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# THE $q$-FOULKES PROBLEM III: ON FOULKES' CONJECTURE WITH A GRADED INNER FACTOR

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ABSTRACT. We show that surjectivity of the canonical map $\mathrm{Sym}^b(\mathrm{Sym}^a V) \to \mathrm{Sym}^a(\mathrm{Sym}^b V)$ survives grading of the inner factor. Consequently $h_b[H_a] - H_a[h_b]$ is Schur positive whenever the canonical map is surjective, where $H_a$ is the (modified) Hall–Littlewood polynomial; the same holds for (modified) Macdonald polynomials. We prove this surjectivity for $b \geq (a-1)^2$, improving the known bound $a(a-1)$. For Bergeron's $q$-analog of Foulkes' conjecture that lets both factors be graded, we prove nonnegativity of every Schur coefficient for certain special cases. We also study the full conjecture under an expansion hypothesis on an explicit space of maps.

## 1. INTRODUCTION

We study the $q$-analog of Foulkes' conjecture proposed by Bergeron (2017). Recall that Foulkes's conjecture (1950) states that the difference of plethysms $h_b[h_a] - h_a[h_b]$ is Schur positive for $a \leq b$, i.e., it expands with nonnegative integer coefficients in the Schur functions $s_\lambda$, $\lambda \vdash ab$; here $h_n$ is the complete homogeneous symmetric function. This positivity is known for $a \leq 6$ (Thrall, 1942; Dent and Siemons, 2000; McKay, 2008; Cheung et al., 2017; OpenAI, 2026a) and for $b$ large with respect to $a$ (Brion, 1993), but it remains open in general.

Bergeron's $q$-analog replaces $h_n$ by the (modified) Hall–Littlewood polynomial

$$H_n(X;q) := (q;q)_n h_n\left[\tfrac{X}{1-q}\right], \quad \text{where} \quad (q;q)_n := \textstyle\prod_{j=1}^{n}(1-q^j),$$

and his analog of Foulkes' conjecture is the following:

**Conjecture 1.1** (Bergeron, 2017)**.** *For all integers $a \leq b$, the divided difference*

$$F_{a,b} := \frac{H_b[H_a] - H_a[H_b]}{1-q}$$

*is Schur positive with coefficients in $\mathbb{N}[q]$.*

Since $H_n(X;1) = h_1^n$, both plethysms equal $h_1^{ab}$ at $q=1$, so the numerator of $F_{a,b}$ is divisible by $1-q$; since $H_n(X;0) = h_n$, the conjecture at $q=0$ reduces to Foulkes' conjecture. Bergeron proved the conjecture at $q=1$ and showed that Foulkes' inequality also orders the coefficients of $q^1$ (Bergeron, 2017, Section 3), so the known cases control only the two lowest powers of $q$.

Recall that $h_b[h_a]$ and $h_a[h_b]$ are characters of the domain and range of the *Foulkes–Howe map*

$$\mathrm{Sym}^b(\mathrm{Sym}^a V) \longrightarrow \mathrm{Sym}^a(\mathrm{Sym}^b V), \tag{1.1}$$

so Foulkes' conjecture follows whenever this map is surjective.

Our first main result shows that this approach survives when the inner factor is graded. The polynomial $H_a$ is the graded Frobenius character of the coinvariant algebra $R_a$ of $S_a$, so replacing

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$\mathrm{Sym}^a W$ by $\mathcal{H}_a(W) = (R_a \otimes W^{\otimes a})^{S_a}$ grades the inner factor. Multiplication then maps $\mathrm{Sym}^b(\mathcal{H}_a(V))$ to $\mathcal{H}_a(\mathrm{Sym}^b V)$, and in coinvariant degree zero this map is (1.1).

**Theorem 1.2** (Graded lift)**.** *Suppose the canonical map (1.1) is surjective for every finite-dimensional complex $V$, for a fixed pair $a, b \geq 1$. Then multiplication gives a natural degree-preserving surjection*

$$\mathrm{Sym}^b(\mathcal{H}_a(V)) \longrightarrow \mathcal{H}_a(\mathrm{Sym}^b V) \tag{1.2}$$

*for every $V$. Consequently*

$$h_b[H_a] - H_a[h_b] \in \mathbb{N}[q]\{s_\lambda\}. \tag{1.3}$$

The proof (Section 2.3) uses derivations that multiply the $i$th tensor row by a power of the $i$th coinvariant variable. They preserve the image of multiplication and carry degree zero to every coinvariant degree, so surjectivity propagates from degree zero. The argument needs only that $R_a$ is generated by variables that $S_a$ permutes, so it applies equally to the Garsia–Haiman modules, whose characters are the modified Macdonald polynomials $\widetilde{H}_\mu$, $\mu \vdash a$. Under the same hypothesis, therefore, $h_b[\widetilde{H}_\mu] - \widetilde{H}_\mu[h_b] \in \mathbb{N}[q,t]\{s_\lambda\}$ (Corollary 2.4).

Theorem 1.2 assumes that (1.1) is surjective, which the annihilator argument of OpenAI (2026b) proves for $b \geq a(a-1)$. Our second main result slightly improves their bound.

**Theorem 1.3** (Surjectivity)**.** *For every $a \geq 2$, every integer $b \geq (a-1)^2$, and every finite-dimensional complex vector space $V$, the map (1.1) is surjective. Consequently $h_b[h_a] - h_a[h_b]$ is Schur positive in this range.*

The annihilator argument shows that every symmetric form killing the image vanishes, by polarizing one variable at a time. We polarize by whole powers of an operator instead. A whole power stays injective in a wider range of degrees than its factors checked one by one (Lemma 2.1), and this lowers the bound from $a(a-1)$ to $(a-1)^2$. Together with Theorem 1.2, Theorem 1.3 makes the graded lift unconditional, so (1.3) holds for every $b \geq (a-1)^2$ (Corollary 2.2).

Our third main result concerns Bergeron's $F_{a,b}$ itself. The lift does not extend to a graded outer factor (Section 4), so we compute instead. The depth of $\lambda$ is $r = ab - \lambda_1$, the number of boxes below its first row. The coefficients of depth $r$ depend on $b$ only through a polynomial of degree at most $r$ in $q^b$ (Theorem 3.1). We expand this polynomial in Gaussian binomial coefficients in $b$, which have nonnegative coefficients and increase with $b$. This Gaussian Newton expansion therefore proves positivity and monotonicity for every $b$ at once whenever its coefficients are nonnegative.

**Theorem 1.4** (Positivity at bounded depth)**.** *For every $b \geq a \geq 4$, every Schur coefficient of $F_{a,b}$ of depth at most six belongs to $\mathbb{N}[q]$. The same conclusion holds through depth seven if $a \geq 5$, through depth eight if $a \geq 6$, and through depth ten if $a = 7$. In each of these ranges, the Gaussian Newton coefficients based at $b = a$ are nonnegative, and for each fixed partition below the first row the coefficient increases coefficientwise with $b$.*

The inner size $a$ is unbounded as well, and a second Gaussian Newton expansion in $q^a$ reduces each range to finitely many identities that are exact over $\mathbb{Q}(q)$ (Appendix A). For each fixed partition below the first row, the monotonicity is Bergeron's $q$-stability conjecture (Bergeron, 2017, Conjecture 2). Corollary A.2 proves it for every $a \geq 2$ at depth three, where the three coefficients are multiples of one explicit polynomial.

Finally, Section 4 returns to maps. With both factors graded, the relations among the coefficients depend on how the $ab$ positions are grouped into blocks, and extending the canonical map polynomially does not respect them (Proposition B.3). Projecting onto harmonic polynomials avoids this

obstruction, giving a space of equivariant maps that never lower degree. It remains open whether these maps together expand every graded subspace; if they do, the conjecture follows (Theorem 4.1).

1.1. **Conventions.** We work over $\mathbb{C}$. Plethysm acts on every grading variable through the Adams operations $\psi_r(p_j) = p_{rj}$, $\psi_r(q) = q^r$, and $\psi_r(t) = t^r$. Set

$$[n]_q = \frac{1-q^n}{1-q}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \quad (0 \le k \le n),$$

with $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $k > n$. Unless stated otherwise, $a, b \ge 2$ are integers and $N = ab$. For $\mu = 1^{m_1} 2^{m_2} \cdots$, put $z_\mu = \prod_r r^{m_r} m_r!$. For $\lambda \vdash N$, $S^\lambda$ is the irreducible $S_N$-module, $\chi^\lambda$ its character, and $d_\lambda = \dim S^\lambda$; for a partition $\mu$, put $n(\mu) = \sum_i (i-1)\mu_i$. For a semiring $K$, $K\{s_\lambda\}$ is the set of symmetric functions whose Schur coefficients lie in $K$.

## 2. The canonical map and its graded lift

Theorem 1.3 supplies the surjectivity that Theorem 1.2 assumes, so we prove it first. Both proofs view the canonical map (1.1) as arranging the tensor positions in a rectangle and averaging over the symmetry groups of its codomain, where we identify a symmetric power with invariant tensors by

$$\mathrm{Av}_a(w_1 \otimes \cdots \otimes w_a) = \frac{1}{a!} \sum\nolimits_{\sigma \in S_a} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(a)}.$$

In this picture the image of (1.1) is the degree-$b$ part of the subalgebra of $((\operatorname{Sym} V)^{\otimes a})^{S_a}$ generated by $\operatorname{Sym}^a V$.

2.1. **Whole powers of polarization.** The proof of Theorem 1.3 keeps the annihilator construction of OpenAI (2026b) and replaces its successive first derivatives by whole powers of polarization operators. The injectivity of these powers is the $\mathfrak{sl}_2$ argument in the proof of OpenAI (2026a, Lemma 2.5); the following lemma records it with a norm bound.

For independent vector variables $U = (U_i)$ and $Y = (Y_i)$, put

$$D = D_{Y,U} := \sum\nolimits_i Y_i \frac{\partial}{\partial U_i}, \qquad E = \sum\nolimits_i U_i \frac{\partial}{\partial Y_i}, \qquad H = N_U - N_Y,$$

where $N_U$ and $N_Y$ count degrees in $U$ and $Y$. We use the Fischer inner product, for which monomials are orthogonal and $\|U^\alpha Y^\beta\|^2 = \alpha!\,\beta!$, also for spectator variables. Then $D^* = E$, $[E, D] = H$, and $[H, D] = -2D$, so each application of $D$ lowers by two the weight $H$, which is the excess of the degree in $U$ over the degree in $Y$. When the variables have at least two coordinates, $D$ alone is injective on a space of fixed bidegree only if the weight is positive. If we checked $D^L$ one factor at a time, we would therefore need weight at least $2L-1$, whereas on whole $\mathfrak{sl}_2$ strings weight $L$ suffices. Write $x^{\underline{L}} = x(x-1)\cdots(x-L+1)$, with empty product one.

**Lemma 2.1** (Power polarization)**.** *Let $f$ have bidegree $(p, s)$ in $(U, Y)$, with arbitrary spectator variables. If $L \ge 0$ and $m = p - s \ge L$, then*

$$\|D^L f\|^2 \ge L!\, m^{\underline{L}}\, \|f\|^2. \tag{2.1}$$

*In particular, $D^L$ is injective on this bidegree space, including the boundary $p - s = L$.*

*Proof.* Decompose each space of polynomials of fixed total degree into orthogonal $\mathfrak{sl}_2$ strings. In a string of highest weight $\ell$, a vector of weight $m$ is at position $j = (\ell - m)/2 \ge 0$. The squared norm multiplier for $D^L$ is

$$\prod\nolimits_{u=0}^{L-1} (j+u+1)(\ell - j - u) = \prod\nolimits_{u=0}^{L-1} (j+u+1)(m+j-u).$$

Since $m \geq L$, this multiplier is at least $\prod_{u=0}^{L-1}(u+1)(m-u) = L!\, m^{\underline{L}} > 0$ when $L > 0$. For $L = 0$ the assertion is the identity. Orthogonality gives the estimate for an arbitrary sum of strings. Spectator variables do not alter the calculation. □

For $L > 0$, the threshold in this lemma is sharp on the full bidegree space when $\dim V \geq 2$. If $0 \leq p - s < L$, then $D^L$ kills the nonzero polynomial

$$(U_1Y_2 - U_2Y_1)^s U_1^{p-s}.$$

If $p < s$, then $D$ already kills $(U_1Y_2 - U_2Y_1)^p Y_1^{s-p}$.

2.2. **The annihilator argument.** Linear functionals on the codomain of (1.1) correspond to symmetric forms on $\mathrm{Sym}^b V$. A functional vanishes on the image exactly when its form vanishes on the diagonal of every product of $b$ vectors, so surjectivity means that every such form is zero. Polarization carries the vanishing from these special arguments to arbitrary ones, one argument at a time. The obstacle is a degree count. Polarization is injective only while the differentiated variable has larger degree than the inserted one, and the decisive step uses up this excess, polarizing $r = a - 1$ arguments by blocks of $r$ derivatives each. A block applied as a whole power needs an excess of only $r$, rather than $2r - 1$, so Lemma 2.1 applies to every block exactly when $b \geq r^2$.

*Proof of Theorem 1.3.* The case $\dim V = 0$ is immediate. We induct on $a$, allowing $a = 1$ and every $b \geq 0$ as the identity base case. The case $(a, b) = (2, 1)$ is also an identity. Otherwise put

$$r = a - 1, \qquad m = b - r.$$

Thus $m \geq 1$ and $b \geq r^2$.

***Forms vanishing on products.*** Pure powers $v^a$ span $\mathrm{Sym}^a V$. Therefore the image of (1.1) is spanned by $P^{\otimes a}$, where $P = u_1 \cdots u_b$ is a product of $b$ vectors: averaging sends $u^a$ to $u^{\otimes a}$, and multiplication sends $u_1^{\otimes a} \cdots u_b^{\otimes a}$ to $P^{\otimes a}$. A functional $\ell$ on the codomain corresponds to the symmetric form $T(w_1, \ldots, w_a) = \ell(\mathrm{Av}_a(w_1 \otimes \cdots \otimes w_a))$. Thus the annihilator of the image consists exactly of the forms satisfying

$$T(P, \ldots, P) = 0 \quad \text{for every such product } P. \tag{2.2}$$

We prove that every such $T$ is zero.

***Freeing the last argument.*** To isolate one argument of $T$, we split a product into $r$ equal factors and one different one. Fix $Q = y_1 \cdots y_m$. The binary form $x^r + ct^r$ factors into $r$ linear factors over $\mathbb{C}$, so $(x^r + ct^r)Q$ is a product of $b$ vectors. Taking the coefficient of $c$ in (2.2) yields

$$T(x^rQ, \ldots, x^rQ, t^rQ) = 0,$$

with $r$ copies of $x^rQ$; the coefficient is $a$ times the displayed expression, and $a \neq 0$ over $\mathbb{C}$.

To replace $t^rQ$ by $t^b$ in the last argument, define

$$F_x(y_1, \ldots, y_m, t) = T(x^rQ, \ldots, x^rQ, t^b).$$

It has degree $b$ in $t$ and degree $r$ in each $y_i$. Moreover,

$$D_{y_m,t} \cdots D_{y_1,t} F_x = b^{\underline{m}}\, T(x^rQ, \ldots, x^rQ, t^rQ) = 0.$$

Before applying the $i$th operator, the relevant degrees are $p = b - i + 1$ and $s = r$. Hence $p - s = m - i + 1 \geq 1$. Lemma 2.1 with $L = 1$ makes every step injective, so

$$T(x^rQ, \ldots, x^rQ, t^b) = 0 \quad \text{for all products } Q = y_1 \cdots y_m.$$

***The other arguments, by induction.*** For fixed $x, t$, the left side is the diagonal of a symmetric $r$-linear form on $\mathrm{Sym}^m V$. The induction hypothesis applies, because

$$m = b - r \geq r^2 - r \geq (r - 1)^2.$$

For $r = 1$ the induction uses only the identity base case. It follows that

$$T(x^r Q_1, \dots, x^r Q_r, t^b) = 0 \quad \text{for all } Q_i \in \operatorname{Sym}^m V. \tag{2.3}$$

***Whole powers of polarization.*** It remains to replace each $x^r Q_i$ by an arbitrary pure power. Set $f = T(v_1^b, \dots, v_r^b, t^b)$, and introduce an independent variable $x$ of initial degree zero. Apply, in order, the $r$ operators $(D_{x,v_i})^r$. Before the $i$th block of $r$ derivatives, $v_i$ has degree $b$ and $x$ has degree $(i-1)r$. Consequently

$$p - s = b - (i-1)r \geq r,$$

so every whole block is injective by Lemma 2.1. Their composition is

$$(b^{\underline{r}})^r \, T(x^r v_1^m, \dots, x^r v_r^m, t^b) = 0$$

by (2.3). Thus $f = 0$. Pure powers span each copy of $\operatorname{Sym}^b V$, so $T = 0$, and the map is surjective. Complete reducibility gives the stated Schur positivity. □

2.3. **The graded lift.** Theorem 1.2 transfers surjectivity from coinvariant degree zero to every degree. Write the coinvariant algebra as $R_a = \mathbb{C}[\xi_1, \dots, \xi_a]/(e_1(\xi), \dots, e_a(\xi))$, so that $\mathcal{H}_a(W) = (R_a \otimes W^{\otimes a})^{S_a}$. The difficulty is that the hypothesis says nothing in positive degree, and a single variable $\xi_j$ cannot be multiplied in, because it is not invariant under $S_a$. The transfer uses derivations that pair $\xi_i$ with the $i$th row instead. They commute with $S_a$, preserve the image of multiplication, and act on the coefficient of a suitable vector as multiplication by $\xi_j$.

*Proof of Theorem 1.2.* ***One algebra for both sides.*** Form the Segre algebra

$$C(V) = \bigoplus_{d \geq 0} (\operatorname{Sym}^d V)^{\otimes a}, \qquad B(V) = (R_a \otimes C(V))^{S_a},$$

with componentwise multiplication and simultaneous permutation of the $a$ rows and coefficient variables. Write $B_d$ for Segre degree $d$. Thus $B_1 = \mathcal{H}_a(V)$ and $B_b = \mathcal{H}_a(\operatorname{Sym}^b V)$. Let $A \subseteq B$ be the subalgebra generated by $B_1$. Multiplication gives (1.2), with image $A_b$. In coinvariant degree zero it is (1.1).

***Current operators.*** Choose basis variables $x_{i,u}$ for the $i$th factor of $\operatorname{Sym} V$. For basis labels $u, v$ and $r \geq 0$, define a derivation

$$\mathcal{D}_{uv,r} = \sum\nolimits_{i=1}^{a} \xi_i^r x_{i,u} \frac{\partial}{\partial x_{i,v}}.$$

It descends modulo the coinvariant ideal, since it differentiates only the $x$ variables. It preserves the degree in each row and commutes with the simultaneous $S_a$ action. Hence it acts on every $B_d$, preserves $B_1$, and raises the coinvariant degree by $r$. The Leibniz rule shows that it preserves $A$ and every $A_d$. We call these derivations current operators, because $E_{uv} \otimes \xi^r \mapsto \mathcal{D}_{uv,r}$ is a representation of the current algebra $\mathfrak{gl}(V) \otimes \mathbb{C}[\xi]$.

***Reduction to one weight space.*** A diagonal current singles out one row only when the label it acts on occurs once, so we reduce to the multilinear weight. For each fixed coinvariant degree, the multiplication map is a natural transformation of homogeneous polynomial functors of degree $N = ab$. Its cokernel is therefore a direct sum of Schur functors $S_\lambda$, with $\lambda \vdash N$, tensored with fixed multiplicity spaces. Each $S_\lambda(\mathbb{C}^N)$ has a nonzero multilinear weight space, isomorphic to the Specht module $S^\lambda$, and taking a weight space is exact. It therefore suffices to show that the cokernel vanishes on the multilinear weight space $(1, \dots, 1)$ of $V = \mathbb{C}^N$; the cokernel functor then vanishes in every dimension, both below and above $N$.

***Orbit vectors.*** At this weight, the monomials of $C_b$ are indexed by ordered partitions of the $N$ basis labels into $a$ blocks of size $b$. Order each unordered partition $P$ as $(P_1, \dots, P_a)$, and put

$$m_P = \prod\nolimits_{i=1}^{a} \prod\nolimits_{\ell \in P_i} x_{i,\ell}, \qquad w_{P,f} = \sum\nolimits_{\sigma \in S_a} \sigma(f \otimes m_P), \quad f \in R_a.$$

Since $b \geq 1$, the orbit of the ordered monomial is free. Its invariant coefficient space is therefore a copy of $R_a$: the coefficient of $m_P$ in $w_{P,f}$ recovers $f$. The vector $w_{P,1}$ lies in $A_b$, because its coinvariant degree is zero, and in degree zero $B_1 = (V^{\otimes a})^{S_a} = \mathrm{Sym}^a V$ and multiplication is the canonical map, which is surjective by hypothesis.

***Currents generate the coefficients.*** Choose one label $\ell_j \in P_j$ for each block. In each orbit summand, this label occurs exactly once, in the row carrying the $j$th block. Consequently

$$\mathcal{D}_{\ell_j \ell_j, 1} w_{P,f} = w_{P,\xi_j f},$$

so the diagonal current $\mathcal{D}_{\ell_j \ell_j, 1}$ multiplies the coefficient by $\xi_j$. Iterating these commuting currents generates $w_{P,\xi_1^{n_1} \cdots \xi_a^{n_a}}$ from $w_{P,1}$, and the monomials span $R_a$. Since $A_b$ is stable under the currents, it contains the entire coefficient space over $P$. Summing over $P$ proves surjectivity in every coinvariant degree on the multilinear weight space.

Finally the domain and codomain characters are $h_b[H_a]$ and $H_a[h_b]$. Their difference is the graded character of the kernel of (1.2), proving (1.3). □

Theorem 1.3 removes the hypothesis of Theorem 1.2 in its range.

**Corollary 2.2.** *For every $a \geq 2$ and $b \geq (a-1)^2$,*

$$h_b[H_a] - H_a[h_b] \in \mathbb{N}[q]\{s_\lambda\}.$$

*Proof.* Combine Theorems 1.3 and 1.2. □

The proof of Theorem 1.2 used only that $R_a$ is generated by variables of one degree that $S_a$ permutes. Quotients generated by several such lists give the same lift, and this brings the modified Macdonald polynomials within reach.

**Proposition 2.3** (Other coefficient algebras)**.** *Let $C$ be a finite-dimensional nonnegatively multigraded quotient of $\mathbb{C}[\xi_{i,r} : 1 \leq i \leq a,\ 1 \leq r \leq d]$ by a homogeneous $S_a$-stable ideal, where the generators in each list have a common multidegree, $S_a$ permutes $i$, and $C_0 = \mathbb{C}$. Put $\mathcal{C}(W) = (C \otimes W^{\otimes a})^{S_a}$. If the canonical map (1.1) is surjective for every finite-dimensional $V$, then multiplication induces a natural multigrading-preserving surjection*

$$\mathrm{Sym}^b(\mathcal{C}(V)) \longrightarrow \mathcal{C}(\mathrm{Sym}^b V).$$

*Proof.* Use $C$ in place of $R_a$ in the proof of Theorem 1.2. For every generator list, the current $\sum_i \xi_{i,r} x_{i,u} \partial_{x_{i,v}}$ descends to the quotient, commutes with $S_a$, and preserves the subalgebra generated in Segre degree one. On the free orbit over an unordered block partition, its diagonal version multiplies the coefficient by the corresponding $\xi_{j,r}$. Monomials in these generators span every homogeneous component of $C$. The degree-zero orbit vector therefore generates its entire coefficient space inside the image. The reduction to one weight space then gives the asserted surjection in every dimension. □

The Garsia–Haiman modules are quotients of this kind, and Haiman's $n!$ theorem identifies their characters with the modified Macdonald polynomials, which we normalize as

$$\widetilde{H}_\mu[X; q, t] = t^{n(\mu)} J_\mu \left[ \frac{X}{1 - t^{-1}}; q, t^{-1} \right],$$

where $J_\mu$ is the integral Macdonald form. This normalization is the one of Bergeron et al. (1999) and Haiman (2001); see also Macdonald (1995, Chapter VI, (8.1') and (8.3)). In particular $\widetilde{H}_{(n)}(X; q, t) = H_n(X; q)$.

**Corollary 2.4** (A bigraded Macdonald lift)**.** *Let $\mu \vdash a$, $a \geq 2$, and $b \geq (a-1)^2$. There is a natural bigrading-preserving surjection between polynomial functors with characters $h_b[\widetilde{H}_\mu]$ and $\widetilde{H}_\mu[h_b]$. Consequently*

$$h_b[\widetilde{H}_\mu(X;q,t)] - \widetilde{H}_\mu[h_b(X);q,t] \in \mathbb{N}[q,t]\{s_\lambda\}.$$

*Both grading variables undergo Adams substitution in the outer plethysm, and $h_b$ is undeformed.*

*Proof.* For the cells $(u_j, v_j)$ of $\mu$, with zero-based column and row coordinates, let $\Delta_\mu = \det(x_i^{u_j} y_i^{v_j})_{1\leq i,j\leq a}$ and use the Garsia–Haiman quotient

$$C_\mu = \mathbb{C}[x_1,\dots,x_a,y_1,\dots,y_a]/I_\mu, \quad I_\mu = \{f : f(\partial_x,\partial_y)\Delta_\mu = 0\}.$$

Assign $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$, recorded by $q$ and $t$, respectively. The ideal is bigraded and $S_a$-stable, and it contains sufficiently high powers of every variable, so $C_\mu$ is finite dimensional. Since $\Delta_\mu \neq 0$, its degree-zero part is $\mathbb{C}$. With these gradings, Haiman's $n!$ theorem identifies its Frobenius character with $\widetilde{H}_\mu(X;q,t)$ (Haiman, 2001, Section 3.9, equations (55)–(56), and Theorem 3.2). Haiman assigns $x$ to row coordinates and $y$ to column coordinates; our interchange of these two lists makes $q$ record column degree. The quotient is generated by the two equivariant lists $x_i, y_i$. Apply Proposition 2.3 and Theorem 1.3. Complete reducibility in each bidegree makes the character of the kernel coefficientwise Schur positive. □

## 3. Coefficients of bounded depth

Theorem 1.4 asserts positivity for every $b$ at once. A finite computation can prove it because, after restriction to a Young subgroup, the coefficients of depth $r$ depend on $b$ only through polynomials in $q^b$ of degree at most $r$.

3.1. **Restriction to a Young subgroup.** Only the shapes of depth at most $r$ survive restriction to $S_{N-r} \times S_r$ followed by invariants of the first factor, each through a symmetric function of degree $r$. The size of the problem is then the depth rather than $N = ab$.

We restrict the module whose character is $P_{a,b} = H_b[H_a]$. Partition $[N]$ into $b$ blocks of size $a$, and let $W_{a,b} = S_a \wr S_b$ act on $R_a^{\otimes b} \otimes R_b$, with $S_a^b$ acting factorwise and $S_b$ permuting the factors and acting on $R_b$. By graded wreath induction, including the Adams operation on $q$ (Sra, 2026a, Lemma 2.1), the induced module and its graded character are

$$M_{a,b} = \operatorname{Ind}_{W_{a,b}}^{S_N}(R_a^{\otimes b} \otimes R_b), \qquad \operatorname{ch}_q M_{a,b} = P_{a,b}. \tag{3.1}$$

Put

$$\mathcal{R}_{a,b}^{(r)}(Y;q) = h_{N-r}^{\perp} P_{a,b}(Y;q).$$

Here $h_k^\perp$ is adjoint to multiplication by $h_k$ under the Hall scalar product. Representation-theoretically, $\mathcal{R}^{(r)}$ is the $S_r$-character of the $S_{N-r}$-invariants in $M_{a,b}$. The graded trace of the coinvariant algebra $R_n$ at a permutation of cycle type $\mu$ is

$$\kappa_n(\mu;q) = \frac{(q;q)_n}{\prod_{r\in\mu}(1-q^r)}, \tag{3.2}$$

and we write

$$\begin{bmatrix} b \\ n_0, n_1, \dots \end{bmatrix}_q = \frac{(q;q)_b}{\prod_j (q;q)_{n_j}}.$$

**Theorem 3.1** (Fixed-depth formula)**.** *For $0 \leq r \leq N$,*

$$\mathcal{R}_{a,b}^{(r)}(Y;q) = \sum_{\substack{n_0+\cdots+n_a=b \\ \sum_j j n_j = r}} \begin{bmatrix} b \\ n_0,\dots,n_a \end{bmatrix}_q \prod_{j=1}^{a} H_{n_j}\left[\begin{bmatrix} a \\ j \end{bmatrix}_q H_j(Y;q)\right]. \tag{3.3}$$

*All plethysms on the right include the Adams action on $q$. The term with $n_j = 0$ is interpreted as* 1.

*Proof.* The proof is a Mackey decomposition, and the grading enters only through the restriction of $R_a$ to a Young subgroup, which (3.2) settles. The double cosets for restriction to $S_{N-r} \times S_r$ are indexed by how the $r$ selected labels occupy the $b$ blocks: $n_j$ blocks contain exactly $j$ selected labels. On a block of size $a$, the graded character of the coinvariant algebra $R_a$ restricts to $S_j \times S_{a-j}$ as

$$\operatorname{Res} R_a = \begin{bmatrix} a \\ j \end{bmatrix}_q (R_j \boxtimes R_{a-j}).$$

This identity follows from (3.2), because the quotient of the trace factors is $(q;q)_a/((q;q)_j(q;q)_{a-j})$. Taking the invariants on the unselected labels leaves a graded multiplicity space of Hilbert series $\begin{bmatrix} a \\ j \end{bmatrix}_q$ tensored with $R_j$.

Similarly, restricting the outer $R_b$ to $\prod_j S_{n_j}$ produces the Gaussian multinomial factor and the outer modules $R_{n_j}$. Blocks with equal occupancy are permuted, so wreath induction gives $H_{n_j}[\begin{bmatrix} a \\ j \end{bmatrix}_q H_j]$. The $j = 0$ factor has only its degree-zero invariant. Inducing the selected labels to $S_r$ multiplies the Frobenius characters. Mackey decomposition over the occupancy orbits gives (3.3). □

The formula stays small when the depth is small. Only partitions of $r$ occur in the occupancy sum, every symmetric function on the right has degree at most $r$, and $a$ and $b$ enter only through Gaussian coefficients. At $q = 0$, (3.3) is the decomposition of the restricted Foulkes module by Giannelli (2013, Proposition 2.11).

The restriction $\mathcal{R}^{(r)}$ mixes the layers of depth at most $r$, where the layer of depth $d$ collects the Schur coefficients of the shapes $(N-d, \nu)$; the following triangular relation separates them.

**Proposition 3.2** (Triangular recovery of the layers)**.** *For $N \geq 2r$, let*

$$C_d(Y;q) = \sum\nolimits_{\nu \vdash d} [s_{(N-d,\nu)}] P_{a,b}\, s_\nu(Y), \qquad 0 \leq d \leq r.$$

*Then the restrictions and the layers are related by*

$$\mathcal{R}^{(r)}_{a,b} = \sum\nolimits_{d=0}^{r} h_{r-d} C_d, \qquad C_r = \sum\nolimits_{j=0}^{r} (-1)^j e_j \mathcal{R}^{(r-j)}_{a,b}.$$

*The same identities hold for differences and for division by $1-q$.*

*Proof.* By the Pieri rule, $s_\nu$ occurs in $h^{\perp}_{N-r} s_\lambda$ precisely when $\lambda/\nu$ is a horizontal strip of size $N-r$. Such a $\lambda$ has $\lambda_1 \geq N-r$. Write $\lambda = (N-d, \tau)$ with $d \leq r$. Since $N \geq 2r$, the horizontal-strip condition is equivalent to $\nu/\tau$ being a horizontal strip of size $r-d$. This equivalence proves the first identity. The second uses $(\sum_{j\geq 0} h_j z^j)^{-1} = \sum_{j\geq 0} (-1)^j e_j z^j$. □

3.2. **Gaussian Newton expansions.** Theorem 3.1 bounds how the coefficients of depth $r$ depend on $b$. We now expand them in a basis that is itself positive and increasing in $b$, so that positive expansion coefficients give positivity for every $b$.

For integers $a \geq 1$ and $r \geq 0$, choose a baseline $B \geq a$ with $aB \geq 2r$. Writing a shape of depth $r$ as $(ab - r, \nu)$, we call $\nu \vdash r$ its tail and collect the coefficients of depth $r$ in

$$\mathcal{G}_{a,r}(b;Y) = \sum\nolimits_{\nu \vdash r} [s_{(ab-r,\nu)}] F_{a,b}\, s_\nu(Y), \qquad b \geq B.$$

The stable-range condition $aB \geq 2r$ ensures that every displayed shape is a partition. Theorem 3.1 and Proposition 3.2 make $\mathcal{G}_{a,r}$ a polynomial of degree at most $r$ in $z = q^b$, by counting occupied blocks. A term of $\mathcal{R}^{(r)}_{a,b}$ with $m$ occupied blocks contains $\begin{bmatrix} b \\ m \end{bmatrix}_q$ and has $z$-degree $m \leq r$. In $\mathcal{R}^{(r)}_{b,a}$, an inner term of degree $j$ contains $\begin{bmatrix} b \\ j \end{bmatrix}_q$, of $z$-degree $j$. An Adams operation of order $t$ replaces $(q, z)$

by $(q^t, z^t)$, so a total occupied degree $r$ still gives $z$-degree at most $r$, and the lower restrictions in Proposition 3.2 have smaller degree. Hence $\mathcal{G}_{a,r}$ has a unique expansion

$$\mathcal{G}_{a,r}(b;Y) = \sum\nolimits_{j=0}^{r} A^{(B)}_{a,r,j}(Y;q) \begin{bmatrix} b-B \\ j \end{bmatrix}_q . \tag{3.4}$$

Evaluation at $b = B, B+1, \dots, B+r$ gives the triangular system

$$A^{(B)}_{a,r,j} = \mathcal{G}_{a,r}(B+j;Y) - \sum\nolimits_{i<j} \begin{bmatrix} j \\ i \end{bmatrix}_q A^{(B)}_{a,r,i}, \tag{3.5}$$

so the coefficients are integer polynomials in $q$.

Through depth three the expansion is explicit. For every $2 \le a < b$, the coefficients of depth at most two vanish, and the three of depth three are $q$, $1+q^2$, and $q$ times one polynomial with nonnegative coefficients (Theorem A.1). Beyond depth three, Theorem 1.4 needs positivity for unbounded $a$ as well as $b$. A second Gaussian Newton expansion in the inner parameter reduces each of its ranges to finitely many exact coefficient arrays (Appendix A.3), which are positive; Appendix A.4 reconstructs them and proves Theorem 1.4.

A positive Gaussian expansion also proves monotonicity, because

$$\begin{bmatrix} n+1 \\ j \end{bmatrix}_q - \begin{bmatrix} n \\ j \end{bmatrix}_q = q^{n+1-j} \begin{bmatrix} n \\ j-1 \end{bmatrix}_q \qquad (1 \le j \le n+1), \tag{3.6}$$

and terms with $j > n+1$ vanish. Every positive Newton expansion is therefore coefficientwise increasing in $n$, which gives the monotonicity assertion of Theorem 1.4.

## 4. Grading both factors

With both factors graded, positivity becomes a comparison of degrees. The modules $\mathcal{P} = M_{a,b}$ and $\mathcal{Q} = M_{b,a}$ of (3.1) have characters $P_{a,b} = H_b[H_a]$ and $Q_{a,b} = H_a[H_b]$, and both are ungraded copies of $\mathbb{C}[S_N]$. Each irreducible $S^\lambda$ therefore occurs equally often in the two modules, and $[s_\lambda q^k]F_{a,b}$ counts how many more of its copies $\mathcal{P}$ places in degrees at most $k$. In the language of Paper II (Sra, 2026a), the conjecture says that on every isotype the degree distribution of $\mathcal{P}$ is stochastically smaller than that of $\mathcal{Q}$. It would suffice to have an injective equivariant map from $\mathcal{P}$ to $\mathcal{Q}$ that never lowers degree, because such a map embeds the copies above each degree $k$ in $\mathcal{P}$ into those above $k$ in $\mathcal{Q}$.

The lift of Theorem 1.2 does not provide such a map. It uses one coefficient algebra for every position, whereas here the relations among the coefficients depend on how the positions are grouped into blocks, and extending the canonical map polynomially does not respect them (Proposition B.3). Projecting onto harmonic representatives repairs the relations and gives an explicit space of maps that never lower degree, built in Appendix B. No single map in this space needs to be injective; it suffices that the maps together expand every graded subspace.

Fix $b \ge a \ge 2$ and $N = ab$, and put

$$U = \mathcal{P}_0 = \mathrm{Ind}_{S_a \wr S_b}^{S_N} \mathbf{1}, \qquad D_P = b\binom{a}{2} + \binom{b}{2}, \quad D_Q = a\binom{b}{2} + \binom{a}{2}.$$

The integers $D_P, D_Q$ are the top degrees of $\mathcal{P}$ and $\mathcal{Q}$. The maps run from $\mathcal{P}$ to $\mathcal{Q}$, and their adjoints run in the direction of the filtered matchings of Paper II (Sra, 2026a, Proposition 2.3). We realize $\mathcal{P}$ and $\mathcal{Q}$ as spaces of harmonic polynomials with coefficients in permutation modules (Appendix B), write $P_{\mathcal{Q}}$ for the orthogonal projection onto $\mathcal{Q}$, and use the operators $A^U_k(d)$ of Proposition B.1, which are positive and monic of degree $k$ in a parameter $d$. These modules and maps have rational forms; subspaces below are taken after extending scalars to $\mathbb{C}$.

4.1. **The current space.** We call an equivariant map from $U = \mathcal{P}_0$ into $\mathcal{Q}_r$ a seed. Appendix B extends each seed to a map $T_d$ on all of $\mathcal{P}$ that never lowers degree and depends polynomially on the parameter $d$. The parameter is auxiliary, so we keep every coefficient of this dependence. Put $\overline{A}_k^U(\varepsilon) = \varepsilon^k A_k^U(1/\varepsilon)$. This reversed factor is a polynomial of degree at most $k$ in $\varepsilon$, including at $\varepsilon = 0$, because $A_k^U(d)$ is monic of degree $k$ in $d$ (Proposition B.1). For an equivariant homogeneous seed $\sigma_r \colon U \to \mathcal{Q}_r$, let $\widehat{\sigma}_r(f \otimes u) = f\,\sigma_r(u)$. On the harmonic modules, define the maps $\mathcal{A}_j(\sigma_r)$ by their degree blocks

$$\mathcal{A}_j(\sigma_r)|_{\mathcal{P}_k} = P_{\mathcal{Q},k+r}\widehat{\sigma}_r[\varepsilon^j]\overline{A}_k^U(\varepsilon)|_{\mathcal{P}_k}. \tag{4.1}$$

The block is zero when $j > k$ or $k + r > D_Q$. By (B.5), these maps are the coefficients of $T_d$ after removing the factor $(d/N)^k$ and setting $\varepsilon = 1/d$. Each map is equivariant and has degree shift $r \geq 0$. The operators $A_k^U$ and the projections in (4.1) are evaluated before selecting an isotypic component.

Taking all rational equivariant seeds gives the current space

$$\mathscr{M}_{a,b} = \operatorname{span}_{\mathbb{Q}}\{\mathcal{A}_j(\sigma_r) : \sigma_r \in \operatorname{Hom}_{\mathbb{Q}[S_N]}(U, \mathcal{Q}_r),\ 0 \leq r \leq D_Q,\ 0 \leq j \leq D_P\}.$$

The module $\mathcal{Q}$ is an ungraded regular module, so $\dim \operatorname{Hom}_{\mathbb{Q}[S_N]}(U, \mathcal{Q}) = \dim U$. Linearity in the seed therefore gives

$$\dim \mathscr{M}_{a,b} \leq (D_P + 1)\dim U = (D_P + 1)\frac{(ab)!}{(a!)^b b!}.$$

In particular, bases of the finite seed Hom spaces give a homogeneous spanning family for $\mathscr{M}_{a,b}$, so for each pair $a, b$ the condition below is a finite problem in linear algebra.

Write the isotypic decompositions as

$$\mathcal{P} = \bigoplus_{\lambda \vdash N} S^\lambda \otimes \mathcal{E}_\lambda^P, \qquad \mathcal{Q} = \bigoplus_{\lambda \vdash N} S^\lambda \otimes \mathcal{E}_\lambda^Q.$$

Both multiplicity spaces have total dimension $d_\lambda$. By Schur's lemma, each map in $\mathscr{M}_{a,b}$ acts on multiplicity spaces; let $\mathscr{M}_\lambda$ be the resulting space of complex matrices from $\mathcal{E}_\lambda^P$ to $\mathcal{E}_\lambda^Q$, and define

$$\mathscr{M}_\lambda(W) = \operatorname{span}\{Mv : M \in \mathscr{M}_\lambda,\ v \in W\}.$$

Call $\lambda$ *neutral* when $\dim \mathcal{E}_{\lambda,k}^P = \dim \mathcal{E}_{\lambda,k}^Q$ for every degree $k$. Such an isotype contributes zero to $F_{a,b}$.

4.2. **A sufficient condition.** We ask that the matrices in $\mathscr{M}_\lambda$ together expand every graded subspace of $\mathcal{E}_\lambda^P$,

$$\dim \mathscr{M}_\lambda(W) \geq \dim W \quad \text{for every nonneutral } \lambda \text{ and every graded } W \subseteq \mathcal{E}_\lambda^P. \tag{4.2}$$

Here graded means $W = \bigoplus_k (W \cap \mathcal{E}_{\lambda,k}^P)$. A subspace violating this inequality is called *shrunk*. On the degree tails of $\mathcal{E}_\lambda^P$, hypothesis (4.2) gives the inequalities that positivity of $F_{a,b}$ requires.

**Theorem 4.1** (A sufficient condition)**.** *For fixed $b \geq a \geq 2$, hypothesis (4.2) implies*

$$F_{a,b}(X; q) = \frac{H_b[H_a] - H_a[H_b]}{1 - q} \in \mathbb{N}[q]\{s_\lambda\}.$$

*Proof.* Fix a nonneutral $\lambda$ and an integer $k \geq 0$. Every generator of $\mathscr{M}_\lambda$ has nonnegative degree shift. For the graded tail $W = \mathcal{E}_{\lambda,\geq k+1}^P$, hypothesis (4.2) therefore gives

$$\dim \mathcal{E}_{\lambda,\geq k+1}^P \leq \dim \mathscr{M}_\lambda(W) \leq \dim \mathcal{E}_{\lambda,\geq k+1}^Q.$$

Subtracting from the common total dimension $d_\lambda$ yields

$$[s_\lambda q^k]F_{a,b} = \sum\nolimits_{j \leq k} (\dim \mathcal{E}_{\lambda,j}^P - \dim \mathcal{E}_{\lambda,j}^Q) \geq 0.$$

Neutral isotypes contribute zero. Equality of the ungraded characters makes the divided difference a polynomial, proving the assertion. □

The proof uses hypothesis (4.2) only on degree tails. We state it for every graded subspace because it then excludes shrunk subspaces of every kind (by homogeneity), which is the condition that operator scaling tests through capacity; the supplement (Sra, 2026b) uses this to certify (4.2) on individual blocks. It remains open whether hypothesis (4.2) holds in general, and Proposition B.3 shows why it does not follow from the one-layer surjection.

## AI USAGE

The results in this paper were developed through extensive use of AI assistants in sustained mathematical discussion, including failed attempts, counterexamples, audits, and repairs. AI assistants also helped with exact computations, proof checks, and exposition. The external representation-theoretic results are cited at their points of use.

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## Appendix A. Positivity at bounded depth

Theorem 1.4 concerns unbounded $a$ and $b$, so its finite certificates must determine the coefficients as polynomials in both $q^a$ and $q^b$. The degree bounds from the restriction formula make this reconstruction finite. At the first nonzero layer, the same formula gives a closed expression valid for every inner size.

A.1. **The first nonzero layer.** The three coefficients at depth three share one positive factor, so a single polynomial governs their positivity, their Newton expansion, and their stability. The only delicate point is the division by $[3]_q$ in this factor, since a quotient of polynomials with nonnegative coefficients can have negative ones; the residues of $a$ and $b$ modulo 3 settle it.

**Theorem A.1** (The first nonzero layer)**.** *Let $2 \le a < b$, and put*

$$\Gamma_{a,b}(q) = \frac{q^{a-1}(1+q)^2}{[3]_q} \begin{bmatrix} a \\ 2 \end{bmatrix}_q \begin{bmatrix} b \\ 2 \end{bmatrix}_q [b-a]_q. \tag{A.1}$$

*Then $\Gamma_{a,b} \in \mathbb{N}[q]$. All Schur coefficients of $F_{a,b}$ with $\lambda_1 > N-3$ vanish. The three coefficients at depth three are*

$$\begin{aligned} [s_{(N-3,3)}]F_{a,b} &= q\Gamma_{a,b}(q), \\ [s_{(N-3,2,1)}]F_{a,b} &= (1+q^2)\Gamma_{a,b}(q), \\ [s_{(N-3,1,1,1)}]F_{a,b} &= q\Gamma_{a,b}(q). \end{aligned}$$

The vanishing concerns the four shapes $(N)$, $(N-1,1)$, $(N-2,2)$, and $(N-2,1,1)$.

*Proof of Theorem A.1.* Theorem 3.1 at $r \le 3$ gives the three restrictions explicitly, and Proposition 3.2 converts their differences into the Schur coefficients of $F_{a,b}$. Abbreviate

$$A = [a]_q, \quad B = [b]_q, \quad C_a = \begin{bmatrix} a \\ 2 \end{bmatrix}_q, \; C_b = \begin{bmatrix} b \\ 2 \end{bmatrix}_q, \quad E_a = \begin{bmatrix} a \\ 3 \end{bmatrix}_q, \; E_b = \begin{bmatrix} b \\ 3 \end{bmatrix}_q.$$

The first three occupancy formulas are

$$\mathcal{R}^{(1)}_{a,b} = AB\, s_1, \tag{A.2}$$

$$\mathcal{R}^{(2)}_{a,b} = BC_a H_2(Y;q) + C_b H_2[AY], \tag{A.3}$$

$$\mathcal{R}^{(3)}_{a,b} = BE_a H_3(Y;q) + (1+q)AC_aC_b\, H_2(Y;q)h_1(Y) + E_b H_3[AY]. \tag{A.4}$$

The middle term uses the identity $[b]_q[b-1]_q = (1+q)C_b$.

***Cancellation through depth two.*** We substitute $p_1[[n]_q] = [n]_q$ and $p_2[[n]_q] = [n]_{q^2}$ in $H_2 = s_2 + qs_{11}$ and obtain the two Schur coefficients

$$[s_2]H_2[[n]_qY] = [n]_q + 2q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}_q,$$

$$[s_{11}]H_2[[n]_qY] = q[n]_q + q(1+q^2) \begin{bmatrix} n \\ 2 \end{bmatrix}_q.$$

Consequently the coefficients of (A.3) are

$$BC_a + AC_b + 2q^2C_aC_b, \qquad q(BC_a + AC_b + (1+q^2)C_aC_b).$$

Both are symmetric under $a \leftrightarrow b$. The invariants at depth zero are 1 in either order, and (A.2) is symmetric as well. Proposition 3.2 proves the stated cancellation of the four top Schur coefficients.

***The three depth-three coefficients.*** The three terms of (A.4) need three families of Schur coefficients. For $\nu \vdash 3$, put

$$K_\nu(q) = [s_\nu]H_3, \qquad \ell_\nu(q) = [s_\nu](H_2h_1), \qquad L_\nu(n;q) = [s_\nu(Y)]H_3[[n]_qY].$$

The coefficient $L_\nu(n;q)$ is a polynomial in $q^n$ of degree at most three that vanishes at $n=0$. The Gaussian coefficients $[n]_q$, $\begin{bmatrix} n \\ 2 \end{bmatrix}_q$, and $\begin{bmatrix} n \\ 3 \end{bmatrix}_q$ form a basis for these polynomials. Therefore

$$L_\nu(n;q) = K_\nu(q)[n]_q + u_\nu(q)\begin{bmatrix} n \\ 2 \end{bmatrix}_q + v_\nu(q)\begin{bmatrix} n \\ 3 \end{bmatrix}_q. \tag{A.5}$$

The coefficient of $[n]_q$ follows by taking $n=1$, and $u_\nu$ follows by taking $n=2$, in

$$L_\nu(n;q) = \sum\nolimits_{\mu\vdash 3} \frac{\kappa_3(\mu;q)\chi^\nu(\mu)}{z_\mu} \prod\nolimits_{r\in\mu}[n]_{q^r},$$

and the three $S_3$ characters give

| $\nu$ | $K_\nu$ | $\ell_\nu$ | $u_\nu$ |
|---|---|---|---|
| $(3)$ | $1$ | $1$ | $q^2(1+q)(q+2)$ |
| $(2,1)$ | $q+q^2$ | $1+q$ | $q(1+q)^2(2q^2+1)$ |
| $(1,1,1)$ | $q^3$ | $q$ | $q^2(1+q)(q^3+q+1)$ |

The coefficients $v_\nu$ cancel from the difference of the two orders. Using (A.4) and (A.5), the $s_\nu$ coefficient of $\mathcal{R}^{(3)}_{a,b} - \mathcal{R}^{(3)}_{b,a}$ is $u_\nu(E_bC_a - E_aC_b) + (1+q)\ell_\nu(A-B)C_aC_b$. We divide by $1-q$ and use the elementary Gaussian identities $E_bC_a - E_aC_b = C_aC_bq^{a-2}[b-a]_q/[3]_q$ and $A - B = -q^a[b-a]_q$; the coefficient becomes

$$C_aC_bq^{a-2}[b-a]_q \frac{u_\nu/[3]_q - (1+q)q^2\ell_\nu}{1-q}.$$

For the three rows of the table, the final factor is respectively $q^2(1+q)^2/[3]_q$, $q(1+q)^2(1+q^2)/[3]_q$, and $q^2(1+q)^2/[3]_q$. The lower layers have already canceled, so Proposition 3.2 identifies the three resulting products with the Schur coefficients of $F_{a,b}$ at depth three. They are therefore $q\Gamma_{a,b}$, $(1+q^2)\Gamma_{a,b}$, and $q\Gamma_{a,b}$, as Theorem A.1 asserts, because $C_aC_bq^{a-2}(1+q)^2[b-a]_q/[3]_q = q^{-1}\Gamma_{a,b}$.

***Positivity of*** $\Gamma_{a,b}$**.** It remains to prove that division by $[3]_q$ in (A.1) does not introduce negative coefficients. If $a \equiv b \equiv 2 \pmod 3$, then 3 divides $b-a$, and $[b-a]_q/[3]_q = [(b-a)/3]_{q^3} \in \mathbb{N}[q]$. Otherwise some $n \in \{a,b\}$ satisfies $n \geq 3$ and $n \equiv 0$ or $1 \pmod 3$, and we show $\begin{bmatrix} n \\ 2 \end{bmatrix}_q/[3]_q \in \mathbb{N}[q]$.

Write $\begin{bmatrix} n \\ 2 \end{bmatrix}_q = [n]_q[n-1]_q/[2]_q$; one of $n$ and $n-1$ is divisible by three. If that integer is odd, the other one is even, so $[3]_q$ divides the first $q$-integer and $[2]_q$ the second, each with a geometric sum as quotient. If it is even, write it as $6r$ and the other as $m = 6r \pm 1$; the quotient becomes $[r]_{q^6}(1-q+q^2)[m]_q$, and the last two factors multiply to $1+\sum_{j=2}^{m-1} q^j + q^{m+1}$, which has nonnegative coefficients. All remaining factors of (A.1) are in $\mathbb{N}[q]$. □

A.2. **Stability of the depth-three coefficients.** Bergeron's $q$-stability conjecture asks that each coefficient with a fixed tail increase with $b$. At depth three the common factor in Theorem A.1 reduces this to the differences of $\Gamma_{a,b}$. The first difference in $b$ still carries the factor $1/[3]_q$. The mixed difference in $a$ and $b$ cancels it and factors positively, and summing mixed differences over $a$ recovers the first difference. Extend $\Gamma$ by $\Gamma_{1,b} = \Gamma_{a,a} = 0$ and write $D^{\Gamma}_{a,b} = \Gamma_{a,b+1} - \Gamma_{a,b}$.

**Corollary A.2** (Depth-three stability)**.** *For integers $1 \le a \le b$,*

$$D^{\Gamma}_{a,b} = [2]_q[b]_q \sum_{i=1}^{a-1} q^{2i+b-3}[i]_q[b-i]_q \in \mathbb{N}[q]. \tag{A.6}$$

*For integers $1 \le a < b$,*

$$D^{\Gamma}_{a+1,b} - D^{\Gamma}_{a,b} = q^{2a+b-3}[2]_q[a]_q[b]_q[b-a]_q \in \mathbb{N}[q]. \tag{A.7}$$

*For $\nu \vdash 3$, write $G_{a,b,\nu} = [s_{(ab-3,\nu)}]F_{a,b}$ for the fixed-tail coefficients. Consequently, for $2 \le a \le b$ with $ab \ge 6$,*

$$G_{a,b+1,\nu} - G_{a,b,\nu} \in \mathbb{N}[q].$$

*For $2 \le a < b$, moreover,*

$$G_{a+1,b+1,\nu} - G_{a+1,b,\nu} - G_{a,b+1,\nu} + G_{a,b,\nu} \in \mathbb{N}[q].$$

*Proof.* Abbreviate $[j] = [j]_q$ and put $d = b - a$. Cancelling the two factors $[2]$ in (A.1) gives

$$\Gamma_{a,b} = \frac{q^{a-1}[a][a-1][b][b-1][d]}{[3]}.$$

Since $[b+1] - [b-1] = q^{b-1}[2]$ and $[d+1] - [d] = q^d$, it follows that

$$D^{\Gamma}_{a,b} = \frac{q^{b-1}[a][a-1][b]}{[3]}([b+1] + q^{a-1}[2][d]).$$

For $a < b$, we subtract this expression from the one with $a+1$ in place of $a$ and obtain

$$D^{\Gamma}_{a+1,b} - D^{\Gamma}_{a,b} = \frac{q^{a+b-2}[2][a][b]}{[3]}([b+1] + q[a+1][d-1] - [a-1][d]).$$

Using $q[a+1] - [a-1] = q^{a-1}[3] - 1$ and $[d-1] + q^{d-1}[a-1] = [b-2]$, the final bracket is

$$[b+1] - [b-2] + q^{a-1}[3][d-1] = q^{a-1}[3][d],$$

which proves (A.7). We sum it with $a$ replaced by $i = 1, \dots, a-1$ and use $D^{\Gamma}_{1,b} = 0$, which proves (A.6); the sum is empty for $a = 1$. The three fixed-tail coefficients are $q\Gamma_{a,b}$, $(1+q^2)\Gamma_{a,b}$, and $q\Gamma_{a,b}$, so both stated comparisons follow from the corresponding differences of $\Gamma$. □

Theorem 1.4 also asserts nonnegative Newton coefficients based at $b = a$. At depth three they are $q$, $1+q^2$, and $q$ times those of $\Gamma_{a,b}$ in the basis $\begin{bmatrix} b-a \\ j \end{bmatrix}_q$, which are positive for every inner size. Put $T_a = [a-1]_q[a]_q[a+1]_q/[3]_q$. One of the three consecutive integers is divisible by three, so $T_a \in \mathbb{N}[q]$. The factor $\Gamma_{a,b}$ in (A.1) satisfies

$$\begin{aligned} \Gamma_{a,b} = {} & q^{a-1}[a]_q T_a \begin{bmatrix} b-a \\ 1 \end{bmatrix}_q + q^{2a-1}[2]_q^2 T_a \begin{bmatrix} b-a \\ 2 \end{bmatrix}_q \\ & + q^{3a+1}[2]_q[a]_q[a-1]_q \begin{bmatrix} b-a \\ 3 \end{bmatrix}_q . \end{aligned} \tag{A.8}$$

To prove the identity, put $n = b - a$ and write

$$\Gamma_{a,b} = \frac{q^{a-1}[a]_q[a-1]_q}{[3]_q}[n]_q([a]_q + q^a[n]_q)([a-1]_q + q^{a-1}[n]_q).$$

Repeatedly apply $[n]_q \begin{bmatrix} n \\ j \end{bmatrix}_q = [j]_q \begin{bmatrix} n \\ j \end{bmatrix}_q + q^j [j+1]_q \begin{bmatrix} n \\ j+1 \end{bmatrix}_q$. The index-two coefficient simplifies by $[a]_q + q[a-1]_q + q^a(2+q) = [2]_q[a+1]_q$, giving (A.8).

A.3. **The two-parameter Gaussian expansion.** To handle every inner size at once, we treat $q^a$ like $q^b$. The point to check is that the second expansion terminates, that is, the coefficients of the first expansion have degree in $q^a$ bounded independently of $a$. For $\nu \vdash r$, the fixed-depth formula writes $[s_{(ab-r,\nu)}]F_{a,b}$ as a polynomial $G_{r,\nu}(x,z;q)$ in $x = q^a$ and $z = q^b$ over $\mathbb{Q}(q)$. Applied to each variable, the occupancy count of Section 3.2 bounds its bidegree by $(r,r)$, and $G(x,z) = -G(z,x)$. Its total degree is therefore at most $2r-1$. If $E$ shifts $n$ to $n+1$, then $\prod_{i=0}^{j-1}(E - q^i)$ extracts the $j$th Gaussian coefficient at $n = 0$. Applied to $x^u z^v$ after $z = xq^n$, it gives $x^{u+v}\prod_{i=0}^{j-1}(q^v - q^i)$. The product vanishes for $v < j$, so the $j$th coefficient is divisible by $x^j$, and its quotient has $x$-degree at most $2r-1-j$. For an integer $a_0 \geq 2$ with $a_0^2 \geq 2r$, a second Newton expansion gives

$$[s_{(ab-r,\nu)}]F_{a,b} = \sum_{j=1}^{r} \sum_{k=0}^{2r-1-j} q^{j(a-a_0)} B_{r,\nu,j,k}(q) \begin{bmatrix} a-a_0 \\ k \end{bmatrix}_q \begin{bmatrix} b-a \\ j \end{bmatrix}_q, \qquad b \geq a \geq a_0. \tag{A.9}$$

The index $j = 0$ is absent because $G(x,x) = 0$. The signs of the $B_{r,\nu,j,k}$ come from the certificates of Section A.4.

A.4. **Exact certificates for depths four to ten.** Each certificate is a finite set of identities over $\mathbb{Q}(q)$ between the fixed-depth formula and a Gaussian Newton expansion whose coefficients are visibly nonnegative. Finitely many identities cover all $a$ and $b$, because the degree bounds of Section 3.2 and Appendix A.3 determine each polynomial from finitely many values, provided every identity holds exactly in $q$. All computations work in the power-sum basis, where multiplication concatenates cycle partitions and an Adams operation sends $p_j$ to $p_{tj}$ and every grading variable to its $t$th power. They start from $h_d = \sum_{\rho \vdash d} p_\rho / z_\rho$ and, by (3.2), $H_d = \sum_{\rho \vdash d} \kappa_d(\rho;q) p_\rho / z_\rho$, and return to Schur functions by $[s_\nu]f = \sum_\rho [p_\rho]f\, \chi^\nu(\rho)$, with the characters computed both by the rim-hook rule and by the Frobenius formula.

For an occupancy $\lambda = (1^{n_1}2^{n_2}\cdots) \vdash k$ of length $m$, the restriction term is

$$\begin{bmatrix} b \\ m \end{bmatrix}_q \begin{bmatrix} m \\ n_1, n_2, \ldots \end{bmatrix}_q \prod_{j \geq 1} H_{n_j}\left[ \begin{bmatrix} a \\ j \end{bmatrix}_q H_j \right]. \tag{A.10}$$

These terms sum to $\mathcal{R}_{a,b}^{(k)}$, and Proposition 3.2 recovers the tail character as

$$\mathcal{G}_{a,r}(b;Y) = \frac{1}{1-q} \sum_{i=0}^{r} (-1)^i e_i(Y) (\mathcal{R}_{a,b}^{(r-i)} - \mathcal{R}_{b,a}^{(r-i)}). \tag{A.11}$$

For fixed $a, r, B$, we evaluate this expression at $b = B, \ldots, B+r$, solve (3.5), and verify exact divisibility by $1-q$, integrality, and the sign of every coefficient and monomial exponent. The $r+1$ formal identities determine the full polynomial, since the result has degree at most $r$ in $q^b$.

For the two-parameter expansion, we recover the $j$th Newton coefficient at baseline $b = a$, divide it by $q^{j(a-a_0)}$, and expand again in $a - a_0$ by (3.5). Its degree is at most $2r-1-j$ in $q^a$, so $2r-j$ distinct formal nodes determine it. The computation uses all these nodes and further nodes as a check, always with $q$ as an indeterminate, so it establishes identities over $\mathbb{Q}(q)$. The certificates uniform in $a$ have these sizes.

| Depth $r$ | Base $a_0$ | Tails | Newton polynomials | Coefficients |
|---|---|---|---|---|
| 5 | 4 | 7 | 245 | 6,830 |
| 6 | 4 | 11 | 561 | 22,658 |
| 7 | 5 | 15 | 1,050 | 62,649 |
| 8 | 6 | 22 | 2,024 | 166,851 |

Every polynomial $B_{r,\nu,j,k}$ is nonzero, and its coefficients are positive integers from its lowest to its highest degree. For $a = B = 7$, the additional depths nine and ten have respectively 270 and 420 nonconstant Newton polynomials, with 23,810 and 42,850 positive coefficients; all 30 and 42 constant terms vanish. These fixed-parameter identities hold for every $b \geq 7$ by the degree-$r$ bound. Rational evaluations with the second character construction confirm these identities inside and beyond the interpolation ranges, including $b = 41$.

At depth four the certificate is symbolic in both parameters. We replace $\begin{bmatrix} a \\ j \end{bmatrix}_q$ and $\begin{bmatrix} b \\ j \end{bmatrix}_q$ in (A.10) by their product formulas in $x = q^a$ and $z = q^b$ and expand twice in Gaussian binomials, as in (A.9). The 110 polynomials $B_{4,\nu,j,k}$ give the expression

$$\sum_{j=1}^{4} q^{-4j} \left( \sum_{k=0}^{7-j} B_{4,\nu,j,k}(q) \frac{\prod_{i=0}^{k-1}(1 - xq^{-4-i})}{(q;q)_k} \right) \frac{\prod_{i=0}^{j-1}(x - zq^{-i})}{(q;q)_j}.$$

This expression equals (A.11) in $\mathbb{Q}(q)[x, z]$ for each of the five tails, and all 1,908 coefficients and all monomial shifts are nonnegative.

*Proof of Theorem 1.4.* For $(r, a_0) = (4, 4), (5, 4), (6, 4), (7, 5), (8, 6)$, the exact identities above give nonnegative coefficients in (A.9). These identities hold in $\mathbb{Q}(q)[x, z]$, so they apply for every $b \geq a \geq a_0$. For $a = 7$ and $r = 9, 10$, the certificates for (3.4) based at $B = 7$ have nonnegative coefficients and zero constant term. The degree bound proves that they hold for every $b \geq 7$. Depths at most three follow from Theorem A.1 and (A.8). Formula (3.6) proves coefficientwise monotonicity in $b$. □

## Appendix B. Construction of the current maps

The current maps of Section 4 must respect the coinvariant relations of both modules, and the polynomial extension of the canonical map does not (Proposition B.3). The construction avoids the relations by differentiating only on the side of $\mathcal{P}$ and multiplying only on the side of $\mathcal{Q}$. Derivatives preserve the harmonic complement of every ideal, and multiplication preserves every ideal, so a single orthogonal projection at the end suffices, which gives the block formula (B.5). The differentiations contribute operators $A_k^U(d)$ on full polynomial modules, which we call free factors. They are positive for $d \geq N$, hence invertible, so the seed and the final projection determine each block.

Fix $N \geq 2$ and $W_0 = \{w \in \mathbb{C}^N : \sum_i w_i = 0\}$. For a finite-dimensional unitary $S_N$-module $U$, set

$$\mathscr{P}_k(U) = \mathrm{Sym}^k(W_0^*) \otimes U, \qquad \mathscr{P}(U) = \bigoplus_{k \geq 0} \mathscr{P}_k(U).$$

The group acts simultaneously on coordinates and on $U$. Use the Euclidean Fischer inner product on the polynomial factor. Let $C_i$ multiply by $w_i$ and let $D_i = C_i^*$ differentiate in the direction $e_i - N^{-1} \sum_j e_j$. If $\rho(g)$ denotes the simultaneous action, then

$$\sum_i C_i D_i = kI \quad \text{on } \mathscr{P}_k(U), \qquad \rho(g) C_i \rho(g)^{-1} = C_{g(i)}, \quad \rho(g) D_i \rho(g)^{-1} = D_{g(i)}. \tag{B.1}$$

Put $Y_i = \sum_{j \neq i} \rho((ij))$. Each transposition is self-adjoint and unitary, so $-(N-1)I \preceq Y_i \preceq (N-1)I$. When $Y_i$ follows $D_i$, it acts on the remaining polynomial degree and on $U$.

Let $V$ be another finite-dimensional unitary $S_N$-module. For an equivariant map $T\colon \mathscr{P}(U) \to \mathscr{P}(V)$, define

$$\mathscr{C}_d(T) = \sum_i C_i T(dI + Y_i) D_i.$$

The operator $\mathscr{C}_d$ differentiates the input once and multiplies the output once, and it models the contraction of a current operator with tensors of dimension $d$. Restoring the differentiated tensor occurrence gives $d$ terms, while restoring a different occurrence exchanges its label with the first, which accounts for $Y_i$. The displayed order of the factors follows from the identity $\rho((ij))C_j = C_i \rho((ij))$

and equivariance of $T$. The formula is polynomial in $d$; its tensor interpretation uses an integer $d \geq N$.

Iterating $\mathscr{C}_d$ on the identity produces the free factors. Their positivity is not automatic, because the operators $Y_i$ need not be positive; they are only bounded below by $-(N-1)I$. Taking $d \geq N$ makes every factor $dI + Y_i$ positive, and equivariance makes these factors commute.

**Proposition B.1** (Positive free factors)**.** *For $k \geq 0$ define*

$$A_0^U(d) = I_U, \qquad A_k^U(d) = \frac{1}{k} \sum\nolimits_i C_i A_{k-1}^U(d)(dI + Y_i)D_i \quad (k \geq 1). \tag{B.2}$$

*The operator $A_k^U(d)$ is $S_N$-equivariant and monic of degree $k$ in $d$. For real $d \geq N$,*

$$A_k^U(d) \succeq (d - N + 1)^k I. \tag{B.3}$$

*If $\mu\colon U \to V$ is equivariant, its polynomial-linear extension $\widetilde{\mu} = I \otimes \mu$ satisfies $A_k^V(d)\widetilde{\mu} = \widetilde{\mu} A_k^U(d)$.*

*Proof.* Covariance in (B.1) proves equivariance inductively. It follows that $A_{k-1}^U(d)$ commutes with every $Y_i$. For $d \geq N$, the induction hypothesis therefore makes $A_{k-1}^U(d)(dI + Y_i)$ a product of commuting self-adjoint operators with

$$A_{k-1}^U(d)(dI + Y_i) \succeq (d - N + 1)^k I.$$

Sandwich this inequality between $D_i$ and $C_i = D_i^*$, sum over $i$, and use (B.1) to obtain (B.3). The same Euler identity makes the leading term of (B.2) equal to $d^k I$. Finally, $\widetilde{\mu}$ intertwines the group actions and commutes with coordinate multiplication and differentiation, so induction in (B.2) proves the last identity. □

To extend a map from degree zero, let $\sigma_r\colon U \to \mathscr{P}_r(V)$ be equivariant for each $r \geq 0$, with only finitely many nonzero. Define

$$\widehat{\sigma}_r(f \otimes u) = f\, \sigma_r(u), \qquad T_d^{\mathrm{free}} = \exp(\mathscr{C}_d/N)\big((\textstyle\sum_r \sigma_r)\Pi_0\big),$$

where $\Pi_0$ projects onto polynomial degree zero. The exponential is locally finite because each contraction differentiates the domain once. The extension $\widehat{\sigma}_r$ commutes with multiplication; it need not commute with differentiation. The asymmetry is harmless, because in every block all differentiations act before the seed.

**Proposition B.2** (Block formula)**.** *For every $k, r \geq 0$ the block from degree $k$ to degree $k + r$ is*

$$(T_d^{\mathrm{free}})^{k+r,k} = N^{-k} \widehat{\sigma}_r A_k^U(d). \tag{B.4}$$

*Proof.* After $k$ contractions, $\Pi_0$ kills $\mathscr{P}_j(U)$ for every $j \neq k$. Induction gives

$$\mathscr{C}_d^k(\sigma_r \Pi_0)|_{\mathscr{P}_k(U)} = k!\, \widehat{\sigma}_r A_k^U(d).$$

Indeed, at the inductive step its left side is

$$(k-1)! \sum\nolimits_i C_i \widehat{\sigma}_r A_{k-1}^U(d)(dI + Y_i)D_i.$$

Move $C_i$ through $\widehat{\sigma}_r$ and apply (B.2). No derivative is moved through the seed. The factor $1/(N^k k!)$ in the exponential proves (B.4). □

B.1. **The two modules as harmonic spaces.** To apply the construction to Bergeron's modules, we realize them as harmonic polynomials. Set $N = ab$ and let the wreath products $W_{a,b}$ and $W_{b,a}$ stabilize the rows and columns of a $b \times a$ rectangle. Put $U = \mathbb{C}[S_N/W_{a,b}]$ and $V = \mathbb{C}[S_N/W_{b,a}]$. For a row partition $\pi$, write $s_r = a^{-1} \sum_c w_{rc}$ and $x_{rc} = w_{rc} - s_r$. In $\mathbb{C}[W_0]$, let $I_\pi$ be generated by

$$e_j(x_{r1}, \ldots, x_{ra}) \quad (2 \leq j \leq a,\ 1 \leq r \leq b), \qquad e_\ell(s_1, \ldots, s_b) \quad (2 \leq \ell \leq b).$$

For a column partition $\kappa$, define $J_\kappa$ in the same way using $\eta_c = b^{-1}\sum_r w_{rc}$ and $y_{rc} = w_{rc} - \eta_c$. These quotients are the nested coinvariant coefficient algebras of (3.1). Their Fischer orthogonal complements give the induced harmonic modules

$$\mathcal{P} = \bigoplus_\pi I_\pi^\perp \otimes e_\pi \subset \mathscr{P}(U), \qquad \mathcal{Q} = \bigoplus_\kappa J_\kappa^\perp \otimes e_\kappa \subset \mathscr{P}(V),$$

with characters $P_{a,b}$ and $Q_{a,b}$, respectively. Here $e_\pi, e_\kappa$ are the coset basis vectors, and all complements use the global Euclidean Fischer metric.

Let $P_\mathcal{Q}$ be the orthogonal projection and suppose $\sigma_r(U) \subset \mathcal{Q}_r$. Define $T_d$ on $\mathcal{P}$ by the same contraction exponential, using compressed codomain multiplication $P_\mathcal{Q} C_i|_\mathcal{Q}$. Its blocks satisfy

$$T_d^{k+r,k} = N^{-k} P_{\mathcal{Q},k+r} \widehat{\sigma}_r A_k^U(d)|_{\mathcal{P}_k}. \tag{B.5}$$

To prove (B.5), note that derivatives preserve the orthogonal complement of every ideal $I$, because $\langle D_i h, f\rangle = \langle h, C_i f\rangle = 0$ whenever $h \perp I$ and $f \in I$. The simultaneous group action also preserves $\mathcal{P}$ and $\mathcal{Q}$. The operators $(dI + Y_i)D_i$ therefore preserve harmonicity. Since multiplication preserves each ideal $J_\kappa$, the identity $P_\mathcal{Q} C_i = P_\mathcal{Q} C_i P_\mathcal{Q}$ removes the intermediate projections onto $\mathcal{Q}$. Applying Proposition B.2 leaves precisely the final projection in (B.5).

B.2. **Why the final projection is needed.** The final projection in (B.5) cannot be omitted by extending the canonical map polynomial-linearly through the two coefficient quotients. Call a row partition and a column partition transverse if every row block meets every column block in exactly one position. Let $\mu\colon U \to V$ be the canonical incidence map, whose nonzero entries correspond to transverse pairs, and put $\widetilde{\mu} = I \otimes \mu$.

**Proposition B.3.** *For $a, b \geq 2$, the extension $\widetilde{\mu}$ does not map $\bigoplus_\pi I_{\pi,2} \otimes e_\pi$ into $\bigoplus_\kappa J_{\kappa,2} \otimes e_\kappa$.*

*Proof.* Fix transverse partitions $\pi, \kappa$, a row $R$ of $\pi$, and two blocks $K_1, K_2$ of $\kappa$. Choose

$$i \in R \cap K_1, \quad j \in R \cap K_2, \quad i' \in K_1 \setminus R, \quad j' \in K_2 \setminus R.$$

These four labels are distinct. Consider the quadratic relation

$$f = \sum\nolimits_{\ell \in R} w_\ell^2 - \frac{1}{a}\Big(\sum\nolimits_{\ell \in R} w_\ell\Big)^2 \in I_{\pi,2}.$$

The relation $f$ satisfies $\Lambda(f) = -2/a$, where

$$\Lambda(h) = [w_i w_j]h - [w_i w_{j'}]h - [w_{i'} w_j]h + [w_{i'} w_{j'}]h.$$

Every inner quadratic generator of $J_\kappa$ has zero cross-column coefficients, and its outer quadratic generator has constant coefficients on each pair of distinct column blocks. Hence $\Lambda$ annihilates both types of quadratic generator of $J_\kappa$. The functional $\Lambda$ also annihilates a multiple $(\sum_\ell w_\ell)(\sum_\ell c_\ell w_\ell)$ of global centering, whose coefficient of $w_\ell w_{\ell'}$ for $\ell \neq \ell'$ is $c_\ell + c_{\ell'}$. Thus $\Lambda$ is well defined on centered quadratics and $f \notin J_{\kappa,2}$. The $e_\kappa$ component of $\widetilde{\mu}(f \otimes e_\pi)$ is a nonzero scalar multiple of $f$, proving the assertion. □