \nu, x\rangle}$; in type A, $\mathtt{P}_\lambda$ is a principal-specialization normalization, up to the central shift and reversal of conventions. They compare $\mathtt{P}_\lambda(\bar\mu)$ at shifted lattice points $\bar\mu = \mu - w_\mu^{-1}(\rho)$, using duality and the positive Ram–Yip expansion, and they do not compare $P_\lambda(X)/P_\lambda(1^n)$ for arbitrary $X \in \mathbb{R}^n_{>0}$. Their positive representation (D.1) lives at $q=1$; it is not a finite-q representation of the anchored ratio (5.3), and the Hankel obstruction of Proposition 5.3 does not bear on it.

D.2. **The lattice theorem of arXiv:2605.12680.** McSwiggen and Sahi (2026b) work with the principal-specialization normalization Ω_λ of Appendix C and assert (their Theorem 2.1) that for all $q,t \in (0,1)$ and $a>0$, $\lambda \mapsto \Omega_\lambda(x;q,t)$ is log-convex and Schur-convex for every x on the lattice $L_n^{q,t,a}$. Both assertions fail at the central point: for $t=q^k$, $1^2 \in L_2^{q,t,1}$ (Lemma C.2), and there $\Omega_{(2,0)} < \Omega_{(1,1)}$ (Proposition C.3) while $\Omega_{(1)}^2 > \Omega_{(0)}\Omega_{(2)}$ (Remark C.4). Their Remark 2.2 notes, citing Chen et al. (2025, Example 5.1), that Schur-convexity of $\Omega_\lambda(x)$ fails for some positive x off the lattice; the point here is that both properties fail on the lattice itself. The proof of their Theorem 2.1 passes from the interlacing sum (their (2.1)) to a positive exponential sum (their (2.3)) by the identity $\Omega^{(N)}_\lambda(y) = (y_1\cdots y_N)^{\lambda_N}\,\Omega^{(N)}_{\lambda-\lambda_N(1^N)}(y)$, which by Lemma C.1 holds only with the additional factor $t^{-\lambda_N N(N-1)/2}$; we have not traced every step at which the argument fails at central points, but Remark C.4 rules out any representation of $\lambda \mapsto \Omega_\lambda(1^2)$ as a positive exponential sum.

In that paper, Corollary 2.4 (the Jack inequality) and the implication (1)⇒(2) of Theorem 2.5 are deduced from Theorem 2.1. The Jack statement is nonetheless correct: it is Theorem 1.1 here, it follows from McSwiggen and Sahi (2026a, Theorem 4.8) through (2.13), and it follows from Theorem 2.8 of McSwiggen and Sahi (2026b) itself, whose proof does not use Theorem 2.1. The

counterexample does not bear on McSwiggen and Sahi (2026a), whose type-A evaluation points $\bar{\nu} \in W_0(\nu + \rho)$ have pairwise distinct coordinates and are never central.

AI USAGE

The results in this paper were developed through extensive use of an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs.

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